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Primely –S cancellation module

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بِسْمِ اللَّهِ الرَّحْمنِ الرَّحِيم

إِنَّمَا يَخْشَى اللَّهَ مِنْ
عِبَادِهِ الْعُلَمَاءُ ﴿28﴾

سورة فاطر / الآية (28)

الإهـــداء

إلى القلب الكبير الذي حمل اسراري...
إلى الحنان الذي منحني الدفئ والأستقرار...
إلى من أخص الله الجنه لمن ارضاها...

أمي الغالية

إلى الذي بذل جهد السنين سخياَ...
وصاغ من الأيام سلالم العلم...
لارتقى بها الى ذرى الحياه...

أبي العزيز

إلى الشمعات المضيئه التي أضاءت لي طريق العلم...

أساتذتي

إلى العيون التي تنظر لي بحب...
إلى من شجعني وغمرني بحنانه...
وهم اعز الناس الى قلبي...

إخوتي

إلى من سطروا معي أروع ذكريات وعشنا معا أحلى اللحظات...

أصدقائي الأعزاء

ذكرى محمود شيحان

الشكروالتقدير

اشكر الله العلي القدير الذي أنعم علي بنعمة العقل والدين فكان واجبا ان اشكر من مد لي يد العون في مجال البحث العلمي حيث اتقدم اليوم بكلمه الشكرلمشرفه البحث أ.م.د بثينه نجاد شهاب التي أستحقت منا كل التقديروالاحترام أعطيت الكثير وما زالت تعطي من وقتها وفكرها دون أنتظار الثناء او الشكر مميزه بحضورها ومبدعه بتقديم المساعده و رائعه بخبراتها لا تتعب ولا تمل عن تقديم المساعده وما هذه الكلمات الا تعبير بسيط عن تقديرنا لك سائلين المولى عز وجل ان يكتب لك التوفيق دوما في خطاك

.

ذكرى

CONTENTS

ABSTRST

Let R be a commutative ring with identity. In this thesis, the concept of P-Sc modules has been introduced. Many properties and characterization have been given. Also the relationship between P-Sc modules with other modules has been submitted.

INTRODUCTON

A.S. Mijbass in [1] has been introduced the concept of cancellation module. A.A Elewi, in [2] has been defined the concept of strongly cancellation module. In this thesis, we will present the concept of P-Sc module as follows:

An R-module M is called a P-Sc module if for each prime ideal I and each ideal J of R such that $IN = JN$, than $I = J$ for every sub module N of M.

Clearly, the class of strongly cancellation modules contains the class of P-Sc modules and we give an example to show that this inclusion is properly. However we shall give conditions under which the two classes are equivalent.

This thesis consists of two chapters.

Chapter one contains three section. In section one, we recalled some definitions and concepts to use them in our next sections.

Section tow we defined the concept of P-Sc modules with some characterization and properties section three, we studied the relationship between P-Sc modules and strongly cancellation modules, some result have been presented. Chapter tow, the Relation between P-Sc modules and certain types of Modules namely, locally projective and flat modules, this chapter consist of tow section. In section one, we presented some results which are the explain behavior of P-Sc modules with another type modules such as locally projective. In section tow we introduced the relation between P-Sc module and Flat modules, some results about this relation are provided.

CHAPTER ONE

Basic Concepts and Results

Basic Concepts and Results

In this chapter we shall introduce the concept of P-Sc modules**,** which is a generalization of strongly cancellation modules. Our con cern in this chapter is to study the relationships between P-Sc modules and strongly cancellation modules.

The chapter contains three sections. In section one, we recalled some definitions and basic results

Section tow, we introduced the definition of P-Sc module and some characterization for a module to be P-Sc module, and so on many properties and results about this concept are presented.

Section three, the relationships between P-Sc modules and strongly cancellation modules are studied.

