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Properties and Application of the Caputo Fractional Operator

A project submitted to the department of Mathematics College of Education for Pure Sciences Ibn Al Haitham in partial fulfillment for the requirement of the Bachelor of Education Degree in Mathematics

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2019 1440

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قد تم بأشرافي في كلية التربية للعلوم الصرفة ابن الهيثم / قسم علوم الرياضيات

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التاريخ:-

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

فَقُلْتُ اسْتَغْفِرُوا رَبَّكُمْ إِنَّهُ كَانَ غَفَّارًا (*) يُرْسِلِ
السَّمَاءَ عَلَيْكُمْ مِدْرَارًا (*) وَيُمْدِدْكُمْ بِأَمْوَالٍ وَبَنِينَ
وَيَجْعَلْ لَكُمْ جَنَّاتٍ وَيَجْعَلْ لَكُمْ أَنْهَارًا (*) مَا لَكُمْ لَا
تَرْجُونَ لِلَّهِ وَقَارًا (*)

الإهداء

إلى من جرع الكأس فارغاً ليسقيني قطرة حب

إلى من كلّت أنامله ليقدم لنا لحظة سعادة

إلى من حصد الأشواك عن دربي ليمهد لي طريق العلم

إلى القلب الكبير (والدي العزيز)

إلى من أرضعتني الحب والحنان

إلى رمز الحب وبلسم الشفاء

إلى القلب الناصع بالبياض (والدتي الحبيبة)

تحية شكر

تتناثر الكلمات حبرا وحبا ..

على صفائح الأوراق ..

لكل من علمني ..

ومن أزال غيمة جهل مررت بها

..

برياح العلم الطيبة ..

ولكل من أعاد رسم ملامحي ..

وتصحيح عثراتي ..

أبعث تحية شكر واحترام .

CONTENTS

Abstract

In this paper, we introduce the definitions and important properties of fractional derivatives namely the Caputo which are necessary for understanding the fractional Calculus's rules and give several Caputo fractional derivative of functions and illustrate examples

Introduction

During the last few years it has been observed in many fields that any phenomena with strange kinetics cannot be described within the framework of classical theory using integer order derivatives. Recently, fractional differential equations have gained much attention since fractional order system response ultimately converges to the integer order system response. For high accuracy, fractional derivatives are then used to describe the dynamics of some structures. An integer order differential operator is a local operator. Whereas the fractional order differential operator is non local in the sense that it takes into account the fact that the future state not only depends upon the present state but also upon all of the history of its previous states. Because of this realistic property, the fractional order systems are becoming increasingly popular. Another reason in support of the use of fractional order derivatives is that these are naturally related to the systems with memory that prevails for most of the physical and scientific system models. Applications and models involving fractional derivatives can be found in probability physics, astrophysics, chemical physics [Oldham and Spanier (1974); Miller and Ross (1993); Podlubny (1999)] and various fields of engineering. Mainardi et al. (2008) provided a fundamental solution for the determination of probability density function for a general distribution of fractional time order system. Magin et al. (2008) solved the Bloch-Torrey equation after incorporating a fractional order Brownian model of diffusivity.

Recently, Chen et al. (2010) have developed a fractal derivative model of anomalous diffusion and the fundamental solution of this model is compared with the existing method to establish its computational efficiency.

After the Introduction, this paper is organized as follows, In Chapter1: Preliminaries we remind some techniques and special functions which are necessary for the understanding of the fractional calculus's rules.

Chapter2: Basic fractional calculus give the definition of Caputo differ integral their most important properties.

Chapter Three, we give several differential of simple functions, such as exponential function, constant function, Power function, sine function, and cosne function. And illustrate examples.

