Functional Analysis

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Chapter 1

Linear Space

Definition 1.1. ***Linear Space***

Let $(F, +, .)$ be a field whose elements are called **scalars**. Let L is a non empty set whose elements are called **vectors**.Then L is a **linear space** (or a **vector space**) over the field F, if

(1) **addition:** There is a binary operation $+$ on L called **addition** (not usual addition) such that $(L, +)$ is a commutative group.

(2) **scalar multiplication:** $\alpha.x \in L \quad \forall x \in L, \ \forall \alpha \in F$.

(3) The scalar multiplication and addition satisfy

  (i) $\alpha.(x+y) = \alpha.x + \alpha.y \quad \forall x, y \in L, \ \forall \alpha \in F$

  (ii) $(\alpha+\beta).x = \alpha.x + \beta.x \quad \forall x \in L, \ \forall \alpha, \beta \in F$

  (iii) $(\alpha.\beta).x = \alpha.(\beta.x) \quad \forall x \in L, \ \forall \alpha, \beta \in F$

  (iv) $1.x = x \quad \forall x \in L$ and 1 is the unity F

Remark 1.2.

When L is a linear space over F, we say that $L(F)$ is a linear space. We also can say L is a linear space.

Examples of Linear Space

Example 1.3.

Let $(\mathbb{R}, +, .)$ be the field of real numbers. Let $\mathbb{R}^n = \{(x_1, ..., x_n) : x_1, ..., x_n \in \mathbb{R}\}$. For any two elements $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$ of $\mathbb{R}^n$, define

$$X + Y = (x_1 + y_1, ..., x_n + y_n).$$

Also, define scalar multiplication in $\mathbb{R}^n$ over $\mathbb{R}$ by

$$\alpha.X = (\alpha.x_1, ..., \alpha.x_n) \quad \forall \alpha \in \mathbb{R},\ \forall X \in \mathbb{R}^n.$$

Show that $\mathbb{R}^n$ is a linear space over $\mathbb{R}$.

Solution: Let us check linear space conditions

(1) We show $(\mathbb{R}^n, +)$ is a commutative group

(a) Let $X = (x_1, ..., x_n), Y = (y_1, ..., y_n \in \mathbb{R}^n)$. Since $x_1 + y_1, ..., x_n + y_n \in \mathbb{R}$, then $X + Y \in \mathbb{R}^n$. Hence, $\mathbb{R}^n$ is closed with respect to usual addition.

(b) For all $X = (x_1, ..., x_n), Y = (y_1, ..., y_n), Z = (z_1, ..., z_n) \in \mathbb{R}^n$

$$\begin{aligned} X + (Y + Z) &= (x_1, ..., x_n) + [(y_1, ..., y_n) + (z_1, ..., z_n)] \\ &= (x_1, ..., x_n) + (y_1 + z_1, ..., y_n + z_n) \\ &= \big(x_1 + (y_1 + z_1), ..., x_n + (y_n + z_n)\big) \\ &= \big((x_1 + y_1) + z_1, ..., (x_n + y_n) + z_n\big) \\ &= (x_1 + y_1, ..., x_n + y_n) + (z_1, ..., z_n) = (X + Y) + Z. \end{aligned}$$

(c) For all $X = (x_1, ..., x_n), Y = (y_1, ..., y_n) \in \mathbb{R}^n$

$$X+Y = (x_1, ..., x_n)+(y_1, ..., y_n) = (x_1+y_1, ..., x_n+y_n) = (y_1+x_1, ..., y_n+x_n) = Y+X.$$

(d) For all $X = (x_1, ..., x_n) \in \mathbb{R}^n$ we have $(0, ..., 0) \in \mathbb{R}^n$ such that $(x_1, ..., x_n) + (0, ...0) = (x_1, ..., x_n)$. Thus, $(0, ..., 0)$ is the additive identity.

(e) If $X = (x_1, ..., x_n) \in \mathbb{R}^n$ then $-X = (-x_1, ..., -x_n) \in \mathbb{R}^n$ such that $X + (-X) = (0, ..., 0)$. Thus, $-X$ is the additive inverse of X.

From (a)-(e) we get $(R^n, +)$ is a commutative group.

(2) Let $X = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Since $\alpha x_1, ..., \alpha x_n \in \mathbb{R}$, then

$$\alpha.X = (\alpha.x_1, ..., \alpha.x_n) \in \mathbb{R}^n.$$

Hence, $\mathbb{R}^n$ is closed with respect to scalar multiplication.

(3) The scalar multiplication and addition satisfy

(i) If $X = (x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}
\alpha.(X + Y) &= \alpha.(x_1 + y_1, ..., x_n + y_n) \\
&= (\alpha.(x_1 + y_1), ..., \alpha.(x_n + y_n)) \\
&= (\alpha.x_1 + \alpha.y_1, ..., \alpha.x_n + \alpha.y_n) \\
&= (\alpha.x_1, ..., \alpha.x_n) + (\alpha.y_1, ..., \alpha.y_n) \\
&= \alpha.(x_1, ..., x_n) + \alpha.(y_1, ..., y_n) = \alpha.X + \alpha.Y
\end{aligned}$$

(ii) If $X = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned}
(\alpha + \beta).X &= \big((\alpha + \beta).x_1, ..., (\alpha + \beta).x_n\big) \\
&= \big(\alpha.x_1 + \beta.x_1, ..., \alpha.x_n + \beta.x_n\big) \\
&= (\alpha.x_1, ..., \alpha.x_n) + (\beta.x_1, ..., \beta.x_n) \\
&= \alpha.(x_1, ..., x_n) + \beta.(x_1, ..., x_n) = \alpha.X + \beta.Y
\end{aligned}$$

(iii) If $X = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned}
(\alpha.\beta).X &= \big((\alpha.\beta).x_1, ..., (\alpha.\beta)x_n\big) \\
&= \alpha.(\beta.x_1, ..., \beta.x_n) = \alpha.\big(\beta.(x_1, ..., x_n)\big) = \alpha.(\beta.X)
\end{aligned}$$

(iv) If $X = (x_1, ..., x_n) \in \mathbb{R}^n$ and 1 is the unity of $\mathbb{R}$, then

$$1.X = (1.x_1, ..., 1.x_n) = (x_1, ..., x_n) = X$$

Hence $\mathbb{R}^n$ is a linear (vector)space over $\mathbb{R}$.

Example 1.4.

Let $(C, +, .)$ be the field of complex numbers. Let $C^n = \{(x_1, ..., x_n) : x_1, ..., x_n \in C\}$. For any two elements $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$ of C^n, define

$$X + Y = (x_1 + y_1, ..., x_n + y_n).$$

Define scalar multiplication in C^n over C by

$$\alpha.X = (\alpha.x_1, ..., \alpha.x_n) \quad \forall \alpha \in C,\ \forall X \in C^n.$$

Show that C^n is a vector space over C. (**Verify that**)

Example 1.5.

The set of real numbers $\mathbb{R}$, with linear operations **ordinary** addition and **ordinary** multiplication, is a **real** linear space (i.e., a linear space over $\mathbb{R}$). Indeed,

(1) $(\mathbb{R}, +)$ is an abelian group

(2) $\alpha.x \in \mathbb{R} \quad \forall x \in \mathbb{R}, \alpha \in \mathbb{R}$

(3) All other conditions are satisfied (Check!)

Example 1.6.

The set of complex numbers C, with linear operations **ordinary** addition and **ordinary** multiplication, is a **complex** linear space (i.e., a linear space over C). Indeed,

(1) $(C, +)$ is an abelian group

(2) $\alpha.x \in C \quad \forall x \in C, \alpha \in C$

(3) All other conditions are satisfied (Check!)

Example 1.7.

Let $C^b(\mathbb{R}) = \{f : f : \mathbb{R} \to \mathbb{R} \ \ f \text{ is bounded and continuous}\}$ set of all bounded continuous functions defined on $\mathbb{R}$. For any $f, g \in C^b(\mathbb{R})$ and for any $\alpha \in \mathbb{R}$, define

$(f + g)(x) = f(x) + g(x) \ \ \forall x \in \mathbb{R}$ and $(\alpha f)(x) = \alpha.f(x) \ \ \forall x \in \mathbb{R}$.

Show that $C^b(\mathbb{R})$ is a linear space over $\mathbb{R}$.

Now, let us check linear space conditions.

(1) We show $(C^b(\mathbb{R}), +)$ is a commutative group

(a) Let $f, g \in C^b(\mathbb{R}))$ such that f, g are continuous and bounded functions. We want to prove $f + g \in C^b(\mathbb{R})$. (i.e., $f + g$ is continuous and bounded)

Since f, g are continuous, the sum $(f + g)$ is a continuous function **(I)**

Also, since f, g are bounded functions, $\exists M_1, M_2 \in \mathbb{R}_+$ such that $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$. Hence, for all $x \in \mathbb{R}$

$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2.$

$|(f + g)(x)| \leq M_1 + M_2$. Thus, $f + g$ is bounded function **(II)**

By **(I)** and **(II)**, $f + g \in C^b(\mathbb{R})$.

(b) For all $f, g, h \in C^b(\mathbb{R})$ and for all $x \in \mathbb{R}$

$$\begin{aligned}[f + (g + h)](x) &= f(x) + [(g + h)(x)] \\ &= [f(x) + g(x)] + h(x) \\ &= (f + g)(x) + h(x) = [(f + g) + h](x).\end{aligned}$$

(c) For all $f, g \in C^b(\mathbb{R})$

$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$

(d) For all $f \in C^b(\mathbb{R})$, define $\hat{0} : \mathbb{R} \to 0$ by $\hat{0}(x) = 0$.

It is clear that $\hat{0}$ is continuous and bounded function. Thus, $\hat{0} \in C^b(\mathbb{R})$ and $(f + \hat{0})(x) = f(x) + \hat{0}(x) = f(x) + 0 = f(x)$. Thus, $f + \hat{0} = f$

$\hat{0}$ is called the additive identity.

(e) For any $f \in C^b(\mathbb{R})$, define $-f : \mathbb{R} \to \mathbb{R}$ by $(-f)(x) = -[f(x)] \;\; \forall x \in \mathbb{R}$.

Since f is continuous, then $-f$ is continuous.

Moreover, $\forall x \in \mathbb{R}, |-f(x)| = |f(x)| \leq M$. Then, $-f$ is bounded.

Thus, $-f \in C^b(\mathbb{R})$ and

$[f + (-f)](x) = f(x) + (-f)(x) = f(x) + (-f(x)) = f(x) - f(x) = 0 = \hat{0}.$

From (a)-(e) we get $(C^b(\mathbb{R}), +)$ is a commutative group.

(2) Let $f \in C^b(\mathbb{R})$ and $\alpha \in \mathbb{R}$. We want to prove $\alpha f \in C^b(\mathbb{R})$. (i.e., αf is continuous and bounded)

Since f is continuous, then αf is a continuous function.

Also, since f is bounded functions, $\exists M \in \mathbb{R}_+$ such that $|f(x)| \leq M$. Hence, for all $x \in \mathbb{R}$

$|(\alpha f)(x)| = |\alpha . f(x)| = |\alpha| \, |f(x)| \leq |\alpha| \, M.$

Thus, αf is bounded function. Therefore, $\alpha f \in C^b(\mathbb{R})$ ($C^b(\mathbb{R})$ is closed with respect to scalar multiplication).

(3) The scalar multiplication and addition satisfy

(i) If $f, g \in C^b(\mathbb{R})$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}\alpha.(f+g)(x) &= \alpha.[(f(x)+g(x)] \\ &= \alpha.f(x)+\alpha.g(x) \\ &= (\alpha f)(x)+(\alpha g)(x) = (\alpha f+\alpha g)(x)\end{aligned}$$

(ii) If $f \in C^b(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned}[(\alpha+\beta)f](x) &= (\alpha+\beta).f(x) \\ &= \alpha.f(x)+\beta.f(x) \\ &= (\alpha f)(x)+(\beta f)(x) = (\alpha f+\beta f)(x)\end{aligned}$$

(iii) If $f \in C^b(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

$[(\alpha.\beta)f](x) = (\alpha.\beta)f(x) = \alpha.(\beta.f(x)) = \alpha.[(\beta f)(x)] = [\alpha(\beta f)](x)$.

Hence, $(\alpha.\beta)f = \alpha(\beta f)$.

(iv) If $f \in C^b(\mathbb{R})$ and 1 is the unity of $\mathbb{R}$, then

$(1f)(x) = 1.f(x) = f(x)$.

Hence, $C^b(\mathbb{R})$ is a linear (vector)space over $\mathbb{R}$.

Exercise 1.8.

(1) Let $C^b[a,b] = \{f : f : [a,b] \to \mathbb{R} \;\; f \text{ is bounded and continuous}\}$ set of all bounded continuous functions defined on $[0,1]$. Show that $C^b[a,b]$ is a linear space over $\mathbb{R}$ where $f+g$ and αf are defined in the same way as in Example 1.7. (**H.W.**)

(2) Let L be the set of all real valued sequences $\langle x_n \rangle$. Define usual addition and multiplication of a sequence as follows: for any $\langle x_n \rangle, \langle y_n \rangle \in L$ and each $\alpha \in \mathbb{R}$

$\langle x_n \rangle + \langle y_n \rangle = \langle x_n + y_n \rangle$ and $\alpha.\langle x_n \rangle = \langle \alpha.x_n \rangle$. Show that $\langle x_n \rangle$ is a linear space over $\mathbb{R}$. (**H.W.**)

Theorem 1.9. ***Properties of Linear Space***

Let $L(F)$ be a linear space and $\mathbf{0}_L$ is a zero vector of L. Then

(1) $\alpha.\mathbf{0}_L = \mathbf{0}_L \ \ \forall \alpha \in F$.

(2) $0.x = \mathbf{0}_L \ \ \forall x \in L$.

(3) $\alpha.(-x) = -(\alpha.x) \ \ \forall x \in L, \ \ \forall \alpha \in F$.

(4) $(-\alpha).x = -(\alpha.x) \ \ \forall x \in L, \ \ \forall \alpha \in F$.

(5) $\alpha.(x - y) = \alpha.x - \alpha.y \ \ \forall x, y \in L, \ \ \forall \alpha \in F$.

(6) If $\alpha.x = 0$ then $\alpha = 0$ or $x = \mathbf{0}_L$.

Definition 1.10. ***Linear Subspace***

Let L be a linear space over a field F and let H be a non empty subset of L. Then H is called a linear subspace of L if H itself is a linear space over F.

Theorem 1.11.