S_1: Basic concepts and Results

In this section many concepts and results which are needed in our next work are recalled

Definition (1, 1, 1):

An R-module M is said to be cancellation if AM=BM, where A and B are tow ideals of R, then A=B. [1]

Definition (1, 1, 2):

Let M be an R-module and N be a sub module of M. then M is called strongly cancellation module, if AN=BN, where A, B are tow ideals of R, then A=B.[2]

Definition (1, 1, 3):

Recall that the element x∈ M over R(R is an integral domain) is called torsion element if there exists 0≠r∈R such that $rm = 0$, and m is called a non-torsion element if $rm \neq 0 \quad \forall r \in R$. [3]

Definition (1, 1, 4):

The annihilator of L is $annL = [0\colon L]$. L is faithful if $annL = 0$. [4]

Definition (1, 1, 5):

If R has only one maximal ideal, then R is called a local ring.[5]

Definition (1, 1, 6):

Let M and N be tow R-module. The trace of N over M denoted by $T_m(N) = \sum_{\lambda \in \Lambda} \theta_\lambda(M)$, where the sum is taken on all θ_λ in Hom (M, N).

In the special case, if N=R, then the trace of M over R written by T (M) instead of $T_M(R)$ [6].

Proposition (1, 1, 7):

If M is an R-module and T(M) is a multiplication ideal of R, which contain a non-zero divisor, then M is strongly cancellation module.[2]

Definition (1, 1, 8):

An R-module M is called flat module if for every monomorphism $f: A \longrightarrow B$, where A and B are tow R-modules, then $f \otimes 1_M : A \otimes M \longrightarrow B \otimes M$ is also a monomorphism [3, P.257].

Definition (1, 1, 9):

Let M be R-module and N⊆M, N is sub module of M, then M is called multiplication module on R if ∀N⊆M ∃ideal I of R such that N=IM. [5]

S_2: P-Sc modules

In this section we state the definition of P-Sc module with some examples about this concept. Moreover we prove some basic properties of P-Sc modules.

Definition (1, 2, 1):

An R-module M is named a primely –Sc module whenever AN=BN, with A prime ideal of R and B is any ideal of R, implies A=B.

Examples (1, 2, 2):

(1) Z_6 as a Z_{12}-module is P-Sc module

—The sub module $(\bar{3})$, There exists tow ideals are $(\bar{2})$, $(\bar{8})$ in Z_{12} such that

$(\bar{2})(\bar{3})= (\bar{8})(\bar{3}) = 0$

$(\bar{2})=(\bar{8})$

—The sub module $(\bar{2})$, There exists tow ideals are $(\bar{2})$,$(\bar{7})$ in Z_{12} such that

$(\bar{2})(\bar{2})= (\bar{7})(\bar{2})$

$(\bar{2})= (\bar{7})$

—The sub module $(\bar{1})$, There exists tow ideals are $(\bar{3})$, $(\bar{9})$ in Z_{12} such that

$(\bar{3})(\bar{1})= (\bar{9})(\bar{1})$

$(\bar{3}) = (\bar{9})$

—The sub module $(\bar{0})$, There exists tow ideals are $(\bar{3})$,$(\bar{9})$ in Z_{12} such that

$(\bar{3})(\bar{0})= (9)(\bar{0})$

$(\bar{3})=(9)$

(2) Z_4 as a Z-module is not P-Sc module

(3) Q as a Z-module is not P-Sc module

Since $< p > Q = Q$ for any prime number P in Z. It is clear $< p > Q \subseteq Q$

Now, Let x∈ Q, Then x=$\frac{a}{b} = \frac{pa}{pb} = p\,\frac{a}{Pb}$ $\in< p > Q$ where a, b ∈Z, Implies $Q \subseteq< p > Q$, There for Q is not P-Sc module, However $< p >\neq Z$.

For cyclic modules we have the following result

Proposition (1, 2, 3):

Every cyclic module generated by a non-torsion element is primely-Sc module.

Proof:

Let M=$< m >$, where m is a non-torsion element and Let A$< m >= B < m >$, where A is prime ideal of R and B is any ideal of R. am∈ $B < m >$ for all a ∈A, Then $am = bm$ where $b \in B$, implies $am - bm = 0$.
Therefore $(a - b)m = 0$, but m is non-torsion element, then a-b=0, implies a=b and hence A⊆B. similarly B⊆A.

We shall show by an example that the condition N (sub module of an R-module M) is generated by a non-torsion element in proposition (1, 2, 3) can not dropped.

