Partial Differential Equations

المعادلات التفاضلية الجزئية

المرحلة الثالثة

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Chapter One

Methods of Solving Partial Differential Equations

Contents

Section(1.1): Origin of Partial Differential Equations

(1.1.1) Introduction:

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

(1.1.2) Definition Partial Differential Equations(P.D.E.)

An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a (P.D.E.).

For examples of partial differential equations we list the following:

1. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy$

2. $(\frac{\partial z}{\partial x})^2+\frac{\partial^3 z}{\partial y^3}=2x(\frac{\partial z}{\partial y})$

3. $z(\frac{\partial z}{\partial x})+\frac{\partial z}{\partial y}=x$

4. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz$

5. $\frac{\partial^2 z}{\partial x^2}=(1+\frac{\partial z}{\partial y})^{\frac{1}{2}}$

6. $y\{(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2\}=z(\frac{\partial z}{\partial y})$

(1.1.3) <u>Definition: Order of a Partial DifferentialEquation (O.P.D.E.)</u>

The order of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation.

The equations in examples (1),(3),(4) and (6) are of the first order ,(5) is of the second order and (2) is of the third order.

(1.1.4)<u>Definition: Degree of a Partial DifferentialEquation (D.P.D.E.)</u>

The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized, i.e made free from radicals and fractions so for as derivatives are concerned. in (1.1.2), equations (1),(2),(3) and (4) are of first degree while equations(5) and(6) are of second degree.

(1.1.5) <u>Definition: Linear and Non-Linear Partial Differential Equations</u>

A partial differential equation is said to be (Linear) if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied . Apartial differential equation which is not linear is called a(nonlinear) partial differential equation.

In (1.1.2), equations (1) and (4) are linear while equation (2),(3),(5) and (6) are non-linear.

(1.1.6) Notations:

When we consider the case of two independent variables we usually assume them to be x and y and assume (z) to be the dependent variable. We adopt the following notations throughout the study of partial differential equations.

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y} \text{ and } t = \frac{\partial^2 z}{\partial y^2}$$

In case there are n independent variables, we take them to be $x_1, x_2, \dots \dots \dots, x_n$ and z is than regarded as the dependent variable. In this case we use the following notations:

$$p_1 = \frac{\partial z}{\partial x_1}, p_2 = \frac{\partial z}{\partial x_2}, \dots \dots \dots, p_n = \frac{\partial z}{\partial x_n}$$

Sometimes the partial differentiations are also denoted by making use of suffixes. Thus we write :

$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

and so on.

(1.1.7) Classification of First Order p.d.es into:

linear, semi-linear ,quasi-linear and non-linear equations

linear equation: A first order equation $f(x, y, z, p, q) = 0$

Is known as linear if it is linear in p , q and z , that is ,if given equation is of the form:

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$$

for example:

1. $yx^2p + xy^2q = xyz + x^2y^3$
2. $p + q = z + xy$

are both first order L.P.D.Es

Semi-linear equation: A first order p.d.e. $f(x, y, z, p, q) = 0$

Is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only. i.e if the given equation is of the form:

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

for example:

1. $xyp + x^2yq = x^2y^2z^2$
2. $yp + xq = \frac{x^2y^2}{z^2}$

are both semi-linear equations

Quasi-linear equation: A first order p.d.e.$f(x, y, z, p, q) = 0$

Is known as quasi-linear equation, if it is linear in p and q. i.e if the given equation is of the form:

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

for example:

1. $x^2zp + y^2zq = xy$

2. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

are both quasi-linear equation.

***Non-linear equation:** A first order p.d.e $f(x, y, z, p, q) = 0$ which does not come under the above three types ,is known as a non-linear equation.

for example:

1. $p^2 + q^2 = 1$

2. $pq = z$

3. $x^2p^2 + y^2q^2 = z^2$

are all non-linear p.d.es.

Section(1.2):Derivation of Partial Differential Equation by the Elimination of Arbitrary Constants

For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a and b,the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations $F(x, y, z, a, b) = 0$,$\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$

The equation free from a and b will be the required partial differential equation.

Three situations may arise:

Situation (1):

When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.

Example: Consider $z = ax + y$ …………(1)

where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. x, we get

$\frac{\partial z}{\partial x} = a$ …………(2)

Differentiating (1) partially w.r.t. y, we get

$\frac{\partial z}{\partial y} = 1$ …………(3)

Eliminating abetween (1) and (2) yields

$$z = x\left(\frac{\partial z}{\partial x}\right) + y \qquad \ldots\ldots\ldots\ldots(4)$$

Since (3) does not contain arbitrary constant, so (3) is also partial diff. equation under consideration thus, we get two p.d.es (3) and (4).

Situation (2):

When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial diff. eq. of order one.

Example: Eliminate a and b from

$$az + b = a^2x + y \qquad \ldots\ldots\ldots\ldots(1)$$

Differencing (1) partially w.r.t. x and y, we have

$$a\left(\frac{\partial z}{\partial x}\right) = a^2 \qquad \ldots\ldots\ldots\ldots(2)$$

$$a\left(\frac{\partial z}{\partial y}\right) = 1 \qquad \ldots\ldots\ldots\ldots(3)$$

Eliminating a from (2) and (3), we have

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$$

which is the unique p.d.e. of order one.

Situation (3):

When the number of arbitrary constants is greater than the number of independent variables. Then the elimination of arbitrary constants leads to a partial differential equations of order usually greater than one.

Example: Eliminate a, b and c from

$z = ax + by + cxy \qquad \ldots\ldots\ldots\ldots(1)$

Differentiating (1) partially w.r.t. x andy we have

$\frac{\partial z}{\partial x} = a + cy \quad \ldots\ldots\ldots\ldots(2) \qquad \frac{\partial z}{\partial y} = b + cx \quad \ldots\ldots\ldots\ldots(3)$

from (2) and (3) $\qquad \frac{\partial^2 z}{\partial x^2} = 0 \; \frac{\partial^2 z}{\partial y^2} = 0 \quad \ldots\ldots\ldots\ldots(4)$

$\frac{\partial^2 z}{\partial x\,\partial y} = c \quad \ldots\ldots\ldots\ldots(5)$

Now, (2) and (3) $x\frac{\partial z}{\partial x} = ax + cxy$ and

$$y\frac{\partial z}{\partial y} = by + cxy$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \underbrace{ax + by + cxy} + cxy$$

from (1) and (5)

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\frac{\partial^2 y}{\partial x\,\partial y} \qquad \ldots\ldots\ldots\ldots(6)$

Thus, we get three p.d.es given by (4) and (6) which are all of order two.

… Examples …

Example1: Find a p.d.e. by eliminating a and b from

$$z = ax + by + a^2 + b^2$$

Sol. Given $z = ax + by + a^2 + b^2 \quad \ldots\ldots\ldots\ldots(1)$

differentiating (1) partially with respect to x and y,

we get $\qquad \frac{\partial z}{\partial x} = a \qquad$ and $\qquad \frac{\partial z}{\partial y} = b$

substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain

$$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2$$

which is required p.d.e.

Example2: Eliminate arbitrary constants a and b from

$z = (x - a)^2 + (y - b)^2$ to form the p.d.e.

Sol. Given $z = (x - a)^2 + (y - b)^2$ …………(1)

differentiating (1) partially with respect to x and y, to get

$\frac{\partial z}{\partial x} = 2(x - a)$, $\frac{\partial z}{\partial y} = 2(y - b)$

Squatring and adding these equations, we have

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4(x - a)^2 + 4(y - b)^2$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4[(x - a)^2 + (y - b)^2]$$

$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4z$ using (1)

Example 3: from p.d.es by eliminating arbitrary constants a and b from the following relations:

(a) $z = a(x + y) + b$ (b) $z = ax + by + ab$

(c) $z = ax + a^2y^2 + b$ (d) $z = (x + a)(y + b)$

Sol. (a) Given $z = a(x + y) + b$ …………(1)

Differentiating (1) w.r.t.x and y, we get

$\frac{\partial z}{\partial x} = a$,$\frac{\partial z}{\partial y} = a$

eliminating a between these, we get

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ which is the required p.d.e.

(b)Try by yourself (c)Try by yourself (d)Try by yourself

… Exercises …

Ex.(1): Eliminate a and b from $z = axe^{y} + \frac{1}{2}a^2e^{2y} + b$ to form the partial differential equation.

Ex.(2): Eliminate h and k from the equation$(x-h)^2 + (y-k)^2 + z^2 = \alpha^2$to form the p.d.e.

Ex.(3): Eliminate a and b from the following equations to form the p.d.es

(a)$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (b)$2z = (ax+y)^2 + b$(c)$\log(az-1) = x + ay + b$

Ex.(4): Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial diff. eqs

(1) $z = Ae^{pt}\sin px$, (p and A)

(2) $z = Ae^{-p^2t}\cos px$, (p and A)

(3) $z = ax^3 + by^3$, (a and b)

(4) $4z = \left[ax + \left(\frac{y}{a}\right) + b\right]^2$,(a and b)

(5) $z = ax^2 + bxy + cy^2$, (a,b,c)

Section (1.3): Methods for solving linear and non-linear partial differential equations of order one

(1.3.1) Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are function of x, y, z.

A quasi-linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are function of x, y, z. Such a partial differential equation is known as (Lagrange equation), for example: * $xyp + yzq = zx$

* $(x - y)p + (y - z)q = z - x$

(1.3.2) Working Rule for solving $Pp + Qq = R$ by Lagrange's method

Step 1. Put the given quasi-linear p.d.e. of the first order in the standard form $Pp + Qq = R$ …………(1)

Step 2. Write down Lagrange's auxiliary equations for (1) namely

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{…………(2)}$$

Step 3. Solve (2) by using the method for solving ordinary differential equation of order one. The equation (2) gives three ordinary diff. eqs. every two of them are independent and give a solution.

Let $u(x,y,z) = a$ and $v(x,y,z) = b$, then the (general solution) is $\emptyset(u,v) = 0$, wher $\emptyset$ is an arbitrary function and the complete solution is $u = \alpha v + \beta$ where α, β are arbitrary constant.

<u>Ex.1:</u> Solve $\mathbf{2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x}$

Sol. Given $2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x$ …………(1)

The Lagrange's auxiliary for (1) are

$\frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{2x}$ …………(2)

Taking the first two fractions of (2), we have

$\frac{dx}{2} = \frac{dy}{-3} \rightarrow -3dx - 2dy = 0$ …………(3)

Integrating (3), $-3x - 2y = a$ …………(4)

a being an arbitrary constant

Next, taking the first and the last fractions of (2), we get

$\frac{dx}{2} = \frac{dz}{2x} \rightarrow xdx = dz \rightarrow xdx - dz = 0$ ……(5)

Integrating (5), $\frac{x^2}{2} - z = b$ …………(6)

b being an arbitrary constant

From (4) and (6) the required general solution is

$$\emptyset(a,b) = 0 \rightarrow \emptyset\left(-3x - 2y, \frac{x^2}{2} - z\right) = 0$$

Where $\emptyset$ is an arbitrary function.

<u>Ex.2</u>: Solve $\mathbf{\left(\frac{y^2 z}{x}\right)p + xzq = y^2}$

Sol. Given $\left(\frac{y^2z}{x}\right)p + xzq = y^2$ …………(1)

The Lagrange's auxiliary equation for (1) are

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \qquad \text{…………(2)}$$

Taking the first two fractions of (2), we have

$$x^2zdx = y^2zdy \quad \rightarrow \quad x^2dx - y^2dy = 0 \qquad \text{…………(3)}$$

Integrating (3), $\frac{x^3}{3} - \frac{y^3}{3} = a \;\rightarrow\; x^3 - y^3 = a_1$ ……(4)

a_1 being an arbitrary constant.

Next, taking the first and the last fractions of (2), we get

$$xy^2dx = y^2zdz \quad \rightarrow \quad xdx - zdz = 0 \quad \text{…………(5)}$$

Integrating (5), $\frac{x^2}{2} - \frac{z^2}{2} = b \;\rightarrow x^2 - z^2 = b_1$ …(6)

$b_1$being an arbitrary constant

From (4) and (6) the general solution is

$$\emptyset(a_1, b_1) = 0 \quad \rightarrow \quad \emptyset(x^3 - y^3, x^2 - z^2) = 0$$

<u>Ex.3:</u>Solve $\mathbf{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt}$

Sol. Given $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$ …………(1)

The Lagrange's auxiliary equation for (1) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{xyt} \qquad \text{…………(2)}$$

Taking the first two fractions of (2), we have

$$\frac{dx}{x} = \frac{dy}{y} \quad \rightarrow \quad \frac{dx}{x} - \frac{dy}{y} = 0 \qquad \text{…………(3)}$$

Integrating (3), $\ln x - \ln y = \ln a \;\rightarrow\; \frac{x}{y} = a$ ……(4)

Taking the second and the third fractions of (2), we get

$\frac{dy}{y} = \frac{dt}{t} \;\; \rightarrow \;\; \frac{dy}{y} - \frac{dt}{t} = 0$ …………(5)

Integrating (5), $\ln y - \ln t = \ln b \;\; \rightarrow \; \frac{y}{t} = b$ ……(6)

Next, taking the second and the last fractions of (2), we get

$\frac{dy}{y} = \frac{dz}{xyt} \;\; \rightarrow \;\; xtdy - dz = 0$ …………(7)

Substituting (4) and (6) in (7), we get

$\frac{a}{b} y^2 dy - dz = 0$ …………(8)

Integrating (8), $\frac{a}{3b} y^3 - z = c$

Using (4) and (6), $\frac{1}{3} xyt - z = c$ …………(9)

Where a, b and c are an arbitrary constant

The general solution is

$$\emptyset(a, b, c) = 0 \;\; \rightarrow \;\; \emptyset\left(\frac{x}{y}, \frac{y}{t}, \frac{1}{3}xyt - z\right) = 0$$

$\emptyset$ being an arbitrary function.

<u>Rule</u>: for any equal fractions, if the sum of the denominators equalto zero,then the sum of the numerators equal to zero also.

Now, Return to the last example depending on the Rule above we will find the constant c.

Multiplying each fraction in Lagrange's auxiliary (2) by $yt, xt, xy, -3$ respectively, we get the sum of the denominators is

$xyt + xyt + xyt - 3xyt = 0$ …………(10)

Then the sum of the numerators equal to zero also:

$ytdx + xtdy + xydt - 3dz = 0 \to \quad d(xyt) - 3dz = 0$……(11)

Integrating (11), $\quad xyt - 3z = c$ …………(12)

Note that (12) and (9) are the same.

<u>Ex.4:</u> Solve $\quad \mathbf{(y - z)p + (z - x)q = x - y}$

Sol. Given$(y - z)p + (z - x)q = x - y$ …………(1)

The Lagrange's auxiliary equations for (1) are

$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$ …………(2)

The sum of the denominators is

$$y - z + z - x + x - y = 0$$

Then, the sum of the numerators is equal to zero also, (by Rule)

$dx + dy + dz = 0$ …………(3)

Integrating (3), $\quad x + y + z = a$ …………(4)

To find b, multiplying (2) by x, y, z resp. the sum of the denominators is

$$x(y - z) + y(z - x) + z(x - y) = xy - xz + yz - xy + zx - yz = 0$$

Then, the sum of the numerators is equal to zero

$xdx + ydy + zdz = 0$ …………(5)

Integrating (5), $\quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b$ …………(6)

Where, a and b are arbitrary constants.

The general solution is

$$\emptyset(a,b)=0 \ \rightarrow \ \emptyset\left(x+y+z, \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}\right)=0$$

… Exercises …

Solve the following partial differential equation:

1. $p\, tanx + q\, tany = tanz$.

2. $zp = -x$.

3. $y^2p - xyq = x(z-2y)$.

4. $(x^2+2y^2)p - xyq = xz$.

5. $xp + yq = z$.

6. $(-a+x)p + (-b+y)q = (-c+z)$.

7. $x^2p + y^2q + z^2 = 0$.

8. $yzp + zxq = xy$.

9. $y^2p + x^2q = x^2y^2z^2$.

10. $p - q = \frac{z}{(x+y)}$

(1.3.2) The equation of the form $\mathbf{f(p,q)=0}$

Here we consider equations in which p and q occur other than in the first degree, that is non-linear equations. To solve the equation $f(p,q)=0$ ………(1)

Taking $p = constant = a$ …………(2)

$q = constant = b$ …………(3)

Substituting (2),(3) in (1), we get

$\mathrm{F}(\mathrm{a},\mathrm{b}) = 0 \;\rightarrow\; b = \mathrm{F}_1(a) \quad or \;\; a = \mathrm{F}_2(b)$………..(4)

From $dz = pdx + qdy$ …………(5)

Using (2),(3) $\rightarrow dz = adx + bdy$ ………..(6)

Integrating (6), $z = ax + by + c$ ………..(7)

Where c is an arbitrary constant

Substituting (4) in (7) to obtain the complete integral (complete solution)

$z = ax + \mathrm{F}_1(a)y + c \quad or \quad z = \mathrm{F}_2(b)x + by + c$ ……….(8)

<u>Ex.1:</u> Solve $p^2 + p = q^2$

Sol. $\mathrm{p}^2 + \mathrm{p} - \mathrm{q}^2 = 0$ ………..(1)

The equation (1) of the form $f(p,q) = 0$

Let $p = a \;, q = b$

Substituting in (1)

$$a^2 + a - b^2 = 0 \rightarrow b^2 = a^2 + a \rightarrow b = \pm\sqrt{a^2 + a}$$

The complete integral is

$$z = ax + by + c$$

$$= ax \pm \sqrt{a^2 + a}y + c$$

Where c is an arbitrary constant.

<u>Ex.2</u>: Solve $\mathbf{pq = k}$, where k is a constant.

Sol. Given that $pq = k$ …………(1)

Since (1) is of the form $f(p,q)=0$, it's solution is

$z=ax+by+c$ …………(2)

Let $p=a$, $q=b$, substituting in (1) , then $ab=k \rightarrow b=\frac{k}{a}$ ...(3)

Putting (3) in (2), to get the complete solution

$z=ax+\frac{k}{a}y+c$; c is an arbitrary constant .

<u>Ex.3</u>: Solve $\frac{\partial z}{\partial x}-3\frac{\partial z}{\partial y}=(\frac{\partial z}{\partial y})^3$

Sol. Given that $p-3q=q^3$ ……….. (1)

Since (1) is of the form $f(p,q)=0$, then

Let $p=a$, $q=b$

Substituting in (1) , $a-3b=b^3 \rightarrow a=b^3+3b$ ………(2)

Putting (2) in the equation $z=ax+by+c$, we get

$$z=(b^3+3b)x+by+c$$

Where c is an arbitrary constant

The equation (3) is the complete integral .

(1.3.3) <u>The Equation of the form $z=px+qy+f(p,q)$</u>

A first order partial differential equation is said to be of **Clariaut** form if it can be written in the form

$$z=px+qy+f(p,q) \qquad \text{…(1)}$$

to solve this equation taking $p=a$, $q=b$ and substituting in (1), so the complete integral is

$$z=ax+by+f(a,b) \qquad \text{…(2)}$$

Example 1: Solve $\boldsymbol{z = px + qy + pq}$

Sol. The given equation is of the form $z = px + qy + f(p, q)$

let $p = a$ and $q = b$ substituting in the given equation, so the complete integral is

$$\boxed{z = ax + by + ab}$$

where a, b being arbitrary constant.

Example 2: Solve $\boldsymbol{x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = z - 5\,\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}}$

Sol. Rearrange the given equation, we have

$$x\,p + y\,q = z - 5p + pq$$

$z = x\,p + y\,q + 5p - pq$ …(3)

Equation (3) is of Clariaut form

let $p = a$ and $q = b$ substituting in (3), then the complete integral is $\boxed{z = ax + by + 5a - ab}$

where a, b being arbitrary constant.

Example 3: Solve $\boldsymbol{px + qy = z - p^3 - q^3}$

Sol. Rearrange the given equation, we have

$z = px + qy + p^3 + q^3$ …(4)

let $p = a$ and $q = b$ substituting in (4)

$\boxed{z = ax + by + a^3 + b^3}$that is the complete integral and a, b being arbitrary constants.

(1.3.4) The Equation of the form $f(z, p, q) = 0$

To solve the equation of the form

$f(z, p, q) = 0 \quad \ldots(1)$

1. Let $\quad u = x + ay \qquad \ldots(2)$

   where a is an arbitrary constant

2. Replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (1) as follows,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} = a\frac{dz}{du} \qquad \ldots(3)$$

   from (2) $\quad \frac{\partial u}{\partial x} = 1 \quad$ and $\quad \frac{\partial u}{\partial y} = a$

3. Substituting (3) in (1) and solve the resulting ordinary differential equation of first order by usual methods.

4. Next, replace u by $x + ay$ in the solution obtained in step 3 to get the complete solution.

Example 1: Solve $\boldsymbol{z = p + q}$

Sol. Given equation is $\quad z = p + q \qquad \ldots(4)$

which is of the form $f(z, p, q) = 0$. Let $u = x + ay$ where a is an arbitrary constant.

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (4), we get

$$z = \frac{dz}{du} + a\,\frac{dz}{du}$$

$$\Rightarrow z = (1+a)\ \frac{dz}{du}$$

$\Rightarrow du = (1+a)\frac{dz}{z}$ …(5)

Integrating (5), $u + c = (1+a)\ln z$

where c is an arbitrary constant

Replacing u,

$$x + ay + c = \ln z^{(1+a)}$$

$\Rightarrow\ e^{x+ay+c} = z^{(1+a)}$

$\Rightarrow\ z = e^{\frac{x+ay+c}{1+a}}$ …(6)

and that is the complete integral.

Example 2: Solve $\left(\frac{\partial z}{\partial x}\right)^2 z - \left(\frac{\partial z}{\partial y}\right)^2 = 1$

Sol. Rearrange the given equation, we have

$p^2 z - q^2 = 1$…(7)

This equation is of the form $\mathrm{f(z, p, q) = 0}$

Let $u = x + ay$, where a is an arbitrary constant

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (7), we get

$$\left(\frac{dz}{du}\right)^2 z - \left(a\ \frac{dz}{du}\right)^2 = 1$$

$$\Rightarrow (z - a^2)\left(\frac{dz}{du}\right)^2 = 1$$

$\Rightarrow \pm\sqrt{z-a^2}\,\frac{dz}{du} = 1$ by taking the square root

$\Rightarrow \pm\sqrt{z-a^2}\ dz = du$…(8)

Integrating (8),

$\pm\frac{2}{3}(z-a^2)^{3/2}=u+c$...(9)

Replacing u in (9) to get the complete integral

$$\boxed{\pm\frac{2}{3}(z-a^2)^{\frac{3}{2}}=x+ay+c}$$

(1.3.5) The Equation of the form $f_1(x,p)=f_2(y,q)=0$

In this form z does not appear and the terms containing x and p are on one side and those containing y and q on the other side.

To solve this equation putting

$f_1(x,p)=f_2(y,q)=a$...(1)

where a is an arbitrary constant

$\therefore\ \ f_1(x,p)=a \quad \Longrightarrow \quad p=g_1(x,a)$...(2)

$f_2(y,q)=a \quad \Longrightarrow \quad q=g_2(y,a)$...(3)

Substituting (2) and (3) in $dz=pdx+qdy$, we get

$dz=g_1(x,a)dx+g_2(y,a)dy$...(4)

Integrating (4),

$$\boxed{z=\int g_1(x,a)dx+\int g_2(y,a)dy+b}$$

which is a complete integral containing two arbitrary constants$a$ and b.

Example 1: Solve $p=2xq^2$

Sol. Separating p and x from q and y, the given equation reduces to$\frac{p}{x}=2q^2$...(5)

Equating each side to an arbitrary constant a, we have

$$\frac{p}{x} = a \qquad \Rightarrow p = ax$$

$$2q^2 = a \quad \Rightarrow q = \pm\sqrt{\frac{a}{2}}$$

Putting these values of p and q in

$dz = pdx + qdy$, we get

$$dz = axdx \pm \sqrt{\frac{a}{2}}dy \qquad \ldots(6)$$

Integrating (6), $z = \frac{a}{2}x^2 \pm \sqrt{\frac{a}{2}}\, y + b$

where a and b are two arbitrary constants.

Example 2: Solve $xq - y^2p - x^2y^2 = 0$

Sol. Separating p and x from q and y, the given equation reduces to $\frac{p+x^2}{x} = \frac{q}{y^2}$…(7)

Equating each side to an arbitrary constant a, we have

$$\frac{p+x^2}{x} = a \qquad \Rightarrow \qquad p = ax - x^2 \qquad \ldots(8)$$

$$\frac{q}{y^2} = a \qquad \Rightarrow \qquad q = a\,y^2 \qquad \ldots(9)$$

Putting (8) and (9) in $dz = pdx + qdy$, we get

$$dz = (ax - x^2)dx + ay^2dy \qquad \ldots(10)$$

Integrating (10) , $\boxed{z = \frac{ax^2}{2} - \frac{x^3}{3} + a\frac{y^3}{3} + b}$

which is a complete integral containing two arbitrary constants a and b.

Example 3: Solve $\boldsymbol{p - 3x^2 = q^2 - y}$

Sol. Equating each side to an arbitrary constant a, we get

$$p - 3x^2 = a \qquad \Rightarrow \qquad p = a + 3x^2 \qquad \ldots(11)$$

$$q^2 - y = a \qquad \Rightarrow \qquad q = \pm\sqrt{a + y} \qquad \ldots(12)$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (a + 3x^2)dx \pm \sqrt{a + y}dy \qquad \ldots(13)$$

Integrating (13) , $\boxed{z = ax + x^3 \pm \frac{2}{3}(a + y)^{3/2} + b}$

which is a complete integral containing two arbitrary constant a and b.

(1.3.6) Charpit's Method (General Method of Solving p.d.es of Order One but of any Degree)

Let the given p.d.e of first order and non- linear in p and q be

$$f(x, y, z, p, q) = 0 \qquad \ldots(1)$$

To solve this equation we will use the following charpit's auxiliary equations.

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

After substituting the partial derivatives in charpit's auxiliary equations select the proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Then, putting p and q in the relation $dz = pdx + qdy$ which on integration gives the complete integral of the given equation.

Example 1: Solve $\boldsymbol{z = px + qy + p^2 + q^2}$ by charpit's method.

Sol. Let $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$…(2)

charpit's auxiliary equation are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

From (2) $f_x = -p$, $f_y = -q$, $f_z = 1$, $f_p = -x - 2p$, $f_q = -y - 2q$

$$\therefore \quad \frac{dp}{-p + p} = \frac{dq}{-q + q} = \frac{dz}{p(x + 2p) + q(y + 2q)} = \frac{dx}{x + 2p}$$

$$= \frac{dy}{y + 2q}$$

Taking the first fraction $dp = 0 \quad \rightarrow \quad p = a$…(3)

Taking the second fraction $dq = 0 \quad \rightarrow \quad q = b$ …(4)

Substituting (3) and (4) in (2) to get the complete integral

$$\boxed{z = ax + by + a^2 + b^2}$$

where a and b are arbitrary constants.

Example 2: Solve $\boldsymbol{2zx - px^2 - 2qxy + pq = 0}$ by charpit's method.

Sol. Let $f(x, y, z, p, q) = 2zx - px^2 - 2qxy + pq = 0$…(5)

$f_x = 2z - 2px - 2qy$, $f_y = -2qx$, $f_z = 2x f_p = -x^2 + q$, $f_q = -2xy + p$

Substituting in charpit's auxiliary equations , we get

$$\frac{dp}{2z - 2px - 2qy + 2px} = \frac{dq}{-2qx + 2qx} = \frac{dz}{-p(-x^2 + q) - q(-2xy + p)} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} \ldots (6)$$

Taking the second fraction of (6)

$dq = 0 \quad \rightarrow \quad q = c \ldots (7)$

Substituting (7) in (5)

$$2zx - px^2 - 2cxy + cp = 0$$

$p = \frac{2xz - 2cxy}{x^2 - c} \quad \rightarrow \quad p = \frac{2x(z - cy)}{x^2 - c} \ldots (8)$

Putting (7) and (8) in $dz = pdx + qdy$

$$dz = \frac{2x(z - cy)}{x^2 - c} dx + cdy \Longrightarrow dz - cdy = \frac{2x(z - cy)}{x^2 - c} dx$$

$\frac{dz - cdy}{(z - cy)} = \frac{2x\, dx}{x^2 - c} \ldots (9)$

Integrating (9), $ln|z - cy| = \ln|x^2 - c| + \ln b$

$$z - cy = b(x^2 - c)$$

$$\boxed{z = b(x^2 - c) + cy}$$

which is a complete integral where b and c are two arbitrary constants.

… Exercises …

Solve the following equations:

1. $q = 3p^2$
2. $zpq = p + q$
3. $p^2 - y^2 q = y^2 - x^2$
4. $(y^2 + 4)xpq - (x^2 + 2) = 0$
5. $q - px - p^2 = 0$
6. $px + qy = pq$
7. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{y}{x}$
8. $p^2 - q^2 = z$

(1.3.7) Using Some Hypotheses in the Solution

Sometimes we need some hypotheses to solve the partial differential equation, here we will give three types of hypotheses.

A) When the equation contains the term (px) or its' powers we use the hypothesis $\boxed{X = \ln x}$

as follows

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial X}{\partial X} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}.\frac{1}{x}(\text{since } X = \ln x \implies \frac{\partial X}{\partial x} = \frac{1}{x})$$

$$\implies xp = \frac{\partial z}{\partial X}$$

Then substituting this result in the given equation and solve it by previous methods.

Example 1: Solve $z = px$ by hypotheses

Sol. From $X = \ln x$ we have $xp = \frac{\partial z}{\partial X}$ …(1)

Substituting (1) in the given equation, we get

$$z = \frac{\partial z}{\partial X} \implies \partial X = \frac{\partial z}{z} \quad \dots(2)$$

Integrating (2), $X = \ln z + \ln \emptyset(y)$ …(3)

where $\emptyset$ is an arbitrary function for y

replacing X in (3) to get the complete integral

$$\ln x = \ln \emptyset(y).z$$

$$\implies \boxed{z = \frac{x}{\emptyset(y)}} \dots(4)$$

Example 2: Solve $q = px + p^2x^2$ by hypotheses

Sol. Given that $q = px + (px)^2$ …(5)

from $X = \ln x$ we have $xp = \frac{\partial z}{\partial X}$ …(6)

Substituting (6) in (5), we get

$$q = \frac{\partial z}{\partial X} + \left(\frac{\partial z}{\partial X}\right)^2 \quad \ldots(7)$$

Let $\frac{\partial z}{\partial X} = t$then (7) will be

$$q = t + t^2 \quad \ldots(8)$$

The equation (8) is of the form $f(t, q) = 0$

Then let $t = a$ and $q = b$, putting in (8) $b = a + a^2$

Substituting in $z = aX + by + c$

$\Longrightarrow z = aX + (a + a^2)y + c$…(9)

where c is an arbitrary constant

replacing X in (9) to get the complete integral

$$\boxed{z = a\,ln\,x + (a + a^2)y + c}$$

B) When the equation contains the term (qy) or its' powers we use the hypothesis $\boxed{Y = \ln y}$

as follows:

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}.\frac{\partial Y}{\partial Y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y}.\frac{1}{y} \text{ (since } Y = \ln y \Longrightarrow \frac{\partial Y}{\partial y} = \frac{1}{y})$$

$$\Longrightarrow qy = \frac{\partial z}{\partial Y}$$

Then solving by the same way in (A).

Example 3: Solve $2p + qy = 4$ by hypotheses

Sol. Given that $2p + qy = 4$ …(10)

from $Y = \ln y$ we have $qy = \frac{\partial z}{\partial Y}$ …(11)

Substituting (11) in (10), we get

$$2p + \frac{\partial z}{\partial Y} = 4$$

Let $\frac{\partial z}{\partial Y} = t$then,

$2p + t = 4$ …(12)

The equation (12) is of the form $f(p,t) = 0$

Then let $p = a$ and $t = b$, putting in (12) $2a + b = 4$

$\Longrightarrow b = 4 - 2a$ …(13)

Substituting (13) in $z = ax + bY + c$

$\Longrightarrow z = ax + (4 - 2a)Y + c$…(14)

where c is an arbitrary constant

replacing Y in (14) to get the complete integral

$\boxed{z = a\,x + (4 - 2a)\ln y + c}$

Example 4: Solve $p^2x^2 = z^2 + q^2y^2$ by hypotheses

Sol. Given that $p^2x^2 = z^2 + q^2y^2$ …(15)

from $X = \ln x$ and $Y = \ln y$ we have

$xp = \frac{\partial z}{\partial X}$ and $qy = \frac{\partial z}{\partial Y}$ …(16)

Substituting (16) in (15), we get

$\left(\frac{\partial z}{\partial X}\right)^2 = z^2 + \left(\frac{\partial z}{\partial Y}\right)^2$ …(17)

Let $t = \frac{\partial z}{\partial X}$ and $r = \frac{\partial z}{\partial Y}$ putting in (17)

$$t^2 - r^2 = z^2 \qquad \ldots(18)$$

Note that (18) is of the form $f(t, r, z) = 0$

Taking $\quad u = X + aY \qquad$ (a is constant)

Then $\quad t = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial X} \cdot \frac{\partial u}{\partial u} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial X} = \frac{dz}{du}$

$$r = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial u}{\partial u} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial Y} = a\frac{dz}{du} \qquad \ldots(19)$$

because ($\frac{\partial u}{\partial X} = 1$ and $\frac{\partial u}{\partial Y} = a$)

putting (19) in (18)

$$\left(\frac{dz}{du}\right)^2 - a^2 \left(\frac{dz}{du}\right)^2 = z^2$$

$$(1 - a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$\pm\sqrt{1 - a^2}\,\frac{dz}{du} = z \qquad$ (taking the square root)

$$\pm\sqrt{1 - a^2}\,\frac{dz}{z} = du \qquad \ldots(20)$$

Integrating (20),

$$\pm\sqrt{1 - a^2}\,\ln z = u + c \quad (\, c \text{ is constant}) \qquad \ldots(21)$$

Now, replacing u in (21) to get the complete integral

$$\pm\sqrt{1 - a^2}\,\ln z = X + aY + \ln c \qquad \ldots(22)$$

Next, replacing X and Y in (22) to get the complete integral

$$\pm\sqrt{1 - a^2}\,\ln z = \ln x + a\ln y + \ln c$$

$\ln z^b = \ln cx\, y^a \qquad$ where $b = \pm\sqrt{1 - a^2}$

$$\Longrightarrow z^b = cx\, y^a \qquad \ldots(23)$$

So, (23) is the complete integral.

