التبـولـوجيـا العـامـــة

General Topology

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المصادر العربية :

[1] أساسيات التبولوجيا العامة – تاليف : وليام بيرفن – ترجمة : د. عطا الله ثامر العاني.
[2] التبولوجيا العامة – تاليف : د. عريبي الزوبعي.

English References

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[3] C. Kuratowski, Topologies, Warsaw,1952.
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ما هو علم التبولوجي ؟

التبولوجي كلمة مترجمة من الكلمة الإنجليزية Topology، وتنقسم كلمة التبولوجي إلى مقطعين المقطع الأول (Topo) التي تعود إلى أصل يوناني إلى (Topos) والتي تعني "مكان" (Place)، والمقطع الثاني هو (logy) والتي تعود لأصل يوناني (Logos) والتي تعني "دراسة" (Study)، فلو قمنا بعملية ربط المعنيين في الكلمة، لوجدنا أن التبولوجي هو الهندسة الحديثة في دراسة جميع التراكيب والمكونات للفضاءات المختلفة.

إذن يعرف علم التبولوجي :

هو أحد فروع علم الرياضيات والذي يهتم في دراسة تراكيب ومكونات وخصائص جميع الفضاءات المختلفة، بحيث تبقى هذه الخصائص متشابهه تحت عمليات التشكيل المتصلة (Smooth Deformations) دون أن يقوم بعملية تمزيق أو يترك فتحات في الإنتقال من أحدهما إلى الآخر وبالعكس أيضاً.

وكأن التعريف يخبرنا أن الهندسة التي يتعامل بها التبولوجي ليست الهندسة التي نعرفها، بل كأنها هندسة مطاطية، ولكي يتضح المفهوم بشكل جيد، لندرس الآتي :

من المعلوم لدينا أن المستوى الإقليدي في الهندسة الإعتيادية التي نعرفها، أنه بإمكاننا أن نقوم بعملية نقل الأشكال من مكان إلى آخر عن طريق الإزاحة ، وبإمكاننا أيضاً أن نقوم بعملية دوران له وعكسه وقلبه، ولكن لا نستطيع القيام بعملية ثني له أو القيام بعملية تمدد بشكل متصل.

مفهوم الهندسة المطاطية :

بشكل موجز أن الأشكال عبارة عن قطع من المطاط قابلة للثني والتمدد، و كل شكلين أو أكثر بإمكاننا أن نحصل على أحدهما من الآخر و بالعكس يكونا متشابهين .

فـمثـــلاً :

المثلث والدائرة والمربع، كلها أشكال موجودة في المستوى الإقليدي بخصائصها، و نقول أن أحدهما كافيء الآخر إذا كان لهما نفس المساحة.

في الهندسة المطاطية جميع هذه الأشكال هي نفسها متشابهه، فالدائرة هي نفسها المثلث، والسبب يعود إلى أنه يمكن تشكل المثلث من الدائرة بثني محيط الدائرة وجعلها كزوايا للمثلث وبالعكس يمكن إعادة تشكل الدائرة من المثلث بعملية تمديد أضلاع المثلث إلى دائرة، وهذا أيضاً ينطبق على المستطيل.

لاحظ أنه عندما قمنا بتشكل أحد هذه الأشكال من الآخر لم نقم بعملية قطع Cut لأحدها ولم نقم بعملية تمزيق للشكل من جهة أي ترك أي نقطة انفصال. وبالتالي في الهندسة المطاطية (التبولوجي) يكون الأشكال متشابهه إذا استطعنا الحصول على أحدهما من الآخر بعمليات متصلة وبالعكس. وبالتالي الدائرة لا تشابه الشكل الذي يشبه الرقم 8 بسبب أنه يمكن الحصول عليه من قبل الدائرة ولكن في العكس لا يمكن، بل سنحتاج إلى فصل منتصف رقم 8 لم نحتاج إلى أي نقطة انفصال من الدائرة إلى الرقم 8، وقيس على ذلك بأمثلة عديدة.

نستطيع القول بأن الأشكال التي تشترك بنفس العدد من الفتحات (نقاط الإنفصال) يكون كلاهما متشابه في الهندسة المطاطية، أي كلاهما يشتركان في نفس التبولوجي، والتي لا تحوي على أي فتحة تدعى مترابط بشكل بسيط Simply connected space.

التبولوجي يدخل تقريباً في جميع فروع الرياضيات بلغته الخاصة والمميزة.

فروع التبولوجي : يتفرع التبولوجي لعدة فروع وهي :

[1] التبولوجي النقطية (point-set Topology) : وهو الفرع الذي يهتم بالتبولوجي العامة من ناحية خصائص الفضاء من ناحية التراكيب كدراسة Compactness التراص و Connectedness (الترابط).

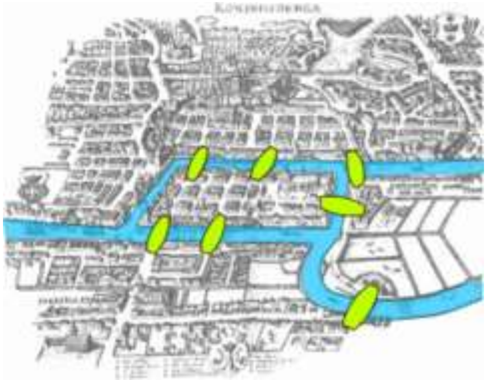
[2] التبولوجي الجبرية (Algebraic Topology) : وهو الفرع الذي يهتم بشكل عام في دراسة درجات الترابط من خلال التراكيب الجبرية، مثل دراسة علم الهومولوجي (Homology).

[3] التبولوجي الهندسية (Geometric Topology) : وهو الفرع الذي يهتم في دراسة Manifolds (بنية رياضية كل نقطة فيها لها جوار يكون هميومورفيك إلى الفضاء الإقليدي) (ويهتم بالأبعاد حسب أبعاد الفضاء الإقليدي).

[4] التبولوجي التفاضلية (Differential Topology)

تأريخ التبولوجي بشكل موجز :

بدأ التفكير في التبولوجي من خلال مشكلة أولير في المسألة المشهورة "السبعة الجسور في مدينة كونسبريك" (Seven Bridges of Königsberg)، وكانت ورقة أويلر عام 1736 أول نتيجة على الفضاء التبولوجي.



أول من قدم مصطلح التبولوجي هم الألمان باسم "Topologie" عام 1847 بواسطة جوهان بندكت، ومن ثم أظهر أصحاب التخصص في اللغة الإنجليزية أن كلمة Topologist هو كل شخص متخصص في التبولوجي.

أما التبولوجي الحديثة فتعمد بشكل قوي جداً على مفاهيم نظرية المجموعات التي أسست من قبل كانتور في أواخر القرن التاسع عشر.

قام عدة علماء بوضع تعاريف محددة له، فقام العالم أسكولي وغيرهم بوضع أول تعريف للفضاء المتري الذي يعتبر حالة خاصة في التبولوجي حالياً في سنة 1906.

و بعدها قام العالم هاوسدورف بوضع تعريف له والذي يعرف حالياً بفضاء هاوسدورف المشهور جداً في سنة 1914. ولكن أتى العالم كزميرز كورتويسكي Kazimierz Kuratowski. سنة 1922 بوضع التعريف المعروف لدينا حالياً.

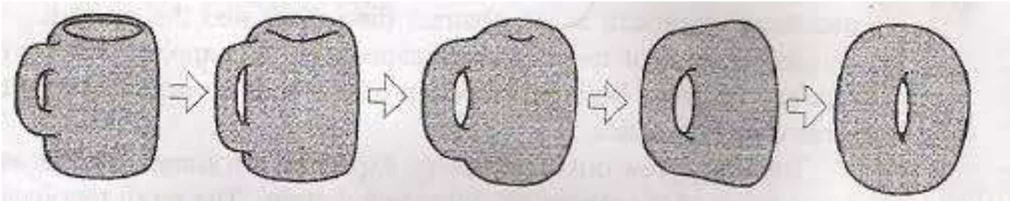
أمثـــــلـــــة :

من أشهر المقولات من باب الدعابة في التبولوجي هي :

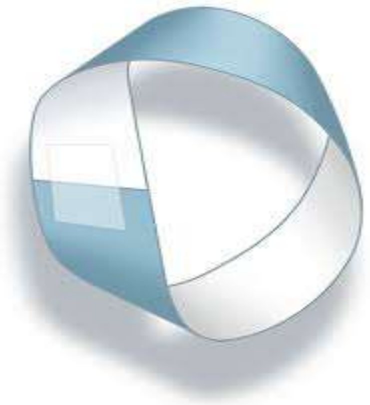
" A topologist is a person who cannot tell a coffee cup from a doughnut"

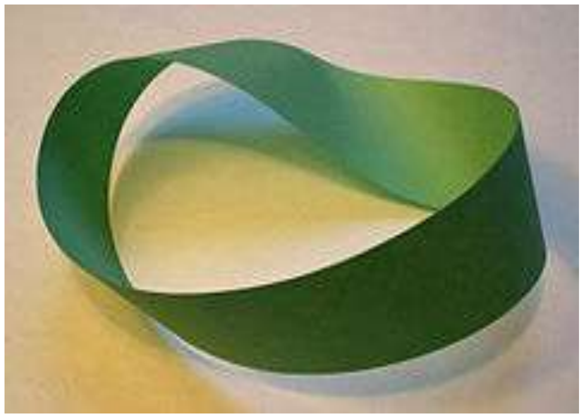
وتقول هذه العبارة أن :

متخصص التبولوجي لا يستطيع التميز بين كوب القهوة (الذي له يد واحدة) مع قطعة الكعكة (الدائرية) و السبب أنه كلاهما له فتحة واحدة ويمكن تشكيل أحدهما إلى الأخر وبالعكس دون وجود أي عملية فصل، وهي أحد تطبيقات علم التبولوجي الجبرية في Homology والشكل الآتي يبن ذلك :

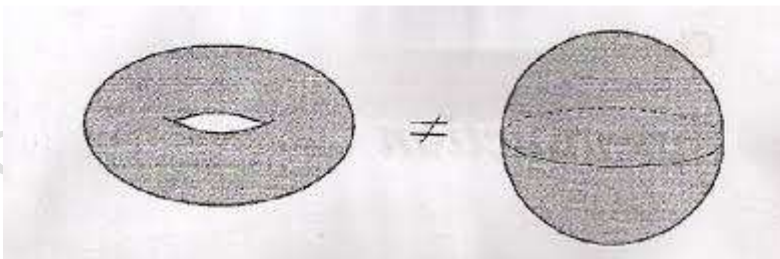


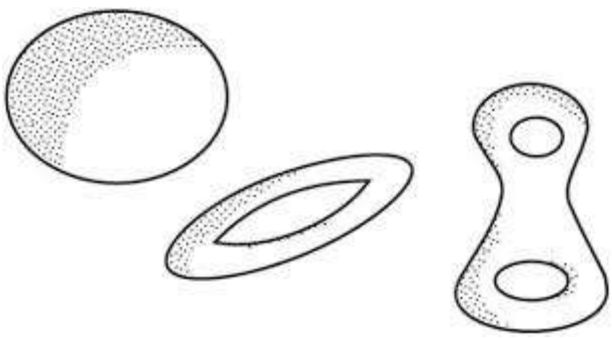
و من أشهر الأشكال في التبولوجي أيضاً هو شريط موبيص (Möbius strip)، وهو شريط له سطح واحد و حافة واحدة، كما في الشكل :





و هنالك الكثير من الأمثلة الجميلة ومنها أيضاً :





Chapter One : Topological Spaces

Definition : Topology & Topological Space

Let X be a nonempty set and τ be a family of subsets of X (i.e., $\tau \subseteq IP(X)$). We say τ is a **topology** on X if satisfy the following conditions :

(1) $X, \phi \in \tau$

(2) If $U, V \in \tau$, then $U \bigcap V \in \tau$

The finite intersection of elements from τ is again an element of τ.

(3) If $U_\alpha \in \tau$; $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau \quad \forall \alpha \in \Lambda$

The arbitrary (finite or infinite) union of elements of τ is again an element of τ.

We called a pair (X, τ) **topological space.**

Remarks :

[1] The topological space (X, τ) is sometimes called the **space** X.

[2] The elements of X are called **points** of the space.

[3] When write τ we said **topology** and when write (X, τ) we said **topological space**.

Definition : Open set & Closed set

Let (X, τ) be a topological space. The subsets of X belonging to τ are called **open sets** in the space X. i.e.,

$$\text{If } A \subseteq X \ \wedge \ A \in \tau \ \Rightarrow \ A \text{ open set}$$

The subset A of X is called a **closed set** in the space X if its complement X\A is open set. We will denoted the family of closed sets by $\mathcal{F}$. i.e.,

$$\text{If } A \subseteq X \ \wedge \ A \in \mathcal{F} \ \Rightarrow \ A \text{ closed set.}$$

Remark : The sets in (X, τ) may be

[1] open and not closed.

[2] closed and not open.

[3] closed and open (clopen).

[4] not open and not closed.

Example : Let X = {a, b, c}, $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi, \{a, c\}\}$, $\tau_3 = \{X, \phi, \{a, b\}, \{a, c\}\}$, $\tau_4 = \{X, \phi, \{a\}, \{b\}, \{a, c\}\}$ and $\tau_5 = \{X, \{a\}, \{b\}, \{a, b\}\}$. Is $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ topology on X.

Solution : Notes that τ_1 and τ_2 is topology on X since its satisfy the three conditions of topology.

τ_3 is not topology on X since $\{a, b\} \bigcap \{a, c\} = \{a\} \notin \tau_3$ (i.e., the condition two is not satisfy).

τ_4 is not topology on X since $\{a\} \bigcup \{b\} = \{a, b\} \notin \tau_4$ (i.e., the condition three is not satisfy).

τ_5 is not topology on X since $\phi \notin \tau_3$ (i.e., the condition one is not satisfy).

Remark : If $X \neq \phi$, then

[1] $\tau = \{X, \phi\}$ is a topology on X and its the smallest topology that we can defined on any set X and called **Indiscrete topology** and denoted by I. (i.e., $I = \{X, \phi\}$).

[2] $\tau = IP(X)$ is a topology on X and its the largest topology that we can defined on any set X and called **Discrete topology** and denoted by D. (i.e., $D = IP(X)$).

[3] If τ any topology on X then $I \subseteq \tau \subseteq D$.

[4] $\tau = D$ if and only if $\{x\} \in \tau \ \forall x \in X$.

Remark : there are 29 different topology on a set X contain only three elements.

If $X = \{1, 2, 3\}$, then all the following is a topology on X.

$\tau_1 = \{X, \phi\}$ Indiscrete Top., $\tau_2 = \{X, \phi, \{1\}\}$, $\tau_3 = \{X, \phi, \{2\}\}$, $\tau_4 = \{X, \phi, \{3\}\}$, $\tau_5 = \{X, \phi, \{1\}, \{1, 2\}\}$, $\tau_6 = \{X, \phi, \{1\}, \{1, 3\}\}$, $\tau_7 = \{X, \phi, \{1\}, \{1, 2\}, \{1, 3\}\}$, $\tau_8 = \{X, \phi, \{2\}, \{1, 2\}\}$, $\tau_9 = \{X, \phi, \{2\}, \{2, 3\}\}$, $\tau_{10} = \{X, \phi, \{2\}, \{1, 2\}, \{2, 3\}\}$, $\tau_{11} = \{X, \phi, \{3\}, \{1, 3\}\}$, $\tau_{12} = \{X, \phi, \{3\}, \{2, 3\}\}$, $\tau_{13} = \{X, \phi, \{3\}, \{1, 3\}, \{2, 3\}\}$, $\tau_{14} = \{X, \phi, \{1, 2\}\}$, $\tau_{15} = \{X, \phi, \{2, 3\}\}$, $\tau_{16} = \{X, \phi, \{1, 3\}\}$, $\tau_{17} = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$, $\tau_{18} = \{X, \phi, \{1\}, \{3\}, \{1, 3\}\}$, $\tau_{19} = \{X, \phi, \{2\}, \{3\}, \{2, 3\}\}$, $\tau_{20} = \{X, \phi, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$, $\tau_{21} = \{X, \phi, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}\}$, $\tau_{22} = \{X, \phi, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}\}$, $\tau_{23} = \{X, \phi, \{1\}, \{2, 3\}\}$, $\tau_{24} = \{X, \phi, \{2\}, \{1, 3\}\}$, $\tau_{25} = \{X, \phi, \{3\}, \{1, 3\}\}$, $\tau_{26} = \{X, \phi, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$, $\tau_{27} = \{X, \phi, \{1\}, \{3\}, \{1, 3\}, \{1,2\}\}$, $\tau_{28} = \{X, \phi, \{2\}, \{3\}, \{2, 3\}, \{1, 2\}\}$, $\tau_{29} = \{X, \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} = IP(X)$ Discrete Top..

Remark : If the number of elements of a set X four elements, then there are more than deference four hundred topology on X.

Theorem : Let (X, τ) be a topological space and $\mathcal{F}$ be a family of closed sets on X, then :

(1) $X, \phi \in \mathcal{F}$

(2) If $A, B \in \mathcal{F}$, then $A \bigcup B \in \mathcal{F} \quad \forall A, B \in \mathcal{F}$

(3) If $A_\alpha \in \mathcal{F}$; $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} A_\alpha \in \mathcal{F} \quad \forall A_\alpha \in \mathcal{F}$

Proof :

(1) $\because\ \phi \in \tau \Rightarrow \phi^c \in \mathscr{F} \Rightarrow X \in \mathscr{F}$

$\because\ X \in \tau \Rightarrow X^c \in \mathscr{F} \Rightarrow \phi \in \mathscr{F}$

(2) Let $A, B \in \mathscr{F} \Rightarrow A^c, B^c \in \tau$ (def. of closed sets)

$\Rightarrow A^c \bigcap B^c \in \tau$ (second condition of def. of top.)

$\Rightarrow (A \bigcup B)^c \in \tau$ (De Morgan's laws)

$\Rightarrow A \bigcup B \in \mathscr{F}$ (def. of closed sets)

(3) Let $A_\alpha \in \mathscr{F} \quad \forall\ \alpha \in \Lambda$

$\Rightarrow A^c{}_\alpha \in \tau \quad \forall\ \alpha \in \Lambda$

$\Rightarrow \bigcup_{\alpha\in\Lambda} A^c{}_\alpha \in \tau$ (third condition of def. of top.)

$\Rightarrow (\bigcap_{\alpha\in\Lambda} A_\alpha)^c \in \tau$ (De Morgan's laws)

$\Rightarrow \bigcap_{\alpha\in\Lambda} A_\alpha \in \mathscr{F}$ (def. of closed sets)

Now we introduce some important examples of topological spaces and show that the open sets and closed sets in this examples :

Example : Usual Topology on $\mathbb{R}$

Let $\tau_u = \{\mathbb{R}, \phi, U\ ;\ \forall\ x \in U\ \exists$ open interval $(a, b)\ ;\ x \in (a, b) \subseteq U\}$
or $\tau_u = \{U \subseteq \mathbb{R}\ ;\ U =$ union of family of open interval$\}$
show that $(\mathbb{R}, \tau_u)$ is a topological space.

Solution :

(1) $\mathbb{R} = (-\infty, \infty) \in \tau_u$ (i.e., $\mathbb{R}$ is open interval and every open interval is open set)

$\phi = (a, a) \in \tau_u$

(2) Let $U, V \in \tau_u$

if U or $V = \phi \Rightarrow U \bigcap V = \phi \in \tau_u$

if U or $V = \mathbb{R} \Rightarrow U \bigcap V = V \in \tau_u$ (if $U = \mathbb{R}$)

$\Rightarrow U \bigcap V = U \in \tau_u$ (if $V = \mathbb{R}$)

Otherwise,

Let $x \in U \bigcap V \Rightarrow x \in U \wedge x \in V$

$\because\ x \in U \Rightarrow \exists$ open interval $(a, b)\ ;\ x \in (a, b) \subseteq U$

$\because\ x \in V \Rightarrow \exists$ open interval $(c, d)\ ;\ x \in (c, d) \subseteq V$

$\Rightarrow x \in (a, b) \bigcap (c, d) \subseteq U \bigcap V$

$\Rightarrow x \in (\max\{a, c\}, \min\{b, d\}) \subseteq U \bigcap V$

$\Rightarrow \exists$ open interval $(\max\{a, c\}, \min\{b, d\})$;

$x \in (\max\{a, c\}, \min\{b, d\}) \subseteq U \bigcap V$

$\Rightarrow U \bigcap V \in \tau_u$

(3) Let $U_\alpha \in \tau_u$; $\alpha \in \Lambda$

if $U_\alpha = \mathbb{R}$ for some $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \mathbb{R} \in \tau_u \ \forall \alpha \in \Lambda$

if $U_\alpha = \phi$ for all $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \phi \in \tau_u \ \forall \alpha \in \Lambda$

if $U_\alpha = \phi$ for some $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcup_{\alpha\in\Lambda} U_\alpha$

Now,

Let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow x \in U_\alpha$ for some α

$\because\ x \in U_\alpha \Rightarrow \exists$ open interval (a, b) ; $x \in (a, b) \subseteq U_\alpha$

$\Rightarrow x \in (a, b) \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_u$

$\therefore$ $(\mathbb{R}, \tau_u)$ is a topological space.

Remarks :

[1] The sets $(0, 1) \cup (2, 4)$, $(-2, 1)$ … etc are open sets in τ_u.

[2] The natural numbers $\mathbb{N}$ is not open set since its cannot represented as a union of open intervals, but its closed set since $\mathbb{N}^c = (-\infty, 1) \cup (1, 2) \cup \ldots$ is open set in τ_u.

[3] Every set contains discrete points is closed set in τ_u.

[4] Every closed interval is closed set in τ_u.

[5] The rational numbers set $\mathbb{Q}$ and the irrational numbers set Irr are not open sets and not closed sets in τ_u.

Example : Let $X = \mathbb{R}$ and $\tau = \{\mathbb{R}, \phi, \mathbb{Q}, \text{Irr}\}$.

τ is a topology on $\mathbb{R}$ and τ is a topology different from τ_u in the previous example. In this example the open intervals is not open sets since it's not contain in τ, while $\mathbb{Q}$, Irr are open and closed in the same time.

How being a topology on any set :

Let X be any nonempty set and A be a proper nonempty subset of X, then

[1] $\tau = \{X, \phi, A\}$ is a topology on X for any X and for any A.

[2] $\tau = \{X, \phi, A, A^c\}$ is a topology on X and this topology has the property that every open set is close set in same time (i.e., $\tau = \mathcal{F}$).

Example : Cofinite Topology

Let X be infinite set and $\tau_{cof} = \{U \subseteq X, U^c = \text{finite set}\} \bigcup \{\phi\}$

Show that (X, τ_{cof}) is a topological space.

Solution :

(1) $\phi \in \tau_{cof}$ (def. of τ_{cof})

$\because$ $X^c = \phi$ and ϕ is finite set, then $X \in \tau_{cof}$

(2) Let $U, V \in \tau_{cof}$

if U or $V = \phi \quad \Rightarrow U \bigcap V = \phi \in \tau_{cof}$

if $U = X \;\Rightarrow U \bigcap V = V \in \tau_{cof}$

if $V = X \;\Rightarrow U \bigcap V = U \in \tau_{cof}$

if U and $V \neq \phi$, X

$\Rightarrow U^c$ and V^c finite set

So,

$(U \bigcap V)^c = U^c \bigcup V^c$ = finite set $\bigcup$ finite set = finite set

$\Rightarrow U \bigcap V \in \tau_{cof}$

(3) Let $U_\alpha \in \tau_{cof}$; $\alpha \in \Lambda$

if $U_\alpha = X$ for some $\alpha \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = X \in \tau_{cof} \quad \forall\ \alpha \in \Lambda$

if $U_\alpha = \phi$ for all $\alpha \quad \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \phi \quad \in \tau_{cof} \;\; \forall\ \alpha \in \Lambda$

if $U_\alpha \neq \phi$ or X for all $\alpha \quad \Rightarrow (\bigcup_{\alpha\in\Lambda} U_\alpha)^c = \bigcap_{\alpha\in\Lambda} U^c_\alpha = \bigcap$ finite sets = finite set

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_{cof}$

$\therefore$ (X, τ_{cof}) is a topological space.

Remarks :

[1] Notes that X is any set, so there are infinite number of the topological spaces that satisfy this definition according to the set which put replace from X which has a condition infinite set, so we can replies X by $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{Q}$ or Irr or $[0, 1]$ or $(-\infty, 2]$ or $\mathbb{C}$ …. etc.

Now : Take a special case when $X = \mathbb{N}$ and study the open and closed sets in the space $(\mathbb{N}, \tau_{cof})$.

Notes that, $\mathbb{N} \setminus \{1\}$ is open set since its complement is $\{1\}$ which is finite and the set of even numbers E^+ and odd numbers O^+ are not open sets since the complement of E^+ is O^+ and the complement of O^+ is E^+ and all E^+ and O^+ are not finite.

[2] In general : every open set in the space (X, τ_{cof}) is infinite set, but if the set is infinite this not mean its open i.e.

$$U \in \tau_{cof} \quad \underset{\not\Leftarrow}{\Rightarrow} \quad U \text{ infinite set}$$

[3] In general : every finite set is closed set and every closed set (except X) is finite set. i.e.,

$$A \in \mathcal{F}_{cof} \quad \Leftrightarrow \quad A \text{ finite set} \quad (A \neq X)$$

Example : Let X be any set contain more than one element and let x_0 any element in X and $\tau = \{U \subseteq X ;\ x_0 \in U\} \bigcup \{\phi\}$. Show that (X, τ) is a topological space.

Solution :

(1) $\phi \in \tau$ (def. of τ)

$X \in \tau$ (since X contain all its elements, therefore its contain x_0)

(2) Let $U, V \in \tau$

if U or $V = \phi \Rightarrow U \bigcap V = \phi \in \tau$

if $U = X \Rightarrow U \bigcap V = V \in \tau$

if $V = X \Rightarrow U \bigcap V = U \in \tau$

if U and $V \neq \phi, X$

$\Rightarrow x_0 \in U \wedge x_0 \in V$ (def. of τ)

$\Rightarrow x_0 \in U \bigcap V$ (def. of intersection)

$\Rightarrow U \bigcap V \in \tau$

(3) Let $U_\alpha \in \tau$; $\alpha \in \Lambda$

$\Rightarrow U_\alpha = \phi \quad \forall \alpha \in \Lambda \quad \vee \quad x_0 \in U_\alpha \ \forall \alpha \in \Lambda$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \phi \quad \vee \quad x_0 \in \bigcup_{\alpha\in\Lambda} U_\alpha$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

$\therefore$ (X, τ) is a topological space.

Remarks :

[1] Notes that any set not contained x_0 is a closed set and any set contained x_0 is open set.

Special case : Suppose that $X = \mathbb{R}$ and $x_0 = 0$, then

the sets $\{0\}$, $(-\infty, 2)$, $\mathbb{Q}$, $[0, 1]$ … etc are open. And

the sets $\{-4\}$, Irr, $\mathbb{N}$, $(6, \infty)$, $[3, 5]$ … etc are closed since it's not contained 0.

[2] In general : we can replace 0 by 2 or $\sqrt{5}$ or any other real number.

And we can replace $\mathbb{R}$ by any other set.

Example : Let X be a nonempty set contain more than one element and let x_0 any element in X and $\tau = \{U \subseteq X ; x_0 \notin U\} \bigcup \{X\}$. Show that (X, τ) is a topological space.

Solution :

(1) $X \in \tau$ (def. of τ)

$\phi \in \tau$ (since $x_0 \notin \phi$ by def. of τ)

(2) Let $U, V \in \tau$

if $U = X \quad \wedge \quad V = X \quad \Rightarrow U \bigcap V = X \in \tau$

if $x_0 \notin U$ or $x_0 \notin V \Rightarrow x_0 \notin U \bigcap V$

$\Rightarrow U \bigcap V \in \tau$ (def. of τ)

(3) Let $U_\alpha \in \tau$; $\alpha \in \Lambda$

if $U_\alpha = X \quad \forall \alpha \in \Lambda \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = X \in \tau$

if $\exists U_\alpha \neq X \Rightarrow x_0 \notin U_\alpha$ (def. of τ)

$\Rightarrow x_0 \notin \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$ (def. of τ)

$\therefore$ (X, τ) is a topological space.

Remarks :

[1] Special case of this example we can take $X = \mathbb{N}$ and $x_0 = 2$, then the open sets are $\mathbb{N}$ and every subset from $\mathbb{N}$ not contain the element 2, while every set contain the element 2 is closed set.

[2] There are infinite number of spaces from this types when replace X by any set and x_0 by any element.

Example : Let $X = \mathbb{N}$ and $\tau = \{A_n \subseteq \mathbb{N} : A_n = \{1, 2, \dots, n\} ;\ n \in \mathbb{N}\} \bigcup \{\mathbb{N}, \phi\}$ show that τ is a topology on $\mathbb{N}$.

Solution : Notes that the elements of τ as follow

$$A_1 = \{1\},\ A_2 = \{1, 2\},\ A_3 = \{1, 2, 3\}, \dots$$

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots\dots \subseteq A_n \subseteq \dots$$

(1) $X, \phi \in \tau$ (def. of τ)

(2) Let $A_i, A_j \in \tau$, then $A_i \bigcap A_j = \begin{Bmatrix} A_i \in \tau & if\ \ i \leq j \\ A_j \in \tau & if\ \ i > j \end{Bmatrix}$

(3) Let $A_\alpha \in \tau$, $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} A_\alpha = \begin{Bmatrix} A_\delta \in \tau & if\ \ \delta \geq \alpha\ and\ \alpha\ finite \\ \mathbb{N} \in \tau & if\ \ \alpha\ infinite \end{Bmatrix}$

$\therefore$ $(\mathbb{N}, \tau)$ is a topological space.

The following sets are open in this example :

$$A_{100} = \{1, 2, 3, \dots, 100\},\ A_{30} = \{1, 2, 3, \dots., 30\}$$

The following sets are closed in this example :

$$\{3, 4, 5, \dots\} = \mathbb{N} \setminus \{1, 2\},\ \ \mathbb{N} \setminus \{1, 2, 3, \dots., 10\},\ \ \mathbb{N} \setminus \{1, 2, 3, 4, 5\}$$

Example : Let X = $\mathbb{N}$ and $\tau = \{B_n \subseteq \mathbb{N} : B_n = \{n, n+1, n+2, \dots\} \; ; \; n \in \mathbb{N}\} \bigcup \{\phi\}$ show that τ is a topology on $\mathbb{N}$.

Solution : Notes that the elements of τ as follow

$$B_1 = \{1, 2, 3, \dots\},\; B_2 = \{2, 3, 4, 5, \dots.\},\; B_3 = \{3, 4, 5, \dots\}$$

$$B_1 = \mathbb{N},\; B_2 = \mathbb{N} \setminus \{1\},\; B_3 = \mathbb{N} \setminus \{1, 2\},\; \dots \text{ etc}$$

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots..$$

(1) $\mathbb{N}, \phi \in \tau$ (def. of τ)

(2) Let $B_i, B_j \in \tau$, then $B_i \bigcap B_j = \begin{Bmatrix} B_i \in \tau & \text{if } i \geq j \\ B_j \in \tau & \text{if } i < j \end{Bmatrix}$

(3) Let $B_\alpha \in \tau$, $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} B_\alpha = \begin{Bmatrix} B_\delta \in \tau & if & \delta \leq \alpha \text{ and } \alpha \text{ finite} \\ \mathbb{N} \in \tau & if & 1 \in \Lambda \quad \alpha \text{ infinite} \end{Bmatrix}$

$\therefore$ $(\mathbb{N}, \tau)$ is a topological space.

Remark : The open sets in this example are a closed sets in the previous example and vise verse.

Definition : Equal Topological Spaces

Let (X, τ), (Y, τ') be two topological spaces, we say that **(X, τ) equal to (Y, τ')** if the sets and topologies are equal, i.e.,

$$(X, \tau) = (Y, \tau') \Leftrightarrow X = Y \wedge \tau = \tau'$$

Definition : Finer Than & Coarser Than

Let τ_1, τ_2 be two topologies on X, we say the topology **τ_2 is Finer than τ_1** if the family τ_1 is subset of the family τ_2 and we say **τ_1 Coarser than τ_2** and denoted by $\tau_1 \leq \tau_2$, i.e.,

$$\tau_2 \text{ Finer than } \tau_1 \text{ or } \tau_1 \text{ Coarser than } \tau_2 \text{ if } \tau_1 \subseteq \tau_2.$$

Remarks : Let τ_1, τ_2 be topologies on X, then

[1] $\tau_1 \bigcap \tau_2$ is a topology on X since

(1) $\because$ $X, \phi \in \tau_1$ and $X, \phi \in \tau_2 \Rightarrow X, \phi \in \tau_1 \bigcap \tau_2$

(2) Let $U, V \in \tau_1 \bigcap \tau_2 \Rightarrow U, V \in \tau_1$ and $U, V \in \tau_2$

$\Rightarrow U \bigcap V \in \tau_1$ and $U \bigcap V \in \tau_2$ (since τ_1, τ_2 are topologies on X)

$\Rightarrow U \bigcap V \in \tau_1 \bigcap \tau_2$

(3) Let $U_\alpha \in \tau_1 \bigcap \tau_2$; $\alpha \in \Lambda$

$\Rightarrow U_\alpha \in \tau_1$ and $U_\alpha \in \tau_2$ $\forall \alpha \in \Lambda$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_1$ and $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_2$ (since τ_1, τ_2 are topologies on X)

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_1 \bigcap \tau_2$

$\therefore$ $\tau_1 \bigcap \tau_2$ is a topology on X.

[2] $\tau_1 \bigcup \tau_2$ is not topology on X in general, for example :
Let X = {1, 2, 3}, τ_1 = {X, ϕ, {1}} and τ_2 = {X, ϕ, {2}}. Notes that τ_1, τ_2 are topologies on X, but $\tau_1 \bigcup \tau_2$ = {X, ϕ, {1}, {2}} is not topology on X.

Remarks :

[1] Intersection of infinite number of open sets need not open set.
Example : Let (X, τ) = ($\mathbb{R}$, τ_u) and $U_n = (-\frac{1}{n}, \frac{1}{n})$ such that n $\in \mathbb{N}$. We know the open intervals is open sets in the space ($\mathbb{R}$, τ_u), so $\{U_n\}_{n\in\mathbb{N}}$ is a family of open sets, but the intersection of this family is not open set since

$$\bigcap_{n\in\mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) = \{0\} \text{ not open.}$$

[2] Union of any family of closed sets need not closed set.
Example : Let (X, τ) = ($\mathbb{N}$, τ_{cof}) and $A_n = \{2n\}_{n\in IN}$ i.e.,

$$A_1 = \{2\}, A_2 = \{4\}, A_3 = \{6\}, \ldots\ldots \ .$$

Notes that every set A_n is closed for all n in this space, but the union of this family is the positive even number {2, 4, 6, …….} and this set is not closed in this space (see example ($\mathbb{N}$, τ_{cof}), page 5).

Definition : Neighborhood

Let (X, τ) be a topological space, x $\in$ X and A $\subseteq$ X. We called A is a **neighborhood** for a point x if there exist an open set U contains x and contain in A and denoted by nbhd. i.e.,

$$A \text{ is a nbhd for } x \iff \exists\, U \in \tau \,;\, x \in U \subseteq A$$

If A is open set and contains x we called A is open neighborhood for a point x.

Example : In the space ($\mathbb{R}$, τ_u), every an open interval is an open nbhd for any point in this interval, while the closed interval or half open interval is nbhd for every point in this intervals except the end point in the closed interval.

Example : In the space ($\mathbb{N}$, τ_{cof}), find three open nbhds for the point 2 and two open nbhds for the point 3.

Solution :

A = {2, 3, 4, 5, …. }, B = {2, 10, 11, 12, …..} and C = {2, 20, 21, ….} are open nbhds for the element 2.
U = {3, 4, 5, ….} and V = {3, 6, 7,8, ….} are open nbhds for the element 3.

<u>Definition :</u> Basis or Base

Let (X, τ) be a topological space and β be a subfamily from τ. We called β is a **<u>basis or base</u>** for τ, if every element in τ is a union numbers of elements of β. i.e.,

β is a basis or base for τ $\Leftrightarrow$ (1) $\beta \subseteq \tau$

(2) $\forall\, U \in \tau$; $U = \bigcup_i B_i$; $B_i \in \beta$ $\forall\, i$

<u>Remark :</u> From the definition of the base we notes that the number of bases are not determined, so the number of bases is open, may be finite number and may be infinite number.

<u>Example :</u> Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, define a base for τ?

<u>Solution :</u> Let $\beta = \{X, \phi, \{a\}, \{b\}\}$

Clear $\beta \subseteq \tau$ and $X, \phi, \{a\}, \{b\} \in \tau$ and also $X, \phi, \{a\}, \{b\} \in \beta$ and

$\{a, b\} \in \tau \;\; \Rightarrow \;\; \{a, b\} = \underset{\in\beta}{\{a\}} \bigcup \underset{\in\beta}{\{b\}}$

$\therefore$ β is a base for τ.

<u>Remark :</u> τ is a baes for τ (i.e., we can chose $\beta = \tau$) and this is a special case and conclude from this case there is not exists topology has no base and this base called **trivial base**.

<u>Example :</u> Let $X = \{1, 2, 3\}$ and $\tau = IP(X) = \{X, \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2,3\}\}$. Define two different bases for τ?

<u>Solution :</u> Let $\beta_1 = \{\phi, \{1\}, \{2\}, \{3\}\}$, $\beta_2 = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$

We can show by a simple way that β_1 and β_2 are bases for τ since every one of them generated the elements of τ.

<u>Example :</u> Define base for the usual topology $(\mathbb{R}, \tau_u)$.

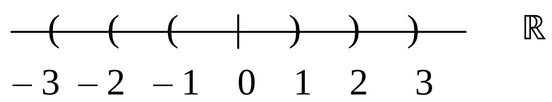
<u>Solution :</u> Let $\beta = \{(a, b) : a \in \mathbb{R} \wedge b \in \mathbb{R}\}$

Notes that β contain every open intervals which end points are real numbers (i.e., $(-3, 2) \in \beta$ while $(-\infty, 5) \notin \beta$)

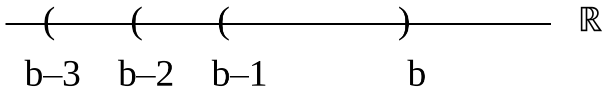
Notes that $\phi \in \beta$ since $\phi = (a, a)$ such that a is real number.

To prove β is a base for τ_u, it's enough to prove $\mathbb{R} = (-\infty, \infty)$, $(-\infty, b)$ and (a, ∞) equal union of family of elements of β. So we introduce this prove :

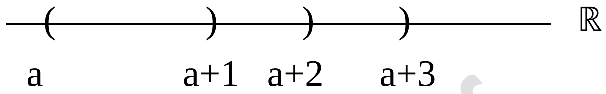
$\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$; $(-n, n) \in \beta$ $\forall\, n \in \mathbb{N}$



$(-\infty, b) = \bigcup_{n=1}^{\infty}(b-n, b)$; $(b-n, b) \in \beta \;\; \forall\, n \in \mathbb{N}$



$(a, \infty) = \bigcup_{n=1}^{\infty}(a, a+n)$; $(a, a+n) \in \beta \;\; \forall\, n \in \mathbb{N}$



This is a prove that β is a base for τ_u.

Remarks :

[1] We can find one topology for more than one base (i.e., the base is not unique).

[2] Every base for any topology must contains the empty set ϕ (i.e., $\phi \in \beta$) since $\phi \in \tau$ must be equal union element from β (by def. of β).

[3] X may be not belong to the base β and the previous example clear that.

[4] If the singleton set $\{x\} \in \tau$, then $\{x\} \in \beta$.

Theorem : Let (X, τ) be a topological space and β be a base for τ, then

(1) X is a union elements of β.

(2) If $B_1, B_2 \in \beta$, then $B_1 \cap B_2$ is a union elements of β.

Proof :

(1) $\because\; X \in \tau \;\Rightarrow\; X$ = union of elements of β (def. of base)

(2) $\because\; B_1 , B_2 \in \beta$ and $\beta \subseteq \tau$

$\Rightarrow\; B_1 , B_2 \in \tau$

$\Rightarrow\; B_1 \cap B_2 \in \tau$ (second condition from def. of top.)

$\Rightarrow B_1 \cap B_2$ = union of elements of β (def. of base)

This theorem clear the properties of base and the next theorem is a new method to get a topology by using a family of sets from β which satisfy the condition of previous theorem.

Theorem : Let X be a nonempty set and β be a family of subsets of X satisfying the following properties:

(1) X = union of family of elements of β

(2) Intersection any two elements of β is a union elements of β.

Then τ which is define as follows:

$\tau = \{U \subseteq X ;\; U = \text{union of elements of } \beta\}$

Is a topology on X and this is the unique on X such that β is a base for τ.

Proof : To prove τ is a topology on X must prove the three condition for topology.

(1) $\phi \in \tau$ (def. of τ)

$X \in \tau$ (from (1))

(2) Let $U, V \in \tau$ to prove $U \cap V \in \tau$

$\because U, V \in \tau \Rightarrow U = \bigcup_i B_i \wedge V = \bigcup_j B_j \ni B_i, B_j \in \beta \ \forall\, i, j$ (def. of τ)

$\Rightarrow U \cap V = (\bigcup_i B_i) \cap (\bigcup_j B_j) = \bigcup_{i,j} (B_i \cap B_j)$

$= \bigcup (\bigcup_k B_k)$; $B_k \in \beta$ (from (2))

(3) Let $U_\alpha \in \tau \ \forall\, \alpha \in \Lambda$ to prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

$\because U_\alpha \in \tau \Rightarrow U_\alpha = \bigcup_i B_i \ni B_i \in \beta \ \forall\, i$ (def. of τ)

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcup_{\alpha\in\Lambda}(\bigcup_i B_i) = \bigcup_k B_k$

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$ (def. of τ)

This prove that τ is a topology on X by define of τ.

To prove that τ is the unique topology generated from β. Suppose there exists another topology say τ' generated from β, this means that τ' = all possible union for elements of β, but τ = all possible union for elements of $\beta \Rightarrow \tau' = \tau$.

Definition : Subbasis

Let (X, τ) be a topological space and β be a base for τ and $\mathcal{S}$ be a subfamily from τ. We called $\mathcal{S}$ is a **subbasis** for τ if every element of basis β equal finite intersection numbers of elements of $\mathcal{S}$. i.e.,

$\mathcal{S}$ is a subbasis of $\tau \Leftrightarrow \forall\ B \in \beta \Rightarrow B = \bigcap_{j=1}^{n} S_j \ni S_j \in \mathcal{S} \ \forall\, j$; $j = 1, 2, \ldots, n$

Example : Let X = {a, b, c}, τ = {X, ϕ, {a}, {b}, {a, b}, {a, c}} and β = {ϕ, {a}, {b}, {a, c}}. Define a subbasis for τ.

Solution : Let $\mathcal{S}$ = {X, {a, c}, {a, b}, {b}}.

Clear, $\mathcal{S} \subseteq \tau$, to prove $\mathcal{S}$ is a subbasis for τ we compute all different intersection for elements of $\mathcal{S}$, if we get β, then $\mathcal{S}$ is subbasis for τ.

