



mathematics

Department of physics level 2

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Chapter One

Sequence:-

A sequence is a function whose domain is the set of positive integer $1, 2, 3, \dots$. We denote the sequence by $\langle f(n) \rangle$. Where $f(n)$ is the function at valued n .

The sequence f is the set $\{(n, f(n)); n \in N\}$. That is, the set of all pairs $(n, f(n))$, with n a positive integer. We also write a_n to denote the sequence whose ordinate at $y = a_n$.

Examples:-

- $\langle \frac{1}{2n} \rangle = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots$

The set $\{(n, \frac{1}{2n}), n = 1, 2, 3, \dots\}$ is a sequence whose value at n is $\frac{1}{2n}$.

- $\langle \frac{1}{n+1} \rangle = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

The set $\{(n, \frac{1}{n+1}), n = 1, 2, 3, \dots\}$ is a sequence whose value at n is $\frac{1}{n+1}$.

- $\langle 1 \rangle = 1, 1, 1, \dots$

The set $\{(n, 1), n = 1, 2, 3, \dots\}$ is a sequence whose value at n is 1.

Finite sequence :-

The sequence $\langle f(n) \rangle$ is said to be finite if there exist two numbers a and b such that $a \leq f(n) \leq b$ or $a < f(n) \leq b$ or $a \leq f(n) < b$ or $a < f(n) < b$.

Note:- every finite sequence is bounded.

Infinite sequence:-

The sequence $\langle f(n) \rangle$ is said to be infinite if both a and b or one of them equal to $\pm\infty$.

Note:- every infinite sequence is unbounded.

Convergent Sequence:-

A sequence is said to be convergent if it approaches some limit.

Formally, a sequence S_n converges to the limit L . A divergent sequence doesn't have a limit. $\lim_{n \rightarrow \infty} S_n = L$

Discuss the following sequences:-

- $\langle \frac{1}{2n} \rangle = \{ \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots \} . n \in N$

The set $\{ (1, \frac{1}{2}), (2, \frac{1}{4}), (3, \frac{1}{6}), \dots \}$ is a finite and bounded sequence when $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{1}{2n} = 0$$

$\langle \frac{1}{2n} \rangle \rightarrow 0$ the sequence converge to zero.

$$0 < \frac{1}{2n} \leq \frac{1}{2}$$

- $\langle \frac{1}{n+1} \rangle = \{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \} . n \in N$

The set $\{ (1, \frac{1}{2}), (2, \frac{1}{3}), (3, \frac{1}{4}), \dots \}$ is a finite and bounded sequence when $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{n}{n} + \frac{1}{n}} = \frac{0}{1+0} = 0$$

$\langle \frac{1}{n+1} \rangle \rightarrow 0$ the sequence is converge to zero.

$$0 < \frac{1}{n+1} \leq \frac{1}{2}$$

- $\langle \mathbf{1} \rangle = \{1, 1, 1, \dots \dots \dots\}$.

The set $\{ (1,1), (2,1), (3,1), \dots \}$ is a finite and bounded sequence when $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} 1 = 1$$

$\langle 1 \rangle \rightarrow 1$ the sequence is converge to 1.

- $\langle \frac{2n+1}{2n} \rangle = \left\{ \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \dots \dots \dots \right\}. n \in N$

The set $\{ (1, \frac{3}{2}), (2, \frac{5}{4}), (3, \frac{7}{6}), \dots \}$ is a finite and bounded sequence since when $n \rightarrow \infty$, there are two Solutions for this sequence:-

First solution:-

$$\lim_{n \rightarrow \infty} \frac{2n+1}{2n} = \lim_{n \rightarrow \infty} \frac{\frac{2n}{n} + \frac{1}{n}}{\frac{2n}{n}} = \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n}}{2} = \frac{2+0}{2} = 1$$

$$\therefore \langle \frac{2n+1}{2n} \rangle \rightarrow 1$$

Second solution:-

$$\lim_{n \rightarrow \infty} \frac{2n+1}{2n} = \lim_{n \rightarrow \infty} \frac{2n}{2n} + \frac{1}{2n} = \lim_{n \rightarrow \infty} 1 + \frac{1}{2n} = 1 + 0 = 1$$

$$\therefore \left\langle \frac{2n+1}{2n} \right\rangle \rightarrow 1$$

The sequence is converge to 1.

$$1 < \frac{2n+1}{2n} \leq \frac{3}{2}$$

- $\langle n \rangle = \{1, 2, 3, \dots\}. n \in N$

The set $\{ (1,1), (2,2), (3,3), \dots \}$ is a infinite and unbounded sequence when $n \rightarrow \infty, \langle n \rangle \rightarrow \infty$. The sequence is diverged.

Note:- Those sequences are also infinite, unbounded and diverge

$\langle \sqrt{n} \rangle, \langle 5n^2 \rangle, \langle \frac{n^3}{6} \rangle, \langle \frac{n^6}{n^3} \rangle$ And so on. Have a same solve like

$\langle n \rangle$.

Example:- Find the Domains for the following sequences

- $\left\langle \frac{1}{n-1} \right\rangle$

$$n - 1 = 0$$

$$n = 1$$

$$n \in N/\{1\}$$

Or $n = \{2, 3, 4, \dots\}$.

- $\langle \sqrt{n-4} \rangle$

$$n - 4 \geq 0$$

$$n \geq 4$$

$$n \in N/\{1, 2, 3\}$$

Or $n = \{4, 5, 6, \dots\}$.

- $\left\langle \frac{1}{\sqrt{n+4}} \right\rangle$

$n \in \mathbb{N}$ since $n + 4 > 0 \rightarrow n > -4$

- $\left\langle \frac{1}{\sqrt{n-3}} \right\rangle$

$n - 3 > 0$

$n > 3$

$n \in \mathbb{N} / \{1, 2, 3\}$

Or $n = \{4, 5, 6, \dots\}$.

- $\left\langle \frac{1}{\sqrt{3-n}} \right\rangle$

$3 - n > 0$

$3 > n$

$n = \{1, 2\}$.

$\therefore \left\langle \frac{1}{\sqrt{3-n}} \right\rangle$ Is finite sequence having two limits.

- $\left\langle \sqrt{7-n} \right\rangle$ How ?

Are the sequences converging or diverge?

1. $\left\langle \frac{n-2}{n^2+1} \right\rangle$

Solution:-

$$\lim_{n \rightarrow \infty} \frac{n-2}{n^2+1} = \lim_{n \rightarrow \infty} \frac{\frac{n}{n^2} - \frac{2}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} - \frac{2}{n^2}}{1 + \frac{1}{n^2}} = \frac{0-0}{1+0} = 0$$

$$\therefore \left\langle \frac{n-2}{n^2+1} \right\rangle \rightarrow 0$$

2. $\left\langle \frac{n^3+1}{n-2} \right\rangle$

Sol.:- $\lim_{n \rightarrow \infty} \frac{n^3+1}{n-2} = \lim_{n \rightarrow \infty} \frac{\frac{n^3}{n} + \frac{1}{n}}{\frac{n}{n} - \frac{2}{n}} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n^3}}{1 - \frac{2}{n}} = \frac{1+0}{0-0} = \frac{1}{0} = \infty$

$$\therefore \left\langle \frac{n^3+1}{n-2} \right\rangle \text{ The sequence is diverge.}$$

3. $\langle (-1)^n \rangle, n \in N$

Sol.:- $\langle (-1)^n \rangle = \{-1, 1, -1, 1, \dots\}$

$$\langle (-1)^n \rangle = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

$$\langle (-1)^n \rangle = \begin{cases} (1, -1), (3, -1), (5, -1), \dots & n \text{ odd} \\ (2, 1), (4, 1), (6, 1), \dots & n \text{ even} \end{cases}$$

$$-1 \leq (-1)^n \leq 1$$

$$\therefore \langle (-1)^n \rangle \text{ Is bounded.}$$

$$\text{But } \lim_{n \rightarrow \infty} (-1)^n = \begin{cases} -1 & n \rightarrow \infty \text{ odd} \\ 1 & n \rightarrow \infty \text{ even} \end{cases}$$

$$\therefore \langle (-1)^n \rangle \text{ Is diverge because it has two limits.}$$

Discuss the following sequences? (Homework)

4. $\langle (-1)^{n+1} \rangle$

5. $\langle (-1)^{n-1} \rangle$

6. $\langle 1 - (-1)^n \rangle$

$$7. \left\langle \frac{1}{(-1)^n} \right\rangle$$

$$8. \langle n - (-1)^n \rangle$$

$$9. \left\langle \frac{n}{(-1)^n} \right\rangle$$

$$10. \left\langle \frac{(-1)^n}{n} \right\rangle$$

$$11. \langle \langle 5n + 4 \rangle \rangle$$

$$12. \left\langle \frac{4}{3n} \right\rangle$$

$$13. \langle \sqrt[5]{n} \rangle$$

$$14. \left\langle \frac{n^2}{n^3+3} \right\rangle$$

Find the values of a_1 , a_2 , a_3 and a_4 for the following sequence

$\langle a_n \rangle$, were given a_n term ?

$$1. a_n = \frac{1-n}{n^2}$$

$$a_1 = \frac{1-1}{1^2} = \frac{0}{1} = 0$$

$$a_2 = \frac{1-2}{2^2} = \frac{-1}{4}$$

$$a_3 = \frac{1-3}{3^2} = \frac{-2}{9}$$

$$a_4 = \frac{1-4}{4^2} = \frac{-3}{16}$$

$$2. a_n = \frac{(-1)^{n+1}}{2^{n-1}}$$

$$a_1 = \frac{(-1)^{1+1}}{2^{1-1}} = \frac{(-1)^2}{2^0} = \frac{1}{1} = 1$$

$$a_2 = \frac{(-1)^{2+1}}{2^{2-1}} = \frac{(-1)^3}{2} = \frac{-1}{2}$$

$$a_2 = \frac{(-1)^{3+1}}{2^{3-1}} = \frac{(-1)^4}{2^2} = \frac{1}{4}$$

$$a_4 = \frac{(-1)^{4+1}}{2^{4-1}} = \frac{(-1)^5}{2^3} = \frac{-1}{8}$$

$$3. a_n = \frac{(-1)^{n+1}}{2n-1} \quad \text{Homework?}$$

$$4. a_n = \left(\frac{n+1}{2n}\right)\left(1 - \frac{1}{n}\right) \quad \text{Homework?}$$

Are the following sequence converge or diverge find the limited of each convergent sequence?

$$1. a_n = 2 + (0.1)^n$$

$$n = 1, 2, 3, \dots$$

$$a_1 = 2.1, a_2 = 2.01, a_3 = 2.001, \dots$$

$$n \rightarrow \infty, \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (2 + (0.1)^n)$$

$$= \lim_{n \rightarrow \infty} \left(2 + \frac{1}{10^n}\right)$$

$$= 2 + \lim_{n \rightarrow \infty} \frac{1}{10^n} = 2 + 0 = 2$$

$$\therefore a_n \rightarrow 2$$

$$2 \leq a_n \leq 2.1$$

$\langle a_n \rangle$ is bounded.

$$2. a_n = 1 + (0.1)^n? \text{ Homework}$$

Not :- All of these sequence are diverge:- $\langle a_n \rangle = \frac{n}{10}$, $\langle a_n \rangle =$

$(n-1)^2$ and $\langle a_n \rangle = (n + (-1)^n)$.

when $n \rightarrow \infty$ the limit sequences $\lim_{n \rightarrow \infty} a_n = \infty$

$$3. \langle a_n \rangle = \frac{\cos n}{n}$$

$$-1 \leq \cos n \leq 1$$

$$\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$$

But $\langle \frac{-1}{n} \rangle \rightarrow 0$ and $\langle \frac{1}{n} \rangle \rightarrow 0$

$$\therefore \langle \frac{\cos n}{n} \rangle \rightarrow 0$$

By theorem if $a_n \leq b_n \leq c_n \forall n$ and $\langle a_n \rangle \rightarrow L$ and $\langle c_n \rangle \rightarrow L$ then $\langle b_n \rangle \rightarrow L$.

But this sequence is diverge show that $\langle a_n \rangle = n + \cos n$?
(Homework).

Mathematical Convergent for Sequence:-

A sequence S_n is said to be converge Mathematically for a real number L if and only if for every positive real number ($\epsilon > 0$), there exist corresponding number ($K \in N$), such that :

$$\langle S_n \rangle \rightarrow L \Leftrightarrow |S_n - L| < \epsilon \quad \forall n > K.$$

Example:- prove that

$$1. \langle \frac{1}{n} \rangle \rightarrow 0$$

$$\text{Proof:- } \left| \frac{1}{n} - 0 \right| < \epsilon \quad \forall n > K$$

$$\left| \frac{1}{n} \right| < \epsilon \quad \forall n > K$$

$$\frac{1}{n} < \epsilon \quad \forall n > K$$

$$n > \frac{1}{\epsilon} \quad \forall n > K$$

$$\therefore K = \frac{1}{\epsilon}.$$

$$2. \left\langle \frac{1}{2^n} \right\rangle \rightarrow 0$$

$$\text{Proof:- } \left| \frac{1}{2^n} - 0 \right| < \epsilon \quad \forall n > K$$

$$\left| \frac{1}{2^n} \right| < \epsilon \quad \forall n > K$$

$$\frac{1}{2^n} < \epsilon \quad \forall n > K$$

$$2^n > \frac{1}{\epsilon} \quad \forall n > K$$

$$\ln 2^n > \ln \frac{1}{\epsilon}$$

$$n \ln 2 > \ln 1 - \ln \epsilon$$

$$n > \frac{0 - \ln \epsilon}{\ln 2}$$

$$\therefore K = \frac{-\ln \epsilon}{\ln 2}.$$

$$3. \left\langle \frac{n+1}{n} \right\rangle \rightarrow 1$$

$$\text{Proof:- } \left| \frac{n+1}{n} - 1 \right| < \epsilon \quad \forall n > K$$

$$\left| \frac{n+1-n}{n} \right| < \epsilon \quad \forall n > K$$

$$\left| \frac{1}{n} \right| < \epsilon \quad \forall n > K$$

$$\frac{1}{n} < \epsilon \quad \forall n > K$$

$$n > \frac{1}{\epsilon} \quad \forall n > K$$

$$\therefore K = \frac{1}{\epsilon}$$

4. $\langle \frac{n}{n+1} \rangle \rightarrow 1$ prove that ?(Homework)

5. $\langle \frac{2}{n} \rangle \rightarrow 0$ prove that ?(Homework)

Converge of sequence by theorem of continuous

If $s_n \rightarrow L$ and f is a continuous function on L ; f is defined on all terms of sequence , then $f(s_n) \rightarrow f(L)$.

Example:- prove that

$$1. \langle \sqrt{\frac{n+1}{n}} \rangle \rightarrow 1$$

Proof:- Let $x = \frac{n+1}{n}$

$$\lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{n \rightarrow \infty} \frac{n}{n} + \frac{1}{n} = \lim_{n \rightarrow \infty} 1 + \frac{1}{n} = 1 + 0 = 1$$

$$\therefore \langle \frac{n+1}{n} \rangle \rightarrow 1$$

Let $f(x) = \sqrt{x}$

f is continuous at 1.

$$\text{Since } f(1) = \lim_{n \rightarrow \infty} f(x) = 1$$

$$\lim_{n \rightarrow +1} f(x) = \lim_{n \rightarrow -1} f(x) = 1$$

The limit from left equal the limit from right equal to 1.

Here, by theorem of converge $f\left(\frac{n+1}{n}\right) \rightarrow f(1), \left\langle \frac{n+1}{n} \right\rangle \rightarrow 1$.

$$2. \left\langle 2^{\frac{1}{n}} \right\rangle \rightarrow 1$$

$$\text{Let } x = \frac{1}{n}, \lim_{n \rightarrow \infty} x = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Let $f(x) = 2^x$ continuous at 0

\therefore By theorem of converge (if $s_n \rightarrow L$ and f is a continuous function on L then $f(s_n) \rightarrow f(L)$).

$$f\left(\frac{1}{n}\right) \rightarrow f(0)$$

$$\rightarrow 2^{\frac{1}{n}} \rightarrow 2^0 = 1$$

$$\therefore \left\langle 2^{\frac{1}{n}} \right\rangle \rightarrow 1$$

$$3. \left\langle \sqrt{\frac{9n+1}{n-2}} \right\rangle \rightarrow 3$$

$$\text{Let } x = \frac{9n+1}{n-2}, \lim_{n \rightarrow \infty} x = \quad \text{H.W.}$$

$$\text{Let } f(x) = \sqrt{x}$$

f is continuous at 1. H.W.

$$4. \left\langle \frac{1}{2^n} \right\rangle \rightarrow \mathbf{1}$$

$$\text{Let } x = \frac{1}{n}, \lim_{n \rightarrow \infty} x = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Let $f(x) = \frac{1}{2^x}$ continuous at 0 H.W.

$$5. \left\langle \sqrt{\frac{1}{n}} \right\rangle \not\rightarrow \mathbf{0}$$

Diverge to zero and the function not continuous at zero because the limit from the right doesn't exist.

The series

Definition:-

If $\langle a_n \rangle = \{a_1, a_2, a_3, \dots\}$ is infinite sequence, then $a_1 + a_2 + a_3 + \dots$ is called an infinite series and it is written as $\sum_{n=1}^{\infty} a_n$.

If series of the form $a_1 + a_2 + a_3 + \dots + a_n$. Then is called a finite series and it is written as $\sum_{k=1}^n a_k = S_n$, which S_n is finite series. The sequence $\langle S_n \rangle$ is called the sequence of partial sums of $\sum_{k=1}^n a_k$, where:-

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

⋮

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

Example :-

$$* \sum_{n=1}^{\infty} 2n = 2(1) + 2(2) + 2(3) + \dots$$

$$** \sum_{n=1}^{\infty} \frac{(n+1)^2}{n} = \frac{(1+1)^2}{1} + \frac{(2+1)^2}{2} + \frac{(3+1)^2}{3} + \dots$$
$$= 4 + \frac{9}{2} + \frac{16}{3} + \dots$$

$$*** \sum_{n=1}^{\infty} \left(\frac{1}{n} + n \right) = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} n = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots \right) + (1 + 2 + 3 + \dots)$$

Theorem:-

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series, and $c \in \mathbb{R}$, then:-

1. $\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n$
2. $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

Definition:-

The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be converging to L i.e.

$\sum_{n=1}^{\infty} a_n = L$, if the sequence of partial sums $\langle S_n \rangle$ converge to L .

$$\lim_{n \rightarrow \infty} S_n = L, L \in \mathbb{R}; \sum_{k=1}^{\infty} a_k = S_n.$$

Example:- Is the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge? find it's

sum if exist?

Sol.:-

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)}$$

$$= \frac{A(n+1) + Bn}{n(n+1)}$$

$$= \frac{An + A + Bn}{n(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{A + n(A+B)}{n(n+1)}$$

$$A = 1, A + B = 0 \longrightarrow 1 + B = 0 \longrightarrow B = -1$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{(n+1)}$$

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

⋮

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{(n+1)}\right) = 1 - \frac{1}{(n+1)}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{(n+1)}\right) = 1 - 0 = 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{(n+1)} = 1$$

Are the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge ? find it's sum if exist?

1. $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

2. $\sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)}$ Homework`

3. $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

4. $\sum_{n=0}^{\infty} (\frac{1}{n+3} - \frac{1}{n+4})$

Geometric Series:-

$$\sum_{n=1}^{\infty} ar^n, \sum_{n=0}^{\infty} ar^n, \sum_{n=1}^{\infty} ar^{n+1}, \sum_{n=0}^{\infty} ar^{n+1} \text{ and } \sum_{n=1}^{\infty} ar^{n-1} \dots etc.$$

Those are Geometric Series and a is positive constant , r is the base of series .