C) When the equation contains the terms $\frac{p}{z}$ or $\frac{q}{z}$ or its' powers we use the hypothesis $\boxed{Z = \ln z}$

as follows:

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial Z}{\partial Z} = \frac{\partial z}{\partial Z}.\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x} \qquad (\text{since } \frac{\partial z}{\partial Z} = z\ )$$

hence $\frac{p}{z} = \frac{\partial Z}{\partial x}$

by the same way we have $\frac{q}{z} = \frac{\partial Z}{\partial y}$

then substituting this terms in the given equation and solve it by the same way in (A) and (B).

Example 5: Solve $\boldsymbol{px + qy = z}$ **by** $\boxed{\boldsymbol{Z = \ln z}}$

Sol. Given that $px + qy = z$ …(24)

Dividing on z, $\frac{p}{z}x + \frac{q}{z}y = 1$ …(25)

using $Z = \ln z$ we have $\frac{p}{z} = \frac{\partial Z}{\partial x}$ and $\frac{q}{z} = \frac{\partial Z}{\partial y}$,substituting in (25)

$$x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = 1 \qquad \ldots(26)$$

Let $t = \frac{\partial Z}{\partial x}$ and $r = \frac{\partial Z}{\partial y}$thus, (26) would be

$$xt + yr = 1 \qquad \ldots(27)$$

Clear that (27) is of the form $f_1(x,t) = f_2(y,r)$

Then putting x, pin one side and y, q in the other side

$xt = 1 - ry = a$ (a is constant)

Then $xt = a \rightarrow t = \frac{a}{x}$ …(28)

$1 - ry = a \quad \rightarrow r = \frac{1-a}{y}$ ...(29)

Substituting (28), (29) in $dZ = tdx + rdy$

$\Longrightarrow dZ = \frac{a}{x}dx + \frac{1-a}{y}dy$ ...(30)

Integrating (30), we get

$Z = a\ln x + (1-a)\ln y + \ln b$ (where b is constant)

Replacing Z from the hypothesis to get the complete integral

$\therefore \ln z = \ln(bx^a y^{(1-a)})$ (by properties of ln)

$\Longrightarrow \boxed{z = b\, x^a\, y^{(1-a)}}$ ...(31)

Then (31) is the complete integral.

Example 6: Solve $p^2 + q^2 = z^2(x+y)$ by hypotheses

Sol. Dividing on z^2,

$\frac{p^2}{z^2} + \frac{q^2}{z^2} = x + y$ ...(32)

using $Z = \ln z$ we have $\frac{p}{z} = \frac{\partial Z}{\partial x}$ and $\frac{q}{z} = \frac{\partial Z}{\partial y}$,substituting in (32)

$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y$ ...(33)

Let $t = \frac{\partial Z}{\partial x}$ and $r = \frac{\partial Z}{\partial y}$putting in (33)

$t^2 + r^2 = x + y$ ...(34)

Then $t^2 - x = a \quad \rightarrow t = \pm\sqrt{a+x}$

$y - r^2 = a \quad \rightarrow r = \pm\sqrt{y-a}$

Substituting in $dZ = tdx + rdy$

$\Longrightarrow dZ = \pm\sqrt{a+x}dx + \pm\sqrt{y-a}dy$ ...(35)

Integrating (35), we get

$Z = \pm\frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c$ (where c is constant)

Replacing Z from the hypothesis to get the complete integral

$$\Rightarrow \boxed{\ln z = \pm\frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c}$$

… Exercises …

1. $p^2x^2 = z(z - qy)$
2. $pq = z^2 y \sec x$
3. $p + q = z\, e^{x+y}$
4. $p^2 + zq = z^2(x - y)$
5. $p^2 + zp = z^2(x - y)$
6. $p^2 + q^2 = z^2$
7. $xp + 4q = \cos y$
8. $p^2 + q^2 = z^2 y$

Section(1.4): Homogeneous linear partial differential equations with constant coefficients and higher order

A linear partial differential equation with constant coefficients is called homogeneous if all it's derivatives are of the same order. The general form of such an equation is

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \ldots\ldots\ldots\ldots(1)$$

Where $A_0, A_1, \ldots, A_n$ are constant coefficients.

For example:

1. $3\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 y}{\partial y^2} = o$ homo. of order 2.
2. $2\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 5\frac{\partial^3 z}{\partial x \partial y^2} - 8\frac{\partial^3 y}{\partial y^3} = x + y$ homo. of order 3.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will be denoted by $D \; or \; D_x$ and D' or D_y respectively. Then (1) can be rewritten as:

$$(A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n)z = f(x,y) \quad \ldots\ldots\ldots\ldots(2)$$

On the other hand, when all the derivatives in the given equation are not of the same order, then it is called a non-homogenous linear partial differential equation with constant coefficients.

In this section we propose to study the various methods of solving homogeneous linear partial differential equation with constant coefficients, namely (2).

Equation (2) may rewritten as:

$$\boxed{F(D_x, D_y)z = f(x,y)} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Where $F(D_x, D_y) = A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n$

Equation (3) has a general solution when $f(x, y) = 0$

i.e $F(D_x, D_y)z = 0$

$\rightarrow \boxed{(A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n)z = 0} \ldots\ldots\ldots\ldots (4)$

And a particular solution (particular integral) when $f(x, y) \neq 0$

❖ Now, we will find the general solution of (4)

Let $z = \emptyset(y + mx)$ be a solution of (4) where $\emptyset$ is an arbitrary function and m is a constant , then

$$D_x z = \emptyset'(y + mx).m$$

$$D_x^2 z = \emptyset''(y + mx).m^2$$

$$\vdots$$

$$D_x^n z = \emptyset^{(n)}(y + mx).m^n$$

$$D_y z = \emptyset'(y + mx)$$

$$D_y^2 z = \emptyset''(y + mx)$$

$$\vdots$$

$$D_y^n z = \emptyset^{(n)}(y + mx)$$

$$D_x D_y z = m\,\emptyset''(y + mx)$$

$$D_x^2 D_y z = m^2 \emptyset^{(3)}(y + mx)$$

$$\vdots$$

$$D_x^r D_y^s z = m^r \emptyset^{(r+s)}(y + mx)$$

$= m^r \emptyset^{(n)}(y + mx)$, where $r + s = n$

Substituting these values in (4) and simplifying, we get :

$(A_0m^n + A_1m^{n-1} + A_2m^{n-2} + \cdots + A_n)\emptyset^{(n)}(y + mx) = 0$ …(5)

Which is true if m is a root of the equation

$A_0m^n + A_1m^{n-1} + A_2m^{n-2} + \cdots + A_n = 0$ ……..(6)

The equation (6) is known as the (characteristic equation) or the (auxiliary equation(A.E.)) and is obtained by putting $D_x = m$ and $D_y = 1$ in $F(D_x, D_y)z = 0$, and it has $n$roots.

Let $m_1, m_2, \dots, m_n$ be n roots of A.E. (6). **Three cases arise**:

Case 1 when the roots are distinct.

If $m_1, m_2, \dots, m_n$ are n distinct roots of A.E. (6) then $\emptyset_1(y + m_1x), \emptyset_2(y + m_2x), \dots \dots \dots, \emptyset_n(y + m_nx)$ are the linear solution corresponding to them and since the sum of any linear solutions is a solution too than the general solution in this case is:

$z = \emptyset_1(y + m_1x) + \emptyset_2(y + m_2x) + \cdots + \emptyset_n(y + m_nx)$ …..(7)

<u>Ex.1</u>: Find the general solution of

$$(\boldsymbol{D_x^3 + 2D_x^2D_y - 5D_xD_y^2 - 6D_y^3)z = 0}$$

Sol. The A.E. is$m^3 + 2m^2 - 5m - 6 = 0$

$$\rightarrow \quad (m + 1)(m^2 + m - 6 = 0$$

$$\rightarrow \quad (m + 1)(m + 3)(m - 2) = 0$$

$$m_1 = -1, \quad m_2 = -3, \quad m_3 = 2$$

Note that m_1, m_2 and m_3 are different roots, then the general solution is

$$z = \emptyset_1(y + m_1x) + \emptyset_2(y + m_2x) + \emptyset_3(y + m_3x)$$

$$\rightarrow \quad z = \emptyset_1(y-x) + \emptyset_2(y-3x) + \emptyset_3(y+2x)$$

Where $\emptyset_1, \emptyset_2, \emptyset_3$ are arbitrary functions.

Ex.2: Find the general solution of $m^2 - a^2 = 0$ where a is a real number.

Sol. Given that $\quad m^2 - a^2 = 0 \rightarrow m^2 = a^2$

$$\rightarrow \quad m = \pm a \quad \text{different root}$$

$$m_1 = a \quad , m_2 = -a$$

The general solution is

$$z = \emptyset_1(y+ax) + \emptyset_2(y-ax)$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Case 2 when the roots are repeated.

If the root m is repeated k times . i.e.$m_1 = m_2 = \cdots = m_k$, then the corresponding solution is :

$$\boxed{z = \emptyset_1(y+m_1x) + x\emptyset_2(y+m_1x) + \cdots + x^{k-1}\emptyset_n(y+m_1x)} \ldots(8)$$

Where $\emptyset_1, \ldots, \emptyset_k$ are arbitrary functions.

Note: If some of the roots $m_1, m_2, \ldots, m_n$ are repeated and the other are not . i.e. $m_1 = m_2 = \cdots = m_k \neq m_{k+1} \neq \cdots \neq m_n$ then the general solution is :

$$z = \emptyset_1(y+m_1x) + x\emptyset_2(y+m_1x) + \cdots + x^{k-1}\emptyset_n(y+m_1x) + \emptyset_{k+1}(y+m_{k+1}x) + \cdots + \emptyset_n(y+m_nx) \quad \ldots\ldots\ldots\ldots(9)$$

Ex.3: **Solve** $(D_x^3 - D_x^2 D_y - 8D_x D_y^2 + 12D_y^3)z = 0$

Sol. The A.E. is $\quad m^3 - m^2 - 8m + 12 = 0$

$$\rightarrow \quad (m-2)(m-2)(m+3) = 0$$

$m_1 = m_2 = 2 \quad , \quad m_3 = -3$

Then, the general solution is

$$z = \emptyset_1(y+2x) + x\emptyset_2(y+2x) + \emptyset_3(y-3x)$$

Where $\emptyset_1, \emptyset_2, \emptyset_3$ are arbitrary functions.

Ex.4: **Find the general solution of the equation that it's A.E. is :**

$$(m-1)^2(m+2)^3(m-3)(m+4) = 0$$

Sol. Given that $(m-1)^2(m+2)^3(m-3)(m+4) = 0$

$m_1 = m_2 = 1 \quad , m_3 = m_4 = m_5 = -2 \quad , m_6 = 3 \quad , m_7 = -4$

The general solution is

$$z = \emptyset_1(y+x) + x\emptyset_2(y+x) + \emptyset_3(y-2x) + x\emptyset_4(y-2x)$$
$$+ x^2\emptyset_5(y-2x) + \emptyset_6(y+3x) + \emptyset_7(y-4x)$$

Where $\emptyset_1, \dots, \emptyset_7$ are arbitrary functions.

Case 3 when the roots are complex.

If one of the roots of the given equation is complex let be m_1 then the conjugate of m_1 is also a root, let be m_2 , so the general solution is:

$$\boxed{z = \emptyset_1(y+m_1x) + \emptyset_2(y+m_2x) + \cdots + \emptyset_n(y+m_nx)}$$

Where $\emptyset_1, \dots, \emptyset_n$ are arbitrary functions.

Ex.5: Solve $(D_x^2 + D_y^2)z = 0$

Sol. The A. E. is $m^2 + 1 = 0 \quad \rightarrow \quad m = \pm i$

$$\therefore \quad m_1 = i \quad , \quad m_2 = -i$$

The general solution is

$$z = \emptyset_1(y + ix) + \emptyset_2(y - ix)$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Ex.6: Solve $(D_x^2 - 2D_xD_y + 5D_y^2)z = 0$

Sol. The A. E. is $m^2 - 2m + 5 = 0$

$$\rightarrow \quad m = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

$$\therefore \quad m_1 = 1 + 2i \quad , \quad m_2 = 1 - 2i$$

$$z = \emptyset_1(y + (1 + 2i)x) + \emptyset_2(y + (1 - 2i)x)$$

That is the general solution where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Ex.7: Solve $(D_x^4 - D_x^3D_y + 2D_x^2D_y^2 - 5D_xD_y^3 + 3D_y^4)z = 0$

Sol. The A.E. is $m^4 - m^3 + 2m^2 - 5m + 3 = 0$

$$\rightarrow \quad (m - 1)^2(m^2 + m + 3) = 0$$

$m_1 = m_2 = 1 \quad , \quad m = \frac{-1 \pm \sqrt{1-12}}{2} = \frac{-1 \pm \sqrt{11}i}{2}$

$$\therefore \; m_3 = \frac{-1 + \sqrt{11}i}{2} \qquad , m_4 = \frac{-1 - \sqrt{11}i}{2}$$

Then, the general solution is

$$z = \emptyset_1(y+x) + x\emptyset_2(y+x) + \emptyset_3\left(y + (\frac{-1+\sqrt{11}i}{2})x\right)$$

$$+ \quad \emptyset_4\left(y + (\frac{-1-\sqrt{11}i}{2})x\right)$$

Where $\emptyset_1 , \dots , \emptyset_4$ are arbitrary functions.

❖ Particular integral (P.I.) of homogeneous linear partial differential equation

When $f(x,y) \neq 0$in the equation (3) which it's$F(D_x, D_y)z = f(x,y)$ multiplying (3) by the inverse operator$\frac{1}{F(D_x,D_y)}$ of the operator $F(D_x, D_y)$ to have

$\frac{1}{F(D_x,D_y)} \cdot F(D_x, D_y)\, z = \frac{1}{F(D_x,D_y)}\, f(x,y)$

$\rightarrow \boxed{z = \frac{1}{F(D_x,D_y)}\, f(x,y)}$ ……… (11)

Which it's the particular integral (P.I.)

The operator $F(D_x, D_y)$ can be written as

$F(D_x, D_y) = (D_x - m_1 D_y)(D_x - m_2 D_y) \dots (D_x - m_n D_y)$ ….(12)

Substituting (12) in (11) :

$z = \frac{1}{(D_x - m_1 D_y)(D_x - m_2 D_y)\dots(D_x - m_n D_y)}\, f(x,y)$ …………(13)

Taking $u_1 = \frac{1}{(D_x - m_n D_y)}\, f(x,y)$

$$\therefore\ (D_x - m_n D_y)u_1 = f(x,y)$$

This equation can be solved by Lagrange's method .

The Lagrange's auxiliary equations are

$\frac{dx}{1} = \frac{dy}{-m_n} = \frac{du_1}{f(x,y)}$ ....(14)

Taking the first two fractions of (14)

$m_n dx + dy = 0 \quad \rightarrow \boxed{m_n x + y = a}$ ....(15)

Taking the first and third fractions of (14)

$dx = \frac{du_1}{f(x,y)} \quad \rightarrow \boxed{f(x,y)dx = du_1}$ ....(16)

Substituting (15) in (16) we have

$$f(x, a - m_n x)dx = du_1$$

Integrating the last one we have

$$u_1 = \int f(x, a - m_n x)dx + b$$

Let $b = 0$, then we have u_1

By the same way , we take

$$u_2 = \frac{1}{D_x - m_{n-1}D_y} u_1$$

And solve it by Lagrange's method to get u_2 , then continue in this way until we get to

$$z = u_n = \frac{1}{D_x - m_1 D_y} u_{n-1}$$

And by solving this equation we get the particular integral (P.I.)

Ex.1: solve $\boldsymbol{(D_x^2 - D_y^2)z = \sec^2(x + y)}$

Sol. Firstly, we will find the general solution of

$(D_x^2 - D_y^2)z = 0$ ....(1)

The A. E. is $m^2 - 1 = 0 \rightarrow m^2 = 1 \quad \rightarrow \quad m = \pm 1$

$$\therefore \quad m_1 = 1 , \quad m_2 = -1$$

$\therefore \quad z = \emptyset_1(y+x) + \emptyset_2(y-x)$ …………(2)

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Second, we will find the particular integral as follows

$$z_2 = \frac{1}{D_x^2 - D_y^2}\sec^2(x+y)$$

$$= \frac{1}{(D_x - D_y)(D_x + D_y)}\sec^2(x+y)$$

Let $u_1 = \frac{1}{(D_x + D_y)}\sec^2(x+y)$

$$(D_x + D_y)u_1 = \sec^2(x+y)$$

The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du_1}{\sec^2(x+y)}$$

Taking the first two fractions

$dx = dy \quad \rightarrow \quad x - y = a$ ……….(3)

Taking the first and third fractions

$dx = \frac{du_1}{\sec^2(x+y)} \quad \rightarrow \quad \sec^2(x+y)\,dx = du_1$ ……….(4)

Substituting (3) in (4), we have

$\sec^2(2x-a)\,dx = du_1$ …………(5)

Integrating (5), we have

$$u_1 = \frac{1}{2}\tan(2x-a) + b$$

Let $b = 0$ and replacing a , we get

$u_1 = \frac{1}{2}\tan(x+y)$ ……..(6)

Putting (6) in z_2

$$z_2 = \frac{1}{(D_x - D_y)} \cdot \frac{1}{2}\tan(x+y)$$

$$\rightarrow \ \left(D_x - D_y\right)z_2 = \frac{1}{2}\tan(x+y)$$

The Lagrange's auxiliary equation are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz_2}{\frac{1}{2}\tan(x+y)}$$

Taking the first two fractions

$dx = -dy \ \rightarrow \ \ x + y = a$ ……….(7)

Taking the first and third fractions

$$dx = \frac{dz_2}{\frac{1}{2}\tan(x+y)}$$

$\frac{1}{2}\tan(x+y)\,dx = dz_2$ ……..(8)

Substituting (7) in (8)

$\frac{1}{2}\tan a \ dx = dz_2$ ………(9)

Integrating (9) , we get

$$\frac{1}{2}x\tan a \ = z_2 + b$$

Let $b = 0$, and replacing a from (7) we get the particular integral

$z_2 = \frac{1}{2}x\tan(x+y)$ ……………(10)

Hence the required general solution is

$$z = z_1 + z_2$$

$$= \emptyset_1(y+x) + \emptyset_2(y-x) + \frac{x}{2}\tan(x+y) \quad \text{……….(11)}$$

Short methods of finding the P.I. in certain cases :

Case 1 **When $f(x,y) = e^{ax+by}$ where a and b are arbitrary constants**

To find the P.I. when $F(a,b) \neq 0$, we derive $f(x,y)$ for x any y n times:

$$D_x e^{ax+by} = a e^{ax+by}$$
$$D_x^2 e^{ax+by} = a^2 e^{ax+by}$$
$$\vdots$$
$$D_x^n e^{ax+by} = a^n e^{ax+by}$$

$$D_y e^{ax+by} = b e^{ax+by}$$
$$D_y^2 e^{ax+by} = b^2 e^{ax+by}$$
$$\vdots$$
$$D_y^n e^{ax+by} = b^n e^{ax+by}$$

$D_x^r D_y^s e^{ax+by} = a^r b^s e^{ax+by}$ where $r + s = n$

So

$$F(D_x, D_y) e^{ax+by} = F(a,b) e^{ax+by}$$

Multiplying both sides by $\frac{1}{F(D_x,D_y)}$, we get

$$e^{ax+by} = \frac{1}{F(D_x,D_y)} F(a,b) e^{ax+by}$$

Since $F(a,b) \neq 0$, then we can divide on it :

$\frac{1}{F(a,b)} e^{ax+by} = \frac{1}{F(D_x,D_y)} e^{ax+by}$ ………………..*

Which it is equal to z , then the P. I. is

$$\boxed{z = \frac{1}{F(D_x,D_y)} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}}$$, where $F(a,b) \neq 0$

when $F(a,b) = 0$, then analyze $F(D_x, D_y)$ as follows

$$F(D_x, D_y) = (D_x - \frac{a}{b} D_y)^r G(D_x, D_y)$$

Where $G(a,b) \neq 0$, we get

$$z = \frac{1}{F(D_x, D_y)} e^{ax+by} = \frac{1}{(D_x - \frac{a}{b} D_y)^r G(D_x, D_y)} e^{ax+by}$$

$$= \frac{1}{(D_x - \frac{a}{b} D_y)^r} \cdot \frac{1}{G(a,b)} e^{ax+by} \quad \text{from } *$$

Since $G(a,b) \neq 0$

$$= \frac{1}{G(a,b)} \cdot \frac{1}{(D_x - \frac{a}{b} D_y)^r} e^{ax+by}$$

Then by Lagrange's method r times , we get

$$\boxed{z = \frac{1}{F(D_x, D_y)} e^{ax+by} = \frac{1}{G(a,b)} \cdot \frac{x^r}{r!} e^{ax+by}}$$

Which it's the P.I. where $F(a,b) = 0$, $G(a,b) \neq 0$

Ex.2: Solve $\left(D_x^2 - D_x D_y - 6D_y^2\right) z = e^{2x-3y}$

Sol.

1) To find the general solution

The A.E. of the given equation is

$$m^2 - m - 6 = 0 \quad \rightarrow \quad (m-3)(m+2) = 0$$

$$\therefore \quad m_1 = 3 \ , \qquad m_2 = -2$$

$\therefore \quad z_1 = \emptyset_1(y+3x) + \emptyset_2(y-2x)$

Where $\emptyset_1$and$\emptyset_2$are arbitrary functions

2) To find the particular Integral (P.I.)

$$a = 2 \ , b = -3$$

$$F(a,b) = a^2 - ab - 6b^2$$

$F(2,-3) = 4 + 6 - 54 = -44 \neq 0$

$$z_2 = \frac{1}{F(a,b)} e^{ax+by} = \frac{1}{-44} e^{2x-3y}$$

$$\therefore \quad z = z_1 + z_2$$

$$= \emptyset_1(y+3x) + \emptyset_2(y-2x) - \frac{1}{44} e^{2x-3y}$$

<u>Ex.3</u>: **Solve** $\boldsymbol{(D_x^2 - D_x D_y - 6D_y^2)z = e^{3x+y}}$

Sol.

1) The general solution is similar to that in Ex.2

2) To find P.I.

$$a = 3 \ , b = 1$$

$$F(a,b) = a^2 - ab - 6b^2$$

$F(3,1) = 9 - 3 - 6 = 0$,

analyze $F(D_x, D_y)$,$F(D_x, D_y) = D_x^2 - D_x D_y - 6D_y^2$

$$= (D_x - 3D_y)(D_x + 2D_y)$$

$(D_x - \frac{a}{b} D_y)^r \rightarrow \therefore r = 1 \ , \quad 3 + 2 = 5 \neq 0 = G$

$$z_2 = \frac{1}{G(a,b)} \cdot \frac{x^r}{r!} e^{ax+by} = \frac{1}{5} \cdot \frac{x}{1} e^{3x+y} = \frac{x}{5} e^{3x+y}$$

$$\therefore \quad z = z_1 + z_2$$

$$= \emptyset_1(y+3x) + \emptyset_2(y-2x) + \frac{x}{5}e^{3x+y}$$

Where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions

Case 2 **when** $\boldsymbol{f(x,y) = \sin(ax+by)}$ ***or*** $\boldsymbol{\cos(ax+by)}$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary constant

Here, we will find the P.I. of (H.L.P.D.E.) of order 2 only, by the same way that in case 1 we will derive $f(x,y)$ for x and y .

Let $f(x,y) = \sin(ax+by)$

$$D_x \sin(ax+by) = a\cos(ax+by)$$

$$\mathrm{D}_x^2 \sin(ax+by) = -a^2\sin(ax+by)$$

$$D_y \sin(ax+by) = b\cos(ax+by)$$

$$\mathrm{D}_y^2 \sin(ax+by) = -b^2\sin(ax+by)$$

$$D_x D_y \sin(ax+by) = D_x[b\cos(ax+by)]$$

$$= -ab\sin(ax+by)$$

$$F(\mathrm{D}_x^2, D_x D_y, \mathrm{D}_y^2)\sin(ax+by) = F(-a^2,-ab,-b^2)\sin(ax+by)$$

Multiplying both sides by $\frac{1}{F(\mathrm{D}_x^2, D_x D_y, \mathrm{D}_y^2)}$

$$\sin(ax+by) = \frac{1}{F(\mathrm{D}_x^2, D_x D_y, \mathrm{D}_y^2)}F(-a^2,-ab,-b^2)\sin(ax+by)$$

If $\boxed{F(-a^2,-ab,-b^2) \neq 0}$ then we can divide on it

$$\rightarrow z = \frac{1}{F(\mathrm{D}_x^2, D_x D_y, \mathrm{D}_y^2)}\sin(ax+by)$$

$$= \frac{1}{F(-a^2,-ab,-b^2)}\sin(ax+by)$$

Which it is the particular integral.

And if $F(-a^2,-ab,-b^2) = 0$ then we write

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad , \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

And follow the solution of the exponential function in case1.

Ex.4: Solve $\left(D_x^2 - D_xD_y - 6D_y^2\right)z = \sin(2x - 3y)$

Sol.

1) The general solution $z_1$is the same in Ex.2

2) The P.I.z_2

$$a = 2 \ , b = -3$$

$$F(-a^2, -ab, -b^2) = -a^2 + ab + 6b^2$$

$$F(-4, 6, -9) = -4 - 6 + 54 = 44 \neq 0$$

$$z_2 = \frac{1}{44}\sin(2x - 3y)$$

The required general solution

$$\therefore \quad z = z_1 + z_2$$

$$= \emptyset_1(y + 3x) + \emptyset_2(y - 2x) + \frac{1}{44}\sin(2x - 3y)$$

Where $\emptyset_1$and$\emptyset_2$are arbitrary functions.

Ex. 5: Solve $\left(D_x^2 - 3D_xD_y + D_y^2\right)z = e^{2x+3y} + e^{x+y} + \sin(x - 2y)$

Sol.

1) Finding the general solution z_1

The A.E. is

$$m^2 - 3m + 2 = 0 \;\Rightarrow\; (m - 2)(m - 1) = 0$$

$$\therefore m_1 = 2 , m_2 = 1$$

$\therefore z_1 = \emptyset_1(y+2x) + \emptyset_2(y+x)$

where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

2) The P.I. of the given equation is

P.I. $= z_2 = \frac{1}{F(D_x,D_y)} e^{2x+3y} + \frac{1}{F(D_x,D_y)} e^{x+y} + \frac{1}{F(D_x,D_y)} \sin(x-2y)$

Let $u_1 = \frac{1}{F(D_x,D_y)} e^{2x+3y}$ $\quad ,a = 2, b = 3$

$F(D_x,D_y) = a^2 - 3ab + 2b^2$

$F(1,1) = 4 - 18 + 18 = 4 \neq 0$

$$\boxed{u_1 = \frac{1}{4} e^{2x+3y}}$$

$u_2 = \frac{1}{F(D_x,D_y)} e^{x+y}$ $\quad ,a = 1, b = 1$

$F(D_x,D_y) = a^2 - 3ab + 2b^2$

$F(1,1) = 1 - 3 + 2 = 0$

Analyze $F(D_x,D_y)$,

$F(D_x,D_y) = (D_x - 2D_y)(D_x - D_y)$

$$u_2 = \frac{1}{G(a,b)} \frac{x^r}{r!} e^{ax+by}$$

$$= \frac{1}{-1} \frac{x}{1} e^{x+y}$$

$$\boxed{u_2 = -x\, e^{x+y}}$$

$$u_3 = \frac{1}{F(D_x,D_y)} \sin(x-2y)$$

$F(-a^2, -ab, -b^2) = -a^2 + 3ab - 2b^2$

$F(-1,2,-4) = -1 - 6 - 8 = -15 \neq 0$

$$\boxed{u_3 = \frac{1}{-15} \sin(x-2y)}$$

Then, the required general solution is

$$z = z_1 + z_2 = \emptyset_1(y+2x) + \emptyset_2(y+x) + \frac{1}{4}e^{2x+3y} - xe^{x+y}$$
$$-\frac{1}{15}\sin(x-2y)$$

where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

Ex. 6: Find the P.I. of the equation

$$\left(D_x^2 - 4D_xD_y + 3D_y^2\right)z = \cos(x+y)$$

Sol. $a = 1, b = 1$

$F(-a^2, -ab, -b^2) = -a^2 + 4ab - 3b^2$

$F(-1,-1,-1) = -1 + 4 - 3 = 0$

Taking $\cos(x+y) = \frac{e^{ix+iy} + e^{-ix-iy}}{2}$

$$z = \frac{1}{2}\left[\frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{ix+iy} + \frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{-ix-iy}\right]$$

Let$u_1 = \frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{ix+iy}$

To find $u_1 a = i, b = i$

$F(a,b) = a^2 - 4ab + 3b^2$

$F(i,i) = i^2 - 4i^2 + 3i^2 = 0$

Analyze $F(D_x, D_y)$,

$F(D_x, D_y) = (D_x - D_y)(D_x - 3D_y)$

$$u_1 = \frac{1}{-2i}xe^{ix+iy}$$

By the same way $u_2 = \frac{1}{2i}xe^{-ix-iy}$

$$\therefore z = \frac{1}{2}\left[\frac{1}{-2i}xe^{ix+iy} + \frac{1}{2i}xe^{-ix-iy}\right]$$

$= \frac{-x}{2}\left[\frac{e^{ix+iy}-e^{-ix-iy}}{2i}\right] = \frac{-x}{2}sin(x+y)$which is the P.I.

Case 3 **When $f(x,y) = x^a y^b$ where a and b are Non- Negative Integer Number**

The particular integral (P.I.) is evaluated by expanding the function$\frac{1}{F(D_x,D_y)}$in an infinite series of ascending powers of D_x or D_y (i.e.) by transfer the function $\frac{1}{F(D_x,D_y)}$ according to the following

$$\frac{1}{1-\theta} = 1+\theta+\theta^2+\cdots$$

<u>Ex.7</u>: Find P.I. of the equation $(D_x^2 - 2D_xD_y)z = x^3y$

Sol.P.I. $= \frac{1}{D_x^2-2D_xD_y}x^3y$

$= \frac{1}{D_x^2(1-2\frac{D_y}{D_x})}x^3y,$ $\qquad D_y^n y^m = 0 \; if \; n > m$

$= \frac{1}{D_x^2}\left[1+2\frac{D_y}{D_x}+\frac{4D_y^2}{D_x^2}+\cdots\right]x^3y$ $\qquad , \frac{4D_y^2}{D_x^2} = 0$

$$= \frac{1}{D_x^2}\left[x^3y+\frac{1}{2}x^4\right]$$

$$= \frac{1}{D_x}\left[\frac{x^4y}{4}+\frac{x^5}{10}\right] = \frac{x^5y}{20}+\frac{x^6}{60}$$

<u>Ex.8</u>: Find P.I. of the equation $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = x^2y$

Sol. P.I. $= \frac{1}{D_x^3-7D_xD_y^2-6D_y^3}x^2y$

$= \frac{1}{D_x^3\left[1-\left(\frac{7D_y^2}{D_x^2}+\frac{6D_y^3}{D_x^3}\right)\right]} x^2y$

$$= \frac{1}{D_x^3}\left[1 + \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right) + \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)^2 + \cdots\right] x^2y$$

$= \frac{1}{D_x^3}[x^2y]$ since $\left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right) = 0 , \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)^2 = 0$

$$= \frac{1}{D_x^2}\frac{x^3y}{3} = \frac{1}{D_x}\frac{x^4y}{12} = \frac{x^5y}{60}$$

<u>Ex.9</u>: Solve $(D_x^3 - a^2D_xD_y^2)z = x$, where $a \in R$

Sol.

1) the general solution z_1

The A.E. of the given equation is

$m^3 - a^2m = 0 \implies m(m^2 - a^2) = 0$

$$\implies m(m-a)(m+a) = 0$$

$\therefore m_1 = 0 , m_2 = a \; , m_3 = -a$ (different roots) $\therefore z_1 = \emptyset_1(y) + \emptyset_2(y + ax) + \emptyset_3(y - ax)$

where $\emptyset_1, \emptyset_2$ and $\emptyset_3$ are arbitrary functions.