Let H be a non empty subset of a linear space $L(F)$. H is called a subspace of L if and only if $\alpha x + \beta y \in H$ for all $x, y \in H$ and for all $\alpha, \beta \in F$.

Examples of Linear Subspace

Example 1.12.

(1) Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$.

(i) $H_1 = \{(0, x_2, x_3) : x_2, x_3 \in \mathbb{R}\}$

(ii) $H_2 = \{(0, 0, x_3) : x_3 \in \mathbb{R}\}$

(iii) $H_3 = \{(x_1, x_2, x_3) : x_1 + 2x_2 = 1\}$

(2) Which of the following subsets of $C^b[-1, 1]$ are subspaces of $C^b[-1, 1]$.

(i) $H_1 = \{f : f(0) = 0\}$

(ii) $H_2 = \{f : f(x) \leq 0, \forall x \in [-1, 1]\}$

(iii) $H_3 = \{f : f(0) = 1\}$

Definition 1.13. ***Linear Transformation***

Let $L(F)$ and $L'(F)$ be two linear spaces over the same field F. A mapping $T : L \to L'$ is called a **linear Operator** or **Linear transformation** if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall x, y \in L, \ \forall \alpha, \beta \in F$.

Example 1.14.

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1, x_2) \quad \forall x_1, x_2, x_3 \in \mathbb{R}$. Show that T is a linear transformation.

Solution: Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}
T(\alpha x + \beta y) &= T[\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3)] \\
&= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) \\
&= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \\
&= (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2) \\
&= \alpha(x_1, x_2) + \beta(y_1, y_2) \\
&= \alpha T(x_1, x_2, x_3) + \beta T(y_1, y_2, y_3).
\end{aligned}$$

Exercise 1.15.

Show that each of the following mappings $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation

(i) $T_1(x_1, x_2) = (ax_1, ax_2)$ where $a \in \mathbb{R}$

(ii) $T_2(x_1, x_2) = (x_2, x_1)$

(iii) $T_3(x_1, x_2) = (0, x_2)$

Theorem 1.16.

Let $T : L(F) \to L'(F)$ be a linear transformation. Then

(i) $T(0_L) = 0_{L'}$ where 0_L is the zero vector of L and $0_{L'}$ is the zero vector of L'

(ii) $T(-x) = -T(x)$

(iii) $T(x - y) = T(x) - T(y)$

Theorem 1.17.

Let L, L' be linear spaces over same field F. Let $T_1, T_2 : L \to L'$ linear transformations.

Define the function $T_1 + T_2 : L \to L'$ as $(T_1 + T_2)(x) = T_1(x) + T_2(x) \quad \forall x \in L$

If $\alpha \in F$, then the function $\alpha T_1 : L \to L'$ is defined as $(\alpha T_1)(x) = \alpha . T_1(x) \quad \forall x \in L$. Then

(i) Show that $T_1 + T_2$ is a linear transformation.

(ii) Show that αT_1 is a linear transformation.

Proof. (i) Let $\alpha, \beta \in F$ and $x, y \in L$. Then

$$\begin{aligned}
(T_1 + T_2)(\alpha x + \beta y) &= T_1(\alpha x + \beta y) + T_2(\alpha x + \beta y) && \text{(Definition of +)}\\
&= \alpha T_1(x) + \beta T_1(y) + \alpha T_2(x) + \beta T_2(y) && \text{(since } T_1, T_2 \text{ linear trans.)}\\
&= \alpha(T_1(x) + T_2(x)) + \beta(T_1(y) + T_2(y))\\
&= \alpha(T_1 + T_2)(x) + \beta(T_1 + T_2)(y).
\end{aligned}$$

Thus, $T_1 + T_2$ is a linear transformation.

(ii) Let $\beta_1, \beta_2 \in F$ and $x, y \in L$. Then

$$\begin{aligned}
(\alpha T_1)(\beta_1 x + \beta_2 y) &= \alpha . T_1(\beta_1 x + \beta_2 y) && \text{(Definition of scalar multiplication)}\\
&= \alpha . \left[\beta_1 . T_1(x) + \beta_2 . T_1(x)\right] && \text{(since } T_1 \text{ linear trans.)}\\
&= \alpha . \beta_1 . T_1(x) + \alpha . \beta_2 . T_1(x)\\
&= \beta_1 . (\alpha T_1)(x) + \beta_2 . (\alpha T_1)(x)
\end{aligned}$$

Thus, αT_1 is a linear transformation. □

Definition 1.18.

Let L be a linear space. A linear transformation $T : L \to F$ is said to be **linear functional**. (note that F can be regarded as a linear space over F).

Example 1.19.

Let $L = F^n = \{(x_1, ..., x_n) : x_1, ..., x_n \in F\}$ be a linear space over the field F. Let $T : F^n \to F$ defined by $T(x_1, ..., x_n) = \alpha_1 x_1 + ... + \alpha_n x_n \;\; \forall (x_1, ..., x_n) \in F^n$. Prove that T is a linear transformation.

Solution: Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in F^n$ and $\alpha, \beta \in F$. Then

$T(\alpha x + \beta y) = T[\alpha(x_1, ..., x_n) + \beta(y_1, ..., y_n)]$

$$= T(\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n)$$

$$= \alpha_1(\alpha x_1 + \beta y_1) + ... + \alpha_n(\alpha x_n + \beta y_n)$$

$$= \alpha(\alpha_1 x_1 + ... + \alpha_n x_n) + \beta(\alpha_1 y_1 + ... + \alpha_n y_n)$$

$$= \alpha T(x_1, ..., x_n) + \beta T(y_1, ..., y_n).$$

Thus, T is a linear transformation (i.e., linear functional).

Chapter 2

Normed Linear Space

Definition 2.1. ***Normed Linear Space***

Let $L(F)$ be a linear space over a field F. A mapping $\| \ \| : L \to \mathbb{R}$ is called **norm** if the following conditions hold

(1) $\|x\| \geq 0 \quad \forall x \in L$.

(2) $\|x\| = 0$ if and only if $x = 0$.

(3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in L$.

(4) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall x \in L, \forall \alpha \in F$.

$(L, \| \ \|)$ is called **normed linear space**.

Remark 2.2.

In this chapter the field F is either $\mathbb{R}$ or C.

Theorem 2.3.

Let $(L, \| \ \|)$ be a normed linear space. Then

(1) $\|\mathbf{0_L}\| = 0 \quad \forall x \in L$.

(2) $\|x\| = \|-x\| \quad \forall x \in L$.

(3) $\|x - y\| = \|y - x\| \quad \forall x, y \in L$.

(4) $| \ \|x\| - \|y\| \ | \leq \|x - y\| \quad \forall x, y \in L$.

(5) $|\ \|x\| - \|y\|\ | \leq \|x+y\| \quad \forall x, y \in L.$

(6) Every subspace of a normed space is itself normed space with respect to the same norm. (**H.W.**)

Proof. (1) $\|\mathbf{0_L}\| = \|0\mathbf{0_L}\|$ (see Theorem 1.9(1))

$= 0\,\|\mathbf{0_L}\| = 0.$

(2) $\|-x\| = |-1|\,\|x\| = \|x\| \quad \forall x \in L.$

(3) $\|x-y\| = \|-(y-x)\| = \|y-x\|$ (by part (2)).

(4)We must prove $-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|$

$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$ (by Definition 2.1(3)).

Hence, $\|x\| - \|y\| \leq \|x-y\|$ (I)

Similarly, $\|y\| = \|y-x+x\| \leq \|y-x\| + \|x\|$ (by Definition 2.1(3)).

Hence, $\|y\| - \|x\| \leq \|x-y\|$ (II)

Hence, by (I) and (II), we get $\|x-y\| \geq |\|x\| - \|y\|| \quad \forall x, y \in L.$

(5) We must prove $-\|x+y\| \leq \|x\| - \|y\| \leq \|x+y\|$

$\|x\| = \|x+y-y\| \leq \|x+y\| + \|-y\|$ (by Definition 2.1(3)).

Hence, $\|x\| - \|y\| \leq \|x+y\|$ (III)

Similarly, $\|y\| = \|y+x-x\| \leq \|y+x\| + \|-x\|$ (by Definition 2.1(3)).

Hence, $\|y\| - \|x\| \leq \|x+y\|$

$\|x\| - \|y\| \geq -\|x+y\|$ (IV)

Hence, by (III) and (IV), we get $-\|x+y\| \leq \|x\| - \|y\| \leq \|x+y\| \quad \forall x, y \in L.$

□

Examples of Normed Linear Space

Example 2.4.

Let $\mathbb{R}$ be a linear space over $\mathbb{R}$ with $\|\ \| : \mathbb{R} \to \mathbb{R}$ such that $\|x\| = |x|$. Show that $(\mathbb{R}, \|\ \|)$ is a normed space.

Solution: We show that

(1) $\|x\| = |x| \geq 0 \quad \forall x \in \mathbb{R}$; hence $\|x\| \geq 0$.

(2) Let $x \in \mathbb{R}$, $\|x\| = 0 \iff |x| = 0 \iff x = 0.$

(3) $\forall x \in \mathbb{R}, \forall \alpha \in \mathbb{R}$,

$\|\alpha x\| = |\alpha x| = |\alpha|\,|x| = |\alpha|\,\|x\|$.

(4) $\|x+y\| = |x+y| \leq |x| + |y| = \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}$.

Example 2.5.

Let C be a complex linear space over C with $\|\ \| : C \to \mathbb{R}$ such that $\|z\| = |z| = \sqrt{a^2+b^2} \quad \forall z = a+ib$. Show that $(C, \|\ \|)$ is a normed space.

Solution: We show that

(1) $\|z\| = |z| = \sqrt{a^2+b^2} \geq 0 \quad \forall z = a+ib \in C$; hence $\|z\| \geq 0$.

(2) Let $z = a+ib \in C$

$\|z\| = |z| = \sqrt{a^2+b^2} = 0 \iff a = b = 0 \iff z = 0 + 0i = 0$.

(3) Let $z, w \in C$

$\|z+w\|^2 = (z+w)(\overline{z+w})$ where $\overline{z+w}$ =conjugate of $z+w$

$$\begin{aligned} &= (z+w)(\overline{z}+\overline{w}) = z\overline{z} + w\overline{w} + w\overline{z} + \overline{w}z \\ &= z\overline{z} + w\overline{w} + w\overline{z} + \overline{w\overline{z}} \\ &= z\overline{z} + w\overline{w} + 2Re\ w\overline{z} \\ &\leq \|z\|^2 + \|w\|^2 + 2\|w\|\,\|z\| = (\|z\| + \|w\|)^2. \end{aligned}$$

Thus, $\|z+w\|^2 \leq (\|z\| + \|w\|)^2$ and hence, $\|z+w\| \leq \|z\| + \|w\|$.

(4) Let $z \in C, \alpha \in C$,

$\|\alpha z\| = |\alpha z| = |\alpha(a+ib)|$

$$= \sqrt{(\alpha a)^2 + (\alpha b)^2} = \sqrt{\alpha^2(a^2+b^2)} = \sqrt{\alpha^2}\sqrt{a^2+b^2} = |\alpha|\,|z| = |\alpha|\,\|z\|.$$

As an application to Example 2.5: Let $z = 2+3i, w = 1-i$, then

$\|z+w\| = \|(2+1) + (3i-i)\| = \|3+2i\| = \sqrt{3^2+2^2} = \sqrt{13}$.

$\|5z\| = \|10+15i\| = \sqrt{10^2+15^2} = \sqrt{325} = 5\sqrt{13}$.

$5\|z\| = 5\sqrt{2^2+3^2} = 5\sqrt{13}$.

Example 2.6.

Show that the linear space $C^b(\mathbb{R})$ is a normed space under the norm

$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}, \quad \forall f \in C^b(\mathbb{R})$.

(1) Since $|f(x)| \geq 0 \ \ \forall x \in \mathbb{R}$. Then, $\|f\| = \sup |f(x)| \geq 0$. Hence, $\|f\| \geq 0$.

(2) $\|f\| = 0 \iff \sup\{|f(x)| : x \in \mathbb{R}\} = 0$

$$\iff |f(x)| = 0 \ \ \forall x \in \mathbb{R}$$

$$\iff f(x) = 0 \ \ \forall x \in \mathbb{R} \iff f = \hat{0} \text{ (zero mapping)}$$

(3) Let $f, g \in C^b(\mathbb{R})$. Then

$$\|f + g\| = \sup\{|f(x) + g(x)| : x \in \mathbb{R}\}$$

$$\leq \sup\{|f(x)| + |g(x)| : x \in \mathbb{R}\}$$

$$\leq \sup\{|f(x)| : x \in \mathbb{R}\} + \sup\{|g(x)| : x \in \mathbb{R}\} = \|f\| + \|g\|.$$

Hence, $\|f + g\| \leq \|f\| + \|g\|$.

(4) Let $f \in C^b(\mathbb{R}), \alpha \in \mathbb{R}$. Then

$$\|\alpha f\| = \sup\{|(\alpha f)(x)| : x \in \mathbb{R}\}$$

$$= \sup\{|\alpha| \, |f(x)| : x \in \mathbb{R}\}$$

$$= |\alpha| \sup\{|f(x)| : x \in \mathbb{R}\} \text{ (By Theorem 2.7 below where } A = |f(x)| \text{ and } \beta = |\alpha|)$$

$$= |\alpha| \, \|f\|.$$

Theorem 2.7.

If A is a bounded above set and $\beta > 0$, then βA is bounded above and $\sup(\beta A) = \beta \sup(A)$.

As an application to Example 2.6: Let $f, g \in C^b(\mathbb{R})$ such that $f(x) = sin(x)$ and $g(x) = 2cos(x) + 1$. Hence,

$\|f\| = \sup\{|sin(x)| : x \in \mathbb{R}\} = 1 \quad$ (since $|sin(x)| \leq 1, \ \ \forall x \in \mathbb{R}$).

$\|f\| = \sup\{|2cos(x) + 1| : x \in \mathbb{R}\}$.

But $|2cos(x) + 1| \leq 2\,|cos(x)| + 1 = 3$. So $\|g\| = 3$.

Example 2.8.

The linear space $C^b[0, 1]$ of all real valued continuous functions on $[0, 1]$ is a normed space under the norm defined in Example 2.6. (**H.W.**)

Example 2.9.

The linear space $C[0, 1]$ of all real valued continuous functions on $[0, 1]$ is a normed space with the norm defined as

$\|f\| = \int_0^1 |f(x)|\ dx\ \ \forall f \in C[0,1]$.

solution: (1) Since $|f(x)| \geq 0,\ \ \forall x \in [0,1]$, then $\int_0^1 |f(x)|\ dx \geq 0$. Thus, $\|f\| \geq 0$.