CHAPTER ONE

Preliminaries

1.1 The Gamma Function

Definition(1.1.1): The extension of a factorial function for the nonnegative integers is known as a gamma function , which is denoted by the symbol Γ. for α>0 , the gamma function Γ(α) is defined as follows; Γ(α)=$\int_0^{\infty} x^{\propto-1}\ e^{-x}\ dx$ where α>0

Properties(1.1.2): Some properties of the gamma function

1.Γ(α)=(α-1)Γ(α-1)

2.Γ(1/2)=$\sqrt{\pi}$

3.Γ(n)=(n-1)

4.Γ(α+1)=αΓ(α)

5.Γ(α)=$\frac{\Gamma(\alpha+1)}{\alpha}$

Examples(1.1.3): Find 1)Γ(7) 2)Γ(2-5) 3)Γ(-0.5) 4)Γ(0.4)

Solution:

1)Γ(α+1)=α! → Γ(7)=Γ(6+1)=6!=720

2)Γ(α+1)=αΓ(α)

Γ(2.5)=Γ(1.5+1)=(1.5)Γ(1.5)=(1.5)Γ(0.5+1)=(1.5)(0.5)Γ(0.5)=0.75 $\sqrt{\pi}$=0.866226

3)Γα=$\frac{\Gamma(\alpha+1)}{\alpha}$ → Γ(-0.5)=$\frac{1}{-0.5}\Gamma(0.5) = -2\sqrt{\pi}$

4)Γα=$\frac{\Gamma(\alpha+1)}{\alpha}$ → Γ(0.4)=$\frac{1}{0.4}\Gamma(1.4)$=$\frac{5}{2}(0.8873) = 2.21825$

Example(1.1.4): Use the Gamma function to calculate some integrals 1)$\int_0^\infty x^6\, e^{-2x} dx$

Solution:

Put 2x=y → dx=$\frac{1}{2}dy$

Then $\int_0^\infty x^6\, e^{-2x} dx = \frac{1}{2}\int_0^\infty (\frac{y}{2})^6 e^{-y} dy$

$$=\frac{1}{2^7}\int_0^\infty y^6 e^{-y}\, dy$$

$$=\frac{1}{2^7}\int_0^\infty y^{7-1}\, e^{-y} dy$$

But Γ(α)=$\int_0^\infty y^{\alpha-1}\, e^{-y} dy = \frac{\Gamma(7)}{2^7} = \frac{45}{8}$

2)$\int_0^\infty \frac{y^c}{c^y} dy$

Solution:

Put $c^y = e^x \rightarrow x = y$ln(c) → dy=$\frac{dx}{\ln(c)}$

Then $\int_0^\infty \frac{y^c}{c^y} dy = \frac{1}{\ln(c)}\int_0^\infty (\frac{x}{\ln(c)})^c\, e^{-x} dx$

$$=\frac{1}{[\ln(c)]^{c+1}}\int_0^\infty x^{c+1-1}\, e^{-x} dx$$

But Γ(α)=$\int_0^\infty x^{\alpha-1} e^{-x} dx = \frac{\Gamma(c+1)}{[\ln(c)]^{c+1}}$

1.2 The Beta Function

Definition(1.2.1): The Beta function denoted by β(m,n) or B(m,n) is defined as β(m,n)=$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, (m>0,n>0)

Properties(1.2.2): Some Proposition of the Beta Function

1)β(m,n)=β(m,n)

2)β(m,n)=2$\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\,\theta d\theta$

3)β(m,n)=$\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

4)β(m,n)=$\int_0^{1} \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

Remark (1.2.3) The relation between the beta function and the function gamma

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\ , x>0\ ,y>0$$

Example(1.2.4): Use the Beta function to calculate some integrals $\int_0^1 x^4(1-\sqrt{x})^5 dx$

Solution:

Let $\sqrt{x}=t\ \rightarrow x=t^2$ so that dx=2t dt

$$\int_0^1 x^4(1-\sqrt{x})^5\,dx=\int_0^1 (t^2)^4(1-t)^5(2t\,dt)$$

$$=2\int_0^1 t^9(1-t)^5 dt$$

=2β(10.6)

$$=\frac{\Gamma(10)\Gamma(6)}{\Gamma(16)} = 2x\frac{9!5!}{15!}$$

$$\therefore \int_0^1 x^4(1\sqrt{x})^5 dx = \frac{1}{15015}$$

<u>1.3 The mittagt-Leffer Function</u>

<u>Definition(1.3.1):</u> In mathematics , the Mittag-Leffler function $E_{\alpha,\beta}$ is a special function , a complex parameters α and β . It may be defined by the following series when the real part of α is strictly positive :$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} = \frac{z^k}{\Gamma(\alpha k+\beta)}$