2) The P.I. of the given equation is

P.I. $= z_2 = \frac{1}{D_x^3 - a^2D_xD_y^2} x$

$= \frac{1}{D_x^3\left[1-\frac{a^2D_y^2}{D_x^2}\right]} x$

$= \frac{1}{D_x^3}\left[1 + \frac{a^2D_y^2}{D_x^2} + \left(\frac{a^2D_y^2}{D_x^2}\right)^2 + \cdots\right] x (\frac{a^2D_y^2}{D_x^2} = 0 , \left(\frac{a^2D_y^2}{D_x^2}\right)^2 = 0)$

$$=\frac{1}{D_x^3}[x]$$

$$=\frac{1}{D_x^2}\left[\frac{x^2}{2}\right]$$

$$=\frac{1}{D_x}\left[\frac{x^3}{6}\right]=\frac{x^4}{24}$$

then, the required general solution is

$$z=z_1+z_2=\emptyset_1(y)+\emptyset_2(y+ax)+\emptyset_3(y-ax)+\frac{x^4}{24}$$

Case 4 **When $f(x,y)=e^{ax+by}V$ where V is a function of x and y**

The P.I. in this case is $z=\frac{1}{F(D_x,D_y)}e^{ax+by}\,V$

$$=e^{ax+by}\frac{1}{F(D_x+a\,,\,D_y+b)}\,V$$

and solving this equation depending on the type of V can get the particular integral (P.I.), as follows:

<u>Ex.10</u>: Find P.I. of the equation $D_xD_yz=e^{2x+3y}x^2y$

Sol. P.I. $=\frac{1}{D_xD_y}e^{2x+3y}x^2y a=2, b=3\ and\ V=x^2y$

$=e^{2x+3y}\frac{1}{(D_x+2)(D_y+3)}x^2y$

$$=e^{2x+3y}\frac{1}{3(D_x+2)(1+\frac{D_y}{3})}x^2y$$

$$=e^{2x+3y}\frac{1}{3(D_x+2)}\left[1-\frac{D_y}{3}+\frac{D_y^2}{9}-\cdots\right]x^2y$$

$$= e^{2x+3y}\frac{1}{3(D_x+2)}\left[x^2y-\frac{x^2}{3}\right]$$

$$= e^{2x+3y}\frac{1}{6(1+\frac{D_x}{2})}\left[x^2y-\frac{x^2}{3}\right]$$

$$=\frac{1}{6}e^{2x+3y}\left[1-\frac{D_x}{2}+\frac{D_x^2}{4}-\frac{D_x^3}{8}+\cdots\right]\left[x^2y-\frac{x^2}{3}\right],\left(\frac{D_x^3}{8}=0\right)$$

$$=\frac{1}{6}e^{2x+3y}\left[x^2y-\frac{x^2}{3}-xy+\frac{x}{3}+\frac{y}{2}-\frac{1}{6}\right]$$

$$= e^{2x+3y}\left[\frac{1}{6}x^2y-\frac{x^2}{18}-\frac{1}{6}xy+\frac{x}{18}+\frac{y}{12}-\frac{1}{36}\right]$$

Ex.11: Find P.I. of the equation $\left(D_x^2-D_xD_y\right)z=e^{x+y}\,xy^2$

Sol.

P.I. $=\frac{1}{D_x^2-D_xD_y}e^{x+y}\,xy^2 a=1, b=1\ and\ V=xy^2$

$=e^{x+y}\frac{1}{(D_x+1)(D_x-D_y)}xy^2$since $D_x^2-D_xD_y=D_x\left(D_x-D_y\right)$

$$= e^{x+y}\frac{1}{(D_x+1)D_x(1-\frac{D_y}{D_x})}xy^2$$

$$= e^{x+y}\frac{1}{(D_x+1)D_x}\left[1+\frac{D_y}{D_x}+\frac{D_y^2}{D_x^2}+\cdots\right]xy^2$$

$$= e^{x+y}\frac{1}{(D_x+1)D_x}\left[xy^2+\frac{2xy}{D_x}+\frac{2x}{D_x^2}\right]$$

$$= e^{x+y}\frac{1}{(D_x+1)D_x}\left[xy^2+x^2y+\frac{x^3}{3}\right]$$

$$= e^{x+y}\frac{1}{(D_x+1)}\left[\frac{x^2y^2}{2}+\frac{x^3y}{3}+\frac{x^4}{12}\right]$$

$$= e^{x+y}[1 - D_x + D_x^2 - D_x^3 + D_x^4 - D_x^5 + \cdots]\left[\frac{x^2y^2}{2} + \frac{x^3y}{3} + \frac{x^4}{12}\right]$$

where $D_x^5 = 0$

$$= e^{x+y}\left[\frac{x^2y^2}{2} + \frac{x^3y}{3} + \frac{x^4}{12} - xy^2 - x^2y - \frac{x^3}{3} + y^2 + 2xy + x^2 - 2y - 2x + 2\right]$$

<u>Ex.12</u>: Find P.I. of the equation $(D_x - D_y)^2 z = e^{x+y}\sin(x + 2y)$

Sol. P.I. $= \frac{1}{(D_x - D_y)^2} e^{x+y}\sin(x + 2y)$, $a_1 = 1, b_1 = 1$

$= e^{x+y}\frac{1}{(D_x + 1 - D_y - \cancel{1})^2}\sin(x + 2y)$

$$= e^{x+y}\frac{1}{(D_x - D_y)^2}\sin(x + 2y)$$

$= e^{x+y}\frac{1}{D_x^2 - 2D_xD_y + D_y^2}\sin(x + 2y)$, $a_2 = 1, b_2 = 2$

$$F(-a_2^2, -a_2b_2, -b_2^2) = -a_2^2 + 2a_2b_2 - b_2^2$$

$$F(-1, -2, -4) = -1 + 4 - 4 = -1 \neq 0$$

$$\therefore z = e^{x+y}.\frac{1}{-1}\sin(x + y) \implies z = -e^{x+y}\sin(x + y)$$

Case 5 **When $f(x, y) = g(ax + by)$ where $F(a, b) \neq 0$**

The particular integral of H.L.P.D.E. of order n is

$$z = \frac{1}{F(a, b)}\underbrace{\int \cdots \int}_{n - times} g(ax + by)\, \underbrace{d(ax + by) \ldots d(ax + by)}_{n - times}$$

<u>Ex.13</u>: Find P.I. of $(D_x^2 + 2D_xD_y - 8D_y^2)z = \sqrt{2x + 3y}$

Sol.

$a = 2, b = 3$, $g(2x + 3y) = \sqrt{2x + 3y}$

$F(a, b) = a^2 + 2ab - 8b^2$

$F(2,3) = 4 + 12 - 72 = -56 \neq 0$, integrating g twice

$\therefore$ P.I. $= z = \frac{1}{-56}\iint \sqrt{2x+3y}\, d(2x+3y)d(2x+3y)$

$$= \frac{1}{-56}\int \frac{2}{3}(2x+3y)^{3/2} d(2x+3y)$$

$$= \frac{4}{-56\ (15)}(2x+3y)^{5/2}$$

$$= \frac{-1}{210}(2x+3y)^{5/2}$$

Case 6 **When $f(x,y) = g(ax+by)$ where $F(a,b) = 0$**

If $F(a,b) = 0$, then $F(D_x, D_y)$ can be written as

$$F(D_x, D_y) = (bD_x - aD_y)^n$$

and the particular solution is $\boxed{Z = \frac{x^n}{n!}\frac{g(ax+by)}{b^n}}$

Ex.14: Find P.I. of $\left(D_x^2 - 6D_xD_y + 9D_y^2\right)z = 3x + y$

Sol. $a = 3, b = 1 \quad , g(3x+y) = 3x + y$

$F(a,b) = a^2 - 6ab + 9b^2$

$$F(3,1) = 9 - 18 + 9 = 0$$

Then $F(D_x, D_x) = D_x^2 - 6D_xD_y + 9D_y^2 = (D_x - 3D_y)^2$, so $n = 2$

$\therefore$ P.I. $= z = \frac{x^2}{2!}\frac{3x+y}{1^2} = \frac{1}{2}x^2(3x+y)$

Ex.15: Find P.I. of $\left(D_x^2 - 4D_xD_y + 4D_y^2\right)z = \tan(2x+y)$

Sol. $a = 2, b = 1 \quad , g(2x+y) = \tan(2x+y)$

$F(a,b) = a^2 - 4ab + 4b^2$

$$F(2,1) = 4 - 8 + 4 = 0$$

Then $F(D_x, D_y) = D_x^2 - 4D_xD_y + 4D_y^2 = (D_x - 2D_y)^2$, so $n = 2$

$\therefore$ P.I. $= z = \frac{x^2}{2!}\frac{\tan(2x+y)}{1^2} = \frac{1}{2}x^2 tan(2x + y)$

<u>Ex.16</u>: Find P.I. of $\boldsymbol{(D_x^2 - D_y^2)z} = \mathbf{sec^2}(\boldsymbol{x + y})$

Sol. $a = 1, b = 1$ $\quad , g(x + y) = \sec^2(x + y)$

$$F(a, b) = a^2 - b^2$$

$$F(1,1) = 1 - 1 = 0$$

Then $F(D_x, D_y) = D_x^2 - D_y^2 = (D_x - D_y)(D_x + D_y)$

$$\therefore z = \frac{1}{(D_x - D_y)(D_x + D_y)}\sec^2(x + y)$$

Let $u_1 = \frac{1}{(D_x + D_y)}\sec^2(x + y)$ by case (5) we have

$u_1 = \frac{1}{F(a,b)}\int g(ax + by)d(ax + by)$ $\quad , \quad F(1,1) = 1 + 1 = 2$

$$= \frac{1}{2}\int \sec^2(x + y)d(x + y)$$

$$= \frac{1}{2}\tan(x + y)$$

$$\Longrightarrow z = \frac{1}{(D_x - D_y)}\frac{1}{2}\tan(x + y)$$

$F(D_x, D_y) = D_x - D_y$

$F(1,1) = 1 - 1 = 0$ $\qquad$ where $n = 1$

$$\therefore z = \frac{x^1}{1!}\frac{1}{2}\frac{\tan(x + y)}{1}$$

$= \frac{x}{2}\tan(x + y)$ which its' the particular integral

…General Exercises …

1- $\left(D_x^4 - D_y^4\right)z = 0$

2- $\left(D_x^3 - 7D_xD_y^2 - 6D_y^3\right)z = \cos(x-y) + x^2 + xy^2 + y^2$

3- $\left(D_x - 2D_y\right)z = e^{3x}(y+1)$

4- $\left(D_x^2 + 3D_xD_y + 2D_y^2\right)z = x + y$

5- $\left(D_x^2 - 5D_xD_y + 4D_y^2\right)z = \sin(4x+y)$

6- $\left(2D_x^2 - D_xD_y - 3D_y^2\right)z = \frac{5\,e^x}{e^y}$

7- $\left(D_x^2 - 3D_xD_y + 2D_y^2\right)z = e^{2x-y} + \cos(x+2y)$

8- $\left(D_x^2 - D_xD_y\right)z = \ln y$

9- $\left(D_x + D_y\right)z = \sec(x+y)$

10- $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

11- $(y^2 + z^2 - x^2)p - 2xyq = -2xz$

12- $pq + 2y(x+1)q + x(x+2)q - 2(x+1) = 0$

13- $(x^2 + 2x)p + (x+1)qy = 0$

14- $\left(D_x^3 - 3D_xD_y^2 + 2D_y^3\right)z = \frac{1}{\sqrt{3x-y}}$

15- $\left(D_x^3 + 2D_x^2D_y - D_xD_y^2 - 2D_y^3\right)z = (y+2)e^x$

16- $\left(4D_x^2 - 4D_xD_y + D_y^2\right)z = (x+2y)^{3/2}$

17- $D_xD_yz = e^{x-y}xy^2$

18- $\left(D_x - D_y\right)z = \tan(x+2y)$

19- $2\left(D_x^3 - 9D_x^2D_y + 27D_xD_y^2 - 27D_y^3\right)z = \tan^{-1}(3x+y)$

20- $(y^3x - 2x^4)\frac{\partial z}{\partial x} + (2y^4 - x^3y)\frac{\partial z}{\partial y} = x^3 - y^3$

Chapter Two

Non-homogeneous Linear Partial Differential Equations

Contents

Section(2.1):Non-homogeneous linear partial differential equations with constant coefficients

Definition: A linear partial differential equation with constant coefficients is known as non-homogeneous l.p.d.e. with constant coefficients if the order of all the partial derivatives involved in the equation are not all equal.

For example:

1) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} + z = x + y$

2) $\frac{\partial^3 z}{\partial x^3} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = e^{x+y}$

Definition: A linear differential operator $F(D_x, D_y)$ is known as (reducible), if it can be written as the product of linear factors of the form $aD_x + bD_y + c$ with a, b and c as constants. $F(D_x, D_y)$ is known as (irreducible), if it is not reducible.

For example:

The operator $D_x^2 - D_y^2$ which can be written in the form $(D_x - D_y)(D_x + D_y)$ isreducible, whereas the operator $D_x^2 - D_y^3$ which cannot be decomposed into linear factors is irreducible.

<u>Note</u>: A l.p.d.e with constant coefficient $F(D_x, D_y)z = f(x, y)$ is known as reducible, if $F(D_x, D_y)$ reducible, and is known as irreducible, if $F(D_x, D_y)$ is irreducible.

(2.1.1)Determination of Complementary function (C.F.)(the general solution) of a reducible non-homo.l.p.d.e. with constant coefficients

(A) let $F(D_x, D_y) = (aD_x + bD_y + c)^k$,where a, b, c are constants and k is anatural number

then the equation $F(D_x, D_y)z = 0$ will by

$(aD_x + bD_y + c)^k z = 0$ and the solution is

$$z = e^{\frac{-c}{a}x}\emptyset(ay - bx) \qquad ; \quad a \neq 0 \ , k = 1$$

Or

$$z = e^{\frac{-c}{b}y}\emptyset(ay - bx) \qquad ; \quad b \neq 0 \ , k = 1$$

For any $k > 1$, the solution is

$$z = e^{\frac{-c}{b}y}[\emptyset_1(ay - bx) + x\emptyset_2(ay - bx) + \cdots + x^{k-1}\emptyset_k(ay - bx); \ b \neq 0$$

Or

$$z = e^{\frac{-c}{a}x}[\emptyset_1(ay - bx) + x\emptyset_2(ay - bx) + \cdots + x^{k-1}\emptyset_k(ay - bx); \ a \neq 0$$

Where $\emptyset_1, \dots, \emptyset_n$ are arbitrary functions.

<u>Ex.1</u>: Solve $(2D_x - 3D_y - 5)z = 0$

Sol. The given equation is linear in $F(D_x, D_y)$

Then $a = 2, b = -3, c = -5, k = 1$

The general solution is

$$z = e^{\frac{5}{2}x}\emptyset(2y + 3x)$$

Where $\emptyset$ is an arbitrary function.

Ex.2: Solve $(D_x - 5)z = e^{x+y}$

Sol. To find the general solution of $(D_x - 5)z = 0$

We have $a = 1\,, b = 0\,, c = -5\,, k = 1$

$\therefore z_1 = e^{5x}\emptyset(y)$, Where $\emptyset$ is an arbitrary function.

To find the P.I. z_2 , we have $a = 1\,, b = 1$

$$F(a,b) = a - 5 \;\rightarrow\; F(1,1) = 1 - 5 = -4 \neq 0$$

$$\therefore z_2 = \frac{1}{-4} e^{x+y}$$

Then the required general solution of the given equation is

$$z = z_1 + z_2 \;\rightarrow\; z = e^{5x}\emptyset(y) - \frac{1}{4} e^{x+y}$$

Ex.3: Solve $(2D_y + 5)^2 z = 0$

Sol. The given equation is reducible, then

$$a = 0\,, b = 2\,, c = 5\,, k = 2\,.$$

The general solution is

$$z = e^{\frac{-5}{2}y}[\emptyset_1(-2x) + x\emptyset_2(-2x)]$$

Where $\emptyset_1$ *and* $\emptyset_2$ are arbitrary functions

Ex.4: Solve $(D_x - 2D_y + 1)^4 z = 0$

Sol. We have $a = 1, b = -2, c = 1, k = 4$

then

$$z = e^{\frac{1}{2}y}[\emptyset_1(y + 2x) + x\emptyset_2(y + 2x) + x^2\emptyset_3(y + 2x) + x^3\emptyset_4(y + 2x)]$$

Where $\emptyset_1 , \dots , \emptyset_4$ are arbitrary functions

(B) when $F(D_x, D_y)$can be written as the product of linear factors of the form $(aD_x + bD_y + c)$, i.e. $F(D_x, D_y)$ is reducible , then the general solution is the sum of the solutions corresponding to each factor.

Ex.5: **solve** $\underbrace{(2D_x - 3D_y + 1)}_{\text{linear}}\underbrace{(D_x + 2D_y - 2)}_{\text{linear}}z = 0$

Sol. The given equation is reducible, then we have

$$a_1 = 2 \ , b_1 = -3 \ , c_1 = 1 \quad , k_1 = 1$$

$$z_1 = e^{\frac{-1}{2}x}\emptyset_1(2y + 3x)$$

$$a_2 = 1 \ , b_2 = 2 \quad , c_2 = -2 \quad , k_2 = 1$$

$$z_2 = e^{2x}\emptyset_2(y - 2x)$$

The general solution is

$$z = z_1 + z_2 \quad \rightarrow \quad z = e^{\frac{-1}{2}x}\emptyset_1(2y + 3x) + e^{2x}\emptyset_2(y - 2x)$$

Where $\emptyset_1$, $\emptyset_2$ are two arbitrary functions.

Ex.6: **solve**$D_x(D_x + D_y + 1)(D_x + 3D_y - 2)z = 0$

Sol. We have

$$a_1 = 1 \ , b_1 = 0 \quad , c_1 = 0 \quad , k_1 = 1$$

$$a_2 = 1 \ , b_2 = 1 \quad , c_2 = 1 \quad , k_2 = 1$$

$$a_3 = 1 \ , b_3 = 3 \quad , c_3 = -2 \quad , k_3 = 1$$

Then the general solution is

$$z = \emptyset_1(y) + e^{-x}\emptyset_2(y - x) + e^{2x}\emptyset_3(y - 3x)$$

Where $\emptyset_1, \dots, \emptyset_3$ are arbitrary functions.

Ex.7: solve $\left(D_x^3 - D_x D_y^2 - D_x^2 + D_x D_y\right)z = 0$

Sol. We have , $\left(D_x^3 - D_x D_y^2 - D_x^2 + D_x D_y\right)z = 0$

$$D_x\left(D_x^2 - D_y^2 - D_x + D_y\right)z = 0$$

$$D_x\left[\left(D_x - D_y\right)\left(D_x + D_y\right) - \left(D_x - D_y\right)\right] = 0$$

$$D_x\left(D_x - D_y\right)\left(D_x + D_y - 1\right)z = 0$$

Then , $a_1 = 1$, $b_1 = 0$, $c_1 = 0$, $k_1 = 1$

$a_2 = 1$, $b_2 = -1$, $c_2 = 0$, $k_2 = 1$

$a_3 = 1$, $b_3 = 1$, $c_3 = -1$, $k_3 = 1$

Then the general solution is

$$z = \emptyset_1(y) + \emptyset_2(y + x) + e^x \emptyset_3(y - x)$$

Where $\emptyset_1, \dots, \emptyset_3$ are arbitrary functions.

(C) When $F(D_x, D_y)$ is irreducible then the general solution is

$$z = \sum_{i=1}^{\infty} A_i\, e^{a_i x + b_i y}$$

Where $F(a_i, b_i) = 0$,A_i , a_i, b_i are all constants.

Ex.8: Solve $\left(D_x - D_y^3\right)z = 0$

Sol. The given equation is irreducible, then

$$F(a, b) = 0 \quad \rightarrow \quad F(a_i, b_i) = 0$$

$$a - b^3 = 0 \quad \rightarrow \quad a_i - {b_i}^3 = 0 \quad \rightarrow \quad a_i = {b_i}^3$$

The general solution is

$$z = \sum_{i=1}^{\infty} A_i\, e^{a_i x + b_i y} = \sum_{i=1}^{\infty} A_i\, e^{{b_i}^3 x + b_i y}$$

Where A_i , $b_i$are constants.

Ex.9: Solve $\left(D_x^2 + D_x + D_y\right)z = 0$

Sol. The given equation is irreducible, then

$$F(a,b) = a^2 + a + b = 0 \quad \rightarrow \; {a_i}^2 + \; a_i + b_i = 0$$

$$\rightarrow \quad b_i = -{a_i}^2 - \; a_i$$

The general solution is

$$z = \sum_{i=1}^{\infty} A_i \, e^{a_i x + b_i y} = \sum_{i=1}^{\infty} A_i \, e^{a_i x + (-{a_i}^2 - \, a_i)y}$$

Where A_i , $a_i$are constants.

Ex.10: Solve $\left(D_x - D_y^2\right)z = e^{2x+3y}$

Sol. (1) we find the general solution of the irreducible eqution

$$\left(\mathrm{D_x} - \mathrm{D_y^2}\right)\mathrm{z} = 0$$

$$F(a,b) = a - b^2 = 0 \;\; \rightarrow \;\; F(a_i, b_i) = \; a_i - {b_i}^2 = 0 \rightarrow \; a_i = {b_i}^2$$

Then

$$z_1 = \sum_{i=1}^{\infty} A_i \, e^{a_i x + b_i y} = \sum_{i=1}^{\infty} A_i \, e^{{b_i}^2 x + b_i y}$$

Where A_i , $b_i$are constants.

(2) The P.I. is

$$F(a,b) = a - b^2$$

$$\therefore F(2,3) = 2 - 9 = -7 \neq 0$$

$$\therefore z_2 = \frac{1}{-7} e^{2x+3y}$$

And the required general solution is

$$z = z_1 + z_2 = \sum_{i=1}^{\infty} A_i\, e^{{b_i}^2 x + b_i y} - \frac{1}{7} e^{2x+3y}$$

(D) When $F(D_x, D_y)$ can be written as the product of linear and non-linear factors the general solution is the sum of the solutions corresponding to each factor.

<u>Ex.11</u>: Solve $\left(\boldsymbol{D_x + 2D_y}\right)\left(\boldsymbol{D_x - 2D_y + 1}\right)\left(\boldsymbol{D_x - D_y^2}\right)\boldsymbol{z = 0}$

Sol:

Factor 1, $a_1 = 1\ , b_1 = 2\ \ , c_1 = 0\ \ , k_1 = 1$

Factor 2, $a_2 = 1\ , b_2 = -2\ \ , c_2 = 1\ \ , k_2 = 1$

Factor 3, $F(a,b) = a - b^2 = 0 \ \rightarrow\ a = b^2 \ \rightarrow\ \ a_i = b_i^2$

$$\therefore\ \mathrm{z} = \emptyset_1(y - 2x) + \mathrm{e}^{\frac{1}{2}\mathrm{y}}\emptyset_2(y + 2x) + \sum_{i=1}^{\infty} A_i\, e^{{b_i}^2 x + b_i y}$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions and $A_i\ , b_i$ are constants.

<u>Ex.12</u>: Solve $\left(\boldsymbol{D_x^2 - D_y^2 + D_x}\right)\boldsymbol{z = x^2 + 2y}$

Sol: (1) The general solution of $\left(\mathrm{D_x^2 - D_y^2 + D_x}\right)\mathrm{z} = 0$ is

$F(a,b) = a^2 - b^2 + a = 0\ \rightarrow b = \pm\sqrt{a^2 + a} \rightarrow b_i = \pm\sqrt{a_i^2 + a_i}$

Then

$$z_1 = \sum_{i=1}^{\infty} A_i\, e^{a_i x \pm \sqrt{a_i^2 + a_i}\, y}$$

(2) The P.I. is

$$z_1 = \frac{1}{D_x^2 - D_y^2 + D_x}(x^2 + 2y)$$

$$= \frac{1}{D_x(1 + D_x - \frac{D_y^2}{D_x})}(x^2 + 2y)$$

$$= \frac{1}{D_x[1 - \left(\frac{D_y^2}{D_x} - D_x\right)]}(x^2 + 2y)$$

$$= \frac{1}{D_x}[1 + \underbrace{\frac{D_y^2}{D_x}}_{=0} - D_x + \left(\frac{D_y^2}{D_x} - D_x\right)^2 + \cdots](x^2 + 2y)$$

$$= \frac{1}{D_x}[x^2 + 2y - 2x + 2] = \frac{x^3}{3} + 2xy - x^2 + 2x$$

The required general solution is

$$z = z_1 + z_2 = \sum_{i=1}^{\infty} A_i \, e^{a_i x \pm \sqrt{a_i^2 + a_i y}} + \frac{x^3}{3} + 2xy - x^2 + 2x$$

<u>Ex.13:</u>Solve$(2D_x + 3D_y)(3D_x - 4D_y + 5)(3D_x - D_y^2)z = 0$

Sol:

Factor 1, $a_1 = 2$, $b_1 = 3$, $c_1 = 0$, $k_1 = 1$

Factor 2, $a_2 = 3$, $b_2 = -4$, $c_2 = 5$, $k_2 = 1$

Factor 3, $F(a,b) = 3a - b^2 = 0 \quad \rightarrow \quad a = \frac{b^2}{3} \quad \rightarrow \quad a_i = \frac{b_i^2}{3}$

The general solution is

$$\therefore z = \emptyset_1(2y - 3x) + e^{\frac{5}{4}y}\emptyset_2(3y + 4x) + \sum_{i=1}^{\infty} A_i \, e^{\frac{b_i^2}{3}x + b_i y}$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions and A_i , $b_i$are constants.

NoteTo determine the P.I. of non-homo.p.d.e. when

$f(x, y) = \sin(ax + by) \quad or \quad \cos(ax + by)$ we put $D^2 = -a^2$, $D_y^2 = -b^2$, $D_x D_y = -ab$, which provided the denominator is non-zero, as follows.

Ex.14: Solve $\left(D_x^2 - D_y\right)z = \sin(x - 2y)$

Sol: (1)The general solution z_1 of $\left(D_x^2 - D_y\right)z = 0$ is

$$F(a, b) = \boxed{a^2 - b = 0} \quad \rightarrow \; a_i^2 = b_i$$

$$z_1 = \sum_{i=1}^{\infty} A_i \, e^{a_i x + a_i^2 y}$$

(2) To find the P.I. of the given equation

$$P.I. = z = \frac{1}{D_x^2 - D_y} \sin(x - 2y)$$

$$a = 1, \qquad b = -2 \quad \rightarrow D_x^2 = -a^2 = -1$$

$$= \frac{1}{-1 - D_y} \sin(x - 2y)$$

Multiplying by $\frac{1}{-1+D_y}$

$$= \frac{-1 + D_y}{1 - D_y^2} \sin(x - 2y)$$

$$D_y^2 = -b^2 = -4$$

$$= \frac{-1 + D_y}{1 + 4} \sin(x - 2y)$$

$$= \frac{1}{5}[-\sin(x-2y) - 2\cos(x-2y)]$$

…Exercises…

Solve the following equations:

1. $\left(D_x^2 + D_x D_y + D_y - 1\right)z = 0$
2. $(D_x + 1)\left(D_x - D_y + 1\right)z = 0$
3. $(D_x^2 + D_x D_y + D_x)z = 0$
4. $\left(D_x^2 + D_y + 4\right)z = e^{4x-y}$
5. $\left(D_x^2 + D_x D_y + D_y - 1\right)z = \sin(x+2y)$
6. $\left(D_x - D_y - 1\right)\left(D_x - D_y - 2\right)z = x$
7. $\left(D_x^2 - D_y^2 + D_x + 3D_y - 2\right)z = x^2 y$
8. $(D_x + 3D_y - 2)^2 z = 2e^{2x}\sin(y+3x)$

Section(2.2): Partial differential equations of order two with variable coefficients

In the present section, we propose to discuss partial differential equations of order two with variable coefficients. An equation is said to be of order two, if it involves at least one of the differential coefficients $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$, but none of higher order, the quantities p and q may also inter into the equation. Thus the general from of a second order partial differential equation is

$$\boldsymbol{R(x,y)\frac{\partial^2 z}{\partial x^2} + S(x,y)\frac{\partial^2 z}{\partial x \partial y} + T(x,y)\frac{\partial^2 z}{\partial y^2} + P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} + V(x,y)z = f(x,y)}$$

...(1)

Or $\quad Rr + Ss + Tt + Pp + Qq + Vz = f$ ...(2)

Where R, S, T, P, Q, V, f are functions of x and y only and not all R, S, T are zero.

We will discuss three cases of the equation (2):

Case 1when one of R, S, T not equal to zero and P, Q, V are equal to zero ,then the solution can be obtained by integrating both sides of the equation directly.

<u>**Ex.15:**</u>**Solve** $\boldsymbol{y\frac{\partial^2 z}{\partial x^2} + 5y - x^2y^2 = 0}$

Sol: Given equation can be written

$\frac{\partial^2 z}{\partial x^2} = yx^2 - 5$...(3)

Integrating (3) w.r.t. x

$\frac{\partial z}{\partial x} = \frac{yx^3}{3} - 5x + \emptyset_1(y)$...(4)

Integrating (4) w.r.t. x

$$z = \frac{yx^4}{12} - \frac{5}{2}x^2 + x\emptyset_1(y) + \emptyset_2(y)$$

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

Ex.16: Solve $xy\frac{\partial^2 z}{\partial x \partial y} - y^2 x = 0$

Sol: Given equation can be written

$\frac{\partial^2 z}{\partial x\, \partial y} = y$ ...(5)

Integrating (5) w.r.t. x

$\frac{\partial z}{\partial y} = xy + \emptyset_1(y)$...(6)

Integrating (6) w.r.t. y

$$z = \frac{xy^2}{2} + \int \emptyset_1(y)\partial y + \emptyset_2(x)$$

$$= \frac{xy^2}{2} + \varphi(y) + \emptyset_2(x)$$

Where $\varphi$and $\emptyset_2$ are two arbitrary functions.

Case2 When all the derivatives in the equation for one independent variable i.e the equation is of the form

$\boldsymbol{Rr + Pp + Vz = f(x, y)}$ **or** $\boldsymbol{Tt + Qq + Vz = f(x, y)}$

Some of these coefficients may be Zeros.

These equations will be treated as a ordinary linear differential equations, a follows:

<u>Ex.17</u>: Solve $\boldsymbol{y\frac{\partial^2 z}{\partial y^2} + 3\frac{\partial z}{\partial y} = 2x + 3}$

Sol: let $\frac{\partial z}{\partial y} = q \rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}$

Substituting in the given equation, we get

$y\frac{\partial q}{\partial y} + 3q = 2x + 3 \quad \rightarrow \quad \frac{\partial q}{\partial y} + \frac{3}{y}q = \frac{2x+3}{y}$ …(7)

Which it's linear diff. eq. in variables q and y , regarding x as a constant.

Integrating factor (I.F.)of (7) $= e^{\int \frac{3}{y}\partial y} = e^{3\ln y} = y^3$

And solution of (7) is

$$y^3 q = \int \frac{2x+3}{y} y^3 \partial y + \emptyset_1(x)$$

$$y^3 q = (2x+3)\frac{y^3}{3} + \emptyset_1(x)$$

$$q = \frac{2x+3}{3} + y^{-3}\emptyset_1(x)$$

$\frac{\partial z}{\partial y} = \frac{2x+3}{3} + y^{-3}\emptyset_1(x)$, integrating w.r.t. y

$$z = \frac{2x+3}{3}y - \frac{1}{2y^2}\emptyset_1(x) + \emptyset_2(x)$$

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

<u>Ex.18:</u> Solve $\frac{\partial^2 z}{\partial x^2} - 2y\frac{\partial z}{\partial x} + y^2 z = (y-3)e^{2x+3y}$

Sol: The given equation can be written as

$$D_x^2 - 2yD_x + y^2 z = (y-3)e^{2x+3y}$$

$\rightarrow (D_x - y)^2 z = (y-3)e^{2x+3y}$...(8)

The A.E. of the equation $(D_x - y)^2 z = 0$ is

$$(m-y)^2 = 0 \rightarrow m_1 = m_2 = y$$

$\therefore z_1 = \emptyset_1(y)e^{yx} + x\emptyset_2(y)e^{yx}$...(9)

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

The P.I. (z_2) is

$$z_2 = \frac{1}{(D_x - y)^2}(y-3)e^{2x+3y} = (y-3)\frac{1}{(2-y)^2}e^{2x+3y}$$

$$\therefore z = z_1 + z_2$$

$$= \emptyset_1(y)e^{yx} + x\emptyset_2(y)e^{yx} + (y-3)\frac{1}{(2-y)^2}e^{2x+3y}$$

Case3 under this type, we consider equations of the form

$$\boxed{\boldsymbol{Rr + Ss + Pp = f(x,y) \rightarrow R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x \partial y} + P\frac{\partial z}{\partial x} = f(x,y)}}$$

And $\boxed{\boldsymbol{Ss + Tt + Qq = f(x,y) \rightarrow S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} + Q\frac{\partial z}{\partial y} = f(x,y)}}$

These can be transform to a linear, p.d.es of order one with p or q as dependent variable and x, y as independent variables. In such situations we shall apply well known Lagrange's method.

<u>Ex.19</u>: Solve $\boldsymbol{x\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} = 0}$

Sol: let $p = \frac{\partial z}{\partial x} \rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}, \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$

Substituting in the given equation , we get

$x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y} - p = 0$…(10)

Which it is in Lagrange's form, the Lagrange's auxiliary equations are : $\frac{dx}{x} = \frac{dy}{-y} = \frac{dp}{p}$ …(11)

Taking the first and second fractions of (11)

: $\frac{dx}{x} = \frac{dy}{-y} \rightarrow lnx = -lny + lna \rightarrow \boxed{xy = a}$…(12)

Taking the first and the third fractions of (11)

$\frac{dx}{x} = \frac{dp}{p} \rightarrow lnx = lnp + lnb \rightarrow \boxed{\frac{x}{p} = b}$…(13)

From (12) &(13) , the general solution is

$$\emptyset(a,b) = 0 \rightarrow \emptyset\left(xy, \frac{x}{p}\right) = 0 \rightarrow \frac{x}{p} = g(xy)$$

$$\rightarrow p = \frac{x}{g(xy)}$$

$\rightarrow \boldsymbol{\frac{\partial z}{\partial x}} = \frac{x}{g(xy)}$…(14)

Integrating (14) w.r.t. x , we get

$z = \int \frac{x}{g(xy)}\, \partial x + \varphi(y)$…(15)

Where g and φ are two arbitrary functions.

Then (15) is the required solution of the given equation.