(2) $\|f\| = 0 \iff \int_0^1 |f(x)|\ dx = 0$

$$\iff |f(x)| = 0\ \ \forall x \in [0,1]$$

$$\iff f(x) = 0\ \ \forall x \in [0,1]$$

$$\iff f = \hat{0} \text{ (zero mapping).}$$

(3) Let $f, g \in C[0,1]$. Then

$\|f+g\| = \int_0^1 |f(x)+g(x)|$

$$\leq \int_0^1 (|f(x)| + |g(x)|)\ dx$$

$$= \int_0^1 |f(x)|\ dx + \int_0^1 |g(x)|\ dx = \|f\| + \|g\|$$

(4) Let $f \in C[0,1], \alpha \in \mathbb{R}$. Then

$\|\alpha f\| = \int_0^1 |\alpha f(x)|\ dx = \int_0^1 |\alpha|\,|f(x)|\ dx = |\alpha| \int_0^1 |f(x)|\ dx = |\alpha|\,\|f\|$.

Example 2.10.

Consider the linear space F^n over F ($F = \mathbb{R}$ or C). Define $\|\ \| : F^n \to \mathbb{R}$ by $\|X\| = \max\{|x_1|, ..., |x_n|\}\ \ \forall X = (x_1, ..., x_n) \in F^n$. Then $(F^n, \|\ \|)$ is a normed space.

solution: (1) For any $X = (x_1, ..., x_n) \in F^n,\ |x_i| \geq 0,\ \forall i = 1, ..., n$.

Then $\max\{|x_1|, ..., |x_n|\} \geq 0$, then $\|X\| \geq 0$.

(2) $\|X\| = 0$, where $X = (x_1, ..., x_n) \in F^n$

$$\iff \max\{|x_1|, ..., |x_n|\} = 0$$

$$\iff |x_1| = ... = |x_n| = 0 \iff x_1 = ... = x_n = 0$$

$$\iff X = (x_1, ..., x_n) = (0, ..., 0) = \mathbf{0}_{F^n}$$

(3) Let $X = (x_1, ..., x_n), Y = (y_1, ..., y_n) \in F^n$

$\|X+Y\| = \max\{|x_1+y_1|, ..., |x_n+y_n|\}$

$$\leq \max\{|x_1| + |y_1|, ..., |x_n| + |y_n|\}$$

$$\leq \max\{|x_1|, ..., |x_n|\} + \max\{|y_1|, ..., |y_n|\} = \|X\| + \|Y\|$$

(4) Let $X = (x_1, ..., x_n) \in F^n$ and $\alpha \in F$

$\|\alpha X\| = \max\{|\alpha x_1|, ..., |\alpha x_n|\}$

$= \max\{|\alpha| |x_1|, ..., |\alpha| |x_n|\} = |\alpha| \max\{|x_1|, ..., |x_n|\} = |\alpha| \|X\|$

As an application to Example 2.10: Consider the linear space $\mathbb{R}^n$ over $\mathbb{R}$. Let $X = (1, 2, -5), Y = (0, -7, 3)$. Then $\|X\| = \max\{|1|, |2|, |-5|\} = 5$ and

$\|Y\| = \max\{|0|, |-7|, |3|\} = 7.$

$\|X + 2Y\| = \max\{|1|, |-12|, |11|\} = 12$

Find $\|2X - Y\|$, $\|2X + 3Y\|$, $\|3X\|$

Exercise 2.11.

(1) Let $L = C^2$ be a linear space over $F = C$. Define $\| \ \| : C^2 \to \mathbb{R}$ such that $\|X\| = a|x_1| + b|x_2|$, $\forall X = (x_1, x_2) \in C^2$ and $a, b > 0$. Show that $\| \ \|$ is a norm on C^2.

(2) Consider the linear space $\mathbb{R}^2$. Let $\|X\| = \min\{|x|, |y|\}$, $\forall X = (x,y) \in \mathbb{R}^2$. Show that $\| \ \|$ is not a norm on $\mathbb{R}^2$.

solution: Let $X = (0,-3) \in \mathbb{R}^2$

$\|X\| = \min\{|0|, |-3|\} = \min\{0,3\} = 0$

Since $X \neq \mathbf{0}_{\mathbb{R}^2}$, but $\|X\| = 0$. Condition (2) of the definition of the norm is not valid. Hence, $\| \ \|$ is not a norm on $\mathbb{R}^2$.

(3) Consider the linear space $\mathbb{R}^2$. Let $\|X\| = |x|^2 + |y|^2$, $\forall X = (x,y) \in \mathbb{R}^2$. Show that $\| \ \|$ does not satisfies condition (4).

solution: Let $X = (1,3), \alpha = 2$

$|\alpha| \|X\| = 2(|x|^2 + |y|^2) = 2(1^2 + 3^2) = 20$

$\|\alpha X\| = \|2(1,3)\| = \|(2,6)\| = 2^2 + 6^2 = 40$

Thus, $|\alpha| \|X\| = 20 \neq \|\alpha X\| = 40$.

(4) Let $(L, \| \ \|)$ be a normed space. Let $\|x+y\| = \|x\| + \|y\| \ \ \forall x,y \in L$.

Show that $\|3x+2y\| = 3\|x\| + 2\|y\|$.

solution: We must show $\|3x+2y\| \geq 3\|x\| + 2\|y\|$ and $\|3x+2y\| \leq 3\|x\| + 2\|y\|$

$$\begin{aligned}
\|3x+2y\| = \|3x+3y-y\| &= \|3(x+y)-y\| \\
&\geq |\,\|3(x+y)\| - \|y\|\,| \quad \text{(By Theorem 2.3(4))} \\
&= |3(\|x+y\|) - \|y\|\,| \quad \text{(By axiom (4))} \\
&= |3(\|x\| + \|y\|) - \|y\|\,| \quad \text{(By assumption)} \\
&= |3\|x\| + 2\|y\|\,| = 3\|x\| + 2\|y\|
\end{aligned}$$

Thus, $\|3x+2y\| \geq 3\|x\| + 2\|y\|$ (1)

On the other hand, $\|3x+2y\| \leq 3\|x\| + 2\|y\|$ (By axioms (3-4)) (2)

From (1) and (2), $\|3x+2y\| = 3\|x\| + 2\|y\|$.

Normed space and Metric space

Definition 2.12.

Let X be a non empty set and $d : X \ timesX \to \mathbb{R}$ be a mapping. Then d is called metric if

(1) $d(x,y) \geq 0 \ \ \forall x,y \in X$

(2) $d(x,y)=0 \iff x=y \quad \forall x,y \in X$

(3) $d(x,y)=d(y,x) \quad \forall x,y \in X$

(4) $d(x,y) \leq d(x,z)+d(z,y) \quad \forall x,y,z \in X.$

(X,d) is called **metric space**

Theorem 2.13.

Let $(L,\| \|)$ be a normed linear space. Let $d : L \times L \to \mathbb{R}$ defined by $d(x,y) = \|x-y\| \quad \forall x,y \in X$. Prove that (L,d) is a metric space. (i.e., every normed space is a metric space). The metric d is called metric **induced** by the norm.

Proof. To prove (L,d) is a metric space.

(i) By definition of norm, $\|x-y\| \geq 0 \quad \forall x,y \in L$. Hence, $d(x,y)=\|x-y\| \geq 0$

(ii) $d(x,y)=\|x-y\|=\|y-x\|=d(y,x)$

(iii) $d(x,y)=0 \iff \|x-y\|=0 \iff x-y=0 \iff x=y$

(iv) $d(x,y)=\|x-y\|=\|x-z+z-y\| \leq \|x-z\|+\|z-y\|=d(x,z)+d(y,z)$ □

Lemma 2.14.

Let d be a metric induced by a normed space $(L,\| \|)$ (i.e., $d(x,y)=\|x-y\|$). Then d satisfies the following:

(i) $d(x+a,y+a)=d(x,y) \quad \forall x,y,a \in L.$

(ii) $d(\alpha x,\alpha y)=|\alpha|\, d(x,y) \quad \forall x,y \in L,\ \forall a \in F.$

Proof. (1) $d(x+a,y+a)=\|x+a-(y+a)\|=d(x,y) \quad \forall x,y,a \in L$

(2) $d(\alpha x,\alpha y)=\|\alpha x-\alpha y\|=\|\alpha(x-y)\|=|\alpha|\,\|x-y\|=|\alpha|\, d(x,y).$ □

Remark 2.15.

Not every metric space is a normed space as we show in the next example

Example 2.16.

Let d be the discrete metric on a space X. Then d can't be obtained from a norm on X (i.e., $(X, \| \, \|)$, where

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & x \neq y \,. \end{cases}$$

Solution: Suppose d induced by a norm on X. Then, by previous Lemma,

$d(\alpha x, \alpha y) = |\alpha| \, d(x,y) \;\; \forall x, y \in X$ and $\forall \alpha \in F$.

Let $x, y \in X$ such that $x \neq y$. Then $\alpha x \neq \alpha y$ such that $d(x,y) = 1, d(\alpha x, \alpha y) = 1$ (1)

But $|\alpha| \, d(x,y) = |\alpha|$ (2)

Hence, $d(\alpha x, \alpha y) = 1 \neq |\alpha| = |\alpha| \, d(x,y)$ for any $\alpha \neq \pm 1$. Thus, d can not be induced by a normed space.

Example 2.17.
Let $d(x,y) = |x| + |y| \quad \forall x, y \in \mathbb{R}$. Then, d is a metric on $\mathbb{R}$ (check!). However, d is not induced by a normed space. To show this, let $x = 1, y = 3, a = 2 \in \mathbb{R}$.

$d(x,y) = d(1,3) = |1| + |3| = 4$

On the other hand, $d(x+a, y+a) = d(3,5) = |3| + |5| = 8$

Thus, $d(x,y) \neq d(x+a, y+a)$. By Lemma 2.14, d is not induced by a norm.

Some Important Inequalities

To give more examples about normed space, it is important to present some inequalities.

If $l_p = \{\langle x_n \rangle : x_n$ is a real number and $\sum_{i=1}^{\infty} |x_i|^p < \infty\}$ be a set of sequence space (see Excercise 1.8(2)). Let $x_n = \langle x_1, x_2, ..., \rangle \in l_p$, $y_n = \langle y_1, y_2, ..., \rangle \in l_q$. Then

(1) Holder's Inequality

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \Big[\sum_{i=1}^{\infty} |x_i|^p \Big]^{\frac{1}{p}} \Big[\sum_{i=1}^{\infty} |y_i|^q \Big]^{\frac{1}{q}},$$

where $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(2) Cauchy Schwarz's Inequality

$$\sum_{i=1}^{\infty}|x_iy_i| \leq \Big[\sum_{i=1}^{\infty}|x_i|^2\Big]^{\frac{1}{2}}\Big[\sum_{i=1}^{\infty}|y_i|^2\Big]^{\frac{1}{2}},$$

Note that Cauchy Schwarz's inequality is a special case of Holder's inequality where $p = q = 2$.

(3) Minkowski's Inequality

If $p \geq 1$

$$\Big[\sum_{i=1}^{\infty}|x_i+y_i|^p\Big]^{\frac{1}{p}} \leq \Big[\sum_{i=1}^{\infty}|x_i|^p\Big]^{\frac{1}{p}} + \Big[\sum_{i=1}^{\infty}|y_i|^p\Big]^{\frac{1}{p}},$$

Remark 2.18.

The three inequalities above hold for finite sum.

Now we can give the following examples

Example 2.19.

Show that the linear space $\mathbb{R}^n$ over $\mathbb{R}$ (or C^n over C) is a normed space with $\|X\| = \sum_{i=1}^{n}\big[|x_i|^2\big]^{\frac{1}{2}}$ $\forall x \in \mathbb{R}^n$ or $C^n, X = (x_1, ..., x_n)$. The space $(\mathbb{R}^n, \|X\|)$ is called **Euclidean space** and $(C^n, \|X\|)$ is called **Unitary space**.

Solution: Let $X = (x_1, ..., x_n), Y = (y_1, ..., y_n) \in \mathbb{R}^n$ (or C^n) and $\alpha \in \mathbb{R}$ (or C^n).

(1) Since $|x_i| \geq 0,\ \ \forall i = 1, ..., n$. Then, $\big[\sum_{i=1}^{n}|x_i|^2\big]^{\frac{1}{2}} \geq 0$; that is $\|X\| \geq 0$.

(2) $\|X\| = 0 \iff \big[\sum_{i=1}^{n}|x_i|^2\big]^{\frac{1}{2}} = 0 \iff \sum_{i=1}^{n}|x_i|^2 = 0$

$$\iff |x_i|^2 = 0,\ \ \forall i = 1, ..., n$$

$$\iff x_i = 0,\ \ \forall i = 1, ..., n$$

$$\iff X = (x_1, ..., x_n) = \mathbf{0}_{\mathbb{R}^n}$$

(3) $\|X + Y\| = \|(x_1 + y_1, ..., x_n + y_n)\|$

$$= \Big[\sum_{i=1}^{n}|x_i+y_i|^2\Big]^{\frac{1}{2}} \leq \Big[\sum_{i=1}^{n}|x_i|^2\Big]^{\frac{1}{2}} + \Big[\sum_{i=1}^{n}|y_i|^2\Big]^{\frac{1}{2}} \quad \text{(Minkoski's Inequality)}$$

$$= \|X\| + \|Y\|$$

(4) $\|\alpha X\| = \|(\alpha x_1, ..., \alpha x_n)\| = \big[\sum_{i=1}^{n}|\alpha x_i|^2\big]^{\frac{1}{2}}$

$$= \left[\, \sum_{i=1}^{n} |\alpha|^2 \, |x_i|^2 \,\right]^{\frac{1}{2}}$$

$$= |\alpha| \left[\, \sum_{i=1}^{n} |x_i|^2 \,\right]^{\frac{1}{2}} = |\alpha| \, \|X\|$$

Product of Normed Spaces

Let $(X, \| \, \|_X), (Y, \| \, \|_Y)$ be normed linear spaces over a field F. Let

$X \times Y = \{(x, y) : x \in X, y \in Y\}$ be the cartesian product of X and Y.

Define $+$ on $X \times Y$ by

$$(x_1, y_1) + (x_2, y_2) = (\underbrace{x_1 + x_2}_{\text{sum on X}}, \underbrace{y_1 + y_2}_{\text{sum on Y}}) \ \ \forall (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Define a scalar multiplication

$$\alpha(x, y) = (\alpha x, \alpha y) \ \ \forall (x, y) \in X \times Y, \forall \alpha \in F.$$

Show that $(X \times Y, +, \times)$ is a linear space over F. (**H. W.**)

This linear space can be made a normed space by different ways. For example, define $\| \, \| : X \times Y \to \mathbb{R}$ such that

(1) $\|(x, y)\|_1 = \|x\|_X + \|y\|_Y$

(2) $\|(x, y)\|_2 = \max\{\|x\|_X, \|y\|_Y\}$

Show that $(X \times Y, \| \, \|_1), (X \times Y, \| \, \|_2)$ are normed spaces.