<u>Properties(1.3.2):</u>Some Proposition of the Mittag-Leffler Function

1)$E_{1,1}(z) = e^z$

2)$E_{2,1}(z^2) = \cosh(z)$

3)$E_{2,2}(z^2) = \frac{\sinh(z)}{z}$

4)$E_{\alpha,1}(z) = E_{\alpha}(z)$

5)$E_{\frac{1}{2},1}(z) = e^{z^2} erfc(-z)$

<u>Example(1.3.3):</u> prove that $E_{1,2}(z) = \frac{e^z-1}{z}$

<u>Solution.</u>

We have $E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)}$

$$=\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)!}$$

$$= \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{1}{z}(e^z - 1)$$

CHAPTER TWO

Basic Fractional Calculus

2.1 The Caputo Differ integral

Definition(2.1.1): Suppose that α>0, t>a, α, a, t ∈ R.

The fractional operator

$$D^{\alpha}f = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{a}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\,d\tau, & n-1<\alpha<n\in N \\ \dfrac{d^{n}}{dt^{n}}f(t) & \alpha=n\ \in N, \end{cases}$$

Is called the Caputo fractional derivative or Caputo fractional differential operator of order α.

Example(2.1.2): Let a=0 , α=$\frac{1}{2}, (n-1), f(t)=t$ Then, applying formula we get $D^{\frac{1}{2}}t = \frac{1}{\Gamma(1/2)}\int_{0}^{t}\frac{1}{(t-\tau)^{1/2}}d\tau$ Taking intoaccount the properties of the Gamma function and using the substitution u=(t-τ)$^{\frac{1}{2}}$ the final result for the Caputo fractional derive of the function f(t)=t is obtained as

$$D^{\frac{1}{2}}t = -\frac{1}{\sqrt{\pi}}\int_{0}^{t}\frac{1}{(t-\tau)^{\frac{1}{2}}}d(t-\tau)$$

$$=-\frac{1}{\sqrt{\pi}}\int_{\sqrt{t}}^{0}\frac{1}{u}du^{2}$$

$$=\frac{1}{\sqrt{\pi}}\int_{o}^{\sqrt{t}}\frac{2u}{u}du$$

$$=\frac{2}{\sqrt{\pi}}(\sqrt{t}-0)$$

Thus, it holds $D^{\frac{1}{2}}t = \frac{2\sqrt{t}}{\sqrt{\pi}}$

2.2 Basic Properties of Caputo Differ integrals

- ***Interpolation***

Lemma(2.2.1). Let n-1<α<n, n $\in N,\ \alpha \in R$ and f(t) be such that $D^{\alpha} f(t)$ exists .Then the following properties for the Caputo operator hold $\lim_{\alpha \to n} D^{\alpha} f(t) = f^{(n)}(t)$,

$$\lim_{\alpha \to n-1} D^{\alpha} f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$$

Proof: The proof uses integration by Parts

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau$$

$$= \frac{1}{\Gamma(n-\alpha)} \left(-f^{(n)}(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} \Big|_{\tau=0}^{t} - \int_0^t -f^{(n+1)}(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} d\tau \right)$$

$$= \frac{1}{\Gamma(n-\alpha+1)} \left(f^{(n)}(0) t^{n-\alpha} + \int_0^t f^{(n+1)}(\tau)(t-\tau)^{n-\alpha} d\tau \right)$$

Now, by taking the limit for α→n and α→n-1, respectively , it follows

$$\lim_{\alpha \to n} D^{\alpha} f(t) = \left(f^{(n)}(0) + f^{(n)}(\tau) \Big|_{\tau=0}^{t} = f^{(n)} \right)$$