ϕ = {a, c} $\cap$ {b}, {a} = {a, c} $\cap$ {a, b}, {b} = {b} $\cap$ {b},

{a, c} = {a, c} $\cap$ {a, c}, {a, b} = {a, b} $\cap$ {a, b}, X = X $\cap$ X,

So, we get all elements of β, this means that $\mathcal{S}$ is a subbasis for τ.

Example : Define a subbasis for a usual topological space $(\mathbb{R}, \tau_u)$.

Solution : From the previous example we prove that $\beta = \{(a, b) ; a, b \in \mathbb{R}\}$ is a basis for τ_u. We must define a subbasis $\mathcal{S}$ for τ_u such that $\mathcal{S}$ generated β.

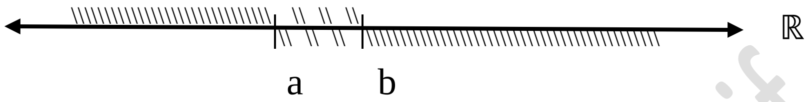
Define $\mathcal{S} = \{(a, b) \; ; \; a = -\infty \;\vee\; b = \infty\}$

Notes that $\mathcal{S} \subseteq \tau_u$ and

$(-1, 4) \in \beta \;\wedge\; (-1, 4) \notin \mathcal{S}$, $(-\infty, 3) \in \mathcal{S} \;\wedge\; (-\infty, 3) \notin \beta \Rightarrow \mathcal{S} \not\subset \beta \wedge \;\; \beta \not\subset \mathcal{S}$

Now, to show that $\mathcal{S}$ is subbasis for τ, we take an element of β and prove that its equal finite intersection numbers of elements of $\mathcal{S}$ as follow :

Let $(a, b) \in \beta \;\; ; \;\; a, b \in \mathbb{R}$



$$(a, b) = \underset{\in \mathcal{S}}{(-\infty, b)} \bigcap \underset{\in \mathcal{S}}{(a, \infty)}$$

$\therefore \;\; \mathcal{S}$ is a subbasis for τ_u.

Remarks :

[1] We can find more than one subbasis for one topology.

[2] May be ϕ not contain in a subbasis.

[3] $X \in \mathcal{S}$.

[4] τ is a subbasis for τ.

Theorem : Let X be a nonempty set and $\mathcal{S}$ be a subfamily of elements of X, then the set $\beta = \{B \subseteq X \; ; \; B = \text{finite intersection numbers of elements of } \mathcal{S}\}$ is a basis for the unique topology on X define as follow :

$$\tau = \{U \subseteq X \; ; \; U = \text{union of elements of } \beta\}.$$

Proof : without prove.

This theorem show that there exists a method to generated a topology on X if we have a set $\mathcal{S}$ such that $\mathcal{S}$ generated β and β generated τ and this topology is unique such that β is a basis for τ and $\mathcal{S}$ is a subbasis for τ.

Definition : Open Neighborhood System

Let (X, τ) be a topological space and $x \in X$ and η_x be a family of open sets (i.e., $\eta_x \subseteq \tau$) and satisfying the following conditions:

(1) $\eta_x \neq \phi$ for all $x \in X$.

(2) $x \in N \quad \forall \quad N \in \eta_x$.

(3) $\forall\, N_1 , N_2 \in \eta_x \;\Rightarrow\; \exists\, N_3 \in \eta_x$ such that $N_3 \subseteq N_1 \bigcap N_2$.

(4) $\forall\, N \in \eta_x \;\; \forall\, y \in N \;\; \exists\, N' \in \eta_y$ such that $N' \subseteq N$.

(5) $U \in \tau \;\Leftrightarrow\; \forall\, x \in U \;\exists\, N \in \eta_x$ such that $N \subseteq U$.

We called the family $\eta = \{\eta_x ; x \in X\}$ **open neighborhood system** for τ and denoted by (o.n.s)

Example : Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define open neighborhood system for τ.

Solution : We must prove for all element in X a family of this element satisfy the five conditions in the definition as follow :

$\eta_a = \{\{a\}, \{a, b\}\}$, $\eta_b = \{\{b\}, \{a, b\}\}$, $\eta_c = \{X\}$, notes that

(1) η_a, η_b and η_c are nonempty and contain of a family of open sets and

(2) $\forall\ N \in \eta_a \Rightarrow a \in N$, $\forall\ N \in \eta_b \Rightarrow b \in N$ and $\forall\ N \in \eta_c \Rightarrow c \in N$,

(3) Intersection of any two element in η_a or η_b or η_c is an element in η_a or η_b or η_c

(4) $\{a, b\} \in \eta_a$; $b \in \{a, b\}$ and $b \in \{b\}$; $\{b\} \subseteq \{a, b\}$

(5) Every open set satisfy the five condition

$\therefore\ \eta = \{\eta_a, \eta_b, \eta_c\}$ is open neighborhood system for τ

Example : Define open neighborhood system for a usual topological space $(\mathbb{R}, \tau_u)$.

Solution : Let $x \in \mathbb{R}$, define η_x as follow : $\eta_x = \{(a, b) ; x \in (a, b)\}$

i.e., η_x is a family of every open sets that contain x.

clear $\eta_x \subseteq \tau_u$ and η_x satisfy the five conditions of open neighborhood system for τ as follow :

(1) $\eta_x \neq \phi$ since $(x - \varepsilon, x + \varepsilon) \in \eta_x$; $\varepsilon > 0$

(2) if $N \in \eta_x$, then $N = (a, b)$ and $x \in (a, b)$ (def. of η_x)

(3) if $N_1 , N_2 \in \eta_x$

$\Rightarrow N_1$ open interval s.t. $x \in N_1$ and N_2 open interval s.t. $x \in N_2$

$\Rightarrow N_1 \bigcap N_2 \neq \phi$

$\Rightarrow N_1 \bigcap N_2$ open interval s.t. $x \in N_1 \bigcap N_2$

$\Rightarrow N_1 \bigcap N_2 \in \eta_x$.

(4) let $N \in \eta_x$ and $y \in N \Rightarrow N$ open interval s.t. $y \in N \Rightarrow N \in \eta_y$.

(5) This condition is satisfy from definition of usual topology on $\mathbb{R}$.

Theorem : Let X be a nonempty and η_x be a family of subsets from X. for all $x \in X$; η_x satisfy the condition (1), (2), (3), (4) in the definition of open neighborhood system , then τ which define as follow :

$$\tau = \{U \subseteq X ;\ \forall\ x \in U\ \ \exists\ N \in \eta_x \text{ such that } N \subseteq U \}$$

Is a topology on X such that η_x is open neighborhood system for τ.

Proof : We must prove τ satisfy the three conditions for topology.

(1) $X \in \tau$ (since X contains all subset of X)

$\phi \in \tau$ (since, $x \in \phi \Rightarrow \exists N \in \eta_x ; N \subseteq \phi$)

$(F \Rightarrow F) = T$

(2) Let $U, V \in \tau$ To proof $U \cap V \in \tau$

Let $x \in U \cap V \Rightarrow x \in U$ and $x \in V$ (def. of $\cap$)

$\Rightarrow \exists N_1 \in \eta_x$ such that $N_1 \subseteq U$ and $\exists N_2 \in \eta_x$ such that $N_2 \subseteq V$ (def. of τ)

$\Rightarrow \exists N_3 \in \eta_x$ such that $N_3 \subseteq N_1 \cap N_2$

(condition (3) from open neighborhood system)

$\Rightarrow N_3 \subseteq U$ and $N_3 \subseteq V$

$\Rightarrow N_3 \subseteq U \cap V$ (def. of $\cap$)

$\Rightarrow U \cap V \in \tau$

(3) Let $U_\alpha \in \tau$; $\alpha \in \Lambda$ To proof $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

Let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow \exists \alpha \in \Lambda ; x \in U_\alpha$ (def. of $\bigcup$)

$\Rightarrow \exists N \in \eta_x$ such that $N \subseteq U_\alpha$ (since $U_\alpha \in \tau$ and def. of τ)

$\Rightarrow N \subseteq U_\alpha \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

$\Rightarrow N \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

$\therefore \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau$

$\therefore$ τ is a topology on X and from prove above we have η_x $\forall x \in X$ is open neighborhood system for τ.

Remark : From information above we have five deference method to define topology on a nonempty set as follow :

[1] Direct definition for τ by write $\tau = \{\ldots.\ \}$.

[2] Using the family $\mathscr{F}$ such that the complement of this family is topology.

[3] Using the family β such that the union of all possible of elements of β is topology.

[4] Using the family $\mathscr{S}$ such that the finite intersection of elements of $\mathscr{S}$ is a basis for topology.

[5] Using the family η_x ; $x \in X$ and τ is the family of every sets that contain open neighborhood for every element.

Derived Sets

Definition : Interior points and Interior set

Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in A$ is called an **interior point** of A iff there exists an open set $U \in \tau$ containing x such that $x \in U \subseteq A$. The set of all interior points of A is called the **interior** of A and is denoted by A^o or Int(A). i.e.,

$$A^o = \{x \in A : \exists\, U \in \tau\ ;\ x \in U \subseteq A\}$$

$$x \in A^o \Leftrightarrow \exists\, U \in \tau\ ;\ x \in U \subseteq A$$

if $x \notin A^o$, we define

$$x \notin A^o \Leftrightarrow \forall\, U \in \tau \text{ such that } x \in U \text{ and } U \not\subset A$$

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $A = \{b\}$, $B = \{a, c\}$ and $C = \{c\}$. Find A^o, B^o and C^o.

Solution :

$A^o = \{b\} = A$ since $b \in U = \{b\} \subseteq A = \{b\}$

$B^o = \{a\}$ since $a \in U = \{a\} \subseteq B = \{a\}$

$C^o = \phi$ since the only open set contain in C is ϕ.

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A^o \subseteq A$

(2) $A \subseteq B \Rightarrow A^o \subseteq B^o$

(3) $A \in \tau$ (i.e., A is open) $\Leftrightarrow A^o = A$

(4) $A^o \bigcap B^o = (A \bigcap B)^o$

(5) $A^o \bigcup B^o \subseteq (A \bigcup B)^o$

(6) $A^o = \bigcup\{U \in \tau\ ;\ U \subseteq A\}$ (this means A^o is the large open set contain in A)

Proof :

(1) From definition of A^o

(2) Suppose that $A \subseteq B$ to prove $A^o \subseteq B^o$

Let $x \in A^o \Rightarrow \exists\, U \in \tau\ ;\ x \in U \subseteq A$ (def of A^o)

$\Rightarrow \exists\, U \in \tau\ ;\ x \in U \subseteq B$ (since $A \subseteq B$)

$\Rightarrow x \in B^o$ (def of B^o)

(3) ($\Rightarrow$) Suppose that A is open, to prove $A^o = A$

From (1) $A^o \subseteq A$ ---------------(1)

Let $x \in A \Rightarrow x \in A \subseteq A$ (since $A \in \tau$)

$\Rightarrow x \in A^o$ (def of A^o)

$\Rightarrow A^o \subseteq A$ ---------------(2)

From (1) and (2), we have $A^o = A$

($\Leftarrow$) Suppose that $A^o = A$, to prove A is open

$\forall\ x \in A \Rightarrow \exists\, U_x \in \tau\, ;\, x \in U_x \subseteq A$ (since $A^o = A$)

$\Rightarrow \bigcup_{x\in A} U_x \subseteq A \quad \wedge \quad A \subseteq \bigcup_{x\in A} U_x$

$\Rightarrow A = \bigcup_{x\in A} U_x$

But, U_x open set $\forall\ x \Rightarrow \bigcup_{x\in A} U_x$ is open

$\Rightarrow A$ is open (by three condition of def. of top.)

(4) To prove $A^o \bigcap B^o = (A \bigcap B)^o$, we must prove

$(A \bigcap B)^o \subseteq A^o \bigcap B^o \ \wedge\ A^o \bigcap B^o \subseteq (A \bigcap B)^o$

$(A \bigcap B) \subseteq A \quad \wedge \quad (A \bigcap B) \subseteq B$ (def. of $\bigcap$)

$\Rightarrow (A \bigcap B)^o \subseteq A^o \quad \wedge \quad (A \bigcap B)^o \subseteq B^o$ (from (2) above)

$\Rightarrow (A \bigcap B)^o \subseteq A^o \bigcap B^o$ ---------------(1)

From (1) $A^o \subseteq A \ \wedge\ B^o \subseteq B$

$\Rightarrow A^o \bigcap B^o \subseteq A \bigcap B$

$\because$ $A^o \bigcap B^o$ open set containing in $A \bigcap B$

and $(A \bigcap B)^o$ large open set containing in $A \bigcap B$

$\Rightarrow A^o \bigcap B^o \subseteq (A \bigcap B)^o$ ---------------(2)

From (1) and (2), we have $(A \bigcap B)^o = A^o \bigcap B^o$

(5) $A \subseteq A \bigcup B \ \wedge\ B \subseteq A \bigcup B$ (def. of $\bigcup$)

$\Rightarrow A^o \subseteq (A \bigcup B)^o \quad \wedge \quad B^o \subseteq (A \bigcup B)^o$

$\Rightarrow A^o \bigcup B^o \subseteq (A \bigcup B)^o$

(6) To prove $A^o = \bigcup\{U \in \tau\, ;\, U \subseteq A\}$

$x \in A^o \Leftrightarrow \exists\, U \in \tau\, ;\, x \in U \subseteq A$ (def of A^o)

$\Leftrightarrow x \in \bigcup\{U \in \tau\, ;\, U \subseteq A\}$

Since the element x belong to one of this sets in the union then its belong to union

$\therefore A^o = \bigcup\{U \in \tau\, ;\, U \subseteq A\}$

Remarks :

[1] The converse of property (2) is not true, i.e.,

$$A^o \subseteq B^o \not\Rightarrow A \subseteq B$$

The following example show that :

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$, $A = \{a\}$, $B = \{b, c\}$.

$A^o = \phi$ and $B^o = \{b, c\} = B$

Notes that, $A^o \subseteq B^o$ but $A \not\subset B$

[2] The converse contains of property (5) is not true in general. i.e.,

$$(A \bigcup B)^o \not\subset A^o \bigcup B^o$$

In the previous example show that :

$A \bigcup B = X \;\Rightarrow\; (A \bigcup B)^o = X$

But, $A^o = \phi$ and $B^o = \{b, c\} \;\Rightarrow\; A^o \bigcup B^o = \{b, c\}$ and $X \not\subset \{b, c\}$.

[3] There exists a special cases of property (3) as follow :

$X \in \tau \;\Rightarrow\; X^o = X$ and $\phi \in \tau \Rightarrow \phi^o = \phi$ and $(A^o)^o = A^o$

In a space (X, I) the only open sets are X and ϕ, so if $A \subsetneq X$, then $A^o = \phi$.

In a space (X, D) every subset of X is open, so $\forall\, A \subseteq X$, then $A^o = A$.

[4] If $\{x\}$ open set in any topological space, then x is interior point of any set contain x, i.e., $\{x\} \in \tau \;\Rightarrow\; x \in A^o \;\; \forall\; A$ such that $x \in A$

Example : In usual topological space $(\mathbb{R}, \tau_u)$, find the interior of the following sets :

$$A = [a, b],\ B = \mathbb{N},\ C = \mathbb{Q},\ D = [0, \infty)$$

Solution :

Interior of any set in this example is the largest open set containing in this set.

$[a, b]^o = [a, b)^o = (a, b]^o = (a, b)^o = (a, b)$

$\mathbb{N}^o = \mathbb{Z}^o = \mathbb{P}^o = E^o = O^o = \phi$

$\mathbb{Q}^o = Irr^o = \phi$

$[a, \infty)^o = (a, \infty)$ and $(-\infty, b]^o = (-\infty, b)$.

Example : In cofinite topological space $(\mathbb{N}, \tau_{cof})$, let $A \subseteq \mathbb{N}$. Find A^o.

Solution : If A is open set, then $A^o = A$. For example $A = \{4, 5, 6, \dots\}$

If A is not open set, so there exists two cases either A closed set or A is not closed set, then $A^o = \phi$ [[since A^o is open set, this means by definition τ_{cof} that the complement of A^o is finite set and since $A^o \subseteq A$ (in general), then the complement of A must be finite if $A^o \neq \phi$. This means the interior of a set in this space either ϕ or A.

Definition : **Exterior points and Exterior set**

Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in A^c$ is called an **exterior point** of A iff there exists an open set $U \in \tau$ containing x such that $x \in U \subseteq A^c$. The set of all exterior points of A is called the **exterior** of A and is denoted by A^x or Ext(A). i.e.,

$$A^x = \{x \in A^c : \exists\, U \in \tau\, ;\, x \in U \subseteq A^c\}$$

$$x \in A^x \Leftrightarrow \exists\, U \in \tau\, ;\, x \in U \subseteq A^c$$

if $x \notin A^x$, we define

$$x \notin A^x \Leftrightarrow \forall\, U \in \tau \text{ such that } x \in U \not\subset A^c$$

Remark : From definition we have $A^x \subseteq A^c$ or $A^x \cap A = \phi$ and $A^x = (A^c)^o$.

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $A = \{b\}$, $B = \{a, c\}$ and $C = \{c\}$. Find A^x, B^x and C^x.

Solution :

$A^x = (A^c)^o = \{a\}$ (largest open set contain in A^c)

$B^x = (B^c)^o = \phi$ and $C^x = \{a, b\}$.

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A^o \bigcap A^x = \phi$

(2) $A \subseteq B \Rightarrow B^x \subseteq A^x$

(3) $(A \bigcup B)^x = A^x \bigcap B^x$

(4) $A^c \in \tau$ (i.e., A closed) $\Leftrightarrow A^x = A^c$

(5) $A^x \bigcup B^x \subseteq (A \bigcap B)^x$

Proof :

(1) From definition of A^o $\Rightarrow A^o \subseteq A$ and $A^x \subseteq A^c$

$\Rightarrow A^o \bigcap A^x \subseteq A \bigcap A^c$

$\Rightarrow A^o \bigcap A^x \subseteq \phi$

$\Rightarrow A^o \bigcap A^x = \phi$

(2) Suppose that $A \subseteq B$ to prove $B^x \subseteq A^x$

Let $x \in B^x \Rightarrow \exists\, U \in \tau\, ;\, x \in U \subseteq B^c$ (def. of B^x)

$\Rightarrow \exists\, U \in \tau\, ;\, x \in U \subseteq A^c$ (since $A \subseteq B \Rightarrow B^c \subseteq A^c$)

$\Rightarrow x \in A^x$ (def. of B^x)

$\therefore B^x \subseteq A^x$

(3) To prove $(A \bigcup B)^x = A^x \bigcap B^x$

$(A \bigcup B)^x = ((A \bigcup B)^c)^o = (A^c \bigcap B^c)^o = (A^c)^o \bigcap (B^c)^o = A^x \bigcap B^x$

(4) ($\Rightarrow$) Suppose that A is closed or A^c is open, to prove $A^x = A^c$

$A^c \in \tau \Rightarrow (A^c)^o = A^c$ (by theorem, $A \in \tau \Leftrightarrow A^o = A$)

$\Rightarrow A^x = A^c$ (since $(A^c)^o = A^x$)

($\Leftarrow$) Suppose that $A^x = A^c$, to prove A is closed or A^c is open

$A^x = A^c \Rightarrow (A^c)^o = A^c$ (since $(A^c)^o = A^x$)

$\Rightarrow A^c \in \tau$ (by theorem, $A \in \tau \Leftrightarrow A^o = A$)

$\Rightarrow$ A is closed

(5) $A \bigcap B \subseteq A$ and $A \bigcap B \subseteq B \;\Rightarrow\; A^x \subseteq (A \bigcap B)^x$ and $B^x \subseteq (A \bigcap B)^x$

$$\Rightarrow\; A^x \bigcup B^x \subseteq (A \bigcap B)^x.$$

Example : In usual topological space $(\mathbb{R}, \tau_u)$, find the exterior of the following sets :

$$\mathbb{N}, \mathbb{Q}, (6, 7), \{-\sqrt{2}, \sqrt{2}\}, (-\infty, 5], [-1, \infty), [2, 4]$$

Solution :

exterior of any set in this example is the largest open set exterior this set.

$\mathbb{N}^x = (\mathbb{N}^c)^o$

$\mathbb{R}$ ——— 1| — 2| — 3| — 4| —

$\mathbb{N}^c = \mathbb{R} - \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup \ldots\ldots$

Clear that $\mathbb{N}^c$ is a union of open interval, so it's open set

$\mathbb{N}^x = (\mathbb{N}^c)^o = \mathbb{R} - \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup \ldots\ldots$

$\mathbb{Q}^x = \phi$ and $(\mathrm{Irr})^x = \phi$

$(6, 7)^x = (-\infty, 6) \cup (7, \infty) \subseteq \mathbb{R} \setminus [6, 7]$

——— 6| ——— 7|

$\{-\sqrt{2}, \sqrt{2}\}^x = (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty) = \mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\}$

$(-\infty, 5]^x = (5, \infty)$ and $[-1, \infty)^x = (-\infty, -1)$

$[2, 4]^x = \mathbb{R} - [2, 4] = (-\infty, 2) \cup (4, \infty)$

Remarks :

[1] In a space (X, τ), every one X, ϕ are closed sets, so property (4) apply of them, i.e., $X^x = \phi$, $\phi^x = X$.

[2] In a space (X, I), if $\phi \neq A \subseteq X$, then $A^x = \phi$ because the only sets in I are X, ϕ and since $A \neq \phi$, then $A^x \neq X$, so the unique open set contain in A^c is ϕ.

[3] In a space (X, D), if $A \subseteq X$, then $A^x = A^c$ because every sets in D are open and closed.

Example : Let $X = \mathbb{R}$ and $\tau = \{X, \phi, \mathbb{N}, \mathbb{P}\}$; $\mathbb{P}$ is prime numbers set and $\mathbb{P} \subseteq \mathbb{N}$. Clear that τ is a topology on $\mathbb{R}$ and the open sets in this space are $\mathbb{R}$, ϕ, $\mathbb{N}$, $\mathbb{P}$ only. Find exterior set of the following sets :

$$\mathbb{Q}, \quad \mathrm{Irr}, \quad [2, 6], \quad \mathbb{N}, \quad \mathbb{Z}, \quad (-\infty, 1]$$

Solution :

$\mathbb{Q}^x = \phi$ since $\mathbb{Q}^c = \mathrm{Irr}$ and there is no open set contain in Irr except ϕ.

$\mathrm{Irr}^x = \mathbb{N}$ since $(\mathrm{Irr})^c = \mathbb{Q}$ and $\mathbb{N}$ is large open set contain in $\mathbb{Q}$.

$[2, 6]^x = \phi$ since $[2, 6]^c = (-\infty, 2) \cup (6, \infty)$ and there is no open set contain in $(-\infty, 2) \cup (6, \infty)$ except ϕ.

$\mathbb{N}^x = \phi$ since $\mathbb{N}^c = \mathbb{R} \setminus \mathbb{N}$ and $\mathbb{R} \setminus \mathbb{N}$ not contain $\mathbb{R}, \mathbb{N}, \mathbb{P}$.

$\mathbb{Z}^x = \phi$ since $\mathbb{Z}^c = \mathbb{R} \setminus \mathbb{Z}$ and $\mathbb{R} \setminus \mathbb{Z}$ not contain $\mathbb{R}, \mathbb{N}, \mathbb{P}$.

$(-\infty, 1]^x = \mathbb{P}$ since $(-\infty, 1]^c = (1, \infty)$ and $\mathbb{P} \subseteq (1, \infty)$ while $\mathbb{N} \not\subset (1, \infty)$.

Definition : Boundary points and boundary set

Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called a **boundary point** of A iff every open set in X containing x contains at least one point of A, and at least one point of A^c. The set of all boundary points of A is called the **boundary** of A and is denoted by A^b or Bd(A) or b(A) or $\partial(A)$. i.e.,

$$A^b = \{x \in X : \forall\, U \in \tau \,;\, x \in U ,\, U \cap A \neq \phi \;\wedge\; U \cap A^c \neq \phi\}$$

$$x \in A^b \Leftrightarrow \forall\, U \in \tau \,;\, x \in U ,\, U \cap A \neq \phi \;\wedge\; U \cap A^c \neq \phi$$

if $x \notin A^b$, we define

$$x \notin A^b \Leftrightarrow \exists\, U \in \tau \,;\, x \in U ,\, U \cap A = \phi \;\vee\; U \cap A^c = \phi.$$

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a, c\}$, $B = \{c\}$ and $C = \{a, b\}$. Find A^b, B^b and C^b.

Solution :

$A^b = ?$

To find the boundary of any set we must choose every open sets for every point in X and notes satisfy the definition or not.

$a \in X$ and the open sets contain a are X, {a}, {a, b}

notes that : $\{a\} \cap A = \{a\} \neq \phi$ while $\{a\} \cap A^c = \{a\} \cap \{b\} = \phi \;\Rightarrow\; a \notin A^b$.

$b \in X$ and the open sets contain a are X, {b}, {a, b}

notes that : $\{b\} \cap A = \phi \Rightarrow b \notin A^b$.

$c \in X$ and the only open set contain c is X.

notes that : $X \cap A \neq \phi$ and $X \cap A^c \neq \phi \;\Rightarrow\; c \in A^b$.

Therefore $A^b = \{c\}$

$B^b = ?$

Since $\{a\} \cap B = \{a\} \cap \{c\} = \phi \;\Rightarrow\; a \notin B^b$

Since $\{b\} \cap B = \{b\} \cap \{c\} = \phi \;\Rightarrow\; b \notin B^b$

Since $X \cap B \neq \phi$ and $X \cap B^c \neq \phi \;\Rightarrow\; c \in B^b$.

Therefore $B^b = \{c\}$

$C^b = ?$. by similar way we have $a \notin C^b$, $b \notin C^b$, $c \in C^b$

Therefore $C^b = \{c\}$.

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, c\}\}$, $A = \{b\}$, $B = \{a, b\}$ and $C = \{b, c\}$. Find A^b, B^b and C^b.

Solution :

$A^b = ?$

$a \in X$ and the open sets contain a are $X, \{a, c\}$

notes that : $\{a, c\} \bigcap A = \phi \Rightarrow a \notin A^b$.

$b \in X$ and the only open set contain b is X

notes that : $X \bigcap A \neq \phi$ and $X \bigcap A^c \neq \phi \Rightarrow b \in A^b$.

$c \in X$ and the open sets contain a are $X, \{a, c\}$

notes that : $\{a, c\} \bigcap A = \phi \Rightarrow c \notin A^b$.

Therefore $A^b = \{b\}$

$B^b = ?$

Since $X \bigcap B \neq \phi$ and $X \bigcap B^c \neq \phi$ also

$\{a, c\} \bigcap B \neq \phi$ and $\{a, c\} \bigcap B^c \neq \phi \Rightarrow a \in B^b$

Since $X \bigcap B \neq \phi$ and $X \bigcap B^c \neq \phi \Rightarrow b \in B^b$.

Since $X \bigcap B \neq \phi$ and $X \bigcap B^c \neq \phi$ also

$\{a, c\} \bigcap B \neq \phi$ and $\{a, c\} \bigcap B^c \neq \phi \Rightarrow c \in B^b$

Therefore $B^b = \{a, b, c\} = X$.

$C^b = ?$

$a \in C^b$, $b \in C^b$, $c \in C^b$

Therefore $C^b = \{a, b, c\} = X$.

Remarks :

[1] Notes that : $A^b \subseteq A$ or $A^b \subseteq A^c$ or $A^b \bigcap A \neq \phi$ or $A^b \bigcap A^c \neq \phi$. i.e., anything possible.

[2] If $\{a\} \in \tau$ in any topological space (X, τ) ; $a \in X$, then a is not boundary point for any set A in X since if $a \in A$, then $\{a\} \bigcap A^c = \phi$ and if $a \notin A$, then $\{a\} \bigcap A = \phi$, so in this two case $a \notin A^b$. Therefore, we can use this idea to have a set contain number of boundary points we determent, for example :

Example : Give an example for a subset A of topological space (X, τ) contains six boundary points.

Solution : Let $X = \{1, 2, 3, 4, 5, 6, 7\}$, $\tau = \{X, \phi, \{1\}\}$ and let $A \subseteq X$; $\phi \neq A \neq \{1\}$, then $A^b = \{2, 3, 4, 5, 6, 7\}$.

We can generalizations this example for any numbers of boundary points.

[3] In a space (X, I), if $\phi \neq A \subseteq X$, then $A^b = X$ because the only open set in I is X for every element in X and $X \bigcap A \neq \phi$ and $X \bigcap A^c \neq \phi$.

[4] In a space (X, D), if $A \subseteq X$, then $A^b = \phi$ because $\{x\} \in D$ for all $x \in X$ and by Remake (2) every point is not boundary.

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A^b \bigcap A^o = \phi$ and $A^b \bigcap A^x = \phi$

(2) $A^b = (A^c)^b$

(3) $(A \bigcup B)^b \subseteq A^b \bigcup B^b$

(4) $A \in \tau \Leftrightarrow A^b \subseteq A^c$ and $A^b \bigcap A = \phi$

(5) $A^c \in \tau \Leftrightarrow A^b \subseteq A$ and $A^b \bigcap A^c = \phi$

(6) $A, A^c \in \tau \Leftrightarrow A^b = \phi$

Proof :

(1) To prove $A^b \bigcap A^o = \phi$, suppose that $A^b \bigcap A^o \neq \phi$

$\Rightarrow \exists\, x \in A^b \bigcap A^o \Rightarrow x \in A^b \wedge x \in A^o$

$\Rightarrow \exists\, U \in \tau \,;\, x \in U \subseteq A$ (def. of A^o)

$\Rightarrow U \bigcap A^c = \phi$ (since $U \subseteq A$)

$\Rightarrow x \notin A^b$ contradiction !!!

$\therefore A^b \bigcap A^o = \phi$

By similar way, to proof $A^b \bigcap A^x = \phi$, suppose that $A^b \bigcap A^x \neq \phi$

$\Rightarrow \exists\, x \in A^b \bigcap A^x \Rightarrow x \in A^b \wedge x \in A^x$

$\Rightarrow \exists\, U \in \tau \,;\, x \in U \subseteq A^c$ (def. of A^x)

$\Rightarrow U \bigcap A = \phi$ (since $U \subseteq A^c$)

$\Rightarrow x \notin A^b$ contradiction !!!

$\therefore A^b \bigcap A^x = \phi$

(2) By definition of A^b, we have

$x \in A^b \Leftrightarrow \forall\, U \in \tau \,;\, x \in U \,,\, U \bigcap A \neq \phi \wedge U \bigcap A^c \neq \phi$

$\Leftrightarrow \forall\, U \in \tau \,;\, x \in U \,,\, U \bigcap (A^c)^c \neq \phi \wedge U \bigcap A^c \neq \phi$ (since $A = (A^c)^c$)

$\Leftrightarrow x \in (A^c)^b$

$\therefore A^b = (A^c)^b$

(3) To prove $(A \bigcup B)^b \subseteq A^b \bigcup B^b$

$x \in (A \bigcup B)^b \Rightarrow \forall\, U \in \tau \,;\, x \in U \,,\, U \bigcap (A \bigcup B) \neq \phi \wedge U \bigcap (A \bigcup B)^c \neq \phi$

$\Rightarrow (U \bigcap A) \bigcup (U \bigcap B) \neq \phi \wedge U \bigcap (A^c \bigcap B^c) \neq \phi$

$\Rightarrow [(U \bigcap A) \bigcup (U \bigcap B) \neq \phi] \wedge [(U \bigcap A^c) \bigcap (U \bigcap B^c) \neq \phi]$

$\Rightarrow [U \bigcap A \neq \phi \vee U \bigcap B \neq \phi] \wedge [U \bigcap A^c \neq \phi \wedge U \bigcap B^c \neq \phi]$

$\Rightarrow [U \bigcap A \neq \phi \wedge U \bigcap A^c \neq \phi] \vee [U \bigcap B \neq \phi \wedge U \bigcap B^c \neq \phi]$

$\Rightarrow x \in A^b \vee x \in B^b$

$\Rightarrow x \in A^b \bigcup B^b$

$\therefore\ (A \bigcup B)^b \subseteq A^b \bigcup B^b$

(4) $(\Rightarrow)$ Suppose that $A \in \tau$, to prove $A^b \subseteq A^c$ and $A^b \cap A = \phi$

Let $x \in A^b \Rightarrow \forall\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap A \neq \phi\ \wedge\ U \bigcap A^c \neq \phi$

$\Rightarrow$ i.e., every open set contain x intersect A and A^c

But, A is open (since $A \in \tau$) and $A \bigcap A^c = \phi$

$\Rightarrow x \notin A\ \Rightarrow x \in A^c\ \Rightarrow A^b \subseteq A^c$ and $A^b \cap A = \phi$.

$(\Leftarrow)$ Suppose that $A^b \subseteq A^c$, to prove A is open ($A \in \tau$)

To prove A is open, we must prove that A contains open nbhd for every point in A

Let $x \in A\ \Rightarrow x \notin A^c\ \Rightarrow x \notin A^b$ (since $A^b \subseteq A^c$)

$\Rightarrow \exists\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap A = \phi\ \vee\ U \bigcap A^c = \phi$

$\Rightarrow U \bigcap A \neq \phi$ (since $x \in U \wedge\ x \in A$)

$\Rightarrow U \bigcap A^c = \phi$

$\Rightarrow U \subseteq A$

$\Rightarrow\ A \in \tau$ (since A contains open nbhd for every point in A)

$\therefore$ A is open

(5) $(\Rightarrow)$ Suppose that $A^c \in \tau$, to prove $A^b \subseteq A$

Let $x \in A^b \Rightarrow \forall\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap A \neq \phi\ \wedge\ U \bigcap A^c \neq \phi$ (def. of A^b)

Since A^c open set $\Rightarrow x \notin A^c$

$\because$ every open set contains x intersect A and A^c , then x cannot in A^c since A^c contains open nbhd for every point in A^c.

$\Rightarrow x \in A$ (since $X = A \bigcup A^c$)

$\therefore\ A^b \subseteq A$

$(\Leftarrow)$ Suppose that $A^b \subseteq A$, to prove A is closed ($A^c \in \tau$)

we will prove A^c open set i.e., A^c contains open nbhd for every point in A^c.

Let $x \in A^c\ \Rightarrow x \notin A\ \Rightarrow x \notin A^b$ (since $A^b \subseteq A$)

$\Rightarrow \exists\ U \in \tau\ ;\ x \in U\ ,\ U \bigcap A = \phi$ (def. of boundary point and since $x \in A^c$)

$\Rightarrow U \subseteq A^c \neq \phi$ (since $X = A \bigcup A^c$)

So, A^c contains open nbhd for every point in A^c.

$\Rightarrow\ A^c \in \tau$ (i.e., A^c open set)

$\Rightarrow$ A closed set.

(6) $(\Rightarrow)$ Suppose that $A, A^c \in \tau$, to prove $A^b = \phi$

$\because$ A open set $\Rightarrow A^b \subseteq A^c$ (By (4))

$\because$ A closed set $\Rightarrow A^b \subseteq A$ (By (5))

$\therefore A^b \subseteq A \bigcap A^c = \phi \quad \Rightarrow A^b \subseteq \phi \quad \Rightarrow A^b = \phi$

($\Leftarrow$) Suppose that $A^b = \phi$, to prove $A, A^c \in \tau$

$\because A^b = \phi$ and $\phi \subseteq A$ and $\phi \subseteq A^c$

$\Rightarrow A^b \subseteq A \Rightarrow A^c \in \tau$ (By (5))

$A^b \subseteq A^c \Rightarrow A \in \tau$ (By (4))

$\Rightarrow A, A^c \in \tau$ (i.e., A is closed and open))

Remarks :

[1] Notes that : $X = A^o \bigcup A^x \bigcup A^b$ and $\phi = A^o \bigcap A^x \bigcap A^b$, this means the sets A^o, A^x, A^b, being a partition for X, also if $x \in X$, then $x \in A^o$ or $x \in A^x$ or $x \in A^b$.

[2] The set A^b is closed set since $A^b = X \setminus (A^o \bigcup A^x)$ and we know that the sets A^o and A^x are open sets, therefore $A^o \bigcup A^x$ is open set, so $X \setminus (A^o \bigcup A^x)$ is closed set, hence A^b closed set.

Example : Let X = {1, 2, 3, 4, 5} and τ be a topology on X and $A \subseteq X$ such that A^o = {1} and A^x = {2, 3}. Find A^b.

Solution : using previous remark

$A^b = X \setminus (A^o \bigcup A^x)$

$A^b = \{1, 2, 3, 4, 5\} \setminus (\{1\} \bigcup \{2, 3\}) = \{4, 5\}$

Notes that we find A^b thought we unknown the topology τ.

Definition : Derived set

Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called a **cluster point** (or **accumulation point or Limit point)** of A iff every open set containing x contains at least one point of A different from x. The set of all cluster points of A is called the **derived set** of A and is denoted by $A^/$. i.e.,

$$A^/ = \{x \in X : \forall\, U \in \tau\, ;\, x \in U \ \wedge\ U \setminus \{x\} \bigcap A \neq \phi\}$$

$$\text{or} \quad x \in A^/ \Leftrightarrow \forall\, U \in \tau\, ;\, x \in U \ \wedge\ U \setminus \{x\} \bigcap A \neq \phi$$

if $x \notin A^/$, we define

$$x \notin A^/ \Leftrightarrow \exists\, U \in \tau\, ;\, x \in U \ \wedge\ U \setminus \{x\} \bigcap A = \phi$$

Example : Let X = {a, b, c}, τ = {X, ϕ, {a}, {a, b}, {a, c}}, A = {b, c}, B = {c}, C = {a, b} and D = {a}. Find $A^/$, $B^/$, $C^/$ and $D^/$.

Solution :

$A^/ = ?$

To find the cluster set of any set must choose every open sets for every point in X and notes satisfy the definition or not.

$a \in X$ and the open sets contain a are X, {a}, {a, b}, {a, c}

$b \in X$ and the open sets contain b are X, {a, b}

$c \in X$ and the open sets contain c are X,{a, c}

notes that : $\{a\} \setminus \{a\} \bigcap A = \phi \bigcap A = \phi \Rightarrow a \notin A'$.

notes that : $\{a, b\} \setminus \{b\} \bigcap A = \{a\} \bigcap A = \phi \Rightarrow b \notin A'$.

notes that : $\{a, c\} \setminus \{c\} \bigcap A = \{a\} \bigcap A = \phi \Rightarrow c \notin A'$.

Therefore $A' = \phi$

By the similar way compute the other sets such that

$B' = \phi$, $C' = \{b, c\}$, $D' = \{b, c\}$.

Remarks :

[1] If $\{x\} \in \tau$ in any topological space (X, τ), then $x \notin A'$ for any subset $A \subseteq X$. Since $\{x\} \in \tau$ this means {x} is open set of X and $\{x\} \setminus \{x\} \bigcap A = \phi \bigcap A = \phi$ so the definition not satisfy (in the previous example take the element a).

[2] If A = {a} singleton set, then $a \notin A'$ since $U \setminus \{a\} \bigcap A = \phi$ (in the previous example take the set B).

[3] Notes that $A' \not\subset A$ and $A \not\subset A'$ and sometime $A' \bigcap A = \phi$ or $A' \bigcap A \neq \phi$ (in the previous example notes that $C' \not\subset C$ and $C \not\subset C'$ and $D' \bigcap D = \phi$).

[4] In a space (X, I), if $A \neq \phi$ and A contains more than one element, then $A' = X$ because the only open set in I is X for every element in X and $X \setminus \{x\} \bigcap A \neq \phi$.

[5] In a space (X, D), if $A \subseteq X$, then $A' = \phi$ because $\{x\} \in D$ for every $x \in X$ and by Remake (1) every point is not cluster point for any set.

[6] In any topological space (X, τ), we have $\phi' = \phi$ since for every open set U any for every element x is $U \setminus \{x\} \bigcap \phi = \phi$.

[7] The derived set of X is change by change the topology may be $X' = \phi$ (remake (5)) or may be $X' \neq \phi$ or $X' \neq X$. In the previous example $X' = \{b, c\}$).

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A \subseteq B \Rightarrow A' \subseteq B'$ (In general the converse is not true)

(2) $(A \bigcup B)' = A' \bigcup B'$

(3) $(A \bigcap B)' \subseteq A' \bigcap B'$ (In general the equality is not true)

(4) $A^c \in \tau \Leftrightarrow A' \subseteq A$

or A is closed $\Leftrightarrow A' \subseteq A$.

Proof :

(1) $x \in A' \Rightarrow \forall\, U \in \tau\,;\, x \in U \;\wedge\; U \setminus \{x\} \bigcap A \neq \phi$

$\Rightarrow \forall\, U \in \tau\,;\, x \in U \;\wedge\; U \setminus \{x\} \bigcap B \neq \phi$ (since $A \subseteq B$)

$\Rightarrow x \in B'$ (def. of cluster point)

$\therefore\ A' \subseteq B'$

(2) To prove $(A \bigcup B)' = A' \bigcup B'$

$A \subseteq A \bigcup B$ (def. of $\bigcup$) $\Rightarrow A' \subseteq (A \bigcup B)'$ (By (1))

$B \subseteq A \bigcup B$ (def. of $\bigcup$) $\Rightarrow B' \subseteq (A \bigcup B)'$ (By (1))

$\Rightarrow\ A' \bigcup B' \subseteq (A \bigcup B)'$ ---------------(1)

Let $x \notin A' \bigcup B' \Rightarrow x \notin A' \wedge x \notin B'$

$\Rightarrow \exists\, U \in \tau\,;\, x \in U \wedge U \setminus \{x\} \bigcap A = \phi \;\wedge\; \exists\, V \in \tau\,;\, x \in V \wedge V \setminus \{x\} \bigcap B = \phi$

(def. of cluster point)

$\Rightarrow U \bigcap V \in \tau\,;\, x \in U \bigcap V \wedge\ (U \bigcap V) \setminus \{x\} \bigcap (A \bigcup B) = \phi$

$\Rightarrow x \notin (A \bigcup B)'$

$(A \bigcup B)' \subseteq A' \bigcup B'$ ---------------(2)

From (1) and (2), we have $(A \bigcup B)' = A' \bigcup B'$

(3) To prove $(A \bigcap B)' \subseteq A' \bigcap B'$

Let $x \in (A \bigcap B)' \Rightarrow \forall\, U \in \tau\,;\, x \in U \wedge U \setminus \{x\} \bigcap (A \bigcap B) \neq \phi$

(def. of cluster point)

$\Rightarrow \forall\, U \in \tau\,;\, x \in U \wedge [(U \setminus \{x\} \bigcap A) \bigcap (U \setminus \{x\} \bigcap B)] \neq \phi$

($\bigcap$ distribution on $\bigcap$)

$\Rightarrow \forall\, U \in \tau\,;\, x \in U \wedge (U \setminus \{x\} \bigcap A) \neq \phi \wedge (U \setminus \{x\} \bigcap B) \neq \phi$

$\Rightarrow x \in A' \wedge x \in B'$ (def. of cluster point)

$\Rightarrow x \in A' \bigcap B'$

$\therefore\ (A \bigcap B)' \subseteq A' \bigcap B'$

(4) ($\Rightarrow$) Suppose that $A^c \in \tau$, to prove $A' \subseteq A$

To prove $A' \subseteq A$, we must prove that $A^c \subseteq (A')^c$

Let $x \notin A \Rightarrow x \in A^c$

$\Rightarrow\ \exists\, U \in \tau\,;\, x \in U \;\wedge\; U \subseteq A^c$ (def. of open set and $A^c \in \tau$)

$\Rightarrow U \subseteq A^c \Rightarrow U \bigcap A = \phi$

$\Rightarrow U \setminus \{x\} \bigcap A = \phi$ (since $x \notin A$)

$\Rightarrow x \notin A'$ (def. of cluster point)

($\Leftarrow$) Suppose that $A' \subseteq A$, to prove A is closed, i.e., A^c is open

Let $x \in A^c \Rightarrow x \notin A$ (def. of complement)

$\Rightarrow x \notin A'$ (since $A' \subseteq A$)

$\Rightarrow \exists U \in \tau ; x \in U \wedge U \setminus \{x\} \cap A = \phi$

$\Rightarrow U \setminus \{x\} \subseteq A^c \wedge x \in A^c$

$\Rightarrow U \subseteq A^c$

$\Rightarrow A^c \in \tau$

$\therefore$ A is closed

Remarks :

[1] The converse of property (1) is not true in general for example :

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, $A = \{a\}$, $B = \{b\}$.