The sequence of partial some for the series $\sum_{n=1}^{\infty} ar^{n-1}$ is:-

$$S_n = \frac{a}{\text{first term}} + ar + ar^2 + \dots + ar^{n-1}$$

1. If $|r| = 1$

$$S_n = a + a + a + \dots + a = na$$

$\langle S_n \rangle = \langle na \rangle$ diverge $\implies \sum_{n=1}^{\infty} ar^{n-1}$ diverge

2. If $|r| > 1$

$\sum_{n=1}^{\infty} ar^{n-1}$ diverge

3. If $|r| < 1$

The series $\sum_{n=1}^{\infty} ar^{n-1}$ is converge and the sum is:-

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{\text{first term}}{1 - \text{base}} = \frac{a}{1 - r}$$

Not:-If the series $\sum_{n=1}^{\infty} ar^n$ and $|r| < 1$ the sum is

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$$

That mean the first term is ar and the partial sequence of the series is

$$S_n = ar + ar^2 + ar^3 + \dots + ar^n$$

Example:- which series are converge and diverge .if it converge find the sum of series?

$$1. \sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right)$$

$$\text{sol. } \sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right) = \frac{7}{4} + \frac{7}{4^2} + \frac{7}{4^3} + \dots$$

$$\sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right) = \sum_{n=1}^{\infty} 7 \cdot \left(\frac{1}{4^n}\right) = \sum_{n=1}^{\infty} ar^n$$

$$a = 7, r = \frac{1}{4} \text{ and } F.T. = \frac{7}{4}$$

$$|r| < 1 \longrightarrow \sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right) \text{ converge and the sum is } = \frac{\frac{7}{4}}{1-\frac{1}{4}} = \frac{\frac{7}{4}}{\frac{3}{4}} = \frac{7}{3}$$

$$\sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right) \rightarrow \frac{7}{3}$$

$$2. \sum_{n=1}^{\infty} \left(\frac{7}{3}\right)^n$$

$$\text{sol. } \sum_{n=1}^{\infty} \left(\frac{7}{3}\right)^n = \frac{7}{3} + \left(\frac{7}{3}\right)^2 + \left(\frac{7}{3}\right)^3 + \dots$$

$$\sum_{n=1}^{\infty} \left(\frac{7}{3}\right)^n = \sum_{n=1}^{\infty} 1 \cdot \left(\frac{7}{3}\right)^n = \sum_{n=1}^{\infty} ar^n$$

$$a = 1, r = \frac{7}{3} \text{ and } F.T. = \frac{7}{3}$$

$|r| > 1 \longrightarrow \sum_{n=1}^{\infty} \left(\frac{7}{3}\right)^n$ and have no summation.

diverge

$$3. \sum_{n=1}^{\infty} \frac{1}{2} (-1)^n = \sum_{n=1}^{\infty} ar^n$$

$$\text{sol. } \sum_{n=1}^{\infty} \frac{1}{2} (-1)^n = -\frac{1}{2} + \frac{1}{2} \cdot (1)^2 + \frac{1}{2} \cdot (-1)^3 + \dots$$

$$a = \frac{1}{2}, |r| = 1 \therefore \sum_{n=1}^{\infty} \frac{1}{2} (-1)^n \text{ diverge.}$$

$$4. \sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right) = \sum_{n=1}^{\infty} \frac{5}{2^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = \sum_{n=1}^{\infty} 5 \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} ar^n, \quad a = 5, |r| = \frac{1}{2}, \quad F.T. = \frac{5}{2}$$

The series is converge and the sum is:-

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = \frac{\frac{5}{2}}{1 - \frac{1}{2}} = \frac{\frac{5}{2}}{\frac{1}{2}} = 5, \therefore \sum_{n=1}^{\infty} \frac{5}{2^n} \rightarrow 5$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} 1 \cdot \frac{1}{3^n} = \sum_{n=1}^{\infty} ar^n, \quad a = 1, |r| = \frac{1}{3}, \quad F.T. = \frac{1}{3}$$

The series is converge and the sum is:-

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}, \therefore \sum_{n=1}^{\infty} \frac{1}{3^n} \rightarrow \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right) = \sum_{n=1}^{\infty} \frac{5}{2^n} + \sum_{n=1}^{\infty} \frac{1}{3^n} = 5 + \frac{1}{2} = \frac{11}{2}, \therefore \sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right) \rightarrow \frac{11}{2}$$

Discuss the following series?

$$1. \ 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots + \frac{2}{3^{n-1}} + \dots$$

$$= \sum_{n=1}^{\infty} 2 \cdot \frac{1}{3^{n-1}} = \sum_{n=1}^{\infty} ar^n, a = 2, r = \frac{1}{3}, F.T. = 2$$

$$|r| < 1 \longrightarrow \sum_{n=1}^{\infty} \left(\frac{2}{3^{n-1}} \right) \text{ converge and the sum is } = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{2}{3^{n-1}} \right) \rightarrow 3$$

$$2. \ 1 + \frac{4}{3} + \left(\frac{4}{3} \right)^2 + \dots$$

$$\sum_{n=0}^{\infty} \left(\frac{4}{3} \right)^n = \sum_{n=0}^{\infty} 1 \cdot \left(\frac{4}{3} \right)^n = \sum_{n=0}^{\infty} ar^n$$

$$a = 1, r = \frac{4}{3} \text{ and } F.T. = 1$$

$$|r| > 1 \longrightarrow \sum_{n=1}^{\infty} \left(\frac{4}{3} \right)^n \text{ diverge.}$$

$$3. \ 2 - 3 + 8 + 7^2 + 7^3 + 7^4 + \dots$$

$$= -1 + 1 + 7 + 7^2 + 7^3 + 7^4 + \dots$$

$$= -1 + \sum_{n=0}^{\infty} (7)^n = \sum_{n=0}^{\infty} ar^n, a = 1, |r| = 7 > 1$$

$$4. \ 1 - 2 + 4 - 8 + \dots$$

$$= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + \dots$$

$$= \sum_{n=0}^{\infty} (-2)^n, \quad a = 1, |r| = 2 > 1$$

The series is diverge.

$$\begin{aligned} 5. & 3 + 6 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \\ &= 8 + 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \\ &= 8 + \sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} 1 \cdot \frac{1}{3^n} = \sum_{n=0}^{\infty} ar^n, \quad a = 1, |r| = \frac{1}{3}, \quad F.T. = 1 \end{aligned}$$

The series is converge and the sum is:-

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}, \quad \therefore 8 + \sum_{n=1}^{\infty} \frac{1}{3^n} \rightarrow 8 + \frac{3}{2} = \frac{19}{2}$$

$$6. 1 + 2(7) + 2(7)^2 + 2(7)^3 + \dots \text{ H.W.}$$

7. $\sum_{n=1}^{\infty} c = c + c + c + \dots$ is constant series and geometric series with

$a = c, |r| = 1$ where c is constant and The series is diverge.

Write the geometric series with (Homework).

$$1. a = \frac{1}{5}, r = \frac{2}{3}$$

$$2. a = 3, r = \frac{1}{2}$$

Method of convergence

1. The P-Series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 0 \quad \text{is P - Series}$$

a) If $p > 1 \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ is converge.

- b) If $P < 1 \implies \sum_{n=1}^{\infty} \frac{1}{n^p}$ is diverge.
- c) If $P = 1 \implies \sum_{n=1}^{\infty} \frac{1}{n^p}$ is diverge and it's called **Harmonic Series** .

Example:-

- 1) $\sum_{n=1}^{\infty} \frac{1}{n}$, Harmonic Series and P-Series diverge because $p=1$.
- 2) $\sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$, Harmonic Series and P-Series diverge because $p=1$.
- 3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, P-Series diverge because $p = \frac{1}{2} < 1$.
- 4) $\sum_{n=1}^{\infty} \frac{1}{n^2}$, P-Series converge because $p = 2 > 1$.
- 5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n}$, Harmonic Series and P-Series diverge because $p=1$.
- 6) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$, P-Series converge because $p = \frac{3}{2} > 1$.

2. The n-Ratio n-terms Test:-

Let $\sum_{n=1}^{\infty} a_n$ be a series ; $a_n > 0$,then:-

If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L \in R$,

- a) If $L < 1 \implies \sum_{n=1}^{\infty} a_n$ is converge.
- b) If $L > 1 \implies \sum_{n=1}^{\infty} a_n$ is diverge.
- c) If $L = 1 \implies \sum_{n=1}^{\infty} a_n$ the series may by converge or diverge.

Examples :- Are the following series converge ?

i. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Sol. $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \lim_{n \rightarrow \infty} \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} = \lim_{n \rightarrow \infty} \frac{n+1}{3n}$

$= \frac{1}{3} \lim_{n \rightarrow \infty} \left(\frac{n}{n} + \frac{1}{n} \right) = \frac{1}{3} (1 + 0) = \frac{1}{3} < 1$

$$\therefore \sum_{n=1}^{\infty} \frac{n}{3^n} \rightarrow \frac{1}{3}$$

$$\text{ii. } \sum_{n=1}^{\infty} \frac{2^{n+5}}{3^n}$$

$$\text{Sol. } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{2^{n+1+5}}{3^{n+1}}}{\frac{2^{n+5}}{3^n}} = \lim_{n \rightarrow \infty} \frac{2^{n+1+5}}{3^{n+1}} \cdot \frac{3^n}{2^{n+5}} =$$

$$\lim_{n \rightarrow \infty} \frac{2^{n+1+5}}{3(2^{n+5})}$$

$$= \frac{1}{3} \lim_{n \rightarrow \infty} \frac{2^n \cdot 2 + 5}{2^{n+5}} = \frac{1}{3} \lim_{n \rightarrow \infty} \frac{\frac{2^n \cdot 2}{2^n} + \frac{5}{2^n}}{\frac{2^n}{2^n} + \frac{5}{2^n}} = \frac{1}{3} \lim_{n \rightarrow \infty} \frac{2 + \frac{5}{2^n}}{1 + \frac{5}{2^n}} = \frac{1}{3} \left(\frac{2+0}{1+0} \right) = \frac{2}{3} < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} \rightarrow \frac{2}{3}$$

$$\text{iii. } \sum_{n=1}^{\infty} \frac{4^n \sqrt{n}}{3^n}$$

$$\text{Sol. } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{4^{n+1} \sqrt{n+1}}{3^{n+1}}}{\frac{4^n \sqrt{n}}{3^n}} = \lim_{n \rightarrow \infty} \frac{4^{n+1} \sqrt{n+1}}{3^{n+1}} \cdot \frac{3^n}{4^n \sqrt{n}} =$$

$$\lim_{n \rightarrow \infty} \frac{4 \sqrt{n+1}}{3 \sqrt{n}}$$

$$= \frac{4}{3} \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n}} = \frac{4}{3} \lim_{n \rightarrow \infty} \sqrt{\frac{\frac{n+1}{n}}{\frac{n}{n}}} = \frac{4}{3} \sqrt{\frac{1+0}{1}} = \frac{4}{3} > 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} \text{ diverge.}$$

$$\text{iv. } \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

$$\text{Sol. } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{(-1)^{n+1} (n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} = \lim_{n \rightarrow \infty} \frac{(-1)^{n+1} (n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^3} =$$

$$\begin{aligned}
&= \frac{1}{3} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{(-1)^n \cdot (-1)}{3^n \cdot 3} \cdot \frac{3^n}{(-1)^n} = \frac{-1}{3} \lim_{n \rightarrow \infty} \left(\frac{n}{n} + \frac{1}{n}\right)^3 \\
&= \frac{-1}{3} (1+0) = \frac{-1}{3} < 1
\end{aligned}$$

$$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n} \rightarrow \frac{-1}{3}$$

3. The Comparison Ratio Test:-

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two infinite series such that:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \neq 0$$

Then:

- 1) $\sum_{n=1}^{\infty} a_n$ is converge if and only if $\sum_{n=1}^{\infty} b_n$ is converge.
- 2) $\sum_{n=1}^{\infty} a_n$ is diverge if and only if $\sum_{n=1}^{\infty} b_n$ is diverge.

Examples :- Are the following series converge ?

i. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

Sol. $a_n = \frac{1}{2^{n-1}}, b_n = \frac{1}{2^n}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2^{n-1}}}{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} = \lim_{n \rightarrow \infty} \frac{\frac{2^n}{2^n}}{\frac{2^n}{2^n} - \frac{1}{2^n}} = \frac{1}{1 - 0} = 1 \neq 0$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{geo. Ser.} \quad |r| = \frac{1}{2} < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ is converge and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} \neq 0$$

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is converge

ii. $\sum_{n=1}^{\infty} \frac{2n}{3n^2 - 4n + 1}$

Sol. $a_n = \frac{\frac{2}{n}}{3 - \frac{4}{n} + \frac{1}{n^2}}, b_n = \frac{2}{n}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\frac{2}{n}}{3 - \frac{4}{n} + \frac{1}{n^2}}}{\frac{2}{n}} = \lim_{n \rightarrow \infty} \frac{1}{3 - \frac{4}{n} + \frac{1}{n^2}} = \frac{1}{3 - 0 + 0} = \frac{1}{3} \neq 0$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n} \quad \text{P- Ser. } p = 1$$

$\therefore \sum_{n=1}^{\infty} \frac{2}{n}$ is diverge and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} \neq 0$$

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n}{3n^2 - 4n + 1}$ is diverge.

iii. $\sum_{n=1}^{\infty} \frac{2n^3 + 100n^2 + 1000}{8n^6 - n + 2}$ Hint let $b_n = \frac{2}{n^3}$

Sol. $a_n = \frac{\frac{2}{n^3} + \frac{100}{n^4} + \frac{1000}{n^6}}{8 - \frac{1}{n^5} + \frac{2}{n^6}}, b_n = \frac{2}{n^3}$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{\frac{2}{n^3} + \frac{100}{n^4} + \frac{1000}{n^6}}{8 - \frac{1}{n^5} + \frac{2}{n^6}}}{\frac{2}{n^3}} = \lim_{n \rightarrow \infty} \frac{1 + \frac{50}{n} + \frac{500}{n^3}}{8 - \frac{1}{n^5} + \frac{2}{n^6}} = \frac{1 + 0 + 0}{8 - 0 + 0} = \frac{1}{8} \neq 0$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^3} \quad \text{P- Ser. } p > 1$$

$\therefore \sum_{n=1}^{\infty} \frac{2}{n^3}$ is converge and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} \neq 0$$

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n^3 + 100n^2 + 1000}{8n^6 - n + 2}$ is converge.

iv. $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is diverge because $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ (Homework)

4. The n-th Root Test

Let $\sum_{n=1}^{\infty} a_n$ be infinite series, such that :-

$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L \in R$, then :-

- a) If $L < 1$ $\implies \sum_{n=1}^{\infty} a_n$ is converge.
- b) If $L > 1$ $\implies \sum_{n=1}^{\infty} a_n$ is diverge.
- c) If $L = 1$ $\implies \sum_{n=1}^{\infty} a_n$ has no information.

Ex. Discuss the convergence for the following series?

i) $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

Sol.

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^n}{3^n}} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{2}{3}\right)^n} = \frac{2}{3} < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n}{3^n} \rightarrow \frac{2}{3}$$

ii) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Sol.

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n^n}} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^n} \rightarrow 0$$

iii) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

Sol.

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{n}}}{\frac{2}{n^n}} = 2 \lim_{n \rightarrow \infty} \frac{1}{\frac{2}{n^n}} = 2 \cdot \frac{1}{\frac{2}{\infty \infty}} = 2 \cdot \frac{1}{\infty^0} = 2 > 1$$

$\therefore \sum_{n=1}^{\infty} \frac{2^n}{n^2}$ the series is diverge.

iv) $\sum_{n=1}^{\infty} \frac{5n}{2(3)^n}$

Sol.

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{5n}{2(3)^n}} = \lim_{n \rightarrow \infty} \left(\frac{5}{2}\right)^{\frac{1}{n}} \cdot \frac{1}{\frac{2}{3^n}} = \left(\frac{5}{2}\right)^{\frac{1}{\infty}} \cdot \frac{1}{\frac{2}{\infty \infty}} = \left(\frac{5}{2}\right)^0 \cdot \frac{\infty^0}{2} = \frac{1}{2} < 1$$

$$= \frac{1}{2} < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{5n}{2(3)^n}$ the series is convergent.

v) $\sum_{n=1}^{\infty} \frac{5^n}{2(3)^n}$

Sol.

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{5^n}{2(3)^n}} = \lim_{n \rightarrow \infty} \left(\frac{5}{2}\right)^{\frac{1}{n}} \cdot \left(\frac{5}{3}\right)^{\frac{n}{n}} = \left(\frac{5}{2}\right)^{\frac{1}{\infty}} \cdot \left(\frac{5}{3}\right) = \left(\frac{5}{2}\right)^0 \cdot \left(\frac{5}{3}\right) = \frac{5}{3} > 1$$

$\therefore \sum_{n=1}^{\infty} \frac{5^n}{2(3)^n}$ the series is diverge.

5. The Comparison inequality Test

If $0 \leq a_n \leq b_n \forall n \in N$, then:

- If $\sum_{n=1}^{\infty} a_n$ diverge, then $\sum_{n=1}^{\infty} b_n$ diverge.
- If $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} a_n$ converge.

Ex. Are the following series converge?

1. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$

Sol.

Is converge since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is converge

$$0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n^2}$$

2. $\sum_{n=1}^{\infty} \frac{1}{n!}$

Sol.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n!}, \quad \sum_{n=1}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{(2)1} + \frac{1}{(3)(2)1} + \frac{1}{(4)(3)(2)1} + \dots$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

$$0 \leq \frac{1}{n!} \leq \frac{1}{2^n}$$

$\sum_{n=1}^{\infty} a_n$ converge since $\sum_{n=1}^{\infty} b_n$ is converge .

6. The Absolutely Converge Test:-

- If $\sum_{n=1}^{\infty} a_n$ is Absolutely converge if $\sum_{n=1}^{\infty} |a_n|$ is converge.
- If $\sum_{n=1}^{\infty} |a_n|$ is not converge, then $\sum_{n=1}^{\infty} a_n$ is not Absolutely converge.

Ex.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ Absolutely converge since $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2^n} \right| = \sum_{n=1}^{\infty} \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ geometric series $|r| = \left| \frac{1}{2} \right| < 1$ converge.

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ is Absolutely converge.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ Absolutely converge since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ P- series $p = 2 > 1$ converge.

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is Absolutely converge.

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not Absolutely converge since

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ the series is diverge.

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is not Absolutely converge since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ P- series $p = \frac{1}{2} < 1$ diverge.

Conditional Converge :-

If $\sum_{n=1}^{\infty} (-1)^n a_n$ is converge, but $\sum_{n=1}^{\infty} |a_n|$ is not converge.

$\sum_{n=1}^{\infty} (-1)^n a_n$ Is converge if it satisfies the following condition:-

- $a_n > 0 \quad \forall n \in N.$
- $a_n \geq a_{n+1} \quad \forall n \in N.$
- $\lim_{n \rightarrow \infty} a_n = 0 \quad \forall n \in N.$

We said that the series was Conditional Converge not absolutely converge.

Ex. Do the following series absolutely converge or conditional converge?

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not Absolutely converge since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ the series is diverge.}$$

$$a_n > 0 \quad \forall n \in \mathbb{N} \rightarrow \frac{1}{n} > 0$$

$$a_n \geq a_{n+1} \rightarrow \frac{1}{n} \geq \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ conditional converge not absolutely converge .

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ Is not Absolutely converge, but conditional converge.
(Homework)

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ Is not Absolutely converge, but conditional converge.
(Homework)

Theorem:-

If $\sum_{n=1}^{\infty} a_n$ is converge, then $\lim_{n \rightarrow \infty} a_n = 0$.

That means if $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is diverge.

4. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{5n+1}$ Is not Absolutely converge

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n}{5n+1} \right| = \sum_{n=1}^{\infty} \frac{n}{5n+1}$$

$$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \lim_{n \rightarrow \infty} \frac{n}{5n+1} = \lim_{n \rightarrow \infty} \frac{1}{5 + \frac{1}{n}} = \frac{1}{5} \neq 0$$

By theorem above $\sum_{n=1}^{\infty} \frac{n}{5n+1}$ is diverge.

i.e. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{5n+1}$ Is not Absolutely converge.

Note: - The converse of the theorem above is not true

If $\lim_{n \rightarrow \infty} a_n = 0 \rightarrow$ this not necessary to be $\sum_{n=1}^{\infty} a_n$ converge.

Ex. Are the series converging or diverging?

1. $\sum_{n=1}^{\infty} \frac{1}{n}$ is diverge

Although $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is diverge

Although $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$.

3. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is converge

And $\lim_{n \rightarrow \infty} \frac{1}{n^3} = 0$.

4. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ Is diverge because

$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$. By theorem $\therefore \sum_{n=1}^{\infty} a_n$ is diverge.

5. $\sum_{n=1}^{\infty} n^2$

$\therefore \lim_{n \rightarrow \infty} n^2 = \infty \neq 0 \rightarrow$ By theorem $\therefore \sum_{n=1}^{\infty} a_n$ is diverge.

6. $\sum_{n=1}^{\infty} \sqrt{n}$

$\therefore \lim_{n \rightarrow \infty} \sqrt{n} = \infty \neq 0 \rightarrow$ By theorem $\therefore \sum_{n=1}^{\infty} a_n$ is diverge.

7. $\sum_{n=1}^{\infty} \frac{n^3}{n\sqrt{n}}$ (Homework)

8. $\sum_{n=1}^{\infty} \frac{n^2+1}{n}$

9. $\sum_{n=1}^{\infty} n$

$$10. \sum_{n=1}^{\infty} \frac{n^2 \sqrt{n}}{n^2 + 1}$$

Ex. Determine whether the given series converge or diverge ?

$$1. \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$$

$$2. 1 + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$3. \sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right)$$

$$4. \sum_{n=1}^{\infty} \frac{(-1)^n}{3}$$

$$5. \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

$$6. \sum_{n=1}^{\infty} \frac{(n+1)!}{3! n! 3^n}$$

Chapter Two

The power series :- are the series of the form:

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots + C_n(x-a)^n + \dots$$

In which the center a and the coefficients C_0, C_1, \dots are constants.

Ex.:- Taking the coefficients to be 1 in the equation of the power series, we get the geometric power series :

$$\sum_{n=0}^{\infty} 1 \cdot x^n = 1 + x + x^2 + \dots + x^n + \dots, \text{ where } a = 1, r = x$$

when $|x| < 1$,

Ex.:- Find the interval of the convergence of the series $\sum_{n=0}^{\infty} x^n$ and the sum of it ?

$$\sum_{n=0}^{\infty} 1 \cdot x^n = \sum_{n=0}^{\infty} a \cdot r^n \quad \text{geometric series}$$

$a = 1, r = x$, is converge when $|r| = |x| < 1 \rightarrow -1 < x < 1 \rightarrow x \in (-1, 1)$.

$$\sum_{n=0}^{\infty} x^n = \frac{F.T.}{1-r} = \frac{1}{1-x}, \quad \forall x \in (-1, 1)$$

Ex. For the following power series: find the interval of convergence and the sum of it?