Example (1, 2, 4):

$Z_4 = (\bar{1})$, $\bar{1}$ is a torsion element in Z_4

$(4)\ (\bar{2}) = (\bar{0})(\bar{2}) = 0$ but (4)≠ (0)

In the following theorem we give some characterizations of P-Sc module.

Theorem (1, 2, 5):

Let M be an R-module, the following statements are equivalent:

(1) M is P-Sc module

(2) If AN⊆ BN such that B is prime ideal and A any ideal Then A⊆B

(3) $< a > N \subseteq BN \Rightarrow a \in B$

(4) (AN: N) =A, A is prime

(5) (AN: BN) = (A: B), B is an ideal of R

Proof: (1) ⇒ (2) Assume that M is primely-Sc module and AN⊆BN

BN= $AN + BN = (A + B)N \Rightarrow B = A + B$, implies A⊆B.

(2)⇒ (3) let $< a > N \subseteq BN$. then $< a > \subseteq B$ by (2) hence a ∈B

(3)⇒ (4) let x∈ (AN: N), Then XN ⊆ AN by (3) x ∈A, Hence (AN: N) ⊆ A.

On the other side, if x ∈A, then xN ⊆ AN ⇒ x∈ (AN: N) and hence

(AN: N) =A.

(4)⇒ (5) let x∈ (A: B). Then x∈((AN:N):B) (since (AN:N)=A by (4), implies x∈(AN:BN) if x∈(AN:BN)=((AN:N):B) and by (4) x∈(A:B)⇒ (AN:BN)=(A:B)

(5)⇒ (1) Let AN=BN, then (AN: BN) =R, implies (A: B) =R ⇒ B⊆A, similarly A⊆B, Then A=B

Proposition (1, 2, 6):

Let M be an R-module and f: M⟶N be epimorphism such that N is P-Sc module, then M is P-Sc module such that $f\ (M)\ = N$

Proof: let A, B are tow ideals of R and K sub module of M, let $AK = BK$ then f$(AK) = f(BK)\ \Rightarrow\ Af(K) = Bf(K)$ since $f(K) \subseteq N$ and N is P-Sc module ⇒$A = B$

Form the last proposition we get the following

Corollary (1, 2, 7):

If M is P-Sc module, then every direct summand of M is P-Sc module.

Proof:

Let N be sub module of module M and let K be sub module of N, to prove N is P-Sc module, suppose that AK=BK where A, B are tow ideal of R, since M is P-Sc module and K sub module of M, implies A=B, then N is P-Sc module.

Proposition (1, 2, 8): If $M_1 \cong M_2$ then M_1 is P-Sc module iff M_2 is P-Sc module.

Proof:(⇒) since$M_1 \cong M_2$, Then there exists an isomorphism function f:$M_1 \longrightarrow M_2$ suppose AN=BN where A is prime ideal of R and B any ideal of R and N be sub module of M_2, as f is onto, there exists a sub module K of M_1 such that f(K)=N and hence Af(K)=Bf(K), but f is homomorphism, thus f(AK)=f(BK), since f is one-to-one then AK=BK and K be sub module of M_1 and M_1 is P-Sc module, Thus A=B.

(⇐) The proof by similar way

S_3: P-Sc modules and strongly cancellation modules.

In this part, the relationship between tow concepts namely, P-Sc module and strongly cancellation modules will be examin more closely and we try to lie some light on this relation.

Proposition (1, 3, 1)

Let M be an R-module then M is strongly cancellation R- module if and only if faithful and P-Sc module.

The Jacobson radical of R is the intersection of all maximal ideals of R, J(R) $=\cap\{I: I$ is maximal ideal of R}. The Jacobson radical of M is the intersection of all maximal sub modules of M, $J(M) = \cap\{N: N$ is maximal sub modules of M}, and $J(M) = M$, if M has no maximal sub modules.

The following proposition and it is Corollary gives a necessary condition for a module to be relatively cancellation module.

Proposition (1, 3, 2):

Let M be a non-zero module on R. If M is P-Sc module. Then (J (N):$_R N) = R$.