Notes that, $A' = \{b, c\}$ and $B' = \{c\}$, so $B' \subseteq A'$, but $B \not\subset A$.

[2] The equality of property (3) is not true in general i.e., $A' \cap B' \not\subset (A \cap B)'$ for example :

Example : In the previous example notes that

$A \cap B = \{a\} \cap \{b\} = \phi \quad \Rightarrow \quad (A \cap B)' = \phi' = \phi$

But, $A' \cap B' = \{b, c\} \cap \{c\} = \{c\}$

$\therefore A' \cap B' \not\subset (A \cap B)'$

Definition : Closure of a set

Let (X, τ) be a topological space and $A \subseteq X$. The **closure** of a set A is $A \cup A'$ and is denoted by $\overline{A}$ or Cl(A). i.e.,

$$\overline{A} = A \cup A'$$

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$, $A = \{a, c\}$. Find $\overline{A}$.

Solution :

$\overline{A} = ?$

To find the closure set of A we must find A'.

$a \in X$ and the open sets contain a are X, {a, b},

$b \in X$ and the open sets contain b are X, {a, b}

$c \in X$ and the open set contain c is X

notes that : $\{a, b\} \setminus \{a\} \cap A = \{b\} \cap A = \phi \Rightarrow a \notin A'$.

notes that : $\{a, b\} \setminus \{b\} \cap A = \{a\} \cap A \neq \phi$ and

$X \setminus \{b\} \cap A = \{a, c\} \cap A \neq \phi \Rightarrow b \in A'$.

notes that : $X \setminus \{c\} \cap A = \{a, b\} \cap A \neq \phi \Rightarrow c \in A'$.

Therefore $A' = \{b , c\}$

$\therefore \overline{A} = A \cup A' = \{a, c\} \cup \{b, c\} = \{a, b, c\} = X$

Theorem : Let (X, τ) be a topological space and $A, B \subseteq X$. Then

(1) $A \subseteq \overline{A}$

(2) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$ (In general the converse is not true)

(3) $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$

(4) $\overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$ (In general the equality is not true)

(5) $\overline{A} = \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$ (i.e., $\overline{A}$ is smallest closed set contains A)

(6) $A^c \in \tau$ (i.e., A is closed) $\Leftrightarrow$ $\overline{A} = A$

(7) $\overline{\overline{A}} = \overline{A}$

Proof :

(1) $\because \overline{A} = A \bigcup A'$ (def. of $\overline{A}$) $\Rightarrow$ $A \subseteq \overline{A}$

(2) Suppose that $A \subseteq B$, to prove $\overline{A} \subseteq \overline{B}$

$\because A \subseteq B \Rightarrow A' \subseteq B'$ (property of cluster set)

$\Rightarrow A \subseteq B$ and $A' \subseteq B'$

$\Rightarrow A \bigcup A' \subseteq B' \bigcup B$

$\Rightarrow \overline{A} \subseteq \overline{B}$ (def. of $\overline{A}$)

(3) To prove $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$

First we prove, $\overline{A \bigcup B} \subseteq \overline{A} \bigcup \overline{B}$

From (1), $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ $\Rightarrow$ $A \bigcup B \subseteq \overline{A} \bigcup \overline{B}$

From (5) , $\overline{A}$, $\overline{B}$ are closed sets

$\therefore \overline{A} \bigcup \overline{B}$ is closed set contain $A \bigcup B$ (i.e., $A \bigcup B \subseteq \overline{A} \bigcup \overline{B}$)

but $\overline{A \bigcup B}$ is smallest closed set contain $A \bigcup B$ (i.e., $A \bigcup B \subseteq \overline{A \bigcup B}$).

$\Rightarrow$ $\overline{A \bigcup B} \subseteq \overline{A} \bigcup \overline{B}$ -------------(1)

Now, we prove, $\overline{A} \bigcup \overline{B} \subseteq \overline{A \bigcup B}$

$\because A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$ (def. of $\bigcup$)

From (2), $\overline{A} \subseteq \overline{A \bigcup B}$ and $\overline{B} \subseteq \overline{A \bigcup B}$

$\Rightarrow \overline{A} \bigcup \overline{B} \subseteq \overline{A \bigcup B}$ --------------(2)

From (1) and (2), we have $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$

(4) To prove, $\overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$

$\because A \bigcap B \subseteq A$ and $A \bigcap B \subseteq B$ (def. of $\bigcap$)

From (2), $\overline{A \bigcap B} \subseteq \overline{A}$ and $\overline{A \bigcap B} \subseteq \overline{B}$

$\Rightarrow \overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$

(5) To prove, $\overline{A} = \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

First we prove, $\overline{A} \subseteq \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

Let $x \in \overline{A} \Rightarrow x \in A \bigcup A'$ (def. of $\overline{A}$)

$\Rightarrow x \in A \vee x \in A'$

if $x \in A \Rightarrow x \in A \subseteq F \Rightarrow x \in F \ \forall F \subseteq X ; F^c \in \tau$

$\Rightarrow x \in \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

if $x \in A'$

suppose $x \notin \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

$\exists F \in \mathscr{F} \ ; x \notin F$

$\Rightarrow x \in F^c = U$ open set containing x.

$\because x \in A' \quad \Rightarrow A \bigcap F^c \backslash \{x\} \neq \phi$

$\Rightarrow A \bigcap F^c \neq \phi$

but $A \subseteq F \quad \Rightarrow A \bigcap F^c = \phi$ C!! contradiction.

$\therefore x \in \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

$\therefore \overline{A} \subseteq \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$ ---------------(1)

Second we prove, $\bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\} \subseteq \overline{A}$

Let $x \in \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$ and suppose $x \notin \overline{A}$

$\Rightarrow x \notin A \bigcup A'$ (since $\overline{A} = A \bigcup A'$)

$\Rightarrow x \notin A \ \wedge \ x \notin A'$

$\Rightarrow \exists U \in \tau ; x \in U \ \wedge \ U \setminus \{x\} \bigcap A = \phi$

$\Rightarrow U \bigcap A = \phi$ (since $x \notin A$)

$\Rightarrow A \subseteq U^c$ (since $U \bigcap A = \phi$)

But U^c is closed (since U open)

$\Rightarrow x \in U^c$ (since $x \in \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$) (say $U^c = F$)

$\Rightarrow x \in U$ and $x \in U^c$ contradiction !!!

$\Rightarrow x \in \overline{A}$

$\therefore \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\} \subseteq \overline{A}$ ---------------(2)

From (1) and (2), we have $\overline{A} = \bigcap\{F \subseteq X : F^c \in \tau \wedge A \subseteq F\}$

(6) To prove, $A^c \in \tau \Leftrightarrow \overline{A} = A$

Suppose that $A^c \in \tau$, to prove $\overline{A} = A$

$A \subseteq \overline{A}$ (from (1)) ---------------(1)

$\because A^c \in \tau \Rightarrow A$ is closed and also $A \subseteq A$ and $A' \subseteq A$ (by theorem)

$\Rightarrow A \bigcup A' \subseteq A$

$\Rightarrow \overline{A} \subseteq A$ ---------------(2)

Hence, from (1) and (2) we have $\overline{A} = A$

Suppose that $\overline{A} = A$, to prove $A^c \in \tau$ (i.e., to prove A is closed set)

$\because \overline{A} = A$ and $\overline{A}$ is closed $\Rightarrow$ A is closed $\Rightarrow A^c \in \tau$.

(7) To prove $\overline{\overline{A}} = \overline{A}$

A is closed $\Leftrightarrow$ $\overline{A} = A$ (by (6))

$\overline{A}$ is closed $\Leftrightarrow$ $\overline{A} = \overline{\overline{A}}$ (by (6))

Remarks :

[1] We can using property (5) to find closure set for any set in topological space instead of definition of closure set such that $\overline{A}$ is smallest closed set contains A.

[2] From property (6) and since X, ϕ are closed sets then $\overline{X} = X$ and $\overline{\phi} = \phi$.

[3] The converse of property (2) is not true in general for example :

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $A = \{b, c\}$ and $B = \{a, b\}$.

Notes that $\mathcal{F} = \{X, \phi, \{b, c\}, \{c\}\}$, then

$\overline{A} = \{b, c\} = A$ and $\overline{B} = X$

$\Rightarrow$ $\overline{A} \subseteq \overline{B}$ but $A \not\subset B$.

[4] The equality of property (4) is not true in general i.e., $\overline{A} \cap \overline{B} \not\subset \overline{A \cap B}$ for example :

Example : In the usual topology $(\mathbb{R}, \tau_u)$, let $A = [1, 2]$ and $B = (2, 3)$

Clear, $A \cap B = \phi \Rightarrow \overline{A \cap B} = \overline{\phi} = \phi \Rightarrow \overline{A \cap B} = \phi$

But, $\overline{A} \cap \overline{B} = [1, 2] \cap [2, 3] = \{2\}$

$\overline{A} \cap \overline{B} \not\subset \overline{A \cap B}$. Hence, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

[5] In a space (X, I), if $\phi \neq A \subseteq X$, then $\overline{A} = X$ since $\overline{A}$ is closed set contain A and the only closed set in I contain A is X.

[6] In a space (X, D), every subsets of X is open and closed in the sometime, then $\overline{A} = A$ for all $A \subseteq X$ (by property (6)).

[7] In the usual topological space $(\mathbb{R}, \tau_u)$, if A is closed interval or open interval or half closed (open) as follow : $A = [a, b]$ or $A = (a, b)$ or $A = [a, b)$ or $A = (a, b]$, then $\overline{A} = [a, b]$ since $[a, b]$ is smallest closed set contains A.

If A is a discreet interval in real number (finite or infinit), then $\overline{A} = A$ since A is closed set for example :

$A = \mathbb{N}$, $A = O$, $A = \mathbb{P}$, $A = E$, $A = \{1, 2, 3\}$, $A = \{-\sqrt{2}, 0, \sqrt{2}\}$

Either the rational numbers $\mathbb{Q}$ and irrational numbers, then $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{Irr} = \mathbb{R}$ since the only closed set in τ_u contain $\mathbb{Q}$ and Irr is $\mathbb{R}$.

[8] In the topological space $(\mathbb{N}, \tau_{cof})$, if A is finite set, then A is closed (def. of τ_{cof}), so $\overline{A} = A$.

If A is infinite, then $\overline{A} = \mathbb{N}$ since $\overline{A}$ closed set contain A and the only closed set in τ_{cof} contain A is $\mathbb{N}$.

Definition : **Dense set**

Let (X, τ) be a topological space. Then $A \subseteq X$ is called **dense** set in X iff $\overline{A} = X$.

Examples :

[1] In the usual topological space $(\mathbb{R}, \tau_u)$, the rational numbers $\mathbb{Q}$ and irrational numbers are dense in $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{Irr} = \mathbb{R}$.

[2] In the cofinite topological space $(\mathbb{N}, \tau_{cof})$, every infinite set is dense in $\mathbb{N}$, for example if $A = \{5, 10, 15, \dots\}$, then $\overline{A} = \mathbb{N}$.

[3] In a space (X, I), every nonempty subset of X is dense.

[4] In a space (X, D), the only dense set is X.

[5] In every topological space (X, τ) is X dense set always. So, every topological space contain at least one dense set.

Topological Space Generated by Metric Space

Definition : **Metric & Metric Space**

Let $X \neq \phi$ a function $d : X\times X \to \mathbb{R}$ is called a **metric** on X if :

(1) $d(x, y) \geqslant 0 \ \forall\, x, y \in X$

(2) $d(x, y) = d(y, x) \quad \forall\, x, y \in X$

(3) $d(x, y) = 0 \Leftrightarrow x = y \ \forall\, x, y \in X$

(4) $d(x, y) + d(y, z) \geqslant d(x, z) \ \forall\, x, y, z \in X$

The pair (X, d) is called a **metric space**.

Definition : **Open Ball**

Let (X, d) be a metric space and let $x \in X$, $\varepsilon > 0$, the set

$$B_\varepsilon(x) = \{y \in X;\ d(y, x) < \varepsilon\}$$

is called an **open ball** in X with center x and radius ε.

Definition : **Open Set in Metric Space**

Let (X, d) be a metric space and let $U \subseteq X$, U is said to be **open** in (X, d) if $\forall\, x \in U\ \exists\ \varepsilon > 0\ ;\ B_\varepsilon(x) \subseteq U$.

Proposition : Let (X, d) be a metric space and $U \subseteq X$, U is open in X iff U is the union of open balls.

Proposition : Let (X, d) be a metric space and let τ_d be the family of all open sets in (X, d). i.e., $\tau_d = \{U \subseteq X; U \text{ is open in } (X, d)\}$. Then τ_d is a topological space on X.

Proof :

(1) $\forall x \in X \ \exists \varepsilon > 0 , B_\varepsilon(x) \subseteq X \Rightarrow X \in \tau_d$.

$\phi \in \tau_d$ since $\nexists\, x \in \phi$.

(2) Let $U, V \in \tau_d$, to prove $U \bigcap V \in \tau_d$.

Let $x \in U \bigcap V \Rightarrow x \in U \wedge x \in V$

$\Rightarrow \exists \ \varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $B_{\varepsilon 1}(x) \subseteq U \wedge B_{\varepsilon 2}(x) \subseteq V$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

$\Rightarrow B_\varepsilon(x) \subseteq B_{\varepsilon 1}(x) \bigcap B_{\varepsilon 2}(x) \subseteq U \bigcap V$.

$\Rightarrow U \bigcap V \in \tau_d$.

(3) Let $U_\alpha \in \tau_d \ \ \forall \alpha \in \Lambda$, to prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_d$

Let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow \exists \ \alpha_0 \in \Lambda \ ; \ x \in U_{\alpha 0}$

$\Rightarrow \exists \ \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U_{\alpha 0}$.

but $U_{\alpha 0} \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

$\Rightarrow B_\varepsilon(x) \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$.

$\Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_d$.

So τ_d is a topology on X induced by d.

Example : Let $X = \mathbb{R}$ and $d = |\ |$, then $(X, d) = (\mathbb{R}, |\ |)$ is a metric space.

Now, let $x \in X$, $\varepsilon > 0$ then

$$\begin{aligned} B_\varepsilon(x) &= \{y \in \mathbb{R} \ ; \ |\, y - x \,| < \varepsilon\} \\ &= \{y \in \mathbb{R} \ ; -\varepsilon < y - x < \varepsilon\} \\ &= \{y \in \mathbb{R} \ ; x - \varepsilon < y < x + \varepsilon\} \\ &= (x - \varepsilon, x + \varepsilon) \text{ open interval} \end{aligned}$$

So the open balls here is an open interval, and hence the open sets is the union of open intervals.

i.e., $\tau_d = \{U \subseteq \mathbb{R} \ ; \ U = \text{union of open intervals}\} = \tau_u$

We shall denote this topology by τ_u = the usual topology on $\mathbb{R}$ = the set of real.

Note that $\mathbb{R} \in \tau_u$ and $\mathbb{R} = (-\infty, \infty)$ which is an open interval and $\phi = (a, a) \ ; \ a \in \mathbb{R}$.

Example : Which of the following subsets of $\mathbb{R}$ is open (closed) in $(\mathbb{R},\tau_u)$??
$(-1, 1)$, $(0, 1) \bigcup (10, 20)$, $\mathbb{N}$, $[2, 3]$, $[-1/2, 3)$, $\mathbb{Q}$, Irr, $\{3, 4, 5\}$.

Solution :
$(-1, 1)$ and $(0, 1) \bigcup (10, 20)$ are open but not closed.
$\mathbb{N}$ is not open , but closed
$[2, 3]$ and $\{3, 4, 5\}$ are closed but not open.
$[-1/2, 3)$, $\mathbb{Q}$ and Irr are not open and not closed.

Remark : We can get a topological space from any metric space, but we cannot get a metric space from any topological space.

Definition : (Metrizable Space)
The topological space (X, τ) is called **Metrizable** iff there exists a metric d for X such that the topology τ_d induced by τ (i.e, $\tau = \tau_d$). Otherwise, X is said to be nonmeterizable.

Remark : (X, D) is a metrizable topological space.
i.e., There is a metric d on X such that $\tau_d = D$.
where $d : X\times X \to \mathbb{R}$; $d(x, y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$
$B_r(x) = \{x\}$ if $r < 1$
$\therefore \ \{x\} \in \tau_d \ \ \forall\, x \in X \Rightarrow \tau_d = D$.

Example : If $X = \{1, 2, 3\}$, τ topology on X, (X, τ) is metrizable $\Rightarrow \tau = D$.
Suppose that $\exists\ d : X\times X \to \mathbb{R}$; $\tau_d = \tau$
$\Rightarrow d(1, 1) = d(2, 2) = d(3, 3) = 0$
$d(1, 2) = d(2, 1) = C_1$
$d(1, 3) = d(3, 1) = C_2$
$d(2, 3) = d(3, 2) = C_3$
$B_\varepsilon(1) = \{1\}$ if $\varepsilon < \min\{C_1, C_2\}$
$B_\varepsilon(2) = \{2\}$ if $\varepsilon < \min\{C_1, C_3\}$
$B_\varepsilon(3) = \{3\}$ if $\varepsilon < \min\{C_2, C_3\}$
$\therefore\ \tau = D$
Therefore, every topology (τ) on X not discrete (D) is space not generated by metric.

Chapter Two : Continuity and Derived Topological Spaces

Definition : **Continuous & discontinuous Functions**

Let (X, τ) and (Y, τ') be a topological spaces and $f : (X, \tau) \to (Y, \tau')$. the function f is called **continuous** if the inverse image for any open set in Y is an open set in X. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is continuous } \Leftrightarrow f^{-1}(V) \in \tau \ \ \forall V \in \tau'$$

and the function f is called **discontinuous** if there exist an open set in Y, but inverse image is not open in X. i.e.,

$$f \text{ is discontinuous } \Leftrightarrow \exists V \in \tau' \wedge f^{-1}(V) \notin \tau$$

Example : Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi, \{b\}\}$

(1) Define $f : (X, \tau) \to (Y, \tau')$; $f(1) = b$, $f(2) = f(3) = a$. Is f continuous??

The open sets in Y are Y, ϕ, $\{b\}$. Now take the inverse image of this sets.

$Y \in \tau' \Rightarrow f^{-1}(Y) = X \in \tau$ the set of all element in X its image in Y

$\phi \in \tau' \Rightarrow f^{-1}(\phi) = \phi \in \tau$ the set of all element in ϕ its image in ϕ

$\{b\} \in \tau' \Rightarrow f^{-1}(\{b\}) = \{x \in X ; f(x) = b\} = \{1\} \in \tau$ the set of all element in X its image in $\{b\}$

Therefore, the inverse image of every element in τ' is element in τ, hence f is continuous.

(2) Define $g : (X, \tau) \to (Y, \tau')$; $g(1) = a$, $g(2) = g(3) = b$. Is g continuous??

$Y \in \tau' \Rightarrow g^{-1}(Y) = X \in \tau$

$\phi \in \tau' \Rightarrow g^{-1}(\phi) = \phi \in \tau$

$\{b\} \in \tau' \Rightarrow g^{-1}(\{b\}) = \{2, 3\} \notin \tau$

Therefore, f is discontinuous.

(3) Define $h : (X, \tau) \to (Y, \tau')$; $h(1) = h(2) = h(3) = a$. Is h continuous??

$Y \in \tau' \Rightarrow h^{-1}(Y) = X \in \tau$

$\phi \in \tau' \Rightarrow h^{-1}(\phi) = \phi \in \tau$

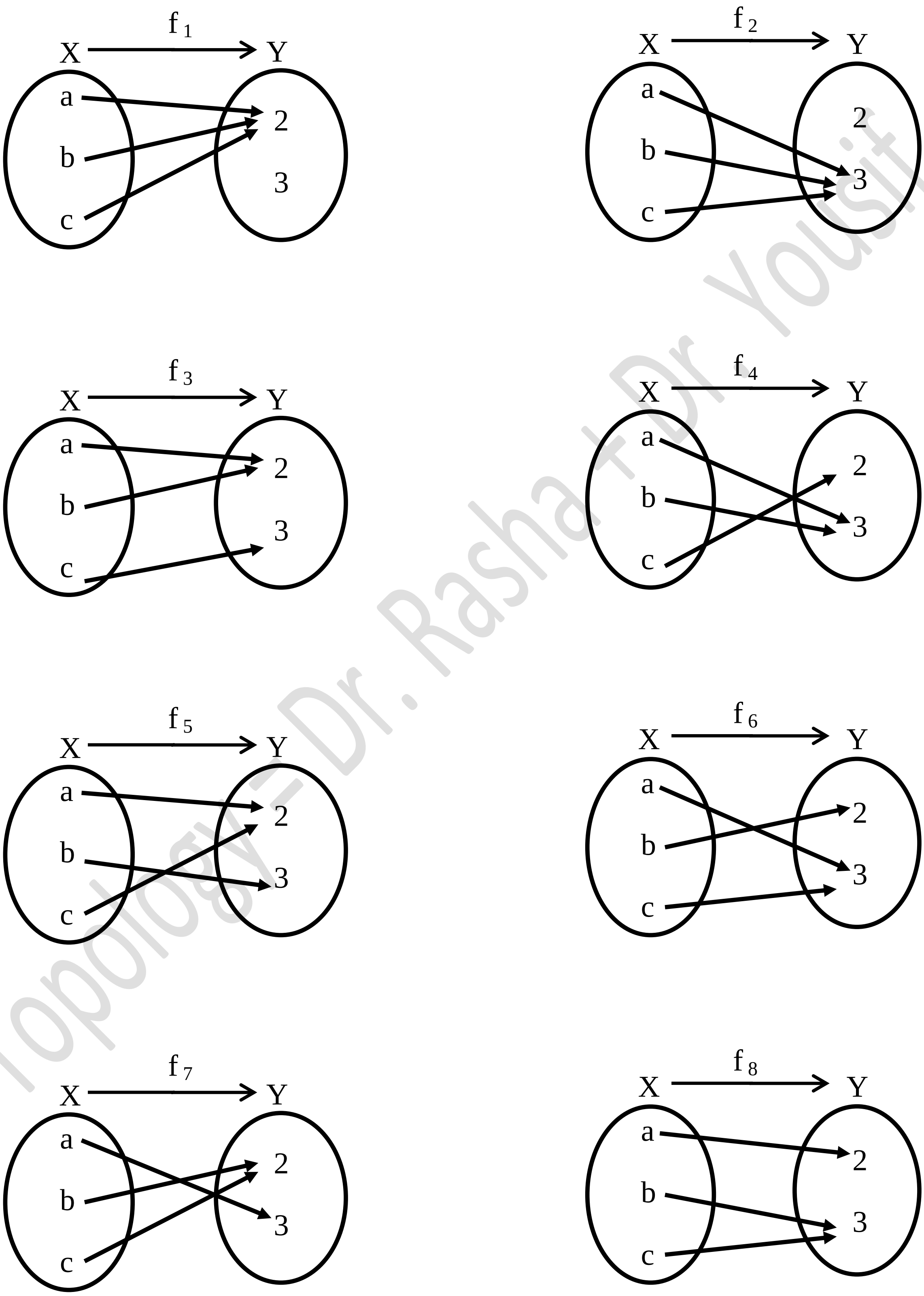
$\{b\} \in \tau' \Rightarrow h^{-1}(\{b\}) = \phi \in \tau$ since there is no element its image is b

Therefore, f is continuous.

Remark : Always the inverse image of Y is X and the inverse image of ϕ is ϕ.

Example : Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}\}$, $Y = \{2, 3\}$ and $\tau' = \{Y, \phi, \{2\}\}$. Find all continuous function define from (X, τ) to (Y, τ').

Solution : There are $2^3 = 8$ from difference functions from X to Y which are some of them continuous and some others of them discontinuous . Now we introduce the figure for all functions from X to Y and discuses there continuous.



From remark above $f_i^{-1}(Y) = X$ and $f_i^{-1}(\phi) = \phi$, i = 1, 2, 3, 4, 5, 6, 7, 8.

f_1 is continuous, since $f_1^{-1}(\{2\}) = \{a, b, c\} = X \in \tau$.

f_2 is continuous, since $f_2^{-1}(\{2\}) = \phi \in \tau$.

f_3 is discontinuous, since $f_3^{-1}(\{2\}) = \{a, b\} \notin \tau$.

f_4 is discontinuous, since $f_4^{-1}(\{2\}) = \{c\} \notin \tau$.

f_5 is discontinuous, since $f_5^{-1}(\{2\}) = \{a, c\} \notin \tau$.

f_6 is discontinuous, since $f_6^{-1}(\{2\}) = \{b\} \notin \tau$.

f_7 is continuous, since $f_7^{-1}(\{2\}) = \{b, c\} \in \tau$.

f_8 is discontinuous, since $f_8^{-1}(\{2\}) = \{a\} \notin \tau$.

Therefore, the continuous functions in this example are f_1, f_2, f_7 only.

Remark : There are special cases of continuous functions.

[1] Every constant function from a space (X, τ) to a space (Y, τ') is continuous. i.e.,

$$f : (X, \tau) \to (Y, \tau') \; ; \; f(x) = c \quad \forall\, x \in X \text{ and } c = \text{constant in } Y.$$

To show that f is continuous.

Let $V \in \tau' \;\Rightarrow\;$ V is open in Y, then

$$f^{-1}(V) = \begin{cases} X & \text{if} \quad c \in V \\ \emptyset & \text{if} \quad c \notin V \end{cases}$$

$\Rightarrow X, \phi \in \tau \Rightarrow$ f is continuous.

[2] If $\tau' = I$, then the function $f : (X, \tau) \to (Y, I)$ is continuous for any set Y and any topological space (X, τ). i.e., $I = \{Y, \phi\}$ and $f^{-1}(Y) = X \in \tau$, $f^{-1}(\phi) = \phi \in \tau$.

Special case : $f : (X, I) \to (Y, I)$ is continues

And the function $f : (X, I) \to (Y, \tau')$; $\tau' \neq I$ is discontinuous in general for example :

$$f : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u) \quad ; \quad f(x) = x$$

f is discontinuous since $(0, 1) \in \tau_u$ and $f^{-1}((0, 1)) = (0, 1) \notin I = \{\mathbb{R}, \phi\}$.

[3] If $\tau = D$, then the function $f : (X, D) \to (Y, \tau')$ is continuous for any set X and any topological space (Y, τ') and for any function f since, if $V \in \tau'$, then $f^{-1}(V) \subseteq X$ this means $f^{-1}(V) \in IP(X)$, but $D = IP(X)$ and this implies $f^{-1}(V) \in D$. Therefore f is continuous.

Special case : $f : (X, D) \to (Y, D)$ is continuous

And the function $f : (X, \tau) \to (Y, D)$; $\tau \neq D$ is discontinuous in general for example :

$$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, D) \quad ; \quad f(x) = x$$

f is discontinuous since $\{1\} \in D$ and $f^{-1}(\{1\}) = \{1\} \notin \tau_u$.

Notes that the function $f : (X, D) \to (Y, I)$ is continuous always for any set X and any set Y since its add the remark [2] and [3] such that $\tau = D$ and $\tau' = I$.

[4] Every identity function from a spaces to the same space is continuous. i.e.,

$$f : (X, \tau) \to (X, \tau) \quad ; \quad f(x) = x \quad \forall\, x \in X$$

is continuous function since $f^{-1}(V) = V$ for any open set V in (X, τ) and this implies $f^{-1}(V)$ is open in (X, τ)

notes that the identity function from a space to another space may be continuous and its clear in example in remark [2] and [3] above.

Theorem : If $f : (X, \tau) \to (Y, \tau')$ and $g : (Y, \tau') \to (Z, \tau'')$ are both continuous functions, then the composition $gof : (X, \tau) \to (Z, \tau'')$ is continuous.

Proof :

Let $W \in \tau'' \Rightarrow g^{-1}(W) \in \tau'$ (since g is continuous)

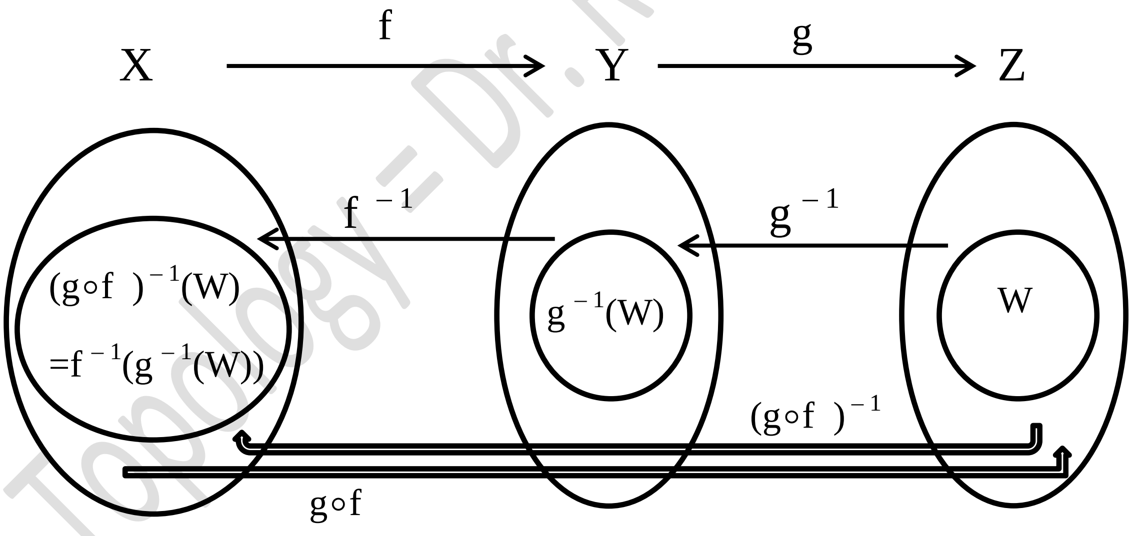
notes that $g^{-1}(w) \subseteq Y$

$\Rightarrow f^{-1}(g^{-1}(W)) \in \tau$ (since f is continuous)

$\Rightarrow (f^{-1}og^{-1})(W) \in \tau$ (by composition of function)

$\Rightarrow (gof)^{-1}(W) \in \tau$ (since $(gof)^{-1} = f^{-1}og^{-1}$)

$\therefore$ gof is continuous. The figure below clear this theorem.



Remake : The composition of finite number of continuous functions is continuous. i.e., the composition of three or five or hundred continuous functions is continuous. For example if f , g, h, k are continuous, then kohogof is continuous …. etc.

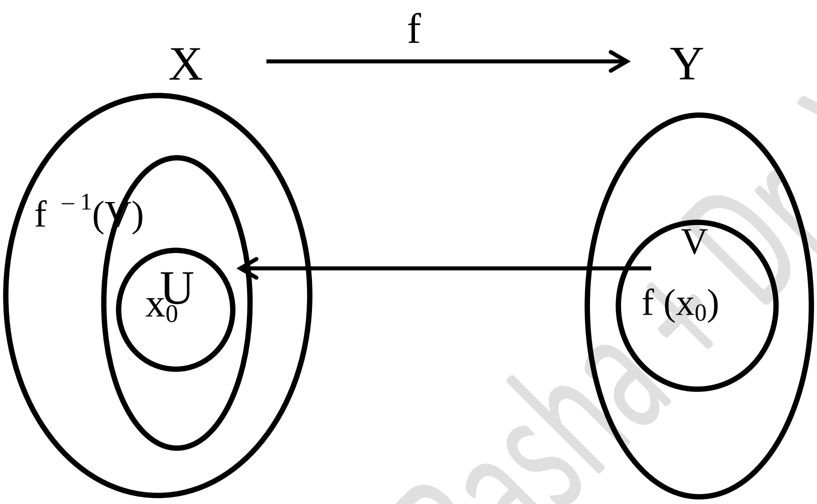
Now we introduce the definition of continuous function at a point :

<u>Definition :</u> Continuous at a Point

Let (X, τ) and (Y, τ') be topological spaces and $f : (X, \tau) \to (Y, \tau')$. the function f is called **<u>continuous at a point</u>** $x_0 \in X$ if the inverse image for any open nbd for $f(x_0)$ in Y contains an open nbd for x_0 in X. i.e.,

$$f \text{ is continuous at } x_0 \in X \Leftrightarrow \forall\, V \in \tau' ;\ f(x_0) \in V\ \exists\, U \in \tau ;\ x_0 \in U \wedge U \subseteq f^{-1}(V)$$

The following figure clear this definition :



Such that V is an open nbd for $f(x_0)$ in Y and $f^{-1}(V)$ is inverse image for V and U is an open nbd for x_0 in X contains in $f^{-1}(V)$.

<u>Remark :</u> If f is continuous function. Then its continuous at every point in the domain. Also, if f is continuous at every point in the domain, then its continuous.

<u>Notes that</u>, if f is continuous at a point in the domain, then its discontinuous in general and the following example show that :

<u>Example :</u> Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, $Y = \{1, 2\}$ and $\tau' = \{Y, \phi, \{1\}\}$. Define f as follow :

$$f : (X, \tau) \to (Y, \tau') \ ;\quad f(a) = f(b) = 2,\ f(c) = 1$$

notes that f is discontinuous since $\{1\} \in \tau'$, but $f^{-1}(\{1\}) = \{c\} \notin \tau$.

On the other hand, thought that f is discontinuous in general, but its continuous at a point a as follow :

$f(a) = 2$ and the open nbd of 2 is Y only and $f^{-1}(Y) = X$ and X is an open nbd

There are several characterizations of continuous functions and, hence, that any one of them may be used to show continuity of a function. These are given in the next theorem :

Theorem : Let $f : (X, \tau) \to (Y, \tau')$ be a function. Then f is continuous iff satisfy one of the following properties :

(1) $f^{-1}(F) \in \mathcal{F} \quad \forall \quad F \in \mathcal{F}'$; $\mathcal{F}$ family of closed sets in X and $\mathcal{F}'$ family of closed sets in Y i.e., The inverse image of every closed set in Y is closed in X.

(2) $f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$; β' is a basis for τ'.

i.e., The inverse image of every element in any basis for τ' is open set in X.

(3) $f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathcal{S}'$; $\mathcal{S}'$ is a subbasis for τ'.

i.e., The inverse image of every element in any subbasis for τ' is open set in X.

(4) $f^{-1}(N_y) \in \tau \quad \forall \quad y \in Y \quad \forall \quad N_y \in \eta_y$; η_y is a family of open nbd for a point y in Y.

i.e., The inverse image of every open nbd for any element Y is open set in X.

(5) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

(6) $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$.

Proof :

(1) To prove, f is continuous $\Leftrightarrow X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$

($\Rightarrow$) Suppose that f is cont. , to prove $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$

Let F closed set in Y $\Rightarrow$ Y – F open set in Y (def. of closed set)

$\Rightarrow f^{-1}(Y - F) \in \tau$ (since f is continuous)

But, $f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F)$

$= X - f^{-1}(F)$ (since $f^{-1}(Y) = X$)

$\because \quad f^{-1}(Y - F) \in \tau \Rightarrow X - f^{-1}(F) \in \tau$

($\Leftarrow$) Suppose that $X - f^{-1}(F) \in \tau \quad \forall \quad Y - F \in \tau'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\therefore$ Y – V closed set in Y since V open

$\Rightarrow f^{-1}(Y - V)$ closed set in X (by hypothesis)

$\Rightarrow X - f^{-1}(Y - V) \in \tau$

But, $X - f^{-1}(Y - V) = X - [f^{-1}(Y) - f^{-1}(V)] = X - [X - f^{-1}(V)] = f^{-1}(V)$

$\Rightarrow f^{-1}(V) \in \tau$

$\therefore$ f is continuous.

(2) To prove, f is continuous $\Leftrightarrow f^{-1}(B') \in \tau \;\; \forall B' \in \beta'$; β' is a basis for τ'.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$

Let β' be a base of τ' and $B' \in \beta'$

$\Rightarrow B' \in \tau'$ (since $\beta' \subseteq \tau'$)

$\Rightarrow f^{-1}(B') \in \tau$ (since f is continuous)

$\Rightarrow f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$

($\Leftarrow$) Suppose that $f^{-1}(B') \in \tau \quad \forall \quad B' \in \beta'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow V = \bigcup_i B_i' \quad ; \quad B_i' \in \beta'$ (def. of basis)

$\Rightarrow f^{-1}(V) = f^{-1}(\bigcup_i B_i') = \bigcup_i f^{-1}(B_i')$

$\Rightarrow \bigcup_i f^{-1}(B_i') \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $f^{-1}(V) = f^{-1}(\bigcup_i B_i') = \bigcup_i f^{-1}(B_i')$)

$\therefore$ f is continuous

(3) To prove, f is continuous $\Leftrightarrow f^{-1}(S') \in \tau \ \forall S' \in \mathcal{S}'$; $\mathcal{S}'$ is a subbasis for τ'.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathcal{S}'$

Let $\mathcal{S}'$ be a subbase of τ' and $S' \in \mathcal{S}'$

$\Rightarrow S' \in \tau'$ (since $\mathcal{S}' \subseteq \tau'$)

$\Rightarrow f^{-1}(S') \in \tau$ (since f is continuous)

$\Rightarrow f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathcal{S}'$

($\Leftarrow$) Suppose that $f^{-1}(S') \in \tau \quad \forall \quad S' \in \mathcal{S}'$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow V = \bigcup_i (\bigcap_{j=1}^{n} S_j')$ (def. of basis and subbasis)

$\Rightarrow f^{-1}(V) = f^{-1}(\bigcup_i (\bigcap_{j=1}^{n} S_j'))$

$= \bigcup_i f^{-1}(\bigcap_{j=1}^{n} S_j')$ (inverse image distribution on union)

$= \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j'))$ (inverse image distribution on intersection)

$\because f^{-1}(S_j') \in \tau \quad \Rightarrow \quad f^{-1}(S_j')$ open in X

$\Rightarrow \bigcap_{j=1}^{n} f^{-1}(S_j') \in \tau$ (by second condition of def. of top.)

$\Rightarrow \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j')) \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $f^{-1}(V) = \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(S_j'))$)

$\therefore$ f is continuous,

(4) To prove, f is continuous $\Leftrightarrow f^{-1}(N_y) \in \tau \quad \forall \quad y \in Y \quad \forall \quad N_y \in \eta_y$

($\Rightarrow$) Suppose that f is continuous, to prove $f^{-1}(N_y) \in \tau \quad \forall \quad y \in Y \quad \forall \quad N_y \in \eta_y$

Let $N_y \in \eta_y$

$\because$ η_y is open nbd system for y, then η_y is a family of open set, therefore

$\Rightarrow N_y \in \tau'$

$\Rightarrow f^{-1}(N_y) \in \tau$ (since f is continuous)

($\Leftarrow$) Suppose that $f^{-1}(N_y) \in \tau \ \forall \ y \in Y \ \forall \ N_y \in \eta_y$, to prove f is continuous

Let V open set in Y i.e., $V \in \tau'$

$\Rightarrow V = \bigcup_{y\in V} N_y$ i.e., V = union of a family of open sets for every point in V by using the fifth condition of def. of o.n.s

$(U \in \tau \Leftrightarrow \exists \ N_y \in \eta_y \ ; N_y \subseteq U \ \forall \ y \in U)$

$\therefore f^{-1}(V) = f^{-1}(\bigcup_{y\in V} N_y) = \bigcup_{y\in V} f^{-1}(N_y)$ (inverse image distribution on union)

$\because f^{-1}(N_y) \in \tau$ (by hypothesis)

$\Rightarrow \bigcup_{y\in V} f^{-1}(N_y) \in \tau$ (by third condition of def. of top.)

$\Rightarrow f^{-1}(V) \in \tau$ (since $(f^{-1}(V) = \bigcup_{y\in V} f^{-1}(N_y))$)

$\therefore$ f is continuous.

(5) To prove, f is continuous $\Leftrightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

($\Rightarrow$) Suppose that f is continuous , to prove $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$.

Let $B \subseteq Y \Rightarrow B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$

$\because \overline{B}$ closed set in Y by (1) $f^{-1}(\overline{B})$ closed set in X

$\Rightarrow \bigcap\{F \subseteq X : F^c \in \tau \ \wedge \ f^{-1}(B) \subseteq F\} \subseteq f^{-1}(\overline{B})$

since $\overline{f^{-1}(B)}$ is intersection of all closed sets that contain $f^{-1}(B)$ and $f^{-1}(\overline{B})$ is one of the closed set that contain $f^{-1}(B)$, then

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ (since $\bigcap\{F : F^c \in \tau \ \wedge \ f^{-1}(B) \subseteq F\} = \overline{f^{-1}(B)}$)

($\Leftarrow$) Suppose that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$. , to prove f is continuous.

We will use part (1) in this theorem

Let F closed set in Y, to prove $f^{-1}(F)$ closed in X i.e., $\overline{f^{-1}(F)} = f^{-1}(F)$.

$f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ (since $A \subseteq \overline{A}$) ---------------(1)

$\because$ F closed $\Rightarrow F = \overline{F} \Rightarrow f^{-1}(F) = f^{-1}(\overline{F})$

By hypnoses, $\overline{f^{-1}(F)} \subseteq f^{-1}(\overline{F}) = f^{-1}(F)$

$\Rightarrow \overline{f^{-1}(F)} \subseteq f^{-1}(F)$ ---------------(2)

From (1) and (2) we have $\overline{f^{-1}(F)} = f^{-1}(F)$

$\therefore f^{-1}(F)$ closed in X.

$\therefore$ f is continuous.

(6) To prove, f is continuous $\Leftrightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$.

($\Rightarrow$) Suppose that f is continuous , to prove $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$

Let $B \subseteq Y \Rightarrow B^o \subseteq B \Rightarrow f^{-1}(B^o) \subseteq f^{-1}(B)$

$\because$ B^o is open in Y $\Rightarrow f^{-1}(B^o)$ is open in X (since f is continuous)

$f^{-1}(B^o) \subseteq \bigcup\{O \subseteq X ; O \in \tau , O \subseteq f^{-1}(B)\}$

since $(f^{-1}(B))^o$ is union of all open sets that contain in $f^{-1}(B)$ and $f^{-1}(B^o)$ is one of the open set that contain in $f^{-1}(B)$, then

$$\Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o \text{ (since } \bigcup\{O \subseteq X ; O \in \tau , O \subseteq f^{-1}(B)\} = (f^{-1}(B))^o \text{)}$$

($\Leftarrow$) Suppose that $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$, to prove f is continuous.

Let V open set in Y i.e., $V \in \tau'$ to prove $f^{-1}(V)$ open in X i.e., $f^{-1}(V) = (f^{-1}(V))^o$.