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-2)^n + \dots$$

$a = 1, r = -\frac{1}{2}(x-2)$, the series is converge when

$$|r| = \left| -\frac{1}{2}(x-2) \right| < 1 \rightarrow \left| \frac{x-2}{2} \right| < 1 \rightarrow -1 < \frac{x-2}{2} < 1$$

$$-2 < x-2 < 2 \rightarrow 0 < x < 4 \quad x \in (0, 4).$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = \frac{F.T.}{1-r} = \frac{1}{1 - \left(-\frac{1}{2}(x-2)\right)}$$

$$= \frac{1}{1 + \frac{x-2}{2}} = \frac{1}{\frac{2+x-2}{2}} = \frac{2}{x}, \quad \forall x \in (0,4)$$

Ex. Find the power series if $C_n = 3$ and $a = 2$? And find the interval of convergence and the sum of it? Homework

Sol.

$$\sum_{n=0}^{\infty} 3(x-2)^n$$

The Taylor Series Expansion of $f(x)$ about a or near $x = a$

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then Taylor series generated by f at a is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

$$+ \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots$$

Remark:-

When $a=0$, then the series said Maclaurin Series.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots$$

Ex. : - Find Taylor Series

Expansion of $f(x) = \frac{1}{x}$ at $a = 2$, and find the convergence sum? or

Find Taylor Series Expansion generated by $f(x) = \frac{1}{x}$ near $a = 2$, and find the convergence sum?

Sol.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots \\ &\quad + \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots \\ &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!} (x-2)^2 + \dots + \frac{f^{(n)}(2)}{n!} (x-2)^n + \dots (*) \end{aligned}$$

Now we found the derivatives of the function

$$f(x) = \frac{1}{x} = x^{-1}, f'(x) = -x^{-2}, f''(x) = 2x^{-3},$$

$$f'''(x) = -6x^{-4}, f^{(4)}(x) = 24x^{-5}, \dots, f^{(n)}(x) = (-1)^n \cdot n! \cdot x^{-(n+1)}, \dots$$

Now we put $a=2$

$$f(2) = \frac{1}{2} = 2^{-1}, f'(2) = -2^{-2}, f''(2) = 2 \cdot 2^{-3}, f'''(2) = -6 \cdot 2^{-4}$$

$$f^{(4)}(2) = 24(2)^{-5}, \dots, f^{(n)}(2) = (-1)^n \cdot n! \cdot 2^{-(n+1)}, \dots$$

$$\frac{f^{(n)}(2)}{n!} = (-1)^n \cdot 2^{-(n+1)} = \frac{(-1)^n}{2^{(n+1)}}$$

Now we put those values in series * and we get

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!} (x-2)^2 + \dots + \frac{f^{(n)}(2)}{n!} (x-2)^n + \dots$$

$$f(x) = \frac{1}{2} + -2^{-2}(x-2) + \frac{2 \cdot 2^{-3}}{2!} (x-2)^2 + \dots + \frac{(-1)^n}{2^{(n+1)}} (x-2)^n + \dots$$

$$f(x) = \frac{1}{2} - \frac{1}{2^2}(x-2) + \frac{1}{2^3}(x-2)^2 - \dots + \frac{(-1)^n}{2^{(n+1)}} (x-2)^n + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{(n+1)}} (x-2)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2} (-1)^n \left(\frac{1}{2}\right)^n (x-2)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2} \left(-\frac{1}{2}(x-2)\right)^n$$

This is a geometric series $a = \frac{1}{2}$, $r = -\frac{1}{2}(x-2)$ and it's converge at the intervals: $|r| < 1 \rightarrow \left|-\frac{1}{2}(x-2)\right| < 1 \rightarrow \left|\frac{1}{2}(x-2)\right| < 1$

$$-1 < \frac{1}{2}(x-2) < 1 \rightarrow -2 < (x-2) < 2 \rightarrow 0 < x < 4 \quad x \in (0,4)$$

$$\sum_{n=0}^{\infty} ar^n = \frac{F.T.}{1-r} = \frac{\frac{1}{2}}{1 + \frac{x-2}{2}} = \frac{\frac{1}{2}}{\frac{2+x-2}{2}} = \frac{1}{x}$$

Ex. Find the Taylor Series for $f(x) = \frac{1}{x}$ near $a = 1$?(Homework)

Ex. Find the converge interval and the sum if exist?

1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$

Sol. We solve this series by using The n-Ratio n-terms test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = \lim_{n \rightarrow \infty} \frac{|x| \cdot n}{n+1}$$

$$= |x| \cdot 1 < 1 \rightarrow -1 < x < 1 \rightarrow x \in (-1,1).$$

2. $\sum_{n=0}^{\infty} 2x^n$

Sol.

$$\sum_{n=0}^{\infty} 2x^n = \sum_{n=0}^{\infty} ar^n = \frac{F.T.}{1-r} = \frac{2}{1-x}$$

$$|r| = |x| < 1 \rightarrow -1 < x < 1, x \in (-1,1).$$

3. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$

Sol.

$$\sum_{n=0}^{\infty} 1 \cdot \frac{(x+2)^n}{3^n} = \sum_{n=0}^{\infty} ar^n = \frac{F.T.}{1-r} = \frac{1}{1-\frac{x+2}{3}} = \frac{1}{\frac{3-x-2}{3}} = \frac{3}{1-x}$$

$$|r| = \left| \frac{x+2}{3} \right| < 1 \rightarrow -1 < \frac{x+2}{3} < 1 \rightarrow -3 < x+2 < 3$$

$-5 < x < 1$, converge interval $x \in (-5, 1)$.

4. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}}$

Sol.

$$\sum_{n=0}^{\infty} \frac{1}{3} \cdot \frac{(x+2)^n}{3^n} = \sum_{n=0}^{\infty} ar^n = \frac{F.T.}{1-r} = \frac{\frac{1}{3}}{1-\frac{x+2}{3}} = \frac{\frac{1}{3}}{\frac{3-x-2}{3}} = \frac{1}{1-x}$$

$$|r| = \left| \frac{x+2}{3} \right| < 1 \rightarrow -1 < \frac{x+2}{3} < 1 \rightarrow -3 < x+2 < 3$$

$-5 < x < 1$, converge interval $x \in (-5, 1)$.

5. $\sum_{n=0}^{\infty} \frac{(2x-5)^n}{n^2}$

Sol.

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(2x-5)^n}{n^2} \right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{|(2x-5)^n|}{n^2}}$$

$$= |2x-5| \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{2}{n}}}$$

$$= |2x-5| < 1 \rightarrow -1 < 2x-5 < 1$$

$4 < 2x < 6 \rightarrow 2 < x < 3$, $x \in (2, 3)$ converge interval

6. $\sum_{n=0}^{\infty} nx^n$ (Homework)

7. $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$

8. $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$

9. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

Ex. Find the Taylor polynomial generated by $f(x) = e^x$ near $x = 0$?

Sol.

$$f(x) = e^x, a = 0$$

$$\sum_{k=0}^n \frac{f^{(k)}(0)}{k!} (x-0)^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) = e^x \rightarrow f(0) = e^0 = 1.$$

$$f'(x) = e^x \rightarrow f'(0) = e^0 = 1.$$

$$f''(x) = e^x \rightarrow f''(0) = e^0 = 1$$

⋮

$$f^{(n)}(x) = e^x \rightarrow f^{(n)}(0) = e^0 = 1$$

$$\sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k = e^0 + e^0 x + \frac{e^0}{2!}x^2 + \dots + \frac{e^0}{n!}x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} = \sum_{k=0}^n \frac{x^k}{k!}$$

Ex. Find the Taylor polynomial generated by

$$f(x) = e^{-x} \text{ near } x = 0? \text{ (Homework).}$$

Ex. Find the Maclaurin Series generated by $f(x) = \cos x$?

Sol.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(x) = \cos x \rightarrow f(0) = \cos 0 = 1$$

$$f'(x) = -\sin x \rightarrow f'(0) = -\sin 0 = 0$$

$$f''(x) = -\cos x \rightarrow f''(0) = -\cos 0 = -1$$

$$f'''(x) = \sin x \rightarrow f'''(0) = \sin 0 = 0$$

∴

$$f^{(2n)}(x) = (-1)^n \cos x \quad n \text{ even} \rightarrow f^{(2n)}(0) = (-1)^n$$

$$f^{(2n+1)}(x) = (-1)^n \sin x \quad n \text{ odd} \rightarrow f^{(2n+1)}(0) = 0$$

$$f(x) = \cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots + \frac{(-1)^n}{(2n)!}x^{2n} + \dots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

Ex. Find the Maclaurin polynomial for $f(x) = \sin x$?

Sol.

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k$$

$$f(x) = \sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots + \frac{(-1)^{2n-1}}{(2n-1)!}x^{2n-1}$$

$$\sin x = \sum_{k=0}^n \frac{(-1)^{2k-1}}{(2k-1)!}x^{2k-1}$$

Computation of Logarithms:-

From Taylor series expansion, we get

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^n}{n} + \dots$$

And

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n} + \dots$$

$$\text{Ln} \frac{(1+x)}{(1-x)} = \text{Ln}(1+x) - \text{Ln}(1-x)$$

$$\text{Ln} \frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} \dots + \frac{x^{2n-1}}{2n-1} + \dots\right)$$

Now put $\frac{N+1}{N} = \frac{(1+x)}{(1-x)}$

$$N(1+x) = (N+1)(1-x)$$

$$N + Nx = N - Nx + 1 - x$$

$$2Nx + x = 1 \rightarrow x(2N+1) = 1 \rightarrow x = \frac{1}{(2N+1)}$$

$$\text{Ln} \frac{N+1}{N} = 2\left(\frac{1}{(2N+1)} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \dots\right)$$

Ex. By approximate find $\text{Ln}2$?

Sol. $\text{Ln}2 = \text{Ln} \frac{1+1}{1} = 2\left(\frac{1}{(2(1)+1)} + \frac{1}{3(2(1)+1)^3} + \frac{1}{5(2(1)+1)^5} + \dots\right)$

$$\text{Ln}2 = 2\left(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5(3)^5} + \dots\right)$$

$$= 2(0.333 + 0.0123 + 0.0008 + \dots)$$

$$\cong 2(0.3464) \cong 0.69314$$

$$\text{Ln}2 \cong 0.69314$$

The logarithms of the number $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, \frac{N+1}{N}$ are ordinarily computed first informing a table of natural logarithms of the integers.

Then it is a matter of simple arithmetic to compute:-

$$\text{Ln}3 = \text{Ln}\frac{2}{1} + \text{Ln}\frac{3}{2}$$

$$\text{Ln}4 = \text{Ln}3 + \text{Ln}\frac{4}{3}$$

$$\text{Ln}5 = \text{Ln}4 + \text{Ln}\frac{5}{4}$$

⋮

$$\text{Ln}(N + 1) = \text{Ln}N + \text{Ln}\frac{N + 1}{N}$$

Ex. Construct a table of natural logarithms in N for $N = 1, 2, 3, \dots, 10$ by the method discussed in connection with equation

$$\text{Ln}\frac{N + 1}{N} = 2\left(\frac{1}{(2N + 1)} + \frac{1}{3(2N + 1)^3} + \frac{1}{5(2N + 1)^5} + \dots\right)$$

But taking advantage of the relationships:-

$$\text{Ln}4 = 2\text{Ln}2, \quad \text{Ln}6 = \text{Ln}2 + \text{Ln}3, \quad \text{Ln}8 = 3\text{Ln}2$$

$$\text{Ln}9 = 2\text{Ln}3, \quad \text{Ln}10 = \text{Ln}2 + \text{Ln}5$$

Properties :-

$$\text{Ln}(a \cdot b) = \text{Ln}a + \text{Ln}b$$

$$\text{Ln}\frac{a}{b} = \text{Ln}a - \text{Ln}b$$

$$\text{Ln}a^k = k\text{Ln}a$$

Fourier Series

A Fourier series is an infinite series expansion in terms of trigonometric functions of $f(x)$ on the interval $(0, 2\pi)$ is

$$f(x) = a_0 + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]$$

Where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$$

and

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx \quad \text{for } k = 0, 1, 2, 3, \dots, n$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx$$

Ex. Find the Fourier series for

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 2 & \pi < x \leq 2\pi \end{cases}$$

Sol.

$$f(x) = a_0 + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx] \quad (1)$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$$

$$a_0 = \frac{1}{2\pi} \left(\int_0^{\pi} 1 dx + \int_{\pi}^{2\pi} 2 dx \right)$$

$$= \frac{1}{2\pi} [x]_0^{\pi} + 2x \Big|_{\pi}^{2\pi}$$

$$= \frac{1}{2\pi} [\pi - 0 + 2(2\pi - \pi)]$$

$$= \frac{1}{2\pi} [\pi + 2\pi] = \frac{3\pi}{2\pi}$$

$$\therefore a_0 = \frac{3}{2}$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx$$

$$a_k = \frac{1}{\pi} \left(\int_0^{\pi} \cos kx \, dx + \int_{\pi}^{2\pi} 2 \cos kx \, dx \right)$$

$$= \frac{1}{\pi} \left[\frac{\sin kx}{k} \Big|_0^{\pi} + 2 \frac{\sin kx}{k} \Big|_{\pi}^{2\pi} \right] \quad ; k = 1, 2, 3, \dots, n$$

$$= \frac{1}{\pi} \left[\frac{1}{k} (\sin k\pi - \sin 0) + \frac{2}{k} (\sin k 2\pi - \sin k\pi) \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{k} (0 - 0) + \frac{2}{k} (0 - 0) \right] = \frac{1}{\pi} (0) = 0$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx$$

$$b_k = \frac{1}{\pi} \left(\int_0^{\pi} \sin kx \, dx + \int_{\pi}^{2\pi} 2 \sin kx \, dx \right)$$

$$= \frac{1}{\pi} \left[-\frac{\cos kx}{k} \Big|_0^{\pi} - 2 \frac{\cos kx}{k} \Big|_{\pi}^{2\pi} \right] \quad ; k = 1, 2, 3, \dots, n$$

$$= \frac{1}{\pi} \left[\frac{1}{k} (-\cos k\pi + \cos 0) + \frac{2}{k} (-\cos k 2\pi + \cos k\pi) \right]$$

$$= \frac{1}{\pi k} [(-\cos k\pi + 1) + 2(-1 + \cos k\pi)]$$

$$= \frac{1}{\pi k} (\cos k\pi - 1) \quad ; k = 1, 2, 3, \dots, n$$

$$\therefore b_k = \frac{(-1)^k - 1}{k\pi}, \quad b_k = \begin{cases} \frac{-2}{k\pi} & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases}$$

$$b_1 = \frac{-2}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{-2}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{-2}{5\pi}$$

Now we put the a_0, a_k, b_k in the series (1), we get:-

$$f(x) = \frac{3}{2} + \sum_{k=1}^{\infty} [0(\cos kx) + b_k \sin kx]$$

$$f(x) = \frac{3}{2} + \sum_{k=1}^{\infty} [b_k \sin kx]$$

$$= \frac{3}{2} + [b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \dots]$$

$$\therefore f(x) = \frac{3}{2} - \frac{2}{\pi} \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right]$$

Ex2. $F(x) = 3, 0 \leq x \leq 2\pi$ find the Fourier series?

Sol.

$$f(x) = a_0 + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx] \quad (*)$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} 3 dx$$

$$= \frac{1}{2\pi} [3x]_0^{2\pi}$$

$$= \frac{1}{2\pi} 3(2\pi - 0)$$

$$= \frac{6\pi}{2\pi}$$

$$\therefore a_0 = 3$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx$$

$$a_k = \frac{1}{\pi} \left(\int_0^{2\pi} 3 \cos kx \, dx \right)$$

$$= \frac{1}{\pi} \left[3 \frac{\sin kx}{k} \Big|_0^{2\pi} \right] \quad ; k = 1, 2, 3, \dots, n$$

$$= \frac{1}{\pi} \left[\frac{3}{k} (\sin k 2\pi - \sin k 0) \right]$$

$$= \frac{1}{\pi} \left[\frac{3}{k} (0 - 0) \right] = \frac{1}{\pi} (0) = 0$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx$$

$$b_k = \frac{1}{\pi} \left(\int_0^{2\pi} 3 \sin kx \, dx \right)$$

$$= \frac{1}{\pi} \left[-3 \frac{\cos kx}{k} \Big|_0^{2\pi} \right] \quad ; k = 1, 2, 3, \dots, n$$

$$= \frac{3}{\pi k} (-\cos k 2\pi + \cos k 0)$$

$$= \frac{3}{\pi k} (-1 + 1) = \frac{3}{\pi k} (0)$$

Now we put the a_0 , a_k , b_k in the series (*), we get:-

$$f(x) = 3 + \sum_{k=1}^{\infty} [0(\cos kx) + 0(\sin kx)]$$

$$\therefore f(x) = 3$$

Ex.3:- $f(x) = x$, $\forall 0 \leq x \leq 2\pi$ find the Fourier series?

Sol.

$$f(x) = a_0 + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx] \quad (*)$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} x dx \\ &= \frac{1}{2\pi} \left[\frac{x^2}{2} \Big|_0^{2\pi} \right] \\ &= \frac{1}{4\pi} (x^2 \Big|_0^{2\pi}) \\ &= \frac{4\pi^2}{4\pi} \end{aligned}$$

$$\therefore a_0 = \pi$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx$$

$$a_k = \frac{1}{\pi} \left(\int_0^{2\pi} x \cos kx dx \right)$$

By using $udv = v \cdot u - \int v du$

$$u = x, \quad du = dx, \quad dv = \cos kx dx, \quad v = \frac{\sin kx}{k}$$

$$\begin{aligned} &= \frac{1}{\pi} \left[x \cdot \frac{\sin kx}{k} \Big|_0^{2\pi} - \int_0^{2\pi} \frac{\sin kx}{k} dx \right] \\ &= \frac{1}{\pi} \left[\left(2\pi \cdot \frac{\sin k2\pi}{k} - 0 \right) + \frac{\cos kx \Big|_0^{2\pi}}{k^2} \right] \end{aligned}$$

$$= \frac{1}{\pi} \left[\left(2\pi \cdot \frac{0}{k} - 0 \right) + \frac{\cos k2\pi - \cos k0}{k^2} \right]$$

$$a_k = 0$$

$$b_k = \frac{1}{\pi} \left(\int_0^{2\pi} x \sin kx \, dx \right)$$

By using $udv = v \cdot u - \int vdu$

$$u = x, \quad du = dx, \quad dv = \sin kx \, dx, \quad v = \frac{-\cos kx}{k}$$

$$= \frac{1}{\pi} \left[-x \cdot \frac{\cos kx}{k} \Big|_0^{2\pi} + \int_0^{2\pi} \frac{\cos kx}{k} \, dx \right]$$

$$= \frac{1}{\pi} \left[\left(-2\pi \cdot \frac{\cos k2\pi}{k} + 0 \right) + \frac{\sin kx \Big|_0^{2\pi}}{k^2} \right]$$

$$= \frac{1}{\pi} \left[\left(-2\pi \cdot \frac{1}{k} \right) + \frac{\sin k2\pi - \sin k0}{k^2} \right] = \frac{1}{\pi} \left(-2\pi \cdot \frac{1}{k} \right) + \frac{0 - 0}{k^2}$$

$$\therefore b_k = \frac{-2}{k}$$

Now we put the a_0, a_k, b_k in the series (*), we get:-

$$f(x) = \pi + \sum_{k=1}^{\infty} \left[0(\cos kx) - \frac{2}{k} (\sin kx) \right]$$

$$\therefore f(x) = \pi - \frac{2}{k} \sum_{k=1}^{\infty} (\sin kx)$$

Ex. Find the Fourier series associated with the following functions. Sketch each functions?

$$1- f(x) = \begin{cases} 0 & 0 \leq x \leq \pi \\ 1 & \pi < x \leq 2\pi \end{cases}$$

$$2- f(x) = \begin{cases} -1 & 0 \leq x \leq \pi \\ 1 & \pi < x \leq 2\pi \end{cases}$$

$$3- f(x) = 3x, \quad 0 \leq x \leq 2\pi$$

$$4- f(x) = \begin{cases} 2 & 0 \leq x \leq \pi \\ x & \pi < x \leq 2\pi \end{cases}$$

$$5- f(x) = \begin{cases} x & 0 \leq x \leq \pi \\ -x & \pi < x \leq 2\pi \end{cases}$$

Chapter Three

The vectors

The vectors

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in \mathbb{R}^2 , then $\overrightarrow{P_1 P_2}$ or $\overrightarrow{P_2 P_1}$ are vectors where :-

$$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1)$$

$$\overrightarrow{P_2 P_1} = (x_1 - x_2, y_1 - y_2)$$

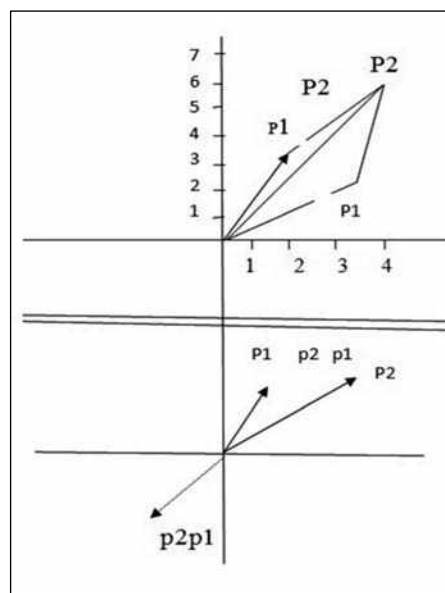
Ex. $P_1 = (2, 3)$, $P_2 = (4, 7)$

$$\overrightarrow{P_1 P_2} = (4 - 2, 7 - 3)$$

$$= (2, 4)$$

$$\overrightarrow{P_2 P_1} = (2 - 4, 2 - 7)$$

$$= (-2, -4)$$



Ex. $P_1 = (4, 2)$, $P_2 = (-3, 5)$ find $\overrightarrow{P_1 P_2}$ and

$\overrightarrow{P_2 P_1}$? (Homework)

- If $P = (x, y)$, $O = (0, 0)$, then

$$\overrightarrow{OP} = (x - 0, y - 0) = (x, y)$$

$$\overrightarrow{PO} = (0 - x, 0 - y) = (-x, -y)$$

$$\vec{V} = (a, b) = (a, 0) + (0, b)$$

$$= a(1, 0) + b(0, 1)$$

$$\therefore \vec{V} = ai + bj$$

$$\text{slop } m = \frac{b}{a} = \tan \theta$$

$$(3,4) = 3i + 4j$$

The Length of \vec{V} :-

We denoted for The Length of \vec{V} by $|\vec{V}|$ for the vector

$\vec{V} = xi + yj$ which equal :-

$$|\vec{V}| = \sqrt{x^2 + y^2}$$

Ex. Find The Length of

1- $\vec{V} = 5i + 7j$

2- $\vec{V} = 4j$

3- $\vec{V} = -3i$ (Homework)

4- $\vec{V} = -i - j$ (Homework)

Sol. 1-

$$|\vec{V}| = \sqrt{5^2 + 7^2} = \sqrt{25 + 49} = \sqrt{74}$$

Sol. 2-

$$|\vec{V}| = \sqrt{0^2 + 4^2} = \sqrt{0 + 16} = \sqrt{16} = 4$$

The Zero Vector:-

$$\vec{0} = 0i + 0j = (0, 0)$$

$$|\vec{0}| = \sqrt{0^2 + 0^2} = 0$$

The Unit Vector:-

$u = xi + yj$ when $|u| = 1$, then u is said to be a unit vector.