And

$$\lim_{\alpha \to n-1} D^{\alpha} f(t) = \left(f^{(n)}(0) t + f^{(n)}(\tau)(t-\tau) \Big|_{\tau=0}^{t} - \int_0^t -f^{(n)}(\tau) d\tau \right)$$

$$= f^{(n-1)}(\tau) \Big|_{\tau=0}^{t}$$

$$= f^{(n-1)}(t) - f^{(n-1)}(0).$$

- ***Linearity***

<u>Lemma(2.2.2.).</u> Let n-1<α<n, n∈ N, α, ƛ ∈ ₵ and the functions f(t) and g(t) be such that both $D^{\alpha}f(t)$ and $D^{\alpha}g(t)$ exist. The Caputo fractional derivative is a linear operator , i.e.,

$$D^{\alpha}\left(ƛf(t)+g(t)\right)=ƛD^{\alpha}f(t)+D^{\alpha}(t)$$

- ***Non-commutation***

<u>Lemma(2.2.3):</u> Suppose that n-1<α<n, m, n ∈ $N, \alpha \in R$ and the functions f(t) is such that $D^{\alpha}f(t)$exists. Then in general

$$D^{\alpha}D^{m}f(t)=D^{\alpha+m}f(t)\neq D^{m}D^{\alpha}f(t).$$

<u>Example(2.2.4):</u> Let f(t)=1 , and m=1 , α=1/2

Then $D^{1}D^{\frac{1}{2}}(1)=D^{1}\left[\frac{1}{\Gamma(1-\frac{1}{2})}t^{-\frac{1}{2}}\right]$

$$=D^{1}\left[\frac{1}{\Gamma(\frac{1}{2})}t^{-\frac{1}{2}}\right]=\frac{1}{\sqrt{\pi}}D^{1}\left[t\frac{-1}{2}\right]$$

$$=\frac{-\frac{1}{2}}{\sqrt{\pi}}t^{-\frac{1}{2}}=\frac{-1}{2\sqrt{\pi}}t^{-\frac{3}{2}}$$

But $D^{\frac{1}{2}}D^{1}[1]=D^{\frac{1}{2}}[0]=0.$

CHAPTER THREE

Examples of Caputo Fractional Derivatives

3.1 The constant Function

Theorem(3.1.1): For the Caputo Fractional derivative it holds $D^{\alpha}C = 0,\ C$ =const

Proof. As usual $0 < n-1 < \alpha < n, n \in N, \alpha \in R$ which means $n \geq 1$. Applying the definition of the Caputo derivative and since the n-th derivative $C^{(n)}$ $(n \in N, n \geq 1)$ of a constant equals 0, it follows

$$D^{\alpha}C = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{C^{(n)}}{(t-\tau)^{\alpha+1-n}}d\tau = 0$$

Example(3.1.2): Find $D^{\frac{1}{2}}2$

Solution:

By

$$D^{\alpha}C = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{C^{(n)}}{(t-\tau)^{\alpha+1-n}}\ d\tau = 0$$

Let n=1 $\qquad D^{\frac{1}{2}}2 = \frac{1}{\Gamma(1-\frac{1}{2})}\int_a^t \frac{2^{(1)}}{(t-\tau)^{\frac{1}{2}+1-1}}d\tau$

$$= \frac{1}{\Gamma(\frac{1}{2})}\int_a^t \frac{2}{(t-\tau)^{\frac{1}{2}}}\ d\tau$$

$$= \frac{1}{\sqrt{\pi}}\int_a^t 0\ d\tau$$

$$= \frac{1}{\sqrt{\pi}}0\Big|_a^t$$

$$= \frac{1}{\sqrt{\pi}}(0-0) = 0$$

Example(3.1.3): Find $D^{\frac{1}{3}}5$

Solution. By

$$D^{\alpha}C = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{C^{(n)}}{(t-\tau)^{\alpha+1-n}}\,d\tau = 0$$