$\because$ $V^o \subseteq V \Rightarrow ((f^{-1}(V))^o \subseteq f^{-1}(V)$ ---------------(1)

$\because$ V open $\Rightarrow V = V^o \Rightarrow f^{-1}(V) = f^{-1}(V^o)$

By hypnoses, $f^{-1}(V) = f^{-1}(V^o) \subseteq (f^{-1}(V))^o$

$\Rightarrow f^{-1}(V) \subseteq (f^{-1}(V))^o$ ---------------(2)

From (1) and (2) we have $f^{-1}(V) = (f^{-1}(V))^o$

$\therefore$ $f^{-1}(V)$ open in X

$\therefore$ f is continuous.

Remark : The six characterizations in the previous theorem for definition of continuity is not unique, but there are another characterizations for example :

$$f \text{ is continuous } \Leftrightarrow f(\overline{A}) \subseteq \overline{f(A)} \quad ; \quad A \subseteq X.$$

$$f \text{ is continuous } \Leftrightarrow (f(A))^o \subseteq f(A^o) \quad ; \quad A \subseteq X.$$

Remarks :

[1] Notes that, if $f : (X, \tau) \to (Y, \tau')$ is continuous function, then it's not necessary that the direct image of open set in X is open set in Y. i.e.,

$$U \in \tau \not\Rightarrow f(U) \in \tau' \text{ (in general not true)}$$

$$V \in \tau' \Rightarrow f^{-1}(V) \in \tau \text{ (is true)}$$

This two statements is difference and the following example show this :

Example : Let $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$ be a function, then f is continuous (see, page 37). Now we will show that the direct image of open set is not open in general :

$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall \ x \in \mathbb{R}$

Let $U = (0, 1)$ open set in $(\mathbb{R}, \tau_u)$ and $f(U) = U$

$\Rightarrow f(U) = U$ is not open in $(\mathbb{R}, I)$ since $U = (0, 1) \notin I = \{\mathbb{R}, \phi\}$

So, we show that $U \in \tau \wedge f(U) \notin \tau'$.

[2] Notes that, if $f : (X, \tau) \to (Y, \tau')$ is continuous function, then it's not necessary that the direct image of closed set in X is closed set in Y. i.e.,

$$F^c \in \tau \not\Rightarrow (f(F))^c \in \tau' \qquad \text{(in general not true)}$$
$$F^c \in \tau' \Rightarrow (f^{-1}(F))^c \in \tau \qquad \text{(is true)}$$

This two statements is difference and the following example show this :

Example : In the previous example :

$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall \ x \in \mathbb{R}$, f is continuous

Let $F = [0, 1]$ closed set in $(\mathbb{R}, \tau_u)$

$\Rightarrow f(F) = F$ is not closed in $(\mathbb{R}, I)$ since $F^c = [0, 1]^c \notin I = \{\mathbb{R}, \phi\}$

So, we show that $[0, 1]^c \in \tau_u \wedge (f([0, 1])^c = [0, 1]^c \notin I$.

Now we will introduce a new definitions for functions satisfy the condition [1] and [2] in the previous remark as follows :

Definition : Open & Closed Functions

Let $f : (X, \tau) \to (Y, \tau')$ be a function.

(1) The function f is called **open** if the direct image for any open set in X is open set in Y. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is open function } \Leftrightarrow \forall\, U \in \tau \Rightarrow f(U) \in \tau'$$
$$f : (X, \tau) \to (Y, \tau') \text{ is not open function } \Leftrightarrow \exists\, U \in \tau \wedge f(U) \notin \tau'$$

(2) The function f is called **closed** if the direct image for any closed set in X is closed set in Y. i.e.,

$$f : (X, \tau) \to (Y, \tau') \text{ is closed function } \Leftrightarrow \forall\, F^c \in \tau \Rightarrow (f(F))^c \in \tau'$$
$$f : (X, \tau) \to (Y, \tau') \text{ is not closed function } \Leftrightarrow \exists\, F^c \in \tau \wedge (f(F))^c \notin \tau'$$

Remark : There are no relation between the concepts continuous, open, closed functions and the following table show that :

f continuous function	f open function	f closed function
T	T	T
T	T	F
T	F	T
F	T	T
T	F	F
F	T	F
F	F	T
F	F	F

Such that T = True (i.e., the function is satisfy) and F = False (i.e., the function is not satisfy). Also, there are eight probability may be taken the function for example (T F T) means that the function f is continuous, not open , closed. Therefore we will introduce an eight examples satisfy this probability :

<u>Example (1) :</u> (T T T) means the function f is continuous, open , closed.
Define the identity function $f : (X, \tau) \to (X, \tau)$; $f(x) = x \quad \forall\ x \in X$.
f is continuous since $\forall\ V \in \tau$ in rang X $\Rightarrow$ $f^{-1}(V) = V \in \tau$ in domain X.
f is open since $\forall\ U \in \tau$ open in domain X $\Rightarrow$ $f(U) = U$ is open in rang X.
f is closed since $\forall$ F closed in domain X $\Rightarrow$ $f(F) = F$ is closed in rang X.

<u>Example (2) :</u> (T T F) means the function f is continuous, open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}\}$
Define the constant function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = f(3) = a$
f is continuous since it is constant.
f is open since : $X \in \tau \Rightarrow f(X) = \{a\} \in \tau'$, $\phi \in \tau \Rightarrow f(\phi) = \phi \in \tau'$ and $\{1\} \in \tau \Rightarrow f(\{1\}) = \{a\} \in \tau'$ (i.e., $\forall\ U \in \tau \Rightarrow f(U) \in \tau'$).
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$)
But, $f(\{2, 3\}) = \{a\} \notin \mathcal{F}'$ since $\{a\}^c = \{b, c\} \notin \tau'$

<u>Example (3) :</u> (T F T) means the function f is continuous, not open , closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a, b\}\}$
Define the constant function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = f(3) = c$
f is continuous since it is constant.
f is not open since $\exists$ open set $\{1\} \in \tau$,but $f(\{1\}) = \{c\} \notin \tau'$.
f is closed since : The family of closed sets in X is $\mathcal{F} = \{X, \phi, \{2, 3\}\}$ and the family of closed set in Y is $\mathcal{F}' = \{Y, \phi, \{c\}\}$, then
$f(X) = \{c\}$, $f(\phi) = \phi$, $f(\{2, 3\}) = \{c\}$ (i.e., $\forall\ F \in \mathcal{F} \Rightarrow f(F) \in \mathcal{F}'$).

<u>Example (4) :</u> (F T T) means the function f is not continuous, open , closed.
Define the function $f : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$; $f(x) = x \quad \forall\ x \in \mathbb{R}$
f is not continuous since $\exists$ (0, 1) open in $(\mathbb{R}, \tau_u)$, but $f^{-1}((0, 1)) = (0, 1)$ not open in $(\mathbb{R}, I)$.
f is open and closed since the only open and closed sets in $(\mathbb{R}, I)$ are $\mathbb{R}$, ϕ and $f(\mathbb{R}) = \mathbb{R}$ and $f(\phi) = \phi$ (i.e., the direct image of open (rep., closed) set is open (rep., closed) set)

Example (5) : (T F F) means the function f is continuous, not open , not closed.
Define the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall x \in \mathbb{R}$
f is continuous since the rang is $(\mathbb{R}, I)$ (see, page 37).
f is not open since $\exists$ open set $(0, 1)$ in $(\mathbb{R}, \tau_u)$, but $f((0, 1)) = (0, 1)$ is not open in $(\mathbb{R}, I)$.
f is not closed since $\exists$ closed set $\{0\}$ in $(\mathbb{R}, \tau_u)$, but $f(\{0\}) = \{0\}$ is not closed in $(\mathbb{R}, I)$.

Example (6) : (F T F) means the function f is not continuous, open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}, \{a, b\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{a\} \in \tau'$,but $f^{-1}(\{a\}) = \{1, 2\} \notin \tau$.
f is open since : $X \in \tau \Rightarrow f(X) = \{a, b\} \in \tau'$, $\phi \in \tau \Rightarrow f(\phi) = \phi \in \tau'$ and $\{1\} \in \tau \Rightarrow f(\{1\}) = \{a\} \in \tau'$ (i.e., $\forall\ U \in \tau \Rightarrow f(U) \in \tau'$).
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$), but, $f(\{2, 3\}) = \{a, b\} \notin \mathcal{F}'$ since $\{a, b\}^c = \{c\} \notin \tau'$.

Example (7) : (F F T) means the function f is not continuous, not open , closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{c\}, \{b, c\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{b, c\} \in \tau'$,but $f^{-1}(\{b, c\}) = \{3\} \notin \tau$.
f is not open since $\exists$ open set $\{1\} \in \tau$,but $f(\{1\}) = \{a\} \notin \tau'$.
f is closed since : The family of closed sets in X is $\mathcal{F} = \{X, \phi, \{2, 3\}\}$ and the family of closed set in Y is $\mathcal{F}' = \{Y, \phi, \{a, b\}, \{a\}\}$, then
$f(X) = \{a, b\}$, $f(\phi) = \phi$, $f(\{2, 3\}) = \{a, b\}$ (i.e., $\forall\ F \in \mathcal{F} \Rightarrow f(F) \in \mathcal{F}'$).

Example (8) : (F F F) means the function f is not continuous, not open , not closed.
Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau' = \{Y, \phi, \{a\}\}$
Define the function $f : (X, \tau) \to (Y, \tau')$; $f(1) = f(2) = a$, $f(3) = b$
f is not continuous since $\exists$ open set $\{a\} \in \tau'$,but $f^{-1}(\{a\}) = \{1, 2\} \notin \tau$.
f is not open since $\exists$ open set $X \in \tau$,but $f(X) = \{a, b\} \notin \tau'$.
f is not closed since $\exists$ closed set $\{2, 3\} \in \mathcal{F}$ (since $\{2, 3\}^c = \{1\} \in \tau$), but, $f(\{2, 3\}) = \{a, b\} \notin \mathcal{F}'$ since $\{a, b\}^c = \{c\} \notin \tau'$.

Remark : The open function is closed and the closed function is open if the function is bijective (injective and surjective). i.e.,

$$f \text{ is bijective function } \Rightarrow (f \text{ open} \Leftrightarrow f \text{ closed})$$

$$f \text{ is bijective function } \Rightarrow (f \text{ not open} \Leftrightarrow f \text{ not closed})$$

This means if we wanted to get a function is open not closed or closed not open must be define a function not bijection (not injective or not surjective) since if we define a bijective function then it's either open and closed or not open and not closed.

Remark : We talking about the function $f : (X, \tau) \to (Y, \tau')$ is continuous or discontinuous. Now we will question if the function f is bijective and continuous this implies that f^{-1} is continuous (i.e., if f^{-1} exists function and f is continuous, then that implies to f^{-1} is continuous ?? or conversaly). The answer of this question is no since we can find a continuous function but your inverse is not continuous and the following example show that :

Example : Define the function $f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall x \in \mathbb{R}$

Notes that f is continuous since the rang is $(\mathbb{R}, I)$ (see, page 37).

Notes that f is bijective , then f^{-1} is exists function and

$f^{-1} : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$, but this function not continuous since

$\exists$ open set $(0, 1)$ in $(\mathbb{R}, \tau_u)$, but $(f^{-1})^{-1}((0, 1)) = (0, 1)$ is not open in $(\mathbb{R}, I)$.

Now the question : Is there are functions is continuous and there inverse is continuous two ??. The answer is yes and the following definition introduce this functions :

Definition : Homeomorphism Functions

Let $f : (X, \tau) \to (Y, \tau')$ be a function. The function f is called **homeomorphism** if its injective, surjective, continuous and f^{-1} continuous. i.e.,

$f : (X, \tau) \to (Y, \tau')$ is homeomorphism $\Leftrightarrow$ f 1-1, onto, continuous and f^{-1} continuous

$f : (X, \tau) \to (Y, \tau')$ is not homeomorphism $\Leftrightarrow$

$$f \text{ not 1-1} \vee f \text{ not onto} \vee f \text{ not continuous} \vee f^{-1} \text{ not continuous}$$

Remark : Clear that every homeomorphism function is continuous, but the converse is not true for example :

$$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, I) \; ; \; f(x) = x \quad \forall \; x \in X$$

The function f is1-1, onto, continuous, but f^{-1} is not continuous. Therefore f is not homeomorphism.

Remark : If $f : (X, \tau) \to (Y, \tau')$ is homeomorphism function, this means :
$(f^{-1})^{-1}(U) \in \tau' \quad \forall \; U \in \tau$ (def of continuity), but $(f^{-1})^{-1}(U) = U$ (since f bijective), so we can said

$$f^{-1} \text{ is continuous} \iff f(U) \in \tau' \quad \forall \; U \in \tau$$

but this is the definition of open function, so if f^{-1} is continuous this means f is open and vise versa with property that f is bijective. i.e.,

$$f^{-1} \text{ is continuous} \iff f \text{ is open}$$

if f is bijective (by previous remark, p. 47, f is open $\iff$ f is closed), so that

$$f^{-1} \text{ is continuous} \iff f \text{ is open} \iff f \text{ is closed}$$

i.e., the three concepts are equivalent and we can replace the definition of homeomorphism as follow :

f is homeomorphism $\iff$ f is 1-1, onto, continuous and open.

f is homeomorphism $\iff$ f is 1-1, onto, continuous and closed.

such that we replace the statement f^{-1} is continuous in definition of homeomorphism by either f open function or f closed function.

Remarks :

[1] If f is homeomorphism function, then f^{-1} is also homeomorphism function.
since f is 1-1 and onto, then f^{-1} is 1-1 and onto
since f is homeomorphism, then f^{-1} is continuous, also, $f = (f^{-1})^{-1}$ is continuous
Therefore, f^{-1} is homeomorphism function.

[2] If $f : (X, \tau) \to (Y, \tau')$ and $g : (Y, \tau') \to (Z, \tau'')$ are both homeomorphism functions, then the composition $gof : (X, \tau) \to (Z, \tau'')$ is homeomorphism.
since f and g are 1-1 and onto, then gof is 1-1 and onto
since f and g are continuous, then gof is continuous (by previous theorem)
since f and g are homeomorphism, then f^{-1} and g^{-1} are continuous also $f^{-1}og^{-1}$ is continuous (by previous theorem)
but, $f^{-1}og^{-1} = (gof)^{-1}$ is continuous.
Therefore, gof is homeomorphism function.

Definition : **Homeomorphic Topologies**

We called two topological spaces (X, τ) and (Y, τ') are **homeomorphic** if there exists a homeomorphism function from (X, τ) to (Y, τ') and denoted by $(X, \tau) \cong (Y, \tau')$ or $(Y, \tau') \cong (X, \tau)$. i.e.,

$$(X, \tau) \cong (Y, \tau') \Leftrightarrow \exists \text{ homeomorphism function } f : (X, \tau) \to (Y, \tau')$$

Theorem : The relation $\cong$ is an equivalent relation on the family of topological spaces.

Proof : We must prove the relation $\cong$ is reflexive, symmetric and transitive.

(1) To prove $\cong$ is reflexive. i.e., $(X, \tau) \cong (X, \tau)$??

Define the identity function $f : (X, \tau) \to (X, \tau)$; $f(x) = x \quad \forall x \in X$

Clear that f is 1-1, onto, continuous and $f = f^{-1}$ so that f^{-1} is continuous

Therefore, $\cong$ is reflexive.

(2) To prove $\cong$ is symmetric. i.e., if $(X, \tau) \cong (Y, \tau') \Rightarrow (Y, \tau') \cong (X, \tau)$??

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists$ homo. funct. $f : (X, \tau) \to (Y, \tau')$

by remark [1] above, we have f^{-1} is homo. funct. and

$f^{-1} : (Y, \tau') \to (X, \tau) \Rightarrow (Y, \tau') \cong (X, \tau)$

Therefore, $\cong$ is symmetric.

(3) To prove $\cong$ is transitive. i.e., if $(X, \tau) \cong (Y, \tau') \cong (Z, \tau'') \Rightarrow (X, \tau) \cong (Z, \tau'')$??

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists$ homo. funct. $f : (X, \tau) \to (Y, \tau')$ and

$\because (Y, \tau') \cong (Z, \tau'') \Rightarrow \exists$ homo. funct. $g : (Y, \tau') \to (Z, \tau'')$ and

by remark [2] above, we have gof is homo. funct. and

$gof : (X, \tau) \to (Z, \tau'') \Rightarrow (X, \tau) \cong (Z, \tau'')$

Therefore, $\cong$ is transitive.

Theorem :

(1) The bijective function $f : (X, \tau) \to (Y, \tau')$ is homeomorphism iff $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$; $B \subseteq Y$.

(2) The bijective function $f : (X, \tau) \to (Y, \tau')$ is homeomorphism iff $f^{-1}(B^o) = (f^{-1}(B))^o$; $B \subseteq Y$.

Proof :

(1) $(\Leftarrow)$ Suppose that $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and f^{-1} is cont.

$\because \overline{f^{-1}(B)} = f^{-1}(\overline{B}) \Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$; $B \subseteq Y$)

and $\because$ $\overline{f^{-1}(B)} = f^{-1}(\overline{B}) \Rightarrow f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)} \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow$ $f(\overline{B}) \subseteq \overline{f(B)}$; $B \subseteq X$ and f replace by f^{-1})

$\therefore$ f is Home.

($\Rightarrow$) Suppose that f is Home., to prove $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$

$\because$ f is home. $\Rightarrow$ f is cont. and f^{-1} is cont.

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ and $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$

$\Rightarrow \overline{f^{-1}(B)} = f^{-1}(\overline{B})$

(2) ($\Leftarrow$) Suppose that $f^{-1}(B^o) = (f^{-1}(B))^o$, to prove f Home.,

$\because$ f is bij., we must prove f is cont. and f^{-1} is cont.

$\because f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o \Rightarrow$ f is cont.

(by theory f is cont $\Leftrightarrow$ $f^{-1}(B^o) \subseteq (f^{-1}(B))^o$; $B \subseteq Y$)

and $\because$ $f^{-1}(B^o) = (f^{-1}(B))^o \Rightarrow (f^{-1}(B))^o \subseteq f^{-1}(B^o) \Rightarrow f^{-1}$ is cont.

(by theory f is cont $\Leftrightarrow$ $(f(B))^o \subseteq f(B^o)$; $B \subseteq X$ and f replace by f^{-1})

$\therefore$ f is Home.

($\Leftarrow$) Suppose that f is Home., to prove $f^{-1}(B^o) = (f^{-1}(B))^o$

$\because$ f is home. $\Rightarrow$ f is cont. and f^{-1} is cont.

$\Rightarrow f^{-1}(B^o) \subseteq (f^{-1}(B))^o$ and $(f^{-1}(B))^o \subseteq f^{-1}(B^o)$

$\Rightarrow f^{-1}(B^o) = (f^{-1}(B))^o$

<u>Definition :</u> Topological Property

A property "P" of a topological space (X, τ) is called a **<u>topological property</u>** iff every topological space (Y, τ') homeomorphic to (X, τ) also has the same property. i.e., if $(X, \tau) \cong (Y, \tau')$ and (X, τ) has a property "P", then (Y, τ') has the same property and vise versa.

<u>Subspace or Induced space</u>

<u>Definition :</u> Let (X, τ) be a topological space and $W \subseteq X$. Define the family τ_W as a family of subset of W as follow :

$$\tau_W = \{W \cap U \ : \ U \in \tau\}$$

Notes that the elements of τ_W are intersection W with every open set in X.

<u>Theorem :</u> Let (X, τ) be a topological space and $W \subseteq X$. Then $\tau_W = \{W \cap U \ : \ U \in \tau\}$ is a topology on W.

<u>Proof :</u> We will satisfy the three conditions in the definition of topological space.

(1) To prove, $W \in \tau_W$ and $\phi \in \tau_W$.

$\because\ X \in \tau \wedge W \subseteq X \Rightarrow W = W \cap X \Rightarrow W \in \tau_W$ (def. of τ_W)

$\because\ \phi \in \tau \wedge \phi \subseteq X \Rightarrow \phi = W \cap \phi \Rightarrow \phi \in \tau_W$ (def. of τ_W)

(2) Let $V_1, V_2 \in \tau_W$, to prove $V_1 \cap V_2 \in \tau_W$

$\because\ V_1 \in \tau_W \Rightarrow \exists\ U_1 \in \tau\ ;\ V_1 = W \cap U_1$ (def. of τ_W)

$\because\ V_2 \in \tau_W \Rightarrow \exists\ U_2 \in \tau\ ;\ V_2 = W \cap U_2$ (def. of τ_W)

$\Rightarrow V_1 \cap V_2 = (W \cap U_1) \cap (W \cap U_2)$

$\Rightarrow V_1 \cap V_2 = W \cap \underbrace{(U_1 \cap U_2)}_{\in\ \tau}$ (since $\cap$ distribution on $\cap$)

$\Rightarrow V_1 \cap V_2 \in \tau_W$ (def. of τ_W)

(3) Let $V_\alpha \in \tau_W\ ;\ \alpha \in \Lambda$, to prove $\bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_W$

$\because\ V_\alpha \in \tau_W \Rightarrow \exists\ U_\alpha \in \tau\ ;\ V_\alpha = W \cap U_\alpha\ ;\ \alpha \in \Lambda$ (def. of τ_W)

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha = \bigcup_{\alpha\in\Lambda}(W \cap U_\alpha)$

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha = W \cap \underbrace{(\bigcup_{\alpha\in\Lambda} U_\alpha)}_{\in\ \tau}$

$\Rightarrow \bigcup_{\alpha\in\Lambda} V_\alpha \in \tau_W$ (def. of τ_W)

Therefore, τ_W is a topology on W.

<u>Definition :</u> Subspace (or Induced) Topology

Let (X, τ) be a topological space and $W \subseteq X$. Then the topology τ_W is called the **<u>subspace</u> (or <u>induced) topology</u>** for W and the pair (W, τ_W) is called **<u>subspace</u>** of (X, τ).

<u>Example :</u> Let X = {a, b, c}, τ = {X, ϕ, {a}, {a, c}}, W = {a, b}, Z = {b} and K = {a, c}. Find τ_W , τ_Z , τ_K .

<u>Solution :</u>

$\tau_W = \{W \cap U\ :\ U \in \tau\}$

$\tau_W = \{W \cap X, W \cap \phi, W \cap \{a\}, W \cap \{a, c\}\}$

$= \{W, \phi, \{a\}\}$

By similar way we compute τ_Z , τ_K .

$\tau_Z = \{Z \cap U\ :\ U \in \tau\}$

$\tau_Z = \{Z \cap X, Z \cap \phi, Z \cap \{a\}, Z \cap \{a, c\}\}$

$= \{Z, \phi\} = I_Z =$ **indiscrete topology on Z**

$\tau_K = \{K \cap U\ :\ U \in \tau\}$

$\tau_K = \{K \cap X, K \cap \phi, K \cap \{a\}, K \cap \{a, c\}\} = \{K, \phi, \{a\}\}$.

Remarks :

[1] Notes that there is an open set in the subspace but it's not open in the space. In the previous example the set $W = \{a, b\}$ is open in the subspace (W, τ_W) but it is not open in the space (X, τ) i.e., $W \notin \tau$, so we have :

$$V \in \tau_w \not\Rightarrow V \in \tau$$

In other word $\tau_W \not\subset \tau$ (in general).

[2] Notes that, in the previous example $\tau_Z = I_Z = \{Z, \phi\}$, but $\tau \neq I_X = \{X, \phi\}$.

[3] There are some example $\tau_W = D_W =$ **discrete topology on W**, but $\tau \neq D_X$ i.e.,

$$(\tau_w = D_W \not\Rightarrow \tau = D_X)$$

For example : Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b\}, \{c, d\}\}$ and $W = \{a, c\}$, then $\tau_W = \{W, \phi, \{a\}, \{c\}\} = D_W$

[4] If $\tau_X = D_X$, then $\tau_W = D_W$ for all $W \subseteq X$. (i.e., $\tau_X = D_X \Rightarrow \tau_W = D_W$)

To prove this property it's enough to prove that every singleton subset of W is open set in W.

Let $y \in W \Rightarrow \{y\} \subseteq W$, to prove $\{y\} \in \tau_W$??

$\because \{y\} \subseteq W \subseteq X \Rightarrow \{y\} \subseteq X \Rightarrow \{y\} \in D_X$

since $\{y\} = W \cap \{y\} \Rightarrow \{y\} \in \tau_W$

$\Rightarrow \tau_W = D_W$.

[5] If $\tau = I_X = \{X, \phi\}$, then $\tau_W = I_W = \{W, \phi\}$. (i.e., $\tau = I_X \Rightarrow \tau_W = I_W$)

To prove this property

$\tau_W = \{W \cap U : U \in \tau\} = \{W \cap X, W \cap \phi\} = \{W, \phi\}$.

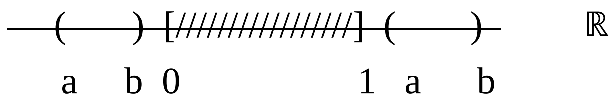
Example : In the usual topological space $(\mathbb{R}, \tau_u)$. Find the induced topology for the following sets : $W = [0, 1]$, $H = \mathbb{N}$, $M = \mathbb{Q}$, $K = [2, 3)$.

Solution : The open sets in $(\mathbb{R}, \tau_u)$ is the union of family of open interval and the family of open interval is a basis for topology τ_u . So we will use the open interval to compute the basis for induce topology for given set as follow :

W = [0, 1] ??

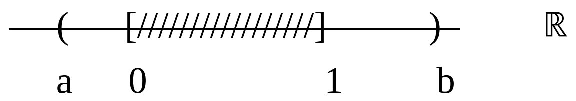
if, $a, b \leq 0 \vee a, b \geq 1$

then, $[0, 1] \cap (a, b) = \phi$



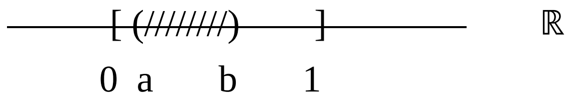
if, $b > 1 \wedge a < 0$

then, $[0, 1] \cap (a, b) = [0, 1]$

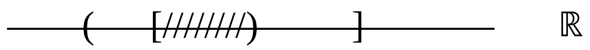


if, $b \leq 1 \wedge a \geq 0$

then, $[0, 1] \cap (a, b) = (a, b)$



if, $a < 0 \wedge 0 < b \leq 1$



then, $[0, 1] \bigcap (a, b) = [0, b)$ a 0 b 1

if, $0 < a \leq 1 \quad \wedge \quad b > 1$ $[\quad (//////] \quad)$ $\mathbb{R}$

then, $[0, 1] \bigcap (a, b) = (a, 1]$ 0 a 1 b

From the probability above the basis for induce topology τ_W is

$$\beta_W = \{[0, 1], \phi, (a, b), [0, b), (a, 1]\}$$

Notes that the elements in this family is infinite since $a, b \in \mathbb{R}$.

H = ℕ ??

The induce topology for $H = \mathbb{N}$ is discrete topology $D_{\mathbb{N}}$ since :

Let (a, b) open interval in $(\mathbb{R}, \tau_u)$; $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$; $n - 1 < a < n$ and $n < b < n+1$.

$n - 1 \quad a \quad n \quad b \quad n+1 \quad n+2 \quad \ldots$

This means that every singleton set (i.e., $\mathbb{N} \bigcap (a, b) = \{n\}$) from $\mathbb{N}$ is open in $\mathbb{N}$ (i.e., $\{n\} \in \tau_{\mathbb{N}}$). Therefore $\tau_{\mathbb{N}} = D$.

M = ℚ ??

We will intersect $\mathbb{Q}$ with every open interval (a, b) in $(\mathbb{R}, \tau_u)$; $a, b \in \mathbb{R}$.

$(|||||||||||||||||||)$ $\mathbb{R}$

a $\mathbb{Q}$ b

$\mathbb{Q} \bigcap (a, b)$ = the rational numbers in (a, b) and the basis for induce topology $\tau_{\mathbb{Q}}$ is

$$\beta_{\mathbb{Q}} = \{\mathbb{Q} \bigcap (a, b) \; ; \; a, b \in \mathbb{R}\}.$$

K = [2, 3) ??

To compute the induce topology τ_K ; $K = [2, 3)$ is similar of compute the induce topology τ_W ; $W = [0, 1]$ above by replace $[0,1]$ by $[2, 3)$ as follow :

if $a, b \leq 2 \; \vee \; a, b \geq 3$ $(\quad) \; [//////////////) \; (\quad)$ $\mathbb{R}$

$\Rightarrow \quad [2, 3) \bigcap (a, b) = \phi$ a b 2 3 a b

if $a < 2 \; \wedge \; b \geq 3$ $(\quad [//////////////) \quad)$ $\mathbb{R}$

$\Rightarrow \quad [2, 3) \bigcap (a, b) = [2, 3)$ a 2 3 b

if $a \geq 2 \; \wedge \; b \leq 3$ $[\,(//////) \quad)$ $\mathbb{R}$

$\Rightarrow \quad [2, 3) \bigcap (a, b) = (a, b)$ 2 a b 3

if $a < 2 \; \wedge \; 2 < b \leq 3$ $(\quad [//////) \quad)$ $\mathbb{R}$

$\Rightarrow \quad [2, 3) \bigcap (a, b) = [2, b)$ a 2 b 3

if $2 < a \leq 3 \; \wedge \; b > 3$ $[\quad (//////) \quad)$ $\mathbb{R}$

$\Rightarrow \quad [2, 3) \bigcap (a, b) = (a, 3)$ 2 a 3 b

From the probability above the basis for induce topology τ_K is

$$\beta_K = \{[2, 3), \phi, (a, b), [2, b), (a, 3]\}.$$

We can compute the induce topology for the intervals [c, d], [c, d), (c, d], (c, d) by similar way by taken this probality and replace W = [0, 1] or K = [2, 3) by [c, d], [c, d), (c, d], (c, d).

Theorem : Let (X, τ) be a topological space and (W, τ_W) be a subspace topology of X. If $W \in \tau$, then τ_W is subfamily of τ. i.e.,
If W open set in X, then every open set in W is open in X.

Proof : We must prove the following statement $\tau_W \subseteq \tau$ (i.e., if $V \in \tau_W \Rightarrow V \in \tau$)

Let $V \in \tau_W \Rightarrow \exists\, U \in \tau \;;\; V = W \cap U$ (def. of τ_W)

$\because\; W \in \tau$ (by hypothesis) $\wedge\; U \in \tau$

$\Rightarrow W \cap U \in \tau$ (def. of Top.)

$\Rightarrow V \in \tau$ (since $V = W \cap U$)

The following example clear this theorem :

Example : Let X = {a, b, c, d}, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and W = {a, b, c}. Find τ_W .

Solution :

Clear $W \in \tau$. To compute τ_W :

$\tau_W = \{W \cap U \;:\; U \in \tau\}$

$\tau_W = \{W \cap X, W \cap \phi, W \cap \{a\}, W \cap \{a, b\}, W \cap \{a, b, c\}\}$

$= \{W, \phi, \{a\}, \{a, b\}\}$

Notes that τ_W is subfamily of τ (i.e., $\tau_W \subseteq \tau$).

Remark : From definition of induce topology τ_W , notes that :

$$V \in \tau_W \Leftrightarrow \exists\, U \in \tau \;;\; V = W \cap U$$

The question now what about the close set, the previous statement satisfy or not ??
The answer **yes** such that :

$$A \in (\tau_W)^c \Leftrightarrow \exists\, F \in \tau^c \;;\; A = W \cap F$$

By other statement :

$$A \in \mathscr{F}_W \Leftrightarrow \exists\, F \in \mathscr{F} \;;\; A = W \cap F$$

Such that $\mathscr{F}$ is the family of closed sets in X and $\mathscr{F}_W$ is the family of closed sets in W.

Example : Let X = {a, b, c}, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and W = {b, c}.
Then $\tau_W = \{W, \phi, \{b\}\}$, $\mathscr{F} = \{X, \phi, \{b, c\}, \{a, c\}, \{c\}\}$ and $\mathscr{F}_W = \{W^c, \phi^c, \{b\}^c\}$ such that :

$\mathcal{F}_W = \{W^c, \phi^c, \{b\}^c\} = \{\phi, W, \{c\}\}$

$= \{W \cap X, W \cap \phi, W \cap \{b, c\}, W \cap \{a, c\}, W \cap \{c\}\}$

<u>Theorem :</u> If K is a subspace from W and W is a subspace from X, then K is a subspace from X.

<u>Proof :</u> Let (X, τ) be a topological space and $W \subseteq X$, $K \subseteq W$, to prove K is a subspace from X, must prove :

(1) $K \subseteq X$

(2) if $A \in (\tau_W)_K \Rightarrow \exists\, U \in \tau$; $A = K \cap U$

Now,

(1) Since $K \subseteq W \subseteq X \Rightarrow K \subseteq X$.

(2) Let $A \in (\tau_W)_K \Rightarrow \exists\, V \in \tau_W$; $A = K \cap V$

(def. of induce top. and K is a sub space from W)

$\because V \in \tau_W \Rightarrow \exists\, U \in \tau$; $V = W \cap U$

(def. of induce top. and W is a sub space from X)

$\because A = K \cap V \Rightarrow A = K \cap (W \cap U)$ (since $V = W \cap U$)

$\Rightarrow A = (K \cap W) \cap U$ ($\cap$ associative)

$\Rightarrow A = K \cap U$ (since $K \subseteq W$ and $K = K \cap W$)

<u>Definition :</u> Restriction Function

Let $f : X \to Y$ be a function and let $A \subseteq X$. We say the function $g : A \to Y$ such that $g(a) = f(a)$ for all $a \in A$ is the **<u>restriction function</u>** on the set A and denoted by $g = f|_A$.

<u>Example :</u> Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 1$ be a function.

Notes that the domain of f is $\mathbb{R}$, take $\mathbb{N} \subseteq \mathbb{R}$ and

$f|_{\mathbb{N}} : \mathbb{N} \to \mathbb{R}$ such that $(f|_{\mathbb{N}})(x) = x + 1$.

$f|_{\mathbb{N}}$ is restriction function on the set $\mathbb{N}$.

We will use this definition to introduce the following theorem :

<u>Theorem :</u> Let $f : (X, \tau) \to (Y, \tau')$ be a continuous function and W be a subspace topology from X. Then $f|_W$ is continuous.

<u>Proof :</u> To prove, $f|_W : (W, \tau_W) \to (Y, \tau')$ is continuous ??

i.e., to prove if $V \in \tau' \Rightarrow (f|_W)^{-1}(V) \in \tau_W$

Let $V \in \tau' \Rightarrow f^{-1}(V) \in \tau$ (since f is continuous)

$\Rightarrow W \cap f^{-1}(V) \in \tau_W$ (By def. of τ_W)

$\Rightarrow (f|_W)^{-1}(V) \in \tau_W$ (since $W \cap f^{-1}(V) = (f|_W)^{-1}(V)$)

$\therefore$ $f|_W$ is cont.

Remark : From previous theorem we can get on an infinite number from continuous functions thought out know one continuous function for example :
$f : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ such that $f(x) = x + 2$ is continuous function.
$\therefore$ every functions of follows is continues function :
$f|_{\mathbb{N}}$, $f|_{\mathbb{Q}}$, $f|_{[2, 3]}$, $f|_{(0,\infty)}$ … etc.

We can get another of an infinite number from new continuous functions by theorem blow :

Theorem : Let (X, τ) be a topological space and (W, τ_W) be a subspace of X. Then the inclusion function $i : (W, \tau_W) \to (X, \tau)$ such that $i(x) = x$ for all $x \in W$ is continuous.

Proof : To prove if $V \in \tau \Rightarrow i^{-1}(V) \in \tau_W$

Let $V \in \tau \Rightarrow i^{-1}(V) = W \cap V$ (since $W \subseteq X$ and def. of i)

$\Rightarrow i^{-1}(V) \in \tau_W$ (since $W \cap V \in \tau_W$ and by def. of τ_W)

$\therefore$ i is cont.

Let (X, τ) be a topological space and (W, τ_W) be a subspace topology of X and $A \subseteq W \subseteq X$. We can compute $\overline{A}$, A^o , A^b in (X, τ) and from the other hand we can compute $\overline{A}$, A^o , A^b in (W, τ_W). The question what relation between ($\overline{A}$ in X and $\overline{A}$ in W), (A^o in X and A^o in W) and (A^b in X and A^b in W) ?? The theorem blow answer on this questions :

Theorem : Let (X, τ) be a topological space and (W, τ_W) be a subspace topology of X and $A \subseteq W \subseteq X$, then

(1) $W \cap \overline{A} = \overline{A}$ in W ; $\overline{A}$ is closure of A in X

(2) $W \cap A^o \subseteq A^o$ in W.

(3) $W \cap A^b \supseteq A^b$ in W.

Proof :

(1) To prove, $W \cap \overline{A} = \overline{A}$ in W

we must prove, $W \cap \overline{A} \subseteq \overline{A}$ in W and $W \cap \overline{A} \supseteq \overline{A}$ in W

$\because A \subseteq W \subseteq X \Rightarrow \overline{A} \in \mathcal{F}$ (by previous theorem $\overline{A}$ is closed in X)

$\Rightarrow \overline{A} \cap W \in \mathcal{F}_W$ ($\overline{A}$ is closed in W)

Now,

$A \subseteq W \quad \wedge \quad A \subseteq \overline{A} \quad \Rightarrow \quad A \subseteq W \cap \overline{A}$

Notes that $W \cap \overline{A}$ is closed set in W and containing A, but $\overline{A}$ in W is the smallest closed set in W contain A, so we get

$\Rightarrow \overline{A}$ in $W \subseteq W \cap \overline{A}$ ---------------(1)

and,

$\because W \cap \overline{A} \in \mathcal{F}_W \Rightarrow \exists F \in \mathcal{F} : \overline{A}$ in $W = W \cap F$

$\Rightarrow A \subseteq F$

$\Rightarrow \overline{A} \subseteq \overline{F} = F \quad \Rightarrow \quad \overline{A} \subseteq F$ ($\overline{F} = F$ since F is closed)

$\Rightarrow W \cap \overline{A} \subseteq W \cap F = \overline{A}$ in W

$\Rightarrow W \cap \overline{A} \subseteq \overline{A}$ in W ----------------(2)

From (1) and (2) we have, $W \cap \overline{A} = \overline{A}$ in W.

(2) To prove, $W \cap A^o \subseteq A^o$ in W

$A^o \in \tau \quad \Rightarrow \quad W \cap A^o \in \tau_W$

$\Rightarrow \quad W \cap A^o \subseteq A^o \subseteq A \quad \Rightarrow \quad W \cap A^o \subseteq A$

$\Rightarrow \quad W \cap A^o \subseteq A^o$ in W (since $W \cap A^o$ open in W contain in A)

i.e., A^o in W must contain all open set in W contain in A.

(3) To prove, A^b in $W \subseteq A^b \cap W$.

Let $x \in A^b$ in W $\Rightarrow \forall V \in \tau_W , x \in V$; $V \cap A \neq \phi \ \wedge \ V \cap A^c \neq \phi$

(By def. of boundary point)

$\because V \in \tau_W \Rightarrow \exists U \in \tau ; V = W \cap U$ (def. of τ_W)

$\Rightarrow \forall U \in \tau, x \in U, U \cap A \neq \phi \ \wedge \ U \cap A^c \neq \phi$

$\Rightarrow x \in A^b \Rightarrow x \in A^b \cap W$ (since $x \in V \subseteq W$)

$\therefore A^b$ in $W \subseteq A^b \cap W$

Remark : The equality of properties (2) and (3) in the previous theorem is not true in general and the following example clear that :

Example : Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $W = \{a, c, d\}$.

Then $\tau_W = \{W, \phi, \{a\}, \{a, c\}\}$, $A = \{a, c\}$

$A = \{a, c\} = A^o$ in W since $A \in \tau_W$ and $A^o = \{a\}$

Since $\{a\}$ is largest open set in X containing in A $\Rightarrow$

$A^o \cap W = \{a\} \cap \{a, c, d\} = \{a\} \Rightarrow A^o$ in $W \neq A^o \cap W$ since $\{a, c\} \neq \{a\}$

On the other hand to compute A^b, A^b in W, we will compute A^x, A^x in W such that

A^x in $W = \phi$, then
A^b in $W = W - (A^o$ in $W \bigcup A^x$ in $W) = W - \{a, c\} = \{d\}$
$\therefore$ A^b in $W = \{d\}$
$A^x = \phi \Rightarrow A^b = X - (A^o \bigcup A^x) = X - \{a\} = \{b, c, d\}$
$\Rightarrow A^b \bigcap W = \{c, d\}$
$\therefore$ A^b in $W \neq A^b \bigcap W$
To check property (1) in the previous theorem we compute $\overline{A}$ and $\overline{A}$ in W as follow :
$\overline{A}$ in $W = W$ and $\overline{A} = X \Rightarrow \overline{A} \bigcap W = X \bigcap W = W$
$\therefore$ $\overline{A}$ in $W = \overline{A} \bigcap W$

Product Space

Definition : **Cartesian Product**
Let X and Y be any two sets. The **Cartesian product** , or simply **product** of X by Y is denoted by X×Y and denoted as :

$$X\times Y = \{(x, y) ; x \in X \wedge y \in Y\}$$

Definition : **Product Space**
Let (X, τ) and (Y, τ') be two topological spaces. We say the topology has a base β ;

$$\beta = \{ U\times V ; U \in \tau \wedge V \in \tau'\}$$

Is the **Product Topology** on the set X×Y and denoted by $\tau_{X\times Y}$ and called the spaces $(X\times Y, \tau_{X\times Y})$ is the **Product Space** of X by Y.

Remark : Notes that β in general not topology since it's not satisfy the three condition of topology, but since β is a base for topology, so we can get the three condition of topology and the following example show that :

Example : Let $X = \{1, 2, 3\}$, $\tau = \{X, \phi, \{1\}\}$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi, \{b\}\}$.
Compute $\tau_{X\times Y}$.
Solution :
$X\times Y = \{(x, y) ; x \in X \wedge y \in Y\}$
$= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
$\beta = \{ U\times V ; U \in \tau \wedge V \in \tau'\}$
$= \{X\times Y, X\times\phi, X\times\{b\}, \phi\times Y, \phi\times\phi, \phi\times\{b\}, \{1\}\times Y, \{1\}\times\phi, \{1\}\times\{b\}\}$
$= \{X\times Y, \phi, X\times\{b\}, \{1\}\times Y, \{1\}\times\{b\}\}$
Since $A\times\phi = \phi$ and $\phi\times A = \phi$ for any set A.

$\therefore\ \beta = \{X\times Y, \phi, \{(1, b), (2, b), (3, b)\}, \{(1, a), (1, b)\}, \{(1, b)\}\}$

Notes that β is not topology since

$\{(1, b), (2, b), (3, b)\} \bigcup \{(1, a), (1, b)\}, \{(1, b)\} \notin \beta$

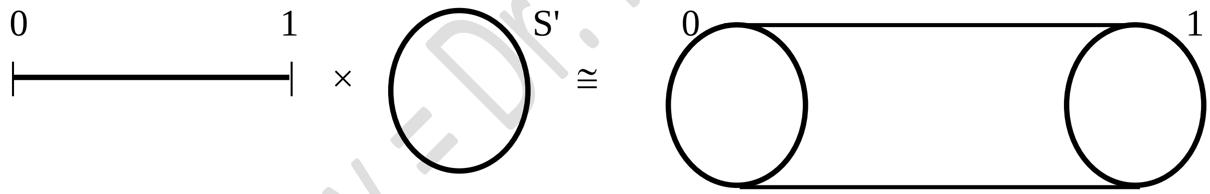
The elements of $\tau_{X\times Y}$ is elements of β and add all possible union of elements of β ;

$\therefore\ \tau_{X\times Y} = \{X\times Y, \phi, \{(1, b), (2, b), (3, b)\}, \{(1, a), (1, b)\}, \{(1, b)\}, \{(1, b), (2, b), (3, b), (1, a)\}\}$.