- $i(1, 0) \Rightarrow |i| = \sqrt{1^2 + 0^2} = 1$, i is a unit vector.
- $j(0, 1) \Rightarrow |j| = \sqrt{0^2 + 1^2} = 1$, j is a unit vector.
- $u = \cos \theta + \sin \theta$

$$\Rightarrow |u| = \sqrt{(\cos \theta)^2 + (\sin \theta)^2} = 1, u \text{ is a unit vector.}$$

To find a unit vector for any vector

We must find the direction of vector V is $\frac{V}{|V|}$ which is the unit

vector such that $\left| \frac{V}{|V|} \right| = 1$.

Ex. $\vec{V} = 3i - 4j$ find a unit vector for \vec{V} ?

Sol.

$$u = \frac{V}{|V|} \quad ; \quad |u| = 1$$

$$|V| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$u = \frac{V}{|V|} = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$\therefore u$ is the unit vector such that:

$$|u| = \left| \frac{V}{|V|} \right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

Propositions:-

If $V_1 = a_1i + b_1j$ & $V_2 = a_2i + b_2j$ are two vectors :

1- $V_1 + V_2 = (a_1 + a_2)i + (b_1 + b_2)j$

$$V_2 + V_1 = (a_2 + a_1)i + (b_2 + b_1)j$$

Where $V_1 + V_2 = V_2 + V_1$

2- $V_1 - V_2 = (a_1 - a_2)i + (b_1 - b_2)j$

$$V_2 - V_1 = (a_2 - a_1)i + (b_2 - b_1)j$$

$$V_1 - V_2 = -(V_2 - V_1)$$

3- $cV_1 = ca_1i + cb_1j = c(a_1i + b_1j)$

4- $c(V_1 \pm V_2) = cV_1 \pm cV_2$

Ex. If $V_1 = 3i + 5j$ & $V_2 = 3i - 2j$, find ?

$$V_1 + V_2 = 6i + 3j$$

$$V_2 + V_1 =$$

$$V_1 + V_1 =$$

$$V_1 - V_2 =$$

$$3V_1 =$$

$$2(5V_1 - V_2) - V_1 =$$

Def.:- Two vectors $V_1 = a_1i + b_1j$ & $V_2 = a_2i + b_2j$ are

equal iff $V_1 = V_2 \Rightarrow a_1i + b_1j = a_2i + b_2j$

$$a_1 = a_2 \text{ and } b_1 = b_2$$

Ex. Show that $V_1 = 3i + 7j$ & $V_2 = \frac{15}{5}i + \left(\frac{12}{2} + 1\right)j$ are equal ?

Sol.

$$3 = \frac{15}{5} \quad \& \quad 7 = \frac{12}{2} + 1$$

$$\therefore V_1 = V_2$$

Ex. If P_1 is the point (1,3) and $P_2(2,-1)$ find $\overrightarrow{P_1P_2}$, $\overrightarrow{P_2P_1}$,

$5\overrightarrow{P_1P_2} - 3\overrightarrow{P_2P_1}$?(Homework).

Ex. Find $\overrightarrow{OP_3}$ if O is the origin point and P_3 is the mid point of the vector $\overrightarrow{P_1P_2}$ joining $P_1(2,-1)$ and $P_2(-4,3)$?

Sol.

$$P_3 = \frac{P_1 + P_2}{2}$$

$$P_1(2,-1) , P_2(-4,3)$$

$$P_3 = \left(\frac{2 - 4}{2} , \frac{-1 + 3}{2} \right)$$

$$P_3 = (-1,1) \text{ \& } O = (0,0)$$

$$\overrightarrow{OP_3} = (-1 - 0, 1 - 0) = (-1,1)$$

Ex. Find the vector from the point A (2,3) to the origin point?

Sol.

$$\overrightarrow{AO} = (0 - 2, 0 - 3) = (-2, -3)$$

Ex. Find the sum of the vector \overrightarrow{AB} and \overrightarrow{CD} , from the four given points A(1,-1), B(2,0), C(-1,3), D(-2,2)?

Sol.

$$\overrightarrow{AB} = (2 - 1, 0 + 1) = (1,1)$$

$$\overrightarrow{CD} = (-2 + 1, 2 - 3) = (-1, -1)$$

$$\overrightarrow{AB} + \overrightarrow{CD} = (1,1) + (-1, -1) = (0,0)$$

Ex. Find the length and direction of the following vectors:-

$$1- i + j \Rightarrow |i + j| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$2- \sqrt{3}i + j \Rightarrow$$

$$3- 2i - 3j \Rightarrow$$

$$4- 5i + 12j \Rightarrow$$

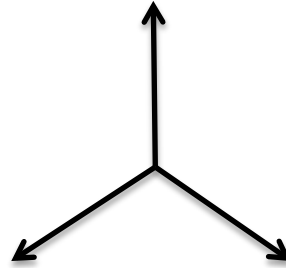
$$5- -2i + 3j \Rightarrow$$

Space Co-ordinates Vectors In Space

The vector in the 3D is $V = (a, b, c)$ or $V = (x, y, z)$

Let V be a vector from the origin point to the point $P(x, y, z)$.

$$\overrightarrow{OP} = \vec{V} = (x, y, z)$$



Ex. Locate (plot) the following points:-

$P_1(1, 2, 3), P_2(1, 2, -3), P_3(1, -2, 3), P_4(-1, 2, 3), P_5(1, -2, -3)$
 $, P_6(-1, 2, -3), P_7(-1, -2, 3), P_8(-1, -2, -3).$

The vector between two points in space

Let $P_1(x_1, y_1, z_1)$ & $P_2(x_2, y_2, z_2)$ be two points

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$\overrightarrow{P_2P_1} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$$

$$\overrightarrow{OP} = (x - 0, y - 0, z - 0) = (x, y, z)$$

$$\vec{V} = (a, b, c) \leftrightarrow V = ai + bj + ck$$

$$|V| = \sqrt{a^2 + b^2 + c^2}, \quad |\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$$

The Direct of Vector in Space

$$\text{dir}(V) = \frac{V}{|V|}$$

The Distant Between Two Point in Space

$$|\vec{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$V_1 = (x_1, y_1, z_1), \quad V_2 = (x_2, y_2, z_2),$$

Prove that $V_1 + V_2 = V_2 + V_1$

Sol.

$$1) \quad V_1 + V_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$= (x_2 + x_1, y_2 + y_1, z_2 + z_1)$$

$$= (x_2, y_2, z_2) + (x_1, y_1, z_1)$$

$$= V_2 + V_1$$

$$\therefore V_1 + V_2 = V_2 + V_1$$

$$2) \quad V_1 - V_2 = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$$

$$V_2 - V_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$3) \quad CV = (cx, cy, cz)$$

$$4) \quad \vec{V}_1 \parallel \vec{V}_2 \rightarrow \exists C \in \mathbb{R} \text{ s.t.} \quad \text{توازي متجهين}$$

$$\mathbf{V}_1 = \mathbf{C}\mathbf{V}_2 \quad \text{OR} \quad \mathbf{V}_2 = \mathbf{C}\mathbf{V}_1$$

$$\text{Such that } c = \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$$

$$5) \text{ If } \mathbf{V} = (a, b, c) = \mathbf{O} (0, 0, 0) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$\rightarrow a = b = c = 0$$

$$6) \vec{V}_1 \perp \vec{V}_2, \theta = \frac{\pi}{2} \& \vec{V}_1 \cdot \vec{V}_2 = 0 \quad \text{تعامد متجهين}$$

(Scalar) Dot Product

الضرب النقطي للمتجهات

$$\mathbf{A} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k} \quad \& \quad \mathbf{B} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$$

$$\mathbf{A} \cdot \mathbf{B} = a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$$

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| \cdot |\mathbf{B}| \cos \theta$$

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|}$$

$$\text{(Ex.) } \mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} \quad \& \quad \mathbf{B} = 8\mathbf{i} - 12\mathbf{j} + 20\mathbf{k}, \text{ find } \mathbf{A} \cdot \mathbf{B} ?$$

Sol. :

$$\mathbf{A} \cdot \mathbf{B} = 2(8) + (-3)(-12) + 5(20)$$

$$= 16 + 36 + 100 = 152$$

$$\text{(Ex.) } \mathbf{A} = a\mathbf{i} - \mathbf{j} + \mathbf{k} \quad \& \quad \mathbf{B} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad \& \quad \mathbf{A} \cdot \mathbf{B} = 3, \text{ find } a = ?$$

Sol :

$$\mathbf{A} \cdot \mathbf{B} = a(1) + (-1)(1) + 1(2)$$

$$3 = a + 1$$

$$\therefore a = 2$$

(Ex.) If A and B are unit vectors and $\theta = 60^\circ$ find $A \cdot B = ??$

Sol .

$$|A| = 1, |B| = 1, \cos \theta = \frac{1}{2}$$

$$A \cdot B = |A| |B| \cos \theta = 1 (1) \left(\frac{1}{2}\right) = \frac{1}{2}$$

Propositions

$$1) A \cdot B = B \cdot A \quad \text{s.t } A = a_1i + b_1j + c_1k, B = a_2i + b_2j + c_2k$$

$$\text{Pf : } A \cdot B = a_1a_2 + b_1b_2 + c_1c_2$$

$$= a_2a_1 + b_2b_1 + c_2c_1$$

$$= B \cdot A$$

Another proof : *برهان اخر*

$$A \cdot B = |A| |B| \cos \theta \quad \left[\cos \theta = \cos (-\theta) \right]$$

لانه الـ Cos دالة زوجية

$$B \cdot A = |B| |A| \cos(-\theta)$$

$$= |A| |B| \cos(\theta)$$

$$2) i \cdot i = 1, j \cdot j = 1, k \cdot k = 1$$

$$\text{pf. } i \cdot i = (1,0,0) \cdot (1,0,0)$$

$$= 1(1) + 0(0) + 0(0) = 1$$

$$j \cdot j = (0,1,0) \cdot (0,1,0)$$

$$= 0(0) + 1(1) + 0(0) = 1$$

$$k \cdot k = (0,0,1) \cdot (0,0,1)$$

$$= 0(0) + 0(0) + 1(1) = 1$$

$$3) A \cdot A = |A|^2$$

$$\text{Pf. : } A = (a,b) = ai + bj \quad \text{Or } A = (a,b,c) = ai + bj + ck$$

$$A \cdot A = ?? \quad |A|^2 = ??$$

$$A \cdot A = (ai + bj) \cdot (ai + bj)$$

$$= a(a) + b(b)$$

$$= a^2 + b^2$$

$$|A| = \sqrt{a^2 + b^2}$$

$$|A|^2 = a^2 + b^2$$

$$\therefore A \cdot A = |A|^2$$

$$4) A(B+C) = A \cdot B + A \cdot C$$

$$A = (a_1, a_2, a_3), B = (b_1, b_2, b_3) \quad \& \quad C = (c_1, c_2, c_3)$$

$$\text{Pf. : } (B + C) = (b_1 + c_1, b_2 + c_2, b_3 + c_3)$$

$$A \cdot (B + C) = (a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3))$$

$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$

$$A \cdot B + A \cdot C = a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2 + a_3b_3 + a_3c_3$$

$$(5) (B + C) \cdot A = B \cdot A + C \cdot A$$

$$(6) A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

$$(7) i \cdot j = j \cdot k = k \cdot i = 0$$

(Ex.) If $A = i + j$, $B = i$ find $A \cdot B = ?$ $\theta = ??$

$$A \cdot B = 1(1) + 1(0) = 1 \quad |A| = \sqrt{2}, \quad |B| = 1$$

$$\cos \theta = \frac{A \cdot B}{|A||B|} = \frac{1}{\sqrt{2}\sqrt{1}} = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \frac{\pi}{4}$$

(Ex.) If $A = i + j - k$, $B = j - k$ & $C = 2i$ compute $A \cdot (B \cdot C)$?

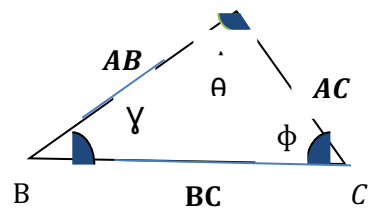
$$\text{Sol.} \quad B \cdot C = 1(2) + (-1)(0) = 2$$

$$A \cdot (B \cdot C) = 2(i + j - k)$$

$$= 2i + 2j - 2k$$

(Ex.) Find the angle of the triangle that $A(-1, 0, 2)$, $B(2, 1, -1)$, $C(1, -2, 2)$ are vertices of it ??

$$\text{Sol.} \quad \cos \theta = \frac{AB \cdot AC}{|AB||AC|}$$



$$AB = (2 - (-1), 1 - 0, -1 - 2)$$

$$\text{متجه بين نقطتين } AB = (3, 1, -3)$$

$$AC = (1 - (-1), -2 - 0, 2 - 2)$$

$$AC = (2, -2, 0)$$

$$AB \cdot AC = 3(2) + 1(-2) + (-3)(0) = 4$$

$$|AB| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19}$$

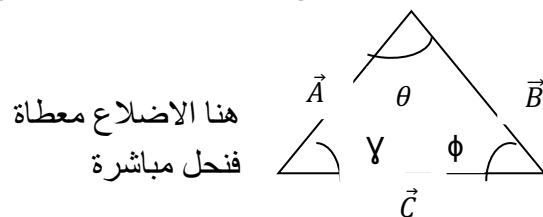
$$|AC| = \sqrt{2^2 + (-2)^2 + (-3)^2} = \sqrt{8}$$

$$\cos \theta = \frac{4}{\sqrt{19}\sqrt{8}} \quad \rightarrow \quad \theta = \cos^{-1} \frac{4}{\sqrt{76}}$$

$$\cos \gamma = \frac{AB \cdot BC}{|AB||AC|}, \quad \cos \phi = \frac{AC \cdot BC}{|AC||BC|}$$

Ex. : If $\vec{A} = i + j$ & $\vec{B} = k$, $\vec{C} = 2j + k$ Find the angles of ΔABC

Sol : (Homework)



$$\cos \theta = \frac{A \cdot B}{|A||B|} =$$

$$\cos \gamma = \frac{A \cdot C}{|A||C|} =$$

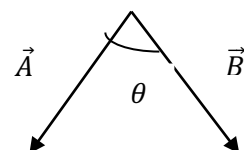
$$\cos \phi = \frac{B \cdot C}{|B||C|} =$$

(Ex.): Let $\vec{A} = 2i - j + k$, $\vec{B} = i + j + 2k$ Find θ between \vec{A} and \vec{B} ?

Sol: $A \cdot B = 2(1) + (-1)(1) + 1(2)$

$$= 2 - 1 + 2 = 3$$

$$|A| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$



$$|B| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\cos \theta = \frac{A \cdot B}{|A||B|}$$

$$= \frac{3}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{2}\right) \rightarrow \therefore \theta = 60^\circ = \frac{\pi}{3}$$

(Ex.): Let $\vec{A} = 2i + j - k$, $\vec{B} = 4j - k$ find $A \cdot B$?

$$\text{Sol.:} - A \cdot B = 2(0) + 1(4) + (-1)(-1) = 4 + 1 = 5$$

(Ex.): If $\vec{A} = 2i + aj + k$, $\vec{B} = 2i + j - 2k$; and $A \perp B$ find a ?

$$\text{Sol.:} - A \perp B \rightarrow \theta = \frac{\pi}{2} \rightarrow \cos \theta = 0$$

$$A \cdot B = |A| |B| \cos \theta = 0$$

$$A \cdot B = 2(2) + a(1) + 1(-2) = 0$$

$$4 + a - 2 = 0 \rightarrow a = -2$$

$$\therefore A = 2i - 2j + k \text{ \& } B = 2i + j - 2k$$

$$A \perp B = A \cdot B = 0$$

الضرب الاتجاهي

Vector Product

الضرب الاتجاهي بين متجهتين هو ناتج المحدد التالي :

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad \begin{cases} \vec{A} = a_1 i + b_1 j + c_1 k \\ \vec{B} = a_2 i + b_2 j + c_2 k \end{cases}$$

$$= i \begin{vmatrix} b1 & c1 \\ b2 & c2 \end{vmatrix} - j \begin{vmatrix} a1 & c1 \\ a2 & c2 \end{vmatrix} + k \begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$$

$$= i (b1 c2 - c1 b2) - j (a2 c2 - a2 c1) + k (a1 b2 - b1 a2)$$

(Ex.) : $\vec{A} = 3i + 7j - k$, $\vec{B} = i + j + 3k$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 3 & 7 & -1 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= i(7(3) - (-1)(1)) - j(3(3) - (-1)(+1)) + k(3(1) - 7(+1))$$

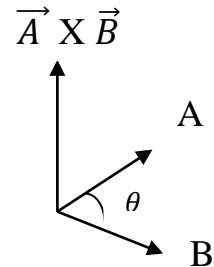
$$= 22i - 10j - 4k$$

تعريف آخر للضرب الاتجاهي

A nother Diffenition for $\vec{A} \times \vec{B}$:-

$$\vec{A} \times \vec{B} = \vec{n} |A| |B| \sin \theta$$

$$\vec{n} = \frac{\vec{A} \times \vec{B}}{|A||B|\sin \theta}$$



Unit Vector متجه وحدة

(Ex) Find the unit vector \vec{n} If $\vec{A} = i + j$ & $\vec{B} = j - k$??

Sol. $\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$

$$= i \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= i (1(-1) - 0(1)) - j (1(-1) - 0(0)) + k (1(1) - 1(0))$$

$$\vec{A} \times \vec{B} = -i + j + k$$

$$A \cdot B = 1(0) + 1(+1) + 0(-1) = 1$$

$$\cos \theta = \frac{A \cdot B}{|A| |B|} = \frac{1}{\sqrt{2} \sqrt{2}} = \frac{1}{2}$$

$$|A| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$|B| = \sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}$$

$$\therefore \theta = \cos^{-1} \left(\frac{1}{2} \right) \rightarrow \theta = \frac{\pi}{3} \quad \sin \theta = \frac{\sqrt{3}}{2}$$

$$\begin{aligned} \vec{n} &= \frac{\vec{A} \times \vec{B}}{|A||B|\sin \theta} \\ &= \frac{(-i+j+k)}{\sqrt{2} \sqrt{2} \frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k \end{aligned}$$

$$\begin{aligned} |\vec{n}| &= \sqrt{\left(\frac{-1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} \\ &= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{3}{3}} = 1 \end{aligned}$$

$\therefore \vec{n}$ is unit vector

(Ex.) If $\vec{A} = i + 4j + k$, $B = -2i + j - 2k$? Is $A \perp B$?

Sol. $A \cdot B$? (If $A \cdot B = 0$ Then $A \perp B$)

$$= 1(-2) + 4(1) + 1(-2)$$

$$= -2 + 4 - 2 = 0$$

$$\therefore \vec{A} \cdot \vec{B} = 0 \rightarrow \text{i.e. } \vec{A} \perp \vec{B}$$

Remark : If $\vec{A} \parallel \vec{B} \rightarrow \theta = \text{Zero} = 0^\circ$

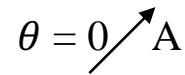
$$\sin 0 = 0 \rightarrow \therefore \vec{A} \times \vec{B} = \text{Zero} \rightarrow \vec{A} \times \vec{B} = \vec{n} |\vec{A}| |\vec{B}| \sin \theta = 0$$

Propositions :-

$$1) \vec{A} \times \vec{B} = \vec{B} \times \vec{A}$$

$$2) \vec{A} \times (\vec{C} + \vec{D}) = (\vec{A} \times \vec{C}) + (\vec{A} \times \vec{D})$$

$$3) \vec{A} \times \vec{A} = 0 \quad \text{Since } \theta = 0$$

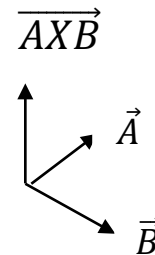


$$4) \vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

$$5) \vec{A} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{B} \times \vec{A})$$

$$= \text{Zero}$$

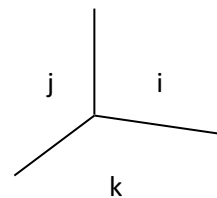
عمودي



Since $\vec{A} \times \vec{B}$ are $\left[\begin{array}{c} \text{Normal} \\ \text{Vertica} \\ \text{perpendicular} \end{array} \right]$ on \vec{A} and on \vec{B}

$$6) \vec{i} \times \vec{j} = \vec{k} \quad \& \quad \vec{j} \times \vec{k} = \vec{i} \quad \& \quad \vec{k} \times \vec{i} = \vec{j}$$

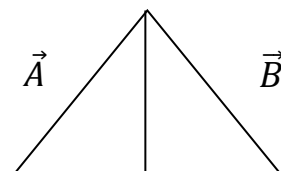
because $\vec{i} \perp \vec{j} \perp \vec{k}$



7) Area of triangle A B C is

$$= \frac{1}{2} |\vec{A} \times \vec{B}|$$

الاضلاع $\vec{C}, \vec{B}, \vec{A}$



\vec{c}

(Ex.) Let $\vec{A} = i+j$ & $\vec{B} = i+k$ Find the Area of a triangle ABC

Where \vec{A} , \vec{B} are two sides of it ??