…Exercises…

Solve the following equations:

1)) $\ln\left(\frac{\partial^2 z}{\partial x \partial y}\right) = x + y$ 2)) $\frac{\partial^2 z}{\partial y^2} - x\frac{\partial z}{\partial y} = x^2$

3)) $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = \frac{x}{y}$ 4)) $y^2\frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} = 1$

Section2.3: Partial differential equations reducible to equations with constant coefficients

In this section, we propose to discuss the method of solving the partial differential equation, which is also called Euler-Cauchy type partial differential equations of the form :

$$a_0 x^n \frac{\partial^n z}{\partial x^n} + a_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + a_n y^n \frac{\partial^n z}{\partial y^n} + \cdots = f(x,y) \ldots(1)$$

i.e. all the terms of the equation of the formula $a_r x^n y^m \frac{\partial^{n+m} z}{\partial x^n \partial y^m}$

To solve this equation ,define two new variables u and v by

$x = e^u$ and $y = e^v$ so that $u = lnx$ and $v = lny$…(2)

Let $D_u = \frac{\partial}{\partial u}$and $D_v = \frac{\partial}{\partial v}$

Now, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{1}{x} \cdot \frac{\partial z}{\partial u}$, using (2)

$\therefore \frac{\partial z}{\partial u} = x \cdot \frac{\partial z}{\partial x} \rightarrow \boxed{D_u z = x D_x z}$…(3)

Again $x^2 \cdot \frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$

$= x^2 \frac{\partial}{\partial x}(\frac{1}{x} \cdot \frac{\partial z}{\partial u})$ from (3)

$$= x^2 \cdot \frac{1}{x} \cdot \frac{\partial^2 z}{\partial x \partial u} - x^2 \cdot \frac{\partial z}{\partial u} \cdot x^{-2}$$

$$= x \cdot \frac{\partial^2 z}{\partial x \partial u} - \frac{\partial z}{\partial u}$$

$$= x \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial z}{\partial u}$$

$$= x \frac{\partial}{\partial u}\left(\frac{1}{x} \cdot \frac{\partial z}{\partial u}\right) - \frac{\partial z}{\partial u}$$

$$= x \cdot \frac{1}{x} \cdot \frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u}$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u}$$

$$\therefore x^2 D_x^2 z = D_u(D_u - 1)z$$

And so on similarly, we have

$yD_y z = D_v z$,$y^2 D_y^2 z = D_v(D_v - 1)z$,…

Hence

$x^n \frac{\partial^n z}{\partial x^n} z = D_u(D_u - 1)(D_u - 2) \dots (D_u - n + 1)z$…(4)

$y^m \frac{\partial^m z}{\partial y^m} z = D_v(D_v - 1)(D_v - 2) \dots (D_v - m + 1)z$…(5)

$$x^n y^m \frac{\partial^{n+m} z}{\partial x^n \partial y^m} z = D_u(D_u - 1) \dots (D_u - n + 1) D_v(D_v - 1) \dots (D_v - m + 1) z \quad \dots(6)$$

Substituting (4),(5),(6) in (1) to get an equation having constant coefficients can easily be solved by the methods of solving homo. And non-homo. Partial differential equations with constant coefficients, Finally , with help of (2), the solution is obtained in terms of old variables x and y.

<u>Ex.20</u>: Solve $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = 0$

Sol: let $x = e^u$, $y = e^v$ then $u = lnx$ and $v = lny$

and $\left.\begin{array}{ll} x \frac{\partial z}{\partial x} = D_u z \quad , & x^2 \cdot \frac{\partial^2 z}{\partial x^2} = D_u(D_u - 1)z \\ y \frac{\partial z}{\partial y} = D_v z \quad , & y^2 \cdot \frac{\partial^2 z}{\partial y^2} = D_v(D_v - 1)z \end{array}\right\} \dots(7)$

Substituting (7) in the given equation,

$$(D_u^2 - D_u - D_v^2 + D_v - D_v + D_u)z = 0$$

$$(D_u^2 - D_v^2)z = 0 \to (D_u - D_v)(D_u + D_v)z = 0$$

The A.E. is $\underbrace{(m-1)}_{m_1=1}\underbrace{(m+1)}_{m_2=-1} = 0$

Then the general solution is

$$z = \emptyset_1(v + u) + \emptyset_2(v - u)$$

$$= \emptyset_1(lny + lnx) + \emptyset_2(lny - lnx)$$

$$= \emptyset_1(lnxy) + \emptyset_2\left(ln\frac{y}{x}\right)$$

$$= h_1(xy) + h_2\left(\frac{y}{x}\right)$$

Where $h_1$andh_2 are two arbitrary functions.

…Exercises…

Solve the following equations:

1)) $(x^2D_x^2 - y^2D_y^2 - yD_y + xD_x)z = xy$

2)) $(x^2D_x^2 - 2xyD_xD_y + y^2D_y^2 + yD_y + xD_x)z = 0$

3)) $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} = lnxy$

Classification of partial differential equations of second order:

Consider a general partial differential equation of second order for a function of two independent variables x and y in the form

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ …(*)

Where A, B, C, D, E, F, G are function of x, y or constants.

The equation (*) is said to be

(i) Hyperbolic at a point (x, y)in domain Dif $B^2 - 4AC > 0$.
(ii) Parabolic at a point (x, y)in domain Dif $B^2 - 4AC = 0$.
(iii)Elliptic at a point (x, y)in domain Dif $B^2 - 4AC < 0$.

Ex.21: Classify the following partial differential equation $2u_{xx} + 3u_{xy} = 0$

Sol:

Comparing the given equation with (*),we get $A = 2, B = 3, C = 0$

$$B^2 - 4AC = 9 - 4(2)(0) = 9 > 0$$

Showing that the given equation is hyperbolic at all points.

Ex.22: Classify the following p.d.eqs.

(1) $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

(2) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol. (1) Re-writing the given equation, we get

$$\alpha^2 u_{xx} - u_t = 0$$

Comparing with (*), we get $A = \alpha^2, B = 0, C = 0$

$$B^2 - 4AC = 0 - 4(\alpha^2)(0) = 0$$

Showing that the given equation is Parabolic at all points.

Sol. (2) Re-writing the given equation, we get

$$c^2 u_{xx} - u_{tt} = 0$$

Comparing with (*), we get $A = c^2, B = 0, C = -1$

$$B^2 - 4AC = 0 - 4(c^2)(-1) = 4c^2 > 0$$

Showing that the given equation is hyperbolic at all points.

Sol. (3) Comparing with (*), we get $A = 1, B = 0, C = 1$

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

Then the equation is an Elliptic at all points.

…Exercises…

Classify the following equations:

1)) $u_x - u_{xy} - u_y = 0$

2)) $u_{rr} - ru_{r\theta} + r^2 u_{\theta\theta} = 0$ $\quad ; u(r,\theta)$

3)) $z_{xx} + z_{xy} + z_y = 2x$

4)) $xyz_{xx} - (x^2 - y^2)z_{xy} - xyz_{yy} + yz_x - xz_y = 2(x^2 - y^2)$

5)) $x^2(y - 1)\frac{\partial^2 z}{\partial x^2} - x(y^2 - 1)\frac{\partial^2 z}{\partial x \partial y} + 4(y - 1)\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

6)) $u_r - u_{\theta\theta} = 5$

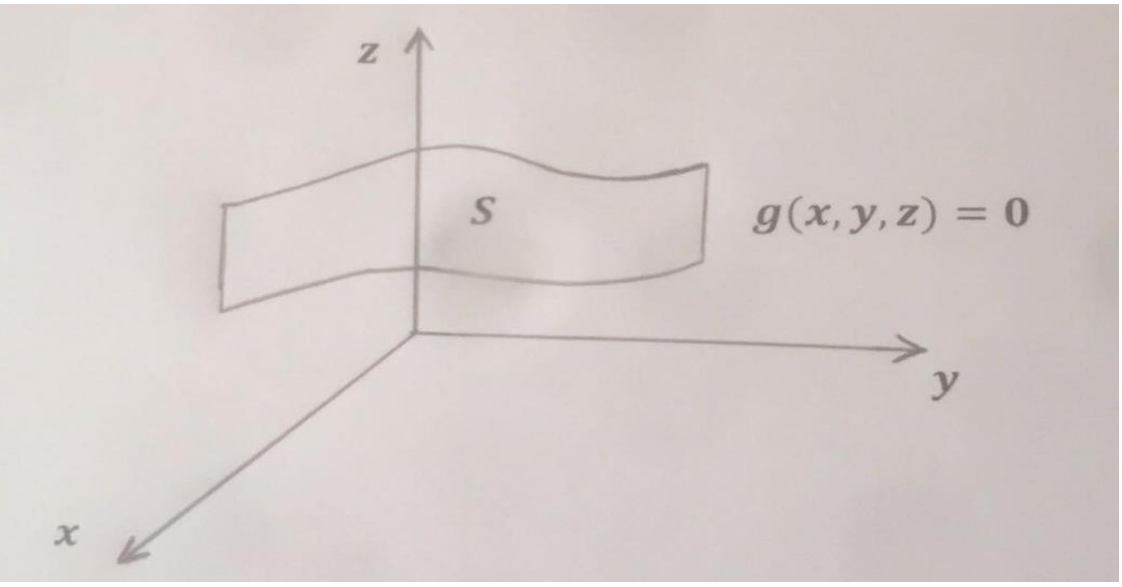
7)) $2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 2$

Section 2.4: Method of Lagrange multipliers

This method applies to minimize (or maximize) a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, construct the auxiliary function

Discussion of the method

Suppose we want to find the minimum (maximum) value of the function $f(x, y, z)$ which represents the distance between the required plane $g(x, y, z) = 0$ and the origin and suppose that f and g having continuous first partial derivatives and ending of f is at the point (x_0, y_0, z_0) which it's on the surface S that defined by $g(x, y, z) = 0$



We said that f has minimum (maximum)value at the point (x_0, y_0, z_0) if it satisfies the following condition

$$\boxed{\nabla f(x, y, z) = \lambda \nabla g(x, y, z)} \quad ...(1)$$

Where λ is Lagrange's multiplier , ∇ denote to the partial derivatives of f and g w.r.t. x, y and z.

<u>Ex.23</u>: by using $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, find the point on the straight $y = 3 - 2x$ that is nearest the origin.

Sol. Let $f(x, y) = x^2 + y^2 \to \nabla f(x, y) =< 2x, 2y >$...(2)

$g(x, y) = y + 2x - 3 = 0 \to \nabla g(x, y) =< 2,1 >$...(3)

Substituting (2) & (3) in $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$,we get

$$< 2x, 2y >= \lambda < 2,1 >$$

$\therefore 2x = 2\lambda \&\ 2y = \lambda \quad \to \quad x = \lambda = 2y$ ...(4)

Substituting (4) in $g(x, y)$, we have

$$y = 3 - 4y \quad \to \quad 5y = 3 \quad \to \quad y = \frac{3}{5}$$

Then form (4),we have $x = \frac{6}{5}$

$$\therefore (x, y) = (\frac{6}{5}, \frac{3}{5})$$

Which it's the point on $y = 3 - 2x$ that is nearest the origin.

Note **The distance between the point (x, y)on a straight and the origin is**

$$\boldsymbol{w = \sqrt{(x - x_0)^2 + (y - y_0)^2} \quad , (x_0, y_0) = (0, 0)}$$

$$\boldsymbol{= \sqrt{x^2 + y^2}}$$

Squaring both sides, we get

$$\boldsymbol{w^2 = x^2 + y^2 = f(x, y)}$$

<u>Ex.24</u>: Find the point on the plane $2x - 3y + 5z = 19$ that is nearest the origin, using the method of Lagrange multiplier.

Sol. As before, let

Let $f(x, y, z) = x^2 + y^2 + z^2 \rightarrow \nabla f(x, y, z) =< 2x, 2y, 2z >$...(5)
$g(x, y, z) = 2x - 3y + 5z - 19 = 0 \rightarrow \nabla g(x, y) =< 2, -3,5 >$...(6)

From the relation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ ...(7)

$$\rightarrow < 2x, 2y, 2z >= \lambda < 2, -3,5 >$$

$$\therefore 2x = 2\lambda \quad , \quad 2y = -3\lambda \quad , \quad 2z = 5\lambda$$

$\rightarrow x = \lambda \quad , \quad y = \frac{-3\lambda}{2} \quad , \quad z = \frac{5\lambda}{2}$ ...(8)

Substituting this values in g , we get

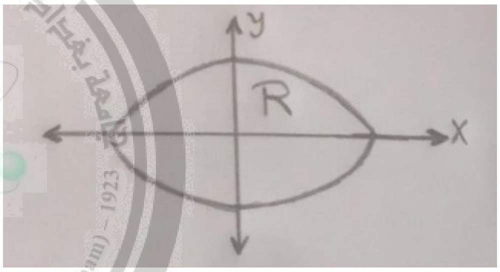
$$2\lambda + \frac{9}{2}\lambda + \frac{25}{2}\lambda = 19 \ \rightarrow 38\lambda = 38 \quad \rightarrow \ \lambda = 1$$

Substituting $(\lambda = 1)$ in (8) , we have

$$x = 1 \quad , \quad y = \frac{-3}{2} \quad , \quad z = \frac{5}{2}$$

$$\therefore p(x, y, z) = (1, \frac{-3}{2}, \frac{5}{2})$$

<u>Ex.25</u>: Suppose that the temperature of metal plate is given by $\boldsymbol{T(x,y) = x^2 + 2x + y^2}$. For the points $\boldsymbol{(x, y)}$ on a plate ellipse defined by $\boldsymbol{x^2 + 4y^2 \leq 24}$. Find minimum and maximum temperature on the plate.



Sol. For the plate in the figure

Firstly, we will find the critical

Points of $T(x, y)$ in R

$$T(x, y) = x^2 + 2x + y^2 \ \ \rightarrow \nabla T(x, y) = < 2x + 2, 2y > = < 0,0 >$$

$$\therefore 2x + 2 = 0 \ \ \& \ 2y = 0 \ \ \rightarrow \ \ x = -1 \ \ \& \ \ y = 0$$

$\therefore (x, y) = (-1,0)$ is in R

Now, using the relation $\nabla f(x, y) = \lambda \nabla g(x, y)$

$$f(x, y) = T(x, y) = x^2 + 2x + y^2 \ \ \rightarrow \nabla T(x, y) = < 2x + 2, 2y >$$

$$g(x, y) = x^2 + 4y^2 - 24 \ \ \rightarrow \nabla g(x, y) = < 2x, 8y >$$

$$\nabla T(x, y) = \lambda \nabla g(x, y)$$

$$< 2x + 2, 2y > = \lambda < 2x, 8y >$$

$\therefore 2x + 2 = 2\lambda x$…(9) $\quad \& 2y = 8\lambda y$…(10)

$y(2-8\lambda)=0$

From (10) $\boxed{y=0}$ *or* $2-8\lambda=0 \rightarrow \boxed{\lambda=\frac{1}{4}}$

*if $y=0 \quad \rightarrow \quad x^2+4(0)=24 \quad \rightarrow \quad x=\pm\sqrt{24}$

$$\therefore (x,y)=\left(\sqrt{24},0\right) \ or \ \left(-\sqrt{24},0\right)$$

*if $\lambda=\frac{1}{4} \quad \rightarrow \quad 2x+2=\frac{1}{2}x \qquad$ from (9)

$$\rightarrow \quad x=\frac{-4}{3}$$

Substituting in g , we have

$\frac{16}{9}+4y^2=24 \quad \rightarrow \quad y=\pm\frac{\sqrt{50}}{3}$

$$\therefore \boxed{(x,y)=\left(\frac{-4}{3},\frac{\sqrt{50}}{3}\right) \quad or \quad \left(\frac{-4}{3},\frac{-\sqrt{50}}{3}\right)}$$

Now , to find the minimum and maximum temperature T substituting all points in T

$$T(-1,0)=-1$$

$$T\left(\sqrt{24},0\right)=24+2\sqrt{24}\cong 33.8$$

$$T\left(-\sqrt{24},0\right)=24-2\sqrt{24}\cong 14.2$$

$$T\left(\frac{-4}{3},\frac{\sqrt{50}}{3}\right)=\frac{14}{3}\cong 4.7$$

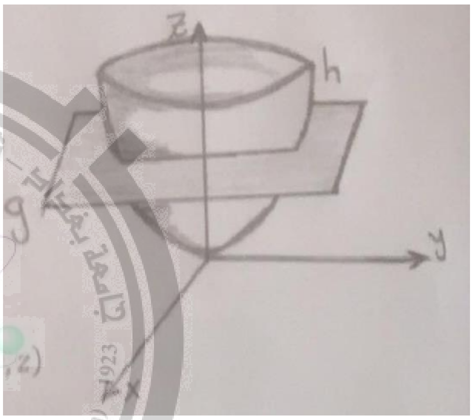
$$T\left(\frac{-4}{3},\frac{-\sqrt{50}}{3}\right)=\frac{14}{3}\cong 4.7$$

Note that the minimum temperature is (-1) at the point $(-1,0)$ and the maximum temperature is (33.8)at the point $(\sqrt{24},0)$.

Remark if there are two constraints intersecting ,say $g(x,y,z) = 0$ and $h(x,y,z) = 0$, we introduce two Lagrange's multipliers $\lambda$and μ and the relation will be

$$\nabla f(x,y,z) = \lambda\nabla g(x,y,z) + \mu\nabla h(x,y,z)$$

<u>Ex.26</u>: the plane $x + y + z = 12$ intersects with the cone $z = x^2 + y^2$by an ellipse. Find the point on the intersection that is nearest to the origin.



Sol. $f(x,y,z) = x^2 + y^2 + z^2$

$g(x,y,z) = x + y + z - 12 = 0$

$h(x,y,z) = x^2 + y^2 - z = 0$

$\nabla f(x,y,z) = \lambda\nabla g(x,y,z) + \mu\nabla h(x,y,z)$

$< 2x, 2y. 2z >= \lambda < 1,1,1 > +\mu < 2x, 2y, -1 >$

$\therefore\ 2x = \lambda + 2\mu x$ ...(11)

$2y = \lambda + 2\mu y$...(12)

$2z = \lambda - \mu$ ...(13)

From (11) and (12)

$$\left.\begin{matrix}\lambda = 2x(1-\mu)\\ \lambda = 2y(1-\mu)\end{matrix}\right\} \rightarrow 2x(1-\mu) = 2y(1-\mu) \rightarrow (2x - 2y)(1-\mu) = 0$$

Then $1-\mu=0 \to \mu=1 \to \lambda=0$ from (11) & (12)

Substituting in (13) we have $z=\frac{-1}{2}$...(14)

Substituting (14) in g and h , we have

$$x+y-\frac{1}{2}-12=0$$

$x^2+y^2=-\frac{1}{2}$ (Contradiction)

$\boxed{\text{Or}}$ $2x-2y=0 \to x=y$, in this case (Substitutingin h and g) we get

In $h x^2+y^2-z=0 \to z=2x^2$

In $g 2x+2x^2-12=0 \to x^2+x-6=0$

$$(x+3)(x-2)=0 \to x=-3 \text{ } or \text{ } x=2$$

$x=y$ & $z=2x^2 \to (x,y,z)=(2,2,8)$ or $(-3,-3,18)$

When $(x,y,z)=(2,2,8) \to f(2,2,8)=72$

When $(x,y,z)=(-3,-3,18) \to f(-3,-3,18)=342$

Then $(2,2,8)$ is the nearest to the origin.

… Exercises …

1)) Find the point on the curve $y=x^2+3$ that is nearest the origin, using the method of Lagrange multipliers.

2)) Find the minimum distance from the surface$x^2+y^2-z^2=1$ to the origin.

3)) Find the point on the surface $z=xy+1$ nearest the origin.

4)) Find the maximum and minimum values of $f(x, y, z) = x - 2y + 5z$on the sphere $x^2 + y^2 + z^2 = 30$.

5)) Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

6)) The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin what are the highest and lowest temperatures encountered by the ant?

7)) Factory produces three types of product x, y, z , the factory's profit (calculated in thousands of dollars) can be formulated in equation $p(x, y, z) = 4x + 8y + 2z$,where the account is bounded by $x^2 + 4y^2 + 2z^2 \leq 800$, find highest profit for the factory.

8)) find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse$\frac{x^2}{8} + \frac{y^2}{2} = 1$.

9)) Find the point on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values .

10)) Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

Chapter Three

Fourier Series

Contents

Section(3.1): The definition of Fourier series and how to find it

In this chapter we will find that we can solve many important problems involving partial differential equations provided that we can express a given function as an infinite sum of sines and (or) cosines. These trigonometric series are called (Fourier Series).

Definition: Let f be defined on $[-\pi,\pi]$, we said

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad \dots(1)$$

Is Fourier Series of f if it converges at all points of f on the interval $[-\pi,\pi]$, where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

a_0 , a_n , b_n are called Fourier coefficients.

Note f is a periodic function with period 2π since sine and cosine are periodic functions with period 2π , as shown :

$$\boldsymbol{f(x+2\pi)=f(x)}$$

$$f(x+2\pi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos n(x+2\pi)+b_n\sin n(x+2\pi)]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos(nx+2n\pi)+b_n\sin(nx+2n\pi)]$$

Since cosine and sine are periodic functions with period 2π then

$$f(x+2\pi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$=f(x) \qquad\qquad \text{from the definition}$$

<u>Ex. 1:</u> Find the Fourier Series for $\boldsymbol{f(x)=\begin{cases}1 & ,-\pi\le x\le 0\\ 2 & ,0<x\le\pi\end{cases}}$

Sol:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}dx+\frac{1}{\pi}\int_{0}^{\pi}2dx$$

$$=\frac{1}{\pi}x|_{-\pi}^{0}+\frac{2}{\pi}x|_{0}^{\pi}=3$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}2\cos nx\,dx$$

$$= \frac{1}{n\pi} \sin nx \Big|_{-\pi}^{0} + \frac{2}{n\pi} \sin nx \Big|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} \sin nx \, dx + \frac{2}{\pi} \int_{0}^{\pi} \sin nx \, dx$$

$$= \frac{-1}{n\pi} \cos nx \Big|_{-\pi}^{0} + \frac{-2}{n\pi} \cos nx \Big|_{0}^{\pi}$$

$$= \frac{-1}{n\pi} + \frac{1}{n\pi} \cos n\pi - \frac{2}{n\pi} \cos n\pi + \frac{2}{n\pi}$$

$$= \frac{1}{n\pi} - \frac{1}{n\pi} \cos n\pi$$

$$= \frac{1}{n\pi} (1 - (-1)^n)$$

Hence

$$b_n = \begin{cases} 0 & if \ n \ is \ even \\ \frac{2}{n\pi} & if \ n \ is \ odd \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{3}{2} + \sum_{\substack{n=1 \\ \text{n is odd}}}^{\infty} \frac{2}{n\pi} \sin nx$$

$$= \frac{3}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \cdots$$

Notes

1. $\cos n\pi = (-1)^n \qquad , \sin n\pi = 0$
2. f is even $\Leftrightarrow f(x) = f(-x)$
   f is odd $\Leftrightarrow f(x) = -f(-x)$

for example:

$f(x) = x^2, \cos x\ , x^4, \ldots \qquad f$ is even

$f(x) = x, x^3, \sin x\ , \ldots \qquad f$ is odd

3. even function × even function = even function
   odd function × odd function = even function
   even function × odd function = odd function

4.

$$\int_{-c}^{c} (even\ function)dx = 2\int_{0}^{c} (even\ function)dx$$

$$\int_{-c}^{c} (odd\ function)dx = 0$$

5. When f is even then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{even}\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx \qquad from\ 4$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{even} \cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx\, dx \qquad from\ 3,4$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{even} \sin nx\, dx = 0 \qquad from\ 3,4$$

When f is odd function then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{odd}\, dx = 0 \qquad\qquad from\ 4$$

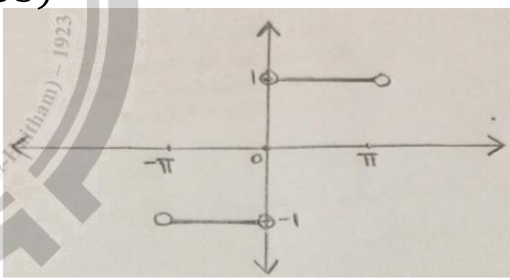
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{odd} \cos nx\, dx = 0 \qquad\qquad from\ 3{,}4$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{odd} \sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx \qquad\qquad from\ 3{,}4$$

<u>Ex. 2</u>: Find the Fourier Series for $f(x) = \begin{cases} -1 & if\ -\pi \le x < 0 \\ 1 & if\quad 0 < x < \pi \end{cases}$

Sol: note that f is odd, then

$$\left.\begin{aligned} a_0 = a_n = 0 \\ b_n = \tfrac{2}{\pi}\textstyle\int_0^{\pi} f(x)\sin nx\, dx \end{aligned}\right\} \quad from\ (note5)$$



$$= \frac{2}{\pi}\int_{0}^{\pi} \sin nx\, dx = \frac{-2}{n\pi}\cos nx \Big|_{0}^{\pi}$$

$$= \frac{-2}{n\pi}[\cos n\pi - \cos 0] = \frac{-2}{n\pi}[(-1)^n - 1] \qquad from\ (note\ 1)$$

$$= \begin{cases} \dfrac{4}{n\pi} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

Then , $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

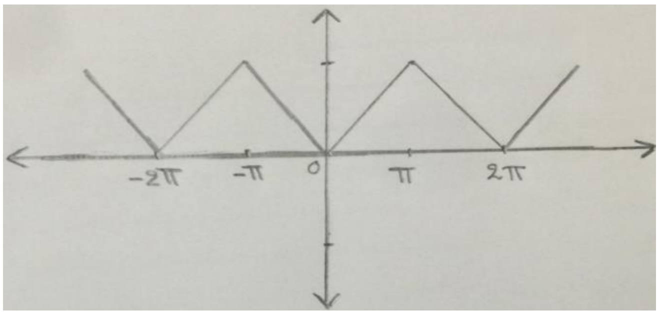
$$= \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{4}{n\pi} sin\, nx$$

Or

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} sin\,(2n-1)x$$

<u>Ex. 3</u>: Find the Fourier Series for $\boldsymbol{f(x) = \begin{cases} -x & if -\pi < x < 0 \\ x & if \quad 0 \le x \le \pi \end{cases}}$

Sol. note that f is even, then



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{\pi}x^2\Big|_0^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx \;= \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi - \frac{0}{n}\sin n0 + \frac{1}{n^2}\cos n0\right]$$

$$= \frac{2}{n^2\pi}[(-1)^n - 1] = \begin{cases} \frac{-4}{n^2\pi} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

$b_n = 0$ since f is even .

Then , $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2\pi} cos(2n-1)x$$

Section(3.2): The Fourier convergence

When we find a Fourier series for a function f we assume that f is defined on $[-\pi,\pi]$ and periodic with period 2π, then f must satisfy $f(\pi)=f(-\pi)$ otherwise , the function becomes discontinuous at the points $\pi+2n\pi \quad ,n=0,1,2,\ldots$, when f discontinuous at x_0 , then the series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

may not be convergent to $f(x_0)$ unless some certain conditions are satisfied. There are many conditions , if at least one of them satisfies, the Fourier series approaching f.

Here we will discuss the (Dirichlet's conditions) in the following theorem.

Dirichlet theorem:

Suppose that f and f' are piecewise continuous on the interval $-\pi\le x<\pi$. Further suppose that f is defined outside the interval $-\pi\le x<\pi$, So that it is periodic with period 2π then f has a Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

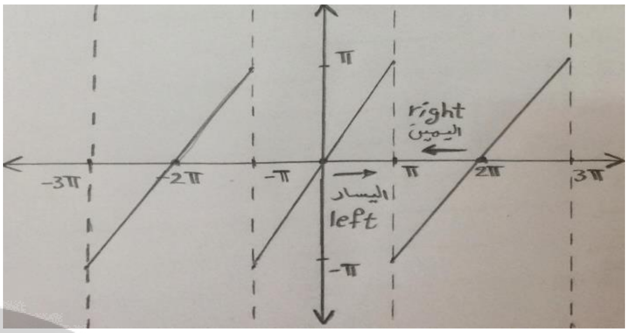
The Fourier series converges to $f(x)$ at all points where f is continuous, and to $\frac{1}{2}[\lim_{x\to x_0^+}f(x)+\lim_{x\to x_0^-}f(x)]$ at all points where f is discontinuous.

Ex. 4: **Find the Fourier series for $f(x) = x$ where $x \in [-\pi, \pi]$ then find:**

(i) The convergence on the interval $[-\pi, \pi]$.

(ii) The convergence of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

(iii) The approximate value of π.



Sol. f is odd function then

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}x \cdot \frac{-1}{n}\cos nx \,\Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\cos nx\,dx$$

$$= \frac{2}{\pi}x\cos nx + \frac{2}{n^2\pi}\sin nx \,\Big|_0^{\pi} = \frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

(i) The convergence on $[-\pi, \pi]$.

$$[-\pi, \pi] = (-\pi, \pi) \cup \{-\pi, \pi\}$$

The convergence on the interval $(-\pi, \pi)$ which it's continuous is

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

The convergence on the point $x = -\pi$ (where f is discount on it) is

$$f(-\pi) = \frac{1}{2}\left[\lim_{x \to -\pi^+} f(x) + \lim_{x \to -\pi^-} f(x)\right] = \frac{1}{2}[-\pi + \pi] = 0$$

The convergence on the point $x = \pi$ (where f is discount on it) is

$$f(\pi) = \frac{1}{2}\left[\lim_{x \to \pi^+} f(x) + \lim_{x \to \pi^-} f(x)\right] = \frac{1}{2}[-\pi + \pi] = 0$$

(ii) $f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$ ... $(*)$

Let $x = \frac{\pi}{2}$ (where f is continuous)

substituting in $(*)$, we get

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin \frac{n\pi}{2}$$

$$\frac{\pi}{4} = \underbrace{\sin\frac{\pi}{2}}_{=1} - \frac{1}{2}\underbrace{\sin\pi}_{=0} + \frac{1}{3}\underbrace{\sin\frac{3\pi}{2}}_{=-1} - \frac{1}{4}\underbrace{\sin 2\pi}_{=0} + \frac{1}{5}\underbrace{\sin\frac{5\pi}{2}}_{=1} - \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

(iii) since $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ (from (ii))

Then $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{1}{7} + \cdots$

let $n = 1 \rightarrow \pi = 4$

let $n = 2 \rightarrow \pi = 4 - \frac{4}{3} = \frac{8}{3} = 2.66$

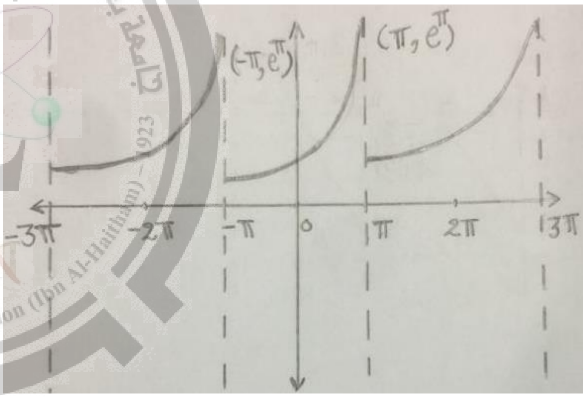
let $n = 3 \rightarrow \pi = 4 - \frac{4}{3} + \frac{4}{5} = \frac{52}{15} = 3.46$

$\vdots$

So, we approaching from the approximate value of π when n increasing .

<u>Ex. 5</u>: Find the Fourier series for $f(x) = e^x$ on $[-\pi, \pi]$, then find The convergence of Fourier series to f .

Sol. f is not odd neither even



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$$

$$= \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} dx = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$= \frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$ (by $\int udv$ twice)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,d$$

$$= \frac{n(-1)^{n+1}(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$$

Then

$$f(x) = \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}\cos nx + \frac{n(-1)^{n+1}(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}\sin nx\right]$$

The convergence

$[-\pi,\pi] = (-\pi,\pi) \cup \{-\pi,\pi\}$

In the interval $(-\pi,\pi)$ the Fourier series converge to e^{x}

at the point $x = -\pi$ the Fourier series converge to

$$f(-\pi) = \frac{1}{2}\left[\lim_{x\to-\pi^{+}} f(x) + \lim_{x\to-\pi^{-}} f(x)\right] = \frac{1}{2}[e^{-\pi} + e^{\pi}]$$

at the point $x = \pi$ the Fourier series converge to

$$f(\pi) = \frac{1}{2}\left[\lim_{x\to-\pi^{+}} f(x) + \lim_{x\to-\pi^{-}} f(x)\right] = \frac{1}{2}[e^{-\pi} + e^{\pi}]$$

Section 3: Extension of functions

A **The odd extension**: Let f be defined on the interval $[0,\pi]$, we will define the function $F(x)$ on the interval $[-\pi,\pi]$ such that

$$F(x)=\begin{cases} f(x) & ; x\in[0,\pi] \\ -f(-x) & ; x\in[-\pi,0) \end{cases}$$

we must prove that $F(x)=-F(-x)$ (i.e: F is odd)

let $x\in[0,\pi]\rightarrow -x\in[-\pi,0)$

$F(-x)=-f\big(-(-x)\big)=-f(x)=-F(x)$

$\therefore F(-x)=-F(x)$

By the same way if $x\in[-\pi,0)$, we get

$F(-x)=-F(x)$

then F is odd.

hence,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{F(x)}_{odd}\,dx=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underbrace{F(x)}_{odd}\ \underbrace{\cos nx}_{even}}_{odd}\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underbrace{F(x)}_{odd}\ \underbrace{\sin nx}_{odd}}_{even}\,dx=\frac{2}{\pi}\int_{0}^{\pi}F(x)\sin nx\,dx$$

and

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ; x \in [0, \pi]$$

but $F(x) = f(x)$ on the interval $[0, \pi]$, then

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ; x \in [0, \pi]$$

This series is called ((Fourier sine series))

<u>Ex. 6</u>: Find the Fourier sine series for the function $f(x) = cosx$ where $x \in [0, \pi]$.

Sol.