Remark : We can compute the product space $(X\times X, \tau_{X\times X})$ depending on (X, τ_X) only, also we can compute $(Y\times Y, \tau_{Y\times Y})$, $(Y\times X, \tau_{Y\times X})$, $(X\times Y\times Z, \tau_{X\times Y\times Z})$, … etc., there are an infinite number from product spaces which can computing from one space known or more than one space. In general $X\times Y \neq Y\times Y$.

From known product spaces which we use always is usual space $\mathbb{R}^n$; $n \in\mathbb{N}$ and the most common one is $\mathbb{R}^2$ which represent the plane and its product space follow from product $(\mathbb{R}, \tau_u)$ by self.

Example : Let $X = [0, 1]$ be a subspace of $(\mathbb{R}, \tau_u)$ and take $S' = \{(x, y) \in \mathbb{R}^2; x^2+y^2 =1\}$ be a subspace of $(\mathbb{R}^2, \tau_u)$ such that S' geometry represented as a circle in plane its center the original point (0, 0). Then $[0, 1] \times S'$ is a cylinder as follow :



Remarks : Let (X, τ) and (Y, τ') be any two topological spaces.

[1] If $\tau = I_X$ and $\tau' = I_Y$, then $\tau_{X\times Y} = I_{X\times Y}$, i.e.,

If $\tau =\{X, \phi\}$ and $\tau' = \{Y, \phi\}$, then $\tau_{X\times Y} = \{X\times Y, \phi\}$ and β is :

$$\beta = \{X\times Y, X\times\phi, \phi\times Y, \phi\times\phi\} = \{X\times Y, \phi\} = I_{X\times Y}$$

[2] If $\tau = I_X$ and $\tau' \neq I_Y$ or $\tau \neq I_X$ and $\tau' = I_Y$, then $\tau_{X\times Y} \neq I_{X\times Y}$, for example :

If $X = \{1, 2\}$, $\tau =\{X, \phi, \{1\}\}$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi\}$, then

$$\beta = \{X\times Y, \phi, \{1\}\times Y\} = \{X\times Y, \phi, \{(1, a), (1, b)\}\} = \tau_{X\times Y} \neq I_{X\times Y}$$

[3] If $\tau = D_X$ and $\tau' = D_Y$, then $\tau_{X\times Y} = D_{X\times Y}$,

Proof. To prove any topology is discrete topology it's enough to prove every singleton set is open i.e., ($\forall$ $\{(x, y)\}$ singleton set $\Rightarrow \{(x, y)\} \in \tau_{X\times Y}$)

$\{(x, y)\} = \{x\} \times \{y\}$ (def. Cartesian product of X by Y)

$\{x\} \in \tau$ (since $\tau = D_X$)

$\{y\} \in \tau'$ (since $\tau' = D_Y$)

$\{(x, y)\} \in \beta_{X\times Y}$

$\Rightarrow \{(x, y)\} \in \tau_{X\times Y}$ (def. product spaces)

[4] If $\tau \neq D_X$ or $\tau' \neq D_Y$, then $\tau_{X\times Y} \neq D_{X\times Y}$, and the following example show that :
If $X = \{1, 2\}$, $\tau =\{X, \phi, \{1\}, \{2\}\} = D_X$, $Y = \{a, b\}$ and $\tau' = \{Y, \phi\} \neq D_Y$, then

$$\beta = \{X\times Y, \phi, \{1\}\times Y, \{2\}\times Y\}$$
$$= \{X\times Y, \phi, \{(1, a), (1, b)\}, \{(2, a), (2, b)\}\} = \beta_{X\times Y} = \tau_{X\times Y} \neq D_{X\times Y}$$

[5] If $A \subseteq X$ and $B \subseteq Y$, then $A\times B \subseteq X\times Y$ and we can compute the closure of $A\times B$ in $X\times Y$ (i.e., $\overline{A \times B}$), on the other hand there are $\overline{A}$ in X and $\overline{B}$ in Y, also we can compute $\overline{A} \times \overline{B}$ and the question what relation between $\overline{A \times B}$ and $\overline{A} \times \overline{B}$ and the answer $\overline{A \times B} = \overline{A} \times \overline{B}$.
Also, by similar way we can conclusion $(A\times B)^o= A^o\times B^o$.

[6] There are two natural projection functions from product space $X\times Y$ to codomain X and others to codomainY and denoted by P_X and P_Y and called the first project $X\times Y$ on X and called the second project $X\times Y$ on Y. We will show that the two functions are surjective , continuous and open as follows :

$P_X : X\times Y \to X$; $P_X((x, y)) = x$ and

$P_Y : X\times Y \to Y$; $P_Y((x, y)) = y$

; the first projection map the order pair (x, y) to first coordinate while the second projection map the order pair (x, y) to the second coordinate.

<u>To prove, P_X is continuous function</u>

We must prove, if $U \in \tau \Rightarrow P_X^{-1}(U) \in \tau_{X\times Y}$

Let $U \in \tau \Rightarrow P_X^{-1}(U) = U\times Y$ (By def. of P_X)

$\because U \in \tau \wedge Y \in \tau' \Rightarrow U\times Y \in \beta_{X\times Y}$

$\Rightarrow U\times Y \in \tau_{X\times Y}$ (since $\beta_{X\times Y} \subset \tau_{X\times Y}$)

$\Rightarrow P_X^{-1}(U) \in \tau_{X\times Y}$

$\therefore P_X$ is continuous functions

By similar way we prove P_Y is continuous functions

Let $V \in \tau' \Rightarrow P_Y^{-1}(V) = X\times V$ (By def. of P_Y)

$\because X \in \tau \wedge V \in \tau' \Rightarrow X\times V \in \beta_{X\times Y}$

$\Rightarrow X\times V \in \tau_{X\times Y}$ (since $\beta_{X\times Y} \subset \tau_{X\times Y}$)

$\Rightarrow P_Y^{-1}(V) \in \tau_{X\times Y}$

$\therefore P_Y$ is continuous functions

<u>To prove, P_X is open function</u>

Let $U\times V \in \beta_{X\times Y} \Rightarrow U\times V$ open set in $X\times Y$; $U \in \tau \wedge V \in \tau'$

$\Rightarrow P_X(U\times V) = U$

$\because U \in \tau \Rightarrow P_X(U\times V) \in \tau$

$\therefore P_X$ is open functions

By similar way we prove P_Y is open functions

Let $U\times V \in \beta_{X\times Y} \Rightarrow U\times V$ open set in $X\times Y$; $U \in \tau \wedge V \in \tau'$

$\Rightarrow P_Y(U\times V) = V$

$\because V \in \tau' \Rightarrow P_Y(U\times V) \in \tau$

$\therefore P_Y$ is open functions.

[7] Notes that $X\times Y \neq Y\times X$ since $(x, y) \neq (y, x)$ in general, but $X\times Y \cong Y\times X$ (i.e., $X\times Y$, $Y\times X$ are Homeomorphic), to prove this :

Define $f : X\times Y \to Y\times X$; $f((x, y)) = (y, x)$

f is 1-1 function since,

Let $f((x_1, y_1)) = f((x_2, y_2)) \Rightarrow (y_1, x_1) = (y_2, x_2)$

$\Rightarrow x_1 = x_2 \wedge y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$.

f is onto function since,

$\forall (y, x) \in Y\times X \; \exists (x, y) \in X\times Y ; f((x, y)) = (y, x)$.

f is continuous function since,

Let β be a base of $X\times Y$ and β' be a base of $Y\times X$

Let $V\times U \in \beta' \Rightarrow V \in \tau' \wedge U \in \tau$

$\Rightarrow V\times U \in \tau_{Y\times X}$

$\Rightarrow f^{-1}(V\times U) = U\times V$ open set in $X\times Y$

f is open function since, the image of every open set in domain is open set in codomain ;

Let $U\times V \in \beta \Rightarrow U\times V$ open set in $X\times Y$; $U \in \tau \wedge V \in \tau'$

$\Rightarrow f(U\times V) = V\times U \in \tau_{Y\times X}$

$\therefore$ f is homeomorphism function.

[8] If $y_0 \in Y$, then the product space $X\times\{y_0\}$ topological equivalent the space X. i.e., $X\times\{y_0\} \cong X$; $X\times\{y_0\} = \{(x, y_0) : x \in X\}$

To prove this :

Define $f : X \to X\times\{y_0\}$; $f(x) = (x, y_0) \quad \forall x \in X$

f is 1-1 function since,

Let $f(x_1) = f(x_2) \Rightarrow (x_1, y_0) = (x_2, y_0)$

$\Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X$

f is onto function since,

$\forall (x, y_0) \in X\times\{y_0\} \; \exists x \in X ; f(x) = (x, y_0)$

f is continuous function, since the sets in the base of the space $X\times\{y_0\}$ is $U\times\{y_0\}$; $U \in \tau$ or ϕ, then

$f^{-1}(U\times\{y_0\}) = U \in \tau$ and $f^{-1}(\phi) = \phi \in \tau$

f is open function, since if U is open in domain X, then $f(U) = U\times\{y_0\}$ and $U\times\{y_0\}$ is open in codomain $X\times\{y_0\}$.

$\therefore$ f is homeomorphism function.

[9] If $x_0 \in X$, then the product space $\{x_0\}\times Y$ topological equivalent the space Y. i.e., $\{x_0\}\times Y \cong Y$; $\{x_0\}\times Y = \{(x_0, y) : y \in Y\}$

To prove this : (By a similar way of [8])

Define $f : Y \to \{x_0\}\times Y$; $f(y) = (x_0, y)$ $\forall\ y \in Y$

f is 1-1 function since,

Let $f(y_1) = f(y_2) \Rightarrow (x_0, y_1) = (x_0, y_2)$

$\Rightarrow y_1 = y_2$ $\forall\ y_1, y_2 \in Y$

f is onto function since,

$\forall\ (x_0, y) \in \{x_0\}\times Y\ \exists\ y \in Y$; $f(y) = (x_0, y)$

f is continuous function, since the sets in the base of the space $\{x_0\}\times Y$ is $\{x_0\}\times V$; $V \in \tau'$ or ϕ, then

$f^{-1}(\{x_0\}\times V) = V \in \tau'$ and $f^{-1}(\phi) = \phi \in \tau'$

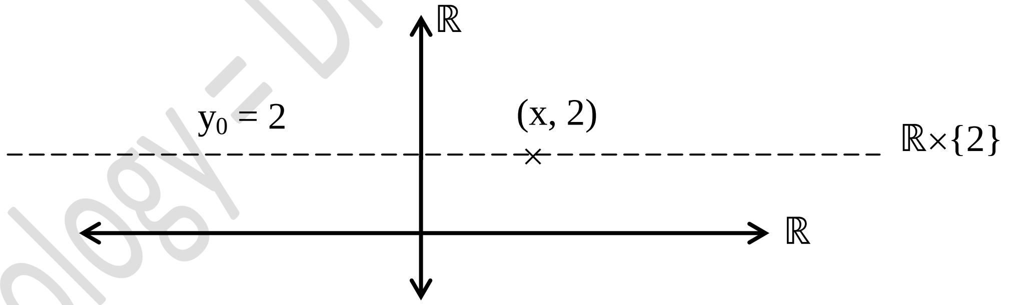
f is open function, since if V is open in domain Y, then $f(V) = \{x_0\}\times V$ and $\{x_0\}\times V$ is open in codomain $\{x_0\}\times V$.

$\therefore$ f is homeomorphism function.

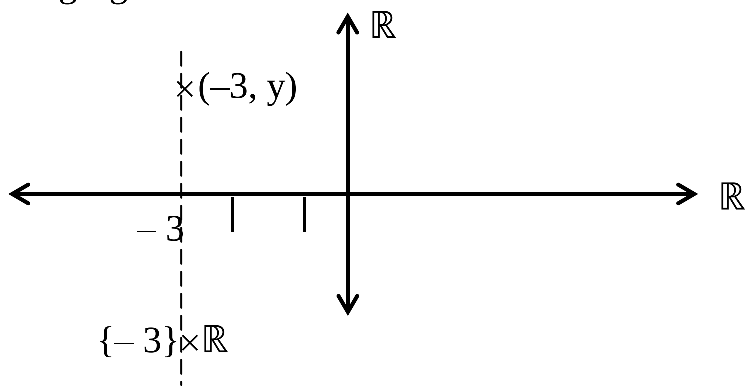
Notes that $X\times\{y_0\}$ is a sub space of the space $X\times Y$ and represented horizontal section in the space $X\times Y$ at the point y_0 . Also, $\{x_0\}\times Y$ is a sub space of the space $X\times Y$ and represented vertical section in the space $X\times Y$ at the point x_0.

For example, take $X = Y = \mathbb{R}$ and $\tau = \tau' = \tau_u$, then the product space $X\times Y$ is the known plane $\mathbb{R}^2$.

Let $y_0 = 2$, then $X\times\{y_0\} = \mathbb{R}\times\{2\}$ is subspace from $\mathbb{R}^2$ and represented as horizontal line segment and the following figure show this :



Let $x_0 = -3$, then $\{x_0\}\times Y = \{-3\}\times\mathbb{R}$ is subspace from $\mathbb{R}^2$ and represented as vertical line segment and the following figure show this :



Definition : Quotient Space

Let (X, τ_X) be a topological space and Y be any set. Let $f : X \to Y$ be a surjective function, then the set

$$\tau_f = \{G \subseteq Y \,;\, f^{-1}(G) \in \tau_X\}$$

Is a topology on Y this topology called **quotient topology** on Y generated by f and (X, τ_X).

Question : The topology $\tau_f = \{G \subseteq Y \,;\, f^{-1}(G) \in \tau_X\}$ is the largest topology on Y make the function f continuous.

Answer : Let τ be another topology on Y making f continuous.

$\Rightarrow$ $f^{-1}(G)$ is open in X $\forall$ $G \in \tau$.

$\Rightarrow$ $G \in \tau_f$ (def. of τ_f)

$\Rightarrow$ $\tau \subseteq \tau_f$

$\Rightarrow$ τ_f is the largest topology on Y making f is continuous.

Theorem : Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous surjective function, if f either open or closed, then $\tau_f = \tau_Y$.

Proof :

Clearly, $\tau_Y \subseteq \tau_f$ (by previous question)

Now, to show that $\tau_f \subseteq \tau_Y$

Let $G \in \tau_f \Rightarrow f^{-1}(G) \in \tau_X$

$\Rightarrow$ $f(f^{-1}(G)) = G$ is open in Y (since f is open)

$\Rightarrow$ $G \in \tau_Y$

$\Rightarrow$ $\tau_f \subseteq \tau_Y$

So, $\tau_f = \tau_Y$.

By similar way if f is closed.

Theorem : Let Y has the quotient space generated by the surjective function $f : X \to Y$, then $g : Y \to Z$ is continuous function if and only if gof is continuous function.

Proof :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gof"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

($\Rightarrow$) The composition of continuous functions is continuous.

($\Leftarrow$) Let G be open set in Z

Since gof is cont. $\Rightarrow$ $(gof)^{-1}(G) = f^{-1}(g^{-1}(G))$ is open in X

But $g^{-1}(G) \subseteq Y \wedge f^{-1}(g^{-1}(G))$ is open in X

$\Rightarrow$ $g^{-1}(G)$ is open in Y (by definition of τ_f , $g^{-1}(G) \in \tau_f$)

$\Rightarrow$ g is continuous.

Remarks :

[1] Let X be a nonempty set. The **partition or decomposition** on X with the relation R is the family of disjoint nonempty subsets of X and their union equal X. The elements of this partition called **equivalence classes** and denoted by [x].

[2] The set of equivalence classes for X is called **quotient set** for X with the relation R and denoted by $X|R = \{[x] : x \in X\}$.

[3] The mapping $p : X \to X/R$; $p(x) = [x]$ is called **quotient mapping**.

Definition : Quotient Space

Let (X, τ) be a topological space and R be equivalence relation on X. Let $p : X \to X/R$; $p(x) = [x]$ be surjective quotient mapping from X to X|R, then the quotient topology on X|R is the largest topology make the function f continuous and the space $(X|R, \tau_{X|R})$ is called **quotient space**.

Chapter Three : Compact Spaces

Definition : Cover & Finite Cover & Open (resp., Closed) Cover

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of subsets of the space (X, τ). We called the family $\{A_\alpha\}_{\alpha\in\Lambda}$ **cover** of X iff X equal the union of elements of the family $\{A_\alpha\}_{\alpha\in\Lambda}$.

$$\text{(i.e., } X = \bigcup_{\alpha\in\Lambda} A_\alpha)$$

If $\{A_\alpha\}_{\alpha\in\Lambda}$ is finite and cover X, then $\{A_\alpha\}_{\alpha\in\Lambda}$ is called a **finite cover** of X.

If each A_α, $\alpha\in\Lambda$, is open (resp., closed) in X and $\{A_\alpha\}_{\alpha\in\Lambda}$ cover X, then $\{A_\alpha\}_{\alpha\in\Lambda}$ is called an **open (resp., closed) cover** of X.

Definition : Subcover

Let $C = \{A_\alpha\}_{\alpha\in\Lambda}$ be a cover of X and $\{B_i\}_{i\in\Lambda}$ be a sub family of C and cover X, then $\{B_i\}_{i\in\Lambda}$ is called **subcover** from C.

Definition : Compact Space

A space X is called **compact** iff each open cover of X has a finite subcover for X.
i.e.,

$$X \text{ is compact } \Leftrightarrow \forall\ C = \{U_\alpha\}_{\alpha\in\Lambda}\ ;\ U_\alpha \in \tau\ \forall\, \alpha\ \wedge\ X = \bigcup_{\alpha\in\Lambda} U_\alpha$$
$$\Rightarrow \exists\ \alpha_1, \alpha_2, \dots, \alpha_n\ ;\ X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$
$$X \text{ is not compact } \Leftrightarrow \exists\ C = \{U_\alpha\}_{\alpha\in\Lambda}\ ;\ U_\alpha \in \tau\ \forall\, \alpha\ \wedge\ X = \bigcup_{\alpha\in\Lambda} U_\alpha$$
$$\Rightarrow \nexists\ \alpha_1, \alpha_2, \dots, \alpha_n\ ;\ X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Example : Take $X = \mathbb{R}$ and $\tau = \{\mathbb{R}, \phi, \mathbb{Q}, Irr\}$

The open set in τ are $\mathbb{R}, \phi, \mathbb{Q}$, Irr.

Take, $C_1 = \{\mathbb{Q}, Irr\}$ is open cover for $\mathbb{R}$ (i.e., $\mathbb{R} = \mathbb{Q} \bigcup Irr$) and it's a finite subcover of $\mathbb{R}$, so this cover satisfy the definition of compact space.

Now, we introduce all open cover for $\mathbb{R}$ as follow :

$$C_2 = \{\mathbb{R}, \phi, \mathbb{Q}\},\ C_3 = \{\mathbb{R}, \phi, Irr\},\ C_4 = \{\mathbb{R}, \mathbb{Q}, Irr\},\ C_5 = \{\phi, \mathbb{Q}, Irr\},$$
$$C_6 = \{\mathbb{R}, \phi\},\ C_7 = \{\mathbb{R}, \mathbb{Q}\},\ C_8 = \{\mathbb{R}, Irr\},\ C_9 = \{\mathbb{R}, \phi, \mathbb{Q}, Irr\} = \tau$$

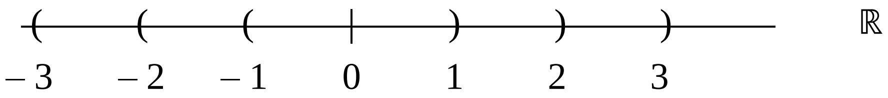
and every cover of them has a finite subcover, hence $(\mathbb{R}, \tau)$ is compact.

Remark : If we want to show that the space is not compact it's enough give one open cover, but has not finite subcover. The following example show this :

Example : Is $(\mathbb{R}, \tau_u)$ compact space ???

Answer : **No**

Take the open cover $C = \{(-n, n)\ ;\ n \in \mathbb{N}\}$ for $\mathbb{R}$ (i.e., $\mathbb{R} = \bigcup_{i=1}^{n}(-n, n)$).



Notes that the open cover C has no finite subcover, because if we assumption there exists a finite open interval cover $\mathbb{R}$, then their union is the large interval, for example $(-m, m)$; $m \in \mathbb{N}$, this means $\mathbb{R} = (-m, m)$, $m \neq \infty$ and this contradiction!!!

Example : Show that $(\mathbb{N}, \tau_{cof})$ is compact space.

Solution : Let $C = \{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover for $\mathbb{N}$, then

$$\mathbb{N} = \bigcup_{\alpha\in\Lambda} U_\alpha\ ;\ U_\alpha \in \tau_{cof}\ \ \forall\ \alpha \in \Lambda$$

since $U_\alpha \in \tau_{cof}$, then $\mathbb{N} - U_\alpha$ if finite , for all $U_\alpha \in \tau_{cof}$

take arbitrary set say U_{α_n}, then $\mathbb{N} - U_{\alpha_n}$ if finite,

let $\mathbb{N} - U_{\alpha_n} = \{x_1, \ldots, x_{n-1}\}$; $x_1, \ldots, x_{n-1} \in \mathbb{N}$

this means that U_{α_n} contains all natural numbers excepts $x_1, \ldots, x_{n-1}$

take another set contains x_1 say U_{α_1} and set contains x_2 say U_{α_2} …. etc set contains x_{n-1} say $U_{\alpha_{n-1}}$. So we have n set which are U_{α_n} ,......, U_{α_1} such that $\mathbb{N} = \bigcup_{i=1}^{n} U_{\alpha_i}$.

therefore, the open cover $C = \{U_\alpha\}_{\alpha\in\Lambda}$ has a finite subcover $\{U_{\alpha_i}\}_{i=1}^{n}$, hence $(\mathbb{N}, \tau_{cof})$ is compact.

Definition : Compact Subspace

Let (X, τ) be a topological space and W be a subspace of X. We called a space W is **compact space** iff every open cover from X cover W has a finite subcover. i.e.,

$$W \text{ is compact} \Leftrightarrow \forall\ \{U_\alpha\}_{\alpha\in\Lambda}\ ; U_\alpha \in \tau\ \ \forall\ \alpha\ \wedge\ W \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$$

$$\Rightarrow\ \exists\ \alpha_1, \alpha_2, \ldots, \alpha_n\ ; W \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Theorem : (Heine-Borel Theorem)

The subset $A \subseteq X$ is compact iff A is closed and bounded.

Remarks : The previous theorem is one of theorems which study in mathematical analysis which is specific to subsets of Euclidean space $\mathbb{R}$ with usual topology and special case in $(\mathbb{R}, \tau_u)$. So, every subset of $\mathbb{R}$ is compact iff its closed and bounded.

Example : In ($\mathbb{R}$, τ_u) show that any set from the following sets is compact by using Heine-Borel Theorem.

$A = (2, 3)$, , $B = [5, 7]$, $C = [-2, 1)$, $D = \mathbb{N}$, , $E = \{2, 3, 4\}$, $F = \mathbb{Q}$

Solution :

A not closed and bounded $\Rightarrow$ not compact.

B closed and bounded $\Rightarrow$ compact.

C not closed and bounded $\Rightarrow$ not compact.

D closed and not bounded $\Rightarrow$ not compact.

E closed and bounded $\Rightarrow$ compact.

F not closed and not bounded $\Rightarrow$ not compact.

In general in usual topology ($\mathbb{R}$, τ_u),

every closed intervals is compact sets.

every half closed (open) interval and open intervals is not compact sets.

every finite sets of points is compact sets.

$\mathbb{Q}$ and Irr not compact.

Definition : Hereditary Property

We call a property "P" of a space (X, τ) **hereditary property** iff every subspace of a space X with the property must have the property.

Notes that if there exists at least one subspace not satisfy this property, then this property not hereditary property.

Remark : Compactness is not hereditary property. For example :

Example : Take $X = [0,1]$ with induce topology from ($\mathbb{R}$, τ_u) and take $W = (0, 1)$. Clear that $W \subseteq X$ and X is compact space, but W is not compact.

Theorem : If A and B are compact sets in a space (X, τ), then $A \bigcup B$ is compact set.

Proof : Let $C = \{U_\alpha\}_{\alpha\in\Lambda}$; $U_\alpha \in \tau \ \forall \alpha$ open cover of $A \bigcup B$.

To prove C has a finite subcover

$\because\ A \bigcup B \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow A \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \wedge B \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

(since $A \subseteq A \bigcup B$, $B \subseteq A \bigcup B$)

$\therefore$ C cover of A and B, but A and B are compact

$\Rightarrow \exists\, \alpha_1, \dots, \alpha_n$; $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ and $\exists\, \alpha_1, \dots, \alpha_m$; $B \subseteq \bigcup_{j=1}^{m} U_{\alpha_j}$

$\Rightarrow A \bigcup B \subseteq \bigcup_{k=1}^{n+m} U_{\alpha_k}$

$\therefore$ C has finite subcovet for $A \bigcup B \Rightarrow A \bigcup B$ compact set.

Remark : If A and B are compact sets in a space (X, τ), then $A \bigcap B$ is not necessary compact set.

Remarks :

[1] If τ is finite set, then (X, τ) is compact space, since every open cover heir being finite, so every open cover has a finite subcover.

Special case : (X, I) is compact space for any X (finite or infinite), for example $(\mathbb{R}, I)$ and $(\mathbb{N}, I)$ are compact spaces … etc.

Another special case : if X is finite, then τ is finite set and $\tau \subseteq IP(X)$, therefore (X, τ) is compact space.

[2] If X is infinite set, then (X, D) is not compact space, since the open cover $C = \{\{x\} ; x \in X\}$ has no finite subcover. If X is finite, then (X, D) is compact space (by Remark [1]).

Theorem : The continuous image of compact space is compact. i.e.,

If $f : (X, \tau) \to (Y, \tau')$ is continuous function and X is compact space, then $f(X)$ is compact.

Proof : Let $f : (X, \tau) \to (Y, \tau')$ be continuous and X compact space.

To prove, $f(X)$ compact in Y

Let $C = \{V_\alpha\}_{\alpha\in\Lambda}$ open cover for $f(X)$

$\Rightarrow f(X) \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$; $V_\alpha \in \tau'$ $\forall \alpha \in \Lambda$

$\Rightarrow f^{-1}(f(X)) \subseteq f^{-1}(\bigcup_{\alpha\in\Lambda} V_\alpha)$

$\Rightarrow X \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$ (since $f^{-1}(f(X)) = X$ and $f^{-1}(\bigcup_{\alpha\in\Lambda} A_\alpha) = \bigcup_{\alpha\in\Lambda} f^{-1}(A_\alpha)$)

Since f is continuous $\Rightarrow f^{-1}(V_\alpha) \in \tau$ $\forall \alpha \in \Lambda$

$\Rightarrow \{f^{-1}(V_\alpha)\}_{\alpha\in\Lambda}$ is open cover for X

$\because$ X is compact $\Rightarrow \exists\, \alpha_1, \dots, \alpha_n$; $X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$

$\Rightarrow f(X) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}))$

$\Rightarrow \quad = \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i}))$ (since $f(A \bigcup B) = f(A) \bigcup f(B)$)

$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$ (since $f(f^{-1}(A)) \subseteq A$)

$\therefore$ $f(X)$ compact set.

Corollary : If the product space $X\times Y$ is compact, then X and Y are compact spaces.

Proof : The projection function $P_X : X\times Y \to X$ is continuous and onto

$\because$ $X\times Y$ compact $\Rightarrow$ $P_X(X\times Y)$ compact (by previous theorem)

$\because$ P_X onto $\Rightarrow$ $P_X(X\times Y) = X$

$\Rightarrow$ X compact

By the similar way we prove Y compact.

The projection function $P_Y : X\times Y \to Y$ is continuous and onto

$\because$ $X\times Y$ compact $\Rightarrow$ $P_Y(X\times Y)$ compact (by previous theorem)

$\because$ P_Y onto $\Rightarrow$ $P_Y(X\times Y) = Y$

$\Rightarrow$ Y compact

Remark : If X and Y are compact spaces, , then X×Y is compact spaces (i.e., the converse of the above theorem is true in general), and it's theorem one of the important theorem in topology called **Tichonov theorem** (without prove) and we will introduce some examples to user of this theorem.

$(\mathbb{R}, \tau_u)\times(\mathbb{N}, \tau_{cof})$ is not compact space, since $(\mathbb{R}, \tau_u)$ not compact.

$(\mathbb{N}, \tau_{cof})\times(\mathbb{N}, \tau_{cof})$ is compact space, since $(\mathbb{N}, \tau_{cof})$ compact.

$(\mathbb{N}, \tau_{cof})\times(\mathbb{R}, I)$ is compact space, since $(\mathbb{N}, \tau_{cof})$ compact and $(\mathbb{R}, I)$ compact.

$(\mathbb{R}, D)\times(\mathbb{R}, I)$ is not compact space, since $(\mathbb{R}, D)$ not compact.

$(\{1, 2, 3\}, \tau)\times(X, I)$ is compact space for any τ and for any X, since {1, 2, 3} is finite set and (X, I) compact for any X.

Definition : Topological Property

A property "P" of a topological space (X, τ) is called a **topological property** iff every topological space (Y, τ') homeomorphic to (X, τ) also has the same property. i.e.,
if $(X, \tau) \cong (Y, \tau')$ and (X, τ) has a property "P", then (Y, τ') has the same property and vise versa.

Theorem : Compactness is a topological property.

Proof : Let (X, τ) and (Y, τ') be topological space ; $X \cong Y$

Suppose that X is compact, To prove Y is compact

$\because$ $X \cong Y \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

$\because$ f continuous, onto and X compact $\Rightarrow$ $f(X) = Y$ compact

(by theorem : The continuous image of compact space is compact)

Now, suppose that Y is compact, To prove X is compact

$\because$ f^{-1} continuous, onto and Y compact $\Rightarrow$ $f^{-1}(Y) = X$ compact

(by same theorem : The continuous image of compact space is compact)

Remark : By using compactness as a topological property, we can decided the known space is equivalent another known space or not. Also, we can decided the space unknown (compact or not compact) if its equivalent another known space compact or not compact. For example :

$(\mathbb{R}, I) \ncong (\mathbb{R}, \tau_u)$ since $(\mathbb{R}, I)$ is compact, but $(\mathbb{R}, \tau_u)$ not compact. Also,

If $(\mathbb{N}, \tau_{cof}) \cong (Y, \tau)$ and since $(\mathbb{N}, \tau_{cof})$ is compact, then (Y, τ) is compact (since compactness is topological property).

Remark : Compactness is not hereditary property (remark p.67), but if we add a condition for the subset from compact space become a compact set and the following theorem show that :

Theorem : A **closed** subset of a compact space is compact.

Proof :

Let (X, τ) compact space and F closed set in X

To prove, F compact set

Let $C = \{U_\alpha\}_{\alpha\in\Lambda}$ open cover of F

$\Rightarrow F \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \; ; U_\alpha \in \tau \;\; \forall \alpha \in \Lambda$

$\because \; X = F \bigcup F^c \Rightarrow X = \bigcup_{\alpha\in\Lambda} U_\alpha \bigcup F^c$ (since $F \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$)

$\because \; U_\alpha \in \tau \;\; \forall \alpha \in \Lambda \;\wedge\; F^c \in \tau$ (since F closed set)

$\Rightarrow \{U_\alpha\}_{\alpha\in\Lambda} \bigcup \{F^c\}$ open cover of X

$\because$ X compact $\Rightarrow \exists\, \alpha_1, \dots, \alpha_n \; ; X = (\bigcup_{i=1}^{n} U_{\alpha_i}) \bigcup F^c$

But, $F \subseteq X \Rightarrow F \subseteq (\bigcup_{i=1}^{n} U_{\alpha_i}) \bigcup F^c$

Since $F \bigcap F^c = \phi \Rightarrow F \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

$\therefore$ F compact set.

Notes that the condition being F closed is very important and the theorem is not true if the condition deleted.

Definition : Finite Intersection Property

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of sets. We call this family satisfy the **finite intersection property** and denoted by (f.i.p) if the intersection of any finite number of elements of this family is nonempty. i.e.,

$$\{A_\alpha\}_{\alpha\in\Lambda} \text{ has f.i.p.} \Leftrightarrow \bigcap_{i=1}^{n} A_{\alpha_i} \neq \phi \;\; \forall\, n$$

Example : Let $A_n = (-\frac{1}{n}, \frac{1}{n})$; $n \in \mathbb{N}$
Notes that, the intersection of any numbers of elements of this family is nonempty, so its satisfy (f.i.p).

Remark : If the family $\{A_\alpha\}_{\alpha\in\Lambda}$ satisfy (f.i.p), but not necessarily intersection every elements of the family is nonempty. i.e., not necessarily $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$.

Example : Let $A_n = \{n, n+1, \ldots\}$; $n \in \mathbb{N}$

$$A_1 = \mathbb{N},\ A_2 = \mathbb{N} \setminus \{1\},\ A_3 = \mathbb{N} \setminus \{1, 2\}, \ldots$$

Notes that, intersection every finite numbers of the family is nonempty while $\bigcap_{n=1}^{\infty} A_n = \phi$ and this family satisfy (f.i.p). i.e., in general

$$\bigcap_{i=1}^{n} A_{\alpha_i} \neq \phi\ \forall\ n \not\Rightarrow \bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$$

Theorem : A space (X, τ) is compact iff every family of closed subsets of X satisfy (f.i.p) being intersection nonempty.

Proof : $(\Rightarrow)$ Suppose that X is compact space and $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of closed sets satisfy (f.i.p.), to prove $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

Suppose $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$

$\Rightarrow (\bigcap_{\alpha\in\Lambda} F_\alpha)^c = \phi^c$

$\Rightarrow \bigcup_{\alpha\in\Lambda} F_\alpha^c = X$

Since F_α is closed $\forall\ \alpha \Rightarrow F_\alpha^c \in \tau\ \forall\ \alpha$

$\Rightarrow \{F_\alpha^c\}_{\alpha\in\Lambda}$ is open cover for X

Since X is compact $\Rightarrow \exists\ \alpha_1, \ldots, \alpha_n$; $X = \bigcup_{i=1}^{n} F_{\alpha i}^c$

$\Rightarrow X^c = (\bigcup_{i=1}^{n} F_{\alpha i}^c)^c$

$\Rightarrow \phi = \bigcap_{i=1}^{n} F_{\alpha i}$ C!!

Since this family satisfy (f.i.p), then $\bigcap_{i=1}^{n} F_{\alpha i} \neq \phi$.

So, $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

$(\Leftarrow)$ Suppose $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of closed sets satisfy (f.i.p.) and $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$. ((for any family of closed sets satisfy (f.i.p.))), to prove X is compact.

Suppose that X is not compact $\Rightarrow \exists$ open cover for X has not finite subcover

i.e., $X = \bigcup_{\alpha\in\Lambda} U_\alpha \wedge X \neq \bigcup_{i=1}^{n} U_{\alpha i}\ \ \forall\ n$

$\Rightarrow X^c \neq (\bigcup_{i=1}^{n} U_{\alpha i})^c \Rightarrow \phi \neq \bigcap_{i=1}^{n} U_{\alpha i}^c$

$U_{\alpha i}^c \in \mathcal{F}$ since $U_{\alpha i} \in \tau$

we have, the family of closed sets $\{U^c_\alpha\}_{\alpha\in\Lambda}$ satisfy (f.i.p), but intersection this family is empty since

$X = \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow X^c = (\bigcup_{\alpha\in\Lambda} U_\alpha)^c$

$\Rightarrow \phi = \bigcap_{\alpha\in\Lambda} U^c_\alpha$ C!! (with hypothesis)

$\therefore$ X compact space.

Definition : Lindelöf Space

A space (X, τ) is called **Lindelöf space** iff each open cover of X has a countable subcover for X. i.e.,

$$X \text{ is Lindelöf} \Leftrightarrow \forall\ C = \{U_\alpha\}_{\alpha\in\Lambda}\ ;\ U_\alpha \in \tau\ \forall\ \alpha\ \wedge\ X = \bigcup_{\alpha\in\Lambda} U_\alpha$$
$$\Rightarrow\ \exists\ \alpha_1, \alpha_2, \ldots\ ;\ X = \bigcup_{i=1}^{\infty} U_{\alpha_i}.$$

Question : Prove or disprove :

(1) Every compact space is Lindelöf space.

(2) Every Lindelöf space is compact space.

(3) Every finite space is Lindelöf space.

(4) Every countable space is Lindelöf space.

Solution :

(1) **Yes, prove,** i.e., Compact $\Rightarrow$ Lindelöf.

Let X be a compact space $\Rightarrow$ every open cover of X has a finite subcover

$\because$ every finite set is countable set

$\Rightarrow$ every open cover of X has a countable subcover

$\Rightarrow$ X is Lindelöf space.

(2) **No, disprove,** i.e., Lindelöf $\not\Rightarrow$ Compact. For example :

$(\mathbb{R}, \tau_u)$ is not compact space (see page 66), but its Lindelöf space :

Since every open cover of $\mathbb{R}$ contains of open intervals (by definition of τ_u) and every open interval contains at least one rational number (since $\mathbb{Q}$ is dense set in $\mathbb{R}$), so we can use this rational numbers to numerical the open intervals, so this cover became countable (since the rational numbers is countable).

$\because$ every set is subset of itself, so the countable open cover we search it is itself,

$\therefore$ $(\mathbb{R}, \tau_u)$ is Lindelöf.

(3) **Yes, prove,**

Since every finite space is compact space and every compact space is Lindelöf.

(4) **Yes, prove,**

Since every open cover is countable, so it's the subcover required.

Example : $(\mathbb{N}, \tau)$ and $(\mathbb{Q}, \tau)$ is Lindelöf for any topology τ.

Notes that, the Lindelöf space not nessarily countable (example : ($\mathbb{R}$, τ_u) is Lindelöf, but not countable)

Example : (X, I) is Lindelöf, since its compact space and the only open sets are X, ϕ. For examples of this space ($\mathbb{R}$, I), ($\mathbb{C}$, I) and ($\mathbb{Q}$, I) are Lindelöf space (we can replace X by any set).

Example : (X, D) is Lindelöf if X is countable and not Lindelöf if X is uncountable. For examples of this space ($\mathbb{N}$, D) and ($\mathbb{Q}$, D) are Lindelöf space, but ($\mathbb{R}$, D) and ($\mathbb{C}$, D) are not Lindelöf space.

Example : (X, τ_{cof}) is Lindelöf if X any infinite set since its compact space. For example of this space ($\mathbb{N}$, τ_{cof}), ($\mathbb{R}$, τ_{cof}) and ($\mathbb{C}$, τ_{cof}) … etc.

Example : (X, τ_{cof}) is Lindelöf if X any uncountable set since its compact space.

Remark : Lindelöfness is not hereditary property. For example (see page 67), but if we add a condition for the subset from Lindelöf space become a Lindelöf set and the following theorem show that :

Theorem : A **closed** subset of a Lindelöf space is Lindelöf.
Proof : Similarly of prove Compactness (see page 70).

Theorem : The continuous image of Lindelöf space is Lindelöf. i.e.,
If f : (X, τ) $\rightarrow$ (Y, τ') is continuous function and X is Lindelöf space, then f(X) is Lindelöf.
Proof : Similarly of prove Compactness (see page 68).

Theorem : Lindelöfness is topological property.
Proof : Let (X, τ) and (Y, τ') be topological spaces ; X $\cong$ Y
Suppose that X is Lindelöf, to prove Y is Lindelöf
$\because$ X $\cong$ Y $\Rightarrow$ $\exists$ f : (X, τ) $\rightarrow$ (Y, τ') ; f 1-1, f onto, f continuous, f^{-1} continuous
$\because$ f continuous, onto and X Lindelöf $\Rightarrow$ f(X) = Y Lindelöf
(by theorem : The continuous image of Lindelöf space is Lindelöf)
Now, suppose that Y is Lindelöf, To prove X is Lindelöf
$\because$ f^{-1} continuous, onto and Y Lindelöf $\Rightarrow$ f^{-1}(Y) = X Lindelöf
(by same theorem : The continuous image of Lindelöf space is Lindelöf)

Examples of this theorems :

Example (1) : Let $f : (\mathbb{R}, \tau_u) \to (X, \tau)$ be continuous onto function. What about the space (X, τ) ??

Solution : Since $(\mathbb{R}, \tau_u)$ is Lindelöf space, then (X, τ) is Lindelöf (by theorem : The continuous image of Lindelöf space is Lindelöf)

Example (2) : Let $f : (\mathbb{R}, D) \to (X, \tau)$ be continuous onto function. What about the space (X, τ) ??

Solution : Since $(\mathbb{R}, D)$ is not Lindelöf space, then we cannot decided the space (X, τ) is not Lindelöf because theorem tell us : The continuous onto image of Lindelöf space is Lindelöf, but the domain is not Lindelöf, for example :

$f : (\mathbb{R}, D) \to (X, I)$ such that $f(x) = x$

f is continuous onto and domain $(\mathbb{R}, D)$ is not Lindelöf, but codomain (X, I) is Lindelöf.

$f : (\mathbb{R}, D) \to (X, D)$ such that $f(x) = x$

f is continuous onto and domain $(\mathbb{R}, D)$ is not Lindelöf and codomain (X, D) is not Lindelöf.

Remark : Let $f : (X, \tau) \to (Y, \tau')$ be continuous onto function and Y is Lindelöf, but not necessarily X is Lindelöf, for example :

Example : Let $f : (\mathbb{R}, D) \to (\mathbb{R}, \tau_u)$; $f(x) = x$, be continuous onto function and $(\mathbb{R}, \tau_u)$ is Lindelöf, but $(\mathbb{R}, D)$ X is not Lindelöf

Corollary : If the product space $X \times Y$ is Lindelöf, then X and Y are Lindelöf spaces.

Proof : The projection function $P_X : X \times Y \to X$ is continuous and onto

$\because$ $X \times Y$ Lindelöf $\Rightarrow$ $P_X(X \times Y)$ Lindelöf

(by theorem : The continuous image of Lindelöf space is Lindelöf)

$\because$ P_X onto $\Rightarrow$ $P_X(X \times Y) = X$

$\Rightarrow X$ Lindelöf

By the similar way we prove Y Lindelöf.

The projection function $P_Y : X \times Y \to Y$ is continuous and onto

$\because$ $X \times Y$ Lindelöf $\Rightarrow$ $P_Y(X \times Y)$ Lindelöf

(by same theorem : The continuous image of Lindelöf space is Lindelöf)

$\because$ P_Y onto $\Rightarrow$ $P_Y(X\times Y) = Y$

$\Rightarrow Y$ Lindelöf

Remark : If X and Y are Lindelöf spaces, then X×Y is Lindelöf space (i.e., the converse of the above corollary is true in general), i.e.,

X and Y are Lindelöf $\Leftrightarrow$ X×Y is Lindelöf

and we will introduce some user of this theorem.

$(\mathbb{R}, \tau_u)\times(\mathbb{R}, D)$ is not Lindelöf space, since $(\mathbb{R}, D)$ not Lindelöf.