Sol. Area = $\frac{1}{2} |\vec{A} \times \vec{B}|$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= i - j - k$$

$$|\vec{A} \times \vec{B}| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}$$

$$\therefore \text{Area} = \frac{1}{2} \sqrt{3} = \frac{\sqrt{3}}{2}$$

(Ex.) : Find the Area of triangle ABC where are vertices

A (1, -1 , 0) , B (2, 1, -1) , (-1, 1, 2)

ملاحظه :- يُفضل للرؤوس احرف كبيرة والمتجهات احرف صغيرة كذلك الرؤوس

نأخذها بشكل احداثيات والمتجهات نأخذها بشكل مجموع ضرب

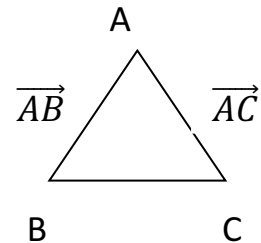
$$A (a, b, c) \quad \& \quad \vec{V} = ai+bj +ck$$

نقطة

متجه

(لكي تُميز بينهما)

Sol: $\vec{AB} = (2 - 1 , 1 - (-1), (-1) - 0)$



$$= (1, 2, -1)$$

$$\overrightarrow{AB} = i+2j+k$$

$$\overrightarrow{AC} = (-1-1, 1-(-1), 2-0)$$

$$= (-2, 2, 2)$$

$$\overrightarrow{AC} = -2i+2j+2k$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= i \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}$$

$$= i(4+2) - j(2-2) + k(2+4)$$

$$= 6i+6k$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{6^2 + 0^2 + 6^2}$$

$$= \sqrt{36 + 36} = \sqrt{72}$$

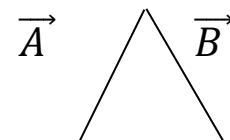
$$\begin{vmatrix} 2 & 72 \\ 6 & 36 \\ 6 & 6 \\ & & 1 \end{vmatrix}$$

$$\therefore \text{Area} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2} \sqrt{72} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

(Ex.) Find Area of triangle where $\vec{A} = i + j$ & $\vec{B} = i + k$ are two sides of it?(Homework)

$$\underline{\text{Sol.}} \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$



$$\therefore \vec{A} \times \vec{B} = \underline{\hspace{2cm}}$$

$$|\vec{A} \times \vec{B}| = \sqrt{\hspace{2cm}}$$

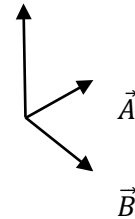
$$\text{Area} = \frac{1}{2} |\vec{A} \times \vec{B}| = \frac{1}{2} (\hspace{2cm})$$

Orthogonal = vertical = Normal = Perpendicular

تعني هذه الكلمات مرادفات كلمة عمودي

(Ex .) Find the normal Vector from $\vec{N} = \vec{A} \times \vec{B} = ?$ Where

$$\vec{A} = (2, 1, -1) \quad \& \quad \vec{B} = (-2, 4, -1) \quad \vec{N} = \vec{A} \times \vec{B}$$



$$\vec{A} = 2i + j - k \quad \vec{B} = -2i + 4j - k$$

Sol.
$$\vec{N} = \vec{A} \times \vec{B} \begin{matrix} i & j & k \\ 2 & 1 & -1 \\ -2 & 4 & -1 \end{matrix} =$$

Direction (Homework)

(Ex) find the direction of \vec{A} and \vec{B} , $\vec{A} = i$, $\vec{B} = j$?

هنا كلمة Dir unit vector, نفس المعنى اي الاتجاه Dir هو متجه الوحدة المطلوب.

Sol.
$$\text{Dir} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} \quad ; \quad \vec{N} = \vec{A} \times \vec{B}$$

$$\text{Dir} = \vec{u} = \frac{\vec{N}}{|\vec{N}|}$$

$$\text{Dir} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i(0) + j(0) + k(1) \\ = \vec{k}$$

$$|\vec{A} \times \vec{B}| = \sqrt{0^2 + 0^2 + 1^2} = (1)$$

$$\therefore \vec{u} = \text{Dir} = \frac{\vec{k}}{1} = \vec{k}$$

$$\text{Where } |\vec{k}| = 1$$

هنا متجهة الوحدة الناتج او (*Dir*) الاتجاه بين المتجهتين نتج المتجه \vec{k} والذي هو متجه وحده وطول (1) و $(0,0,1)$ وهذا تطبيق للخاصية $i \times j = k$

(Ex) : Find the Dir (direction) between $\vec{A} = 2i-2j+k$ &

$$\vec{B} = i + j + k$$

Sol. (Homework)

(Ex) : If $\vec{V} = 2i - j$ & $\vec{w} = i + 3j - 2k, \vec{u} = 3i-4j+2k$??

Compute أحسب

$$(1) (\vec{V} \times \vec{W}) \cdot \vec{U} = ?? \text{ (Homework)}$$

$$\text{Sol(1)} \vec{V} \times \vec{W} = ?$$

$$(\vec{V} \times \vec{W}) \cdot \vec{U} = ?$$

$$(2) (\vec{W} \times \vec{U}) \cdot \vec{V}$$

$$\text{Sol(2)} \vec{W} \times \vec{U} = ?$$

$$(\vec{W} \times \vec{U}) \cdot \vec{V} = ?$$

$$(3) (\vec{U} \times \vec{V}) \cdot \vec{W}$$

$$\text{Sol(2)} \vec{U} \times \vec{V} = ?$$

$$(\vec{U} \times \vec{V}) \cdot \vec{W} = ?$$

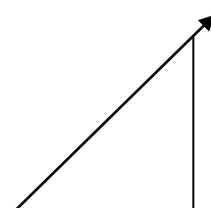
Ex.: If $\vec{A} = 4i - 8j + k$ & $\vec{B} = 2i + j - 2k$

هل تحقق المساواة $(\vec{A} \times \vec{B}) \cdot \vec{A} = \vec{B} \cdot (\vec{A} \times \vec{B})$??

Vector projection

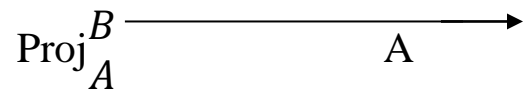
الاسقاط الاتجاهي

وهو يعرف حسب القيمة التالية



$$\text{Proj}_A^B = \frac{B \cdot A}{A \cdot A} \cdot \vec{A}$$

B



Praposition:

$$|A|^2 = A \cdot A ; \vec{A} = ai + bj + ck$$

Proof: $A \cdot A = a(a) + b(b) + c(c)$

$$= a^2 + b^2 + c^2 \dots\dots\dots*$$

$$|A| = \sqrt{a^2 + b^2 + c^2}$$

$$|A|^2 = a^2 + b^2 + c^2 \dots\dots\dots*$$

$$\therefore A \cdot A = |A|^2$$

$$\therefore \text{Proj}_A^B = \frac{B \cdot A}{|A|^2} \cdot \vec{A}$$

Scalar projection

الاسقاط الدوري

هو يمثل طول الاسقاط الاتجاه اي:-

$$\left[\begin{array}{l} sc.pro. \\ for \vec{B} on \vec{A} \end{array} \right] = |Proj_A^B|$$

Ex:- a) Find vektor projection for \vec{B} on \vec{A} and scalar projection of it ?

b) Find vector projection for \vec{B} on \vec{A} and scalar projection $|Proj_A^B| = ?$

Where $\vec{A} = 10i + 11j - 2k$

$$\vec{B} = 2i + 3j + 4k$$

Sol :- a) $|Proj_A^B| = ?$

$$\& Proj_A^B = \frac{B \cdot A}{|A|^2} \vec{A} = ?$$

$$b) Proj_B^A = \frac{A \cdot B}{|B|^2} \vec{B} =$$

$$\& |Proj_B^A| = ?$$

Equation of sphere

معادلة الكرة

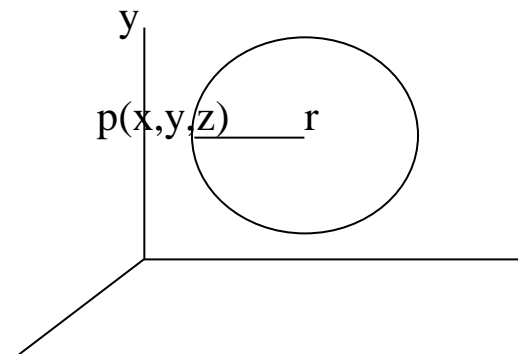
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

حيث ان $C(z_0, y_0, x_0)$ نقطة مركز الكرة و r نصف القطر حيث p اية نقطة

على محيط الكرة $p(x, y, z)$

$$r = |p_0 p|$$

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$



Ex.:- find the center c and the radius r for the following sphere

$$(x-3)^2 + (y+7)^2 + (z-5)^2 = 4 ?$$

Sol. :- c (3, -7, 5), r = 2

Exc:- find c = 2 and r = 2 where $x^2 + (y-2)^2 + (z+2)^2 = 36$

Sol.:- (Homework)

Ex.:- find the center and radius from the following sphere equation $x^2 + y^2 + z^2 + 4x - 3z = 4$?

Sol.:-

$$x^2 + y^2 + z^2 + 4x - 3z = 4$$

لاكمال المربع نضيف (1/2) معامل x

$$x^2 + 4x + 4 + y^2 + z^2 - 3z + (1/2(3))^2 = 4 + 4 + \frac{9}{4}$$

$$(x+2)^2 + y^2 + (z - \frac{3}{2})^2 = \frac{16+16+4}{4}$$

$$\therefore C(-2, 0, \frac{3}{2}), r = \frac{\sqrt{41}}{2}$$

Ex.:- find the center and radius from sphere eq.

$$x^2 + y^2 - 4x + z^2 + 3z = 4 ? \text{ (Homework)}$$

Ex.:- find the center and radius from sphere eq.

$$x^2 + y^2 + z^2 - 2az = 0 ?$$

$$\text{Sol :- } (x-0)^2 + (y-0)^2 + z^2 - 2az + (\frac{-2a}{2})^2 = a^2$$

$$(x-0)^2 + (y-0)^2 + (z-a)^2 = a^2$$

$$C(0,0,a) \quad r = a$$

Ex.:- find the center and radius from $x^2 - 4x + y^2 + 2ay = z^2$?

(Homework).

معادلة المستقيم

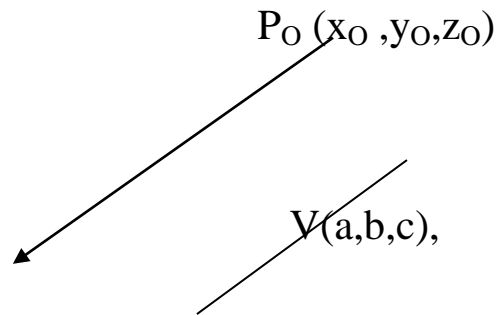
Equation of the line

$$V(a,b,c), P_0(x_0, y_0, z_0)$$

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$



هذه المعادلات لوسيطه للمستقيم

$$t = \frac{X-X_0}{a} = \frac{Y-Y_0}{b} = \frac{Z-Z_0}{c}$$

هذه معادلة المستقيم

اشتقاق القانون اعلاه حسب :-

هنا $L = P_0P$ حيث P اي نقطة على L

$$P(x, y, z), P_0(x_0, y_0, z_0)$$

ويكون L موازي للمتجه $V(a, b, c)$

$$V \parallel PP_0 \rightarrow PP_0 = tV$$

$$(x-x_0)i + (y-y_0)j + (z-z_0)k = t(ai+bj+ck)$$

$$\rightarrow at = x - x_0, bt = y - y_0, ct = z - z_0$$

EX:- Find the equation of line L, which pass through $P_0(1,2,3)$
And Parallel to the line $\overleftrightarrow{V}(4,7,-1) = \mathbf{v} = 4\mathbf{i} + 7\mathbf{j} - \mathbf{k}$??

$$\text{Sol:- } L = t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$x = x_0 + at \quad P_0(x_0, y_0, z_0)$$

$$y = y_0 + bt \quad \mathbf{V}(a, b, c)$$

$$z = z_0 + ct$$

$$L = \frac{x-1}{4} = \frac{y-2}{7} = \frac{z-4}{-1}$$

$$x = 1 + t(4) \quad x = 1 + 4t$$

$$y = 2 + t(7) \quad y = 2 + 7t$$

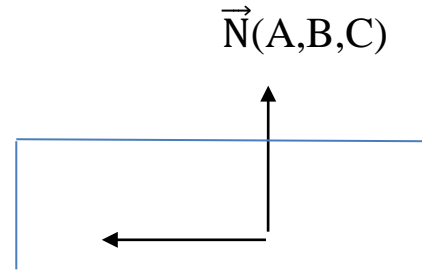
$$z = 4 + t(-1) \quad z = 4 - t$$

Ex.:- find line equation for $p_0(0, 1, -2)$ $\mathbf{v} = (7, 6, 2)$?(Homework)

$$Ax + By + Cz = D \quad \text{معادلة المستوي} \quad D = Ax_0 + By_0 + Cz_0$$

$$P_0(x_0, y_0, z_0)$$

$$\vec{N}(A, B, C) \quad \&$$



$$p(x, y, z) \quad P_0(x_0, y_0, z_0)$$

اشتقاق القانون :- هنا لدينا نقطة $P_0(x_0, y_0, z_0)$

ومستقيم N عمودي على المستوى على P_0 لذا P اية نقطة المستوى المتكون منها حيث سيكون التعامد $N \perp PP_0$ هنا $N \cdot PP_0 = 0$ اي التعامد يعني

$$\cos \frac{\pi}{2} = 0 \quad \phi = \frac{\pi}{2}$$

$$\vec{PP_0} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

$$0 = \vec{N} \cdot \vec{PP_0} = A(x - x_0) + B(y - y_0) + C(z - z_0)$$

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$

$$\therefore Ax + By + Cz = D$$

Ex :- find plan equation for $p_0(1, 4, 0)$ and $\vec{N} = 4\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$?

$p_0(x_0, y_0, z_0)$ and $\vec{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$??

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0 = D$$

$$4X + 7Y - 2Z = 4(1) + 7(4) + (-2)(0)$$

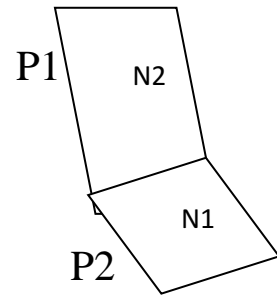
$$\underline{4X + 7Y + 2Z = 32}$$

EXE : find plane equation for \vec{N} (3,5,7) and $p_0(-3,4,5)$??

تعامد مستويات

$$P_1 \perp P_2 \text{ ----- } N_2 \perp N_1$$

$$N_2 \cdot N_1 = 0$$



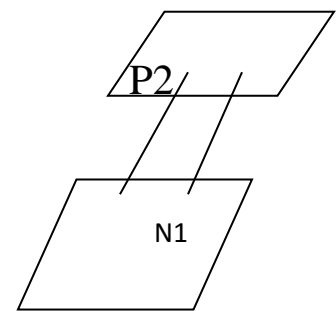
حيث p_1 & p_2 مستويات و \vec{N}_1 , \vec{N}_2 المستقيمت العموديات عليها على التوالي

$$\vec{N}_1 (A_1, B_1, C_1) \text{ \& } \vec{N}_2 (A_2, B_2, C_2)$$

توازي المستويات :-

$$P_1 \parallel P_2 \rightarrow \vec{N}_2 \parallel \vec{N}_1 \rightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

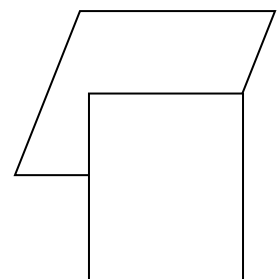
$$\vec{N}_1 (A_1, B_1, C_1) \text{ \& } \vec{N}_2 (A_2, B_2, C_2)$$



تقاطع المستويات

$$P_1 \cap P_2 \rightarrow \vec{N}_2 \cap \vec{N}_1 = \theta$$

$$\cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1| |\vec{N}_2|}$$



المسافة من نقطة الى مستوى حيث P مستوى Q نقطة

$$|PQ| = \sqrt{(X_1 - X_0)^2 + (Y_1 - Y_0)^2 + (Z_1 - Z_0)^2}$$

Q (X ₀ , Y ₀ , Z ₀)	P(X ₁ , Y ₁ , Z ₁)
---	--

حيث p مستوى و Q نقطة خارج المستوى و P1 اية نقطة داخل المستوى فتكون المسافة من المستوى P الى النقطة Q هي نفسها المسافة من النقطة Q الى اية نقطة P1 داخل المستوى .

CYLINDRICAL COORDINATES ↔ CARTESIAN COORDINATES

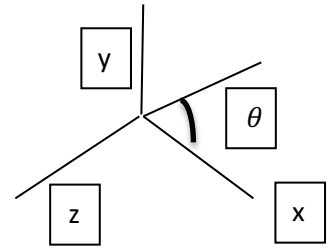
$$(r, \theta, z) \rightarrow (x, y, z)$$

$$\tan \theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$



$$\text{Ex. :- } (r, \theta, z) = (1, \frac{\pi}{2}, 1) \rightarrow (x, y, z) \rightarrow x = \cos \frac{\pi}{2} = 0$$

$$Y = 1 \sin \frac{\pi}{2} = 1$$

$$Z = 1$$

$$\therefore (x, y, z) = (0, 1, 1)$$

$$\text{Ex. :- } (x, y, z) = (1, 0, 1) \text{ find } (r, \theta, z)?$$

$$\text{Sol.:- } r = \sqrt{x^2 + y^2} = 1$$

$$\text{Tan } \theta = \frac{0}{1} = 0$$

$$\theta = 0$$

$$\therefore (r, \theta, z) = (1, 0, 1)$$

Chapter four

Partial Differential

Partial Differential

1- Functions of two and more variables

- $y = f(x)$ Two variables x and y .