Let n=1 $\quad D^{\frac{1}{3}}5 = \frac{1}{\Gamma(1-\frac{1}{3})}\int_a^t \frac{5^{(1)}}{(t-\tau)^{\frac{1}{3}+1-1}}\ d\tau$

$$= \frac{1}{\Gamma(\frac{2}{3})}\int_a^t 0\ d\tau$$

$$= \frac{1}{\Gamma(\frac{2}{3})} 0\Big|_a^t$$

$$= \frac{1}{\Gamma\left(\frac{2}{3}\right)}(0-0) = 0$$

3.2 The Power Function

Theorem(3.2.1): The Caputo fractional derivative of the power function satisfies

$$D^{\alpha}t^{\rho} = \begin{cases} \dfrac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}t^{\rho-\alpha} = D^{\alpha}t^{p} & n-1<\alpha<n, \rho>n-1, \rho\in R \\ 0 & n-1<\alpha<n, \rho\leq\ n-1, \rho\in N \end{cases}$$

Proof. Let $-1<\alpha<n, \rho>n-1, \rho\in R\ , \alpha\in R\ , n\in N$

The direct way reads $D^{\alpha}t^{\rho} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{(\tau^{\rho})^{(n)}}{(t-\tau)^{\alpha+1-n}}d\tau$

$=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\Gamma(\rho+1)}{\Gamma(\rho-n+1)}(\tau^{\rho-n})(t-\tau)^{n-\alpha-1}d\tau,$

And using the substitution $\tau = ƛt, 0\leq ƛ\leq 1$

$$D^{\alpha}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(n-\alpha)\Gamma(\rho-n+1)}\int_0^1 (ƛt)^{\rho-n}((1-ƛ)t)^{n-\alpha-1}t\ dƛ$$

$$= \frac{\Gamma(\rho+1)}{\Gamma(n-\alpha)\Gamma(\rho-n+1)}t^{\rho-\alpha}\int_0^1 ƛ^{\rho-n}(1-ƛ)^{n-\alpha-1}t\ dƛ$$

$$= \frac{\Gamma(\rho+1)}{\Gamma(n-\alpha)\Gamma(\rho-n+1)}t^{\rho-\alpha}\beta(\rho-n+1, n-\alpha)$$

$$= \frac{\Gamma(\rho+1)}{\Gamma(n-\alpha)\Gamma(\rho-n+1)}t^{\rho-\alpha}\frac{\Gamma(\rho-n+1)\Gamma(n-\alpha)}{\Gamma(\rho-\alpha+1)}$$

$$= \frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}t^{\rho-\alpha}$$

Example(3.2.2): Find $D^{\frac{1}{2}}t^2$

Solution. By $D^{\alpha}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}t^{\rho-\alpha}$

$$\rightarrow D^{\frac{1}{2}}t^2 = \frac{\Gamma(2+1)}{\Gamma(2-\frac{1}{2}+1)}t^{2-\frac{1}{2}}$$

$$= \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}$$

$$= \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}$$

Example(3.2.3): Find $D^{\frac{1}{4}}t^1$

Solution. By $D^{\alpha}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}t^{\rho-\alpha}$

$$D^{\frac{1}{4}}t^1 = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{4}+1)}t^{1-\frac{1}{4}}$$

$$= \frac{\Gamma(2)}{\Gamma(\frac{7}{4})}t^{\frac{3}{4}}$$

3.3 The Exponential Function

Theorem(3.3.1): Let $\alpha \epsilon R, n-1 < \alpha < n, n \in N, v \epsilon ₵$ Then the Caputo fractional derivative of the exponential function has the form

$$D^{\alpha} e^{vt} = v^{n} t^{n-\alpha} E_{1,n-\alpha+1}(vt),$$

Proof. $D^{\alpha} e^{vt} = D^{\alpha} \sum_{k=0}^{\infty} \frac{(vt)^k}{k!}$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t \left(\sum_{k=0}^{\infty} \frac{(v\tau)^k}{k!} \right)^{(n)} (t-\tau)^{n-\alpha-1} d\tau$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t \sum_{k=0}^{\infty} \frac{v^{n+k} \tau^k}{k!} (t-\tau)^{n-\alpha-1}\, d\tau$$

$$= \frac{v^n}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{v^k}{k!} \int_0^t \tau^k (t-\tau)^{n-\alpha-1} d\tau$$