(1) To show $(X \times Y, \| \, \|_1)$ is a normed space,

(i) Since $\|x\|_X \geq 0$ and $\|y\|_Y \geq 0 \ \ \forall x \in X, \forall y \in Y$, then $\|x\|_X + \|y\|_Y = \|(x, y)\| \geq 0$.

(ii) $\|(x, y)\| = 0 \iff \|x\|_X + \|y\|_Y = 0$

$\iff \|x\|_X = \|y\|_Y$

$\iff x = y = 0 \quad ((X, \| \, \|_X), (Y, \| \, \|_Y)$ are normed spaces)

$\iff (x, y) = (0, 0)$

(iii) For each $(x_1, y_1), (x_2, y_2) \in X \times Y$

$\|(x_1, y_1) + (x_2, y_2)\| = \|(x_1 + x_2, y_1 + y_2)\|$

$$= \|x_1 + x_2\|_X + \|y_1 + y_2\|_Y$$

$$\leq \|x_1\|_X + \|x_2\|_X + \|y_1\|_Y + \|y_2\|_Y$$

$$= (\|x_1\|_X + \|y_1\|_Y) + (\|x_2\|_X + \|y_2\|_Y)$$

$$= \|(x_1, y_1)\| + \|(x_2, y_2)\|$$

(iv) For each $(x, y) \in X \times Y$ and for each $\alpha \in F$

$$\|\alpha(x, y)\| = \|(\alpha x, \alpha y)\| = \|\alpha x\|_X + \|\alpha y\|_Y$$

$$= |\alpha| \, \|x\|_X + |\alpha| \, \|y\|_Y = |\alpha| \, (\|x\|_X + \|y\|_Y) = |\alpha| \, \|(x, y)\|$$

(2) Now, we show that $\| \|_2$ is a norm on $X \times Y$

(i) Since $\|x\|_X \geq 0$ and $\|y\|_Y \geq 0 \ \ \forall x \in X, \forall y \in Y$, then

$\max\{\|x\|_X, \|y\|_Y\} = \|(x, y)\| \geq 0.$

(ii) $\|(x, y)\| = 0 \iff \max\{\|x\|_X, \|y\|_Y\} = 0$

$$\iff \|x\|_X = \|y\|_Y$$

$$\iff x = y = 0 \quad ((X, \| \|_X), (Y, \| \|_Y) \text{ are normed spaces})$$

$$\iff (x, y) = (0, 0)$$

(iii) For each $(x_1, y_1), (x_2, y_2) \in X \times Y$

$$\|(x_1, y_1) + (x_2, y_2)\| = \|(x_1 + x_2, y_1 + y_2)\|$$

$$= \max\{\|x_1 + x_2\|_X, \|y_1 + y_2\|_Y\}$$

$$\leq \max\{\|x_1\|_X + \|x_2\|_X, \|y_1\|_Y + \|y_2\|_Y\}$$

$$\leq \max\{\|x_1\|_X, \|y_1\|_Y\} + \{\|x_2\|_X, \|y_2\|_Y\}$$

$$= \|(x_1, y_1)\| + \|(x_2, y_2)\|$$

(iv) For each $(x, y) \in X \times Y$ and for each $\alpha \in F$

$$\|\alpha(x, y)\| = \|(\alpha x, \alpha y)\| = \max\{\|\alpha x\|_X, \|\alpha y\|_Y\}$$

$$= \max\{|\alpha| \, \|x\|_X, |\alpha| \, \|y\|_Y\} = |\alpha| \max\{\|x\|_X, \|y\|_Y\} = |\alpha| \, \|(x, y)\|$$

Generalizations of Some Concepts from Metric Space

In what follow, we give generalizations of some known concepts from metric space such as open (closed) ball, open (closed) set, interior set, closure of a set, convergent sequence, Cauchy sequence, and bounded sequence.

Definition 2.20.

Let $(L, \| \|)$ be a normed linear space. Let $x_0 \in L, r \in \mathbb{R}, r > 0$. Then the set

$$B_r(x_0) = \{x \in L : \|x - x_0\| < r\}$$

is called an **open ball** with center x_0 and radius r. Similarly,

$$\bar{B}_r(x_0) = \{x \in L : \|x - x_0\| \leq r\}$$

is called an **closed ball** with center x_0 and radius r.

Definition 2.21.

Let $(L, \| \|)$ be a normed space and $A \subseteq L$. Then A is said to be

- **open set** if $\forall x \in A, \exists r > 0$ such that $B_r(x) \subseteq A$.
- **closed set** if $A^c = L \setminus A$ is open set

Remark 2.22.

Let $(L, \| \|)$ be a normed space. Then

(1) L, ϕ are closed and open.

(2) The union of any family of open sets is open

(3) The union of finite family of closed sets is closed

(4) The intersection of finite family of open sets is open

(5) The intersection of any family of closed sets is closed.

Theorem 2.23.

Any finite subset of a normed space is closed.

Proof. Let L be a normed space and $A \subseteq L$.

If $A = \phi$, then A is closed (by Remark 2.22(1))

If $A = \{x\}$ to prove A is closed (i.e., to prove $L \setminus A$ is open)

Let $y \in L \setminus A = L \setminus \{x\}$ so that $y \neq x$. Put $\|x - y\| = r > 0$. Thus, $x \notin B_r(y)$ and hence $B_r(y) \subseteq A^c = L \setminus \{x\}$. Thus, A^c is open and thus A is closed.

If $A = \{x_1, .., x_n\}, n \in Z_+, n > 1$ then $A = \cup_{i=1}^n \{x_i\}$. By Remark 2.22(3), A is closed $\square$

Exercise 2.24.

Let $(L, \| \|)$ be a normed space. Prove that

(i) The set $A_1 = \{x \in L : \|x\| \leq 1\}$ is closed

(ii) The set $A_2 = \{x \in L : \|x\| < 1\}$ is open

(iii) The set $C = \{x \in L : \|x\| = 1\}$ is closed

Solution: (i) $A_1 = \{x \in L : \|x\| \leq 1\} = \overline{B}_1(0)$. So, A_1 is a closed set (by Definition **??**)

(ii) $A_2 = \{x \in L : \|x\| < 1\} = B_1(0)$. So, A_1 is an open set (by Definition **??**)

(iii) $C = \{x \in L : \|x\| = 1\}$

$L \setminus C = \{x \in L : \|x\| < 1\} \cup \{x \in L : \|x\| > 1\}$

Let $C_1 = \{x \in L : \|x\| < 1\}$ is open set

Let $C_2 = \{x \in L : \|x\| > 1\}$

So, $L \setminus C_2 = \{x \in L : \|x\| \leq 1\}$ which is closed set. Hence, C_2 is an open set.

Thus, $L \setminus C = C_1 \cup C_2$ is an open set (by Remark 2.22(2)).

Definition 2.25.

Let L be a normed space and $A \subseteq L$. A point $x \in L$ is called **limit point** of A if for each open set G containing x, we have $(G \cap A) \setminus \{x\} \neq \phi$.

The set of all limit points of A is denoted by A' and is called **derived set**.

The closure of A is denoted by $\overline{A}$ and is defined as $\overline{A} = A \cup A'$.

Proposition 2.26.

Let L be a normed linear space and $A \subseteq L$. Then $x \in \overline{A}$ if and only if $\forall r > 0, \exists y \in A, \|x - y\| < r$.

Proof. $(\Rightarrow)$ Let $x \in \overline{A} = A \cup A'$

If $x \in A'$ then for each open set G, $x \in G, (G \cap A) \setminus \{x\} \neq \phi$.

Since $B_r(x)$ is an open set then $\forall r > 0$, we have $B_r(x) \cap A \setminus \{x\} \neq \phi$. Thus, $\exists y \in B_r(x) \cap A, y \neq x \implies \|y - x\| < r$ **(I)**

If $x \in A$ then $\exists y = x$ such that $\|y - x\| = 0 < r$ **(II)**

From **(I)** and **(II)**, we get the required result.

$(\Leftarrow)$ If for each $r > 0, \exists y \in A$ such that $\|y - x\| < r$; that is $\forall r > 0, \exists y \in A, y \in B_r(x)$ $\implies \forall r > 0, (B_r(x) \cap A) \setminus \{x\} \neq \phi \implies x \in A'$. Thus, $x \in \overline{A}$. □

Convergence in Normed Space

Definition 2.27.

Let $\langle x_n \rangle$ be a sequence in a normed space $(L, \| \|)$. Then $\langle x_n \rangle$ is said to be **convergent** in L if $\exists x \in L$ such that $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_n - x\| < \epsilon, \ \forall n > k$

We write $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty}(x_n) = x$; that is

$\|x_n - x\| \to 0 \iff x_n \to x$.

$\langle x_n \rangle$ is **divergent** if it is not convergent.

Theorem 2.28.

If $\langle x_n \rangle$ is a convergent sequence in $(L, \| \ \|)$, then its limit is unique. i.e.,

If $\langle x_n \rangle \to x$ and $\langle x_n \rangle \to y$ then $x = y$.

Proof. Let $\epsilon > 0$. Since $\langle x_n \rangle \to x$ and $\langle x_n \rangle \to y$, then $\exists k_1, k_2 \in Z_+$ such that

$\|x_n - x\| < \dfrac{\epsilon}{2}, \ \ \forall n > k_1$ and $\|x_n - y\| < \dfrac{\epsilon}{2}, \ \ \forall n > k_2$

Let $k = \max\{k_1, k_2\}$, so $\forall n > k$

$$\begin{aligned} \|x - y\| = \|x_n - y - x_n + x\| &= \|(x_n - y) - (x_n - x)\| \\ &\leq \|x_n - y\| + \|x_n - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\implies \|x - y\| < \epsilon, \ \ \forall \epsilon > 0$. Thus, $\|x - y\| = 0$, so $x = y$. □

Theorem 2.29.

Let $A \subseteq L$ where L is a normed space, let $x \in L$. Then

$x \in \overline{A} \iff \exists \langle x_n \rangle$ a sequence in A such that $\langle x_n \rangle \to x$.

Proof. $(\Rightarrow)$ Let $x \in \overline{A} = A \cup A'$

If $x \in A$ then the sequence $\langle x, x, x, ... \rangle \to x$ **(I)**

If $x \notin A, i.e., x \in A'$ then for each open set G, $x \in G, (G \cap A) \setminus \{x\} \neq \phi$.

Since $B_r(x)$ is an open set then $\forall r > 0$, we have $B_r(x) \cap A \setminus \{x\} \neq \phi$. Set $0 < r = \dfrac{1}{n} \in Z_+$.

Then $\forall n \in Z_+, (B_{\frac{1}{n}}(x) \cap A) \setminus \{x\} \neq \phi$

Let $x_n \in B_{\frac{1}{n}}(x) \cap A$, such that $x_n \neq x$ and hence $\|x_n - x\| < \dfrac{1}{n}, \ \ \forall n \in Z_+$ **(*)**

Thus, $\exists \langle x_n \rangle \in A$ such that $\|x_n - x\| < \frac{1}{n}, \ \ \forall n \in Z_+$.

To show $\langle x_n \rangle \to x$; that is $\|x_n - x\| < \epsilon, \ \ \forall \epsilon > 0$

Let $\epsilon > 0$ so by Archmedian theorem $\exists k \in Z_+$ such that $\dfrac{1}{k} < \epsilon$

Hence, $\forall n > k, \dfrac{1}{n} < \dfrac{1}{k} < \epsilon$

From **(*)**, $\forall n > k, \|x_n - x\| < \dfrac{1}{n} < \dfrac{1}{k} < \epsilon$. Thus, $x_n \to x$ **(II)**

From **(I)** and **(II)**, we get the required result.

$(\Leftarrow)$ If $\exists \langle x_n \rangle$ a sequence in A such that $\langle x_n \rangle \to x$. To prove $x \in \overline{A} = A \cup A'$

If $x \in A$ then $x \in \overline{A}$

If $x \notin A$. Let G be an open set in L such that $x \in G$. Then $\exists r > 0$ such that $B_r(x) \subseteq G$.

Since $r > 0$ and $x_n \to x, \exists k \in Z_+$ such that $\|x_n - x\| < r, \quad \forall n > k$.

This implies, $x_n \in B_r(x) \quad \forall n > k$ and since $x_n \in A \;\; \forall n \in Z_+$. Then $(B_r(x) \cap A) \setminus \{x\} \neq \phi$. Since $B_r(x) \subseteq G$, then $(G \cap A) \setminus \{x\} \neq \phi$. So, $x \in A'$, and therefore $x \in \overline{A}$. □

Theorem 2.30.

Let $\langle x_n \rangle, \langle y_n \rangle$ be two sequences in normed space $(L, \| \, \|)$ such that $x_n \to x$ and $y_n \to y$. Then

(1) $\langle x_n \rangle \pm \langle y_n \rangle \to x \pm y$

(2) $\lambda \langle x_n \rangle \to \lambda x \quad$ for any scalar λ

(3) $\| \langle x_n \rangle \| \to \|x\|$

Proof. **(1)** Since $x_n \to x$, then for $\epsilon > 0, \exists k_1 \in Z_+$ such that $\|x_n - x\| < \frac{\epsilon}{2}, \;\; \forall n > k_1$
Also since $y_n \to y$, then for $\epsilon > 0, \exists k_2 \in Z_+$ such that $\|y_n - y\| < \frac{\epsilon}{2}, \;\; \forall n > k_2$
Let $k = \max\{k_1, k_2\}$. Then, for each $n > k$
$\|x_n - x\| < \frac{\epsilon}{2}$ and $\|y_n - y\| < \frac{\epsilon}{2} \qquad$ **(I)**
Now, for each $n > k$,

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (\text{from } \textbf{(I)})$$

Thus, $x_n + y_n \to x + y$ as required.
(2) Let $\epsilon > 0$. Since $x_n \to x, \exists k \in Z_+$ such that $\|x_n - x\| < \frac{\epsilon}{|\lambda|}, \quad \forall n > k \qquad$ **(I)**
But $\|\lambda x_n - \lambda x\| = |\lambda| \underbrace{\|x_n - x\| < \frac{\epsilon}{|\lambda|}}_{\text{using } \textbf{(I)}} |\lambda| = \epsilon$
Thus, $\lambda \langle x_n \rangle \to \lambda x$
(3) Let $\epsilon > 0$. Since $x_n \to x, \exists k \in Z_+$ such that $\|x_n - x\| < \epsilon, \quad \forall n > k \qquad$ **(I)**
But $| \, \|x_n\| - \|x\| \, | \leq \underbrace{\|x_n - x\| < \epsilon}_{\text{using } \textbf{(I)}} \quad \forall n > k$. Hence, $\|x_n\| \to \|x\|$. □

Definition 2.31.