$(\mathbb{N}, D)\times(\mathbb{N}, \tau_{cof})$ is Lindelöf space, since $(\mathbb{N}, D)$ Lindelöf and $(\mathbb{N}, \tau_{cof})$ Lindelöf.

Example : If $(\mathbb{R}, I)\times(\mathbb{R}, \tau)$ is Lindelöf space, what about the space $(\mathbb{R}, \tau)$??

Solution : The space $(\mathbb{R}, \tau)$ must be Lindelöf.

Corollary : Every quotient space from a Lindelöf space is Lindelöf.

Proof : let (X, τ) be a Lindelöf space and ~ be equivalent relation on X, then the quotient space is $(X/\sim, \tau_p)$ such that $p : X \to X/\sim$; $p(x) = [x]$.

clear that p is continuous onto (see quotient space)

since X is Lindelöf $\Rightarrow$ X/~ is Lindelöf

(by theorem : The continuous image of Lindelöf space is Lindelöf)

Examples :

Every quotient space from (X, I) is Lindelöf space.

Every quotient space from $(\mathbb{R}, \tau_u)$ is Lindelöf space.

Every quotient space from (X, τ_{cof}) is Lindelöf space.

Every quotient space from $(\mathbb{R}, D)$ is not necessarily Lindelöf space.

Chapter Four : Separation Axioms

Definition : **T_0 – Space**

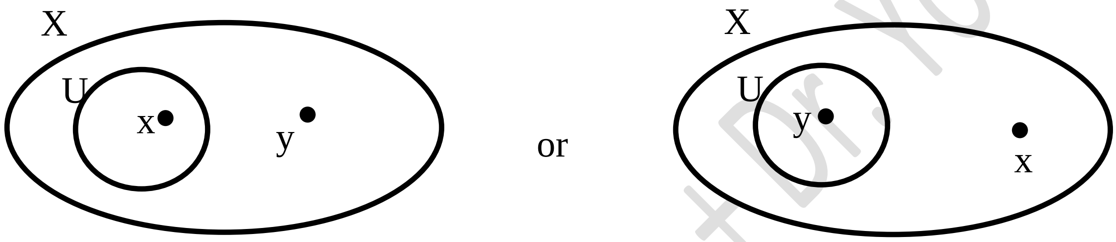
Let (X, τ) be a topological space. Then the space $(X, \tau)$is called **T_0 – space** iff for each pair of distinct points $x, y \in X$, there is either an open set containing x but not y or an open set containing y but not x. i.e.,

X is T_0 – space $\Leftrightarrow \forall\, x, y \in X\ ;\ x \neq y\ \ \exists\, U \in \tau\ ;\ (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$.

If (X, τ) is not T_0 – space, we define,

X is not T_0 – space $\Leftrightarrow \exists\, x, y \in X\ ;\ x \neq y\ \ \forall\, U \in \tau\ ;\ (x, y \in U) \vee (x, y \notin U)$.

The following figure show the definition of T_0 – Space :



Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{1, 2\}\}$. Is (X, τ), T_0 – space.

Solution : Must test every deference elements in X, satisfy the definition or not as follows :

$1 \neq 2 \Rightarrow \exists$ open set $\{1\} \in \tau\ ;\ 1 \in \{1\}\ \wedge\ 2 \notin \{1\}$

$1 \neq 3 \Rightarrow \exists$ open set $\{1\} \in \tau\ ;\ 1 \in \{1\}\ \wedge\ 3 \notin \{1\}$

$2 \neq 3 \Rightarrow \exists$ open set $\{1, 2\} \in \tau\ ;\ 2 \in \{1, 2\}\ \wedge\ 3 \notin \{1, 2\}$

$\therefore$ The definition is satisfy $\Rightarrow (X, \tau)$ is T_0 – space.

Example : Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Is (X, τ), T_0 – space.

Solution : No, since $b \neq c \Rightarrow \nexists$ open set containing b but not c or an open set containing c but not b.

Clear that X containing b, c and ϕ open set not containing b, c and $\{a\}$ open set not containing b, c. Therefore, (X, τ) is not T_0 – space.

Example : Is $(\mathbb{N}, \tau_{cof})$ T_0 – space.

Solution : Yes, we prove that in general since $\mathbb{N}$ containing infinite numbers :

Let $n, m \in \mathbb{N}\ ;\ n \neq m$, to prove $\exists$ open set $U \in \tau_{cof}\ ;\ m \in U\ \wedge\ n \notin U$ or vice versa.

Take $U = \mathbb{N} \setminus \{n\}\ \Rightarrow m \in U \wedge\ n \notin U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

By similar way we can take : $U = \mathbb{N} \setminus \{m\} \Rightarrow m \notin U \wedge n \in U$

and $U \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

The two cases similar and satisfy the definition $\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_0 – space.

<u>Example :</u> In the space (X, I) if X is any set containing more than one element, then (X, I) is not T_0 – space, since X contains more than one element we take $x, y \in X$; $x \neq y$ and $\nexists$ open set containing x but not y or an open set containing y but not x and ϕ open set not containing x, y.

<u>Example :</u> The space (X, D) is T_0 – space.

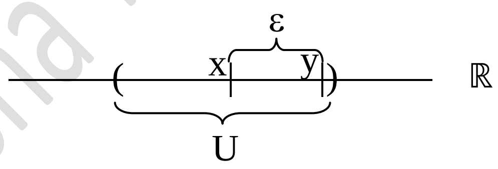
<u>Solution :</u> Let $x, y \in X$; $x \neq y$, then $\{x\} \in D$ i.e., $\{x\}$ open set (by definition D since $D = IP(X)$), hence $x \in \{x\}$ and $y \notin \{x\}$. We can take $\{y\}$ replace of $\{x\}$ and $y \in \{y\}$ and $x \notin \{y\}$. Therefore, (X, D) is T_0 – space.

<u>Example :</u> The usual topological space $(\mathbb{R}, \tau_u)$ is T_0 – space.

<u>Solution :</u> Let $x, y \in \mathbb{R}$; $x \neq y$

Take $U = (x - \varepsilon , x + \varepsilon)$; $\varepsilon = | x - y |$

$\Rightarrow U \in \tau_u \wedge x \in U \wedge y \notin U$



$\therefore$ $(\mathbb{R}, \tau_u)$ is T_0 – space.

<u>Theorem :</u> Every metric space is T_0 – space.

<u>Proof :</u> Let (X, d) be a metric space and $x, y \in X$; $x \neq y$

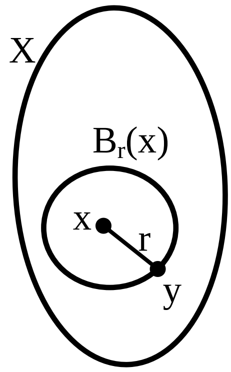
Let $r = d(x, y)$; $d(x, y)$ is the distinct between x and y

Take $U = B_r(x)$; $U = B_r(x)$ is open ball with center x and radius r

$\therefore$ $U \in \tau_d$; $x \in U \wedge y \notin U$;

τ_d is a topology on X induced by d (see page 33)

$\therefore$ (X, d) is T_0 – space.



Now, we introduce theorem gave an equivalent modules for definition T_0 – space.

<u>Theorem :</u> (X, τ) is T_0 – space iff $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall x, y \in X$; $x \neq y$.

i.e., (X, τ) is T_0 – space iff the closure of singleton sets is deference if the elements are deference.

<u>Proof :</u> ($\Rightarrow$) Suppose that X is T_0 – space, to prove $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall$ $x, y \in X$; $x \neq y$

$\because$ X is T_0 – space and $x \neq y \Rightarrow \exists U \in \tau$; $(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

Suppose that $(x \in U \wedge y \notin U) \Rightarrow (x \in U \wedge y \in X - U)$

$X - U$ closed set since U open set $\Rightarrow \{y\} \subseteq X - U$

$\Rightarrow \overline{\{y\}} \subseteq \overline{X - U} = X - U$

(since $X - U$ closed and $\overline{X - U} = X - U$)

$\Rightarrow \overline{\{y\}} \subseteq X - U \wedge x \in U$

$\Rightarrow \{x\} \not\subseteq X - U$

$\Rightarrow \overline{\{x\}} \not\subseteq X - U$

$\therefore \overline{\{x\}} \neq \overline{\{y\}}$

By similar way if we take $(x \notin U \wedge y \in U)$

$(\Leftarrow)$ suppose that $\overline{\{x\}} \neq \overline{\{y\}}$ $\forall$ $x \neq y \in X$, to prove X is T_0 – space

Suppose that X is not T_0 – space $\Rightarrow (\exists\, x, y \in X ; \forall\, U \in \tau ; x \in U \Rightarrow y \in U)$ (def of T_0 – space) (i.e., every open set containing x its containing y)

Let $z \in X$; $z \in \overline{\{x\}}$ ---------------(*)

$\Rightarrow \forall\, U \in \tau$; $z \in U \wedge U \cap \{x\} \neq \phi$

(since by true : $z \in \overline{A} \Leftrightarrow \forall\, U \in \tau$; $z \in U \wedge U \cap A \neq \phi$)

But, $U \cap \{x\} \neq \phi \Rightarrow x \in U$ (since the only element in $\{x\}$ is x)

$\therefore$ every set contains z must contains x. So, we have the following two statements : every open set contains z must contains x and every open set contains x must contains y.

$\therefore$ every open set contains z must contains y.

$\Rightarrow \forall\, U \in \tau$; $z \in U \wedge U \cap \{y\} \neq \phi$

$\Rightarrow z \in \overline{\{y\}}$ ---------------(*)

$\Rightarrow \forall\, z \in \overline{\{x\}} \Rightarrow z \in \overline{\{y\}} \Rightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$

By similar way we prove $\overline{\{y\}} \subseteq \overline{\{x\}}$

$\therefore$ $\overline{\{x\}} = \overline{\{y\}}$ C!! contribution (since $\overline{\{x\}} \neq \overline{\{y\}}$)

$\therefore$ X is T_0 – space.

Theorem : The property of being a T_0 – space is a hereditary property.

Proof :

Let (X, τ) T_0 – space and (W, τ_W) subspace of X, to prove (W, τ_W) is T_0 – space

Let $x, y \in W$; $x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_0 – space $\Rightarrow \exists\, U \in \tau$; $(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$

$\Rightarrow U \cap W \in \tau_W$ (by def. of τ_W)

; $(x \in U \cap W \wedge y \notin U \cap W) \vee (x \notin U \cap W \wedge y \in U \cap W)$

$\therefore$ (W, τ_W) is T_0 – space.

<u>Theorem :</u> The property of being a T_0 – space is a topological property.

<u>Proof :</u>

Let $(X, \tau) \cong (Y, \tau')$ and suppose that X is T_0 – space, to prove Y is T_0 – space

$\because$ $(X, \tau) \cong (Y, \tau') \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow f^{-1}(y_1), f^{-1}(y_2) \in X$

$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi,\ f^{-1}(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\, x_1 \in X ; f^{-1}(y_1) = x_1$ and $\exists!\, x_2 \in X ; f^{-1}(y_2) = x_2$

and $x_1 \neq x_2$ and $x_1, x_2 \in X$

$\because$ X is T_0 – space $\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ f^{-1} is cont. or f open $\Rightarrow f(U) \in \tau'$; $(\, f(x_1) \in f(U) \wedge f(x_2) \notin f(U)\,)$

$\vee\ (\, f(x_1) \notin f(U) \wedge f(x_2) \in f(U)\,)$

$\therefore$ Y is T_0 – space

By similar we prove, if Y is T_0 – space, then X is T_0 – space.

<u>Theorem :</u> Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_0 – space iff each X and Y are T_0 – space.

<u>Proof :</u>

$(\Rightarrow)$ Suppose that $X \times Y$ is a T_0 – space, to prove that X and Y are T_0 – space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow (x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because$ $X \times Y$ is a T_0 – space $\Rightarrow \exists$ a basic open set $U \times V \in \tau_{X \times Y}$;

$(\, (x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V\,) \vee (\, (x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V\,)$

$\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U) \Rightarrow$ X is a T_0 – space

and $\exists\, V \in \tau'$; $(y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V) \Rightarrow$ Y is a T_0 – space.

$(\Leftarrow)$ Suppose that X and Y are T_0 – space, to prove $X \times Y$ is a T_0 – space

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow (x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_0 – space $\Rightarrow \exists\, U \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \vee (x_1 \notin U \wedge x_2 \in U)$

$\because$ Y is a T_0 – space $\Rightarrow \exists\, V \in \tau'$; $(y_1 \in V \wedge y_2 \notin V) \vee (y_1 \notin V \wedge y_2 \in V)$

$\Rightarrow \exists\, U \times V$ is a basic open set ; $(\, (x_1, y_1) \in U \times V \wedge (x_2, y_2) \notin U \times V\,)$

$\vee\ (\, (x_1, y_1) \notin U \times V \wedge (x_2, y_2) \in U \times V\,)$

$\therefore$ $X \times Y$ is a T_0 – space.

Definition : **T_1 – Space**

Let (X, τ) be a topological space. Then the space (X, τ)is called **T_1 – space** iff for each pair of distinct points x, y ∈ X, there exists an open set in X containing x but not y, and an open set in X containing y but not x. i.e.,

X is T_1 – space $\Leftrightarrow \forall\, x, y \in X ; x \neq y\ \exists\, U, V \in \tau ; (x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

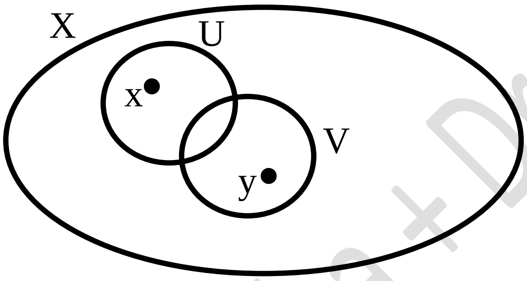
If (X, τ) is not T_1 – space, we define,

X is not T_1 – space $\Leftrightarrow \exists\, x, y \in X ; x \neq y\ \forall\, U, V \in \tau$

$$; (x \in U \wedge y \in U) \vee (x \notin U \wedge y \notin U)$$

$$(x \in V \wedge y \in V) \vee (x \notin V \wedge y \in V).$$

The following figure show the definition of T_1 – space :



Remark : Every T_1 – space is T_0 – space (i.e., $T_1 \Rightarrow T_0$). But the reverse implications does not hold (i.e., $T_0 \not\Rightarrow T_1$) and the following example show that :

Example : Let (X, τ) be a topological space such that X = {1, 2, 3} and τ = {X, ϕ, {1}, {1, 2}}.

Solution : Clear (X, τ) is T_0 – space (see page 76).

But, (X, τ) is not T_1 – space, since 2 ≠ 3 and ∃ open set {1, 2} in X containing 2 but not 3, but ∄ open set in X containing 3 but not 2, since the only open set containing 3 is X and X containing 2 too.

Remark : If (X, τ) is T_1 – space, then its not necessary to test that the space is T_0 – space, since every T_1 – space is a T_0 – space.

Example : The space (X, D) is T_1 – space.

Solution :

Let $x, y \in X ; x \neq y \Rightarrow \exists\, \{x\}, \{y\} \in D ; (x \in \{x\} \wedge y \notin \{x\}) \wedge (x \notin \{y\} \wedge y \in \{y\})$

$\Rightarrow$ (X, D) is T_1 – space.

Example : Is $(\mathbb{N}, \tau_{cof})$ T_1 – space ??

Solution : Yes,

Let $n, m \in \mathbb{N}$; $n \neq m$, take $U = \mathbb{N} \setminus \{m\}$, $V = \mathbb{N} \setminus \{n\}$

$\Rightarrow U, V \in \tau_{cof}$ (since $U^c = (\mathbb{N} \setminus \{m\})^c = \{m\}$ finite set by def. of τ_{cof})

(since $V^c = (\mathbb{N} \setminus \{n\})^c = \{n\}$ finite set by def. of τ_{cof})

$\Rightarrow (n \in U = \mathbb{N} \setminus \{m\} \wedge m \notin U) \wedge (n \notin V = \mathbb{N} \setminus \{n\} \wedge m \in V)$

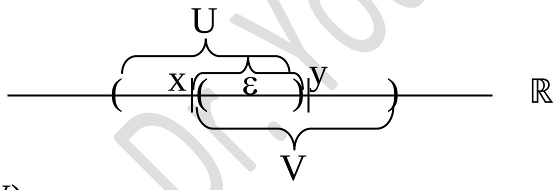
$\Rightarrow (\mathbb{N}, \tau_{cof})$ is T_1 – space.

Example : The usual topological space $(\mathbb{R}, \tau_u)$ is a T_1 – space.

Solution :

Let $x, y \in \mathbb{R}$; $x \neq y$, $\varepsilon = | x - y |$

Take $U = (x - \varepsilon , x + \varepsilon)$, $V = (y - \varepsilon, y + \varepsilon)$



$\therefore$ $U, V \in \tau_u$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.

$\therefore$ $(\mathbb{R}, \tau_u)$ is T_1 – space.

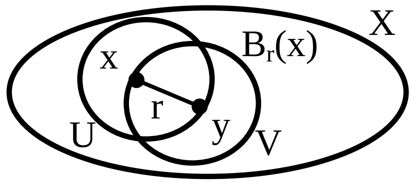
Theorem : Every metric space is a T_1 – space.

Proof : Let (X, d) be a metric space and $x, y \in X$; $x \neq y$

Take $U = B_r(x)$, $V = B_r(y)$; $r = d(x, y)$

$\therefore$ $U, V \in \tau_d$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$

$\therefore$ (X, d) is T_1 – space.



Theorem : (X, τ) is T_1 – space iff $\{x\}$ is closed $\forall\, x \in X$.

i.e., (X, τ) is T_1 – space iff every singleton set in X is closed.

Proof : ($\Rightarrow$) Suppose that X is T_1 – space, to prove $\{x\}$ closed $\forall\, x \in X$

i.e., $X - \{x\}$ open set, we must prove $X - \{x\}$ contains a nbhd $\forall\, y \in X - \{x\}$

Let $y \in X - \{x\} \Rightarrow x \neq y$

$\because$ X is T_1 – space $\Rightarrow \exists\, U, V_y \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V_y \wedge y \in V_y)$

$\Rightarrow y \in V_y \wedge x \notin V_y$

$\Rightarrow \{x\} \bigcap V_y = \phi$

$\Rightarrow V_y \subseteq X - \{x\} \wedge y \in V_y$

$\Rightarrow V_y \subseteq X - \{x\} \;\; \forall\, y \in X - \{x\}$

$\therefore$ $X - \{x\}$ contains a nbhd $\forall\, y \in X - \{x\}$.

$\therefore$ $X - \{x\}$ open set $\Rightarrow \{x\}$ closed $\forall\, x \in X$.

($\Leftarrow$) suppose that $\{x\}$ closed $\forall\ x \in X$, to prove X is T_1 – space

Let $x, y \in X$; $x \neq y \Rightarrow \{x\}, \{y\}$ are closed sets

$\Rightarrow X - \{x\}, X - \{y\}$ are open sets

Say $U = X - \{y\}, V = X - \{x\} \Rightarrow (x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$

$\therefore$ (X, τ) is T_1 – space.

<u>Corollary :</u> If X is a T_1 – space, then every finite set is closed.

<u>Proof :</u> Let A be a finite set in X

$\Rightarrow A = \{x_1, \dots . x_n\} = \bigcup_{i=1}^{n}\{x_i\}$

$\because$ (X, τ) is T_1 – space $\Rightarrow \{x_i\} \in \mathscr{F}\ \forall\ i$

$\Rightarrow \bigcup_{i=1}^{n}\{x_i\}$ closed

$\Rightarrow$ A is a closed

<u>Corollary :</u> If X is finite set and (X, τ) is a T_1 – space, then $\tau = D$.

<u>Proof :</u> To prove $\tau = D$ must we prove $(\forall\ x \in X \Rightarrow \{x\} \in \tau)$, i.e., every singleton set $\{x\}$ is open.

Let $x \in X$

$\because$ X finite set $\Rightarrow X - \{x\}$ finite set

$\because$ X is T_1 – space $\Rightarrow X - \{x\}$ closed set

(by previous corollary : If X is T_1 – space, then every finite set is closed)

$\Rightarrow \{x\}$ open

$\therefore$ $\tau = D$.

<u>Remark :</u> From the previous corollary the only topology that make the space (X, τ) is T_1 – space when X is finite set is D. For example if $X = \{1, 2, 3\}$ and we know there is 29 deference topology on X (see page 2) so that there is 28 topology on X is not T_1 – space except one topology is D. Therefore, we not try to give an example for space is T_1 – space on finite set and the topology not D.

Now, we introduce some corollaries on the theorem in page 81 and your proves is directed from theorem.

<u>Corollary (1) :</u> (X, τ) is T_1 – space iff $\overline{\{x\}} = \{x\}\ \forall\ x \in X$.

<u>Corollary (2) :</u> (X, τ) is T_1 – space iff $\{x\} = \bigcap\{F\ ; F \in \mathscr{F} \wedge x \in F\}\ \forall\ x \in X$.

Corollary (3) : (X, τ) is T_1 – space iff $\{x\}^b = \phi \ \ \forall x \in X$.

Corollary (4) : (X, τ) is T_1 – space iff $\{x\}^b \subseteq \{x\} \ \ \forall x \in X$.

Corollary (5) : (X, τ) is T_1 – space iff $\{x\}^/ \subseteq \{x\} \ \ \forall x \in X$.

Corollary (6) : (X, τ) is T_1 – space iff $\{x\}^/ = \phi \ \ \forall x \in X$.

Theorem : The property of being a T_1 – space is a hereditary property.
Proof :
Let (X, τ) T_1 – space and (W, τ_W) subspace of X, to prove (W, τ_W) T_1 – space
Let $x, y \in W$; $x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)
$\because$ X is T_1 – space $\Rightarrow \exists\, U, V \in \tau$; $(x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V)$.
$\Rightarrow U \bigcap W \wedge V \bigcap W \in \tau_W$ (by def. τ_W)
$\Rightarrow (x \in U \bigcap W \wedge y \notin U \bigcap W) \wedge (x \notin V \bigcap W \wedge y \in V \bigcap W)$.
$\therefore$ (W, τ_W) is a T_1 – space.

Theorem : The property of being a T_1 – space is a topological property.
Proof :
Let $(X, \tau) \cong (Y, \tau')$ and X is a T_1 – space, to prove Y is a T_1 – space
$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous
Let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow f^{-1}(y_1), f^{-1}(y_2) \in X$
$\because$ f onto function $\Rightarrow f^{-1}(y_1) \neq \phi,\ f^{-1}(y_2) \neq \phi$
$\because$ f 1-1 function $\Rightarrow \exists!\, x_1 \in X$; $f^{-1}(y_1) = x_1$ and $\exists!\, x_2 \in X$; $f^{-1}(y_2) = x_2$
and $x_1 \neq x_2$ and $x_1, x_2 \in X$
$\because$ X is T_1 – space $\Rightarrow \exists\, U, V \in \tau$; $(x_1 \in U \wedge x_2 \notin U) \wedge (x_1 \notin V \wedge x_2 \in V)$
$\because f^{-1}$ is cont. or f open $\Rightarrow f(U), f(V) \in \tau'$; $(f(x_1) \in f(U) \wedge f(x_2) \notin f(U))$
$\wedge\ (f(x_1) \notin f(V) \wedge f(x_2) \in f(V))$
$\therefore$ Y is T_1 – space
By similar way we prove, if Y is T_1 – space, then X is T_1 – space.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_1 – space iff each X and Y are T_1 – space.

Proof :

$(\Leftarrow)$ Suppose that X and Y are T_1 – space, to prove $X \times Y$ is a T_1 – space

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow (x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_1 – space $\Rightarrow \exists\, U_1, U_2 \in \tau$; $(x_1 \in U_1 \wedge x_2 \notin U_1) \wedge (x_1 \notin U_2 \wedge x_2 \in U_2)$

$\because$ Y is a T_1 – space $\Rightarrow \exists\, V_1, V_2 \in \tau'$; $(y_1 \in V_1 \wedge y_2 \notin V_1) \wedge (y_1 \notin V_2 \wedge y_2 \in V_2)$

$\Rightarrow \exists$ basic open sets $U_1 \times V_1, U_2 \times V_2$;

$((x_1, y_1) \in U_1 \times V_1 \wedge (x_2, y_2) \notin U_1 \times V_1) \wedge ((x_1, y_1) \notin U_2 \times V_2 \wedge (x_2, y_2) \in U_2 \times V_2)$

$\therefore$ $X \times Y$ is a T_1 – space.

$(\Rightarrow)$ Suppose that $X \times Y$ is a T_1 – space, to prove X and Y are T_1 – space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow (x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because X \times Y$ is a T_1–space $\Rightarrow \exists\, U, V \in \tau_{X \times Y}$; $(x_1, y_1) \in U \wedge (x_2, y_2) \notin U \wedge (x_2, y_2) \in V \wedge (x_1, y_1) \notin V$ that is mean $\exists$ basic open sets $U_1 \times V_1, U_2 \times V_2 \in \tau_{X \times Y}$;

$((x_1, y_1) \in U_1 \times V_1 \wedge (x_2, y_2) \notin U_1 \times V_1) \wedge ((x_1, y_1) \notin U_2 \times V_2 \wedge (x_2, y_2) \in U_2 \times V_2)$

$\Rightarrow \exists\, U_1, U_2 \in \tau$; $(x_1 \in U_1 \wedge x_2 \notin U_1) \wedge (x_1 \notin U_2 \wedge x_2 \in U_2) \Rightarrow$ X is a T_1 – space

and $\exists\, V_1, V_2 \in \tau'$; $(y_1 \in V_1 \wedge y_2 \notin V_1) \wedge (y_1 \notin V_2 \wedge y_2 \in V_2) \Rightarrow$ Y is a T_1 – space.

Definition : T_2 – Space or Hausdorff Space

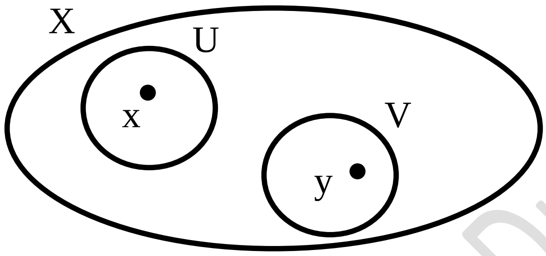
Let (X, τ) be a topological space. Then the space (X, τ) is called a **T_2 – space or Hausdorff space** iff for each pair of distinct points $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \phi$. i.e.,

$$X \text{ is } T_2 - \text{space} \Leftrightarrow \forall\, x, y \in X \,;\, x \neq y\ \exists\, U, V \in \tau \,;\, (x \in U \wedge y \in V),\ U \cap V = \phi$$

If (X, τ) is not T_2 – space, we define,

$$X \text{ is not } T_2\text{–space} \Leftrightarrow \exists\, x, y \in X \,;\, x \neq y\ \forall\, U, V \in \tau;\ U \cap V = \phi,\ (x, y \in U \vee x, y \in V)$$

The following figure show the definition of T_2 – space :



Remark : Every T_2 – space is T_1 – space (i.e., $T_2 \Rightarrow T_1$). But the reverse implications do not hold (i.e., $T_1 \not\Rightarrow T_2$) and the following example show that :

Example : Take cofinite topology $(\mathbb{N}, \tau_{cof})$.

Solution : Clear $(\mathbb{N}, \tau_{cof})$ is T_1 – space (see page 81).

But, $(\mathbb{N}, \tau_{cof})$ is not T_2 – space, since if $n \neq m$, take $U = \mathbb{N} \setminus \{m\}$, $V = \mathbb{N} \setminus \{n\}$, but $U \cap V \neq \phi$. Therefore, $T_1 \not\Rightarrow T_2$.

Remark : If (X, τ) is T_2 – space, then not necessary test that the space is T_1 – space and T_0 – space, since every T_2 – space is T_1 – space and every T_1 – space is T_0 – space i.e., $(T_2 \Rightarrow T_1 \Rightarrow T_0)$.

Example : In the space (X, I) if X is any set containing more than one element, then (X, I) is not T_0 – space (see page 77), so that it's not T_1 – space and not T_2 – space.

Example : The space (X, D) is T_2 – space.

Solution :

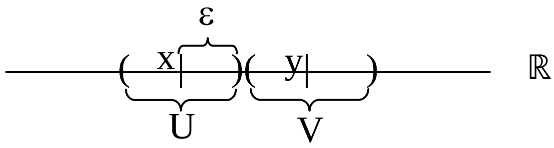
Let $x, y \in X \,;\, x \neq y \Rightarrow \{x\}, \{y\} \in D \,;\, \{x\} \cap \{y\} = \phi\ , (x \in \{x\} \wedge y \in \{y\})$

$\Rightarrow$ (X, D) is T_2 – space.

Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_2 – space.

Solution : Let $x, y \in \mathbb{R}$; $x \neq y$, $\varepsilon = \frac{1}{2} | x - y |$

Take $U = (x - \varepsilon, x + \varepsilon)$, $V = (y - \varepsilon, y + \varepsilon)$



$\therefore$ $U, V \in \tau_u$; $U \cap V = \phi$, $(x \in U \wedge y \in V)$

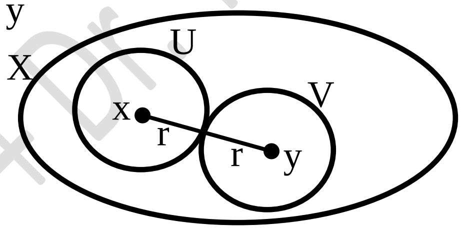
$\Rightarrow$ $(\mathbb{R}, \tau_u)$ is T_2 – space.

Remark : In the previous remark (p. 82) we show that if X is finite set and $\tau \neq D$, then (X, τ) is not T_1 – space and we say her is not T_2 – Space. i.e., the only topology make (X, τ) is T_2 – space if X is finite set is D.

Theorem : Every metric space is T_2 – space.

Proof : Let (X, d) be a metric space and $x, y \in X$; $x \neq y$

Take $U = B_r(x)$, $V = B_r(y)$; $r = \frac{1}{2}$ $d(x, y)$



$\therefore$ $U, V \in \tau_d$; $U \cap V = \phi$, $(x \in U \wedge y \in V)$

$\therefore$ (X, d) is T_2 – space.

Theorem : (X, τ) is a T_2 – space iff the diagonal $\Delta = \{(x, x) \in X \times X \ ; \ x \in X\}$ is a closed subset of the product $X \times X$.

Proof : $(\Rightarrow)$ Suppose that X is T_2 – space, to prove Δ closed in $X \times X$

i.e., $X \times X - \Delta$ open set, we must prove $X \times X - \Delta$ contains a nbhd $\forall$ $(x, y) \in X \times X - \Delta$

Let $(x, y) \in X \times X - \Delta \Rightarrow (x, y) \notin \Delta$ (def. of deference)

$\Rightarrow x \neq y$ (since Δ has equal coordinate)

$\because$ X is a T_2 – space $\Rightarrow \exists\, U, V \in \tau$; $U \cap V = \phi$, $(x \in U \wedge y \in V)$

$\Rightarrow U \times V \in \beta_{X \times X} \subseteq \tau_{X \times X}$ (by def. product space)

$\Rightarrow U \times V$ open set in $X \times X$ and

$U \times V \subseteq X \times X - \Delta \ \wedge \ (x, y) \in U \times V$ (since $U \cap V = \phi$)

Since, if $U \times V \not\subseteq X \times X - \Delta \Rightarrow \exists\, (x, x) \in \Delta \Rightarrow x \in U \wedge x \in V$ C!! (contridition)

$\therefore$ $X \times X - \Delta$ contains a nbhd $\forall$ $x \in X \times X - \Delta$

$\Rightarrow X \times X - \Delta \in \tau_{X \times X}$

$\Rightarrow \Delta$ closed in $X \times X$

$(\Leftarrow)$ Suppose that Δ closed in $X \times X$, to prove X is T_2 – space

Let $x, y \in X$; $x \neq y \Rightarrow (x, y) \notin \Delta$ (by def. of Δ)

$\Rightarrow (x, y) \in \Delta^c = X \times X - \Delta$

$\because$ Δ closed set $\Rightarrow X \times X - \Delta$ open set

$\Rightarrow \exists\, U\times V\ ;\ U, V \in \tau \wedge (x, y) \in U\times V\ ,\ U\times V \subseteq X\times X - \Delta,\ x \in U,\ y \in V$

$\Rightarrow U\times V \bigcap \Delta = \phi$ (i.e., $\nexists$ element in $U\times V$ has equal coordinate)

$\Rightarrow U \bigcap V = \phi$

$\therefore$ (X, τ) is T_2 – space.

Theorem : The property of being a T_2 – space is a hereditary property.

Proof :

Let (X, τ) T_2 – space and (W, τ_W) subspace of X, to prove (W, τ_W) T_2 – space

Let $x, y \in W\ ;\ x \neq y \Rightarrow x, y \in X$ (since $W \subseteq X$)

$\because$ X is T_2 – space $\Rightarrow \exists\, U, V \in \tau\ ;\ U \bigcap V = \phi\ ,\ (x \in U \wedge y \in V)$.

$\Rightarrow U \bigcap W \wedge V \bigcap W \in \tau_W$, (by def. of τ_W)

$(U \bigcap W) \bigcap (U \bigcap W) = (U \bigcap V) \bigcap W = \phi \bigcap W = \phi$

and $(x \in U \bigcap W \wedge y \in V \bigcap W)$.

$\therefore$ (W, τ_W) is T_2 – space.

Theorem : The property of being a T_2 – space is a topological property.

Proof :

Let $(X, \tau) \cong (Y, \tau')$ and suppose that Y is T_2 – space, to prove X is T_2 – space

$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')\ ;$ f 1-1, f onto, f continuous, f^{-1} continuous

Let $x_1, x_2 \in X\ ;\ x_1 \neq x_2 \Rightarrow f(x_1), f(x_2) \in Y$

$\because$ f onto function $\Rightarrow f(x_1) \neq \phi,\ f(y_2) \neq \phi$

$\because$ f 1-1 function $\Rightarrow \exists!\, y_1 \in Y\ ;\ f(x_1) = y_1$ and $\exists!\, y_2 \in Y\ ;\ f(x_2) = y_2$

and $y_1 \neq y_2$, $y_1, y_2 \in Y$

$\because$ Y is T_2 – space $\Rightarrow \exists\, V_1, V_2 \in \tau'\ ;\ V_1 \bigcap V_2 = \phi\ ,\ (y_1 \in V_1 \wedge y_2 \in V_2)$

$\because$ f is continuous $\Rightarrow f^{-1}(V_1) = U_1,\ f^{-1}(V_2) = U_2 \in \tau\ ;$

$U_1 \bigcap U_2 = f^{-1}(V_1) \bigcap f^{-1}(V_2) = f^{-1}(V_1 \bigcap V_2) = f^{-1}(\phi) = \phi$,

$(x_1 \in U_1 \wedge x_2 \in U_2)$

$\therefore$ X is T_2 – space

By similar way we prove, if X is T_2 – space, then Y is T_2 – space.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a T_2 – space iff each X and Y are T_2 – space.

Proof :

($\Leftarrow$) Suppose that X and Y are T_2 – space, to prove $X \times Y$ is T_2 – space

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

By def. product space $\Rightarrow$ $(x_1, x_2 \in X \wedge x_1 \neq x_2) \wedge (y_1, y_2 \in Y \wedge y_1 \neq y_2)$

$\because$ X is a T_2 – space $\Rightarrow \exists\, U_1, U_2 \in \tau$; $U_1 \cap U_2 = \phi$, $(x_1 \in U_1 \wedge x_2 \in U_2)$

$\because$ Y is a T_2 – space $\Rightarrow \exists\, V_1, V_2 \in \tau'$; $V_1 \cap V_2 = \phi$, $(y_1 \in V_1 \wedge y_2 \in V_2)$

$\Rightarrow \exists$ basic open sets $U_1 \times V_1, U_2 \times V_2$;

$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \phi \times \phi = \phi$,

$((x_1, y_1) \in U_1 \times V_1 \wedge (x_2, y_2) \in U_2 \times V_2)$

$\therefore$ $X \times Y$ is a T_2 – space.

($\Rightarrow$) Suppose that $X \times Y$ is a T_2 – space, to prove X and Y are T_2 – space

Let $x_1, x_2 \in X$; $x_1 \neq x_2$ and $y_1, y_2 \in Y$; $y_1 \neq y_2$

$\Rightarrow$ $(x_1, y_1), (x_2, y_2) \in X \times Y$; $(x_1, y_1) \neq (x_2, y_2)$

$\because X \times Y$ is T_2 – space $\Rightarrow \exists\, U, V \in \tau_{X \times Y}$; $(x_1, y_1) \in U \wedge (x_2, y_2) \in V \wedge U \cap V = \phi$,

that is mean $\exists$ basic open sets $U_1 \times V_1, U_2 \times V_2 \in \tau_{X \times Y}$;$U_1 \times V_1, U_2 \times V_2 \in \tau_{X \times Y}$;

$(U_1 \times V_1) \cap (U_2 \times V_2) = \phi$, $((x_1, y_1) \in U_1 \times V_1 \wedge (x_2, y_2) \in U_2 \times V_2)$

$\Rightarrow \exists\, U_1, U_2 \in \tau$; $U_1 \cap U_2 = \phi$, $(x_1 \in U_1 \wedge x_2 \in U_2)$ $\Rightarrow$ X is a T_2 – space

and $\exists\, V_1, V_2 \in \tau'$; $V_1 \cap V_2 = \phi$, $(y_1 \in V_1 \wedge y_2 \in V_2)$ $\Rightarrow$ Y is a T_2 – space.

Remark : We study in mathematical analysis real sequences and their convergence and we know if the real sequences is convergence, then has a **unique limit** and this is especial case since their domain is real numbers with usual topology ($\mathbb{R}$, τ_u). But, if we go to the topological spaces in general and define the convergence sequences in topological space we found the sequences is convergence but their **limit not unique**. In mathematical analysis (3^{th} stage) we use the usual topology only and this T_2 – space and we will introduce illustrate to this problem with definitions and theorems.

Definition : Let (X, d) be a metric space. We called the function $S : \mathbb{N} \to X$ is a sequences in X and denoted to image of element n in $\mathbb{N}$ which is S(n) by S_n , so that the sequence is $S_1, S_2, \ldots, S_n, \ldots$; $n \in \mathbb{N}$ or $(S_n)_{n\in \mathbb{N}}$.
We called the sequence is convergence to element x_0 in X (denoted by $S_n \longrightarrow x_0$) if the following condition satisfy :

$$\forall\ \varepsilon > 0 \quad \exists\ k \in \mathbb{N}\ ;\ S_n \in N_\varepsilon(x_0)$$

such that $N_\varepsilon(y)$ is open ball in X with center x_0 and radius ε.
this means that the sequences (S_n) convergence to limit x_0 in X if every open ball with center x_0 contains all elements of sequences except finite numbers of elements.

Notes that the definition especial of metric space, so we now generalization this definition to topological space such that we replaces open ball by open set since the open ball no exist in topological space because there is not distant.

Definition : Let $(S_n)_{n\in\mathbb{N}}$ be a sequences in topological space (X, τ) (i.e., $S : \mathbb{N} \longrightarrow X$). We called the sequence $(S_n)_{n\in\mathbb{N}}$ is convergence to x_0 in X (denoted by $S_n \longrightarrow x_0$) if the following condition satisfy :

$$S_n \longrightarrow x_0 \Leftrightarrow \forall\ U \in \tau\ ;\ x_0 \in U \quad \exists\ k \in \mathbb{N} ; S_n \in U \quad \forall\ n \geq k$$

i.e., every open nbhd of x_0 contains all elements of sequences except finite numbers.

Remark : If (X, τ) is not T_2 – space, then the convergence sequences may be has more than one limit point.

Example : Let X = {1, 2, 3} and τ = I = {X, ϕ} such that (X, τ) is topological space. Let $(S_n)_{n\in\mathbb{N}}$ be a sequences in X such that $S_n = 1$ for all n

Solution : Clean $S_n \longrightarrow 1$, $S_n \longrightarrow 2$, and $S_n \longrightarrow 3$??
To clear that apply the definition ; $S_n \longrightarrow 1$, since the open nbhds of 1 is X only because it's the unique open set contains 1 and $S_n \longrightarrow 1$ since X contains all elements so its contain the sequence. Therefore the definition satisfy.

By similar way $S_n \longrightarrow 2$ and $S_n \longrightarrow 3$ since X the unique open set that contains 2 and also contains 3 and X contains the sequences too.

Question : Give an example to convergence sequence in topological space has five deference limit point.
Answer : Let X = {1, 2, 3, 4, 5, 6} and $\tau = \{X, \phi, \{6\}\}$ such that (X, τ) is topological space.
Define $(S_n)_{n\in\mathbb{N}}$ as follows $S_n = 3 \ \forall\, n \in \mathbb{N}$, so that

$$S_n \longrightarrow 1,\ S_n \longrightarrow 2,\ S_n \longrightarrow 3,\ S_n \longrightarrow 4,\ S_n \longrightarrow 5, \text{ but } S_n \not\longrightarrow 6$$

Since {6} is open nbhd for 6 , but $S_n \notin \{6\}\ \forall\, n \in \mathbb{N}$, because $S_n = 3$ and $3 \notin \{6\}$.
On the other hand, $S_n \longrightarrow 1, 2, 3, 4, 5$ since X the unique open set that contains 1, 2, 3, 4, 5 and X contains 3 i.e., $S_n \in X\ \forall\, n \in \mathbb{N}$.

We can change the previous question to make give an example to convergence sequence has ten or seven or any known number deference limit point. By taken X has 11 element (if require 10 deference limit point) and define topology on X contains X, ϕ, and singleton set contain one of the elements of X and define constant sequence such that the constant number is one number of elements of X not in singleton set. Therefore, we have the require.

Question : Give an example to convergence sequence in topological space has infinite number of deference limit point.
Answer : Let $X = \mathbb{R}$ (or X = any infinite set) and take $\tau = I = \{X, \phi\}$ and take any sequence in $\mathbb{R}$ for example :

$$S_n = \begin{cases} \sqrt{2} & \text{if} \quad n \in E \\ 0 & \text{if} \quad n \in O \end{cases}$$

Notes that the sequence $(S_n)_{n\in\mathbb{N}}$ is not convergence in $(\mathbb{R}, \tau_u)$.
But in $(\mathbb{R}, I)$ is convergence and has infinite numbers of limit points, since every real number is limit point because $\mathbb{R}$ the unique open set and $\mathbb{R}$ contains $\sqrt{2}$ and 0 and $\mathbb{R}$ contains all elements of the sequences (i.e., every open nbhd for all element in $\mathbb{R}$ contains all elements of the sequences) and this means in the definition of convergence in metric space that for all real number is limit point of the sequence $(S_n)_{n\in\mathbb{N}}$.

Remark : If (X, τ) is T_2 – space, then every convergence sequence in X has unique limit point and this illustrate that consider the limit point if exist, then its unique in mathematical analysis (3^{th} stage), because we study the metric space only and special case ($\mathbb{R}$, | |) and we prove that every metric space is T_2 – space (see page 86), so that every convergence sequence in metric space has unique limit point and we will introduce the theorem show this :

Theorem : If (X, τ) is T_2 – space, then every convergence sequence in X has unique limit point.