Proof: Let AN=N for some prime ideal A. Then A=R, which is contradiction! (Since M is P-Sc module). Therefore AN$\neq N$ for all maximal ideals A of R. Now (J(N):N)=$(\cap_{\lambda\in\wedge} A_\lambda N: N) = \cap_{\lambda\in\wedge} (A_\lambda N: N)$ by [6,Ex.14,P.240]. But $A_\lambda \subseteq (A_\lambda N: N)$, then$(A_\lambda N: N) = A_\lambda$ $\forall\lambda\in\wedge$, Therefore (JN: N)=$\cap_{\lambda\in\wedge} A_\lambda$=J(R).

Corollary (1, 3, 3):

If M is P-Sc module on R, then $ann_R(N) \subseteq J(R)$, for every sub module N of M

Corollary (1, 3, 4):

If M is P-Sc module over R, then $ann_R(N) \subseteq J(R)$, and therefore $ann_R(N)$ is a small ideal of R.

Proof: By proposition (1,3,4), (J(N):N)=J(R), $ann_R(N) \subseteq (J(N): N)$, then $ann(N) \subseteq J(R)$.

Corollary (1, 3, 5):

If M is a P-Sc module over R, and $ann_R(N)$ is a maximal ideal of R, then R is a local ring.

Proof: It is clear, so it is omitted.

The two classes of strongly cancellation modules are coincides over rings which have zero Jacobson radical. This is what we shall show in the following result.

Proposition (1, 3, 6):

Let M be an R-module and J(R) =0, M is P-Sc module iff M is strongly Cancellation module.

Proof:

Suppose M is relatively cancellation module. Then $ann_R(N) \subseteq J(R) = 0$. Therefore ann(N) =0 by proposition(1,3,2). M is strongly cancellation module.

The converse is clear, since every cancellation is P-Sc module.

Corollary (1, 3, 7):

Every P-Sc module over the ring of all integers is strongly cancellation module.

Proof: It is clear, so it is omitted.

Proposition (1, 3, 8):

If every cyclic R-module is P-Sc module, then R is a local ring.

Proof:

Let A and B tow maximal ideals of R. since R/A is a simple module and $ann(R/A) = A$. R/A is cyclic module, then R/A is P-SC module.

Therefore A⊆J(R) by Corollary (1, 3, 4). But A is maximal of R, then $J(R) = A$.

In the same way we can prove that$J(R) = B$. Hence R is a local ring.

CHAPTER TOW

The Relation between P-Sc modules and certain types of Modules

The Relation between P-Sc modules and certain types of Modules

The purpose of this chapter is to investigate the relation between P-Sc modules and some types of modules such as locally projective modules and flat modules.

The chapter, included tow section .in section one, we shall discuss the relation between P-Sc modules and a locally projective module that is beside proving under certain condition, a locally projective module is P-Sc module, see proposition (2.1.5).

Next, in the second section we shall consider the validity of strongly cancellation and P-Sc properties in the class of flat modules. Namely we shall show that the tow properties of strongly cancellation and P-Sc equivalent in this class.

S_1: P-Sc modules and locally projective modules.

Azumaya in [6] has been defined an R-module M to be locally projective module if for all epimorphism f; $f: A \longrightarrow B$ where $A\ and\ B$ are tow R-modules, for all homomorphism g ; $g: M \longrightarrow B$ and $\forall\ x \in M$, a homo h ; $h: M \longrightarrow A$ such that $foh(x) = g(x)$ that means the following diagram is commutative.

It is clear that every projective module is locally projective module, but the converse is not true.

In this section the P-Sc property for locally projective modules will be studied.

First we need to recall the following concept, [7]. The homomorphism α, α:A⟶B is called split point wise, where A and B are tow R-modules if for each b ∈B, there exists a homomorphism β, $\beta: B \longrightarrow A$ depends on b, such $that\ (\alpha o\ \beta)(b) = b$.

Next the following proposition is also needed in [7].

<u>**Proposition (2.1.1):**</u>[7]

Let M be an R-module. Then the following statements are equivalent:

(1) M is locally projective module.

(2) For all R-modules A, all epimorphosis α:A⟶B split point wise.