Ex1. $y = 3x^2$

$$y = f(x) = 3x^2$$

$$\frac{dy}{dx} = 3(2x) = 6x$$

- $Z = F(x, y)$ Three variables x , y and z

Ex2. $z = 3x^2 + y^2$

$$\frac{\partial z}{\partial x} = 3(2x) + 0$$

$$\frac{\partial z}{\partial x} = 6x$$

$$\frac{\partial z}{\partial y} = 2y$$

- $W = f(x, y, z)$ Four variables x , y , z and w

⋮

- $W = f(x_1, x_2, x_3, \dots, x_n)$

$$\text{Ex3. } W = \frac{x^2 - z^2}{y^2}$$

There are two ways to solve that example:-

First way

$$\frac{\partial w}{\partial x} = \frac{(2x - 0) \cdot y^2 - (x^2 - z^2) \cdot (0)}{(y^2)^2}$$

$$\frac{\partial w}{\partial x} = \frac{2xy^2}{y^4} = \frac{2x}{y^2}$$

Second way

$$W = \frac{1}{y^2} (x^2 - z^2)$$

$$W = \frac{x^2}{y^2} - \frac{z^2}{y^2}$$

$$\frac{\partial w}{\partial x} = \frac{1}{y^2} (2x) - 0 = \frac{2x}{y^2}$$

$$\frac{\partial w}{\partial y} = \frac{(0) \cdot y^2 - 2y \cdot (x^2 - z^2)}{(y^2)^2}$$

$$= \frac{-2y(x^2 - z^2)}{y^4} = \frac{-2(x^2 - z^2)}{y^3}$$

$$\frac{\partial w}{\partial z} = \frac{(-2z) \cdot y^2 - 0}{y^4} = \frac{-2z}{y^2}$$

$$\text{Ex4. } W = e^x \text{Ln}(x^2 + y^2 + 1)$$

$$\frac{\partial w}{\partial x} = e^x \cdot \frac{1}{x^2 + y^2 + 1} \cdot (2x) + e^x \text{Ln}(x^2 + y^2 + 1)$$

$$\frac{\partial w}{\partial y} = e^x \cdot \frac{1}{x^2 + y^2 + 1} \cdot (2y)$$

$$\text{Ex5. } z = f(x, y) = \text{Ln} \sqrt{x^2 + y^2}$$

$$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \cdot 2x$$

$$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial w}{\partial y} = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2y = \frac{y}{x^2 + y^2}$$

$$\text{Ex6. } W = \frac{(x^2 + y^2)}{(u^2 + v^2)}$$

$$\frac{\partial w}{\partial x} = \frac{2x(u^2 + v^2) - (0)(x^2 + y^2)}{(u^2 + v^2)^2} = \frac{2x(u^2 + v^2)}{(u^2 + v^2)^2}$$

$$= \frac{2x}{(u^2 + v^2)}$$

$$\frac{\partial w}{\partial y} = \frac{2y(u^2 + v^2) - (0)(x^2 + y^2)}{(u^2 + v^2)^2} = \frac{2y}{(u^2 + v^2)}$$

$$\frac{\partial w}{\partial u} = \frac{(0)(u^2 + v^2) - 2u(x^2 + y^2)}{(u^2 + v^2)^2} = -\frac{2u(x^2 + y^2)}{(u^2 + v^2)^2}$$

$$\frac{\partial w}{\partial v} = \frac{(0)(u^2 + v^2) - 2v(x^2 + y^2)}{(u^2 + v^2)^2} = -\frac{2v(x^2 + y^2)}{(u^2 + v^2)^2}$$

Ex7. Find all Partial Differential of the following blow :-

$$1. Z = \frac{(u^2+v^2)}{2x}$$

$$2. f(x, y) = x^2 - y$$

$$3. f(x, y) \frac{3y}{x}$$

$$4. W = \frac{z+x}{y}$$

Differential by definition

$Z = f(x, y)$ and the partial differential of this function are:

Z_x , Z_y , f_x and f_y .

$$f_x = \frac{\partial f}{\partial x} = \lim_{Dx \rightarrow 0} \frac{f(x_0 + Dx, y_0) - f(x_0, y_0)}{Dx}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{Dy \rightarrow 0} \frac{f(x_0, y_0 + Dy) - f(x_0, y_0)}{Dy}$$

Ex. Find f_x and f_y for the following function:-

$$f(x, y) = 100 - x^2 - y^2$$

Sol.

$$f_x = \frac{\partial f}{\partial x} = \lim_{Dx \rightarrow 0} \frac{f(x_0 + Dx, y_0) - f(x_0, y_0)}{Dx}$$

$$f(x_0, y_0) = 100 - x_0^2 - y_0^2$$

$$f(x_0 + Dx, y_0) = 100 - (x_0 + Dx)^2 - y_0^2$$

$$f_x = \frac{\partial f}{\partial x} = \lim_{Dx \rightarrow 0} \frac{100 - (x_0 + Dx)^2 - y_0^2 - (100 - x_0^2 - y_0^2)}{Dx}$$

$$= \lim_{Dx \rightarrow 0} \frac{100 - x_0^2 - 2x_0Dx - Dx^2 - y_0^2 - 100 + x_0^2 + y_0^2}{Dx}$$

$$= \lim_{Dx \rightarrow 0} \frac{-2x_0Dx - Dx^2}{Dx} = \lim_{Dx \rightarrow 0} \frac{(-2x_0 - Dx)Dx}{Dx}$$

$$= \lim_{Dx \rightarrow 0} (-2x_0 - Dx) = -2x_0 - 0 = -2x_0$$

$$f_x = \frac{\partial f}{\partial x} = -2x_0$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{Dy \rightarrow 0} \frac{f(x_0, y_0 + Dy) - f(x_0, y_0)}{Dy}$$

$$f(x_0, y_0 + Dy) = 100 - x_0^2 - (y_0 + Dy)^2$$

$$\begin{aligned}
f_y &= \frac{\partial f}{\partial y} = \lim_{Dy \rightarrow 0} \frac{100 - x_0^2 - (y_0 + Dy)^2 - (100 - x_0^2 - y_0^2)}{Dy} \\
&= \lim_{Dy \rightarrow 0} \frac{100 - x_0^2 - y_0^2 - 2y_0Dy - Dy^2 - 100 + x_0^2 + y_0^2}{Dy} \\
&= \lim_{Dy \rightarrow 0} \frac{-2y_0Dy - Dy^2}{Dy} = \lim_{Dy \rightarrow 0} \frac{(-2y_0 - Dy)Dy}{Dy} \\
&= \lim_{Dy \rightarrow 0} (-2y_0 - Dy) = -2y_0 - 0 = -2y_0
\end{aligned}$$

$$f_y = \frac{\partial f}{\partial y} = -2y_0$$

Ex. Find f_x and f_y for the function $f(x, y) = x + 3y$ by using definition?(Homework).

The Chain Rule

$$1. w = f(x, y), \quad x = x(t) \quad \& \quad y = y(t)$$

$$\frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \frac{dx}{dt}, \quad \frac{dy}{dt}$$

$$\therefore \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

Ex. Find $\frac{dw}{dt}$ for the function:-

$$w = x^2 + 2y^2, \quad x = \sin t \quad \text{and} \quad y = \cos t$$

$$\frac{\partial w}{\partial x} = 2x + 0, \quad \frac{dx}{dt} = \cos t$$

$$\frac{\partial w}{\partial y} = 0 + 4y, \quad \frac{dy}{dt} = -\sin t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2x \cos t - 4y \sin t$$

$$= 2 \sin t \cos t - 4 \cos t \sin t$$

$$\frac{dw}{dt} = -2 \sin t \cos t$$

2. $w = f(x, y)$, $x = x(t, z)$ & $y = y(t, z)$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial z}$$

Ex.

$$z = f(v, w), \quad v = v(n, u) \text{ \& } w = w(n, u)$$

$$z = vw, \quad v = un, \quad w = n^2 + u^2$$

$$\frac{\partial z}{\partial n} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial n} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial n}$$

$$= wu + 2vn$$

$$\frac{\partial z}{\partial n} = (n^2 + u^2)u + 2un^2$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial u} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial u}$$

$$= wn + 2vu$$

$$\frac{\partial z}{\partial u} = (n^2 + u^2)u + 2nu^2$$

Ex. Find $\frac{dz}{dt}$ for the functions:- (Homework)

1. $z = f(x, y)$, $x = x(t)$ and $y = y(t)$

2. $z = \sin(x \cdot y)$, $x = \frac{1}{t}$ and $y = 3t$

Ex. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial w}$ for the functions:- (Homework)

$$z = f(x, y) = y \sin x \text{ , } x = w - u^2 \text{ and } y = \frac{w}{u}$$

The gradate & vector differential

We denoted for gradate function $f(x, y, z)$ at the point $p_0(x_0, y_0, z_0)$ by:-

$$\Delta f|_{p_0} = \frac{\partial f}{\partial x}|_{p_0} \cdot i + \frac{\partial f}{\partial y}|_{p_0} \cdot j + \frac{\partial f}{\partial z}|_{p_0} \cdot k$$

Ex. Find the gradate for $f(x, y, z)$ where

$$f(x, y, z) = x^3 - xy^2 - z \text{ at } p_0(1,1,0)?$$

Sol.

$$\Delta f|_{p_0} = \frac{\partial f}{\partial x}|_{p_0} \cdot i + \frac{\partial f}{\partial y}|_{p_0} \cdot j + \frac{\partial f}{\partial z}|_{p_0} \cdot k$$

$$\frac{\partial f}{\partial x} = 3x^2 - y^2 \text{ , } \quad \frac{\partial f}{\partial y} = -2xy \text{ , } \quad \frac{\partial f}{\partial z} = -1$$

$$\frac{\partial f}{\partial x}|_{p_0(1,1,0)} = 3 \cdot (1)^2 - (1)^2 = 2$$

$$\frac{\partial f}{\partial y}|_{p_0(1,1,0)} = -2 \cdot (1) \cdot (1) = -2$$

$$\frac{\partial f}{\partial z}|_{p_0(1,1,0)} = -1$$

$$\Delta f|_{p_0} = 2i - 2j - k$$

Ex. Find the gradate for $f(x, y, z)$ where

$$f(x, y, z) = \sin xyz \text{ at } p_0(0,0,0) \text{ (Homework)}$$

Vector differential

We denote for vector differential by $Du\Delta f|p_0$ which equals the gradate multiplying by unit vector.

$$Du\Delta f|p_0 = \Delta f|p_0 \cdot \vec{u}$$

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|}$$

Ex. Find vector differential for the previous example

$f(x, y, z) = x^3 - xy^2 - z$ at $p_0(1,1,0)$ in the direction of

$$\vec{A} = 2i - 3j + 6k$$

Sol.

$$\therefore \Delta f|p_0 = 2i - 2j - k, \quad \vec{A} = 2i - 3j + 6k$$

$$|\vec{A}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k$$

$$Du\Delta f|p_0 = \Delta f|p_0 \cdot \vec{u}$$

$$= (2i - 2j - k) \cdot \left(\frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k \right)$$

$$= (2) \cdot \left(\frac{2}{7} \right) + (-2) \cdot \left(-\frac{3}{7} \right) + (-1) \cdot \left(\frac{6}{7} \right)$$

$$\therefore Du\Delta f|p_0 = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$$

Higher order differentials

$$z = f(x, y)$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x$$

$$f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xxx} = \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial x^2} \right) = \frac{\partial^3 f}{\partial x^3}$$

⋮

$$f_{\underset{n \text{ times}}{x \dots x}} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \left(\dots \frac{\partial f}{\partial x} \right) \right) \right) = \frac{\partial^n f}{\partial x^n}$$

$$f_y = \frac{\partial f}{\partial y} = z_y$$

$$f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{yyy} = \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial y^2} \right) = \frac{\partial^3 f}{\partial y^3}$$

⋮

$$f_{\underset{n \text{ - times}}{y \dots y}} = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \left(\dots \frac{\partial f}{\partial y} \right) \right) \right) = \frac{\partial^n f}{\partial y^n}$$

Now, if we have

$$f_x = \frac{\partial f}{\partial x}$$

$$f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{xyx} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$

⋮

And so on .

Ex. $f(x, y) = x^5 - 7x + y^2$ Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right), \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$

$$\frac{\partial f}{\partial x} = 5x^4 - 7 = f_x$$

$$\frac{\partial f}{\partial y} = 2y = f_y$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 0 = f_{xy}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = 0 = f_{yx}$$

$$f_{xy} = f_{yx}$$

Ex. Prove that $f_{xy} = f_{yx}$ for ? (Homework)

1. $f(x, y) = \sin(xy)$

2. $f(x, y) = \frac{x}{y}$

3. $z = x^2 + 2y^2$

Ex. Prove that $f_{xxy} = f_{xyx} = f_{yxx}$ for $f(x, y) = x \cos y + ye^x$

$$\frac{\partial f}{\partial x} = \cos y + ye^x = f_x$$

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = ye^x = f_{xx}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \right) = e^x = f_{xxy}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = -\sin y + e^x = f_{xy}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right) = e^x = f_{xyx}$$

$$\frac{\partial f}{\partial y} = -x \sin y + e^x = f_y$$

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = -\sin y + e^x = f_{yx}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \right) = e^x = f_{yxx}$$

$$\therefore f_{xxy} = f_{xyx} = f_{yxx} = e^x$$

Ex. Prove that $f_{xxy} = f_{xyx} = f_{yxx}$ for $f(x, y) = \sin xy - \frac{y}{x}$?

(Homework).

Definition: Local Extrema

We call $f(a, b)$ a **local maximum** of f if there is an open disk R centered at point (a, b) for which $f(a, b) \geq f(x, y)$ for all $(x, y) \in R$. Similarly, we call $f(a, b)$ a **local minimum** of f if there is an open disk R centered at point (a, b) for which $f(a, b) \leq f(x, y)$ for all $(x, y) \in R$.

Definition: Critical Point

The point (a, b) is a critical point of the function $f(x, y)$ if (a, b) is

in the domain of f and either $\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$ or one or both of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not exist at (a, b) .

Saddle Point

The point $P(a, b, f(a, b))$ is a saddle point of $z = f(x, y)$ if (a, b) is a critical point of f and if every open disk centered at (a, b) contains points (x, y) in the domain of f for which $f(x, y) < f(a, b)$ and points (x, y) in the domain of f for which $f(x, y) > f(a, b)$.

Maximum and Local Minmum Points

To find the function if it has a Local Maximum or Local Minmum Points by using some steps which are :-

1. We find the differential $f_x = ? , f_y = ?$
2. $f_x = 0 , f_y = 0$ and solve it to find the critical points.
3. We find the differential $f_{xx} = ? , f_{xy} = ? , f_{yy} = ?$ And put the critical points in it .
4. We put it into the relation

$$D = f_{xx}|_c \cdot f_{yy}|_c - f_{xy}|_c^2$$

5. $D > 0$ or $D < 0$ or $D = 0$.

Here compared with zero .

- a. If $D > 0 , f_{xx} > 0 \Rightarrow$ Local Min.
- b. If $D > 0 , f_{xx} < 0 \Rightarrow$ Local Max.
- c. If $D < 0 \Rightarrow$ Saddle Points.
- d. If $D = 0 \Rightarrow$ No Conclusion.

Ex. Find Local max or min if exist for

$$1- f(x, y) = 6xy - x^2 - 3y^2$$

$$\text{Sol. } f_x = 6y - 2x = 0$$

$$f_y = 6x - 6y = 0$$

$$\frac{\quad}{4x = 0 \Rightarrow x = 0}$$

$$6y = 0 \Rightarrow y = 0$$

$$\therefore c(0,0)$$

$$f_{xx} = -2 \Rightarrow f_{xx}|_{(0,0)} = -2$$

$$f_{yy} = -6 \Rightarrow f_{yy}|_{(0,0)} = -6$$

$$f_{xy} = 6 \Rightarrow f_{xy}|_{(0,0)} = 6$$

$$D = f_{xx}|_{(0,0)} \cdot f_{yy}|_{(0,0)} - f_{xy}|_{(0,0)}^2$$

$$= (-2) \cdot (-6) - 36 = -24$$

$$\therefore D = -24 < 0$$

$\therefore c(0,0)$ is Saddle point.

$$2- f(x, y) = 3 - 6x - 6y + x^2 + y^2$$

$$\text{Sol. } f_x = -6 + 2x = 0 \Rightarrow x = 3$$

$$f_y = -6 + 2y = 0 \Rightarrow y = 3$$

$\therefore c(3,3)$ is critical point.

$$f_{xx} = 2 \Rightarrow f_{xx}|_{(3,3)} = 2$$

$$f_{yy} = 2 \Rightarrow f_{yy}|_{(3,3)} = 2$$

$$f_{xy} = 0 \Rightarrow f_{xy}|_{(3,3)} = 0$$

$$D = f_{xx}|_{(3,3)} \cdot f_{yy}|_{(3,3)} - f_{xy}|_{(3,3)}^2$$

$$= (2) \cdot (2) - 0 = 4$$

$$\therefore D = 4 > 0, f_{xx} = 2 > 0$$

$\therefore c(3,3)$ is Local Min. Point.

$$\mathbf{3-} \mathbf{f(x, y) = 6x + 3y - x^2 - y^3} \quad \mathbf{(Homework).}$$

Chapter Five

The Differential Equation

the Differential Equation

المعادلات التفاضلية

المعادلة التفاضلية :- هي اي علاقة بين متغيرات وتفاضلاتها او مشتقاتها (Differential Equation) هناك نوعان منها :-

١- المعادلة التفاضلية الاعتيادية Ordinary Differential Equation

هي المعادلة التي تحتوي على متغير مستقل واحد فقط مع تفاضلة ومشتقة.

٢- المعادلة التفاضلية الجزئية Partial Differential Equation

هي المعادلة التي تحتوي على متغيرين مستقلين أو أكثر .

رتبة المعادلة التفاضلية The Order of Differential Equation

هي اعلى مشتقة (تفاضل) يظهر في المعادلة .

درجة المعادلة التفاضلية The Degree of Differential Equation

هي اس (قوة) اعلى مشتقة تظهر في المعادلة وترمز ب ∂ لتفاضل او المشتقة الجزئية وترمز ب d للتفاضل او المشتقة الاعتيادية .

Examples:-

$$\text{Ex1 :- } Qy'' + 3y = x^2$$

- معادلة تفاضلية اعتيادية من الرتبة الثانية والدرجة الاولى

$$\text{Ex2 :- } (3x + 2y)^2 \cdot y^2 = 1$$

- معادلة تفاضلية اعتيادية من الرتبة الاولى والدرجة الاولى

$$\text{Ex3 :- } \left(\frac{\partial^2 P}{\partial x^2}\right)^4 + \left(\frac{\partial^2 P}{\partial y^2}\right)^3 + \frac{\partial^2 P}{\partial z^2} = 0$$

- معادلة تفاضلية جزئية من الرتبة الثانية والدرجة الرابعة

$$\text{Ex4:- } x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$$

- معادلة تفاضلية جزئية من الرتبة الاولى والدرجة الاولى

$$\text{Ex 5 :- } \left(\frac{dy}{dx}\right)^4 - x^5 = 0$$

- معادلة تفاضلية اعتيادية من الرتبة الاولى والدرجة الرابعة

$$\text{Ex 6 :- } y'' - y^2 = 0$$

- معادلة تفاضلية اعتيادية من الرتبة الثانية والدرجة الاولى

$$\text{Ex 7 :- } y' + y = 2$$

- معادلة تفاضلية اعتيادية من الرتبة الاولى والدرجة الاولى

The Solution of The Differential Equation حل المعادلة التفاضلية

هي اي علاقة بين متغيرات المعادلة التفاضلية وتكون خالية من المشتقات او التفاضلات وتحقق المعادلة التفاضلية .

$$\text{Ex:- show that } y = x \ln x - x$$

Is a solution for the following diff . eq. $xy' = x+y$ ***

$$\text{Sol. :- } y = x \ln x - x \dots *$$

$$y' = 1 \cdot \ln x + x \frac{1}{x} - 1$$

$$y' = \ln x \dots **$$

Put (*) and (**) in (***)

$$xy' = x + y$$

$$x(\ln x) = x + (x \ln x - x)$$

$$\therefore x \ln x = x \ln x$$

The general solution

الحل العام

هو مجموعه كل الحلول للمعادلة حيث ان عدد الثوابت الاختيارية تكون بقدر رتبة المعادلة .

Ex :- prove that $y = x^4 + c_1 x + c_2 \dots\dots *$ Is a general solution for the following differentail equation $y'' = 12 x^2 \dots\dots\dots ***$

Sol. :- $y' = 4x^3 + c_1 \dots\dots **$

$y'' = 12 x^2 \dots\dots ***$

$\therefore *$ is a general sol. For $***$

The special solution

الحل الخاص

هو حل واحد فقط او اكثر ينتج من فرض وتعويض قيم ثوابت الاختيارية

Ex :- show that $y = x^4 + 5x + 4$ is a special sol. for the differentail equation $y'' = 12x^2$?

Sol. $y = x^4 + 5x + 4 \dots\dots\dots *$

$y' = 4x^3 + 5 \dots\dots\dots **$

$y'' = 12 x^2 \dots\dots\dots ***$

$\therefore *$ is a sp.sol . for $***$

$$y = x^4 + c_1 x + c_2$$

هنا كان الحل عام

حيث الثوابت (هي حل للمعادلة y'') وتم فرض $c_1 = 5$, $c_2 = 4$ فنتج حل خاص للمعادلة y'' .

Ex1:- find the ordred and degree for the following differential equation ?

1. $(y')^2 + x y' = y^2$

2. $y'' = (3y' + x)^2$
3. $(y'')^2 = (1 + y')^3$

Ex2 :- prove that the following values are a general solution for the equations bested them ??

a- $y'' + 4y = 0$, $y = c_1 \sin 2x + c_2 \cos 2x$

b- $y'' + 3y' + 2y = 0$, $y = c^1 e^{-x} + c^2 e^{-2x}$

حلول المعادلات التفاضلية الرتبة الاولى والدرجة الاولى
first order and first degree diff. eq.

$\frac{dy}{dx} = f(x,y)$ تكون المعادلة التفاضلية من الرتبة الاولى والدرجة الاولى بالشكل

او بالشكل $m(x,y)dx + n(x,y)dy = 0$

وتكون على عدة انواع :-

Seperable Differential المعادلة التفاضلية القابلة للفصل

Equation

وهي المعادلة التفاضلية التي نستطيع فصل متغيراتها مع تفاضلها ونجد التكامل لها فنحصل على حلها اي اولا نضغطها بالشكل $A(x)dx + B(y)dy = 0$ ثم نكامل الاطراف فنحصل على $f(x,y)=c$ وهي مجموعه الحل للمعادلة.

Ex:- solve the following diff. eq. ?

$$x^2 (1 - y^2) dx + y(1 - x^2)dy = 0$$

Sol. :- $\frac{x^2}{1+x^2} dx + \frac{y}{1-y^2} dy = 0$

$$\int (1 - \frac{1}{1-x^2}) dx + \left(\frac{1}{-2}\right) \int \frac{-2dy}{1-y^2} = c$$

$$\frac{\sqrt{x^2+1} \sqrt{x^2}}{-1}$$

بالطرح

$$\int dx - \int \frac{dx}{1+x^2} - \frac{1}{2} \ln|1 - y^2| = c$$

$$x - \tan^{-1} x - \frac{1}{2} \ln|1 - y^2| = c$$

$\therefore f(x,y) = c$ general sol.

Ex : solve the following diff.eq. $xy \, dy + (2yx^2 + 4x^2 - y - 2) \, dx = 0$

Sol. :- $xy \, dy + 2x^2(y+2) - (y+2) \, dx = 0$

$$xy \, dy + (2x^2 - 1)(y+2) \, dx = 0$$

$$\frac{2x^2-1}{x} \, dx + \frac{y}{y+2} \, dy = 0$$

$$(2x - \frac{1}{x}) \, dx + (1 - \frac{2}{y+2}) \, dy = 0$$

$$\int 2x \, dx - \int \frac{dx}{x} + \int dy - 2 \int \frac{dy}{y+2} = c$$

$\frac{y+2}{-2} \sqrt{y}$

 بالشرح

$$X^2 - \ln|x| + y - 2\ln|y + 2| = c$$

$\therefore f(x,y) = c$ g . sol . for the diff. eq.