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \; \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} [\sin(1+n)\, x + \sin(1-n)\, x] dx$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)\, x - \sin(n-1)\, x] dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos(n+1)\, x}{n+1} + \frac{\cos(n-1)\, x}{n-1} \right]_0^{\pi} \qquad ; n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(n-1) - (-1)^n(n+1) + n - 1 - n - 1}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n n - (-1)^n - (-1)^n n - (-1)^n - 2}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2(-1)^n - 2}{n^2-1}\right]$$

$$= \frac{-2}{\pi}\left[\frac{(-1)^n + 1}{n^2-1}\right]$$

$$= \begin{cases} 0 & if\ n\ is\ odd\ \ n \neq 1 \\ \dfrac{-4}{\pi(n^2-1)} & if\ n\ is\ even \end{cases}$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x\ \sin x\ dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x\ dx = \frac{-1}{2\pi}[cos2x]_0^{\pi}$$

$$= \frac{-1}{2\pi}[cos2\pi - \cos 0] = \frac{-1}{2\pi}[1-1] = 0$$

Then the Fourier sine series is

$$f(x) = \sum_{\substack{n=2 \\ n\ is\ even}}^{\infty} \frac{-4}{\pi(n^2-1)} \sin nx$$

B **The even extension**: Let f be defined on the interval $[0,\pi]$, we will define the function $F(x)$ on the interval $[-\pi,\pi]$, such that:

$$F(x)=\begin{cases} f(x) & ; x\in[0,\pi] \\ f(-x) & ; x\in[-\pi,0] \end{cases}$$

we must prove that F is even i.e $F(x)=F(-x)$

$$x\in[0,\pi] \quad \rightarrow F(x)=f(x)$$

let $-x\in[-\pi,0] \ \rightarrow F(-x)=f\big(-(-x)\big)=f(x)=F(x)$

$$\therefore F(-x)=F(x)$$

by the same way when $x\in[-\pi,0]$, we get $F(x)=F(-x)$

then F is even , hence the Fourier series of F is

$$a_0=\frac{2}{\pi}\int_0^{\pi} F(x)\,dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} F(x)\cos nx\,dx$$

$$b_n=0$$

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx \qquad ; x\in[0,\pi]$$

But on the interval $[0,\pi]$ the function $F(x)$ is equal to $f(x)$ then

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \ a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad ; \ b_n=0$$

and

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx$$

This series is called ((Fourier cosine series))

<u>Ex.7</u>: Find the Fourier cosine series for the function $\boldsymbol{f(x) = \sin x \ \ ; x \in [0, \pi]}$

Sol.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{-2}{\pi}\cos x \Big|_0^{\pi} = \frac{-2}{\pi}[\cos\pi - \cos 0] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x\,]dx$$

$$= \frac{1}{\pi}\left[\frac{-1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\right]_0^{\pi}$$

(by the same way in Ex.6)

$$= \begin{cases} 0 & if\ n\ is\ odd\ \ , n \neq 1 \\ \dfrac{-4}{\pi(n^2-1)} & if\ \ n\ is\ even \end{cases}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = 0$$

$$\therefore f(x) = \frac{2}{\pi} + \sum_{\substack{n=2 \\ n\ is\ even}}^{\infty} \frac{-4}{\pi(n^2-1)}\cos nx$$

Note 1. $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

2. $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

3. $\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$

Section 4: Fourier series on the interval [-L,L]

Let f be defined on the interval $[-L, L]$, we assume that $z = \frac{\pi x}{L}$ to transform f on the interval $[-\pi, \pi]$, hence

$f(x) = F(z)$ where $-L \leq x \leq L$ & $-\pi \leq z \leq \pi$, *and*

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nz + b_n \sin nz] \qquad \ldots (*)$$

s.t.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z)\, dz$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \cos nz\, dz$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \sin nz\, dz$$

Replacing z from the hypothesis, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

s.t.

$$a_0 = \frac{1}{\pi} \int_{-L}^{L} f(x) \frac{\pi}{L} dx = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \cdot \frac{\pi}{L} dx = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{\pi} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \cdot \frac{\pi}{L} dx = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

<u>Ex. 8</u>: Find the Fourier series and the convergence on $[-\mathbf{1},\mathbf{1}]$ **for the function** $\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} \mathbf{-3} & ; \ \mathbf{-1 \le x \le 0} \\ \mathbf{2} & ; \mathbf{0 < x \le 1} \end{cases}$

Sol. $[-L, L] = [-1,1]$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_{-1}^{0} -3\,dx + \int_{0}^{1} 2\,dx$$

$$= -3x\Big|_{-1}^{0} + 2x\Big|_{0}^{1} = -3 + 2 = -1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \int_{-1}^{0} -3\cos n\pi x\ dx + \int_{0}^{1} 2\cos n\pi x\ dx$$

$$= \frac{-3}{n\pi}\sin n\pi x\Big|_{-1}^{0} + \frac{2}{n\pi}\sin n\pi x\Big|_{0}^{1} = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \int_{-1}^{0} -3\sin n\pi x\ dx + \int_{0}^{1} 2\sin n\pi x\ dx$$

$$= \frac{3}{n\pi}\cos n\pi x\Big|_{-1}^{0} - \frac{2}{n\pi}\cos n\pi x\Big|_{0}^{1}$$

$$= \frac{3}{n\pi}\cos 0 - \frac{3}{n\pi}\cos n\pi - \frac{2}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos 0$$

$$= \frac{5}{n\pi} + \frac{5(-1)^n}{n\pi} = \frac{5}{n\pi}(1-(-1)^n) = \begin{cases} 0 & if\ n\ is\ even \\ \dfrac{10}{n\pi} & if\ n\ is\ odd \end{cases}$$

Then

$$f(x) = \frac{-1}{2} + \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{10}{n\pi}\sin n\pi x$$

* the convergence on $[-1,1]$

$[-1,1] = (-1,0) \cup (0,1) \cup \{-1,0,1\}$

(i) on the interval $(-1,0)$ the Fourier series converge to -3
(ii) on the interval $(0,1)$ the Fourier series converge to 2
(iii) at the point $x = -1$ the Fourier series converge to

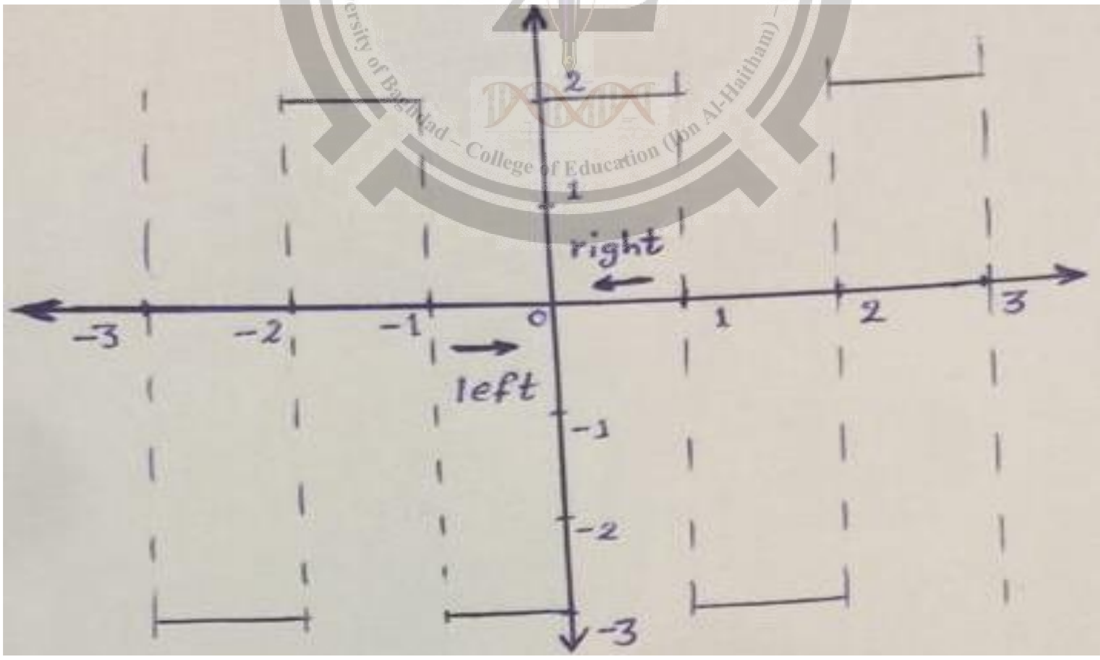
$$f(-1) = \frac{1}{2}\left[\lim_{x \to -1^+} f(x) + \lim_{x \to -1^-} f(x)\right] = \frac{1}{2}[-3+2] = \frac{-1}{2}$$

at the point $x = 0$ the Fourier series converge to

$$f(0) = \frac{1}{2}\left[\lim_{x \to 0^+} f(x) + \lim_{x \to 0^-} f(x)\right] = \frac{1}{2}[2+(-3)] = \frac{-1}{2}$$

at the point $x = 1$ the Fourier series converge to

$$f(1) = \frac{1}{2}\left[\lim_{x \to 1^+} f(x) + \lim_{x \to 1^-} f(x)\right] = \frac{1}{2}[-3+2] = \frac{-1}{2}$$



… Exercises …

(i) Find the Fourier cosine series for $f(x) = x$ where $x \in [0, \pi]$.

(ii) For the given functions:

(a) Find the Fourier series.

(b) Find the convergence on the whole interval.

(c) Sketch the graph of the function.

1. $f(x) = x^2 \quad ; x \in [-\pi, \pi]$

2. $(x) = 1 - x^2 \quad ; x \in [-1,1]$

3. $f(x) = \begin{cases} 0 & ; -\pi \le x < 0 \\ x^2 & ; 0 \le x < \pi \end{cases}$

4. $f(x) = \begin{cases} x & ; -\pi \le x < 0 \\ 0 & ; 0 \le x \le \pi \end{cases}$

Then find the sum of the series $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$

5. $f(x) = \begin{cases} x + \pi & ; -\pi \le x < 0 \\ \pi - x & ; 0 \le x < \pi \end{cases}$

(iii) Find the Fourier cosine series for $f(x) = \pi - x \; ; x \in [0, \pi]$.

(iv) Find the Fourier sine series for $f(x) = \pi - x \; ; x \in [0, \pi]$.

(v) Find the Fourier sine and Fourier cosine series for

$$f(x) = \begin{cases} 1 & ; 0 < x \le \frac{2}{3} \\ 0 & ; \frac{2}{3} < x < 1 \end{cases}$$

Then find the sum of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \cdots$

Section 5: Derivation of Fourier series

We know that the Fourier series for the function $f(x) = x$ is

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \qquad ; -\pi \leq x \leq \pi$$

$$= 2(sinx - \frac{sin2x}{2} + \frac{sin3x}{3} - \cdots)$$

If we derive the both sides of this series , we get

$$1 = 2(cosx - \cos 2x + \cos 3x - \cdots)$$

When $x = 0$, we get

$$1 = 2(1 - 1 + 1 - \cdots)$$

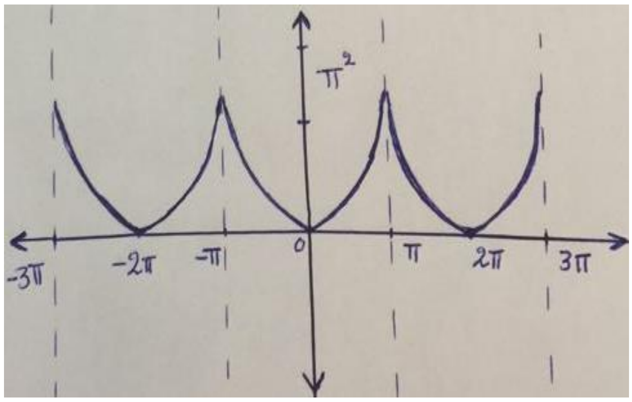
and this is not true because this series is not convergent this means that the derivative of f not necessary equal to the derivative of the series.

But there are conditions indicate whether it was possible to derive a Fourier series end to end and be equal to the derivative of the function f or not, which are described in the following theorem.

Theorem : If f' is bounded and periodic with a finite number of relative minimum and relative maximum points in each period, and f is continuous on $[-L, L]$, (i.e.) $f(L) = f(-L)$ then we can derive the Fourier series end to end and be equal to the derivative of f .

<u>Ex. 9</u>: Find the Fourier series for $f(x) = x^2$ where $x \in [-\pi, \pi]$, then find the derivative (if it exist).

Sol. since f is even $b_n = 0$, and



$$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{n}\sin nx\,\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx \cdot 2x dx\right]$$

$$= \frac{2}{\pi}\left[\underbrace{\frac{x^2}{n}\sin nx\,\Big|_0^\pi}_{=0} - \frac{2}{n}\left\{\frac{-x}{n}\cos nx\,\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx \cdot dx\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_0^\pi$$

$$= \frac{2}{\pi}\left(\frac{2\pi}{n^2}\cos n\pi\right) = \frac{4(-1)^n}{n^2}$$

So, the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

The derivative of Fourier series:

$f(x) = x^2 \rightarrow f'(x) = 2x$

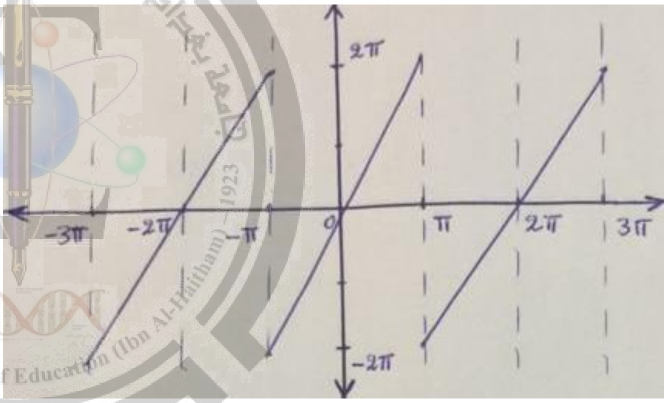
Note that f' is bounded from above by (2π) and from bottom by (-2π) and periodic , then the first condition is satisfied .

Next, $f(x) = \pi^2$ & $f(-\pi) = (-\pi)^2 = \pi^2$

Then $f(x) = f(-\pi)$

The second condition is satisfied too.

So, the derivative is exist and equal to the following:



$$2x = \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2} \sin nx \cdot n$$

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

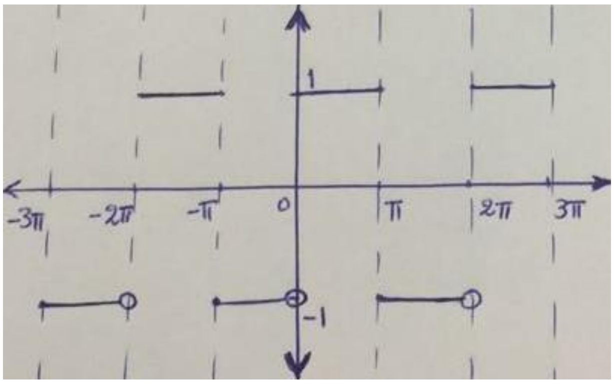
<u>**Ex. 10**</u>**: Find the derivative of Fourier series for the function**

$$\boldsymbol{f(x) = \begin{cases} -x & ; -\pi \le x < 0 \\ x & ; \ 0 \le x \le \pi \end{cases}}$$ **if it exist.**

Sol. From example 3, the Fourier series is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2\pi} cos(2n-1)x$$

To find the derivative :



$$f'(x) = \begin{cases} -1 & ; \ -\pi \leq x < 0 \\ 1 & ; \ 0 \leq x \leq \pi \end{cases}$$

Note that f' is periodic and bounded from above by the number (1) and from bottom by (-1), then the first condition is satisfied

Next, $f(x) = \pi$ and $f(-\pi) = (-(-\pi)) = \pi$

Then $f(\pi) = f(-\pi)$

The second condition is satisfied too.

Then we can find the derivative of Fourier series as follows:

$$f'(x) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^2} \sin(2n-1)x \cdot (2n-1)$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)} \sin(2n-1)x$$

Section 6: Integration of Fourier series

We can find the integration of Fourier series if the following theorem is satisfied.

Theorem : If f is bounded and periodic with a finite number of relative minimum and maximum and discontinuous points at each period then the integration of Fourier series will be equal to the integration of the function f.

Ex. 11: Find the integration of the Fourier series for $f(x) = x$ on the interval $(-\pi, \pi)$ then prove that

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}$$

Sol. From example (4) , the Fourier series of f is

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \qquad \ldots (1)$$

Note that f is periodic with period 2π , bounded from above by the number (π) and from bottom by $(-\pi)$ and it has two discontinuous points in every period which it is equal (π) and $(-\pi)$ in the original period.

hence, by integrating (1), we get

$$\frac{x^2}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} (-\cos nx) + c$$

Where c is an arbitrary constant.

$$\Rightarrow x^2 = \sum_{n=1}^{\infty} \frac{4(-1)^{n+2}}{n^2} \cos nx + 2c$$

$$\Rightarrow x^2 = 2c + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \qquad ; (-1)^{n+2} = (-1)^n \underbrace{(-1)^2}_{=1} = (-1)^n$$

Note that this series is a (Fourier cosine series) then

$$2c = \frac{a_0}{2} \qquad ; a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$\Rightarrow 2c = \frac{\pi^2}{3}$$

hence, the integration of Fourier series is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

To prove the given series, take $\underbrace{x = 0}_{cont.point}$, then

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \underbrace{\cos 0}_{=1}$$

$$\frac{-\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{-\pi^2}{12} = \frac{-1}{1} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

Ex. 12: Find the integration of Fourier series for

$$\boldsymbol{f(x)=\begin{cases}-1 & ; -\pi \le x < 0\\ 1 & ; \ 0 \le x \le \pi\end{cases}}$$

Sol. From example (2), the Fourier series for f is

$$f(x)=\sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)}\sin(2n-1)x \qquad \ldots(1)$$

Integrating (1) , we get

$$h(x)=\int f(x)dx=\sum_{n=1}^{\infty}\frac{-4}{\pi(2n-1)^2}\cos(2n-1)x+c \qquad \ldots(2)$$

where

$$h(x)=\begin{cases}-x & ; -\pi \le x < 0\\ x & ; \ 0 \le x \le \pi\end{cases}$$

and c is an arbitrary constant

since (2) is a Fourier cosine series then $c=\frac{a_0}{2}$

$$a_0=\frac{2}{\pi}\int_0^{\pi} xdx=\frac{1}{\pi}x^2\Big|_0^{\pi}=\frac{\pi^2}{\pi}=\pi$$

hence $c=\frac{\pi}{2}$ then (2) will be

$$h(x)=\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{-4}{\pi(2n-1)^2}\cos(2n-1)x$$

which it is the integration of Fourier series.

… Exercises …

1. Find the Fourier series for $f(x) = \begin{cases} 2 & ; -\pi \leq x < 0 \\ x & ; \ 0 \leq x \leq \pi \end{cases}$

   Then find the sum of $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

2. Find the Fourier series for $f(x) = \begin{cases} 0 & ; -\pi < x < \frac{-\pi}{2} \\ \sin x & ; \frac{-\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & ; \frac{\pi}{2} < x < \pi \end{cases}$

3. By using the Fourier cosine series of $f(x) = \sin x \ \ on \ [0, \pi]$ prove that

   $$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

4. Find the Fourier series for $f(x) = \begin{cases} x & ; -1 \leq x \leq 0 \\ x + 2 & ; \ 0 < x \leq 1 \end{cases}$

   Then use $x = \frac{1}{2}$ to find the sum of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

5. Use the function $f(x) = \begin{cases} x(\pi - x) & ; \ 0 < x < \pi \\ x(\pi + x) & ; \ -\pi < x < 0 \end{cases}$

   To prove that

   $$f(x) = \frac{8}{\pi} \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{\sin nx}{n^3}$$

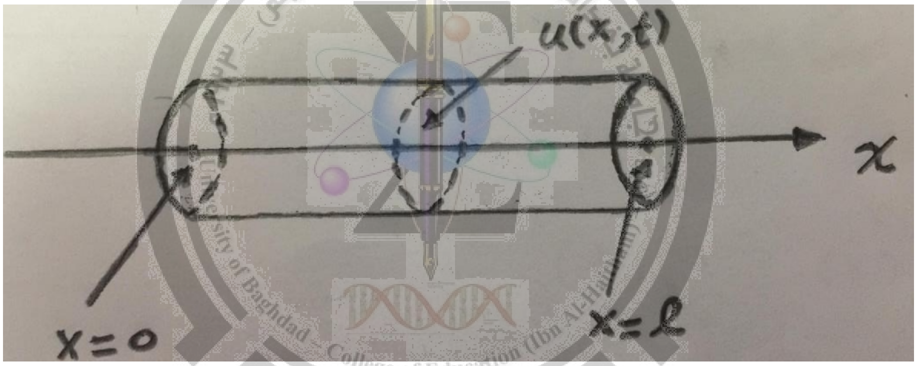
Chapter Four

The Heat Conduction Equation

Contents

The Heat Conduction Equation

Let us consider a heat conduction problem for a straight bar of uniform cross section and homogeneous material. Let the x-axis be chosen to lie along the axis of the bar, and let $x = 0$ and $x = l$ denote the ends of the bar (as shown in the figure) suppose further that the sides of the bar are perfectly insulated so that no heat passes through them. We also assume that the cross-sectional dimensions are so small that the temperature u can be considered as constant on any given cross section. Then u is a function only of the axial coordinate x and the time t.



The variation of temperature in the bar is governed by partial differential equation called the (heat conduction equation), and has the form

$$\boxed{u_t = \alpha^2 u_{xx} \quad ,0 < x < l \quad ,0 < t < \infty} \qquad \ldots(1)$$

Where α^2 is a constant known as the thermal diffusivity, the parameter α^2 depends only on the material from which the bar is made, and is defined by $\alpha^2 = \frac{k}{ps}$ where k is the thermal conductivity , p is the density, and s is the specific heat of the material in the bar.

Now, to find the solution of equation (1), we start by making a basic assumption about the form of the solutions that has far-reaching, and perhaps unforeseen, consequences. The assumption is that $u(x,t)$ is a product of two other functions, one depending only on x and the other depending only on t.

This method is called (Separation of Variables).

----

1 Separation of Variables

Let $u(x,t) = X(x).T(t)$ ...(2)

Differentiating (2) w.r.t. t and x, we get

$$\frac{\partial u}{\partial t} = X(x).T'(t) \qquad ...(3)$$

and

$$\frac{\partial^2 u}{\partial x^2} = X''(x).T(t) \qquad ...(4)$$

Substituting (3) and (4) in (1), we get

$$X.T' = \alpha^2 X''.T \qquad ...(5)$$

Equation (5) is equivalent to

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = \lambda \qquad \text{(where } \lambda \text{ is a constant)}$$

Hence $T' - \alpha^2 \lambda T = 0$ ...(6)

and $X'' - \lambda X = 0$ ...(7)

where (6) and (7) are two ordinary differential equations can be solved as follows:

The (A.E.) of (6) is $m - \alpha^2\lambda = 0 \Rightarrow m = \alpha^2\lambda$

$$\therefore T(t) = c_1 e^{\alpha^2 \lambda t} \qquad ...(8)$$

Where c_1 is an arbitrary constant .

The (A.E.) of (7) is $m^2 - \lambda = 0 \Rightarrow m = \pm\sqrt{\lambda}$

$$\therefore X(x) = c_2 e^{\sqrt{\lambda}x} + c_3 e^{-\sqrt{\lambda}x} \qquad ...(9)$$

Where c_2 and c_3 are two arbitrary constants.

Substituting (8) , (9) in (2), we get

$$u(x,t) = c_1 e^{\alpha^2 \lambda t}\left[c_2 e^{\sqrt{\lambda}x} + c_3 e^{-\sqrt{\lambda}x}\right]$$

$$\Rightarrow u(x,t) = e^{\alpha^2 \lambda t}\left[A e^{\sqrt{\lambda}x} + B e^{-\sqrt{\lambda}x}\right] \qquad \ldots(10)$$

Where $A = c_1 c_2$, $B = c_1 c_3$

There are three possibilities to choose the constant λ

1. $\lambda > 0$, This is contrary to reality because the temperature increases infinitely with the passage of time.
2. $\lambda = 0$, This is also contrary to reality because the temperature will remain constant over time.
3. $\lambda < 0$, and this is the correct case because the temperature will increase slightly and be restrained with the passage of time.

Let $\lambda = -w^2 \Rightarrow \sqrt{\lambda} = wi$

Then equation (10) will be :

$$u(x,t) = e^{-\alpha^2 w^2 t}\left[A e^{iwx} + B e^{-iwx}\right]$$

from $e^{i\theta} = \cos\theta + i\sin\theta$, we get

$$u(x,t) = e^{-\alpha^2 w^2 t}[A(\cos wx + i\sin wx) + B(\cos wx - i\sin wx)]$$

$$= e^{-\alpha^2 w^2 t}[(A+B)\cos wx + (Ai - Bi)\sin wx]$$

$$u(x,t) = e^{-\alpha^2 w^2 t}[K\cos wx + L\sin wx] \qquad \ldots(11)$$

Where $K = A + B$, $L = Ai - Bi$

So equation (11) is the general solution of the heat equation.

2 General Solution of heat equation with homogeneous boundary Conditions

If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $\emptyset(x)$ in the bar. (i.e.) the boundary conditions are

$u(0,t) = 0^0 \;, u(l,t) = 0^0$ (homo. Boundary conditions)

and the initial condition is $u(x,0) = \emptyset(x)$.

To find the general solution of heat equation under this conditions we will substitute the boundary and initial conditions one after the other in equation (11) as follows :

$$u(x,t) = e^{-\alpha^2 w^2 t}[K\cos wx + L\sin wx] \qquad \ldots(11)$$

Substituting the boundary condition $u(0,t) = 0$ in (11), we get

$$u(0,t) = e^{-\alpha^2 w^2 t}[K\cos 0 + L\sin 0]$$

$$0 = \underbrace{e^{-\alpha^2 w^2 t}}_{\neq 0} K \Rightarrow \quad \boxed{K = 0} \qquad \ldots(12)$$

Butting (12) in (11), we get

$$u(x,t) = e^{-\alpha^2 w^2 t} L\sin wx \qquad \ldots(13)$$

Substituting the second boundary condition $(u(l,t) = 0)$ in (13), we get

$$u(l,t) = e^{-\alpha^2 w^2 t} L\sin wl$$

$$0 = \underbrace{e^{-\alpha^2 w^2 t}}_{\neq 0} \underbrace{L}_{\neq 0} \sin wl \;\Rightarrow \sin wl = 0 \Rightarrow wl = n\pi$$

$$\Rightarrow w = \frac{n\pi}{l}, n = 0, \pm 1, \pm 2, \dots$$

Then

$$u_n(x,t) = e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} L_n \sin\frac{n\pi x}{l} \qquad \dots(14)$$

There are infinitely many functions in (14) so a general combination of them is an infinite series.

Thus we assume that

$$u(x,t) = \sum_{-\infty}^{\infty} u_n(x,t) = \sum_{-\infty}^{\infty} e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} L_n \sin\frac{n\pi x}{l}$$

$$\boxed{\Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \sin\frac{n\pi x}{l}} \qquad \dots(15)$$

Where $A_n = L_n - L_{-n}$

Now, substituting the initial condition $(u(x,0) = \emptyset(x))$in equation (15), we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n e^0 \sin\frac{n\pi x}{l}$$

$$\emptyset(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

Which it is Fourier sine series, so the constants A_n are given by

$$A_n = \frac{2}{l}\int_0^l \emptyset(x)\sin\frac{n\pi x}{l}\,dx\ ; n = 1,2,3,\ldots$$

…(16)

Hence ,(15) is the required solution where A_n is given by (16).

<u>Ex 1:</u> Find the temperature $u(x,t)$ at any time in a metal rod (2 cm) long, homogeneous and insulated, which initially has a uniform temperature of $3x$, and it's ends are maintained at 0^0c for all $t > 0$. Then find the temperature of the middle of the rod at $t = 4$.

Sol. $l = 2\ ,\emptyset(x) = 3x\ ,u(0,t) = 0\ ,u(2,t) = 0$

$$A_n = \frac{2}{l}\int_0^l \emptyset(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^2 3x\sin\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{-6x}{n\pi}\cos\frac{n\pi x}{2} + \frac{12}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2 = \frac{12(-1)^{n+1}}{n\pi}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{n\pi}e^{-\alpha^2\frac{n^2\pi^2}{4}t}\sin\frac{n\pi x}{2}$$

$$u(1,4) = \sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{n\pi}e^{-\alpha^2\frac{n^2\pi^2}{4}4}\sin\frac{n\pi}{2}$$

$$= \sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{n\pi}e^{-\alpha^2 n^2\pi^2}\sin\frac{n\pi}{2}$$

<u>Ex 2:</u> A rod of length 50 cm is homogeneous and insulated, is initially at a uniform temperature 20^0c, and it's ends are maintained at 0^0c for all $t > 0$, find the temperature $u(x,t)$.

Sol.

$l = 50\,, \emptyset(x) = 20^0\,, u(0,t) = 0\,, u(50,t) = 0$

$$A_n = \frac{2}{l}\int_0^l \emptyset(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{50}\int_0^{50} 20\sin\frac{n\pi x}{50}\,dx$$

$$= \frac{4}{5}\int_0^{50} \sin\frac{n\pi x}{50}\,dx$$

$$= \frac{-40}{n\pi}\cos\frac{n\pi x}{50}\Big|_0^{50}$$

$$= \frac{-40}{n\pi}[(-1)^n - 1] = \begin{cases} \dfrac{80}{n\pi} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

Finally, by substituting in $u(x,t)$, we get

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l}$$

$$= \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{80}{n\pi} e^{-\alpha^2\frac{n^2\pi^2}{2500}t}\sin\frac{n\pi x}{50}$$

3 General Solution of heat equation with non-homogeneous boundary Conditions

Suppose now that one end of the bar is held at a constant temperature k_1 and the other is maintained at a constant temperature k_2 , then the boundary conditions are $u(0,t) = k_1$, $u(l,t) = k_2$, $t > 0$ the initial condition $u(x,0) = \emptyset(x)$ remain unchanged we can solve it by reducing it to a problem having homogeneous boundary conditions, which can then be solved as in previous case, thus we write

$$u(x,t) = k_1 + \frac{x}{l}(k_2 - k_1) + U(x,t) \qquad \ldots(17)$$

We will prove that (17) is satisfy equation (1) we derive the equation (17) w.r.t. t and , we have

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} \quad , \quad \frac{\partial u}{\partial x} = \frac{k_1 - k_2}{l} + \frac{\partial U}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 U}{\partial x^2}$$

Substituting in (1) we get :

$$\boxed{\frac{\partial U}{\partial t} = \alpha^2 \frac{\partial^2 U}{\partial x^2}} \qquad \ldots(18)$$

Substituting the boundary and initial conditions in (18), we have

$$k_1 = k_1 + 0 + U(0,t) \Rightarrow \boxed{U(0,t) = 0} \qquad \text{(from cond.1)}$$

$$k_2 = k_1 + \frac{l}{l}(k_2 - k_1) + U(l,t) \Rightarrow k_2 - k_1 - k_2 + k_1 = U(l,t)$$

$\Rightarrow \boxed{U(l,t) = 0}$ (from cond. 2)

$\emptyset(x) = k_1 + \frac{x}{l}(k_2 - k_1) + U(x,0)$ (from the initial cond.)

$$\boxed{U(x,0) = \emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)}$$

Hence the new equation $U(x,t)$ represent a heat conduction equation with homo. boundary conditions and initial condition

$$U(0,t) = 0\,, U(l,t) = 0\,, U(x,0) = \emptyset(x) - k_1 + \frac{x}{l}(k_2 - k_1)$$

Then the general solution can be found as follows:

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \sin \frac{n\pi}{l} x} \qquad \dots(19)$$

(from (15))

Where

$$\boxed{A_n = \frac{2}{l} \int_0^l \left[\emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)\right] \sin \frac{n\pi}{l} x \, dx} \qquad \dots(20)$$

Substituting (19) in (17), we get

$$\boxed{u(x,t) = k_1 + \frac{x}{l}(k_2 - k_1) + \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \sin \frac{n\pi}{l} x} \qquad \dots(21)$$

So (21) is the general solution with A_n shown in (20).

<u>Ex. 3:</u> A rod of length 1, is initially at a uniform temperature x, the end $x = 0$ is heated to 2^0 and the end $x = 1$ is heated to 3^0, Find the temperature distribution in the rod at any time t.

Sol.

$$u(0,t) = 2 \;, u(1,t) = 3 \;, u(x,o) = \emptyset(x) = x$$

$$A_n = \frac{2}{l}\int_0^l \left[\emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)\right] \sin\frac{n\pi}{l} x \, dx$$

$$= \frac{2}{1}\int_0^1 \left[x - 2 - \frac{x}{1}(3-2)\right] \sin\frac{n\pi}{1} x \, dx$$

$$= 2\int_0^1 [x - 2 - x] \sin n\pi x \, dx$$

$$= -4\int_0^1 \sin n\pi x \, dx$$

$$= \frac{4}{n\pi}\cos n\pi x \,\Big|_0^1 = \frac{4}{n\pi}((-1)^n - 1) = \begin{cases} \dfrac{-8}{n\pi} & nis\ odd \\ 0 & n\ is\ even \end{cases}$$

$$u(x,t) = 2 + \frac{x}{1}(3-2) + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-8}{n\pi} e^{-\alpha^2 \frac{n^2\pi^2}{1} t} \sin\frac{n\pi}{1} x$$

$$= 2 + x + \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{-8}{n\pi} e^{-\alpha^2 n^2 \pi^2 t} \sin n\pi x$$

<u>Ex. 4:</u> A copper rod of length 50 cm, is initially at a uniform temperature $\frac{x}{2}$, the end $x = 0$ is heated to 10^0c and the end $x = 50$ is heated to 35^0c, Find the temperature distribution in the rod at any time.