Let $\tau = \lambda t, 0 < \lambda < 1 \rightarrow d\tau = t\, d\lambda$

$$\rightarrow = \frac{v^n}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{v^k}{k!} \int_0^1 (\lambda t)^k (t - \lambda t)^{n-\alpha-1} t\, d\lambda$$

$$= \frac{v^n}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{v^k}{\Gamma(k+1)} t^{n+k-\alpha} \int_0^1 \lambda^k (1-\lambda)^{n-\alpha-1}\, d\lambda$$

$$= \Gamma \frac{v^n t^{n-\alpha}}{(n-\alpha)} \sum_{k=0}^{\infty} \frac{v^k t^k}{(k+1)} \quad \beta(k+1, n-\alpha)$$

$$= \frac{v^n t^{n-\alpha}}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{(vt)^k}{\Gamma(k+1)} \cdot \frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(n+k-\alpha+1)}$$

$$= v^n t^{n-\alpha} \sum_{k=0}^{\infty} \frac{(vt)^k}{\Gamma(n+k-\alpha+1)}$$

Example(3.3.2): Find $D^{\frac{1}{2}} e^{1t}$

Solution: By

$$D^{\alpha} e^{vt} = v^n t^{n-\alpha} \sum_{k=0}^{\infty} \frac{(vt)^k}{\Gamma(n+k-\alpha+1)}$$

Let n=1 →

$$D^{\frac{1}{2}} e^{1t} = 1^1 t^{1-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(1t)^k}{\Gamma(1+k-\frac{1}{2}+1)}$$

$$= t^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\frac{3}{2}+k)}$$

Example(3.3.3): Find $D^{\frac{1}{3}} e^{2t}$

Solution: By

$$D^{\alpha} e^{vt} = v^n t^{n-\alpha} \sum_{k=0}^{\infty} \frac{(vt)^k}{\Gamma(n+k-\alpha+1)}$$

Let n=1→

$$D^{\frac{1}{3}} e^{2t} = 2^1 t^{1-\frac{1}{3}} \sum_{k=0}^{\infty} \frac{(2t)^k}{\Gamma(1+k-\frac{1}{3}+1)}$$

$$= 2t^{\frac{2}{3}} \sum_{k=0}^{\infty} \frac{(2t)^k}{\Gamma(\frac{5}{3}+k)}$$

3.4 Other Frequently Used Functions

Theorem(3.4.1): Let $\lambda \in ₵, \alpha \in R, n \epsilon N, n-1 < \alpha < n.$

Then

$$D^{\alpha} \sin\lambda\, t = -\frac{1}{2} i(i\lambda)^n t^{n-\alpha} (E_{1,n-\alpha+1}\big(i\lambda t\big) - (-1)^n (E_{1,n-\alpha+1}(-i\lambda t)^2)$$

Proof. The following representation of the sine function used

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in ₵$$

Now, using the linearity property of the Caputo fractional derivative and formula for the exponential function it can be shown that

$$D^{\alpha} \sin \lambda t = D^{\alpha} \frac{e^{i\lambda t} - e^{-i\lambda t}}{2i}$$

$$= \tfrac{1}{2i} (D^{\alpha} e^{i\lambda t} - D^{\alpha} e^{-i\lambda t})$$

$$= \frac{1}{2i} \Big(\big(i\lambda\big)^n t^{n-\alpha} E_{1,n-\alpha+1}\big(i\lambda t\big) - \big(-\lambda i\big)^n t^{n-\alpha} E_{1,n-\alpha+1}(-i\lambda t) \Big)$$

$$=-\tfrac{1}{2} i(i\lambda)^n t^{n-\alpha} \big(E_{1,n-\alpha+1}\big(i\lambda t\big) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t)\big)$$

In the same manner a formula for the Caputo derivative of the cosine function is received.

The corresponding representation is

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}$$

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