Ex : Homework solve the following diff. eq. ??

$$xy \, dy + (cx^2 - 2yx^2 + 4 - 8y) \, dx = 0$$

Homogeneous diff. eq.

٢- المعادلات التفاضلية المتجانسة

الدالة المتجانسة هي الدالة التي تحقق الشرط التالي $f(tx,ty) = t^n f(x,y)$

$$\text{Ex :- } f(x,y) = x^3 - 2xy^2 + x^2y + y^3$$

$$f(tx, ty) = (tx)^3 - 2(tx)(ty)^2 + (tx)^2(ty) + (ty)^3$$

$$= t^3x^3 - 2t^3xy^2 + t^3x^2y + t^3y^3$$

$$= t^3(x^3 - 2xy^2 + x^2y + y^3)$$

$$= t^3 f(x,y)$$

f متجانسة من الدرجة الثالثة $\therefore f$

$$\text{Ex :- } f\left(\frac{y}{x}\right) = e^{\frac{y}{x}} + \sin\frac{y}{x} - \frac{1}{\left(\frac{y}{x}\right)} + \left(\frac{y}{x}\right)^2$$

$$f\left(\frac{ty}{tx}\right) = e^{\frac{ty}{tx}} + \sin\frac{ty}{tx} - \frac{1}{\left(\frac{ty}{tx}\right)} + \left(\frac{ty}{tx}\right)^2$$

$$= e^{\frac{y}{x}} + \sin\frac{y}{x} - \frac{1}{\frac{y}{x}} + \left(\frac{y}{x}\right)^2$$

$$= t^0 \left(f\left(\frac{y}{x}\right)\right)$$

Ex. Is $f(x, y) = \frac{x^3+y^3}{xy^2}$ Homo. Function?(Homework)

المعادلة التفاضلية المتجانسة **Homogeneous differential equation**

هي المعادلة التي تحقق فيها كون كل من M & N دالة متجانسة من نفس الدرجة وطريقة حلها كالآتي:-

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(tx,ty) = t^n M(x,y)$$

$$N(tx,ty) = t^n N(x,y)$$

وكل من M و N دالة متجانسة من نفس الدرجة.

خطوات حل هذا النوع من المعادلات:-

١. نضع المعادلة بالشكل $\frac{dy}{dx} = f(x, y)$

٢. ونضعها بالشكل $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

٣. نفرض $v = \frac{x}{y} \rightarrow \frac{dy}{dx} = f(v)$

٤. اصبحت لدينا معادلة تفاضلية قابلة للفصل بالنسبة ل x, v

$$\frac{dx}{x} + \frac{dv}{v - f(v)} = 0$$

٥. نحل الصيغة اعلاه ونعوض ب $\frac{x}{y}$ بدل v .

Ex. Solve the following differential equation

$$x dy - y dx = \sqrt{x^2 + y^2} dx?$$

Sol.

$$(y + \sqrt{x^2 + y^2}) dx - x dy = 0$$

$$M(x, y) = (y + \sqrt{x^2 + y^2})$$

$$M(tx, ty) = (ty + \sqrt{(tx)^2 + (ty)^2})$$

$$M(tx, ty) = (ty + \sqrt{t^2(x^2 + y^2)})$$

$$M(tx, ty) = (ty + t\sqrt{(x^2 + y^2)})$$

$$M(tx, ty) = t(y + \sqrt{(x^2 + y^2)})$$

$\therefore M(tx, ty) = t M(x, y)$ دالة متجانسة من الدرجة الأولى

$$N(x, y) = -x$$

$$N(tx, ty) = -(tx)$$

$$= t(-x)$$

$$= t N(x,y)$$

∴ N is homo . function of degree one

1- we put $\frac{dy}{dx} = f(x,y)$

$$\frac{dy}{dx} = \frac{\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}}{\frac{x}{x}} = \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}}$$

$$= \frac{y}{x} + \sqrt{\frac{x^2}{x^2} + \frac{y^2}{x^2}}$$

$$= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

$$\therefore \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right)$$

$$\text{Put } v = \frac{y}{x} \rightarrow f(v) = v + \sqrt{1 + v^2} \quad -$$

$$\text{Solve } \frac{dx}{x} + \frac{dv}{v - f(v)} = 0 \quad -$$

$$\frac{dx}{x} + \frac{dv}{v - (v + \sqrt{1 + v^2})} = 0$$

$$\frac{dx}{x} + \frac{dv}{-\sqrt{1 + v^2}} = 0$$

اصبحت لدينا معادلة تفاضلية قابلة للفصل بالنسبة ل x, y , نكامل للطرفين فنحصل على

$$\int \frac{dx}{x} - \int \frac{dv}{\sqrt{1 + v^2}} = \int 0$$

$$\ln|x| - \int \frac{dv}{\sqrt{1 + v^2}} = c \quad \dots\dots\dots*$$

$$\int \frac{dv}{\sqrt{1+v^2}} \quad ??$$

هنا نحل التكامل التالي بطريقة التعويضات المثلثية

$$\int \frac{dv}{\sqrt{1+v^2}}$$

$$\text{Let } v = \tan \theta$$

$$dv = \sec \theta \, d\theta$$

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{\sec \theta \, d\theta}{\sqrt{\tan^2 \theta + 1}}$$

$$= \int \frac{\sec \theta \, d\theta}{\sin \theta} \qquad \tan^2 \theta + 1 = \sec^2 \theta$$

$$= \int \sec \theta \, d\theta$$

$$= \text{Ln} |\sec \theta + \tan \theta|$$

$$= \text{Ln} |\sqrt{1 + \tan^2 \theta} + \tan \theta|$$

$$= \text{Ln} |\sqrt{1 + v^2} + v|$$

$$= \text{Ln} \left| \sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x} \right| \dots\dots\dots **$$

$$\text{Ln} |x| - \text{Ln} \left| \sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x} \right| = c \quad \text{نعوض في}$$

$$F(x,y)=c$$

Ex :- Homework :- Solve $xy \, dx + (cx^2 - 2y^2)dy = 0$

المعادلات التفاضلية الغير المتجانسة non – homogeneous differential eq.

المعادلة التفاضلية من الرتبة الاولى والدرجة الاولى تكون غير متجانسة اذا لم تحقق

$$M(tx, ty) \& N(tx, ty)$$

هنا ستكون بالشكل التالي

$$(ax + by + c) dx + (\alpha x + \beta Y + \Upsilon) dy = 0$$

$$\frac{dy}{dx} = \frac{ax+by+c}{\alpha x+\beta Y+\Upsilon}$$

نصفها بالشكل التالي

الان نأخذ الحالتان

الحالة الاولى :- في حالة المشتقات متقاطعات نحلها انيا ونجد نقطة التقاطع

$$ax + by + c = 0$$

$$ax + \beta Y + \Upsilon = 0$$

$$\frac{a}{\alpha} \neq \frac{d}{\beta} \text{ ولتكن}$$

$$x = x_1 + h \& \quad y = y_1 + k \quad \text{نفرض الفرضية التالية}$$

$$\& \frac{dy}{dx} = \frac{dy_1}{dx_1} \quad \text{ونعوض في المعادلة اعلاه}$$

فحصل على معادلة متجانسة بدلالة y_1, x_1 ونحلها حسب الطريقة في الحالة الثانية

(من نوع الثاني) بعدها نعوض بدلالة x, y ونحصل على الحل العام

Ex :- solve

$$(2x - 3y + 4) dx + (3x - 2y + 1) dy = 0$$

$$\text{Sol:-} \quad \frac{dy}{dx} = \frac{-2x+3y-4}{3x-2y+1}$$

$$\frac{a}{\alpha} = \frac{-2}{3} + \frac{d}{\beta} = \frac{3}{-2}$$

المستقيمان متقاطعان نحلها انيا

$$-2x + 3y - 4 = 0$$

$$3x - 2y + 1 = 0$$

$$-6x + 9y - 12 = 0$$

$$6x - 4y + 2 = 0$$

$$(h, k) = ?(1, 2)$$

$$5y = -10$$

$$Y = -2$$

$$3x - 2(-2) + 1 = 0$$

$$3x - 3 = 0 \rightarrow x = 1$$

Let $x = x_1 + h$ & $y = y_1 + k$

$$x = x_1 + 1 \quad \& \quad y = y_1 + 2$$

$$\frac{dy_1}{dx_1} = \frac{-2(x_1+1)3(y_1+2)-4}{3(x_1+1)-2(y_1+2)+1}$$

$$\frac{dy_1}{dx_1} = \frac{-2x_1-2+3y_1+6-4}{3x_1+3-2y_1-4+1}$$

$$\frac{dy_1}{dx_1} = \frac{-2x_1+3y_1}{3x_1-2y_1}$$

$$\frac{dy_1}{dx_1} = \frac{-\frac{2x_1}{x_1} + \frac{3y_1}{x_1}}{\frac{3x_1}{x_1} - \frac{2y_1}{x_1}}$$

$$\frac{dy_1}{dx_1} = \frac{-2+3\left(\frac{y_1}{x_1}\right)}{3-2\left(\frac{y_1}{x_1}\right)}$$

$$\text{Let } v = \frac{y_1}{x_1} \rightarrow \frac{dy_1}{dx_1} = \frac{-2+3v}{3-2v} = f(v)$$

$$\frac{dx_1}{x_1} + \frac{dv}{v-f(v)} = 0$$

$$\frac{dx_1}{x_1} + \frac{dv}{v - \frac{-2+3v}{3-2v}} = 0$$

$$\frac{dx_1}{x_1} + \frac{dv}{\frac{3v-2v^2+2-3v}{3-2v}}$$

$$\frac{dx_1}{x_1} + \frac{dv}{\frac{-2(v^2-1)}{(3-2v)}}$$

$$\frac{dx_1}{x_1} - \frac{(3-2v)dv}{2(v^2-1)} = 0$$

$$\frac{dx_1}{x_1} - \frac{3}{2} \frac{dv}{v^2-1} + \frac{2v dv}{2(v^2-1)} = 0$$

$$\int \frac{dx_1}{x_1} - \frac{3}{2} \int \frac{dv}{v^2-1} + \frac{1}{2} \int \frac{2v dv}{v^2-1} = \int 0$$

$$\text{Ln } |x_1| - \frac{3}{2} \int \frac{dv}{(v-1)(v+1)} + \frac{1}{2} \text{Ln} |v^2 - 1| = c$$

هنا تجزئة كسور

$$\int \frac{dv}{(v-1)(v+1)} \quad ? \quad \int \frac{\frac{1}{2}}{v-1} - \frac{\frac{1}{2}}{v+1} dv$$

$$= \frac{1}{2} \int \frac{dv}{v-1} - \frac{1}{2} \int \frac{dv}{v+1}$$

$$= \frac{1}{2} \text{Ln} |v - 1| - \frac{1}{2} \text{Ln} |v + 1|$$

$$\text{Ln} |x_1| - \frac{3}{2} \left(\frac{1}{2} \text{Ln} |v - 1| - \frac{1}{2} \text{Ln} |v + 1| \right) + \frac{1}{2} \text{Ln} |v^2 - 1| = c$$

$$\text{Ln} |x_1| - \frac{3}{4} \text{Ln} \left| \frac{y_1}{x_1} - 1 \right| + \frac{3}{4} \text{Ln} \left| \frac{y_1}{x_1} + 1 \right| + \frac{1}{2} \text{Ln} \left| \left(\frac{y_1}{x_1} \right)^2 - 1 \right|$$

$$\text{Ln} |x - 1| - \frac{3}{4} \text{Ln} \left| \frac{y-2}{x-1} - 1 \right| + \frac{3}{4} \text{Ln} \left| \frac{y-2}{x-1} + 1 \right| + \frac{1}{2} \text{Ln} \left| \frac{(y-2)^2}{(x-1)^2} - 1 \right| = c$$

$$x = x_1 + 1 \rightarrow x_1 = x - 1, y = y_1 + 2 \rightarrow y_1 = y - 2 \quad \therefore f(x, y) = c$$

الحالة الثانية :- في حالة المستقيمان متوازيان

$$\frac{dy}{dx} = \frac{ax + by + c}{ax + \beta y + \gamma}$$

هنا نختبر $\frac{a}{\alpha} = \frac{b}{\beta}$ في حالة التساوي نستنتج ان مستقيمان متوازيان.

ونحلها باستخدام الفرضية التالية الطريقة
حيث تصبح قابلة للفصل بعدها نعوض بدلالة x, z فنحصل على معادلة متجانسة
بدلالة z

Ex :- solve

$$(2x + y)dx - (4x + 2y - 1)dy = 0$$

Sol:- $\frac{dy}{dx} = \frac{2x+y}{4x+2y-1}$

$$\frac{a}{\alpha} = \frac{2}{4} + \frac{b}{\beta} = \frac{1}{2} \text{ المستقيمان متوازيان}$$

$$z = 2x + y$$

$$dz = 2dx + dy$$

$$dy = dz - 2dx$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 2$$

$$\frac{dy}{dx} = \frac{2x+y}{4x+2y-1}$$

$$\frac{dz}{dx} - 2 = \frac{2x+y}{2(2x+y)-1}$$

$$\frac{dz}{dx} = \frac{z}{2z-1} + 2$$

$$\frac{dz}{dx} = \frac{z+4z-2}{2z-1}$$

$$\frac{dz}{dx} = \frac{5z-2}{2z-1}$$

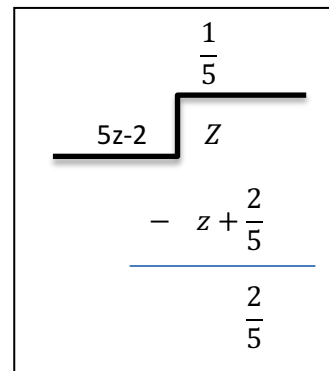
$$\int \frac{2z-1}{5z-2} dz - \int dx = \int 0$$

$$2 \int \frac{z}{5z-2} dz - \int \frac{1}{5z-2} dz - \int dx = \int 0$$

$$2 \int \frac{z}{5z-2} dz - \frac{1}{5} \text{Ln}|5z-2| - x = c \dots\dots\dots 1)$$

$$2 \int \frac{z}{5z-2} dz = 2 \left(\frac{1}{5} \int dz + \frac{2}{5} \int \frac{1}{5z-2} dz \right)$$

$$= 2 \left(\frac{1}{5} z + \frac{2}{25} \text{Ln}|5z-2| \right) \dots\dots\dots *$$



Put * in equation 1 we get:-

$$2 \left(\frac{1}{5} z + \frac{2}{25} \text{Ln}|5z-2| \right) - \frac{1}{5} \text{Ln}|5z-2| - x = c \dots\dots\dots 1)$$

$$\frac{2}{5} z + \frac{4}{25} \text{Ln}|5z-2| - \frac{1}{5} \text{Ln}|5z-2| - x = c$$

$$\frac{2}{5} z + \frac{4}{25} \text{Ln}|5z-2| - \frac{1}{5} \text{Ln}|5z-2| - x = c$$

$$\frac{2}{5} z + \frac{4-5}{25} \text{Ln}|5z-2| - x = c$$

$$\frac{2}{5} z - \frac{1}{25} \text{Ln}|5z-2| - x = c$$

Put $z = 2x+y$ we get

$$\frac{2}{5} (2x+y) - \frac{1}{25} \text{Ln}|5(2x+y)-2| - x = c$$

$$\therefore f(x,y) = c$$

Exact differential equation

(٤) المعادلة التفاضلية تامة

The differential equation $M(x, y)dx + N(x, y)dy = 0$

become Exact if it satisfy the condition :-

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and the general solution is $f(x, y) = c$

To find the function f ,we used the relation

$$f(x, y) = \int M(x, y)dx + \phi(y) \text{ -----*}$$

حيث $\phi(y)$ دالة اختيارية بالنسبة ل y ونجدها عن طريق اشتقاق الدالة f_y ومساواتها بدالة $N(x, y)$

$$f_y = \frac{\partial f}{\partial y} = N(x, y) \text{ -----**}$$

نجد $\phi(y)$ ونعوضها في* ونجد بذلك الحل العام للمعادلة التفاضلية التامة.

Ex: solve $(3x^2 + 3xy^2) dx + (3x^2 y - 3y^2 + 2y)dy = 0$

Sol.

$$M(x, y) = 3x^2 + 3xy^2 , N(x, y) = 3x^2 y - 3y^2 + 2y$$

$$\frac{\partial M}{\partial y} = 6xy = \frac{\partial N}{\partial x} = 6x y$$

Exact diff. eq.

$$f(x, y) = \int M(x, y)dx + \phi(y)$$

$$f(x, y) = \int (3x^2 + 3xy^2) dx + \phi(y)$$

$$f(x, y) = x^3 + \frac{3}{2}x^2y^2 + \phi(y)$$

$$\text{Now } f_y = \frac{\partial f}{\partial y} = N(x, y)$$

$$0 + \frac{3}{2}x^2(2y) + \phi'(y) = 3x^2y - 3y^2 + 2y$$

$$\phi'(y) = \frac{d\phi}{dy} = -3y^2 + 2y$$

$$\int d\phi = \int (-3y^2 + 2y)dy$$

$$\phi(y) = -y^3 + y^2 - c$$

$$\therefore F(x, y) = x^3 + \frac{3}{2}x^2y^2 - y^3 + y^2 = c$$

$$\therefore F(x, y) = c$$

Ex. Solve $3x(xy-2)dx + (x^3 + 2y)dy = 0$ (Homework)

Non – Exact diff. eq.

(٥) المعادلة التفاضلية الغير تامة

هنا في هذه الحالة تكون المشتقة غير متساوية أي انه

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

ولكي نجعلها تساوي تجري العمليات التالية :-

١- نجد عامل التكامل للمعادلة وذلك كالآتي :

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial M}{\partial x} \right) = f(x)$$

نختبر

إذا كان الناتج دالة لـ x معامل التكامل سيكون

$$I(x, y) = e^{\int f(x) dx}$$

٢- نضرب طرفي المعادلة به فتصبح المعادلة تامة ونحلها حسب الحالة السابقة

Remark :

$$\frac{1}{M} \left(\frac{\partial M}{\partial y} - \frac{\partial M}{\partial x} \right) = g(y) \quad \text{في حالة كون الاختبار حقق تساوي دالة (y)}$$

يكون عامل التكامل

$$I(x, y) = e^{-\int g(y) dy}$$

Ex. : solve $(3x y^3 + 4y) dx + (3x^2 y^2 + 2x) dy = 0$

Sol.

$$\frac{\partial M}{\partial y} = 3x y^2 + 4 \neq \frac{\partial N}{\partial x} = 6xy^2 + 2$$

Not Exact

$$\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = ? f(x)$$

$$\frac{1}{3x^2 y^2 + 2x} \left((3x y^2 + 4) - (6xy^2 + 2) \right) = \frac{1}{x(3y^2 + 2)} (3xy^2 + 2) = \frac{1}{x} = f(x)$$

$$I(x, y) = e^{\int f(x) dx}$$

$$I(x, y) = e^{\int \frac{dx}{x}} = e^{\ln |x|} = x$$

$$x(3x y^3 + 4y) dx + x(3x^2 y^2 + 2x) dy = 0$$

$$(3x^2 y^3 + 4xy) dx + (3x^3 y^2 + 2x^2) dy = 0$$

M

N

$$\frac{\partial M}{\partial y} = 9x^2 y^2 + 4x = \frac{\partial N}{\partial x} = 9x^2 y^2 + 4x$$

∴ Exact diff. eq.

The Sol. Is :

$f(x,y) = c$ where

$$f(x,y) = \int M(x,y) dx + \phi(y)$$

$$f(x,y) = \int (3x^2y^3 + 4xy) dx + \phi(y)$$

$$f(x,y) = x^3y^3 + 2x^2y + \phi(y)$$

$$f_y = \frac{-\partial f}{\partial y} = N(x,y)$$

$$\frac{\partial f}{\partial y} = 3x^3y^2 + 2x^2 + \phi'(y) = N(x,y)$$

$$\rightarrow \cancel{3x^3y^2} + \cancel{2x^2} + \frac{d\phi}{dy} = \cancel{3x^3y^2} + \cancel{2x^2}$$

$$\rightarrow \frac{d\phi}{dy} = 0 \rightarrow \int d\phi = \int 0 dy$$

$$\rightarrow \phi = c y^\circ \leftarrow \text{constant}$$

$$\therefore \phi(y) = 0$$

$$\therefore F(x,y) = x^3y^3 + 2x^2y + c$$

$$\therefore F(x,y) = c \quad \text{g. s. for diff eq.}$$

Linear diff.eq

(٦) المعادلة التفاضلية الخطية

$$\frac{dy}{dx} + p(x)y = q(x)$$

تكون على الصيغة

حيث p و q دوال لـ x وقد تكون ثوابت خطوات حل هكذا معادلة :-

^١ نجد عامل التكامل كالآتي

$$I(x,y) = e^{\int P(x)dx}$$

$$f(x,y) = c$$

٢- نجد الحل بالخطوة التالية

$$y.I = \int q(x) \cdot I dx + c$$

Ex. Solve $(1+x^2) y' + x y = x^{-1} \sqrt{x^2 + 1}$

Sol. $\frac{dy}{dx} + \frac{x}{x^2+1} \cdot y = \frac{\sqrt{x^2+1}}{x(x^2+1)}$

$$\frac{dy}{dx} + \frac{x}{x^2+1} \cdot y = \frac{1}{x\sqrt{x^2+1}}$$

$$I(x, y) = e^{\int \frac{x}{x^2+1} dx}$$

$$= e^{\frac{1}{2} \ln|x^2+1|}$$

$$I = e^{\ln \sqrt{x^2 + 1}} = \sqrt{x^2 + 1}$$

g.s. $f(x,y) = c$

$$y \cdot I = \int q(x) \cdot I dx + c$$

$$y \cdot \sqrt{x^2 + 1} = \int \frac{1}{x\sqrt{x^2+1}} \sqrt{x^2 + 1} dx + c$$

$$y \cdot \sqrt{x^2 + 1} = \int \frac{1}{x} dx + c$$

$$y \sqrt{x^2 + 1} = \ln|x| + c$$

$$f(x,y) = c$$

ملاحظه : هنا اذا كان y بالنسبة لـ x اما اذا كان بالعكس x بالنسبة لـ y

$$\frac{dx}{dy} + p(y) \cdot x = q(y)$$

$$I(x, y) = e^{\int P(y) dy}$$

$$x \cdot I = \int q(y) \cdot I dy + c$$

Ex. $y \frac{dx}{dy} + 2x = y^3$

Sol. $y \frac{dx}{dy} + 2x = y^3$

$$\frac{dx}{dy} + \frac{2}{y}x = y^2$$

$$\frac{dx}{dy} + p(y).x = q(y)$$

$$I(x, y) = e^{\int P(y)dy}$$

$$= e^{\int \frac{2}{y}dy} = e^{2\ln y} = e^{\ln y^2} = y^2$$

$$x.I = \int q(y).I dy + c$$

$$x.y^2 = \int y^2 . y^2 dy + c$$

$$x.y^2 = \frac{1}{5} y^5 + c$$

$$\therefore f(x,y) = c$$

Special Types of Second order eq.