Let $\langle x_n \rangle$ be a sequence in a normed space $(L, \| \, \|)$. Then $\langle x_n \rangle$ is said to be **Cauchy sequence** if $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_n - x_m\| < \epsilon, \;\; \forall n, m > k$.

Theorem 2.32.

Every convergent sequence in a normed space $(L, \| \ \|)$ is a Cauchy sequence.

Proof. Let $\langle x_n \rangle$ be a convergent sequence in L. Then $\exists x \in L$ such that $x_n \to x$ and so $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_n - x\| < \dfrac{\epsilon}{2} \quad \forall n > k \qquad \textbf{(I)}$

Now, for $n, m > k$,

$$\|x_n - x_m\| = \|(x_n - x) + (x - x_m)\| \le \underbrace{\|x_n - x\| + \|x_m - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}}_{\text{using } \textbf{(I)}} = \epsilon.$$

Thus, $\langle x_n \rangle$ is a Cauchy sequence. □

Definition 2.33.

Let $\langle x_n \rangle$ be a sequence in a normed space $(L, \| \ \|)$. Then $\langle x_n \rangle$ is said to be **bounded sequence** if $\exists k \in \mathbb{R}, k > 0$ such that $\|x_n\| \le k, \ \forall n \in Z_+$.

Theorem 2.34.

Every Cauchy sequence $\langle x_n \rangle$ in a normed space $(L, \| \ \|)$ is bounded.

Proof. Let $\epsilon = 1$. Since $\langle x_n \rangle$ is a Cauchy sequence, $\exists k \in Z_+$ such that $\|x_n - x_m\| < 1, \ \forall n, m > k$. Hence, $\|x_n - x_{k+1}\| < 1, \ \forall n > k$ (by considering $m = k + 1$) $\qquad \textbf{(I)}$

By Theorem 2.3(4), we have $| \ \|x_n\| - \|x_{k+1}\| \ | \le \underbrace{\|x_n - x_{k+1}\| < 1}_{\text{using } \textbf{(I)}} \ \forall n > k$

Thus, $\|x_n\| - \|x_{k+1}\| < 1 \quad \forall n > k$

Then, $\|x_n\| < 1 + \|x_{k+1}\| \qquad \forall n > k$

Let $M = \max\{\|x_1\|, \|x_2\|, ..., \|x_k\|, 1 + \|x_{k+1}\|\}$

Hence, $\|x_n\| \le M \qquad \forall n \in Z_+$. So, $\langle x_n \rangle$ is bounded. □

Corollary 2.35.

Every convergent sequence in a normed space $(L, \| \ \|)$ is bounded.

Proof. From Theorem 2.32, Every convergent sequence in a normed space $(L, \| \ \|)$ is Cauchy, and from Theorem 2.34, every Cauchy sequence in a normed space $(L, \| \ \|)$ is bounded. □

Convexity in Normed Linear Space

Definition 2.36.

A subset A of a linear space L is said to be **convex** if $\forall x, y \in A, \lambda \in [0,1]$ then $\lambda x + (1-\lambda)y \in A$.

Example 2.37.

Let $A = (1,3) \subset \mathbb{R}$. Is A convex set?

Solution: Let $x, y \in A, \lambda \in [0,1]$

Since $1 < x < 3 \implies 1\lambda < \lambda x < 3\lambda \quad \text{(I)}$

Since $1 < y < 3 \implies 1(1-\lambda) < (1-\lambda)y < 3(1-\lambda) \quad \text{(II)}$

By summing up (I) and (II)

$$\lambda + (1-\lambda) < \lambda x + (1-\lambda)y < 3\lambda + 3(1-\lambda)$$

$$1 < \lambda x + (1-\lambda)y << 3$$

Thus, $\lambda x + (1-\lambda)y \in A$. Hence, A is convex set.

Proposition 2.38.

Let L linear space. Then

(1) Every subspace of L is convex

(2) If $A, B \subset L$ are convex sets then $A \cap B$ is convex (**H.W.**)

(3) If $A, B \subset L$ are convex sets then $A + B$ is convex

Proof. **(1)** Let L be a linear space over a field $F = \mathbb{R}$ or C, let A be a subspace of L. Hence, by Theorem 1.11, $\forall x, y \in A$ and $\forall \alpha, \beta \in F$ we have $\alpha x + \beta y \in A$.

Take $\alpha = \lambda \in [0,1]$ and $\beta = 1 - \lambda$. Hence, $\alpha x + \beta y = \lambda x + (1-\lambda)y \in A$. Thus, A is a convex set.

(3) Let $a_1 + b_1, a_2 + b_2 \in A + B$, then $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

To prove $\lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B, \ \ \forall \lambda \in [0,1]$.

Since A convex and $a_1, a_2 \in A \implies \lambda a_1 + (1-\lambda)a_2 \in A \quad \forall \lambda \in [0,1] \qquad \text{(I)}$

Since B convex and $b_1, b_2 \in B \implies \lambda b_1 + (1-\lambda)b_2 \in B \quad \forall \lambda \in [0,1] \qquad \text{(II)}$

By summing up (I) and (II) we get

$\lambda a_1 + (1-\lambda)a_2 + \lambda b_1 + (1-\lambda)b_2 \in A + B$

i.e., $\lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$. Thus, $A + B$ is a convex set. $\square$

Remark 2.39.

The union of two convex sets is not necessary convex. For example, let

$A = (3,7) \cup (7,12)$. Then A is not convex. To show this, take $x = 6, y = 8, \lambda = \dfrac{1}{2}$

$\lambda x + (1-\lambda)y = \dfrac{1}{2}(6) + \dfrac{1}{2}(8) = 7 \notin A \cup B$.

Proposition 2.40.

Let $(L, \| \ \|)$ be a normed linear space, let $x_0 \in L$. Then $B_r(x_0)$ and $\overline{B}_r(x_0)$ are convex sets.

Proof. To prove $B_r(x_0)$ is a convex set. Let $x, y \in B_r(x_0)$, and let $\lambda \in [0,1]$. Then,

$\|x - x_0\| < r$ and $\|y - x_0\| < r$ (I)

We must prove $\lambda x + (1-\lambda)y \in B_r(x_0)$; that is we must prove $\|\lambda x + (1-\lambda)y - x_0\| < r$

$$\begin{aligned}
\|\lambda x + (1-\lambda)y - x_0\| &= \|\lambda x + \lambda \mathbf{x_0} - \lambda \mathbf{x_0} + (1-\lambda)y - x_0\| \quad \text{(adding and subtracting } \lambda x_0\text{)} \\
&= \|\lambda(x - x_0) + (1-\lambda)(y - x_0)\| \\
&\leq |\lambda| \, \|(x - x_0)\| + |1-\lambda| \, \|y - x_0\| \\
&< \lambda r + (1-\lambda)r = r \quad \text{(by (I) and since } \lambda > 0 \text{ then } |\lambda| = \lambda, |1-\lambda| = 1-\lambda\text{)}
\end{aligned}$$

This implies $\lambda x + (1-\lambda)y \in B_r(x_0)$ and hence $B_r(x_0)$ is convex. Similarly, $\overline{B}_r(x_0)$ is a convex set. $\square$

Proposition 2.41.

Let $(L, \| \ \|)$ be a normed linear space and $A \subset L$ and convex then $\overline{A}$ is a convex set.

Proof. Let $x, y \in \overline{A}$ and $\lambda \in [0,1]$. To prove $\lambda x + (1-\lambda)y \in \overline{A}$

Let $r > 0$. Since $x, y \in \overline{A}$ then by Proposition 2.26, $\exists a, b \in A$ such that

$\|x - a\| < r$ and $\|x - b\| < r$ (I)

Since A is convex then $\lambda a + (1-\lambda)b \in A$

Now, $\left\|\lambda x + (1-\lambda)y - \big(\lambda a + (1-\lambda)b\big)\right\| = \|\lambda(x - a) + (1-\lambda)(y - b)\|$

$$\leq \lambda \|x - a\| + (1-\lambda) \|y - b\|$$

$$< \lambda r + (1-\lambda) r \qquad \text{(from (I))}$$

$$= r$$

$$\text{Thus,} \left\| \big(\lambda x + (1-\lambda) y\big) - \big(\underbrace{\lambda a + (1-\lambda) b}_{\in A}\big) \right\| < r$$

Thus, from Proposition 2.26, $\lambda x + (1-\lambda)y \in \overline{A}$. □

Remark 2.42.

The converse of the above proposition is not true. For example, let $A = [1,2) \cup (2,5] \subset (\mathbb{R}, |\,|)$ then $\overline{A} = [1,5]$ is a convex set. But A is not convex, since if $x = 1, y = 3, \lambda = \dfrac{1}{2}$ then $\lambda x + (1-\lambda) y = \frac{1}{2} + \frac{1}{2}(3) = 2 \notin A$.

Continuity in Normed Linear Space

Definition 2.43.

Let X, Y be normed linear spaces. A mapping $f : X \to Y$ is called **continuous** at $x_0 \in X$ if for each $\epsilon > 0, \exists \delta > 0$ (depend on x_0) such that

$$\forall x \in X, \text{ if } \|x - x_0\| < \delta \text{ then } \|f(x) - f(x_0)\| < \epsilon.$$

Theorem 2.44.

Let X, Y be normed linear spaces. A mapping $f : X \to Y$ is continuous at $x_0 \in X$ iff $\forall \langle x_n \rangle \in X$ with $x_n \to x_0$ implies that $f(x_n) \to f(x_0)$.

Proof. ($\Rightarrow$) Let f be a contiunuous mapping at x_0 and let $x_n\rangle$ be a sequence in X such that $x_n \to x_0$. To prove $f(x_n) \to f(x_0)$.

Let $\epsilon > 0$, then $\exists \delta > 0$ such that $\forall x \in X$

if $\|x - x_0\| < \delta$ then $\|f(x) - f(x_0)\| < \epsilon$ (From continuity of f at x_0).

Since $x_n \to x_0$ and $\delta > 0, \exists k \in Z_+$ such that $\|x_n - x_0\| < \delta, \quad \forall n > k$.

Hence, $\|f(x_n) - f(x_0)\| < \epsilon, \quad \forall n > k$; that is $f(x_n) \to f(x_0)$.

($\Leftarrow$) Suppose that $x_n \to x_0$ implies that $f(x_n) \to f(x_0)$. To prove f is continuous at x_0.

Assume that f is not continuous at x_0, so $\exists \epsilon > 0$ such that $\forall \delta > 0, \exists x \in X$ and

$\|x - x_0\| < \delta$ but $\|f(x) - f(x_0)\| \geq \epsilon$.

Now, $\forall n \in Z_+, \dfrac{1}{n} > 0$, then $\exists x_n \in X$ such that

$\|x_n - x_0\| < \dfrac{1}{n}$ but $\|f(x_n) - f(x_0)\| \geq \epsilon$. This means $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$ in Y which is a contradiction. Thus, f is continuous at x_0. $\square$

Theorem 2.45.

Let $(X, \|\ \|)$ be a normed space and let $f : (X, \|\ \|) \to (\mathbb{R}, |\ |)$ such that $f(x) = \|x\| \quad \forall x \in X$. Then f is continuous.

Proof. Let $x_n \to x_0$ in X. Then $\forall \epsilon > 0, \exists k \in Z_+$ such that

$\|x_n - x_0\| < \epsilon \quad \forall n > k \qquad$ **(I)**

But $|\ \|x_n\| - \|x_0\|\ | \leq \|x_n - x_0\| \quad \forall n > k$

$\implies |\ \|x_n\| - \|x_0\|\ | < \epsilon \quad \forall n > k \quad$ (Using **(I)**)

$\implies |f(x_n) - f(x_0)| < \epsilon \quad \forall n > k \quad$ (Using (since $f(x) = \|x\|$))

$f(x_n) \to f(x_0)$; that is f is continuous. $\square$

Remark 2.46.

Let X, Y and Z be normed spaces and let $f : X \times Y \to Z$ be a mapping. Then f is continuous at $(x_0, y_0) \in X \times Y$ iff whenever $\langle (x_n, y_n) \rangle \to (x_0, y_0)$ then $f(x_n, y_n) \to f(x_0, y_0)$.

Theorem 2.47.

Let X be a normed space over a field F. Then

(1) The mapping $f : X \times X \to X$ such that $f(x, y) = x + y \quad \forall x, y \in X$ is continuous.

(2) The mapping $g : F \times X \to X$ such that $g(\lambda, x) = \lambda x \quad \forall x \in X, \forall \lambda \in F$ is continuous.

Proof. **(1)** Let $(x_n, y_n) \to (x_0, y_0)$. Then, $x_n \to x_0$ and $y_n \to y_0$ such that

$\|x_n - x_0\| \to 0$ and $\|y_n - y_0\| \to 0$ as $n \to \infty$.

$$\begin{aligned} \text{Now, } \|f(x_n, y_n) - f(x_0, y_0)\| &= \|(x_n + y_n) - (x_0 + y_0)\| \\ &= \|(x_n - x_0) + (y_n - y_0)\| \\ &\leq \|x_n - x_0\| + \|y_n - y_0\| \end{aligned}$$

Thus, $\|f(x_n, y_n) - f(x_0, y_0)\| \to 0$ as $n \to \infty$; that is f is continuous at (x_0, y_0). Since

(x_0, y_0) is arbitrary, f is continuous.

(2) Let $(\lambda_n, x_n) \to (\lambda, x_0)$. Then, $\lambda_n \to \lambda$ and $x_n \to x_0$.

Hence, $|\lambda_n - \lambda| \to 0$, $\|x_n - x\| \to 0$ as $n \to \infty$.

$$\begin{aligned}
\|g(\lambda_n, x_n) - g(\lambda, x_0)\| &= \|\lambda_n x_n - \lambda x_0\| \\
&= \|\lambda_n x_n - \boldsymbol{\lambda_n}\mathbf{x_0} + \boldsymbol{\lambda_n}\mathbf{x_0} - \lambda x_0\| \\
&= \|\lambda_n (x_n - x_0) + (\lambda_n - \lambda) x_0\| \\
&\leq |\lambda_n| \, \|x_n - x_0\| + |\lambda_n - \lambda| \, \|x_0\|
\end{aligned}$$

But $\|x_n - x_0\| \to 0$ and $|\lambda_n - \lambda| \to 0$ so that

$\|g(\lambda_n, x_n) - g(\lambda, x_0)\| \to 0$ as $n \to \infty$; that is $g(\lambda_n, x_n) \to g(\lambda, x_0)$. Thus, g is continuous.