Sol.

$$l = 50\ , u(0,t) = 10^0\ \ , u(50,t) = 35^0\ \ , \emptyset(x) = \frac{x}{2}$$

$$A_n = \frac{2}{l}\int_0^l \left[\emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)\right]\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{50}\int_0^{50} \left[\frac{x}{2} - 10 - \frac{x}{50}(35 - 10)\right]\sin\frac{n\pi}{50}x\,dx$$

$$= \frac{1}{25}\int_0^{50} \left[\frac{x}{2} - 10 - \frac{x}{2}\right]\sin\frac{n\pi}{50}x\,dx\ = \frac{-10}{25}\int_0^{50} \sin\frac{n\pi}{50}x\,dx$$

$$= \frac{20}{n\pi}\cos\frac{n\pi}{50}x\,\Big|_0^{50} = \frac{20}{n\pi}\left((-1)^n - 1\right) = \begin{cases} \dfrac{-40}{n\pi} & n is\ odd \\ 0 & n\ is\ even \end{cases}$$

$$u(x,t) = k_1 + \frac{x}{l}(k_2 - k_1) + \sum_{n=1}^{\infty} A_n e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi}{l}x$$

$$u(x,t) = 10 + \frac{x}{50}(35 - 10) + \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{-40}{n\pi}e^{-\alpha^2\frac{n^2\pi^2}{2500}t}\sin\frac{n\pi}{50}x$$

<u>Ex.5</u>:Solve the equation $u_t = \alpha^2 u_{xx}$, $0 < x < l$, $0 \le t < \infty$ that satisfies the conditions $u(0,t) = 0, u_t(l,t) = 0, u(x,0) = \emptyset(x)$.

Sol. Using the equation

$$u(x,t) = e^{-\alpha^2 w^2 t}[K \cos wx + L \sin wx] \qquad \ldots(1)$$

Substituting the first condition $(u(0,t) = 0)$ in (1), we get

$$u(0,t) = e^{-\alpha^2 w^2 t}[K \cos 0 + L \sin 0]$$

$$0 = e^{-\alpha^2 w^2 t} K \Rightarrow \boxed{K=0} \qquad \ldots(2)$$

Butting (2) in (1), we get

$$u(x,t) = e^{-\alpha^2 w^2 t} L \sin wx \qquad \ldots(3)$$

Differentiating (3) w. r. t. (t), we get

$$u_t(x,t) = -\alpha^2 w^2 e^{-\alpha^2 w^2 t} L \sin wx \qquad \ldots(4)$$

Substituting the second condition $(u_t(l,t) = 0)$ in (4), we get

$$u_t(l,t) = -\alpha^2 w^2 e^{-\alpha^2 w^2 t} L \sin wl$$

$$0 = -\alpha^2 w^2 e^{-\alpha^2 w^2 t} L \sin wl \Rightarrow \sin wl = 0$$

$$\Rightarrow wl = n\pi \quad , n = 0, \pm 1, \pm 2, \ldots \qquad \Rightarrow w = \frac{n\pi}{l}$$

Substituting in (3), we have

$$u_n(x,t) = e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} L_n \sin \frac{n\pi}{l} x \qquad \ldots(5)$$

Since n has infinite values, there are infinitely many functions in (5) so a general combination of them is an infinite series, thus

$$u(x,t)=\sum_{-\infty}^{\infty} L_n\, e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi}{l}x$$

$$\boxed{u(x,t)=\sum_{n=1}^{\infty} A_n\, e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi}{l}x}$$

…(6)

Where $A_n = L_n - L_{-n}$

Now, substituting the third condition $(u(x,0)=\emptyset(x))$ in (6), then

$$u(x,0)=\sum_{n=1}^{\infty} A_n\, e^{0}\sin\frac{n\pi}{l}x$$

$$\emptyset(x)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x$$

Which it is Fourier since series, so the constants A_n are given by

$$\boxed{A_n=\frac{2}{l}\int_0^l \emptyset(x)\sin\frac{n\pi}{l}x\,dx}$$

…(7)

Hence (6) is the required solution where A_n is given in (7).

4 Bar with Insulated Ends

A slightly different problem occurs if the ends of the bar are insulated so that there is no passage of heat through them. Thus in this case of no heat flow the boundary conditions are

$u_x(0,t) = 0\,, u_x(l,t) = 0\,, t > 0$

<u>Ex :</u> Solve the equation $\boldsymbol{u_t = \alpha^2 u_{xx}}$, $\boldsymbol{0 < x < l\,, 0 \le t < \infty}$ that satisfies the conditions $\boldsymbol{u_x(0,t) = 0,}$ $\boldsymbol{u_x(l,t) = 0\,,}$ $\boldsymbol{u(x,0) = \emptyset(x)}$

Sol. Using the equation

$$u(x,t) = e^{-\alpha^2 w^2 t}[K\cos wx + L\sin wx] \qquad \ldots(1)$$

Differentiating (1) for x

$$u_x(x,t) = e^{-\alpha^2 w^2 t}[-Kw\sin wx + Lw\cos wx] \qquad \ldots(2)$$

Butting the condition (1) in (2)

$$u_x(0,t) = e^{-\alpha^2 w^2 t}[-Kw\sin 0 + Lw\cos 0]$$

$$0 = e^{-\alpha^2 w^2 t} Lw \Rightarrow \boxed{L = 0} \qquad \ldots(3)$$

Butting (3) in (1)

$$u(x,t) = e^{-\alpha^2 w^2 t} K\cos wx \qquad \ldots(4)$$

Differentiating (4) for x

$$u_x(x,t) = -e^{-\alpha^2 w^2 t} Kw\sin wx \qquad \ldots(5)$$

Substituting condition 2 in (5), we get

$$u_x(l,t) = -e^{-\alpha^2 w^2 t} Kw\sin wl$$

$$0 = -e^{-\alpha^2 w^2 t} K w \sin wl \Rightarrow \sin wl = 0$$

$$\Rightarrow wl = n\pi \quad , n = 0, \pm 1, \pm 2, \dots \qquad \Rightarrow w = \frac{n\pi}{2}$$

Substituting this in equation (4), we get

$$u(x,t) = e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} K \cos \frac{n\pi}{l} x \qquad \dots(6)$$

Since n has infinite values, there are infinitely many functions in (6) so a general combination of them is an infinite series, thus

$$u_0(x,t) = K_0$$

$$u_n(x,t) = e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} K_n \cos \frac{n\pi}{l} x$$

$$\boxed{u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \, e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \cos \frac{n\pi}{l} x} \qquad \dots(7)$$

Where $K_0 = \frac{c_0}{2}$ $\;, c_n = K_n + K_{-n}$

substituting the third condition in (7), then

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \, e^0 \cos \frac{n\pi}{l} x$$

$$\emptyset(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi}{l} x$$

Which it is Fourier cosine series, so the constants c_0 and c_n are given by

$$c_0 = \frac{2}{l} \int_0^l \emptyset(x)\, dx$$

$$\boxed{c_n = \frac{2}{l}\int_0^l \emptyset(x)\cos\frac{n\pi}{l}x\,dx \quad ,n = 1,2,3,\dots} \quad \dots(7)$$

Hence (7) is the required solution.

… Exercises …

1. A rod of length (10 cm), is initially at a uniform temperature $2x$, and it's ends are maintained at 0^0c , find the temperature $u(x,t)$ at any time.

2. Find the solution of the heat problem $u_t = 100u_{xx}$, $0 < x < 1$, $0 < t < \infty$, $u(0,t) = 0$, $u_t(1,t) = 0$, $u(x,0) = 5^0$, $0 \le x \le 1$.

3. Find the solution of the heat problem $u_t = \alpha^2 u_{xx}$, $0 < x < 2$, $0 < t < \infty$, $u_x(0,t) = 0$, $u_x(2,t) = 0$, $u(x,0) = f(x)$, $0 \le x \le 2$

4. Find the temperature $u(x,t)$ in a metal rod of length (25 cm) that is insulated on the ends as well as on the sides and whose initial temperature distribution is $u(x,0) = x$ for $0 < x < 25$.

5. A rod of length (30 cm) , is initially at a uniform temperature $(60 - 2x)$, the end $x = 0$ is heated 20^0c and the end $x = 30$ is heated to 50^0c . find the temperature distribution in the rod at any time.

6. Find the solution of the heat problem $U_t = U_{xx}$, $0 < x < 1$, $0 < t < \infty$, $u(0,t) = 1^0$, $u_t(1,t) = 2^0$, $u(x,0) = \sin\frac{3\pi x}{l}$, $0 \le x \le l$.

7. A rod of length 1 unit, is initially at a uniform temperature , the temperature of one ends is equal to zero and the rate of change of temperature in the other end is equal to zero too. find the temperature distribution in the rod at any time.

8. A rod of length (3 cm), is initially at a uniform temperature e^l , and it's ends are maintained at 0^0c , find the temperature distribution in the rod at any time.

9. Find the solution of the heat problem $u_t = u_{xx}$, $0 < x < 3$, $0 < t < \infty$, $u_x(0,t) = 0$, $u(3,t) = 0$, $u(x,0) = f(x)$.

Chapter Five

One Dimensional Wave Equation

Contents

Section(5.1): The Wave Equation: Vibration of an Elastic string

A second partial differential equation occurring frequent in applied mathematics is the wave equation. Some form of this equation, or a generalization of it, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves, electromagnetic waves, and seismic waves are all based on this equation.

Perhaps the easiest situation to visualize occurs in the investigation of mechanical vibrations. Suppose that an elastic string of length l is tightly stretched between two supports at the same horizontal level, so that the x-axis lies along the string (see figure 1). The elastic string may be thought of as a violin string, a guy wire, or possibly an electric power line.

Suppose that the string is set in motion(by plucking, for example) so that it vibrates in a vertical plane and let $u(x,t)$ denote the vertical displacement experienced by the string at the point x at time t. If damping effects, such as air resistance, are neglected, and if the amplitude of the motion is not too large, then $u(x,t)$ satisfies the partial differential equation:

$$u_{tt} = c^2 u_{xx} \qquad \dots (1)$$

In the domain $0 < x < l,\ 0 < t \leq \infty$. Equation (1) is known as the (wave equation), where the constant c^2 is given by $c^2 = \frac{T}{p}$

where T is the tension of the string and p is the mass per length of the string material.

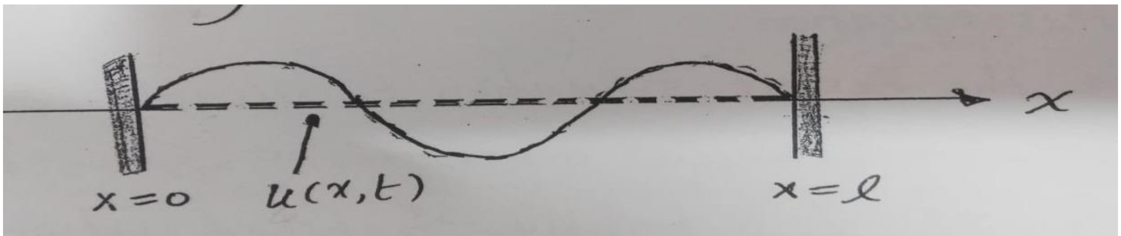


Figure .1

By the same way that in the heat equation we will solve equation (1) by the separation of variables method.

Let $u(x,t) = X(x).T(t)$ $\qquad\qquad$ … … (2)

deriving eq.(2) twice w.r.t x and t and substituting in (1), we get

$$\frac{T^{''}(t)}{c^2\,T(t)} = \frac{X^{''}(x)}{X(x)}$$

Equating this equation to a constant, sayλ:

$$\frac{T^{''}(t)}{c^2\,T(t)} = \frac{X^{''}(x)}{X(x)} = \lambda \qquad \dots\dots (3)$$

Then, we get

$$T^{''} - \lambda c^2\,T = 0 \qquad \dots\dots.(4)$$

$$X^{''} - \lambda X = 0 \qquad \dots\dots.(5)$$

Solving this two ordinary differential equations, we obtain

$$T(t) = c_1 e^{c\sqrt{\lambda}t} + c_2 e^{-c\sqrt{\lambda}t} \qquad \dots\dots (6)$$

$$X(x) = c_3 e^{\sqrt{\lambda}x} + c_4 e^{-\sqrt{\lambda}x} \qquad \dots\dots (7)$$

Substituting (6), (7) in (2), we get

$$u(x,t) = \left(c_1 e^{c\sqrt{\lambda}t} + c_2 e^{-c\sqrt{\lambda}t}\right)\left(c_3 e^{\sqrt{\lambda}x} + c_4 e^{-\sqrt{\lambda}x}\right) \quad \dots\dots (8)$$

Where c_1, c_2, c_3 and c_4 are constants

There are three cases to choose λ:

1- $\lambda > 0$, this leads to an elastic string will vibrate without stopping, and this is contrary to reality.

2- $\lambda = 0$, this leads $u(x,t)$ is constant and this is contrary to reality too.

3- $\lambda < 0$, this is the right situation, let $\lambda = -w^2 \Rightarrow \sqrt{\lambda} = wi$

Substituting in (8), we get

$u(x,t) = [k_1 \cos cwt + k_2 \sin cwt][k_3 \cos wx + k_4 \sin wx]$ ... (9)

Where $k_1 = c_1 + c_2, k_2 = c_1 i - c_2 i, k_3 = c_3 + c_4, k_4 = c_3 i - c_4 i$

The equation (9) is the general solution of (1)

Section (5.2): General solution of one- dimensional wave equation satisfying the given boundary and initial conditions.

Suppose that we have an elastic string of length l, its ends are fixed at $x = 0$ and $x = l$, then we have the two boundary conditions

$u(0,t) = 0, u(l,t) = 0$ … … (10).

The form of the motion of the string will depend on the initial deflection (deflection at t=0) and on the initial velocity (velocity at t=0). Denoting the initial deflection by f(x) and the initial velocity by g(x), we arrive at two initial conditions

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $\quad 0 \le x \le l$ … … (11)

Our problem now is to find a solution of (1) satisfying the conditions (10), (11).

Substituting the condition $u(0,t)$ in (9), we get

$u(0,t) = [k_1 \cos cwt + k_2 \sin cwt][k_3 \cos 0 + k_4 \sin 0]$

$0 = [k_1 \cos cwt + k_2 \sin cwt]k_3 \Rightarrow k_3 = 0$

Substituting in (9), we obtain

$$u(x,t) = [k_1 \cos cwt + k_2 \sin cwt]\, k_4 \sin wx \quad \ldots\ldots(12)$$

Substituting the condition $\mathrm{u}(l,\mathrm{t}) = 0$ in (12), we get

$$u(l,t) = [k_1 \cos cwt + k_2 \sin cwt]\, k_4 \sin wl$$

$$0 = \underbrace{[k_1 \cos cwt + k_2 \sin cwt]}_{\neq 0}\, \underbrace{k_4}_{\neq 0} \sin wl$$

$\therefore \sin wl = 0 \longrightarrow wl = n\pi; \quad n = 1,2,3, \ldots.$

$$\boxed{w = \frac{n\pi}{l}}$$

Substituting in (12), we get

$$u(x,t) = \left[k_1 \cos\frac{cn\pi t}{l} + k_2 \sin\frac{cn\pi t}{l}\right] k_4 \sin\frac{n\pi x}{l}$$

$$\Longrightarrow u(x,t) = \left[R_1 \cos\frac{cn\pi t}{l} + R_2 \sin\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l} \quad \ldots\ldots(13)$$

Where $R_1 = k_1 k_4, R_2 = k_2 k_4$

Since n has infinite values, then there are non-zero solutions

$u_n(x,t)$ of (13)

$$u_n(x,t) = \left[R_{1n} \cos\frac{cn\pi t}{l} + R_{2n} \sin\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l}$$

We consider more general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left[R_{1n} \cos\frac{cn\pi t}{l} + R_{2n} \sin\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l} \quad \ldots\ldots(14)$$

Substituting the initial condition $\mathrm{u}(\mathrm{x},0) = f(x)$ in (14), we get

$$\mathrm{u}(\mathrm{x},0) = \sum_{n=1}^{\infty} [R_{1n} \cos 0 + R_{2n} \sin 0]\, \sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty} R_{1n} \quad \sin\frac{n\pi x}{l}$$

Which is Fourier sine series, then

$$R_{1n}=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \ldots\ldots(15)$$

Differentiating (14) partially w.r.t t we get:

$$u_t(x,t)=\sum_{n=1}^{\infty}\left[\frac{-cn\pi}{l}R_{1n} \quad \sin\frac{cn\pi t}{l}+\frac{cn\pi}{l}\,R_{2n}\cos\frac{cn\pi t}{l}\right]\sin\frac{n\pi x}{l}$$

Substituting the initial velocity condition $u_t(x,0)=g(x)$, we get

$$u_t(x,0)=\sum_{n=1}^{\infty}\left[\frac{-cn\pi}{l}R_{1n} \quad \sin 0+\frac{cn\pi}{l}\,R_{2n}\cos 0\right]\sin\frac{n\pi x}{l}$$

$$g(x)=\sum_{n=1}^{\infty}\frac{cn\pi}{l}R_{2n} \quad \sin\frac{n\pi x}{l}$$

Which is Fourier sine series, then

$$\frac{cn\pi}{l}R_{2n}=\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \longrightarrow R_{2n}=\frac{2}{cn\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \quad (16)$$

Hence the required solution is given by (14) where

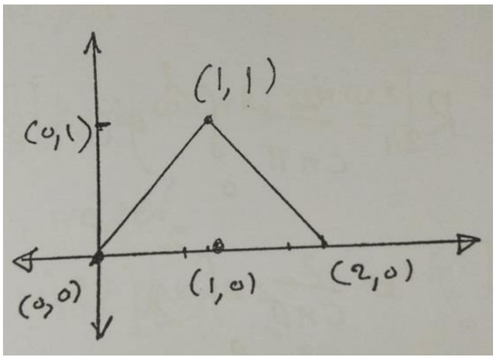
R_{1n} and R_{2n} are given in (15) and (16).

<u>**Ex. 1**</u>**: A taut string of length 2 has its ends x=0 and x=2 fixed. The mid-point is taken to height 1 and released from rest at time t=0. Find the displacement function $\mathbf{u(x,t)}$.**

Sol.:

$l = 2, g(x) = 0$ (from rest)

Then $R_{1n} = 0$ because $f(x) = 0$



$\mathrm{u}(\mathrm{x}, 0) = f(x) = \begin{cases} x & if \quad 0 \leq x \leq 1 \\ 2 - x & if \;\; 1 \leq x \leq 2 \end{cases}$

So $R_{2n} = 0$ because $g(x) = 0$

$$R_{1n} = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{2} \left[\int_0^1 x \sin \frac{n\pi x}{2} \, dx + \int_1^2 (2 - x) \sin \frac{n\pi x}{2} \, dx \right]$$

$= \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2}$ (integrating by parts)

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \cos \frac{cn\pi t}{2} \sin \frac{n\pi x}{2}$$

<u>Ex. 2</u>: An elastic string of length (2 cm) has its ends x=0 and x=2 fixed with no initial displacement. The string is released with initial velocity equal x Find the displacement function $\mathbf{u}(\mathbf{x}, \mathbf{t})$.

<u>Sol.:</u>

$l = 2, f(x) = 0, g(x) = x$

Then $R_{1n} = 0$ because $f(x) = 0$

$$R_{2n} = \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{cn\pi}\int_0^2 x \sin\frac{n\pi x}{2}\ dx = \frac{8\,(-1)^{n+1}}{c\, n^2\pi^2}$$

$$u(x,t) = \sum_{n=1}^{\infty} R_{2n} \quad sin\frac{cn\pi t}{l}\ \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} \frac{8\,(-1)^{n+1}}{c\, n^2\pi^2} \quad sin\frac{cn\pi t}{2}\ \sin\frac{n\pi x}{2}$$

<u>Ex. 3</u>: An elastic string of length (2) has its ends x=0 and x=2 fixed. Its initial deflection looks like a parabola with vertex at the point (1,1). It is set in motion with initial velocity equal x Find the displacement function $\mathbf{u(x,t)}$, then find the displacement $\mathbf{u(1,2)}$.

Sol.:

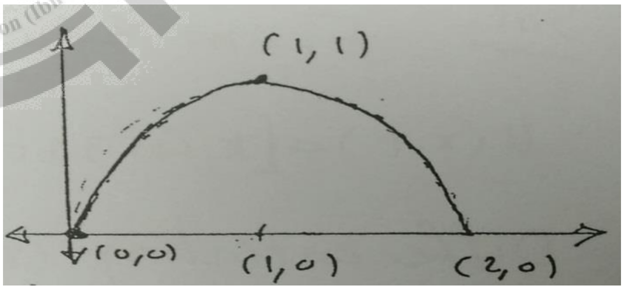
$$\mathrm{l} = 2, \mathrm{g}(x) = \mathrm{x}\ , \mathrm{u}(0,\mathrm{t}) = 0, u(2,t) = 0$$

f(x) is a parabola, the vertex is (1,1) then

$y - k = 4p(x-h)^2$ or

$(x-h)^2 = -4p(y-k)$ … … $(*)$

$\Longrightarrow (x-1)^2 = -4p(y-1),\ (h,k) = (1,1)$ … … $(**)$



Putting$(0,0)$ in $(**)$, we get

$$1 = 4p \longrightarrow p = \frac{1}{4}$$

Then $(**)$ will be

$(x-1)^2 = -(y-1)$

$\longrightarrow \boxed{y = f(x) = -x^2 + 2x}$

So $R_{1n} = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$= \int_0^2 (-x^2+2x)\sin\frac{n\pi x}{2}\,dx = \frac{16[(-1)^n-1]}{n^3\pi^3}$

$$R_{2n} = \frac{2}{cn\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

$= \frac{2}{cn\pi}\int_0^2 x\sin\frac{n\pi x}{2}\,dx = \frac{8(-1)^{n+1}}{c\,n^2\pi^2}$

Then the displacement is

$$u(x,t) = \sum_{n=1}^{\infty}\left[R_{1n}\cos\frac{cn\pi t}{l} + R_{2n}\sin\frac{cn\pi t}{l}\right]\sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty}\left[\frac{16[(-1)^n-1]}{n^3\pi^3}\cos\frac{cn\pi t}{2} + \frac{8(-1)^{n+1}}{c\,n^2\pi^2}\sin\frac{cn\pi t}{2}\right]\sin\frac{n\pi x}{2}$$

$$u(1,2) = \sum_{n=1}^{\infty}\left[\frac{16[(-1)^n-1]}{n^3\pi^3}\cos cn\pi + \frac{8(-1)^{n+1}}{c\,n^2\pi^2}\sin cn\pi\right]\sin\frac{n\pi}{2}$$

<u>Ex.4:</u> Solve the wave equation $u_{tt} = 9u_{xx},\ 0 \le x \le l,\ 0 \le t < \infty$ subject to the following conditions:

$$\boldsymbol{u_t(0,t) = 0,\ u(l,t) = 0,\ u(x,0) = \varphi(x),\ u_t(x,0) = g(x).}$$

Sol.: From equation (9), we have

$u(x,t) = [k_1\cos 3wt + k_2\sin 3wt][k_3\cos wx + k_4\sin wx]$ ... (4.1)

Differentiating (4.1) w.r.t (t), we get

$$u_t(x,t) = [-3wk_1\sin 3wt + 3wk_2\cos 3wt][k_3\cos wx + k_4\sin wx]$$

Substituting the first condition, we get

$$u_t(0,t) = [-3wk_1 \ \sin 3wt + 3wk_2 \cos 3wt]\left[k_3 \underbrace{\cos 0}_{=1} + k_4 \underbrace{\sin 0}_{=0}\right]$$

$$0 = \underbrace{[-3wk_1 \ \sin 3wt + 3wk_2 \cos 3wt]}_{\neq 0} \ k_3 \ \Longrightarrow \boxed{k_3 \ = 0}$$

Substituting in (4.1), we get

$$u(x,t) = [k_1 \ \cos 3wt + k_2 \ \sin 3wt] k_4 \ \sin wx \ \ldots \ldots (4.2)$$

Substituting the second condition in (4.2)

$$u(l,t) = [k_1 \ \cos 3wt + k_2 \ \sin 3wt] k_4 \ \sin wl$$

$$0 = \underbrace{[k_1 \ \cos 3wt + k_2 \ \sin 3wt]}_{\neq 0} \underbrace{k_4}_{\neq 0} \sin wl$$

$\therefore \sin wl = 0 \longrightarrow \ wl = n\pi, \quad n = 1,2,3, \ldots \ldots$

$$\boxed{w = \frac{n\pi}{l}}$$

Substituting in (4.2)

$$u(x,t) = \left[R_{1n} \ \cos \frac{3n\pi t}{l} + R_{2n} \ \sin \frac{3n\pi t}{l}\right] \sin \frac{n\pi x}{l} \ \ldots \ldots (4.3)$$

Where $R_1 = k_1 \, k_4 \, , R_2 = k_2 \ k_4$

Since n has infinite values, then there are non-zero solutions $u_n(x,t)$, we will take a linear combination of them for more general solution

$$u(x,t) = \sum_{n=1}^{\infty} \left[R_{1n} \ \cos \frac{3n\pi t}{l} + R_{2n} \ \sin \frac{3n\pi t}{l}\right] \sin \frac{n\pi x}{l} \ \ldots \ldots (4.4)$$

Substituting the third condition in (4.4)

$$u(x,0) = \sum_{n=1}^{\infty} \left[R_{1n} \ \underbrace{\cos 0}_{=1} + R_{2n} \ \underbrace{\sin 0}_{=0}\right] \sin \frac{n\pi x}{l}$$

$$\phi(x) = \sum_{n=1}^{\infty} R_{1n} \quad \sin\frac{n\pi x}{l}$$

Which is Fourier sine series, then

$$R_{1n} = \frac{2}{l}\int_{0}^{l} \phi(x)\sin\frac{n\pi x}{l}\, dx \ldots\ldots (4.5)$$

Now, substituting the forth condition in (4.4) but first we will derive the equation (4.4) w.r.t (t)

$$u_t\,(x,t) = \sum_{n=1}^{\infty}\left[\frac{-3n\pi}{l}R_{1n}\ \sin\frac{3n\pi t}{l} + \frac{3n\pi}{l}\,R_{2n}\,\cos\frac{3n\pi t}{l}\right]\sin\frac{n\pi x}{l}$$

$$u_t\,(x,0) = \sum_{n=1}^{\infty}\left[\frac{-3n\pi}{l}R_{1n}\ \ \sin 0 + \frac{3n\pi}{l}\,R_{2n}\,\cos 0\right]\sin\frac{n\pi x}{l}$$

$$g(x) = \sum_{n=1}^{\infty}\frac{3n\pi}{l}R_{2n}\ \ \sin\frac{n\pi x}{l}$$

Which is Fourier sine series, then

$$R_{2n} = \frac{2}{3n\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \qquad\qquad \ldots\ldots(4.6)$$

Hence, the required solution is given by (4.4) where R_{1n} and R_{2n} are given in (4.5) and (4.6)

… Exercises …

1- An elastic string of length (5) has its ends x=0 and x=5 fixed is initially in the position equal $\frac{x}{2}$. The string is set in motion with initial velocity equal $\frac{1}{2}$. Find the displacement function $u(x,t)$ for all t.

2- An elastic string of length (2 cm) has its ends x=0 and x=2 fixed with no initial displacement. The string is set in motion with initial velocity equal 3. Find the displacement $u(x,t)$ for all t.

3- An elastic string of length (4 cm) has its ends x=0 and x=4 fixed. It is released from rest in the position 2x. Find the displacement of the string $u(x,t)$.

4- An elastic string of length (2 cm) has its ends x=0 and x=2 fixed. It is released from rest in the position x^2. Find the displacement of the string at any time.

5- Solve the wave equation $u_{tt} = u_{xx},\ 0 \le x \le 1,\ 0 \le t < \infty$ under the following conditions:

$$u(0,t) = 0,\ u(1,t) = 0, u(x,0) = 3, u_t\,(x,0) = 5.$$

6- Solve the wave equation $u_{tt} = \alpha^2 u_{xx},\ 0 \le x \le 5,\ 0 \le t < \infty$ under the following conditions:

$$u(0,t) = 0, u_t\,(5,t) = 0, u(x,0) = 2x, u_t\,(x,0) = 4$$

7- Solve the wave equation $u_{tt} = u_{xx},\ 0 \le x \le 1,\ 0 \le t < \infty$ under the following conditions:

$$u_x\,(0,t) = 0, u(1,t) = 0, u(x,0) = f(x), u_t\,(x,0) = 0.$$

8- Solve the wave equation $u_{tt} = u_{xx},\ 0 \le x \le 3,\ 0 \le t < \infty$ under the following conditions:

$$u_x\,(0,t) = 0, u_x\,(3,t) = 0, u(x,0) = 2x,\ u_t\,(x,0) = 3.$$

Section 5.3: The D'Alembert Solution of the Wave Equation

In the case of the free vibration of an infinite string, the required function $u(x, t)$ must satisfy equation (1):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

With the initial conditions (11): $u(x, 0) = f(x), u_t\,(x, 0) = g(x)$

Where f(x) and g(x) must be specified in the interval $(-\infty, \infty)$ since the string is infinite.

The general solution of (1) can in fact be found, and in such a form that conditions (11) can easily be satisfied.

For this, we transform (1) to the new independent variables:

$$\xi = x + ct, \eta = x - ct \qquad \ldots\ldots (17)$$

These variables are called the (canonical coordinates).

On taking u as depending on x and t indirectly via ξ and η

We can express the derivatives with respect to the first variables in term of the derivatives with respect to the new variables:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial u}{\partial \eta} \cdot \underbrace{\frac{\partial \eta}{\partial x}}_{=1} \quad \text{(from (17))}$$

$$= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\therefore \boxed{u_x = u_\xi + u_\eta} \ \ldots\ldots (18)$$

Also, $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \cdot \underbrace{\frac{\partial \xi}{\partial t}}_{=c} + \frac{\partial u}{\partial \eta} \cdot \underbrace{\frac{\partial \eta}{\partial t}}_{=c}$ (from (17))

$$= c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

$\therefore \boxed{u_t = c(u_\xi - u_\eta)}$ ……(19)

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} . \frac{\partial u}{\partial x}$$

$$= \frac{\partial}{\partial x} . \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) \quad \text{(from (18))}$$

$$= \frac{\partial}{\partial \xi} . \frac{\partial u}{\partial x} + \frac{\partial}{\partial \eta} . \frac{\partial u}{\partial x}$$

$$= \frac{\partial}{\partial \xi} . \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta} . \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$\therefore \boxed{u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}}$ …….(20)

By the same way, we get

$\boxed{u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})}$ …….(21)

Substituting (20) and (21) in (1), we get

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2\,(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$c^2 u_{\xi\xi} - c^2 2u_{\xi\eta} + c^2 u_{\eta\eta} - c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} - c^2 u_{\eta\eta} = 0$$

$-4c^2 u_{\xi\zeta} = 0 \xrightarrow{\div -4c^2} \boxed{u_{\xi\zeta} = 0}$ ……(22)

Integrating (22) w.r.t ξ and η, we get $\quad u_\eta = \phi_1(\eta)$

$$u = \int \phi_1(\eta)\,\partial\eta + \phi_2(\eta)$$

$u(\xi,\eta) = \Phi(\eta) + \Psi(\xi)$

Where $\Phi(\eta) = \int \phi_1(\eta)\,\partial\eta$ and $\Psi(\xi) = \phi_2(\xi)$

Φ and Ψ are two arbitrary functions.

Now, returning to the old variables x and t, we get

$$\boxed{u(x,t) = \Phi(x-ct) + \Psi(x+ct)} \qquad \ldots..(23)$$

Substituting the initial condition $u(x,0) = f(x)$ in (23)

We get $u(x,0) = \Phi(x) + \Psi(x)$

$$\therefore \boxed{f(x) = \Phi(x) + \Psi(x)} \qquad \ldots\ldots(24)$$

Differentiating (23) w.r.t (t), $u_t(x,t) = \Phi'(x-ct)(-c) + c\Psi'(x+ct)$

Substituting the second initial condition $u_t(x,0) = g(x)$

$g(x) = -c\Phi'(x) + c\Psi'(x)$

Integrating this equation from 0 to x:

$$k + \int_0^x g(z)\,dz = -c\Phi(x) + c\Psi(x)$$

$$\Longrightarrow \boxed{\frac{k}{c} + \frac{1}{c}\int_0^x g(z)\,dz = -\Phi(x) + \Psi(x)} \qquad \ldots\ldots(25)$$

From (24) and (25), we can easily find $\Phi(x)$ and $\Psi(x)$, as follows:

$\Phi(x) = \frac{1}{2}f(x) - \frac{k}{2c} - \frac{1}{2c}\int_0^x g(z)\,dz$

$$\Psi(x) = \frac{1}{2} f(x) + \frac{k}{2c} + \frac{1}{2c} \int_0^x g(z)\, dz$$

Then replacing x by $(x - ct)$ in Φ and by $(x + ct)$ in Ψ and substituting in (23)

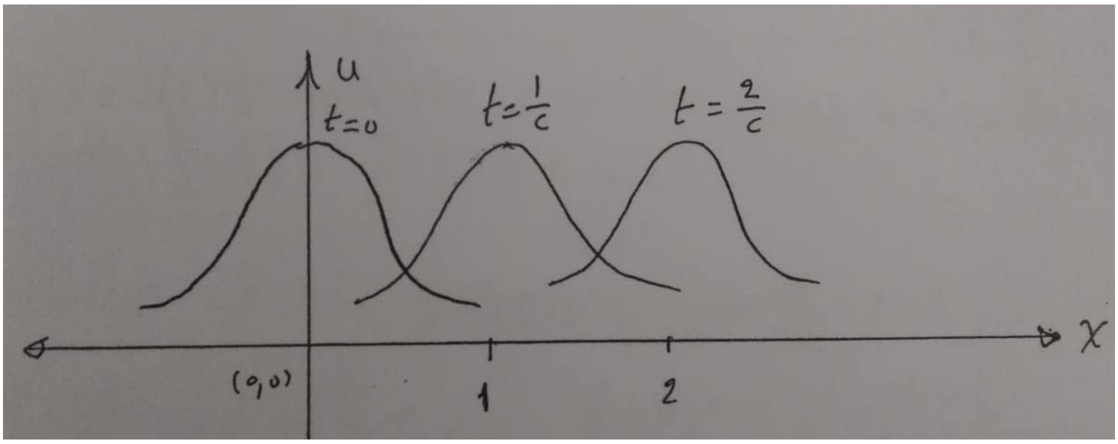
$$u(x,t) = \frac{1}{2} f(x - ct) - \frac{k}{2c} - \frac{1}{c} \int_0^{x-ct} g(z)\, dz + \frac{1}{2} f(x + ct) + \frac{k}{2c}$$
$$+ \frac{1}{c} \int_0^{x+ct} g(z)\, dz$$

$$\Longrightarrow \boxed{u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\, dz} \dots\dots (26)$$

This is D'Alembert 's solution to (1) subject to (11), on the interval $(-\infty, \infty)$.