(3) For each $m \in M$, there exists a family $\{x_i\}_{i \in I}$, $x_i \in M, \forall i$ and a family $\{\theta_i\}$, $\theta_i \in M^*, \forall i$ Such that:

(a) $\theta_i(m) \neq 0$ For only finite number of $i \in I$

(b) $m = \sum \theta_i(m)x_i$.

In following example we explain that the projective modules and the locally projective modules are not necessary P-Sc modules.

Example (2, 1, 2):

Let $M = Z_6$ be a module over R=Z_6 and $Z_6 = A \oplus B$, where $A = \{\bar{0}, \bar{2}, \bar{4}\}$ and $B = \{\bar{0}, \bar{3}\}$ B is projective module over R. But B is not P-Sc module over R. In fact $< \bar{3} > B =< \bar{3} >$ and $< \bar{1} > B =< \bar{3} >$, where $< \bar{3} >$ is prime ideal in Z_6. But $< \bar{3} > \neq < \bar{1} >$

In order to give sufficient condition for locally projective module to be P-Sc module, we need the following lemmas and definition.

Recall that I is called pure ideal of R, if $I \cap B = BI$, for all ideals B in R, [5].

Lemma (2, 1, 3): [7].

Let M be a locally projective R-module. If $a \in M$, than a ∊ aT(M).

Lemma (2,1,4):

If M is locally projective R-module, than $T(M)$ is a pure ideal of R.

Proof:

Let $x \in T(M)$. Then $x = \sum_{i \in I} \theta_i(m_i), m_i \in$M, $\theta_i \in M^* = Hom(M, R)$, implies $m_{i,} = ym_i$ by lemma (2, 1, 3), where $y \in T(m)$. Therefore $x = \sum_{\theta_i}(ym_i) = y(\sum_{i \in I} \theta_i(m_i)) = yx.$,Hence T(M) is pure ideal of R.

Proposition (2, 1, 5):

If M is locally projective module and T (M) contains non-zero divisor of R, then M is P-Sc module.

Proof: T (M) is a pure ideal in R by lemma (2, 1, 4). Then T(M) is multiplication ideal [8 , corollary (1.8)].

Therefore it is easily to show that M is P-Sc module.

Corollary (2, 1, 6):

Every non-zero locally projective module over an integral domain is P-Sc module.

Proof: let M be a locally projective module over an integral domain R. T (M)$\neq$ 0 and R is an integral domain, then T (M) contains non-zero divisor of R. Therefore M is P-Sc module by proposition (2, 1, 5).

S_2: P-Sc modules and Flat modules

In this section we shall study the relation between the P-Sc modules and flat modules. We shall show that an R-module M is P-Sc module if and only if M is faithfully flat see proposition (2, 2, 2), where an R-module M is called faithfully flat if one of the tow following equivalent statements are satisfied [9, proposition (2, 3, 1),p_{24}].

(1) M is flat module and for all modules N of R, if M⊗N=0, then N=0

(2) M is flat module and PM⊆ M for all maximal ideals P of R.

Let us start by the following remark:

Remark (2, 2, 1):

Every faithfully flat module is faithful module.

Proof:

Let x∈ $ann_R(M)$. Then $xM = 0$ Now, $M \otimes Rx = xM \otimes R = 0 \otimes R = 0$. Then Rx=0 (since M is faithfully flat module).Therefore x=0 and hence $ann(M) = 0$

The converse of remark (2, 2, 1) is not true in general. For example:
Q is flat module over Z. But is not faithfully flat module over Z. Because (p) Q=Q for all prime number P in Z.

The following proposition proves the P-Sc modules and strongly cancellation modules are equivalent in the class of flat modules.

Proposition (2, 2, 2):

If M is a faithfully flat R-module, then M is P-Sc module

Proof:

Let M be a P-Sc module. Then $PN \subset N$ for some ideal p of R. Therefore $PN \subset N$ for all maximal ideals P of R. Hence M is faithfully flat by definition of faithfully flat.

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المستخلص

لتكن R حلقه أبداليه ذات عنصر محايد. في هذا البحث مفهوم موديول حذف اولي من نوع S قد تم تقديمه . العديد من الخواص و المميزات قد اعطيت.ايضا العلاقه بين موديول حذف اولي من النوع S وانواع اخرى من الموديولات قد تم تقديمه.