$\square$

Theorem 2.48.
Let X, Y be normed spaces and let $f : X \to Y$ be a linear transformation. If f is continuous at 0 the f is continuous at any point.

Proof. Let $x_0 \in X$ be an arbitrary point and let $x_n \to x_0$.

To prove $f(x_n) \to f(x_0)$ (using Theorem 2.44).

Since $x_n \to x_0$, then $x_n - x_0 \to 0$

But f is continuous at 0, thus $f(x_n - x_0) \to f(0)$

Since f is a linear transformation, then $f(x_n) - f(x_0) \to f(0)$

It follows that $f(x_n) \to f(x_0)$. □

Remark 2.49.
The condition f is a linear transformation in the above theorem is necessary condition. For example: consider the normed space $(\mathbb{R}, |\,|)$. Let f is defined as

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } \quad x > 1\ . \end{cases}$$

It is clear that f is continuous at 0 and discontinuous at 1.

Also f is not linear transformation because

$f(5+6) = f(11) = 11+1 = 12$

and $f(5) + f(6) = (5+1) + (6+1) = 13$

Hence $f(5+6) \neq f(5) + f(6)$

Theorem 2.50.
Let X and Y be normed spaces and let $f : X \to Y$ be a linear transformation. Then either f is continuous at each point or discontinuous at each point.

Proof. Let $x_1 \in X$ and assume that f is continuous at x_1. Let $x_2 \in X$ be any point. To prove that f is continuous at x_2. Let $x_n \to x_2$ in X. Then, $x_n - x_2 \to 0$ and hence $x_n - x_2 + x_1 \to x_1$. Since f is continuous at x_1 then $f(x_n - x_2 + x_1) \to f(x_1)$.

Since f is a linear transformation, then $f(x_n) - f(x_2) + f(x_1) \to f(x_1)$.

Hence, $f(x_n) - f(x_2) \to 0$, and thus, $f(x_n) \to f(x_2)$.

Therefore, f is continuous at x_2. Thus, f can not be continuous at some points and discontinuos at some points. □

Example 2.51.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \ . \end{cases}$$

Show that f is not continuous at $(0,0)$.

<u>**Solution**</u>**:** Let $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{-1}{n} \quad \forall n \in N$.

Then, $x_n \to 0$ and $y_n \to 0$. Thus, $(x_n, y_n) \to (0,0)$. But

$$f(x_n,y_n) = \frac{\frac{1}{n}(\frac{-1}{n})}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{\frac{-1}{n^2}}{\frac{2}{n^2}} = \frac{-1}{2}$$

Hence, $f(x_n,y_n) \to \frac{-1}{2}$ but $f(0,0) = (0,0)$. Thus, $f(x_n,y_n) \nrightarrow f(0,0)$. Thus, f is not continuous at $(0,0)$.

Boundedness in Normed Linear Space

Definition 2.52. ***Bounded Set***

Let X be a normed space and let $A \subset X$. A is called a **bounded set** if there exists $k > 0$ such that $\|x\| \leq k \quad \forall x \in A$.

Example 2.53.

Consider $(\mathbb{R}, \| \ \|)$ and let $A = [-1, 1)$. Since $\|x\| \leq 1$, then A is bounded.

Example 2.54.

Consider $(\mathbb{R}^2, \| \ \|)$ be a normed space such that

$\|X\| = \left[\sum_{i=1}^{2} |x_i|^2\right]^{\frac{1}{2}}$ be the Eucledian norm, for each $X = (x_1, x_2) \in \mathbb{R}^2$.

Let $A = \{(x_1,x_2) \in \mathbb{R}^2 : -1 \leq x \leq 1, \ \ y \geq 0\}$. Then, A is unbounded.

Theorem 2.55.

Let X be a normed space and let $A \subseteq X$. Then the following statements are equivalent.

(1) A is bounded.

(2) If $\langle x_n \rangle$ is a sequence in A and $\langle \alpha_n \rangle$ is a sequence in F such that $\alpha_n \to 0$ then $\alpha_n x_n \to 0$.

Proof. (1)$\Rightarrow$(2) Since A is bounded, $\exists k > 0$ such that $\|x_n\| \leq k \quad \forall x_n \in A$.

Since $\alpha_n \to 0$ as $n \to \infty$, then $|\alpha_n| \to 0$. Hence,

$\|\alpha_n x_n - 0\| = \|\alpha_n x_n\| = |\alpha_n| \|x_n\| \leq |\alpha_n| k \quad$ (since $\|x_n\| \leq k$)

But $|\alpha_n| \to 0$, thus $|\alpha_n| k \to 0$. Therefore, $\|\alpha_n x_n - 0\| \to 0$ and hence $\alpha_n x_n \to \mathbf{0}_X$.

(2)$\Rightarrow$(1) Suppose A is not bounded. Then, $\forall k \in Z_+, \exists x_k \in A$ such that $\|x_k\| > k$.

Put $\alpha_k = \dfrac{1}{k}$. Hence, $\alpha_k \to 0$. But

$\|\alpha_k x_k\| = \left\|\frac{1}{k} x_k\right\| = \dfrac{1}{k} \|x_k\| > \dfrac{1}{k}.k = 1$

Then, $\|\alpha_k x_k\| > 1$, thus $\alpha_k x_k \nrightarrow 0$ which contradicts (2). $\square$

Definition 2.56. ***Bounded Function***

Let X, Y be two normed space and $f : X \to Y$ be a linear transformation. f is called **bounded function** if A is bounded and $f(A)$ is bounded set in $Y \quad \forall A \subseteq X$.

Example 2.57.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = x + y \quad \forall (x, y) \in \mathbb{R}^2$. Show that f is a linear transformation (**H.W.**). Let $A \subseteq \mathbb{R}^2$ and A is bounded. Show that $f(A)$ is bounded.

Solution: Let $A \subseteq \mathbb{R}^2$ and A is bounded, then $\exists k \geq 0$ such that $\|(x, y)\| \leq k \quad \forall (x, y) \in A$

$\implies (x^2 + y^2)^{\frac{1}{2}} \leq k \implies x^2 + y^2 \leq k^2$

Since $x^2 \leq x^2 + y^2 \leq k^2$, then $x^2 \leq k^2 \implies |x| \leq k \qquad$ **(I)**

Similarly, $y^2 \leq x^2 + y^2 \leq k^2$, then $y^2 \leq k^2 \implies |y| \leq k \qquad$ **(II)**

Note that $\forall (x, y) \in A \implies f(x, y) = x + y \in f(A)$

$|f(x, y)| = |x + y| \leq \underbrace{|x| + |y| \leq k + k}_{\text{by } \mathbf{(I)} \text{ and } \mathbf{(II)}} = 2k$

i.e., $|f(x, y)| \leq 2k$. Thus, $f(A)$ is bounded, and hence, f is bounded.

Theorem 2.58.

Let X, Y be normed spaces and $f : X \to Y$ be a linear transformation. Then f is bounded iff $\exists k > 0$ such that $\|f(x)\| \leq k \|x\| \qquad \forall x \in X$.

Proof. ($\Rightarrow$) If f is bounded and let $A = \{x \in X : \|x\| \leq 1\}$.

It is clear A is bounded, and hence, $f(A)$ is bounded in $Y \quad$ (by definition of bnd function).

Thus, $\exists k > 0$ such that $\|f(x)\| \leq k \qquad \forall x \in A \qquad$ **(I)**

(1) If $x = \mathbf{0_X}$ then $f(\mathbf{0_X}) = \mathbf{0_Y}$, and thus, $\|f(\mathbf{0_X})\| = 0 \leq k \|\mathbf{0_X}\| = 0$.

(2) If $x \neq \mathbf{0_X}$, put $y = \dfrac{x}{\|x\|}$ such that $\|y\| = \left\|\dfrac{x}{\|x\|}\right\| = \dfrac{1}{\|x\|}.\|x\| = 1$.

Hence, $y \in A$. Thus, $\|f(y)\| \leq k$ **(II)**

$\|f(y)\| = \left\|f(\frac{x}{\|x\|})\right\| = \left\|\frac{1}{\|x\|}f(x)\right\| = \frac{1}{\|x\|}\|f(x)\|$

By **(II)**, $\|f(y)\| \leq k$, thus $\frac{1}{\|x\|}\|f(x)\| \leq k$. i.e., $\|f(x)\| \leq k.\frac{1}{\|x\|}$ as required.

($\Leftarrow$) Let A be a bounded set. Then, $\exists k_1 > 0$ such that $\|x\| \leq k_1 \quad \forall x \in A$

Since $\|f(x)\| \leq k\|x\| \quad \forall x \in X$, then we get $\|f(x)\| \leq kk_1$

Thus, $\|f(x)\| \leq k_2 \quad \forall x \in A$ where $k_2 = kk_1$; that is, $f(A)$ is a bounded set. □

Theorem 2.59.

Let X, Y be normed spaces and $f : X \to Y$ be a linear transformation. Then f is bounded iff f is continuous.

Proof. ($\Rightarrow$) Suppose that f is not bounded, hence $\forall n \in Z_+, \exists x_n \in X$ such that

$\|f(x_n)\| > n\|x_n\|$

Let $y_n = \frac{x_n}{n\|x_n\|}$. Then, $\|f(y_n)\| = \left\|f(\frac{x_n}{n\|x_n\|})\right\| = \frac{\|f(x_n)\|}{n\|x_n\|} > \frac{n\|x_n\|}{n\|x_n\|} = 1$

Thus, $\|f(y_n)\| > 1$ **(I)**

but $\|y_n\| = \left\|\frac{x_n}{n\|x_n\|}\right\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n}$

as $n \to \infty$, we get $\|y_n\| \to 0$, and hence, $y_n \to \mathbf{0}_X$.

It follows that $f(y_n) \to \underbrace{f(\mathbf{0}_X) = \mathbf{0}_Y}_{\text{By Theorem 1.16(i)}}$ (Since f is a linear transformation)

This contradicts **(I)**, thus, f is bounded.

($\Leftarrow$) Assume that f is bounded to prove f is continuous. Let $x_0 \in X$ and $\epsilon > 0$, to find $\delta > 0$ such that

$\forall x \in X, \|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$.

$\|f(x) - f(x_0)\| = \|f(x - x_0)\|$ (f is linear transformation)

Since f is bounded, then $\exists k > 0$ such that $\|f(x)\| \leq k\|x\| \quad \forall x \in X$ **(I)**

Hence, $\|f(x) - f(x_0)\| = \underbrace{\|f(x - x_0)\| \leq k\|x - x_0\|}_{\text{By } \textbf{(I)}}$

$< k\delta$ (Since $\|x - x_0\| < \delta$)

$= k.\frac{\epsilon}{k}$ (By choosing $\delta = \frac{\epsilon}{k} = \epsilon$)

Thus, $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$.

Hence, f is continuous at $x_0 \in X$. Since x_0 is an arbitrary, then f is cont. $\forall x \in X$. □

Theorem 2.60.

Let X, Y be normed spaces and $f : X \to Y$ be a linear transformation. If X is a finite dimensional space then f is bounded (hence, continuous).

Example 2.61.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = x + y \quad \forall (x, y) \in \mathbb{R}^2$.

f is a linear transformation function (check!)

and $dim(\mathbb{R}^2) = 2$. Hence, f is bounded (hence, continuous).

Bounded Linear Transformation

Definition 2.62.

Let X, Y be normed spaces over a field F. The set of all bounded linear transformation operators from X to Y is defined as

$B(X, Y) = \{T : T : X \to Y \text{ is a linear bounded (hence, cont.) transformation}\}$

Theorem 2.63.

Prove that $B(X, Y)$ is a linear subspace (over a field F) of the space of linear transformation operators with respect to usual addition and usual scalar multiplication.

Proof. Let $\alpha, \beta \in F$ and $T_1, T_2 \in B(X, Y)$. To prove $\alpha T_1 + \beta T_2 \in B(X, Y)$

Since T_1, T_2 are linear transformations, then by Theorem 1.17(ii), $\alpha T_1, \beta T_2$ are linear trans.

Now, $\alpha T_1, \beta T_2$ are linear trans., by Theorem 1.17(i),$\alpha T_1 + \beta T_2$ is linear transformation.

Next, we show $\alpha T_1 + \beta T_2$ is bounded.

Since T_1, T_2 are bounded, then $\exists k_1, k_2 > 0$ such that $\forall x \in X$ we have

$\|T_1(x)\| \leq k_1 \|x\|$ and $\|T_2(x)\| \leq k_2 \|x\| \qquad$ **(I)**

Then,
$$\begin{aligned}
\|(\alpha T_1 + \beta T_2)(x)\| &= \|(\alpha T_1)(x) + (\beta T_2)(x)\| \\
&= \|\alpha.T_1(x) + \beta.T_2(x)\| \qquad \text{(Definition of scalar multiplication)} \\
&\leq \|\alpha.T_1(x)\| + \|\beta.T_2(x)\| \\
&= |\alpha| \|T_1(x)\| + |\beta| \|T_2(x)\| \\
&\leq |\alpha| k_1 \|x\| + |\beta| k_2 \|x\|
\end{aligned}$$

$$= (|\alpha|\, k_1 + |\beta|\, k_2)\, \|x\| = k\, \|x\| \qquad (k = |\alpha|\, k_1 + |\beta|\, k_2)$$

Hence, $\alpha T_1 + \beta T_2$ is bounded.

Since $\alpha T_1 + \beta T_2$ is bounded and linear transformation, then $\alpha T_1 + \beta T_2 \in B(X, Y)$ □

Theorem 2.64.

Let X, Y be normed space. Prove that $B(X, Y)$ is a normed space such that $\forall T \in B(X, Y)$ we have

$\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$

Proof. To prove $\| \ \|$ is a norm on $B(X, Y)$

(1) since $\|T(x)\| \geq 0 \ \ \forall x \in X, \|x\| \leq 1$, then $\|T\| \geq 0$.