حالات خاصة من المعادلات ذات الرتبة الثانية

هنا تخفيض الرتبة الثانية للرتبة الاولى له حالتان :-

$$(1) f(x, y', y'') = 0$$

اذا كانت المعادلة بالشكل

نفرض الفرضية التالية

$$P = y' \rightarrow p' = \frac{dp}{dx} = \frac{d}{dx}(y')$$

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = y''$$

$$\therefore P = y' \rightarrow p' = y''$$

اما اذا كانت بالشكل التالي :

$$(2) F(y, y', y'') = 0$$

فتكون الفرضية بالاشتقاق بالنسبة لـ y

$$P = y' \rightarrow \frac{dp}{dy} = \frac{d}{dy}(y')$$

$$= \frac{d}{dy} \left(\frac{dy}{dy} \right) = \frac{d^2y}{dy^2} = y''$$

$$P = y' \rightarrow p' = y''$$

$$\underline{\text{Ex}} : x^2 y'' - (y')^2 2x y' = 0 \quad \text{معادلة تخفيض رتبة}$$

$$\text{Sol. (1) } F(x, y', y'') = 0$$

$$P = y' \rightarrow p' = \frac{dp}{dx} = y''$$

$$x^2 \frac{dp}{dx} - p^2 - 2xp = 0$$

$$\frac{dp}{dx} - \frac{2x}{x^2} \cdot p = \frac{1}{x^2} p^2$$

Bernolly eq.

$$\frac{dy}{dx} + p(x) \cdot y = q(x) \cdot y^n \quad n \in \mathbb{R} \setminus \{0, 1\}$$

معادلة برنولي

$$\frac{dy}{dx} + p(x) y = q(x) \quad \leftarrow \text{هذه خطية عادية}$$

الآن وجود y^n مضروب في $q(x)$ تمثل حالة أخرى من المعادلات الخطية وتسمى معادلة برنولي (Bernolly eq)

* هنا اذا كانت $(0 = n)$ تصبح خطية و $(1-n)$ تكون عامل مشترك مع الطرف الايسر ولا تصبح خطية لذا $(n = 2)$ فما فوق) وهي تدعى معادلة برنولي

Berolly Equation

$$\frac{dy}{dx} + p(x) \cdot y = q(x) \cdot y^n$$

$$(1) \quad \text{Let } Z = y^{1-n} \quad \text{نفرض الفرضية التالية}$$

$$(2) \quad \frac{dz}{dx} + (1-n) p(x) \cdot Z = (1-n) q(x)$$

(٢) نحل المعادلة بالنسبة لـ Z و x حيث ستكون خطية

نعود الان لحل السؤال

$$\frac{dy}{dx} - \frac{2}{x}p = \frac{1}{x^2} p^2, n=2$$

$$(1) \text{ Let } Z = p^{1-n} = p^{1-2} = p^{-1} = \frac{1}{p}$$

$$(2) \frac{dz}{dx} + (1-n) p(x).z = (1-n)q(x)$$

$$\frac{dz}{dx} + (1-2) \left(-\frac{2}{x}\right) \cdot Z = (1-2) \left(\frac{1}{x^2}\right)$$

$$\frac{dz}{dx} + \frac{2}{x} \cdot Z = \frac{-1}{x^2}$$

$$I(x,z) = e^{\int \frac{2}{x} dx} = x^2$$

$$z \cdot I(x, z) = \int q(x) \cdot I(x, z) dx + c$$

$$z \cdot x^2 = \int -\frac{1}{x^2} \cdot x^2 dx + c$$

$$Z \cdot x^2 = -x + c$$

$$Z = \frac{-x+c}{x^2}, \quad Z = \frac{1}{p}$$

$$\frac{1}{p} = \frac{-x+c}{x^2}$$

$$p = \frac{x^2}{-x+c}, \quad p = \frac{dy}{dx} = y'$$

$$\frac{dy}{dx} = \frac{x^2}{-x+c}$$

$$-x - c$$

$$\int dy = \int \frac{x^2}{-x+c} dx$$

$$-x+c \sqrt{x^2}$$

$$\text{بالطرح} \frac{+x^2 - c}{cx - c^2} \\ \frac{cx - c^2}{c^2}$$

$$y = \int \left(-x - c + \frac{c^2}{-x+c} \right) dx$$

$$y = -x^2 - cx - c^2 \ln |-x+c|$$

$$\therefore f(x, y) = c$$

المعادلات التفاضلية الخطية من الرتبة الثانية

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a' y' + a_0 y = 1$$

معادلة تفاضلية خطية من الرتبة n ومعاملاتها دوال لـ x والطرف الايمن دالة لـ f(x)

في حالة a_n, \dots, a_0 تكون ثوابت تسمى المعادلة تفاضلية خطية من الرتبة n ذات معاملات ثابتة واذا كان الطرف الايمن $f(x) = 0$

سُميت معادلة تفاضلية خطية متجانسة

$$y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, \dots, y^{(n)} = \frac{d^n y}{dx^n}$$

$$y'' + p(x)y' + q(x)y = f(x) \quad \text{هنا في حالة } n=2 \text{ تكون بالشكل}$$

عندما $f(x) = 0$ تكون معادلة متجانسة بالشكل التالي

$$y'' + p(x)y' + q(x)y = 0$$

الآن سنأخذ المعادلة الخطية المتجانسة من الرتبة الثانية ذات المعاملات الثانية اي (p, q, a_0, a_1, a_2) تكون ثوابت وهنا تمثل (ثوابت)

$$y'' + p y' + q y = 0$$

الحل العام لها بطريقة جبرية حيث سنفرض

$$y'' = m^2 \& y' = m \& y = 1$$

$$m^2 + pm + q = 0$$

نحلها ونجد جذور المعادلة والتي ستأخذ ثلاث حالات

(1) عندما $m_1 \& m_2$ جذور حقيقية مختلفة سيكون الحل للمعادلة التفاضلية

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

(٢) عندما تكون $(m_1 = m_2)$ حقيقية ومتساوية

$$y = c_1 e^{mx} + x \cdot c_2 e^{mx}$$

(٣) ثالثاً عندما تكون m_1, m_2 جذور خيالية الحل للمعادلة سيكون $m = a \mp ib$

$$y = e^{ax} (c_1 \cos bx \mp c_2 \sin bx)$$

Ex. solve (find the general solution)

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

هنا نلاحظ من الرتبة الثانية وطرفها الايمن صفر تعني حالة الـ m اي نفرض

$$y = 1, \frac{dy}{dx} = m, \frac{d^2 y}{dx^2} = m^2$$

$$m^2 + m - 6 = 0$$

$$(m+3)(m-2) = 0$$

$$m_1 = -3, m_2 = 2$$

$$\text{g.s. } y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y = c_1 e^{-3x} + c_2 e^{2x}$$

* هنا للتحقق من صحة الحل نشتق y' و y'' ونفوق في الطرف الايسر سيكون الناتج = صفر اي الطرف الايمن

$$\text{Ex. } y'' - 6y' = -9y$$

$$\text{Sol. } y'' + 6y' + 99 = 0$$

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m_1 = m_2 = m = -3$$

$$\therefore y = c_1 e^{-3x} + c_2 x e^{-3x}$$

Ex.: $y'' + qy = 0??$

Sol. $m^2 + q = 0$

$$m^2 = -9$$

$$m_{1,2} = \mp \sqrt{-9}$$

$$= \mp 3\sqrt{-1}$$

$$= \mp 3i$$

$$m_1, m_2 = a \mp bi \rightarrow a = 0, b = 3$$

$$y = e^{ax}(c_1 \cos bx \mp c_2 \sin bx)$$

$$y = c_1 \cos 3x \mp c_2 \sin 3x$$

Exc. (1) $y'' - 3y' + 2y = 0$ (3) $y'' + qy = 0$

(2) $y'' = 4y' + 4y = 0$ (Homework)

Laplace Trans for motions

تحويلات لابلاس

تحويل لابلاس : يُعرف تحويل لابلاس للدالة $f(x)$ بأنه التكامل التالي :-

$$L\{f(x)\} = \int_0^{\infty} f(x)e^{-px} dx; \quad p \text{ عدد حقيقي}$$

خواص تحويل لابلاس :-

(1) $L\{f(x) \mp g(x)\} = L\{f(x)\} \mp L\{g(x)\}$

(2) $L\{A f(x)\} = AL\{f(x)\}$

(3) $L\{A f(x) \mp B g(x)\} = AL\{f(x)\} \mp BL\{g(x)\}$

هنا خاصية رقم واحد تمثل توزيع تحويل لابلاس على الجمع والطرح لانه يمثل متكامل لذا يحقق هذه الخاصية للتكامل اي التوزيع على الجمع والطرح .

والخاصية الثانية تمثل الثابت يخرج خارج التحويل حيث ان التكامل بحاصل ضرب ثابت في دالة يساوي الثابت في تامل الدالة .

اما الشرط الثالث فهو جامع للشرطين السابقين ويسمى الخاصية الخطية (اي انه التكامل يحقق الخاصية الخطية) وبالتالي فإن تحويل لابلاس يحقق الخاصية الخطية .

Proof :

$$(1) \quad L \{ f(x) \mp g(x) \} \stackrel{??}{=} L \{ f(x) \} \mp L \{ g(x) \}$$

$$\text{Pf: } L \{ f(x) \mp g(x) \} = \int_0^{\infty} (f(x) \mp g(x)) e^{-px} dx$$

$$= \int_0^{\infty} f(x) \cdot e^{-px} dx \mp \int_0^{\infty} g(x) e^{-px} dx$$

$$= L \{ f(x) \mp L \{ g(x) \}$$

$$(2) \quad L \{ A f(x) \} = A L \{ f(x) \}$$

$$\text{Df: } L \{ A f(x) \} = \int_0^{\infty} (A f(x)) e^{-px} dx$$

$$= \int_0^{\infty} A (f(x) e^{-px}) dx$$

$$= A \int_0^{\infty} f(x) e^{-px} dx$$

$$= A L \{ f(x) \}$$

$$(3) \quad L \{ A f(x) \mp B g(x) \} \stackrel{??}{}$$

$$= \int_0^{\infty} (A f(x) e^{-px}) dx \mp \int_0^{\infty} (B g(x) e^{-px}) dx$$

$$= A \int_0^{\infty} f(x) e^{-px} dx \mp B \int_0^{\infty} g(x) e^{-px} dx$$

$$= A \{ f(x) \} \mp B L \{ g(x) \}$$

Pf (3) H.W . برهن خاصية (٣) بالاعتماد على خاصية (١) و (٢) .

$$L \{ A f(x) \mp B g(x) \} \stackrel{??}{}$$

from propo. (1)

from propo. (2)

NOW if (1) $f(x) = 1 \rightarrow$ find $L \{f(x)\} = ?$

$$\begin{aligned}
 (1) \quad L \{1\} &= \int_0^{\infty} 1 \cdot e^{-px} dx = \frac{-1}{p} e^{-px} \Big|_0^{\infty} \\
 &= \frac{-1}{p} (e^{-p(\infty)} - e^{-p(0)}) \\
 &= -\frac{1}{p} \left(\frac{1}{e^{\infty}} - e^0 \right) \\
 &= -\frac{1}{p} (0-1)
 \end{aligned}$$

$$\therefore L \{1\} = +\frac{1}{p} \leftarrow$$

تحويل لابلاس للواحد هو

$$(2) \quad \text{If } f(x) = e^{ax}$$

$$\begin{aligned}
 (2) \quad L \{e^{ax}\} &= \int_0^{\infty} e^{ax} e^{-px} dx \\
 &= \int_0^{\infty} e^{(a-p)x} dx = \frac{1}{(a-p)} \int_0^{\infty} (a-p) e^{(a-p)x} dx \\
 &= \frac{1}{a-p} e^{(a-p)x} \Big|_0^{\infty} = \frac{1}{a-p} (e^{(a-p)\infty} - e^0) \\
 &= \frac{1}{a-p} (e^{a\infty} (e^{-p\infty}) - 1) = \frac{1}{a-p} (\infty(0) - 1)
 \end{aligned}$$

وهكذا نجد تحويلات لابلاس لكل دالة تعطى لنا

Ex. : (**Homework**) Find Laplace transformation for the following function:-

$$1. f(x) = 2x^4$$

$$2. f(x) = \sin 3x e^{-x}$$

$$3. f(x) = \frac{1}{2} \sinh 5x$$

$$4. f(x) = \frac{1}{3} \cos 2x e^{3x}$$

$$5. f(x) = 4 \cosh 7x$$

$$6. f(x) = 3x^2 e^x$$

F(t)	F(s)=L(f(t))	
1		
t^n	$\frac{n!}{s^{n+1}}$	$s > 0, n = 0, 1, 2, \dots$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\text{Sinh}(at)$	$\frac{a}{s^2 - a^2}$	$s > a $
$\text{Cosh}(at)$	$\frac{s}{s^2 - a^2}$	$s > a $
$\text{Sin}(at)$	$\frac{a}{s^2 + a^2}$	$s > 0$
$\text{Cos}(at)$	$\frac{s}{s^2 + a^2}$	$s > 0$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$	
$f'(t)$	$sF(s) - f(0)$	
$\frac{1}{t} f(t)$	$\int F(u) du$	
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$n = 1, 2, 3, \dots$

جدول تحويلات لابلاس لبعض الدوال

ملاحظه :- الجدول ليس للحفظ حيث يعطى التحويل المطلوب استعماله في السؤال.

Inverse Laplace transformation

معكوس تحويل لابلاس

$$1. L^{-1} \left\{ \frac{1}{P} \right\} = 1$$

$$2. L^{-1} \left\{ \frac{1}{P-a} \right\} = e^{ax}$$

اي انه معكوس تحويلات لابلاس للمقدار هو ارجاع الدالة الاصلية التي كان تحويل لابلاس لها هو المقدار المراد ايجاد معكوس التحويل له.

$$\text{Ex. Find } L^{-1} \left\{ \frac{3}{P(P+3)} \right\} = ?$$

$$\text{Sol. } L^{-1} \left\{ \frac{3}{P(P+3)} \right\} = L^{-1} \left\{ \frac{1}{P} - \frac{1}{P+3} \right\}$$

$$= L^{-1} \left\{ \frac{1}{P} \right\} - L^{-1} \left\{ \frac{1}{P+3} \right\} \quad \text{by prop. 1)}$$

$$= 1 - e^{-3x} \quad \text{حسب جدول معكوس}$$

Ex. Find Laplace inverse transformation for the following:-

$$1. L^{-1} \left\{ \frac{3P+2}{P^2+4} \right\}$$

$$\text{Sol. } = L^{-1} \left\{ \frac{3P}{P^2+4} + \frac{2}{P^2+4} \right\}$$

$$= L^{-1} \left\{ \frac{3P}{P^2+4} \right\} + L^{-1} \left\{ \frac{2}{P^2+4} \right\}$$

$$= 3L^{-1} \left\{ \frac{P}{P^2+2^2} \right\} + 2L^{-1} \left\{ \frac{1}{P^2+2^2} \right\}$$

$$= 3 \cos 2x + 2 \sin 2x \quad \text{من الجدول}$$

$$2. L^{-1} \left\{ \frac{P+5}{(P+2)^2+9} \right\}$$

$$\text{Sol. } = L^{-1} \left\{ \frac{P+2}{(P+2)^2+3^2} \right\} + L^{-1} \left\{ \frac{3}{(P+2)^2+3^2} \right\}$$

$$= \cos 3x e^{-2x} + \sin 3x e^{-2x} \quad \text{من الجدول}$$

Solution of differential Equation by using Laplace transformation and inverse

$$\text{Let } a_2 y'' + a_1 y' + a_0 y = f(x)$$

معادلة تفاضلية خطية من الرتبة الثانية وتكون ذات معاملات ثابتة وغير متجانسة.

ولحل المعادلة نتبع الخطوات الآتية:-

١- نأخذ تحويل لابلاس لطرفي المعادلة المطلوب حلها.

٢- بعد ترتيب المعادلة حسب التحويل $L\{y\}$ نأخذ معكوس تحويل لابلاس

للطرفين فنحصل على الحل العام للمعادلة.

لحل المعادلة نحتاج الى تحويلات لابلاس للمشتقة الاولى والثانية

وكالاتي:-

$$L\{y'(x)\} = P L\{y(x)\} - y(0)$$

$$L\{y''(x)\} = P^2 L\{y(x)\} - Py(0) - y'(0)$$

Pf.1:-

$$\begin{aligned} L\{y'(x)\} &= \int_0^{\infty} y'(x) e^{-px} dx \\ &= \int_0^{\infty} e^{-px} \cdot y'(x) dx = y(x) e^{-px} \Big|_0^{\infty} + \int_0^{\infty} y(x) p e^{-px} dx \\ &= (y(\infty) e^{-p\infty} - y(0) e^{-p0}) + p \int_0^{\infty} y(x) e^{-px} dx \\ &= (0 - 1 \cdot y(0)) + pL\{y(x)\} \\ &= pL\{y(x)\} - y(0) \end{aligned}$$

Pf 2:-

$$L\{y''(x)\} = P^2 L\{y(x)\} - Py(0) - y'(0)$$

$$L\{y''(x)\} = \int_0^{\infty} y''(x) e^{-px} dx$$

Ex:- Solve $y' - y = 1$; $y(0) = 0$, $L(1) = \frac{1}{p}$

$$, L(e^{ax}) = \frac{1}{p - a}$$

Sol. $L\{y'\} - L\{y\} = L\{1\}$

$$p L\{y\} - y(0) - L\{y\} = \frac{1}{p}$$

$$(1-p) L\{y\} - 0 = \frac{1}{p}$$

$$(1-p) L\{y\} = \frac{1}{p}$$

$$L\{y\} = \frac{1}{(1-p)p} = \frac{1}{(1-p)} - \frac{1}{p}$$

$$\begin{aligned} L^{-1}\{L\{y\}\} &= L^{-1}\left\{\frac{1}{(1-p)} - \frac{1}{p}\right\} \\ &= L^{-1}\left\{\frac{1}{(1-p)}\right\} - L^{-1}\left\{\frac{1}{p}\right\} \end{aligned}$$

$$\therefore y = e^x - 1$$

Ex. :- Solve $y'' + 2y' + 5y = 0$; $y(0) = 1$, $y'(0) = 5$

$$L\{\sin bx e^{ax}\} = \frac{b}{(P + a)^2 + b^2}$$

$$L\{\cos bx e^{ax}\} = \frac{P + a}{(P + a)^2 + b^2}$$

Sol. $L\{y'' + 2y' + 5y\} = L\{0\}$

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = 0$$

$$P^2 L\{y\} - p y(0) - y'(0) + 2(p L\{y\} - y(0)) + 5L\{y\} = 0$$

$$(P^2 + 2p + 5) L\{y\} - p - 5 - 2 = 0$$

$$(P^2 + 2p + 5) L\{y\} = p + 7$$

$$L\{y\} = \frac{p+7}{(p^2 + 2p+5)}$$

$$L^{-1}\{L\{y\}\} = L^{-1}\left\{\frac{p+7}{(p^2 + 2p+5)}\right\}$$

$$y = L^{-1}\left\{\frac{p+1}{(p^2 + 2p+1)+4} + \frac{6}{(p^2 + 2p+1)+4}\right\}$$

$$y = L^{-1}\left\{\frac{p+1}{(p^2 + 2p+1)+4}\right\} + L^{-1}\left\{\frac{6}{(p^2 + 2p+1)+4}\right\}$$

$$y = L^{-1}\left\{\frac{p+1}{(p+1)^2+4}\right\} + 3L^{-1}\left\{\frac{2}{(p+1)^2+4}\right\}$$

$$\therefore y = \cos 2x \cdot e^{-x} + 3 \sin 2x \cdot e^{-x}$$

Ex. Solve (Homework)

1. $y' + 2y = e^x; y(0) = 0$
2. $\frac{dy}{dx} + 2y = \cos x; y(0) = 1$
3. $y' - 4y = 2; y(0) = 2$
4. $y'' - 4y' + 3y = 6e^{4x}; y(0) = 2, y'(0) = 6$
5. $y'' - 9y = 6 \cos 3x; y(0) = 0, y'(0) = 3$