The importance of the general solution of the wave equation (D'Alembert solution) lie that physically represents the sum of two travelling waves in opposite directions with velocity c for example:

1- $u(x,t) = \sin(x - ct)$ (a wave is travelling to the right).
2- $u(x,t) = (x + ct)^2$ (a wave is travelling to the left).
3- $u(x,t) = \sin(x - ct) + (x + ct)^2$ (two waves in opposite directions).
4- $u(x,t) = e^{-(x-ct)^2}$ (a wave is travelling to the right).



$$u(x,t) = e^{-(x-ct)^2}$$

<u>Ex.5:</u> Solve the equation $\mathbf{u_{tt} = c^2 u_{xx}}$, $\mathbf{-\infty < x < \infty}$, $\mathbf{0 \le t < \infty}$ under the following conditions: $\mathbf{u(x,0) = \sin x}$, $\mathbf{u_t(x,0) = 0}$

sol: from D'Alembert solution

$$u(x,t) = \frac{1}{2}[f\,(x+ct) + f\,(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

Since $f(x) = \sin x$ and $g(x) = 0$, then

$u(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)]$

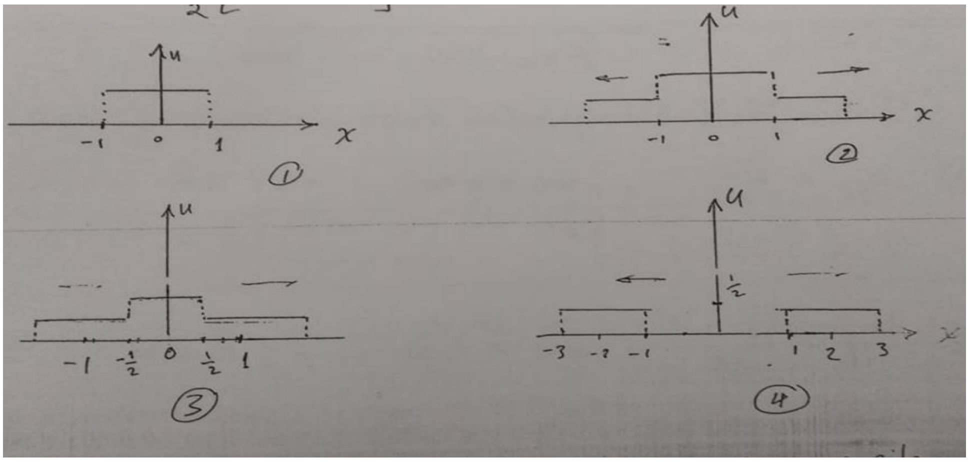
This solution is a two waves in opposite directions

<u>Ex.6:</u> Solve the equation $\mathbf{u_{tt} = c^2 u_{xx}}$, $\mathbf{-\infty < x < \infty}$, $\mathbf{0 \le t < \infty}$ under the following conditions: $\mathbf{u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}}$, $\mathbf{u_t(x,0) = 0}$

sol: from D'Alembert solution

$$u(x,t) = \frac{1}{2}[f\,(x+ct) + f\,(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

$$= \frac{1}{2}[1+1] + 0 = 1$$



Note that the solution $u(x,t)$ is two waves in opposite directions each one equal to $\frac{1}{2}$.

<u>Ex.7:</u> A string is set in motion its equilibrium position with an initial velocity $\mathbf{u}_t(\mathbf{x},\mathbf{0}) = \sin x$ Find the displacement $\mathbf{u}(\mathbf{x},\mathbf{t})$ of the string.

sol:

$\mathrm{u}(\mathrm{x},0) = \mathrm{f}\,(x) = 0$ (since the string is an equi. position)

$\mathrm{u}_t(\mathrm{x},0) = \sin x$

$$u(x,t) = \frac{1}{2}[f\,(x-ct) + f\,(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

$$= \frac{1}{2}[0+0] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin z\,dz$$

$$= \frac{1}{2c}[cos\,(x-ct) - cos\,(x+ct)]$$

… Exercises …

1- Solve the following initial value problems:

a- $u_{tt} = u_{xx},\ -\infty < x < \infty,\ 0 \le t < \infty$, where $\mathrm{u}(\mathrm{x},0) = e^{-x^2}$,

$\mathrm{u}_t(\mathrm{x},0) = 0$

b- $u_{tt} = u_{xx},\ -\infty < x < \infty,\ 0 \le t < \infty$, where $\mathrm{u}(\mathrm{x},0) = 0$,

$\mathrm{u}_t(\mathrm{x},0) = xe^{-x^2}$

2- A string with free ends is set motion with no initial velocity from an initial position $\mathrm{u}(\mathrm{x},0) = x^2$. Find the displacement of the string.

3- Sketch the following solution of the wave equation $\mathrm{u}(\mathrm{x},\mathrm{t}) = (x + ct)^2$

4- Using the canonical coordinates to show that the solution of the equation $u_{tt} = c^2 u_{xx},\ -\infty < x < \infty,\ 0 \le t < \infty$ is $u(x,t) = \Phi(x - ct) + \Psi(x + ct)$

5- If you know that $u(x,t) = \Phi(x - ct) + \Psi(x + ct)$ satisfy the conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, show that

$u(x,t) = \frac{1}{2}[f\ (x + ct) + f\ (x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\, dz$

6- By using the canonical coordinates, show that the equation $u_{tt} = c^2 u_{xx},\ -\infty < x < \infty,\ 0 \le t < \infty$ can be reduced to the form $u_{\xi\eta} = 0$

7- Prove that the equation $u\ (\xi,\eta) = \sin\xi +\ e^{\eta}$ is a solution of the equation $u_{\xi\eta} = 0$

8- A string with free ends is set in motion with no initial velocity from an initial position $\mathrm{u}(\mathrm{x},0) = x^3$. Find the displacement of the string.

9- Solve the wave equation $u_{tt} = c^2 u_{xx},\ -\infty < x < \infty,\ 0 \leq t < \infty$

under the conditions: $\mathrm{u}(\mathrm{x}, 0) = \frac{x}{3}$, $\mathrm{u}_t(\mathrm{x}, 0) = \frac{1}{3}$

10- Solve the wave equation $u_{tt} = c^2 u_{xx},\ -\infty < x < \infty,\ 0 \leq t < \infty$

under the conditions: $\mathrm{u}(\mathrm{x}, 0) = \sin x$, $\mathrm{u}_t(\mathrm{x}, 0) = \cos x$

11- Prove that the solution of the equations

$\Phi(x) + \Psi(x) = f(x)$, $-\Phi'(x) + \Psi'(x) = 0$

Is $u(x,t) = \frac{1}{2}[f\ (x - ct) + f\ (x + ct)]$

Section 5.4: Method of characteristics

The principle purpose in this section is to introduce the notion of characteristics is more important for theoretical purposes and for an understanding of the nature of solutions of certain types of partial differential equations than they are as practical methods for obtaining solutions.

In this section we will discuss the solution of first order linear and quasilinear PDEs by the characteristics method which is based on finding the characteristics curve of PDE .

The method of characteristics can be used only for hyperbolic problems which possess the right number of characteristics families. Recall that for second order parabolic problems we have only one family of characteristics and for elliptic PDEs. no real characteristics curves exist.

5.4.1 : Advance equation(first order wave equation)

The one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{.......(27)}$$

Can be rewritten as either of the following

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = 0 \qquad \text{.....(28)}$$

Since the mixed derivative terms cancel. If we let

$$v = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} \qquad \text{.....(29)}$$

Then (28) becomes

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0 \qquad \text{.....(30)}$$

We now show to solve (30) which is called the first order wave equation or advection equation

Remark Although (30) can be used to solve the one dimensional second order wave equation (27).

To solve (30) we note that if we consider an observer moving on a curve $x(t)$ then by the chain rule we get

$$\frac{dv(x(t),t)}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{dx}{dt}$$

If the observer is moving at a rate $\frac{dx}{dt} = c$, then by comparing (31) and (30) we find

$$\frac{dv}{dt} = 0 \qquad \text{.....(32)}$$

Therefore (30) can be replaced by a set of two ODEs

$$\frac{dx}{dt} = c \qquad \text{....(33)}$$

$$\frac{dv}{dt} = 0 \qquad \text{....(34)}$$

These 2ODEs are easy to solve. Integrating of (33) yields

$x(t) = ct + c_o \qquad c_o = x(0)$

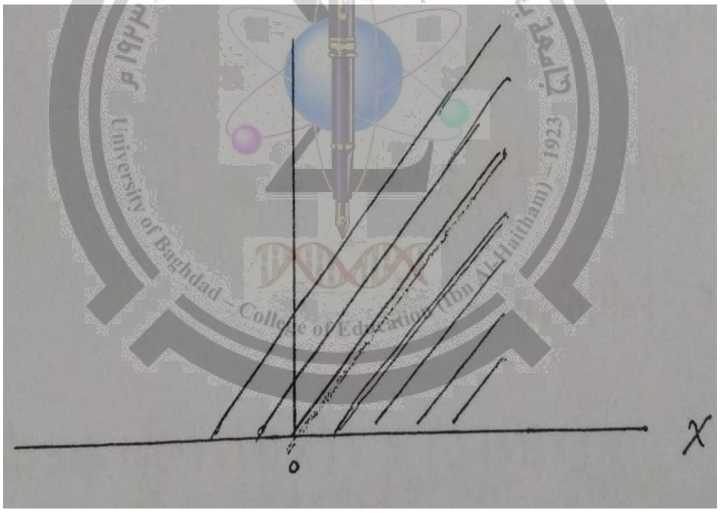
$$\therefore x(t) = ct + x(0) \qquad \text{......(35)}$$

And the other one has a solution

$\boxed{v = b}$

Where b is a constant a long the curve given in (35)

The curve (35) is a straight line. In fact, we have a family of parallel straight lines, called characteristics, as follows in the figure



Characteristic s $t = \frac{1}{c}x - \frac{1}{c}x(0)$

In order to obtain the general solution of the one dimensional equation (30) subject to the initial value

$$v(x(0), 0) = f\big(x(0)\big) \qquad \text{.....(36)}$$

We note that V= constant a long $x(t) = x(0) + ct$

$\Rightarrow x(0) = x(t) - ct \Rightarrow f\big(x(0)\big) = f(x(t) - ct) = v(x, t)$

Then the general solution is $v(x,t) = f(x(t) - ct)$ ....(37)

<u>Ex.8:</u> Solve $\frac{\partial v}{\partial t} + 3\frac{\partial v}{\partial x} = 0$**, where** $v(x,0) = \begin{cases} \frac{1}{2}x & 0 < x < 1 \\ 0 & otherwise \end{cases}$

Sol: The two ODEs are $\frac{dx}{dt} = 3$ ....(8.1)

$\frac{dv}{dt} = 0$ ....(8.2)

The solution of (8.1) is $x(t) = 3t + x(0) \Rightarrow x(0) = x(t) - 3t$

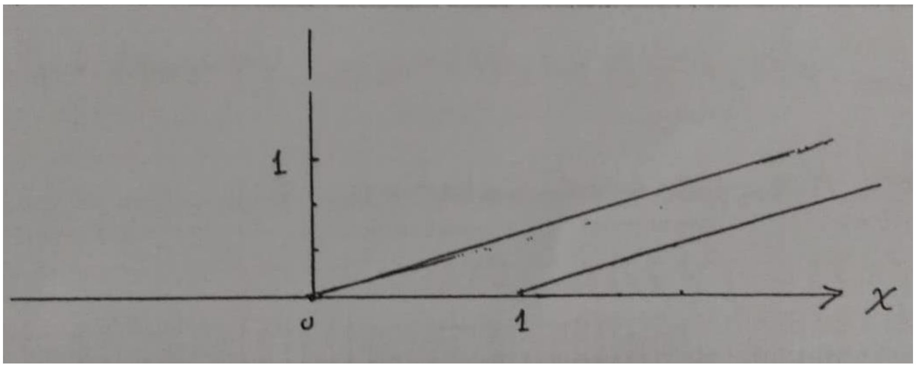
And the solution of (8.2) is $v(x(t),t) = v(x(0),0) = constant$

$$v(x(0),0) = \begin{cases} \frac{1}{2}x(0) & 0 < x(0) < 1 \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \frac{1}{2}(x - 3t) & 0 < x - 3t < 1 \\ 0 & otherwise \end{cases}$$

$$= v(x(t),t)$$

Let's sketch the characteristics through the points $x = 0$, $x = 1$



Characteristics for $x(0) = 0$ and $x(0) = 1$

Ex.9: **Solve using the method of characteristics** $\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} = e^{2x}$**,subject to** $\mathbf{u(x,0)} = \boldsymbol{f(x)}$**.**

Sol: The system of ODEs is

$$\frac{du}{dt} = e^{2x} \qquad \text{....(9.1)}$$

$$\frac{dx}{dt} = -2 \qquad \text{....(9.2)}$$

Solve (9.2) to get the characteristic curve

$$x(t) = x(0) - 2t \qquad \text{....(9.3)}$$

Substituting the characteristic equation in (9.1) yields

$$\frac{du}{dt} = e^{2(x(0)-2t)} \longrightarrow du = e^{2x(0)-4t}dt$$

Then $u = k - \frac{1}{4}e^{2x(0)-4t}$ ....(9.4)

When $t = 0, u(x,0) = f(x) \longrightarrow u(x(0),0) = f(x(0))$

Substituting in (9.4) to find the constant k

$$u(x(0),0) = f(x(0)) = k - \frac{1}{4}e^{2x(0)}$$

$$\therefore k = f(x(0)) + \frac{1}{4}e^{2x(0)}$$

Substitute k in (9.4) we have

$$u(x,t) = f(x(0)) + \frac{1}{4}e^{2x(0)} - \frac{1}{4}e^{2x(0)-4t}$$

now substitute for $x(0)$ from (9.3) we get

$$\boxed{u(x,t) = f(x+2t) + \frac{1}{4}e^{2x}(e^{4t}-1)} \qquad \text{....(9.5)}$$

Hence (9.5) is the general solution with the characteristics in (9.3)

… Exercises …

Solve the following equations using the method of characteristic subject to the initial condition corresponding

1- $\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0, w(x,0) = \sin x$

2- $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = e^{2x}, u(x,0) = f(x)$

3- $\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} = e^{2x}, u(x,0) = \cos x$

4- $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 1, u(x,0) = f(x)$

Chapter six

Laplace' s Equation and Laplace Transform

Contents

Section(6.1): 1- Laplace's Equation in Two Dimensions

One of the most important of all partial differential equations occurring in applied mathematics in that associated with the name of Laplace, in two dimensions

$$u_{xx} + u_{yy} = 0 \qquad \ldots..(1)$$

Laplace's equation appears in many branches of mathematical physics, for example in a stable heat problems (i.e. the problems which the temperature does not depend on time), as well as in stable electrical problems,….

We denote to Laplace's equation by $\nabla^2 u = 0$ where ∇^2 is Laplace's operator.

2- General solution of two- dimensional Laplace's equation

To solve equation (1), we assume that (by separation of variables)

$$u(x,t) = X(x).Y(y) \qquad \ldots\ldots(2)$$

Where X and Y are functions of x and y respectively

From(2), $\frac{\partial^2 u}{\partial x^2} = X''(x).Y(y)$ and $\frac{\partial^2 u}{\partial y^2} = X(x).Y''(y)$

Hence (2) reduces to

$$X''(x).Y(y) + X(x).Y''(y) = 0 \Longrightarrow \frac{X''}{X} = -\frac{Y''}{Y} \qquad \ldots..(3)$$

Since the left hand side of (3) depends only on x and the right hand side depend only on y, both sides of (3) must be equal to same constant, say μ. This leads to two ordinary differential equations.

$$X'' - \mu X = 0 \text{ and } Y'' + \mu Y = 0 \qquad \ldots\ldots.(4)$$

Whose solutions depends only on the value of μ. Three cases arise:

Case 1- When $\mu = 0$, then reduces to $X'' = 0$ and $Y'' = 0$

Solving these, $X = Ax + B$and $Y = Cy + D$, then a solution of (1) is

$$\boxed{u(x,y) = (Ax + B)(Cy + D)} \quad \ldots..(5)$$

When x or $y = 0$ then $u(x,y) = 0$ and this will be a trivial solution

Case 2- When $\mu = \lambda^2$ i.e. positive. Here $\lambda \neq 0$, then (4) reduces to

$X'' - \lambda^2 X = 0$ and $Y'' = +\lambda^2 Y = 0$

Solving these, we get

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ and $Y(y) = Ccos\lambda y + Dsin\,\lambda y$

Then a solution of (1) is

$$\boxed{u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)(Ccos\lambda y + Dsin\,\lambda y)} \quad \ldots.(6)$$

Where A, B, C,D are constant

Case 3- When $\mu = -\lambda^2$ i.e. negative. Here $\lambda \neq 0$, then (4) reduces to

$X'' + \lambda^2 X = 0$ and $Y'' - \lambda^2 Y = 0$

Solving these, we get

$X(x) = Acos\lambda x + Bsin\,\lambda x$ and $Y(y) = Ce^{\lambda y} + De^{-\lambda y}$

Then a solution of (1) is

$$\boxed{u(x,y) = (Acos\lambda x + Bsin\,\lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right)} \quad \ldots.(7)$$

Where A, B, C, and D are arbitrary constants

3- Dirichlet problem in a rectangle

Suppose that we have a rectangular metal plate isolated ends not depend on time, as follows

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ \ 0 \leq x \leq a, \qquad 0 \leq y \leq b$$

With boundary conditions:

$$u(0,y) = f_1(y), u(a,y) = f_2(y), u(x,0) = g_1(x), u(x,b) = g_2(x)$$

here we will study the situation where one of the boundary conditions is a function of x and the other conditions are equal to zero, as shown in the following example.

Ex.1: **Find the solution $\mathbf{u(x,y)}$ of Laplace's equation in the rectangle $\mathbf{0 < x < a,\quad 0 < y < b}$ also satisfying the boundary conditions:**

$$\mathbf{u(0,y) = 0\ ,\ u(a,y) = 0\ ,\ u(x,0) = 0\ ,\ u(x,b) = f(x)}$$

Sol: From equation (7)

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right) \qquad \dots\dots(7)$$

Substituting the condition $u(0,y) = 0$ in (7)

$$u(0,y) = (A\cos 0 + B\sin 0)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$0 = A\underbrace{\left(Ce^{\lambda y} + De^{-\lambda y}\right)}_{\neq 0} \Rightarrow \boxed{A = 0}$$

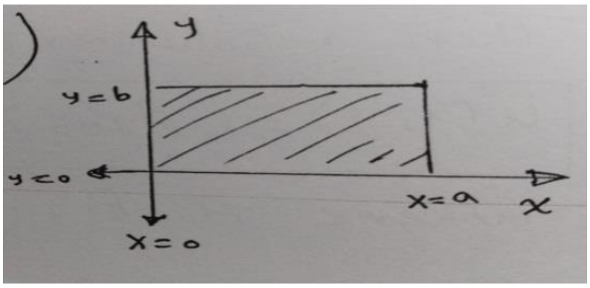
Substituting in (7)

$$u(x,y) = B\sin\lambda x\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$u(x,y) = \sin\lambda x\left(Ee^{\lambda y} + Fe^{-\lambda y}\right) \qquad \dots\dots(8)$$

Substituting the condition $u(x,0) = 0$

$$u(x,0) = \sin\lambda x\,(Ee^{0} + Fe^{0})$$



$$0 = \underbrace{\sin\lambda x}_{\neq 0}\ (E + F) \implies E + F = 0 \implies \boxed{F = -E}$$

Substituting in (8)

$$u(x, y) = \sin\lambda x \left(Ee^{\lambda y} - Ee^{-\lambda y}\right)$$

$$\implies u(x, y) = E \sin\lambda x \left(e^{\lambda y} - e^{-\lambda y}\right) \qquad \dots\dots(9)$$

Substituting the condition $u(a, y) = 0$ in (9)

$$\underbrace{u(a, y)}_{=0} = \underbrace{E}_{\neq 0}\ \sin\lambda a\ \underbrace{\left(e^{\lambda y} - e^{-\lambda y}\right)}_{\neq 0} \implies \sin\lambda a = 0$$

$$\therefore \lambda a = n\pi,\ \ n = 1,2,3,\dots\dots \quad \implies \lambda = \frac{n\pi}{a}$$

Substituting in (9), hence non zero solutions $u_n(x, y)$ of (9) are given by

$$u_n(x, y) = E_n\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)\sin\frac{n\pi x}{a}$$

For more general solution, we take

$$u(x, t) = \sum_{n=1}^{\infty} u_n\,(x, y)$$

$$\implies u(x, t) = \sum_{n=1}^{\infty} E_n\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)\sin\frac{n\pi x}{a} \qquad \dots\dots(10)$$

Substituting the condition $u(x, b) = f(x)$ in (10)

$$\underbrace{u(x, b)}_{=f(x)} = \sum_{n=1}^{\infty} E_n\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)\sin\frac{n\pi x}{a}$$

Which is Fourier sine series of $f(x)$, hence we get

$$E_n\left(e^{\frac{n\pi b}{a}}-e^{-\frac{n\pi b}{a}}\right)=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

$$\Rightarrow E_n=\frac{2}{a\left(e^{\frac{n\pi b}{a}}-e^{-\frac{n\pi b}{a}}\right)}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx \ldots\ldots(11)$$

Hence (10) is the required solution where E_n is given in (11)

Now, we will study the situation where one of the boundary conditions is a function of y and the other conditions are equal to zero, as shown in the following example.

Ex.2: Find the solution $\mathbf{u(x,y)}$ of Laplace's equation in the semi- infinite plate $\mathbf{0<x<\infty}$, $\mathbf{0<y<b}$ also satisfying the boundary conditions:

$$u(0,y)=f(y)\,, u(\infty,y)=0\,, u(x,0)=0\,, u(x,b)=0$$

Sol: Rearrange conditions

1- $u(x,0)=0$, 2- $u(\infty,y)=0$, 3- $u(x,b)=0$, 4- $u(0,y)=f(y)$

From equation (6)

$$u(x,y)=\left(Ae^{\lambda x}+Be^{-\lambda x}\right)(C\cos\lambda y+D\sin\lambda y)$$

Substituting the first condition, we get

$$u(x,0)=\left(Ae^{\lambda x}+Be^{-\lambda x}\right)(C\cos 0+D\sin 0)$$

$$0=\underbrace{\left(Ae^{\lambda x}+Be^{-\lambda x}\right)}_{\neq 0}C \quad\Rightarrow \boxed{C=0}$$

Butting in equation (6)

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)D\sin\lambda y$$

$$u(x,y) = \left(Ee^{\lambda x} + Fe^{-\lambda x}\right)\sin\lambda y \qquad \dots\dots(12)$$

Substituting the second condition in (12)

$$u(\infty,y) = \lim_{x\to\infty}\left(Ee^{\lambda x} + Fe^{-\lambda x}\right)\sin\lambda y$$

$$0 = \left(E\sin\lambda y \lim_{x\to\infty} e^{\lambda x} + F\sin\lambda y \underbrace{\lim_{x\to\infty} e^{-\lambda x}}_{=0}\right)$$

$$\therefore 0 = E \underbrace{\sin\lambda y}_{\neq 0} \underbrace{\lim_{x\to\infty} e^{\lambda x}}_{=0} \Longrightarrow \boxed{E = 0}$$

Putting in (12)

$$u(x,y) = F\, e^{-\lambda x} \sin\lambda y \qquad \dots\dots(13)$$

Substituting the third condition in (13)

$$u(x,b) = F\, e^{-\lambda x} \sin\lambda b$$

$$0 = \underbrace{F}_{\neq 0} \underbrace{e^{-\lambda x}}_{\neq 0} \sin\lambda b \Longrightarrow \sin\lambda b = 0$$

$$\therefore \lambda b = n\pi, \; n = 1,2,3,\dots.. \quad \Longrightarrow \lambda = \frac{n\pi}{b}$$

Substituting in (13), hence non zero solutions $u_n(x,y)$ are given by

$$u_n(x,y) = F_n e^{-\frac{n\pi x}{b}} \sin\frac{n\pi y}{b}$$

For more general solution, we take the sum of $u_n\,(x, y)$

$$u(x,y) = \sum_{n=1}^{\infty} F_n e^{-\frac{n\pi x}{b}} \sin\frac{n\pi y}{b} \qquad \ldots.(14)$$

Substituting the fourth condition in (14)

$$u(0,y) = \sum_{n=1}^{\infty} F_n e^{0} \sin\frac{n\pi y}{b}$$

$$f(y) = \sum_{n=1}^{\infty} F_n \ \sin\frac{n\pi y}{b}$$

Which is the Fourier sine series, then F_n is the Fourier coefficient in the form

$$F_n = \frac{2}{b}\int_0^b f(y) \sin\frac{n\pi y}{b}\ dy \qquad \ldots..(15)$$

Then the equation (14) is the required solution with F_n that given in (15)

Ex.3: Find the steady state temperature distribution in a rectangular plate of sides a and b isolated at the lateral surface and satisfying the boundary conditions:

$\mathrm{u}(0, \mathrm{y}) = \ \mathrm{u}(\mathrm{a}, \mathrm{y}) = 0$ for $0 \le y \le b$, and $\ \mathrm{u}(\mathrm{x}, \mathrm{b}) = 0\ $ and , $\mathrm{u}(\mathrm{x}, 0) = \mathrm{x}(a - x)$ for $0 \le x \le a$

sol:

The boundary conditions is

$\mathrm{u}(0, \mathrm{y}) = 0$, $\mathrm{u}(\mathrm{x}, \mathrm{b}) = 0$, $\mathrm{u}(\mathrm{a}, \mathrm{y}) = 0$, $\ \mathrm{u}(\mathrm{x}, 0) = \mathrm{x}(a - x)$, then we begin with equation (7)

$$u(x,y) = (A cos\lambda x + B sin\, \lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

Where A, B, C, and D are arbitrary constants

Substituting the first condition, we get

$$0 = A\left(Ce^{\lambda y} + De^{-\lambda y}\right) \Longrightarrow \boxed{A = 0}$$

Substituting in (7) , we get

$$u(x,y) = sin\lambda x\left(Ee^{\lambda y} + Fe^{-\lambda y}\right) \qquad \dots..(16)$$

Where $E = BC, F = BD$

Substituting the second condition in (16), we get

$$\underbrace{u(x,b)}_{=0} = \underbrace{sin\lambda x}_{\neq 0}\left(Ee^{\lambda b} + Fe^{-\lambda b}\right) \Longrightarrow Ee^{\lambda b} + Fe^{-\lambda b} = 0$$

$$\Longrightarrow \boxed{F = -Ee^{2\lambda b}}$$

Substituting in (16), we get

$$u(x,y) = E\, sin\lambda x\left(e^{\lambda y} - e^{2\lambda b}\, e^{-\lambda y}\right) \qquad \dots..(17)$$

Substituting the third condition in (17),

$$0 = \underbrace{E}_{\neq 0}\; sin\lambda a\; \underbrace{\left(e^{\lambda y} - e^{2\lambda b}\, e^{-\lambda y}\right)}_{\neq 0} \Longrightarrow sin\lambda a = 0$$

$$\therefore \lambda a = n\pi,\ n = 1,2,3, \dots.. \quad \Longrightarrow \lambda = \frac{n\pi}{a}$$

Putting in (17), hence non zero solutions $u_n(x,y)$ are given by

$$u_n(x,y) = E_n \sin\frac{n\pi x}{a}\left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}}\, e^{-\frac{n\pi y}{a}}\right)$$

For more general solution, we take the sum of $u_n\,(x,y)$

$$u(x,y) = \sum_{n=1}^{\infty} E_n\ \sin\frac{n\pi x}{a}\ \left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}}\ e^{-\frac{n\pi y}{a}}\right) \qquad \ldots.(18)$$

Substituting the fourth condition in (18)

$$u(x,0) = \sum_{n=1}^{\infty} E_n\ \sin\frac{n\pi x}{a}\ \left(e^{0} - e^{\frac{2n\pi b}{a}}\ e^{0}\right)$$

$$x(a-x) = \sum_{n=1}^{\infty} E_n\ \left(1 - e^{\frac{2n\pi b}{a}}\right)\sin\frac{n\pi x}{a}$$

Which is the Fourier sine series, then E_n is given by

$$E_n = \frac{2}{a\left(1 - e^{\frac{2n\pi b}{a}}\right)}\int_0^a x(a-x)\sin\frac{n\pi x}{a}\ dx$$

$$= \frac{4a^2}{n^3\pi^3\left(1 - e^{\frac{2n\pi b}{a}}\right)}[1-(-1)^n] = \begin{cases} 0 & , if\ n\ is\ even \\ \dfrac{8a^2}{n^3\pi^3\left(1 - e^{\frac{2n\pi b}{a}}\right)}, & if\ n\ is\ odd \end{cases}$$

$$u(x,y) = \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{8a^2}{n^3\pi^3\left(1 - e^{\frac{2n\pi b}{a}}\right)} \sin\frac{n\pi x}{a}\ \left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}}\ e^{-\frac{n\pi y}{a}}\right)\ldots.(19)$$

The equation (19) is the required solution

4-The problem of the steady potential

By the same way in section 3 where we find the steady temperature distribution in a rectangular plate when the temperature on its perimeter is known, so too we will find the steady potential on the plate if the potential on the perimeter is known.

<u>Ex.4</u>: A thin uniformly semi- infinite plate is in the form of area enclosed by the lines $\mathbf{x = 0, x = \infty, y = 0}$ and $\mathbf{y = b}$. The potential on the edge $\mathbf{x = 0}$, is $\left(\boldsymbol{5\sin \frac{3\pi y}{b} - 2\sin \frac{\pi y}{b}}\right)$, and on the other edge is zero. Find the steady potential at the points of the plate.