(2) $\|T\| = 0 \iff \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\} = 0$

$$\iff \|T(x)\| = 0 \ \ \forall x \in X, \|x\| \leq 1$$

$$\iff T(x) = 0 \ \ \forall x \in X, \|x\| \leq 1$$

$$\iff T = \hat{0}$$

(3) Let $T_1, T_2 \in B(X, Y)$

$\|T_1 + T_2\| = \sup\{\|(T_1 + T_2)(x)\| : x \in X, \|x\| \leq 1\}$

$$\leq \sup\{\|(T_1(x)\| + \|T_2(x)\| : x \in X, \|x\| \leq 1\}$$

$$\leq \sup\{\|(T_1(x)\| : x \in X, \|x\| \leq 1\} + \sup\{\|(T_2(x)\| : x \in X, \|x\| \leq 1\}$$

$$= \|T_1\| + \|T_2\|$$

(4) $\|\alpha T\| = \sup\{\|(\alpha.T(x)\| : x \in X, \|x\| \leq 1\}$

$$= |\alpha| \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$$

$$= |\alpha|\, \|T\|$$ □

Chapter 3

Banach Space

Definition 3.1. ***Banach Space***

Let X be a normed space. Then, X is **complete** if every Cauchy sequence in X is convergent to a point in X. The complete normed space is called **Banach space**.

Example 3.2.

The space $F^n = \mathbb{R}^n$ (or C^n) with the norm $\|X\| = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \quad \forall X = (x_1, ..., x_n)$ is a Banach space.

<u>**Solution:**</u> Let $\langle X_m \rangle$ be a Cauchy sequence in F^n

$\langle X_m \rangle = \langle X_1, X_2, ..., X_m, ... \rangle$

$\qquad = \langle (x_{11}, x_{12}, ..., x_{1n}), (x_{21}, x_{22}, ..., x_{2n}), ..., (x_{m1}, x_{m2}, ..., x_{mn}), ... \rangle$

Then $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|X_m - X_j\| < \epsilon \quad \forall m, j > k$ **(I)**

Since $X_m, X_j \in F^n$, then

$X_m = (x_{m1}, x_{m2}, ..., x_{mn}), \;\; x_{mi} \in F, \;\; i = 1, ..., n$

$X_j = (x_{j1}, x_{j2}, ..., x_{jn}), \qquad x_{ji} \in F, \;\; i = 1, ..., n$

$X_m - X_j = (x_{m1} - x_{j1}, x_{m2} - x_{j2}, ..., x_{mn} - x_{jn})$

From **(I)**, $\|X_m - X_j\| < \epsilon \quad \forall m, j > k$

$$\|X_m - X_j\|^2 < \epsilon^2 \quad \forall m, j > k$$

$$\textstyle\sum_{i=1}^{n} (x_{mi} - x_{ji}) \leq \epsilon^2 \quad \forall m, j > k$$

$$|x_{mi} - x_{ji}|^2 \leq \epsilon^2 \quad \forall m, j > k, \;\; \forall i = 1, ..., n$$

$|x_{mi} - x_{ji}| \leq \epsilon \quad \forall m, j > k, \;\; \forall i = 1, ..., n$

Hence, $\langle x_{mi} \rangle$ is a Cauchy sequence in F, $\forall i = 1, ..., n$.

Then, $\langle x_{mi} \rangle$ is convergent to x_i $\forall i = 1, ..., n$.

Thus, for any $\epsilon > 0, \exists k_i \in Z_+$ such that $|x_{mi} - x_i| < \frac{\epsilon}{\sqrt{n}} \quad \forall m_i > k_i$

Put $l = \max\{k_1, ..., k_n\}$. Then

$|x_{mi} - x_i| < \dfrac{\epsilon}{\sqrt{n}} \quad \forall m > l, \;\; \forall i = 1, ..., n$

$|x_{mi} - x_i|^2 < \dfrac{\epsilon^2}{n} \quad \forall m > l, \;\; \forall i = 1, ..., n$

$\|X_m - X\|^2 = \sum_{i=1}^{n} |x_{mi} - x_i|^2 < n\dfrac{\epsilon^2}{n} \quad \forall m > l$

$\|X_m - X\| < \epsilon, \quad \forall m > l.$

Thus, $\langle X_m \rangle$ be a Cauchy sequence in F^n and $X_m \to X$. Thus, F^n is a Banach space.

Example 3.3.

The space $F^n = \mathbb{R}^n$ (or C^n) with the norm $\|X\| = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} \quad \forall X = (x_1, ..., x_n) \in \mathbb{R}^n$ (or C^n), $p \geq 1$ is a Banach space. (**H.W.**)

Example 3.4.

The space $\mathbb{R}^n$ (or C^n) with the norm $\|X\| = \max\{|x_1|, ..., |x_n|\} \quad \forall X = (x_1, ..., x_n) \in \mathbb{R}^n$ (or C^n) is a Banach space.

__**Solution:**__ Let $\langle X_m \rangle$ be a Cauchy sequence in F^n

$\langle X_m \rangle = \langle X_1, X_2, ..., X_m, ... \rangle$

$\quad = \langle (x_{11}, x_{12}, ..., x_{1n}), (x_{21}, x_{22}, ..., x_{2n}), ..., (x_{m1}, x_{m2}, ..., x_{mn}), ... \rangle$

Then $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|X_m - X_j\| < \epsilon \quad \forall m, j > k$ **(I)**

Since $X_m, X_j \in F^n$, then

$X_m = (x_{m1}, x_{m2}, ..., x_{mn}), \;\; x_{mi} \in F, \;\; i = 1, ..., n$

$X_j = (x_{j1}, x_{j2}, ..., x_{jn}), \quad x_{ji} \in F, \;\; i = 1, ..., n$

$X_m - X_j = (x_{m1} - x_{j1}, x_{m2} - x_{j2}, ..., x_{mn} - x_{jn})$

Then, $\|X_m - X_j\| = \max\{|x_{m1} - x_{j1}|, |x_{m2} - x_{j2}|, ..., |x_{mn} - x_{jn}|\} < \epsilon \quad \forall m, j > k$

It follows that $|x_{mi} - x_{ji}| < \epsilon \quad \forall i = 1, ..., n$ and $\forall m, j > k$.

Hence, $\langle x_{mi} \rangle$ is a Cauchy sequence in $\mathbb{R}$ (or C). So it is convergent to x_i in $F \quad \forall i = 1, ..., n$.

Hence, for any $\epsilon > 0, \exists k_i \in Z_+$ such that $|x_{mi} - x_i| < \epsilon \quad \forall m_i > k_i$

Put $l = \max\{k_1, ..., k_n\}$. Then, for each $\epsilon > 0$

$|x_{mi} - x_i| < \epsilon \quad \forall m > l, \ \forall i = 1, ..., n$

for each $\epsilon > 0$, $\|X_m - X\| = \max\{|x_{m1} - x_1|, |x_{m2} - x_2|, ..., |x_{mn} - x_n|\} < \epsilon \qquad \forall m > l$

Thus, $\langle X_m \rangle$ be a Cauchy sequence in $\mathbb{R}^n$ (or C^n) and $X_m \to X$. Thus, $\mathbb{R}^n$ (or C^n) is a Banach space.

Example 3.5.

The space $C[a, b]$ with the norm $\|f\| = \max\{|f(x)| : x \in [a, b]\} \quad \forall f \in C[a, b]$ is a Banach space.

Solution: Let $\langle f_n \rangle$ be a Cauchy sequence in $C[a, b]$

Then $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|f_m - f_n\| < \epsilon \quad \forall m, n > k \qquad$ **(I)**

Hence, $\forall \epsilon > 0, \exists k \in Z_+$ such that $\max\{|f_m(x) - f_n(x)| : x \in C[a, b]\} < \epsilon \quad \forall m, n > k$

It follows that $|f_m(x) - f_n(x)| < \epsilon \quad \forall x \in C[a, b] \quad \forall m, n > k$

Hence, $\langle f_n(x) \rangle$ is a Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is a Banach space, then $\langle f_n(x) \rangle$ is convergent to $f(x)$ in $\mathbb{R}$. Thus,

$\forall \epsilon > 0, \exists k \in N$ such that $|f_m(x) - f(x)| < \epsilon \quad \forall m \geq k$

Thus, $\|f_m - f\| = \max\{|f_m(x) - f(x)| : x \in [a, b]\} < \epsilon \quad \forall m \geq k$

Hence, $f_m \to f$ as $m \to \infty$. Thus, $C[a, b]$ ia a Banach space.

Example 3.6.

The space $C[0, 1]$ with the norm $\|f\| = \int_0^1 |f(x)|\, dx$ is not a Banach space.

Solution: The space $(C[0, 1], \|\ \|)$ is a normed space (see Example 2.9). Let

$$f_n(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ -nx + \frac{1}{2}n + 1 & \text{if} \quad \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \text{if} \quad \frac{1}{2} + \frac{1}{n} < x \leq 1 \end{cases}$$

where $n \geq 2$. Then, f_n is continuous function on $[0, 1]$. Now, for all $m, n \geq 2$ we have

$$\|f_m - f_n\| = \int_0^1 |f_m(x) - f_n(x)|\ dx$$
$$= \int_0^{\frac{1}{2}} |f_m(x) - f_n(x)|\ dx + \int_{\frac{1}{2}}^1 |f_m(x) - f_n(x)|\ dx$$

$$= \int_0^{\frac{1}{2}} |1-1|\ dx + \int_{\frac{1}{2}}^1 |f_m(x) - f_n(x)|\ dx$$

$$= \int_{\frac{1}{2}}^1 |f_m(x) - f_n(x)|\ dx$$

$$\leq \int_{\frac{1}{2}}^1 |f_m(x)|\ dx + \int_{\frac{1}{2}}^1 |f_n(x)|\ dx \qquad (\mathbf{I})$$

But $\displaystyle\int_{\frac{1}{2}}^1 |f_m(x)|\ dx = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} (-mx + \frac{m}{2} + 1)\ dx + \int_{\frac{1}{2}+\frac{1}{m}}^1 0\ dx$

$$= \frac{-mx^2}{2} + \frac{mx}{2} + x\Big]_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}}$$

$$= \frac{-m}{2}(\frac{1}{2} + \frac{1}{m})^2 + \frac{m}{2}(\frac{1}{2} + \frac{1}{m}) + (\frac{1}{2} + \frac{1}{m}) = \frac{1}{2m} \qquad (\mathbf{II})$$

Similarly, $\displaystyle\int_{\frac{1}{2}}^1 |f_n(x)|\ dx = \frac{1}{2n} \qquad (\mathbf{III})$

Substitute (**II**) and (**III**) in (**I**) to get $\|f_m - f_n\| \leq \frac{1}{2m} + \frac{1}{2n} \to 0$ as $m, n \to 0$

Thus, $\langle f_m \rangle$ is a Cauchy sequence. From the definition of f_n, we note that $f_n \to g$ where

$$g(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}.$$

But g is not continuous. Thus, $\langle f_m \rangle$ does not converge in $C[0,1]$. Then $(C[0,1], \|\ \|)$ is not a Banach space.

Theorem 3.7.

Let X be a Banach space and let M be a subspace of X. Then, M is a Banach space if and only if M is a closed set in X.

Proof. $\Rightarrow$) If M is a Banach space T.P. $M = \overline{M}$. We know that $M \subseteq \overline{M}$

Let $x \in \overline{M}$, then by Theorem 2.29, $\exists \langle x_n \rangle \in M$ such that $x_n \to x$

Hence, $\langle x_n \rangle$ is a Cauchy sequence in M. Then, $\exists y \in M$ such that $x_n \to y$

Thus, $x_n \to x$ and $x_n \to y$, so $x = y$. Thus, $x \in M$ (i.e., $\overline{M} \subseteq M$). Therefore, $M = \overline{M}$ (i.e., M is closed).

$\Leftarrow$) If M is a closed set. Let $\langle x_n \rangle$ be a Cauchy sequence in M, so that $\langle x_n \rangle$ is a Cauchy sequence in X. Hence, it converges; that is $\exists x \in X$ such that $x_n \to x$. But $\langle x_n \rangle$ is a sequence in M. By Theorem 2.29, $x \in \overline{M} = M$. i.e., $x \in M$. Thus, M is a Banach space. $\square$

Theorem 3.8.

Every finite dimensional normed space is a Banach space.

Corollary 3.9.

Every finite dimensional subspace of a Banach space is closed set.

Proof. Let X be a Banach space and let Y be a finite dimensional subspace of X. Then, by Theorem 3.8, Y is a Banach space. From Theorem 3.7, Y is a closed set. $\square$

Definition 3.10. ***Quotient Space***

Let X be a linear space over F. Let Y be a subspace of X. Let $X/Y = \{x + y : x \in X\}$

Define addition and scalar multiplication by

$(x_1 + Y) + (x_2 + Y) = (x_1 + x_2) + Y \qquad \forall x_1 + Y, x_2 + Y \in X/Y$

$\alpha.(x_1 + Y) = \alpha.x_1 + Y \qquad \forall x_1 + Y \in X/Y$ and $\forall \alpha \in F$.

Proposition 3.11.

Prove that $(X/Y, +, .)$ is a linear space over F. (**H.W.**)

Theorem 3.12.

Let Y be a closed set. Then $(X/Y, +, .)$ is a normed space with $\| \|_1$ where

$\|x + Y\|_1 = \inf\{\|x + y\| : y \in Y\}$.

Proof. (1) T.P. $\|x + Y\|_1 \geq 0$

For any $x + Y \in X/Y$

$\|x + y\| \geq 0 \ \ \forall y \in Y$

$\{\|x + y\| : y \in Y\} \geq 0$

$\|x + Y\|_1 = \inf\{\|x + y\| : y \in Y\} \geq 0$

(2) T.P. $\|x + Y\|_1 = 0 \iff x + Y = Y = 0_{X/Y}$

($\Rightarrow$) If $\|x + Y\|_1 = 0 \implies \inf\{\|x + y\| : y \in Y\} = 0$

Hence, $\exists\langle y_n\rangle \in Y$ such that $\|x + y_n\| \to 0$ as $n \to 0$. Hence, $x + y_n \to 0$ as $n \to 0$.

Thus, $y_n \to -x$. Thus, $\exists\langle y_n\rangle \in Y$ such that $y_n \to -x$. Thus, by Theorem 2.29, $-x \in \overline{Y}$.

Since Y is closed, then $-x \in \overline{Y} = Y$, i.e., $-x \in Y$.

Since Y is a subspace then $x \in Y$ and $x + Y = Y$, that is, $x + Y = 0_{X/Y}$.

($\Leftarrow$) If $x + Y = Y = 0_{X/Y}$ then $x \in Y$. i.e., $x + Y \in Y \quad \forall y \in Y$

Hence, $\|x + Y\|_1 = \inf\{\|x + y\| : y \in Y\} = \inf\{\|z\| : z \in Y\}$

Since $0 \in Y$ and $\|0\| = 0$, so $\inf\{\|z\| : z \in Y\} = 0$. Thus, $\|x + Y\|_1 = 0$.