Proof : Let $(S_n)_{n\in\mathbb{N}}$ be convergence sequences in X

Suppose that $S_n \longrightarrow x$ and $S_n \longrightarrow y$; $x \neq y \in X$

$\because$ X is T_2 – space $\Rightarrow \exists\, U, V \in \tau$; $U \bigcap V = \phi$, $(x \in U \wedge y \in V)$.

$\because$ $S_n \longrightarrow x$ and $x \in U \in \tau \Rightarrow \exists\, k_1 \in \mathbb{N}$; $S_n \in U \;\; \forall\, n \geq k_1$

$\because$ $S_n \longrightarrow y$ and $y \in V \in \tau \Rightarrow \exists\, k_2 \in \mathbb{N}$; $S_n \in V \;\; \forall\, n \geq k_2$

$\because$ elements of sequence is infinite (since domain = $\mathbb{N}$), then there are elements common between U and V (i.e., $U \bigcap V \neq \phi$). C!! contradiction

$\therefore$ every convergence sequence in X has unique limit point.

We will use idea of convergence sequence in topological space and idea axiom of T_2 – space with important continuous concept in topology :

Theorem : If f : (X, τ) $\rightarrow$ (Y, τ') is continuous function and $(S_n)_{n\in\mathbb{N}}$ is convergence sequence in X such that $S_n \longrightarrow x$, then $f(S_n) \longrightarrow f(x)$.

i.e., The continuous function maps convergence sequence in domain to convergence sequence in codomain and their limit point is image of limit point in domain.

Proof : To prove $f(S_n) \longrightarrow f(x)$ in Y

Let V be an open nbhd of f(x) in Y, i.e., $f(x) \in V \in \tau'$

$\because$ f continuous $\Rightarrow f^{-1}(V) \in \tau$, i.e., $f^{-1}(V)$ open set in X

$\because$ $f(x) \in V \quad \Rightarrow x \in f^{-1}(V)$

$\Rightarrow f^{-1}(V)$ be an open nbhd of x in X

$\because$ $S_n \longrightarrow x$ and $x \in f^{-1}(V) \in \tau \Rightarrow \exists\, k \in \mathbb{N}$; $S_n \in f^{-1}(V) \;\; \forall\, n \geq k$

Take image for S_n and for $f^{-1}(V)$, we have

$\Rightarrow \exists\, k \in \mathbb{N}$; $f(S_n) \in V \;\; \forall\, n \geq k$

i.e., V contains all elements of sequence except k of these element.

$\therefore$ $f(S_n) \longrightarrow f(x)$.

Now, we prove one of the important theorem which connected between the concept axiom T_2 – space and concept compactness.

Theorem : Every compact set in T_2 – space is closed.

Proof : Let (X, τ) be T_2 – space and $A \subseteq X$; A compact in X

To prove A is closed in X, i.e., X – A open set in X

we must prove X – A contains an open nbhd $\forall\ x \in X - A$

(i.e., $\forall\ x \in X - A\ \exists\ U \in \tau$; $x \in U \subseteq X - A$)

Let $x \in X - A \Rightarrow x \notin A \Rightarrow x \neq a\ \forall\ a \in A$

$\because$ X is T_2 – space $\Rightarrow \exists\ U_a, V_a \in \tau$; $U_a \bigcap V_a = \phi$, $(x \in U_a \wedge a \in Va)\ \forall\ a \in A$.

We have two family of open sets are $\{U_a\}_{a\in A}$ (every element in this family contains x) and $\{V_a\}_{a\in A}$ (every element in this family contains one of the elements A) and every U_a corresponding V_a such that $U_a \bigcap V_a = \phi$.

$$\Rightarrow \{V_a\}_{a\in A} \text{ open cover of A, i.e., } A \subseteq \bigcup_{a\in A} V_a$$

$\because$ A is compact set $\Rightarrow \exists\ a_1, \dots , a_n$; $A \subseteq \bigcup_{i=1}^{n} V_{a_i}$

Therefore, there is a finite family $\{U_a\}_{a\in A}$ corresponding the finite family $\{V_{a_i}\}_{i=1}^{n}$ which is $\{U_{a_i}\}_{i=1}^{n}$.

$\because$ every U_a contain x, then $x \in \bigcap_{i=1}^{n} U_{a_i}$

$\because$ every U_{a_i} is open set, then $\bigcap_{i=1}^{n} U_{a_i}$ is open set contain x

(second condition of def of top.)

Say, $U = \bigcap_{i=1}^{n} U_{a_i} \Rightarrow x \in U \in \tau$

On the other hand $\bigcup_{i=1}^{n} V_{a_i}$ is open set (third condition of def. of top.)

Say, $V = \bigcup_{i=1}^{n} V_{a_i} \Rightarrow A \subseteq V \in \tau$

Notes that,

$U \bigcap V = \phi$ (since $U_a \bigcap V_a = \phi$)

$\Rightarrow A \bigcap U = \phi$ (since $A \subseteq V$ and $U \bigcap V = \phi$)

$\Rightarrow U \subseteq X - A$

$\Rightarrow x \in U \subseteq X - A \wedge U \in \tau$

$\Rightarrow$ X – A open set in X $\forall\ x \in X - A$

$\Rightarrow$ A closed set in X.

Finally, we introduce one of important theorem which connected the concepts continuous, T_2 – space, and compactness with homomorphism concept.

Theorem : If $f : (X, \tau) \rightarrow (Y, \tau')$ is continuous bijective function and X is compact space and Y is T_2 – space, the f is homomorphism.

Proof : It is enough to prove f is closed function

Let F be a closed set in X

$\because$ X is compact and F closed in X $\Rightarrow$ F compact in X

(by theorem : A closed subset of compact space is compact)

$\because$ f is continuous and F is compact in X $\Rightarrow$ f(F) is compact in Y

(by theorem : A continuous image of compact set is compact set)

$\because$ Y is T_2 – space and f(F) compact in Y $\Rightarrow$ f(F) is closed in Y

(by previous theorem : Every compact set in T_2 – space is closed)

$\therefore$ f is closed function $\Rightarrow$ f is Homeomorphism.

In chapter three (compact space) we say the intersection of two compact sets not necessary compact set (see page 68) and intersection closed set and compact sets not necessary compact set, but this statements are satisfy if we add the condition that T_2 – space and the following theorems show this.

Theorem : If A is closed set and B is compact set in T_2 – space (X, τ), then $A \bigcap B$ is compact.

Proof :

$\because$ X is T_2 – space and B compact in X $\Rightarrow$ B is closed in X

(by theorem : Every compact set in T_2 – space is closed)

$\because$ A and B closed sets $\Rightarrow A \bigcap B$ closed set i.e., $A \bigcap B \in \mathscr{F}$

(second condition of def of top.)

$\because A \bigcap B \subseteq B$ i.e., $A \bigcap B$ subspace of B and

$A \bigcap B$ closed in B and B compact $\Rightarrow$ $A \bigcap B$ compact

(by theorem : A closed subset of compact space is compact)

Corollary : If A and B are compact sets in T_2 – space (X, τ), then $A \bigcap B$ is compact.

Proof :

$\because$ X is T_2 – space and A compact in X $\Rightarrow$ A is closed in X

(by theorem : Every compact set in T_2 – space is closed)

$\because$ X is T_2 – space and A closed and B compact in X $\Rightarrow A \bigcap B$ compact

(by previous theorem)

Remark : If X is T_2 – space and compact, then $X \supseteq A$ closed $\Leftrightarrow$ A compact.

<u>Definition :</u> Regular Space

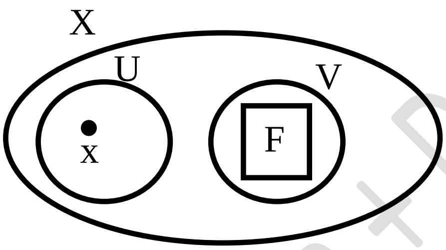
Let (X, τ) be a topological space. Then the space (X, τ) is called a **<u>Regular Space</u>** iff for each closed set $F \subset X$ and each point $x \notin F$, there exist open sets U and V such that $x \in U$, $F \subset V$, and $U \cap V = \phi$ (denoted by R – space). i.e.,

X is R – space $\Leftrightarrow \forall\, x \in X\ \forall\, F \in \mathscr{F};\ x \notin F\ \exists\, U, V \in \tau\ ;\ U \cap V = \phi\ ,\ (x \in U \wedge F \subset V)$.

If (X, τ) is not R – space, we define,

$$X \text{ is not } R\text{ – space} \Leftrightarrow \exists\, x \in X\ \exists\, F \in \mathscr{F}\ ;\ x \notin F \wedge \forall\, U, V \in \tau\ ;\ U \cap V = \phi\ ,$$
$$(x \in U \wedge F \subseteq U) \vee (x \in V \wedge F \subseteq V) \vee$$
$$(x \notin U \wedge F \not\subseteq V) \vee (x \notin V \wedge F \not\subseteq V)\ .$$

The following figure show the definition of R – space :



<u>Example :</u> Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$. Is (X, τ) R – space.

<u>Solution :</u> First, find the family of closed sets :

$$\mathscr{F} = \{X, \phi, \{2, 3\}, \{1, 3\}, \{3\}\}$$

Second, take every closed set and every element not belong to it as follow :

Take X , but every element belong to X

Take ϕ and $1, 2, 3 \notin \phi \Rightarrow \exists\, U = X$ and $V = \phi$ such that $1, 2, 3 \in X = U$ and $\phi \subseteq \phi = V$ and $X \cap \phi = \phi$, so the definition satisfy.

Take $F = \{2, 3\}$ and $1 \notin F \Rightarrow$ the only open set that contains F is X, but $X \cap U \neq \phi$, so the definition not satisfy. $\therefore$ (X, τ) is not R – space.

<u>Example :</u> Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$. Is (X, τ) R – space.

<u>Solution :</u> First, find the family of closed sets :

$$\mathscr{F} = \{X, \phi, \{2, 3\}, \{1\}\} = \tau$$

and X, ϕ in previous example satisfy the definition (in general X, ϕ satisfy the definition in every example).

Take $F = \{1\}$ closed set ; $2, 3 \notin F \Rightarrow \exists\, U = \{2, 3\}$ and $V = \{1\} = F$; $U, V \in \tau$

$\Rightarrow\ U \cap V = \phi\ ;\ 2, 3 \in U\ \wedge\ F \subseteq V$ (the definition satisfy)

Take $F = \{2, 3\}$ closed set ; $1 \notin F \Rightarrow \exists\, U = \{1\}$ and $V = \{2, 3\} = F$; $U, V \in \tau$

$\Rightarrow\ U \cap V = \phi\ ;\ 1 \in U\ \wedge\ F \subseteq V$ (the definition satisfy)

$\therefore$ (X, τ) is R – space.

Remark : In previous example notes that X is not T_0 – space, not T_1 – space, and not T_2 –space, so the R –space is not necessarily T_0 –space or T_1 –space or T_2 – space. i.e.,

$$(R \not\Rightarrow T_0 \ \wedge \ R \not\Rightarrow T_1 \ \wedge \ R \not\Rightarrow T_2)$$

Remark : If $\tau = \mathcal{F}$ in any topological space (X, τ), then its R – space, since :

If F closed set in X and $x \notin F \Rightarrow \exists\ U = F$ and $V = F^c$; $F \subseteq F = U$, $x \in V = F^c$, $F \cap F^c = \phi$. So the definition of R – space satisfy.

From this remark we have (X, I) and (X, D) are R – space.

Example : Is cofinite topology $(\mathbb{N}, \tau_{cof})$ R – space.

Solution : No. Since $\nexists$ two nonempty disjoint open sets satisfy the definition satisfy of R – space, for example :

If $X = \mathbb{N}$ and $F = \{1, 2, 3\}$ and $x = 4$, then $x \notin F$ and if we assume there exists U, V $\in \tau_{cof}$ and $U \cap V = \phi$, then

$$(U \cap V)^c = \phi^c \ \Rightarrow \ \underbrace{U^c}_{\text{finite}} \cup \underbrace{V^c}_{\text{finite}} = \underbrace{X}_{\text{finite}} \quad \text{C!! contradiction}$$

Theorem : The space (X, τ) is regular (R – space) iff for each $x \in X$ and each open set W containing x, there exists an open set U such that $x \in U \subseteq \overline{U} \subseteq W$.

Proof : ($\Rightarrow$) Suppose that X is R – space.

Let $x \in X$, $W \in \tau$; $x \in W \Rightarrow x \notin X - W$ and $X - W \in \mathcal{F}$

$\because$ X is R – space $\Rightarrow \exists\ U, V \in \tau$; $U \cap V = \phi$, $(x \in U \wedge X - W \subseteq V)$

$\because\ U \cap V = \phi \quad \Rightarrow \ U \subseteq X - V$

We have, $U \subseteq X - V$ and $X - V \subseteq W$

$\Rightarrow \ \overline{U} \subseteq \overline{X - V}$ (since $A \subseteq B \ \Rightarrow \ \overline{A} \subseteq \overline{B}$)

$\Rightarrow \ \overline{U} \subseteq X - V$ (since $X - V$ closed $\Rightarrow X - V = \overline{X - V}$)

$\Rightarrow \ \overline{U} \subseteq X - V \ \wedge \ X - V \subseteq W$

$\Rightarrow \ \overline{U} \subseteq W$

$\Rightarrow x \in U \subseteq \overline{U} \subseteq W$ (since $A \subseteq \overline{A}$)

($\Leftarrow$) Suppose the condition of theorem satisfy, to prove X is R – space

Let $x \in X$ and F closed set in X ; $x \notin F$

$\Rightarrow x \in X - F \in \tau$ (since F closed)

$\Rightarrow \exists\ U \in \tau$; $x \in U \subseteq \overline{U} \subseteq X - F$ (by hypothesis)

$\Rightarrow \ \overline{U} \subseteq X - F$

$\Rightarrow \ F \subseteq X - \overline{U}$ (since $A \subseteq B \ \Leftrightarrow \ B^c \subseteq A^c$)

But, $X - \overline{U}$ open since $\overline{U}$ closed, say $X - \overline{U} = V$

$\Rightarrow x \in U \wedge F \subseteq V \wedge U \cap V = \phi$, (since $U \subseteq \overline{U}$ and $\overline{U} \cap X - \overline{U} = \phi \Rightarrow U \cap V = \phi$)

$\therefore$ X is R – space.

Theorem : Every metric space is R – space.

Proof : Let (X, d) be a metric space $\Rightarrow$ (X, τ_d) the topology derivative from this metric. We will prove this theorem by using previous theorem :

Let $x \in X$ and W open set ; $x \in W$

To prove $\exists$ U open set ; $x \in U \subseteq \overline{U} \subseteq W$ or $x \in U \subseteq Cl(U) \subseteq W$

$\because$ $x \in W \Rightarrow \exists$ open ball of x contains in W (since $W \in \tau_d$)

$\Rightarrow \exists\, p \in \mathbb{R}^+$; $N(x, p) \subseteq W$

(since the set is open $\Leftrightarrow$ contains open nbhd for every element)

Take $q \in \mathbb{R}^+$; $0 < q < p$ $\Rightarrow N(x, q) \subseteq N(x, p)$

(since the first half ball similar than second half ball and the center is unique)

$\Rightarrow N(x, q) \subseteq Cl(N(x, q)) \subseteq N(x, p)$ (since $q < p$)

since $N(x, q) = \{y \in X ; d(x, y) < q\}$ and $Cl(N(x, q)) = \{y \in X ; d(x, y) \leq q\}$

$\Rightarrow N(x, q) \subseteq Cl(N(x, q)) \subseteq W$ (since $N(x, p) \subseteq W$ and $Cl(N(x, q) \subseteq N(x, p))$

$\because$ every open ball in metric space is open set $\Rightarrow N(x, q)$ open set, say $U = N(x, q)$

$\Rightarrow x \in U \subseteq Cl(U) \subseteq W$

$\therefore$ (X, d) is R – space.

Remark : There is a method to prove previous theorem by using definition of R – space, but this prove is shorter.

Remark : Since $(\mathbb{R}, \tau_u)$ is metric space, then $(\mathbb{R}, \tau_u)$ is R – space.

Theorem : The property of being a R – space is a hereditary property.

Proof : Let (X, τ) R – space and (W, τ_W) subspace of X, to prove (W, τ_W) R – space

Let $x \in W$ and E closed set in W ; $x \notin E$

$\Rightarrow x \in X$ (since $W \subseteq X$) $\wedge$ $\exists\, F \in \mathcal{F}$; $E = F \cap W$ (i.e., F closed in X)

$\because$ X is R – space $\Rightarrow \exists\, U, V \in \tau$; $U \cap V = \phi$, $(x \in U \wedge F \subseteq V)$

$\Rightarrow$ $U \cap W$ $\wedge$ $V \cap W \in \tau_W$, (by def of τ_W)

$(U \cap W) \cap (U \cap W) = (U \cap V) \cap W = \phi \cap W = \phi$

and $(x \in U \cap W$ (since $x \in U$ $\wedge$ $x \in W$)

$\wedge$ $E \subseteq V \cap W)$. (since $E = F \cap W \Rightarrow E \subseteq F$ $\wedge$ $E \subseteq W$

$\Rightarrow$ $E \subseteq V \wedge E \subseteq W$

$\Rightarrow$ $E \subseteq V \cap W$

$\therefore$ (W, τ_W) is R – space.

Theorem : The property of being a R – space is a topological property.

Proof : Let $(X, \tau) \cong (Y, \tau')$ and suppose that X is R – space, to prove Y is R – space

$\because$ $(X, \tau) \cong (Y, \tau')$ $\Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f open

Let $y \in Y$ and $F \in \mathscr{F}'$ i.e., F closed in Y ; $y \notin F$

$\because$ f onto function $\Rightarrow \exists\, x \in X$; $f(x) = y$

$\because$ f continuous $\Rightarrow f^{-1}(F) \in \mathscr{F}$ i.e., $f^{-1}(F)$ closed in X ; $x \notin f^{-1}(F)$ (since $f(x) = y \notin F$)

$\because$ X is R – space $\Rightarrow \exists\, U, V \in \tau$; $U \cap V = \phi$, $(x \in U \wedge f^{-1}(F) \subseteq V)$

$\because$ f open $\Rightarrow f(U), f(V) \in \tau'$

$\because$ f is 1-1 $\wedge$ onto $\Rightarrow f(x) \in f(U) \wedge f(f^{-1}(F) \subseteq f(V)$

$\Rightarrow y \in f(U) \wedge F \subseteq f(V)$ (since $y = f(x) \wedge f(f^{-1}(F)) = F$)

$\because$ $U \cap V = \phi$ $\Rightarrow f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$

$\therefore$ X is R – space.

By similar way we prove, if Y is R – space, then X is R – space.

Remark : The continuous image of R – space is not necessarily R – space. i.e., if $f : (X, \tau) \to (Y, \tau')$ is continuous onto function and X is R – space, then Y not necessarily R – space and the following example show this :

Example : Let $f : (\mathbb{R}, D) \to (\mathbb{R}, \tau_{cof})$; $f(x) = x \quad \forall\, x \in \mathbb{R}$.

f is continuous function since the domain $(\mathbb{R}, D)$ is discrete topology (see page 37) and clear f is onto and in general (X, D) is R – space (i.e., $(\mathbb{R}, D)$ is R – space), but in general (X, τ_{cof}) is not R – space (see page 95) (i.e., $(\mathbb{R}, \tau_{cof})$ is not R – space).

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X \times Y$ is a R – space iff each X and Y are R – space.

Proof : $(\Leftarrow)$ Suppose that X and Y are R – space, to prove $X \times Y$ is R – space

Let $(x, y) \in X \times Y$ and A closed set in $X \times Y$; $(x, y) \notin A$

$\Rightarrow \exists$ F closed set in X and F' closed set in Y ; $F \times F' \subseteq A$ and $(x, y) \notin F \times F'$

$\Rightarrow x \notin F \in \mathscr{F} \wedge y \notin F' \in \mathscr{F}'$

$\because$ X is a R – space $\Rightarrow \exists\, U, V \in \tau$; $U \cap V = \phi$, $(x \in U \wedge F \subseteq V)$

$\because$ Y is a R – space $\Rightarrow \exists\, U', V' \in \tau'$; $U' \cap V' = \phi$, $(y \in U' \wedge F' \subseteq V')$

$\Rightarrow$

$U \times Y, V \times Y \in \tau_{X \times Y}$; $(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \phi \times Y = \phi$,

$((x, y) \in U \times Y \wedge F \times F' \subseteq V \times Y)$

Or

$X\times U', X\times V' \in \tau_{X\times Y}$; $(X\times U') \bigcap (X\times V') = X \times (U' \bigcap V') = X \times \phi = \phi$,

$$((x, y) \in X\times U' \wedge F\times F' \subseteq X\times V')$$

In both cases we have $X\times Y$ is R – space.

($\Rightarrow$) Suppose that $X\times Y$ is R – space, to prove X and Y are R – space

Let $x \in X$ and $F \in \mathcal{F}$; $x \notin F$ and $y \in Y$ and $F' \in \mathcal{F}'$; $y \notin F'$

$$\Rightarrow (x, y) \in X\times Y \text{ and } F\times F' \in \mathcal{F}_{X\times Y} ; (x, y) \notin F\times F'$$

$\because$ $X\times Y$ is a R – space $\Rightarrow \exists\, U\times V, U'\times V' \in \tau_{X\times Y}$; $(U\times V) \bigcap (U'\times V') = \phi$,

$$((x, y) \in U\times V \wedge F\times F' \subseteq \mathcal{F}\times\mathcal{F}')$$

$\Rightarrow \exists\, U, U' \in \tau$; $U \bigcap U' = \phi$, $(x \in U \wedge F \subseteq U')$ $\Rightarrow$ X is R – space

and $\exists\, V, V' \in \tau'$; $V \bigcap V' = \phi$, $(y \in V \wedge F' \subseteq V')$ $\Rightarrow$ Y is R – space.

Remark : Notes that : $(T_0 \not\Rightarrow R \;\wedge\; T_1 \not\Rightarrow R \;\wedge\; T_2 \not\Rightarrow R)$

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$; (X, τ) topological space.

Solution : Clear (X, τ) is not R – space (see page 94)

On the other hand, (X, τ) is T_0 – space (Check that !!)

So, we have T_0 – space, but not R – space.

Example : Take cofinite topology $(\mathbb{N}, \tau_{cof})$.

Solution : Clear $(\mathbb{N}, \tau_{cof})$ not R – space (see page 95).

On the other hand, $(\mathbb{N}, \tau_{cof})$ is T_1 – space (see page 81).

So, we have T_1 – space, but not R – space.

Definition : T_3 – Space

Let (X, τ) be a topological space. Then the space (X, τ) is called a **T_3 – Space** iff its regular and T_1 – space. i.e.,

$$T_3 \text{ – space} = T_1 \text{ – space} + R \text{ – space}$$

Example : The space (X, D) is T_3 – space, since its T_1 – space and R – space.

Example : The space (X, I) ; X contains more than one element is not T_3 – space, since its not T_1 – space and R – space.

Example : In the cofinite topology (X, τ_{cof}), if X is infinite set, then its not T_3 – space, since its T_1 – space and not R – space.

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$. The space (X, τ) is not T_3 – space, since its not T_1 – space and R – space.

Example : The usual topological space $(\mathbb{R}, \tau_u)$ is T_3 – space, since its T_1 – space and R – space.

Theorem : Every metric space is T_3 – space.
Proof : Since every metric space is T_1 – space and R – space.

Theorem : The property of being a T_3 – space is a hereditary property.
Proof : Since the property T_1 – space and R – space are a hereditary property.
Then T_3 – space is a hereditary property.

Theorem : The property of being a T_3 – space is a topological property.
Proof : Since the property T_1 – space and R – space are a topological property.
Then T_3 – space is a topological property.

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. Then the product space $X\times Y$ is a T_3 – space iff each X and Y are T_3 – space.
Proof : We prove in previous theorems : That the product space $X\times Y$ is T_1 – space and R –space iff each X and Y is T_1 –space (see page 84) and R –space (see page 97). Hence, we have the product space $X\times Y$ is a T_3 – space iff each X and Y are T_3 – space.

Remark : The continuous image of T_3 – space is not necessarily T_3 – space. i.e., if $f : (X, \tau) \to (Y, \tau')$ is continuous onto function and X is T_3 – space, then Y not necessarily T_3 – space and the following example show this :

Example : Let $f : (\mathbb{R}, D) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall\, x \in \mathbb{R}$.
f is continuous function since the domain $(\mathbb{R}, D)$ is discrete topology (see page 37) and clear f is onto and in general (X, D) is T_3 – space (i.e., $(\mathbb{R}, D)$ is T_3 – space), but (X, I) is not T_3 – space (see page 98).

Theorem : If (X, τ) is a T_3 – space, then X is a T_2 – space.

Proof : Suppose that X is a T_3 – space (i.e., ,T_1 – space and R – space), to prove X is T_2 – space.

Let $x, y \in X$; $x \neq y$

$\because$ X is T_1 – space $\Rightarrow \{x\}, \{y\} \in \mathscr{F}$ (by theorem, X is T_1-space $\Leftrightarrow$ $\{x\}$ closed $\forall\, x \in X$)

$\Rightarrow x \notin \{y\}$ (since $x \neq y$)

$\because$ X is R – space $\Rightarrow \exists$ U, V $\in \tau$; $U \bigcap V = \phi$, $(x \in U \ \wedge\ \{y\} \subseteq V)$

$\Rightarrow x \in U \ \wedge\ y \in V$

$\therefore$ (X, τ) is a T_2 – space.

In previous theorem we take $x \notin \{y\}$ and by the similar way we can take $y \notin \{x\}$ and we have the same result.

Remark : From above theorem we have :

$$T_3\text{– space} \Rightarrow T_2\text{– space} \Rightarrow T_1\text{– space} \Rightarrow T_0\text{– space}$$
$$\quad\not\Leftarrow \qquad\qquad \not\Leftarrow \qquad\qquad \not\Leftarrow$$

Remark : There is another method to express on the above theorem as follows :

If (X, τ) is R – space and every singleton set in X is closed, then X is T_3 – space.

Example : Let $X = \mathbb{N}$ and $\tau = \{U \subseteq X ; 1 \in U\} \bigcup \{\phi\}$. Is $(\mathbb{N}, \tau)$ T_3 – space ??

Solution : We test $(\mathbb{N}, \tau)$ is T_1 – space ?? and R – space ??

Let $x, y \in \mathbb{N}$; $x \neq y$, to find open set containing x but not y, and open set containing y but not x.

Suppose x = 1, then for any $y \in \mathbb{N}$ such that $x \neq y$ there is no open set contains y but not x (since definition τ is every open set must contains 1 if it's not empty set), so X not T_1 – space. Furthermore, X is not R – space.

Definition : Normal Space

Let (X, τ) be a topological space. Then the space (X, τ) is called a **Normal Space** (denoted by N – space) iff for each pair of closed disjoint subsets F and E of X, there exist open sets U and V such that $F \subseteq U$, $E \subseteq V$, and $U \cap V = \phi$. i.e.,

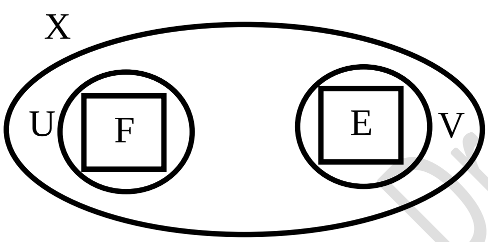
X is N – Space $\Leftrightarrow \forall\, F, E \in \mathscr{F}\,;\, F \cap E = \phi \;\; \exists\, U, V \in \tau\,;\, U \cap V = \phi\,,\, (F \subseteq U \wedge E \subseteq V)$

If (X, τ) is not N – space, we define,

X is not N – Space $\Leftrightarrow \exists\, F, E \in \mathscr{F}\,;\, F \cap E = \phi \;\; \forall\, U, V \in \tau\,;\, U \cap V = \phi\,,$

$$(F \not\subseteq U \wedge E \not\subseteq V) \vee (F \not\subseteq V \wedge E \not\subseteq U)\,.$$

The following figure show the definition of N – space :



Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2\}, \{1, 2\}\}$. Is (X, τ) N – space.

Solution : First, find the family of closed sets :

$$\mathscr{F} = \{X, \phi, \{2, 3\}, \{1, 3\}, \{3\}\}$$

Second, take every two closed sets there intersection is empty :

Notes that any two closed sets their intersection is nonempty, since all closed sets contains 3 except ϕ. Therefore, the definition of N – space is satisfy.

We can prove this as follows :

$$[\forall\, F, E \in \mathscr{F}\,;\, F \cap E = \phi] \Rightarrow [\exists\, U, V \in \tau\,;\, U \cap V = \phi\,,\, (F \subseteq U \wedge E \subseteq V)]$$

False statement either false statement or true statement

In two cases we have $(F \Rightarrow F = T)$ and $(F \Rightarrow T = T)$, therefore the definition of N – space is satisfy.

Remark : In the previous example notes that (X, τ) is not R – space and its N – space so that :

$$(\text{N – space} \not\Rightarrow \text{R – space})$$

Also, in this example (X, τ) is not T_1 – space and not T_2 – space so that :

$$(\text{N – space} \not\Rightarrow T_1\text{ – space}) \wedge (\text{N – space} \not\Rightarrow T_2\text{ – space})$$

Furthermore,

$$(\text{R – space} \not\Rightarrow \text{N – space}) \wedge (T_1\text{ – space} \not\Rightarrow \text{N – space}) \wedge (T_2\text{ – space} \not\Rightarrow \text{N – space})$$

Remark : $(T_0\text{ – space} \not\Rightarrow \text{N – space}) \wedge (\text{N – space} \not\Rightarrow T_0\text{ – space})$

Example : The space $(\mathbb{N}, \tau_{cof})$ is T_0 – space and not N – space, since there is two nonempty disjoint closed sets, but there is no two nonempty disjoint open sets.
Notes that too $(\mathbb{N}, \tau_{cof})$ is T_1 – space and not N – space.

Example : The space $(\mathbb{R}, I)$ is not T_0 – space, since $\mathbb{R}$ is the only open set contains elements and its contains all elements. But $(\mathbb{R}, I)$ is N – space since the closed sets are $F = \mathbb{R}$ and $E = \phi$ only, and $\mathbb{R} \bigcap \phi = \phi$ and the open sets are $\mathbb{R}$ and ϕ and $\mathbb{R} \subseteq \mathbb{R}$ and $\phi \subseteq \phi$.

Example : The space (X, D) is N – space, since every sets her is open and closed then: If $F, E \in \mathcal{F}$; $F \bigcap E = \phi$, then $F, E \in \tau$; $(F \subseteq F \wedge E \subseteq E)$.

Remark : If (X, τ) is topological space and $\tau = \mathcal{F}$ (i.e., every closed set is open) or $U \in \tau \iff U \in \mathcal{F}$, then (X, τ) is N – space.
The spaces (X, I) and (X, D) is special case from this spaces.
We can use this remark to have infinite numbers of N – space simply by take any set X and make topology τ on X as follows : $\tau = \{X, \phi, A, A^c\}$; $A \subseteq X$.

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}\}$. Show that (X, τ) is N – space.
Solution : First, find the family of closed sets :

$$\mathcal{F} = \{X, \phi, \{2, 3\}\}$$

Second, take every two closed sets there intersection is empty as follows :
Take, $\phi, X \in \mathcal{F}$; $\phi \bigcap X = \phi \Rightarrow \exists\ U = \phi \wedge V = X \in \tau$; $U \bigcap V = \phi$, $(\phi \subseteq U \wedge X \subseteq V)$.
Take, $\phi, \{2, 3\} \in \mathcal{F}$; $\phi \bigcap \{2, 3\} = \phi \Rightarrow \exists\ U = \phi \wedge V = X \in \tau$; $U \bigcap V = \phi$,
$(\phi \subseteq U \wedge \{2, 3\} \subseteq V)$.
$\therefore$ (X, τ) is N – space.
Notes that this space not T_0 – space, not T_1 – space, not T_2 – space, not R – space, and not T_3 – space.

Remark : The continuous image of N – space is not necessarily N – space. i.e., if $f : (X, \tau) \to (Y, \tau')$ is continuous onto function and X is N – space, then Y not necessarily N – space and the following example show this :

Example : Let f : $(\mathbb{N}, D) \to (\mathbb{N}, \tau_{cof})$; $f(x) = x \quad \forall x \in \mathbb{N}$.
f is continuous function since the domain $(\mathbb{N}, D)$ is discrete topology (see page 37) and clear f is onto and $(\mathbb{N}, D)$ is N – space, but $(\mathbb{N}, \tau_{cof})$ is not N – space.

Remark : The property of being a N – space is not a hereditary property and the following example show that :

Example : Let X = {1, 2, 3, 4, 5} and $\tau = \{X, \phi, \{1, 4, 5\}, \{1, 3, 5\}, \{1, 5\}, \{1, 3, 4, 5\}\}$. Clear that (X, τ) is N – space, since : the family of closed sets :

$$\mathcal{F} = \{X, \phi, \{2, 3\}, \{2, 4\}, \{2, 3, 4\}, \{2\}\}$$

Notes that any two closed sets except ϕ contains 2, so there is no nonempty disjoint closed sets and the sets are disjoint are ϕ and any other closed. Also, there is disjoint open sets to solve this case which are ϕ and X.
Now, take the subspace $W \subseteq X$; W = {3, 4, 5} such that :

$$\tau_W = \{ W \cap U ; U \in \tau\} = \{W, \phi, \{4, 5\}, \{5\}, \{3, 5\}\}$$

$$\text{and, } \mathcal{F}_W = \{W, \phi, \{3\}, \{4\}, \{3, 4\}\}$$

notes that : {3} and {4} closed sets in W and $\{3\} \cap \{4\} = \phi$. But, there is no disjoint open sets in τ_W such that one of them contains {3} and the other contain {4}, so that W is not N – space, while X is N – space.

Theorem : The property of being a N – space is a topological property.
Proof : Let $(X, \tau) \cong (Y, \tau')$ and suppose that X is N – space, to prove Y is N – space
$\because (X, \tau) \cong (Y, \tau') \Rightarrow \exists f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f open
Let $F, E \in \mathcal{F}'$; $F \cap E = \phi$
$\because$ f continuous $\Rightarrow f^{-1}(F), f^{-1}(E) \in \mathcal{F}$ and $f^{-1}(F) \cap f^{-1}(E) = f^{-1}(F \cap E) = f^{-1}(\phi) = \phi$
(by theorem : the function f is continuous $\Leftrightarrow$ the inverse image of every closed set in codomaun is closed in domain)
$\because$ X is N – space $\Rightarrow \exists U, V \in \tau$; $U \cap V = \phi$, $(f^{-1}(F) \subseteq U \wedge f^{-1}(E) \subseteq V)$
$\because$ f open $\Rightarrow f(U), f(V) \in \tau'$
$\because$ f is onto $\Rightarrow f(f^{-1}(F)) \subseteq f(U) \wedge f(f^{-1}(E)) \subseteq f(V)$
$\Rightarrow F \subseteq f(U) \wedge E \subseteq f(V)$ (since $f(f^{-1}(F)) = F \wedge f(f^{-1}(E)) = E$)
$\because U \cap V = \phi \Rightarrow f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$
$\therefore$ Y is N – space.
By similar way we prove, if Y is N – space, then X is N – space.

Theorem : The space (X, τ) is normal (N – space) iff for each closed subset $F \subseteq X$ and open set W containing F (i.e., $F \subseteq W$), there exists an open set U such that $F \subseteq U \subseteq \overline{U} \subseteq W$.

Proof : ($\Rightarrow$) Suppose that X is N – space and $F \subseteq X$; $F \in \mathscr{F}$.

Let $W \in \tau$; $F \subseteq W \Rightarrow F \bigcap X - W = \phi \wedge X - W \in \mathscr{F}$ (since $W \in \tau$)

$\because$ X is N – space $\Rightarrow \exists\, U, V \in \tau$; $U \bigcap V = \phi$, ($\underline{\underline{F \subseteq U}} \wedge X - W \subseteq V$)

$\Rightarrow \underline{\underline{X - V \subseteq W}}$ (since $A \subseteq B \Rightarrow B^c \subseteq A^c$)

$\because U \bigcap V = \phi \quad \Rightarrow U \subseteq X - V$

$\Rightarrow \overline{U} \subseteq \overline{X - V}$ (since $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$)

$\Rightarrow \overline{U} \subseteq X - V$ (since $X - V$ closed $\Rightarrow X - V = \overline{X - V}$)

$\Rightarrow \underline{\underline{U \subseteq \overline{U} \subseteq X - V}}$ (since $A \subseteq \overline{A}$)

$\Rightarrow F \subseteq U \wedge U \subseteq \overline{U} \subseteq X - V \wedge X - V \subseteq W$

$\Rightarrow F \subseteq U \subseteq \overline{U} \subseteq W$

($\Leftarrow$) Suppose the condition of theorem satisfy, to prove X is N – space

Let $F, E \in \mathscr{F}$; $F \bigcap E = \phi \Rightarrow F \subseteq X - E \in \tau$ (since $E \in \mathscr{F}$).

$\Rightarrow \exists\, U \in \tau$; $F \subseteq U \subseteq \overline{U} \subseteq X - E$ (by hypothesis)

$\Rightarrow \overline{U} \subseteq X - E$

$\Rightarrow E \subseteq X - \overline{U}$ (since $A \subseteq B \Leftrightarrow B^c \subseteq A^c$)

But, $X - \overline{U}$ open since $\overline{U}$ closed, say $X - \overline{U} = V$

$\Rightarrow E \subseteq V = X - \overline{U} \wedge F \subseteq U$

$\wedge\, U \bigcap V = \phi$, (since $U \subseteq \overline{U}$ and $\overline{U} \bigcap X - \overline{U} = \phi \Rightarrow U \bigcap V = \phi$)

$\therefore$ X is N – space.

Remark : Let (X, τ) and (Y, τ') be two topological spaces. If the product space $X \times Y$ is a N – space, then each X and Y are N – space.
But, the conversely is not true in general i.e.,. If each X and Y are N – space, then not necessary that the product space $X \times Y$ is a N – space.

Remark : Every metric space is N – space. Therefore, since $(\mathbb{R}, \tau_u)$ is metric space, then its N – space.

Theorem : A **closed** subspace of N – space is N – space.

Proof :

Let (X, τ) N – space and (W, τ_W) closed subspace of X, to prove (W, τ_W) N – space

Let F_W, E_W are closed sets in W ; $F_W \bigcap E_W = \phi$

$$\Rightarrow \exists\, F, E \in \mathscr{F}\ ;\ F_W = F \cap W \ \wedge\ E_W = E \cap W\ ;\ F \cap E = \phi$$

$\because$ X is N – space $\Rightarrow \exists\, U, V \in \tau\ ;\ U \cap V = \phi$, $(F \subseteq U \wedge E \subseteq V)$

$\Rightarrow U \cap W \ \wedge\ V \cap W \in \tau_W$, (by def. of τ_W)

$(U \cap W) \cap (V \cap W) = (U \cap V) \cap W = \phi \cap W = \phi$,

since $F_W = F \cap W \Rightarrow F_W \subseteq F \ \wedge\ F_W \subseteq W \Rightarrow F_W \subseteq U \wedge F_W \subseteq W \Rightarrow F_W \subseteq U \cap W$

since $E_W = E \cap W \Rightarrow E_W \subseteq E \ \wedge\ E_W \subseteq W \Rightarrow E_W \subseteq V \wedge E_W \subseteq W \Rightarrow E_W \subseteq V \cap W$

$\therefore$ (W, τ_W) is N – space.

<u>Definition :</u> T_4 – Space

Let (X, τ) be a topological space. Then the space (X, τ) is called a **<u>T_4 – Space</u>** iff its normal and T_1 – space. i.e.,

$$T_4 \text{ – space} = T_1 \text{ – space} + N \text{ – space}$$

<u>Example :</u> Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$. Then the space (X, τ) is not T_4 – space, since its N – space but not T_1 – space.

<u>Remark :</u> If X is finite space, then (X, τ) is T_4 – space iff $\tau = D$, (because if X is finite space, then its T_1 – space iff $\tau = D$ and if $\tau = D$, then X is N – space).

<u>Example :</u> The space (X, D) is T_4 – space, since its T_1 – space and N – space.

<u>Example :</u> The space (X, I) ; X contains more than one element is not T_4 – space, since its not T_1 – space.

<u>Remark :</u> The property of being a T_4 – space is not a hereditary property, since the normality is not a hereditary property.

<u>Example :</u> The space $(\mathbb{N}, \tau_{cof})$ is not T_4 – space, since its T_1 – space but not N – space.

<u>Theorem :</u> The property of being a T_4 – space is a topological property.

<u>Proof :</u> Since the property T_1 – space and N – space are a topological property. Then T_4 – space is a topological property.

Theorem : A **closed** subspace of T_4 – space is T_4 – space.

Proof : Let (X, τ) T_4 – space and W closed set in X, to prove W is T_4 – space

$\because$ X is T_1 – space $\Rightarrow$ W is T_1 – space (since T_1 is hereditary property)

$\because$ W is closed in X and X is N – space $\Rightarrow$ W is N – space

(by theorem : A closed subspace of N – space is N – space)

$\therefore$ W is T_4 – space.

Remark : Let (X, τ) and (Y, τ') be a topological spaces. If the product space $X \times Y$ is T_4 – space, then each X and Y is T_4 – space.

But, the conversely is not true in general i.e.,. If each X and Y is T_4 – space, then not necessary that the product space $X \times Y$ is T_4 – space.

Remark : Every metric space is T_4 – space. Since every metric space is T_1 – space and N – space.

Theorem : Every T_4 – space is R – space.

Proof : Let (X, τ) be T_4 – space $\Rightarrow$ X is T_1 – space and N – space

Let $x \in X$ and F closed set in X ; $x \notin F$

$\Rightarrow \{x\} \in \mathcal{F}$ (since X is T_1 – space $\Leftrightarrow$ $\{x\}$ closed $\forall\, x \in X$}

$\Rightarrow \{x\} \bigcap F = \phi$ (since $x \notin F$)

$\because$ X is N – space $\Rightarrow \exists\, U, V \in \tau\,;\ U \bigcap V = \phi\,,\ (\{x\} \subseteq U \wedge F \subseteq V)$

$\Rightarrow x \in U \wedge F \subseteq V$

$\therefore$ X is R – space.

Corollary : Every T_4 – space is T_3 – space.

Proof : Every T_4 – space is R – space (by the above theorem)

Every T_4 – space is T_1 – space and N – space (by def of T_4 – space)

We have, X is T_1 – space R – space

$\therefore$ X is T_3 – space.