Sol:

$$u(0, y) = 5\sin \frac{3\pi y}{b} - 2\sin \frac{\pi y}{b}, , u(\infty, y) = 0, \; u(x, 0) = 0, u(x, b) = 0$$

by the same way in Ex.2, we start with equation (6) and get (14)

$$u(x, y) = \sum_{n=1}^{\infty} F_n e^{-\frac{n\pi x}{b}} \sin \frac{n\pi y}{b},$$

Where $F_n = \frac{2}{b}\int_0^b f(y) \sin \frac{n\pi y}{b}\, dy$

$$= \frac{2}{b}\int_0^b \left(5\sin \frac{3\pi y}{b} - 2\sin \frac{\pi y}{b}\right) \sin \frac{n\pi y}{b}\, dy$$

$$= \frac{10}{b}\int_0^b \sin \frac{3\pi y}{b} \sin \frac{n\pi y}{b}\, dy \; - \frac{4}{b}\int_0^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b}\, dy$$

$$= \frac{10}{b}\int_0^b \frac{1}{2}\left[\cos \frac{(3-n)\pi y}{b} - \cos \frac{(3+n)\pi y}{b}\right] dy$$

$$- \frac{4}{b}\int_0^b \frac{1}{2}\left[\cos \frac{(1-n)\pi y}{b} - \cos \frac{(1+n)\pi y}{b}\right] dy$$

$$=\frac{5}{b}\left[\frac{b}{(3-\mathrm{n})\pi}\sin\frac{(3-\mathrm{n})\pi\mathrm{y}}{\mathrm{b}}-\frac{b}{(3+\mathrm{n})\pi}\sin\frac{(3+\mathrm{n})\pi\mathrm{y}}{\mathrm{b}}\right]_0^b$$

$$-\frac{2}{b}\left[\frac{b}{(1-\mathrm{n})\pi}\sin\frac{(1-\mathrm{n})\pi\mathrm{y}}{\mathrm{b}}-\frac{b}{(1+\mathrm{n})\pi}\sin\frac{(1+\mathrm{n})\pi\mathrm{y}}{\mathrm{b}}\right]_0^b=0;$$

$$n\neq 1,3$$

$\therefore F_n=0; n\neq 1,3$

Then we must find F_1 and F_3

$$F_1=\frac{2}{b}\int_0^b\left(5\sin\frac{3\pi\mathrm{y}}{\mathrm{b}}-2\sin\frac{\pi\mathrm{y}}{\mathrm{b}}\right)\sin\frac{\pi y}{b}\,dy$$

$$=\frac{2}{b}\int_0^b 5\underbrace{\sin\frac{3\pi\mathrm{y}}{\mathrm{b}}\sin\frac{\pi y}{b}\,dy}_{=0}-\frac{4}{b}\int_0^b\sin^2\frac{\pi\mathrm{y}}{\mathrm{b}}\,dy$$

$$=-\frac{4}{b}\int_0^b\frac{1}{2}\left(1-\cos\frac{2\pi\mathrm{y}}{\mathrm{b}}\right)\,dy=-\frac{2}{b}\left(y-\frac{b}{2\pi}\sin\frac{2\pi\mathrm{y}}{\mathrm{b}}\right)_0^b=-2$$

$$F_3=\frac{2}{b}\int_0^b 5\sin\frac{3\pi\mathrm{y}}{\mathrm{b}}\sin\frac{3\pi y}{b}\,dy\;\underbrace{-\frac{4}{b}\int_0^b\sin\frac{\pi\mathrm{y}}{\mathrm{b}}\sin\frac{3\pi y}{b}\,dy}_{=0}$$

$$=\frac{10}{b}\int_0^b\sin^2\frac{3\pi\mathrm{y}}{\mathrm{b}}\,dy=\frac{5}{b}\int_0^b\left(1-\cos\frac{6\pi\mathrm{y}}{\mathrm{b}}\right)\,dy=\frac{5}{b}\left(y-\frac{b}{6\pi}\sin\frac{6\pi\mathrm{y}}{\mathrm{b}}\right)_0^b=5$$

Substituting in equation (14), we get

$$u(x,y) = \sum_{n=1}^{\infty} F_n e^{-\frac{n\pi x}{b}} \sin\frac{n\pi y}{b} = 5e^{-\frac{3\pi x}{b}} \sin\frac{3\pi y}{b} - 2e^{-\frac{\pi x}{b}} \sin\frac{\pi y}{b} \dots (20)$$

Then (20) is the required solution

Section6.2:Laplace's Equation in polar co-ordinates (Dirichlet problem for a circle)

While solving boundary value problems, appropriate choice of coordinate system is very useful. In physical problems that involve a circular geometry (for example, the problem of evaluating the temperature in a circular disc or a sector of a circular disc), it is clear from the nature of the situation that the problem will be simplified if we can use polar coordinates (r,θ) instead of rectangular coordinates (x,y)

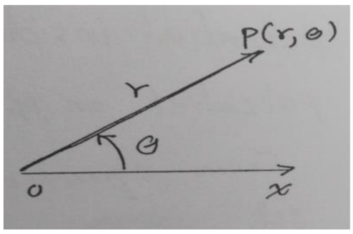
Polar coordinates are defined by means of equations

$x = rcos\theta\ , x = rsin\theta$, where $r \geq 0, 0 \leq \theta \leq 2\pi$ $\qquad \dots .(21)$

The Laplace's equation in polar coordinates is given by

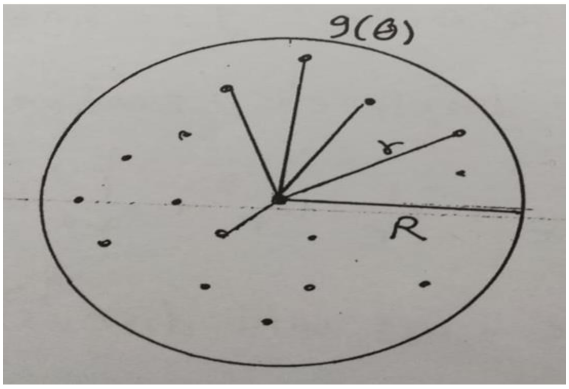
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{\theta}u_{\theta\theta} = 0 \qquad \dots ..(22)$$

Here, we will discuss three cases:



1- Dirichlet problem inside a circle
2- Dirichlet problem outside a circle
3- Dirichlet problem on a circle annulus

Case(1): Dirichlet problem inside a circle



Here we can find the steady temperature and steady potential inside a circle if the steady temperature and stead potential on the circumference is known.

From equations (21), (22) and the separation of variables, we get

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \ (a_n \cos n\theta + b_n \sin n\theta) \qquad \ldots..(23)$$

Which is the steady temperature inside a circle where the steady temperature on the circumference is given by the boundary condition

$u(R,\theta) = g(\theta); 0 \leq \theta \leq 2\pi,\ 0 < r < R$, and R is the radius of the circle

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta \ d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta \ d\theta$$

Note If R=1 then the equation (23) will be:

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n \ (a_n \cos n\theta + b_n \sin n\theta) \qquad \ldots.(24)$$

Where $u(1,\theta) = g(\theta); 0 \leq \theta \leq 2\pi,\ 0 < r < 1$

Ex.5: A steady potential on the circumference of a circle with a radius equal 1 is $(\mathbf{2 + 4\sin 2\theta - 3\cos 4\theta})$, find the steady potential inside the circle.

Sol: $g(\theta) = 2 + 4\sin 2\theta - 3\cos 4\theta\,,\;\; R = 1$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}(2 + 4\sin 2\theta - 3\cos 4\theta)\,d\theta$$

$$= \frac{1}{2\pi}\left(2\theta - 2\cos 2\theta - \frac{3}{4}\sin 4\theta\right)_0^{2\pi} = 2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}(2 + 4\sin 2\theta - 3\cos 4\theta)\cos n\theta\;\,d\theta$$

$$= \frac{1}{\pi}\left(\underbrace{\int_0^{2\pi} 2\cos n\theta d\theta}_{=0} + \underbrace{\int_0^{2\pi} 4\sin 2\theta \cos n\theta d\theta}_{=0} - \int_0^{2\pi} 3\cos 4\theta \cos n\theta d\theta\right)$$

$$= 0; n \neq 4$$

$$a_4 = \frac{1}{\pi}\left(\underbrace{\int_0^{2\pi} 2\cos 4\theta d\theta}_{=0} + \underbrace{\int_0^{2\pi} 4\sin 2\theta \cos 4\theta d\theta}_{=0} - 3\int_0^{2\pi} \cos^2 4\theta\, d\,\theta\right)$$

$$= \frac{-3}{2\pi}\int_0^{2\pi}(1 + \cos 8\theta)\;\,d\theta = \frac{-3}{2\pi}\left(\theta + \frac{1}{8}\sin 8\theta\right)_0^{2\pi} = -3$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}(2 + 4\sin 2\theta - 3\cos 4\theta)\sin n\theta\;\,d\theta$$

$$\frac{1}{\pi}\left(\underbrace{\int_0^{2\pi} 2\sin n\theta d\theta}_{=0} + 4\int_0^{2\pi} \sin 2\theta \sin n\theta d\theta - 3\underbrace{\int_0^{2\pi} \cos 4\theta \ \sin n\theta d\theta}_{=0}\right)$$

$$b_n = 0; n \neq 2$$

$$b_2 = \frac{1}{\pi}\int_0^{2\pi} 4\sin^2 2\theta \ d\theta = \frac{4}{\pi}\int_0^{2\pi} \frac{1}{2}(1-\cos 4\theta) \ d\theta = \frac{2}{\pi}\left(\theta - \frac{1}{4}\sin 4\theta\right)_0^{2\pi} = 4$$

$$\therefore u(r,\theta) = \sum_{n=0}^{\infty} r^n \ (a_n \cos n\theta + b_n \sin n\theta)$$

$$= 2 - 3r^4 \cos 4\theta + 4r^2 \sin 2\theta$$

Which is the required solution.

Case(2): Dirichlet problem outside a circle

It is similar to the first case but the steady heat and steady potential will be calculated outside the circle, the solution is

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^{-n} \ (a_n \cos n\theta + b_n \sin n\theta) \ldots\ldots (25)$$

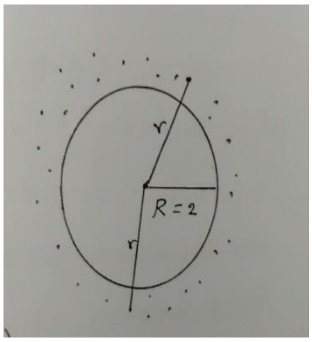
Where R = the radius of the circle, $R < r < \infty, \ \ 0 \leq \theta \leq 2\pi$

a_0, a_n, b_n are Fourier coefficients that in case (1) and $g(\theta)$ is the steady temperature or the steady potential on the circumference

Ex.6: Solve the following Dirichlet proble $\nabla^2 u = 0$, $u(2,\theta) = 2 - \sin\theta + 4\cos 3\theta, 2 < r < \infty, \ \ 0 \leq \theta \leq 2\pi$

Sol:

This problem is outside the circle, then



$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^{-n} \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi}(2 - \sin\theta + 4\cos 3\theta)d\theta$$

$$= \frac{1}{2\pi}\left(2\theta + \cos\theta + \frac{3}{4}\sin 3\theta\right)_0^{2\pi} = \frac{1}{2\pi}(4\pi) = 2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta \; d\theta = \frac{1}{\pi}\int_0^{2\pi} (2 - \sin\theta + 4\cos 3\theta)\cos n\theta \; d\theta$$

$$a_n = 0; n \neq 3$$

$$a_3 = \frac{1}{\pi}\int_0^{2\pi} 4(\cos 3\theta\,)^2 d\theta = \frac{4}{\pi}\int_0^{2\pi} \frac{1}{2}(1 + \cos 6\theta) \; d\theta = \frac{2}{\pi}\left(\theta + \frac{1}{6}\sin 6\theta\right)_0^{2\pi} = 4$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta \; d\theta = \frac{1}{\pi}\int_0^{2\pi} (2 - \sin\theta + 4\cos 3\theta)\sin n\theta \; d\theta$$

$$b_n = 0; n \neq 1$$

$$b_1 = \frac{-1}{\pi}\int_0^{2\pi} \sin^2\theta \; d\theta = \frac{-1}{\pi}\int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta) \; d\theta = \frac{-1}{2\pi}\left(\theta - \frac{1}{2}\sin 2\theta\right)_0^{2\pi}$$

$$= \frac{-1}{2\pi}(2\pi) = -1$$

Then

$$u(r,\theta) = 2 + \left(\frac{r}{2}\right)^{-3} . 4\cos 3\theta - \left(\frac{r}{2}\right)^{-1} \sin\theta$$

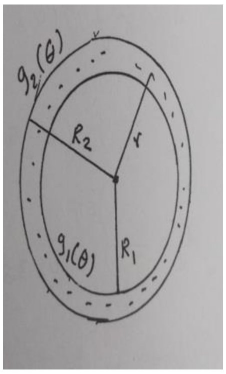
Which is the required solution.

Case(3): Dirichlet problem on a circular annulus

Consider a circular annulus of inner radius R_1 and outer radius R_2.

Let the surface of the annulus be insulated and the temperature distribution along the inner circle $r = R_1$ and the outer circle $r = R_2$ are maintained as $u(R_1, \theta) = g_1(\theta)$ and $u(R_2, \theta) = g_2(\theta)$, so the solution is:

$u(r,\theta) = a_0 + b_0 \ln r + \sum_{n=0}^{\infty}[(a_n r^n + b_n r^{-n}) \cos n\theta + (c_n r^n + d_n r^{-n}) \sin n\theta]$ ... (26)



Where

$a_0 + b_0 \ln R_1 = \frac{1}{2\pi}\int_0^{2\pi} g_1(\theta) d\theta$ ... (27)

$a_n {R_1}^n + b_n {R_1}^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_1(\theta) \cos n\theta \; d\theta$ ... (28)

$c_n {R_1}^n + d_n {R_1}^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_1(\theta) \sin n\theta \; d\theta$ ... (29)

Which it is the coefficients of the inner circle, the coefficients of the outer circle is:

$a_0 + b_0 \ln R_2 = \frac{1}{2\pi}\int_0^{2\pi} g_2(\theta) d\theta$ ... (30)

$a_n {R_2}^n + b_n {R_2}^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_2(\theta) \cos n\theta \; d\theta$ ... (31)

$c_n {R_2}^n + d_n {R_2}^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_2(\theta) \sin n\theta \; d\theta$ ... (32)

Solving (27), (30) to get a_0 and b_0, and (28), (31) to get a_n and b_n, and (29), (32) to get c_n and d_n, then substituting in (25) to get the solution where $R_1 < \mathrm{r} < R_2$

Ex.7: Consider a circular annulus of inner radius 1 and outer radius 3. Let the surface of the annulus be insulated. Find the steady state potential at any point $(\mathbf{r}, \boldsymbol{\theta})$ in the annuls given that the potential distribution along the inner circle and the outer circle are maintained as $\mathbf{u(1, \theta) = 0}$ and $\mathbf{u(3, \theta) = \sin\theta}$

Sol: $g_1(\theta) = 0, g_2(\theta) = \sin\theta,\ R_1 = 1, R_2 = 3$

$$a_0 + b_0 \ln R_1 = \frac{1}{2\pi}\int_0^{2\pi} g_1(\theta) d\theta$$

$$a_0 + b_0 \ln 1 = \frac{1}{2\pi}\int_0^{2\pi} 0\, d\theta \Longrightarrow a_0 = 0 \qquad \ldots\ldots(1*)$$

$$a_0 + b_0 \ln R_2 = \frac{1}{2\pi}\int_0^{2\pi} g_2(\theta) d\theta$$

$$a_0 + b_0 \ln 3 = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta\, d\theta$$

$$a_0 + b_0 \ln 3 = 0 \qquad \ldots\ldots(2*)$$

Then from $(1*)$ and $(2*)$ we get

$$\boxed{a_0 = b_0 = 0} \qquad \ldots\ldots(3*)$$

$$a_n {R_1}^n + b_n {R_1}^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_1(\theta)\cos n\theta\ d\theta$$

$$a_n + b_n = \frac{1}{\pi}\int_0^{2\pi} 0 \cos n\theta\ d\theta$$

$$\therefore a_n + b_n = 0 \qquad \ldots\ldots(4*)$$

$$a_n {R_2}^n + b_n {R_2}^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_2(\theta)\cos n\theta \; d\theta$$

$$a_n 3^n + b_n 3^{-n} = \frac{1}{\pi}\int_0^{2\pi} \sin\theta \cos n\theta \; d\theta = 0$$

$$\therefore\ a_n 3^n + b_n 3^{-n} = 0 \qquad \ldots..(5\,*)$$

From $(4\,*)$ and $(5\,*)$, we get

$$a_n = b_n = 0 \qquad \ldots.(6\,*)$$

$$c_n {R_1}^n + d_n {R_1}^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_1(\theta)\sin n\theta \; d\theta$$

$$c_n + d_n = \frac{1}{\pi}\int_0^{2\pi} 0 \sin n\theta \; d\theta = 0$$

$$\therefore\ c_n + d_n = 0 \qquad \ldots.(7\,*)$$

$$c_n {R_2}^n + d_n {R_2}^{-n} = \frac{1}{\pi}\int_0^{2\pi} g_2(\theta)\sin n\theta \; d\theta$$

$$c_n 3^n + d_n 3^{-n} = \frac{1}{\pi}\int_0^{2\pi} \sin\theta \; \sin n\theta \; d\theta = 0\,; n \neq 1$$

If $n = 1$

$$3c_1 + 3^{-1}d_1 = \frac{1}{\pi}\int_0^{2\pi} (\sin^2\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} (1-\cos 2\theta)\; d\theta$$

$$= \frac{1}{2\pi}\left(\theta - \frac{1}{2}\sin 2\theta\right)_0^{2\pi} = 1$$

$$\therefore\ 3c_n + 3^{-1}d_n = \begin{cases} 0, & n \neq 1 \\ 1, & n = 1 \end{cases} \qquad \ldots.(8\,*)$$

Solving $(7\,*)$ and $(8\,*)$ as follows:

$$\left.\begin{aligned} c_n + d_n &= 0 \\ 3c_n + 3^{-n}d_n &= 0 \end{aligned}\right\} \Longrightarrow c_n = d_n = 0\,;n \neq 1$$

If $n = 1$

$$\left.\begin{aligned} c_1 + d_1 &= 0 \\ 3c_1 + 3^{-1}d_1 &= 0 \end{aligned}\right\} \Longrightarrow c_1 = \frac{3}{8}, d_1 = \frac{-3}{8}$$

$$c_n = \begin{cases} 0 \ if\ n \neq 1 \\ \frac{3}{8}\ if\ n = 1 \end{cases} \text{ and } \quad d_n = \begin{cases} 0 \ if\ n \neq 1 \\ \frac{-3}{8}\ if\ n = 1 \end{cases} \qquad \ldots.(9\,*)$$

Substituting $(3\,*)$, $(6\,*)$ and $(9\,*)$ in the solution

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_{n=0}^{\infty}[(a_n r^n + b_n r^{-n})\cos n\theta + (c_n r^n + d_n r^{-n})\sin n\theta]$$

We get $u(r,\theta) = \frac{3}{8}\left(r - \frac{1}{r}\right)\sin\theta$

Which is the required solution

… Exercises …

1- Find the solution of Laplace's equation in the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 5$ also satisfying the boundary conditions:

$u(0, y) = 0, u(x, 0) = 0,\ u(x, 5) = 0, u(3, y) = y.$

2- Find the steady temperature distribution $u(x, y)$ in the uniform unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ when the edge $y = 1$ is kept a temperature u_o and the remaining three sides are kept at temperature zero.

3- Find the steady temperature distribution in a thin rectangular plate bounded by the lines $x = 0, x = l, y = 0, y = h$, assuming that the heat cannot escape from either surface; the edge $x = 0, x = l, y = 0$ are kept at a temperature zero while the edge $y = h$ is kept at a temperature f(x).

4- Solve the differential equation $u_{xx} + u_{yy} = 0$ which satisfies the conditions $u(0, y) = u(1, y) = u(x, 0) = 0$,and $u(x, 1) = \sin n\pi x$

5- Find the steady temperature in a circular plate of radius a which it's steady temperature on the circumference is θ

6- Solve the following Dirichlet problem $\nabla^2 u = 0$, where

$u(2, \theta) = 6\cos\theta + 10\sin\theta, u(4, \theta) = 15\cos\theta + 17\sin\theta, 2 < r < 4,$
$0 \leq \theta \leq 2\pi$

7- A plate in the form of a ring is bounded by the circle $r = 1$ and $r = 3$. Its surfaces are insulated and the temperature $u(r, \theta)$ along the boundary are $u(1, \theta) = \cos\theta$ and $u(3, \theta) = \sin\theta$. Find the steady temperature $u(r, \theta)$ in the ring.

8- Solve the following Dirichlet problem $\nabla^2 u = 0$, where

$u(7, \theta) = \theta, 7 < r < \infty,\ 0 \leq \theta \leq 2\pi$

Section 6.3: The Laplace Transform

1- **Introduction:** The knowledge of "integral transform" is an essential part of mathematical background required by scientists and engineers. This is because the transform methods provide an easy and effective means for the solution of many problems arising in science and engineering. For example, the Laplace Transformation replaces a given function f(t) by another function F(s). Then Laplace Transformation convert an ordinary differential equation with some given initial conditions into an algebraic equation in terms of F(s) and partial differential equation with two independent variables into ordinary differential equation. Finally, using inverse Laplace Transformation we recover the original function f(t). Thus the method of Laplace Transformation is especially useful for initial value problems, as it enables us to solve the problem without the trouble of finding the general solution first and then evaluating the arbitrary constants. The use of Laplace transforms provide a powerful technique of solving differential and integral equations.

2- **Laplace Transform definition :** Given a function for all $t \geq 0$, the Laplace transform of f(t) is a function of a new variable s given by

$$L[\mathrm{f(t)}] = \mathrm{F(s)} = \int_0^\infty \mathrm{f(t)}\, e^{-st} dt \qquad \ldots\ldots(33)$$

The Laplace transform of f(t) is said to exist if the improper integral (33) converges for some value of s, otherwise it does not exist.

The inverse Laplace transform is

$$L^{-1}[\mathrm{F}] = \mathrm{f(t)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{F(s)}\, e^{st} ds \qquad \ldots\ldots(34)$$

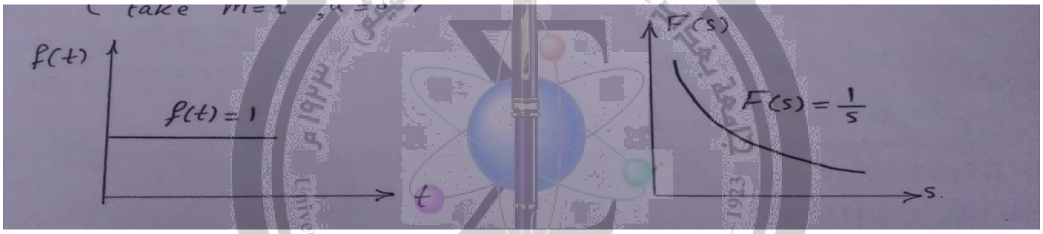
3- Sufficient conditions for existence of Laplace transform:

Theorem: If f(t) is a function which is piecewise continous on every finite interval in the range $t \geq 0$ and satisfies all $|f(t)| \leq me^{at}$ for all $t \geq 0$ and for constants a and m, then the Laplace transform of $f(t)$exists.

4- Laplace transforms of some elementary functions:

Here we will show the figures of some functions that have Laplace transforms:

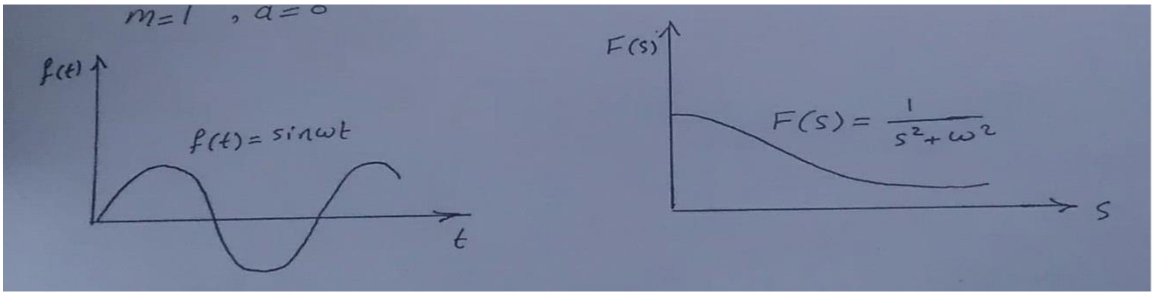
1- $\mathrm{f(t)} = 1,\ 0 < \mathrm{t} < \infty \Rightarrow \mathrm{F(s)} = \int_0^\infty e^{-st}dt = \frac{1}{s}$, (take m=1, a=0)



2- $\mathrm{f(t)} = e^{2t},\ 0 < \mathrm{t} < \infty \Rightarrow \mathrm{F(s)} = \frac{1}{s-2}, s > 2$, (take m=1, a=2)

3- $\mathrm{f(t)} = \sin wt \quad \Rightarrow \mathrm{F(s)} = \frac{w}{s^2+w^2}, s > 2,$

m=1, a=0



5- **Transformation of partial derivatives:**

Let $u(x,t)$ be a function of two independent variables x and t. The Laplace transformation of the partial derivatives u_t, u_{tt}, u_x, u_{xx} is :

$$L[u_t] = \int_0^\infty u_t(x,t)\, e^{-st} dt = sU(x,s) - u(x,0) \qquad \ldots.(35)$$

$$L[u_{tt}\,] = \int_0^\infty u_{tt}(x,t)\, e^{-st} dt = s^2 U(x,s) - su(x,0) - u_t(x,0) \ldots.(36)$$

$$L[u_x] = \int_0^\infty u_x(x,t)\, e^{-st} dt = \frac{\partial U}{\partial x}(x,s) \qquad \ldots.(37)$$

$$L[u_{xx}] = \int_0^\infty u_{xx}(x,t)\, e^{-st} dt = \frac{\partial^2 U}{\partial x^2}(x,s) \qquad \ldots.(38)$$

Where $U(x,s) = L[u(x,t)] = \int_0^\infty u(x,t)\, e^{-st} dt$

6- **Convolution property:**

If f and g are two functions of t, the finite convolution is given by

$$\left.\begin{aligned}(f * g)(t) &= \int_0^t f(\tau).\, g(t-\tau) d\tau \\ &= \int_0^t f(t-\tau).\, g(\tau) d\tau\end{aligned}\right\} \qquad \ldots.(39)$$

For example: let $f(t) = t, g(t) = t^2$

$$(f * g)(t) = \int_0^t f(t-\tau).\, g(\tau) d\tau = \int_0^t (t-\tau).\, \tau^2 d\tau = \int_0^t (t\tau^2 - \tau^3) d\tau$$

$\left(\frac{t\tau^3}{3} - \frac{\tau^4}{4}\right)_0^t = \frac{t^4}{3} - \frac{t^4}{4} = \frac{1}{12}t^4$

As in the infinite convolution the following important property is true

$L[f * g] = L[f].L[g]$ ....(40)

Then $L^{-1}\{L[f].L[g]\} = f * g$ ....(41)

(41) allows us to find the inverse The Laplace transform of multiplying two functions $L[f].L[g]$ by finding the inverse Laplace transform for each of $L[f]$ and $L[g]$ to get f and g.

For example: Find $L^{-1}\left[\frac{1}{s}\ \frac{1}{s^2+1}\right]$

Sol: $L^{-1}\left[\frac{1}{s}\ .\frac{1}{s^2+1}\right] = \int_0^t 1.\sin\tau\, d\tau = \cos\tau_0^t = 1 - \cos t$

Where $\mathrm{F(s)} = \frac{1}{s} \xrightarrow{L^{-1}} f(t) = 1$, $\mathrm{G(s)} = \frac{1}{s^2+1} \xrightarrow{L^{-1}} g(t) = \sin t$

<u>Ex.8</u>: Solve the following initial value problem by Laplace transform $u_t = u_{xx}, 0 < x < \infty, 0 < t < \infty$ where $\mathrm{u(0,t)} = 0, \mathrm{u(x,0)} = u_\circ$

<u>Sol:</u> $u_t = u_{xx}$

$L[u_t] = L[u_{xx}]$

$sU(x,s) - u(x,0) = \frac{\partial^2 U}{\partial x^2}(x,s)$

$(D^2 - s)U(x,s) = -u(x,0)$

$(D^2 - s)U(x,s) = -u_\circ$ (from the initial condition)

The A.E. is $m^2 - s = 0 \longrightarrow m = \pm\sqrt{s}$

The general solution U_1 is

$U_1(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$, where c_1 and c_2 are arbitrary constants

Since x goes to ∞ then $c_1 = 0$

$\therefore U_1(x,s) = c_2 e^{-\sqrt{s}x}$

The particular solution is

$$U_2 = \frac{-u_\circ}{D^2 - s} = \frac{u_\circ}{s}$$

$$\therefore U(x,s) = c_2 e^{-\sqrt{s}x} + \frac{u_\circ}{s} \qquad \ldots\ldots(*)$$

Substituting the boundary condition in $U(x,s)$ to find c_2

$$L[u(0,t)] = U(0,s)$$

$$U(0,s) = c_2 e^0 + \frac{u_\circ}{s}$$

$$0 = c_2 + \frac{1}{s}u_\circ \longrightarrow c_2 = -\frac{1}{s}u_\circ \qquad \ldots.(**)$$

Substituting $(**)$ in $(*)$, we get

$$U(x,s) = -\frac{1}{s}u_\circ e^{-\sqrt{s}x} + \frac{u_\circ}{s}$$

Taking inverse Laplace transformation, we get

$$L^{-1}[U(x,s)] = -u_\circ\, L^{-1}\left[\frac{1}{s}e^{-\sqrt{s}x}\right] + u_\circ L^{-1}\left[\frac{1}{s}\right]$$

From table of Laplace transforms, we get

$u(x,s) = -u_\circ erfc\left(\frac{x}{2\sqrt{t}}\right) + u_\circ$, and this is the general solution

Ex.9: Solve the following initial value problem using Laplace transform $u_t = u_{xx}, \ -\infty < x < \infty, 0 < t < \infty$ where $\mathrm{u}(\mathrm{x}, 0) = \sin x$

Sol: $u_t = u_{xx}$

$L[u_t] = L[u_{xx}]$

$$sU(x,s) - u(x,0) = \frac{\partial^2 U}{\partial x^2}$$

$$sU(x,s) - \sin x = \frac{\partial^2 U}{\partial x^2}$$

$(D^2 - s)U(x,s) = -\sin x$

1- We must find the general solution U_1

The A.E. is $m^2 - s = 0 \longrightarrow m = \pm\sqrt{s}$

$U_1(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$

When $x \longrightarrow \infty \Longrightarrow c_1 = 0$

When $x \longrightarrow -\infty \Longrightarrow c_2 = 0$

$\therefore U_1(x,s) = 0$

2- The particular solution U_2 is

$U_2 = \frac{-1}{D^2 - s} \sin x = \frac{-1}{-1-s} \sin x = \frac{1}{s+1} \sin x \quad , \quad D^2 = -a^2 = -1$

The general solution for the given equation is

$U(x,s) = U_1 + U_2$

$$\Longrightarrow U(x,s) = \frac{1}{s+1} \sin x$$

Taking inverse Laplace transformation, we get

$$L^{-1}[U(x,s)] = L^{-1}\left[\frac{1}{s+1}\sin x\right]$$

$\Longrightarrow u(x,s) = \sin x. e^{-t}$, (from table of Laplace transforms)

<u>Ex.10</u>: Prove that $L[u_t] = sU(x,s) - u(x,0)$

<u>Sol:</u> from the definition of Laplace transform, we get

$L[u_t(x,t)] = \int_0^\infty u_t(x,t)\, e^{-st} dt$ (integration by parts)

$$= e^{-st}u(x,t)|_0^\infty + s\int_0^\infty u(x,t)\, e^{-st} dt = \left.\frac{u(x,t)}{e^{st}}\right|_0^\infty + sU(x,s)$$

$$= \underbrace{\lim_{t\to\infty}\frac{u(x,t)}{e^{st}}}_{\to 0} - \frac{u(x,0)}{\underbrace{e^0}_{=1}} + sU(x,s)$$

$$= -u(x,0) + sU(x,s) = sU(x,s) - u(x,0)$$

<u>Ex.11</u>: Solve the following equation using Laplace transform $u_t = u_{xx}$, $0 \le x < \infty, 0 \le t < \infty$where $u(0,t) = 2, \mathrm{u(x,0)} = 0$

<u>Sol:</u> $u_t = u_{xx}$

$L[u_t] = L[u_{xx}]$

$$sU(x,s) - \underbrace{u(x,0)}_{=0} = D^2U(x,s)$$

$(D^2 - s)U(x,s) = 0$

The A.E. is $m^2 - s = 0 \longrightarrow m = \pm\sqrt{s}$

$\therefore U(x,s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$

Since $x \longrightarrow \infty \Longrightarrow c_1 = 0$

$\therefore U(x,s) = c_2 e^{-\sqrt{s}x}$ …. . $(*)$

The boundary condition $u(0,t) = 2$ will be

$$U(0,s) = L[u(0,t)] = L[2] = \frac{2}{s}$$

Substituting in $(*)$, we get

$$U(0,s) = c_2 e^0 \Longrightarrow \boxed{\frac{2}{s} = c_2}$$

Butting in $(*)$, we get $U(x,s) = \frac{2}{s} e^{-\sqrt{s}x}$

TakingL^{-1} for both sides, we get

$u(x,s) = 2erfc\left(\frac{x}{2\sqrt{t}}\right)$, which is the general solution

… Exercises …

Solve the following equations (using Laplace transform)

1- $u_{tt} = 4\, u_{xx},\ \ \infty < x < \infty, 0 \le t < \infty$where , $\mathrm{u}(\mathrm{x}, 0) = e^{x}$ and $u_t(x, 0) = 0$

2- $u_{tt} = u_{xx},\quad 0 < x < \infty, 0 < t < \infty$where , $\mathrm{u}(\mathrm{x}, 0) = u_t(x, 0) = 0$ and $\mathrm{u}(0, \mathrm{t}) = f(t)$

3- $u_t = u_{xx},\ 0 < x < \infty, 0 < t < \infty$ where,$\mathrm{u}(\mathrm{x}, 0) = 0$ and $\mathrm{u}(0, \mathrm{t}) = t$

Table of Laplace transform

	$f(t) = L^{-1}[F(s)]$	$F(s) = L[f(t)]$
1-	1	$\frac{1}{s},\ \ s > 0$
2-	e^{at}	$\frac{1}{s-a},\ \ s > a$
3-	$\sin at$	$\frac{a}{s^2+a^2}, s > 0$
4-	$\cos at$	$\frac{s}{s^2+a^2}, s > 0$
5-	$\sinh at$	$\frac{a}{s^2-a^2}, s > \lvert a \rvert$
6-	$\cosh at$	$\frac{s}{s^2-a^2}, s > \lvert a \rvert$
7-	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, s > a$
8-	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
9-	t^n, n is natural number	$\frac{n!}{s^{n+1}}, s > 0$
10-	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, s > a$

11-	$H(t-a)$	$\frac{e^{-as}}{s}, s>0$
12-	$H(t-a)\,f(t-a)$	$e^{-as}\,F(s)$
13-	$e^{at}f(t)$	$F(s-a)$
14-	$f(t)*g(t)$	$F(s)\,.\,G(s)$
15-	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
16-	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right), a>0$
17-	$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
18-	$\text{erf}\left(\frac{t}{2a}\right)$	$\frac{1}{s}e^{a^2s^2}\,\text{erfc}(as)$
19-	$\text{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{1}{s}e^{-a\sqrt{s}}$
20-	$J_o(at)$	$(s^2+a^2)^{-\frac{1}{2}}$
21-	$\delta(t-a)$	e^{-as}
22-	$\frac{1}{\sqrt{\pi t}}\exp\left(\frac{-a^2}{4t}\right)$	$\frac{e^{-as}}{\sqrt{s}}, a\geq 0$
23-	$\frac{1}{\sqrt{\pi t}}-a\,\mathrm{e}^{a^2t}\,\text{erfc}(a\sqrt{t})$	$\frac{1}{\sqrt{s}+a}$

Note

$\delta(x) = delta\ function=$ دالة دلتا

$H(x-a)=\begin{cases}0, & x<a\\ 1, & x\geq a\end{cases}$ Heaviside function دالة هافيسايد

$H(a-x)=\begin{cases}1, & x\le a\\ 0, & x>a\end{cases}$ Reflected Heaviside function دالة هافيسايدالمنعكسه

$\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}d\xi$ دالة الخطأ