Remark : Every T_4 – space is T_2 – space since every T_4 – space is T_3 – space and every T_3 – space is T_2 – space so that :

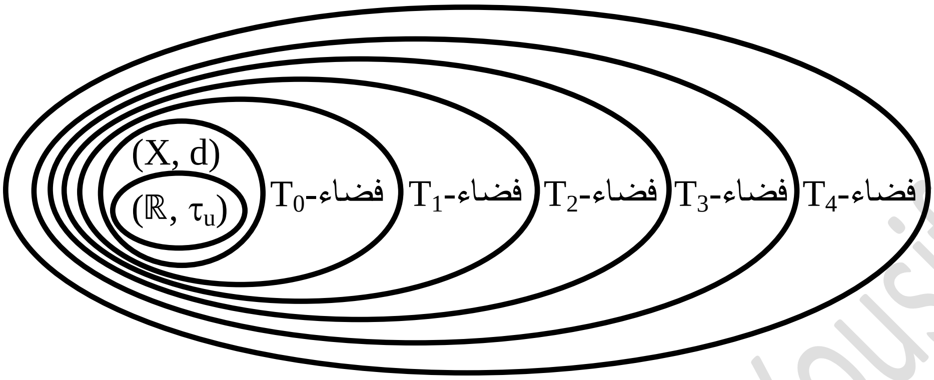
$$T_4\text{ – space} \Rightarrow T_3\text{ – space} \Rightarrow T_2\text{ – space} \Rightarrow T_1\text{ – space} \Rightarrow T_0\text{ – space}$$
$$\not\Leftarrow \qquad\qquad \not\Leftarrow \qquad\qquad \not\Leftarrow \qquad\qquad \not\Leftarrow$$

Notes that : N – space $\not\Rightarrow$ R – space and N – space $\not\Rightarrow$ T_1 – space,

but N – space + T_1 – space $\Rightarrow$ T_3 – space

and N – space + T_1 – space $\Rightarrow$ R – space

Remark : the following figure show that the relation between separation axioms T_0, T_1, T_2, T_3, T_4, and metric space and usual topology $(\mathbb{R}, \tau_u)$:



In previous illustrate we notes that a T_2 – space not necessarily R – space. Also, if X is T_2 – space, then its not necessarily N – space, but this statements are satisfy if we add the condition that compaceness space and the following theorems show this :

Theorem : Every compact T_2 – space is R – space.

Proof : Let (X, τ) be T_2 – space and compact, to prove X is R – space.

Let $x \in X$, $F \in \mathscr{F}$; $x \notin F \Rightarrow x \neq y \ \forall \ y \in F$

$\because$ X is T_2 – space $\Rightarrow \exists \ U_y, V_y \in \tau$; $U_y \bigcap V_y = \phi$, $(x \in U_y \wedge y \in V_y)$.

We have two family of open sets are $\{U_y\}_{y\in F}$ and $\{V_y\}_{y\in F}$ such that every element in F exists in one element of the family $\{V_y\}_{y\in F}$ and every element in the family $\{U_y\}_{y\in F}$ contains the element x and every U_y corresponding V_y such that $U_y \bigcap V_y = \phi$. Therefore, the family $\{V_y\}_{y\in F}$ open cover for F

$$\Rightarrow \{V_y\}_{y\in F} \text{ open cover of F, i.e., } F \subseteq \bigcup_{y\in F} V_y$$

$\because$ F closed in the compact space (by hypothesis), so that F compact space and we have : $\Rightarrow \exists \ y_1, y_2, \dots , y_n$; $F \subseteq \bigcup_{i=1}^{n} V_{y_i}$

Therefore, $\{V_{y_i}\}_{i=1}^{n}$ is a finite family of open sets cover F (let $V = \bigcup_{i=1}^{n} V_{y_i}$),

On the other hand, $\{U_{y_i}\}_{i=1}^{n}$ is a finite family of open sets and every element in this family contains x (let $U = \bigcap_{i=1}^{n} U_{y_i}$)

$\Rightarrow$ U and V are open sets (by second and third condition of def of top.)

Such that $x \in U$ and $F \subseteq V$

Notes that, $U \bigcap V = \phi$ (since $U = \bigcap_{i=1}^{n} U_{y_i} \Rightarrow U \subseteq U_{yi} \ \forall$ i and $U_{yi} \bigcap V_{yi} = \phi$)

$\therefore$ X is R – space.

Remark : There is a theorem similar the above theorem and their prove is similar too and we introduce this theorem but without prove.
((In T_2 – space we can separated any point x and compact subset not contains x by disjoint open sets))

Corollary : Every compact T_2 – space is T_3 – space.
Proof : Every T_2 – space is T_1 – space
Every T_2 – space and compact is R – space (by the above theorem)
We have, X is T_1 – space and R – space
$\therefore$ X is T_3 – space.

Theorem : Every compact T_2 – space is N – space.
Proof : Let (X, τ) be T_2 – space and compact, to prove X is N – space.
Let $F, E \in \mathscr{F}$; $F \bigcap E = \phi \Rightarrow F, E$ are compact.
(by theorem : Every closed set in compact space is compact)
Choose, $x \in F \Rightarrow x \notin E \Rightarrow \exists U_x, V_E \in \tau$; $U_x \bigcap V_E = \phi$, $(x \in U_x \wedge E \subseteq V_E)$.
(by previous remark, since X is T_2 – space and E compact set and $x \notin E$)
Now, repeated this method on every element in F, we have a family of open sets cover F as follows :
$\{U_x ; x \in F \wedge U_x \in \tau\} \Rightarrow F \subseteq \bigcup_{x\in F} U_x$
$\because$ F is compact set $\Rightarrow \exists x_1, x_2, \ldots, x_n$; $F \subseteq \bigcup_{i=1}^{n} U_{x_i}$
Also, we have a family of open sets every elements in this family contain E as follows :
$\{V_i ; i = 1, 2, \ldots, n \wedge E \subseteq V_i \ \forall \ i \wedge V_i \in \tau\}$
$V_i \bigcap U_{xi} = \phi \ \ \forall \ i$; $i = 1, 2, \ldots, n$
Say, $U = \bigcup_{i=1}^{n} U_{x_i} \Rightarrow F \subseteq U \in \tau$ (by third condition of def. of top.)
Say, $V = \bigcap_{i=1}^{n} V_i \Rightarrow E \subseteq V \in \tau$ (by second condition of def of top.)
Notes that, $U \bigcap V = \phi$ (since $[U = \bigcup_{i=1}^{n} U_{xi}] \bigcap [V = \bigcap_{i=1}^{n} V_i] = \phi$)
$\therefore$ X is N – space.

Chapter Five : Connected Spaces

Definition : **Disconnected & Connected Spaces**

The space (X, τ) is **disconnected** iff there exist two open disjoint nonempty sets A and B such that $A \bigcup B = X$. i.e.,

$$X \text{ is disconnected } \Leftrightarrow X = A \bigcup B \quad ; \quad A, B \in \tau \ , \ A \bigcap B = \phi \ , A \neq \phi \neq B.$$

The sets A and B form a **separation** of X.

The space (X, τ) is **connected** iff it is not disconnected.

$$X \text{ is connected } \quad \Leftrightarrow X \neq A \bigcup B \quad ; \quad A, B \in \tau \ , \ A \bigcap B = \phi \ , A \neq \phi \neq B.$$

Remark : The connected spaces is the spaces required in topology, but the definition dependent on disconnected spaces for simply.

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}\}$. Is X is connected or disconnected space ??.

Solution : X is connected, since the only one case to represented on X as a union of nonempty open sets is $X = X \bigcup \{1\}$, but this sets is joint.

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$. Is X is connected or disconnected space ??.

Solution : X is disconnected, since :

$X = \{1\} \bigcup \{2, 3\}$ and $\{1\}, \{2, 3\} \in \tau$ and $\{1\} \bigcap \{2, 3\} = \phi$ and $\{1\} \neq \phi, \{2, 3\} \neq \phi$.

Remark : There are 29 difference topology on set contains 3 elements, we can testes this topologies connected or disconnected.

Remarks :

[1] (X, D) is disconnected if X contains more than one element, since :
$\exists A \ ; \ \phi \neq A \subsetneq X \ \Rightarrow \ X = A \bigcup A^c \ , \ A, A^c \in D, \ A \bigcap A^c = \phi \ , A \neq \phi$ and $A^c \neq \phi$ since $A \neq X$.

[2] (X, I) is connected spaces always since the only open sets are X and ϕ and this sets not make the space is disconnected.

[3] If $\tau = \mathscr{F}$ and $\tau \neq I$, then (X, τ) is disconnected since :

$$\exists A \in \tau \ ; \ \phi \neq A \subsetneq X \ \Rightarrow \ X = A \bigcup A^c.$$

Example : (X, τ_{cof}) is connected space, if X is infinite set since there are not exist nonempty disjoint open sets.

Example : $(\mathbb{R}, \tau_u)$ is connected space, since $\mathbb{R}$ not equal union of two nonempty disjoint open sets,
but $\mathbb{R} \setminus \{x_0\}$; $x_0 \in \mathbb{R}$ is separated since $\mathbb{R} \setminus \{x_0\} = (-\infty, x_0) \bigcup (x_0, \infty)$.

Example : The metric space is either connected or is disconnected, for example :
$(\mathbb{R}, \tau_u)$ generated from metric space $(\mathbb{R}, |\,|)$ is connected space.
But, (X, D) ; X contains more than one element is disconnected space which is generated from metric space.

Example : Let X = {a, b, c, d}. Define a topology τ on X and another topology τ' on X such that (X, τ) is connected and (X, τ') is disconnected.
Solution : Let $\tau = \{X, \phi, \{a, b\}\}$ and $\tau' = \{X, \phi, \{a, b\}, \{c, d\}\}$, then (X, τ) is connected and (X, τ') is disconnected.

We introduce some theorems to equivalent properties for a space being connected :

Theorem : (X, τ) is connected space iff X cannot be written as a union of two nonempty disjoint closed sets.
Proof : $(\Rightarrow)$ Suppose that X is connected
To prove $X \neq A \bigcup B$; $A, B \in \mathcal{F}$, $A \bigcap B = \phi$, $A \neq \phi \neq B$
Suppose that $X = A \bigcup B$; $A, B \in \mathcal{F}$, $A \bigcap B = \phi$, $A \neq \phi \neq B$

$\Rightarrow A = B^c \wedge B = A^c$

$\Rightarrow A \in \tau \wedge B \in \tau$ (since $A = B^c \wedge B \in \mathcal{F}$ and $B = A^c \wedge A \in \mathcal{F}$)

$\Rightarrow X = A \bigcup B$; $A, B \in \tau$, $A \bigcap B = \phi$, $A \neq \phi \neq B$

$\Rightarrow$ X disconnected C!! Contradiction !! since X is connected

$\Rightarrow X \neq A \bigcup B$; $A, B \in \mathcal{F}$, $A \bigcap B = \phi$, $A \neq \phi \neq B$.

$(\Leftarrow)$ Suppose that $X \neq A \bigcup B$; $A, B \in \mathcal{F}$, $A \bigcap B = \phi$, $A \neq \phi \neq B$
To prove X is connected
Suppose that X is disconnected

$\Rightarrow X = U \bigcup V$; $U, V \in \tau$, $U \bigcap V = \phi$, $U \neq \phi \neq V$

$\Rightarrow U = V^c \wedge V = U^c$

$\Rightarrow U, V \in \mathcal{F}$ C!! Contradiction !!

since the complement of every one of them is open set and this contradiction with hypotheses

$\therefore$ X connected space.

Theorem : (X, τ) is connected space iff the only subsets of the space X which are open and closed are X and ϕ.

Proof : ($\Rightarrow$) Suppose that X is connected

To prove, if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$.

Suppose that $A, A^c \in \tau$ and $A \neq X$ and $A \neq \phi$

$$\Rightarrow X = A \bigcup A^c \ \wedge \ A, A^c \in \tau \ \wedge \ A \bigcap A^c = \phi \ , A \neq \phi \neq A^c \ \text{(since } A \neq X)$$

$$\Rightarrow X \text{ disconnected} \qquad \text{C!! Contradiction !!}$$

$\therefore$ if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$.

($\Leftarrow$) Suppose that, if $A \subseteq X$, $A, A^c \in \tau$, then $A = X$ or $A = \phi$, i.e., $\tau \bigcap \mathcal{F} = \{X, \phi\}$.

To prove X is connected

Suppose that X is disconnected

$$\Rightarrow X = U \bigcup V \ ; U, V \in \tau \ , U \bigcap V = \phi \ \wedge \ U \neq \phi \neq V$$

$$\Rightarrow U = V^c \ \wedge \ V = U^c \quad \Rightarrow U, V \in \mathcal{F}$$

$$\Rightarrow U, V \in \tau \bigcap \mathcal{F} \qquad \text{C!! Contradiction !!}$$

Since U and V are open and closed and not equal X and ϕ.

$\therefore$ X connected space.

Theorem : (X, τ) is connected space iff the only subsets of the space X which have empty boundary sets are X and ϕ. i.e.,

$$[X \text{ is connected} \Leftrightarrow (\phi \neq A \subsetneq X \ \Rightarrow \ A^b \neq \phi)]$$

on the other hand :

$$[X \text{ is connected} \Leftrightarrow (A^b = \phi \ \Rightarrow \ A = \phi \ \vee \ A = X)]$$

Proof : ($\Rightarrow$) Suppose that X is connected

To prove, if $A^b = \phi \ \Rightarrow \ A = \phi \ \vee \ A = X$.

Suppose that $A \subseteq X$ and $A^b = \phi \ \wedge \ A \neq \phi \ \wedge \ A \neq X$

$$\Rightarrow \phi \subseteq A \ \Rightarrow \ A^b \subseteq A \ \Rightarrow \ A \in \mathcal{F} \qquad \text{(by theorem : } A \in \mathcal{F} \Leftrightarrow A^b \subseteq A)$$

On the other hand :

$$\phi \bigcap A = \phi \ \Rightarrow \ A^b \bigcap A = \phi \ \Rightarrow \ A \in \tau \qquad \text{(by theorem : } A \in \tau \Leftrightarrow A^b \bigcap A = \phi)$$

$$\Rightarrow A, A^c \in \tau \ \text{ and } \ A \neq \phi \ \wedge \ A \neq X$$

$$\Rightarrow X \text{ disconnected}$$

(by theorem : (X, τ) is connected space iff the only subsets of the space X which are open and closed are X or ϕ)

(this contradiction with hypotheses : X is connected $\Rightarrow A^b \neq \phi$ if $\phi \neq A \neq X$)

$\therefore$ if $A^b = \phi \Rightarrow A = \phi \vee A = X$.

($\Leftarrow$) Suppose that, if $A^b = \phi \Rightarrow A = \phi \vee A = X$.

To prove X is connected

Suppose that X is disconnected

$$\Rightarrow X = U \bigcup V \ ; U, V \in \tau \ , U \bigcap V = \phi \wedge U \neq \phi \neq V$$

$$\Rightarrow U = V^c \wedge V = U^c \Rightarrow U, V \in \mathcal{F}$$

$$\Rightarrow U^b = \phi \wedge V^b = \phi \qquad \text{(by theorem : } A, A^c \in \tau \Leftrightarrow A^b = \phi)$$

(this contradiction since $U^b = \phi$, but $U \neq X$ and $U \neq \phi$)

$\therefore$ X connected space.

Theorem : (X, τ) is connected space iff every continuous function from domain X to codomain $(\{1, 2\}, D)$ is constant function.

Proof : Let $f : (X, \tau) \to (\{1, 2\}, D)$ be continuous function.

($\Rightarrow$) Suppose that X connected space

To prove, f is constant function

Suppose that f not constant

$$\Rightarrow \exists A \subseteq X \ ; f(a) = 1 \quad \forall a \in A$$

$$\text{and } \exists B \subseteq X \ ; f(b) = 2 \quad \forall b \in B$$

Notes that,

(1) $X = A \bigcup B$, since if $X \neq A \bigcup B \Rightarrow \exists x \in X$; x has no image $\Rightarrow$ f not funct.

(2) $A \neq \phi$, since if $A = \phi \Rightarrow$ f constant (the prove end)

and, $B \neq \phi$, since if $B = \phi \Rightarrow$ f constant (the prove end)

(3) $A \bigcap B = \phi$, since if $A \bigcap B \neq \phi \Rightarrow \exists x \in X$; x has two image $\Rightarrow$ f not funct.

Now,

$\{1\}, \{2\} \in D$ (by def. of D) $\Rightarrow \{1\}, \{2\}$ open set in $(\{1, 2\}, D)$

$\because$ f continuous $\Rightarrow A = f^{-1}(\{1\}) \in \tau \wedge B = f^{-1}(\{2\}) \in \tau$

$$\Rightarrow X = A \bigcup B \wedge A, B \in \tau \wedge A \bigcap B = \phi \wedge A \neq \phi \neq B$$

$$\Rightarrow X \text{ disconnected} \qquad \text{C!! Contradiction !!}$$

$\therefore$ f is constant function

($\Leftarrow$) Suppose that f is constant function

To prove, X connected space

Suppose that X disconnected

$\Rightarrow X = A \bigcup B$; $A, B \in \tau$, $A \bigcap B = \phi$, $A \neq \phi \neq B$

Define $f : (X, \tau) \to (\{1, 2\}, D)$; $f(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 2 & \text{if} \quad x \in B \end{cases}$

Clear that f is continuous, since $D = \{\{1, 2\}, \phi, \{1\}, \{2\}\}$ and

$f^{-1}(\{1, 2\}) = \{1, 2\} \in \tau$, $f^{-1}(\phi) = \phi \in \tau$, $f^{-1}(\{1\}) = A \in \tau$, $f^{-1}(\{2\}) = B \in \tau$

$\Rightarrow$ f not constant C!! Contradiction !!

$\therefore$ X connected space.

<u>Remark :</u> If (X, τ) is topological space and (W, τ_W) is a subspace of X, then the space W being disconnected or connected not directly relation by X and the open sets in X, but dependent on the open sets in W. i.e., its dependent on τ_W so that : W is connected space iff there exist two open disjoint nonempty sets A and B in W such that $A \bigcup B = W$.

i.e., W is disconnected $\Leftrightarrow$ $W = A \bigcup B$; $A, B \in \tau_W$, $A \bigcap B = \phi$, $A \neq \phi \neq B$.

The space (W, τ_W) is **<u>connected</u>** iff it is not disconnected.

W is connected $\Leftrightarrow$ $W \neq A \bigcup B$; $A, B \in \tau_W$, $A \bigcap B = \phi$, $A \neq \phi \neq B$.

<u>Remark :</u> The property of being a connected space is not a hereditary property and the following example show that :

<u>Example :</u> Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1, 2\}, \{1, 3\}, \{1\}\}$. Let $W \subseteq X$; $W = \{2, 3\}$. Is W is connected space ??

Solution : Compute τ_W :

$$\tau_W = \{ W \bigcap U ; U \in \tau\} = \{W, \phi, \{2\}, \{3\},\}$$

Notes that $\tau_W = D$, then W is disconnected space but not connected since :

$W = \{2\} \bigcup \{3\}$ and $\{2\}, \{3\} \in \tau_W$ and $\{2\} \bigcap \{3\} = \phi$ and $\{2\} \neq \phi$, $\{3\} \neq \phi$.

Notes that X is connected space but not disconnected, while it's have disconnected subspace.

<u>Theorem :</u> continuous image of connected space is connected. i.e.,

If $f : (X, \tau) \to (Y, \tau')$ is continuous and onto function and X is connected, then Y is connected.

<u>Proof :</u> Suppose that Y is disconnected

$\Rightarrow \exists A, B \in \tau'$; $A \bigcap B = \phi \wedge A \neq \phi \neq B \wedge Y = A \bigcup B$

$\Rightarrow f^{-1}(Y) = f^{-1}(A \bigcup B)$

$\Rightarrow X = f^{-1}(A) \bigcup f^{-1}(B)$ (since f onto $\Rightarrow f^{-1}(Y) = X \wedge f^{-1}(A \bigcup B) = f^{-1}(A) \bigcup f^{-1}(B)$)

$\Rightarrow\ f^{-1}(A), f^{-1}(B) \in \tau$ (since f continuous)

$\wedge\ f^{-1}(A) \bigcap f^{-1}(B) = f^{-1}(A \bigcap B) = f^{-1}(\phi) = \phi$

$\wedge\ A \neq \phi\ \wedge B \neq \phi \ \Rightarrow\ f^{-1}(A) \neq \phi \neq f^{-1}(B)$

$\therefore$ X is disconnected C!! Contradiction !!

$\therefore$ Y is connected.

Remark : If $f : (X, \tau) \to (Y, \tau')$ is continuous and onto function and Y is connected space, then X not necessary connected space and the following example show that :

Example : Let $f : (\mathbb{R}, D) \to (\mathbb{R}, I)$; $f(x) = x \quad \forall\ x \in \mathbb{R}$.

Clear that f is continuous and onto function and $(\mathbb{R}, I)$ is connected, but $(\mathbb{R}, D)$ is not connected (disconnected).

Corollary (1) : The property of being a connected space is a topological property.

Proof :

Let (X, τ) and (Y, τ') be topological space ; $X \cong Y$

$\because\ X \cong Y \Rightarrow \exists f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Suppose that X is connected, to prove Y is connected

$\because$ f continuous, onto and X connected $\Rightarrow f(X) = Y$ connected

(by theorem : continuous image of connected space is connected)

Now, suppose that Y is connected, to prove X is connected

$\because\ f^{-1}$ continuous, onto and Y connected $\Rightarrow f^{-1}(Y) = X$ connected

(by same theorem : continuous image of connected space is connected)

Corollary (2) : Let (X, τ) and (Y, τ') be two topological spaces. If the product space $X\times Y$ is a connected space then each X and Y are connected spaces.

Proof :

The projection function $P_X : X\times Y \to X$ is continuous and onto

$\because\ X\times Y$ connected and P_X continuous $\Rightarrow P_X(X\times Y)$ connected

(by theorem : continuous image of connected space is connected)

$\because\ P_X$ onto $\Rightarrow P_X(X\times Y) = X$

$\Rightarrow$ X connected

By the similar way we prove Y connected.

The projection function $P_Y : X\times Y \to Y$ is continuous and onto

$\because\ X\times Y$ connected and P_Y continuous $\Rightarrow P_Y(X\times Y)$ connected

(by theorem : continuous image of connected space is connected)

$\because$ P_X onto $\Rightarrow$ $P_Y(X\times Y) = Y$

$\Rightarrow$ Y connected

We can use preview theorem and their corollaries to known some spaces either connected or disconnected as the following examples :

<u>Example :</u> If we know the function $f : (\mathbb{R}, \tau_u) \to (Y, \tau)$ is continuous onto. Is Y connected space ??

<u>Solution :</u> **Yes**, since $(\mathbb{R}, \tau_u)$ is connected space and f is continuous onto function, then $f(\mathbb{R}) = Y$ is connected space.

<u>Example :</u> If we know that $(\mathbb{R}, D) \cong (Y, \tau)$. Is Y connected space ??

<u>Solution :</u> **No**, since the equivalent topological spaces either connected spaces or not connected spaces, because the property of being a connected space is a topological property and since $(\mathbb{R}, D)$ is disconnected space, then $f(\mathbb{R}) = Y$ is disconnected space.

<u>Example :</u> If we know the product space $X\times Y$ for spaces X and Y is indiscreet topology i.e., $\tau_{X\times Y} = I$. What we say about X and Y ??

<u>Solution :</u> **Yes**, since $(X\times Y, I)$ is connected space, then X and Y is connected space.

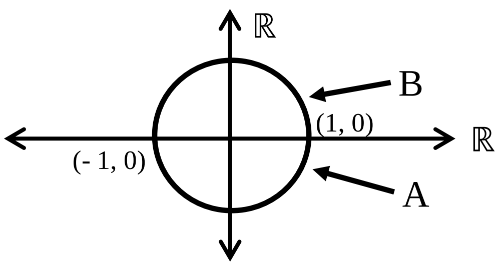
<u>Remark :</u> If A and B are connected subsets of a space (X, τ), then $A \cap B$ is not necessary connected and the following example show that :

<u>Example :</u> Let $X = \mathbb{R}^2$ and its product space $(\mathbb{R}, \tau_u) \times (\mathbb{R}, \tau_u)$ such that the open neighhoods in this space is a disc her center is a point (for example $\{(x, y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ is an open neighhood for the point (0, 0) and geometrical it's a disc with center (0, 0) and radius 1.

Let A and B define as follows :

$A = \{(x, y) \in \mathbb{R}^2 \ ; \ x^2+y^2 = 1 \wedge y \le 0\}$

$B = \{(x, y) \in \mathbb{R}^2 \ ; \ x^2+y^2 = 1 \wedge y \ge 0\}$



such that the geometric representation of A as lower half for circle circumference with radius 1 and center (0, 0) as the above figure.

and the geometric representation of B as upper half for circle circumference with radius 1 and center (0, 0) as the above figure.

$A \bigcap B = \{(1, 0), (-1, 0)\}$

Notes that every one of A and B are connected, but $A \bigcap B$ not connected set.

Remark : If A and B are connected subsets of a space (X, τ), then $A \bigcup B$ is not necessary connected and the following example show that :

Example : Let $(X, \tau) = (\mathbb{R}, \tau_u)$ and $A = (1, 2)$ and $B = (3, 4)$

Clear that A and B are connected sets since we cannot represented as a union of nonempty disjoint open intervals subsets of A or B, but $A \bigcup B$ is not connected (disconnected) since

$A \bigcup B = (1, 2) \bigcup ((3, 4)$ and clear that every one of $(1, 2)$ and $(3, 4)$ nonempty disjoint open sets in $A \bigcup B$.

Notes that in general in a space $(\mathbb{R}, \tau_u)$: If A is subset of $\mathbb{R}$, then A is connected iff A is interval i.e.,

$$A \text{ is connected } \Leftrightarrow A = (a, b) \text{ or } A = [a, b] \text{ or } A = (a, b] \text{ or } A = [a, b)$$

Theorem : Let (X, τ) be a topological space. If W is a connected subsets of X and $X = A \bigcup B$ such that $A, B \in \tau$ and $A \bigcap B = \phi$ and $A \neq \phi \neq B$, then $W \subseteq A$ or $W \subseteq B$.

Proof :

Suppose that $W \not\subseteq A$ and $W \not\subseteq B$

$\Rightarrow W \bigcap A \neq \phi$ and $W \bigcap B \neq \phi$

$\because A, B \in \tau \quad \Rightarrow W \bigcap A, W \bigcap B \in \tau_W$ (by def. of subspace topology)

Notes that,

$W \bigcap A \neq \phi$ (since, if $W \bigcap A = \phi \Rightarrow W \subseteq B$)

$W \bigcap B \neq \phi$ (since, if $W \bigcap B = \phi \Rightarrow W \subseteq A$)

Also,

$(W \bigcap A) \bigcap (W \bigcap B) = W \bigcap (A \bigcap B) = W \bigcap \phi = \phi$

$\Rightarrow$ W is disconnected C!! Contradiction !!

$\therefore W \subseteq A \vee W \subseteq B$.

Remark : Notes that $A \bigcup B$ may be not connected in spite of A connected and B connected, but if we add a condition $A \bigcap B \neq \phi$, then $A \bigcup B$ is connected set and this show in the next theorem :

Theorem : If A and B are connected subsets of a space (X, τ) and $A \bigcap B \neq \phi$, then $A \bigcup B$ is connected.

Proof :

Let (X, τ) topological space and $A, B \subseteq X$; A, B connected and $A \bigcap B \neq \phi$

To prove $A \bigcup B$ connected ??

Suppose that $A \bigcup B$ disconnected

$$\Rightarrow A \bigcup B = U \bigcap V \quad ; \quad U, V \in \tau_{A\bigcup B}, U \bigcap V = \phi , U \neq \phi \neq V$$

$$\Rightarrow A \subseteq A \bigcup B \Rightarrow A \subseteq U \bigcap V \text{ and A connected}$$

$$\Rightarrow A \subseteq U \vee A \subseteq V \qquad \text{(by previous theorem)}$$

By similar way $\Rightarrow B \subseteq A \bigcup B \Rightarrow B \subseteq U \bigcap V$ and B connected

$$\Rightarrow B \subseteq U \vee B \subseteq V \qquad \text{(by previous theorem)}$$

Now,

$\Rightarrow$ either $A \subseteq U \wedge B \subseteq U \Rightarrow A \bigcup B \subseteq U \Rightarrow V = \phi$ C!!

or $A \subseteq V \wedge B \subseteq V \Rightarrow A \bigcup B \subseteq V \Rightarrow U = \phi$ C!!

or $A \subseteq U \wedge B \subseteq V \Rightarrow A \bigcap B \subseteq U \bigcap V = \phi \Rightarrow A \bigcap B = \phi$ C!!

or $A \subseteq V \wedge B \subseteq U \Rightarrow A \bigcap B \subseteq U \bigcap V = \phi \Rightarrow A \bigcap B = \phi$ C!!

$\therefore A \bigcup B$ connected.

Remark : We can generalization the above theorem to family of connected sets as follows :

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of connected subsets of a space (X, τ) and $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is connected set.

We prove previously, if the product space $X\times Y$ is connected space, then each X and Y is connected space and we question the converse is true (i.e., Let (X, τ) and (Y, τ') be a topological spaces. If each X and Y is connected space, then the product space $X\times Y$ is connected space) and the answer is true and we postpone the prove for this problem until the availability of basic prove from previous theorem as follows :

Theorem : Let (X, τ) and (Y, τ') be two topological spaces. If each X and Y are connected space, then the product space $X\times Y$ is a connected space.

Proof :

Let (X, τ) and (Y, τ') connected spaces, to prove $(X\times Y, \tau_{X\times Y})$ connected

$\because X \cong X\times\{y\}$; $y \in Y$ and X connected $\Rightarrow X\times\{y\}$ connected

(connected is topological property)

$\because Y \cong \{x\}\times Y$; $x \in X$ and Y connected $\Rightarrow \{x\}\times Y$ connected

(connected is topological property)

Fixed $y_0 \in Y \Rightarrow X\times\{y_0\}$ connected (this is true $\forall\, y \in Y$)

Clear, $X\times\{y_0\} \bigcap \{x\}\times Y \neq \phi$ (since $(x, y_0) \in X\times\{y_0\} \bigcap \{x\}\times Y$)

$\Rightarrow X\times\{y_0\} \bigcap \{x\}\times Y$ connected $\forall\, x \in X$

(by previous theorem: If A and B are connected and $A\bigcap B\neq\phi$, then $A\bigcup B$ is connected)

The family $\{X\times\{y_0\} \bigcap \{x\}\times Y\}_{x\in X}$ of connected sets in $X\times Y$

Clear that $X\times Y = \bigcup_{x\in X} (X\times\{y_0\} \bigcap \{x\}\times Y)$ and $\bigcap_{x\in X} (X\times\{y_0\} \bigcap \{x\}\times Y) \neq \phi$

$\therefore$ $X\times Y$ connected

(by previous theorem since it's a union of family of intersection connected sets)

Remark : Take the following example to show the Cartesian product which are $X\times\{y\}$ and $\{x\}\times Y$ and the union and intersection in the previous theorem :

Example : Let X = {1, 2, 3} and Y = {a, b, c}, then

$X\times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$ and,

$X\times\{a\} = \{(1, a), (2, a), (3, a)\}$

$X\times\{b\} = \{(1, b), (2, b), (3, b)\}$

$X\times\{c\} = \{(1, c), (2, c), (3, c)\}$ also,

$\{1\}\times Y = \{(1, a), (1, b), (1, c)\}$

$\{2\}\times Y = \{(2, a), (2, b), (2, c)\}$

$\{3\}\times Y = \{(3, a), (3, b), (3, c)\}$

Clear that the intersection any two sets her is nonempty and

$X\times Y = \bigcup_{x\in X} (X\times\{a\} \bigcap \{x\}\times Y)$.

Definition : Component of x

Let (X, τ) be a topology space and $x \in X$. We say the set which is a union of every connected sets that contains x is a **component of x** and denoted by C(x). i.e.,

$$C(x) = \bigcup\{A \subseteq X : x \in A \ \wedge\ A \text{ is connected}\}$$

This means that C(x) is the large connected set contains x.

Example : Let X = {1, 2, 3} and $\tau = \{X, \phi, \{3\}, \{1, 2\}\}$. Compute the component for every element in the space (X, τ).

Solution :

C(1) = {1, 2}, since {1, 2} is a large connected set contains 1.

Notes that {1, 3} contains 1 too, but it's not connected since the induce topology on {1, 3} is $\tau_{\{1, 3\}} = \{\{1, 3\}, \phi, \{3\}, \{1\}\}$, then $\{1, 3\} = \{1\} \bigcup \{3\}$ this means it's not connected.
On the other hand {1} is connected set (since every singleton set in any topology is connected set) and its contains 1 but not largest set.
By similar way : C(2) = {1, 2} while C(3) = {3} since any set contains 3 except {3} is not connected.

Example : Let X = {a, b, c} and $\tau = \{X, \phi, \{a\}\}$. Compute the component for every element in the space (X, τ).
Solution :
C(a) = C(b) = C(c) = X, since X is connected space and its large connected set contains any element in this space.

Remark : (X, τ) is connected space iff C(x) = X for all $x \in X$.
If X is connected, then X is large connected set for every element in X and C(x) = X.
On the other hand, if C(x) = X for every element $x \in X$, then X is connected set since C(x) is connected set (by definition).

Remarks :
[1] The space $(\mathbb{R}, \tau_u)$ is connected space, then $C(x) = \mathbb{R}$ for all $x \in \mathbb{R}$.
[2] In the space (X, D), if X contains more than one element, then C(x) = {x} for all $x \in X$, since the only unique sets her is singleton sets.
[3] The space (X, I) is connected space, then C(x) = X for all $x \in X$.
[4] The space (X, τ_{cof}); X infinite set is connected space, then C(x)= X for all $x \in X$.

Theorem : The component C(x) for every element x is closed set.
Proof :
$C(x) \subseteq \overline{C(x)}$ (since $A \subseteq \overline{A}$)
$\because$ C(x) is connected set $\Rightarrow$ $\overline{C(x)}$ is connected set
(by theorem : If $A \subseteq B \subseteq \overline{A}$ and A connected, then B connected and so $\overline{A}$ connected)
$\because$ C(x) the large connected set that contains x and $\overline{C(x)}$ connected ; $C(x) \subseteq \overline{C(x)}$,
So $C(x) = \overline{C(x)}$ $\Rightarrow$ C(x) is closed set (by theorem : A closed $\Leftrightarrow$ $A = \overline{A}$)

Remark : $C(x) \neq \phi$ for all $x \in X$, since $x \in C(x)$.

Theorem : If $C(x) \bigcap C(y) \neq \phi$, then C(x) = C(y).

Proof :

$\because$ $C(x) \bigcap C(y) \neq \phi$ and $C(x)$, $C(y)$ the connected sets $\Rightarrow C(x) \bigcup C(y)$ is connected set (by theorem : Union of connected sets is connected if their intersection nonempty)

We get : $C(x) \bigcup C(y)$ connected set ; $C(x) \subseteq C(x) \bigcup C(y)$ and $C(y) \subseteq C(x) \bigcup C(y)$

Since $C(x)$ and $C(y)$ are the largest connected sets ; $x \in C(x)$ and $y \in C(y)$

$\Rightarrow C(x) \bigcup C(y) = C(x) = C(y)$.

Remark : Family of components elements in the space (X, τ) being a partition for X.

(1) $C(x) \neq \phi \ \ \forall x \in X$.

(2) if $C(x) \neq C(y)$, then $C(x) \bigcap C(y) = \phi$ (by previous theorem)

(3) $X = \bigcup_{x\in X} C(x)$ i.e., $\bigcup_{x\in X} C(x) \subseteq X$ and $X \subseteq \bigcup_{x\in X} C(x)$

since $C(x) \subseteq X \ \ \forall x \ \Rightarrow \ \bigcup_{x\in X} C(x) \subseteq X$

since $\forall x \in X \Rightarrow x \in C(x)$

$\Rightarrow x \in \bigcup_{x\in X} C(x)$

So, $X \subseteq \bigcup_{x\in X} C(x)$

Example : Let $X = \{1, 2, 3\}$ and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$.

Her $C(1) = \{1\}$ and $C(2) = C(3) = \{2, 3\}$

Notes that $C(1) \neq \phi$, $C(2) \neq \phi$ and $C(3) \neq \phi$

Also, $C(1) \neq C(2) \Rightarrow C(1) \bigcap C(2) = \phi$

and $C(1) \neq C(3) \Rightarrow C(1) \bigcap C(3) = \phi$

on the other hand, the union of components equal X i.e., $X = C(1) \bigcup C(2) \bigcup C(3)$ and this clear since $C(1) = \{1\}$ and $C(2) = C(3) = \{2, 3\}$, then $X = \{1\} \bigcup \{2, 3\}$.

Definition : Locally Connected

The space (X, τ) is **locally connected at a point** $x \in X$ iff there exists an open connected of a point x. If (X, τ) is locally connected at each point $x \in X$, then X is called a **locally connected space.** i.e.,

X is locally connected $\Leftrightarrow \ \forall x \in X \ \exists U \in \tau$; $x \in U$ and U is connected.

Remark : There is no relation between the concepts connected and locally connected i.e., connected and locally connected are independent concepts and we show that by the following example :

Example : (A space that is locally connected but not connected)

Let $X = (-3, 0) \bigcup (3, 8)$

X is a subspace of $(\mathbb{R}, \tau_u)$: Clear that X is not connected since it's a union of nonempty disjoint open intervals. On the other hand X is locally connected space since every element in X either in (– 3, 0) and its connected intervals and it's a connected open neighhood for every element in (– 3, 0) and by similar way if the element contain in (3, 8).

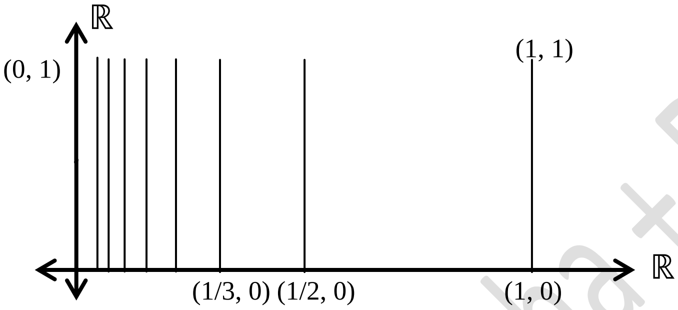
i.e., Locally Connected $\not\Rightarrow$ Connected.

<u>Example :</u> Comb Space (A space that is connected but not locally connected)

$X = A \bigcup B$; $A = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant y \leqslant 1 \ , x = 0 \ \text{ or } \ x = \frac{1}{n} \ , n \in \mathbb{N}\}$

$B = \{(x, 0) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 1\}$

The following figure show the comb space :

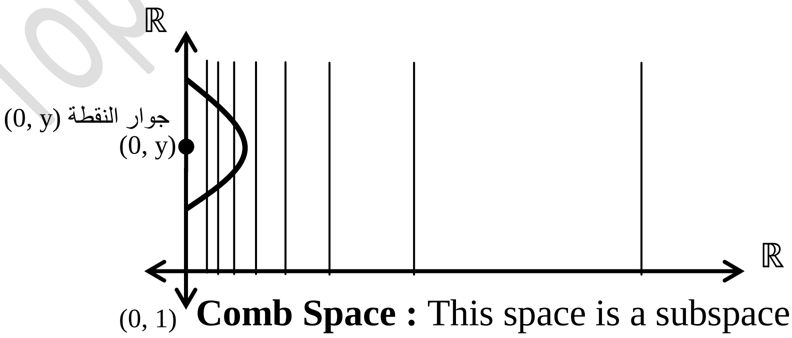


Comb Space : This space is a subspace of $(\mathbb{R}^2, \tau_u)$

Notes that this space consist of infinite sets of line segment length every one 1 and when converge to y-axis , then this line segment is adherent to each other i.e., the distant between them is reduced when they converge from Y.

Notes that the shape this space similarity the comb such that the line segment on x-axis represented the base of the comb while the vertical line represented teeth comb.

We know this space is **<u>connected space</u>** since its consist of one piece, but it's **<u>not locally connected</u>** and in true its not locally connected only on every point on y-axis since every neighhood for any point (0, y) on y-axis consist of sets of disjoint line segment to each other, so that the open neighhood is not connected i.e., we cannot find connected open neighhood for any point from type (0, y) ; $y \neq 0$ and $0 < y \leqslant 1$ and this show by the following figure which is intersection open ball in $\mathbb{R}^2$ with the space X.



Comb Space : This space is a subspace of $(\mathbb{R}^2, \tau_u)$

Connected $\not\Rightarrow$ Locally Connected

Remark : The property of being a locally connected space is not a hereditary property and the following example show that :

Example : In the above example, $(\mathbb{R}^2, \tau_u)$ is locally connected space while the **comb space W** is a sub space of $(\mathbb{R}^2, \tau_u)$ which is not locally connected space.

Example : the space (X, D) is locally connected space since every element x in X, then {x} is a connected open neighhood.

Remark : continuous image of locally connected space is not necessary locally connected and the following example show that :

Example : Take comb space W in previous example and take difference topologies one of them discrete topology D and the other is the induce topology τ_W from $(\mathbb{R}^2, \tau_u)$ and define the function f as follows :

$$f : (W, D) \to (W, \tau_w) \;\; ; \;\; f(x) = x \;\; \forall \;\; x \in W$$

clear that f is continuous since its domain D and its onto since its identity function. Also, we know (W, D) is locally connected space while (W, τ_W) is not locally connected space (we clear that in illustrate previous).

Example : the space (X, I) is locally connected space since the only unique for every element x in X is itself X and X her is a connected set.

Example : $(\mathbb{R}^2, \tau_u)$ is locally connected space since there is always open internals corresponding every real number contains it and since every open interval is connected and it's aa open neighthood for a point, then $(\mathbb{R}^2, \tau_u)$ is locally connected space.

Example : (X, τ_{cof}) is locally connected space since every open set in this space is connected set, therefore there is an open neighthood for every element contains it.

Example : Let X = {1, 2, 3} and $\tau = \{X, \phi, \{1\}, \{2, 3\}\}$ such that τ is a topology on X. Is (X, τ) connected space ?? locally connected space ??

Solution :

(X, τ) is not connected (disconnected) space since $X = \{1\} \bigcup \{2, 3\}$ such that {1}, {2, 3} are nonempty disjoint open sets.

(X, τ) is locally connected space since every element have connected open neighthood ; $1 \in \{1\} \in \tau$ and $\{1\}$ is connected set, also $2, 3 \in \{2, 3\} \in \tau$ and $\{2, 3\}$ is connected set.

Theorem : If f $f : (X, \tau) \to (Y, \tau')$ is onto, continuous and open function and X is locally connected, then Y is locally connected.

Proof :

Let $y \in Y \Rightarrow \exists\, x \in X$; $f(x) = y$ (since f is onto)

$\because$ X is locally connected $\Rightarrow \exists$ connected open nbd for x

i.e., $\exists\, U \in \tau$; $x \in U \wedge U$ is connected

$\because$ f is continuous $\Rightarrow f(U)$ is connected

(by theorem : continuous image of connected space is connected)

$\because$ f is open $\Rightarrow f(U)$ is open i.e., $f(U) \in \tau'$ also $y \in f(U)$

We get, $f(U)$ is connected open nbd for y $\Rightarrow$ Y is locally connected.

Corollary : The property of being a locally connected space is a topological property.

Proof :

Let (X, τ) and (Y, τ') be topological spaces ; $X \cong Y$

$\because X \cong Y \Rightarrow \exists\, f : (X, \tau) \to (Y, \tau')$; f 1-1, f onto, f continuous, f^{-1} continuous

Suppose that X is locally connected, to prove Y is locally connected

$\because$ f onto, continuous, open and X locally connected $\Rightarrow f(X) = Y$ locally connected

(by previous theorem)

Now, suppose that Y is locally connected, to prove X is locally connected

$\because f^{-1}$ onto, continuous, open and Y locally connected $\Rightarrow f^{-1}(Y) = X$ locally connected

(by previous theorem)

Remark : Let (X, τ) and (Y, τ') be a topological spaces. If the product space $X \times Y$ is locally connected space then each X and Y is locally connected space.