(3) T.P. $\|\alpha(x + Y)\|_1 = |\alpha| \, \|x + Y\|_1$

If $\alpha = 0$ then (3) holds

If $\alpha \neq 0$ then

$$\|\alpha(x + Y)\|_1 = \inf\{\|\alpha(x + y)\| : y \in Y\}$$

$$= \inf\{|\alpha| \, \|x + y\| : y \in Y\}$$

$$= |\alpha| \inf\{\|x + y\| : y \in Y\}$$

(If A is bounded below, then $\inf(\alpha A) = \alpha \inf(A)$)

$$= |\alpha| \, \|x + Y\|_1$$

(4) Let $x_1 + Y, x_2 + Y \in X/Y$

$$\|(x_1 + Y) + (x_2 + Y)\|_1 = \|(x_1 + x_2) + Y\|_1$$

$$= \inf\{\|x_1 + x_2 + y\| : y \in Y\}$$

$$= \inf\{\|x_1 + x_2 + z_1 + z_2\| : z_1, z_2 \in Y\}$$

$$\leq \inf\{\|x_1 + z_1\| + \|x_2 + z_2\| : z_1, z_2 \in Y\}$$

$$= \inf\{\|x_1 + z_1\| : z_1 \in Y\} + \inf\{\|x_2 + z_2\| : z_2 \in Y\}$$

$$= \|x_1 + +Y\|_1 + \|x_2 + Y\|_1$$

Thus, X/Y is a normed space. □

Proposition 3.13.

If X is a Banach space and Y is a closed subspace of X. Then X/Y is a Banach space.

Proof. $X/Y = \{x + Y : x \in X\}$. Let $\langle X_n \rangle$ be a Cauchy sequence in X/Y.

Then, $X_n = x_n + Y$, where $x_n \in X$, $\forall n \in N$

$\forall \epsilon > 0, \exists k \in Z_+$ such that $\|X_m - X_n\| < \epsilon \ \forall n, m > k$

so, $\forall \epsilon > 0, \exists k \in Z_+$ such that $\|x_m - x_n + Y\| < \epsilon \ \forall n, m > k$

Then, $\forall \epsilon > 0, \exists k \in Z_+$ such that $\inf\{\|x_m - x_n + Y\| : y \in Y\} < \epsilon \ \forall n, m > k$

This implies $\forall y \in Y$, $\langle x_n + y \rangle$ is a Cauchy in X

Since X is a Banach space,then $\exists z \in X$ such that $x_n + y \to z = (z - y) + y$

$$= w + y \ \ \forall y \in \ Y$$

Thus, $x_n + Y \to w + Y$. Thus, X/Y is a Banach space. □

Chapter 4

Inner Product Space

Definition 4.1. ***Inner Product Space***

Let X is a linear space over F. A mapping $\langle \, , \rangle : X \times X \to F$ is called an **inner product on** X if the following axioms hold

(1) $\langle x, x \rangle \geq 0 \quad \forall x \in X$.

(2) $\langle x, x \rangle = 0 \iff x = 0$.

(3) $\overline{\langle x, y \rangle} = \langle y, x \rangle \quad \forall x, y \in X$, where $\overline{\langle x, y \rangle}$ =conjugate of $\langle x, y \rangle$.

(4) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.

$(X, \langle \, , \rangle)$ is called **inner product space** (briefly, I.P.S) or **Pre-Hilbert space**.

Remark 4.2.

(1) If $F = \mathbb{R}$ then axiom (3) becomes $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in X$.

(2) Every subspace of inner product space is an inner product space.

Examples of Inner Product Space

Example 4.3.

Let $L = \mathbb{R}^2$, which of the following is an inner product on L. and let $\langle\, ,\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to F$ defined as $\langle X, Y\rangle = x_1y_1 + x_2y_2 \quad \forall X, Y \in\ \mathbb{R}^2$ where $X = (x_1, x_2), Y = (y_1, y_2)$. Show that $\langle\, ,\rangle$ is an inner product on $\mathbb{R}^n$.

Solution: (i)We check the I.P axioms

(1) $\langle X, X\rangle = x_1^2 + x_2^2 \geq 0 \quad \forall X = (x_1, x_2) \in \mathbb{R}^2$

(2) $\langle X, X\rangle = 0 \iff x_1^2 + x_2^2 = 0 \iff x_1 = x_2 = 0 \iff X = (0, 0)$

(3) $\langle X, Y\rangle = x_1y_1 + x_2y_2 = \overline{\langle X, Y\rangle}$ (since $F = \mathbb{R}$)

(4) Let $\alpha, \beta \in \mathbb{R}$ and let $X = (x_1, x_2), Y = (y_1, y_2), Z = (z_1, z_2)$

$$
\begin{aligned}
\langle \alpha X + \beta Y, Z\rangle &= \langle(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2), (z_1, z_2)\rangle \\
&= (\alpha x_1 + \beta y_1)z_1 + (\alpha x_2 + \beta y_2)z_2 \\
&= (\alpha x_1 z_1 + \alpha x_2 z_2) + (\beta y_1 z_1 + \beta y_2 z_2) \\
&= \alpha(x_1 z_1 + x_2 z_2) + \beta(y_1 z_1 + y_2 z_2) \\
&= \alpha\langle X, Z\rangle + \beta\langle Y, Z\rangle
\end{aligned}
$$

Thus, $\langle\, ,\rangle$ is an inner product on $\mathbb{R}^2$.

As an application to Example 4.3:

Let $X = (2, 1), Y = (0, -3), Z = (3, 4)$. Find $\langle X, Z\rangle, \langle X, X\rangle, \langle X + Y, Z\rangle$.

Example 4.4.

Let $L = \mathbb{R}^2$, which of the following is an inner product on L.

(i) $\langle X, Y \rangle = 3x_1y_1 + x_2y_2$ **(H.W.)**

(ii) $\langle X, Y \rangle = x_1^2y_1^2 + x_2^2y_2^2$

where $X = (x_1, x_2), Y = (y_1, y_2)$

<u>Solution:</u> (i)We check the I.P axioms

(ii) The first three axioms of the definition of inner product hold but the forth condition does not satisfy.

If $\alpha = \beta = 1$ and let $X = (1, -1), Y = (-1, 0), Z = (-2, 2)$. Then

$\langle \alpha X + \beta Y, Z \rangle = \langle (0, -1), (-2, 2) \rangle = 0^2(-2)^2 + (-1)^2 2^2 = 4$

and $\alpha\langle X, Z \rangle + \beta\langle Y, Z \rangle = \langle (1, -1), (-2, 2) \rangle + \beta\langle (-1, 0), (-2, 2) \rangle$

$$= 1^1.(-2)^2 + (-1)^2.2^2 + (-1)^2.2^2 + 0^2.2^{=}12$$

Thus, $\langle \alpha X + \beta Y, Z \rangle \neq \alpha\langle X, Z \rangle + \beta\langle Y, Z \rangle$.

Example 4.5.

Let $L = F^n$ be a linear space and let $\langle\ ,\rangle : F^n \times F^n \to F$ defined as $\langle X, Y \rangle = \sum_{i=1}^n x_i\overline{y}_i \quad \forall X, Y \in F^n$ where $X = (x_1, ..., x_n), Y = (y_1, ..., y_n)$. Show that $\langle\ ,\rangle$ is an inner product on F^n.

<u>Solution</u>:

(1) $\langle X, X \rangle = \sum_{i=1}^n x_i\overline{x}_i = \sum_{i=1}^n |x_i|^2 \geq 0$

(2) $\langle X, X \rangle = 0 \iff \sum_{i=1}^n |x_i|^2 = 0 \iff x_i = 0 \ \forall i = 1, ..., n$

$$\iff X = (x_1, ..., x_n) = (0, ..., 0) = 0_{F^n}$$

(3) $\overline{\langle X, Y \rangle} = \overline{\sum_{i=1}^n x_i\overline{y}_i} = \sum_{i=1}^n \overline{x}_i y_i = \sum_{i=1}^n y_i\overline{x}_i = \langle Y, X \rangle$

(4) Let $\alpha, \beta \in F$ and let $X, Y, Z \in F^n$

$\alpha X + \beta Y = (\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n)$

$\langle \alpha X + \beta Y, Z \rangle = \sum_{i=1}^n (\alpha x_i + \beta y_i)\overline{z}_i = \alpha \sum_{i=1}^n x_i\overline{z}_i + \beta \sum_{i=1}^n y_i\overline{z}_i = \alpha\langle X, Z \rangle + \beta\langle Y, Z \rangle$.

Thus, $\langle\ ,\rangle$ is an inner product on F^n.

As an application to Example 4.5:

Let $L = C^2$ and $\langle X, Y \rangle = \sum_{i=1}^{2} x_i \overline{y}_i \quad \forall X, Y \in \; C^2$ where $X = (x_1, x_2), Y = (y_1, y_2)$. If $X = (2 + 3i, 1 + i), Y = (1 + i, 1 - i), Z = (2, 1 + i)$

Find $\langle X, X \rangle, \langle X + Y, Z \rangle, \langle X, Y + Z \rangle$

Solution: $\langle X, X \rangle = (2 + 3i)(\overline{2 + 3i}) + (1 + i)(\overline{1 + i})$

$= (2 + 3i)(2 - 3i) + (1 + i)(1 - i)$

$= (4 + 9) + (1 + 1) = 15$

$X + Y = (3 + 4i, 2)$

$\langle X + Y, Z \rangle = (3 + 4i)2 + 2(\overline{1 + i}) = (6 + 8i) + 2(1 - i) = 8 + 6i$

$\langle X, Y + Z \rangle$

Example 4.6.

Let $L = C[0, 1]$ be a linear space over $\mathbb{R}$, and let $\langle \; , \; \rangle : L \times L \to \mathbb{R}$ is defined by $\langle f, g \rangle = \displaystyle\int_0^1 f(x)g(x)\, dx$. Prove that $\langle \; , \; \rangle$ is an inner product on L.

Solution: (1) $\langle f, f \rangle = \displaystyle\int_0^1 f(x)f(x)\, dx = \int_0^1 [f(x)]^2\, dx \geq 0$

(2) $\langle f, f \rangle = 0 \iff \displaystyle\int_0^1 [f(x)]^2\, dx = 0 \iff [f(x)]^2 = 0 \;\; \forall x \in [0, 1]$

$\iff f(x) = 0 \;\; \forall x \in [0, 1] \iff f = \hat{0}$

(3) Let $\alpha, \beta \in \mathbb{R}$ and $f, g, h \in L$

$$\langle \alpha f + \beta g, h \rangle = \int_0^1 (\alpha f + \beta g)(x) h(x)\, dx$$

$$= \int_0^1 (\alpha f(x) + \beta g(x)\ h(x)\, dx$$

$$= \alpha \int_0^1 f(x)\ h(x)\, dx + \beta \int_0^1 g(x)\ h(x)\, dx$$

$$= \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

(4) $\langle f, g \rangle = \displaystyle\int_0^1 f(x) g(x)\, dx = \int_0^1 g(x) f(x)\, dx = \langle g, f \rangle$

As an application to Example 4.6:

Let $f(x) = x + 1$, $g(x) = x^2$, $h(x) = 3x + 2 \quad \forall x \in [0, 1]$

Find $\langle f, f \rangle, \langle f + g, h \rangle, \langle f, h \rangle, \langle 2f + 3g, h \rangle, \langle f - g, h - g \rangle$

Example 4.7.

Let $X = \mathbb{R}$ and $\langle\ ,\ \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\langle x, y \rangle = |xy| \quad \forall x, y \in \mathbb{R}$. Is $(X, \langle\ ,\ \rangle)$ I.P.S?

(**H.W.**)

Theorem 4.8.

Let $(X, \langle \ , \ \rangle)$ be an inner product space (I.P.S). Then

(1) $\langle x, \mathbf{0_X} \rangle = \langle \mathbf{0_X}, x \rangle = 0$

(2) $\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle \quad \forall x, y, z \in X$

Proof. (1) $\langle \mathbf{0_X}, x \rangle = \langle \mathbf{0_X} + \mathbf{0_X}, x \rangle$

$$= \langle \mathbf{0_X}, x \rangle + \langle \mathbf{0_X}, x \rangle$$

Hence, $\langle \mathbf{0_X}, x \rangle + 0 = \langle \mathbf{0_X}, x \rangle + \langle \mathbf{0_X}, x \rangle$

Thus, $0 = \langle \mathbf{0_X}, x \rangle$ (I)

Now, $\overline{\langle \mathbf{0_X}, x \rangle} = \langle x, \mathbf{0_X} \rangle$

$$\overline{0} = \langle x, \mathbf{0_X} \rangle$$

$$0 = \langle x, \mathbf{0_X} \rangle$$

(2) $\langle x, \alpha y + \beta z \rangle = \overline{\langle \alpha y + \beta z, x \rangle}$

$$= \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle}$$

$$= \overline{\alpha} \ \overline{\langle y, x \rangle} + \overline{\beta} \ \overline{\langle z, x \rangle}$$

$$= \overline{\alpha} \ \langle x, y \rangle + \overline{\beta} \ \langle x, z \rangle$$ □

Corollary 4.9.

If $(X, \langle \ , \ \rangle)$ is an I.P.S. Then

(i) $\left\langle \sum_{i=1}^{n} \alpha_i x_i, y \right\rangle = \sum_{i=1}^{n} \alpha_i \left\langle x_i, y \right\rangle$ where $x_1, ..., x_n \in X, y \in X$

(ii) $\left\langle x, \sum_{i=1}^{n} \beta_i y_i \right\rangle = \sum_{i=1}^{n} \overline{\beta}_i \left\langle x, y_i \right\rangle$ where $x \in X, y_1, ..., y_n \in X$

(iii) $\left\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j \right\rangle = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{m} \overline{\beta}_j \langle x_i, y_j \rangle \right)$ where $x_1, ..., x_n, y_1, ..., y_m \in X$

Proof. (i) We proof using induction.

If $n = 1$ then $\langle \alpha_1 x_1, y \rangle = \alpha_1 \langle x_1, y \rangle$ (by definition of norm)

If $n = 2$ then $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$ (by definition of norm)

Suppose (i) hold when $n = k$

$\left\langle \sum_{i=1}^{k} \alpha_i x_i, y \right\rangle = \sum_{i=1}^{k} \alpha_i \left\langle x_i, y \right\rangle$ **(I)**

To prove (i) hold when $n = k+1$

T.p. $\left\langle \sum_{i=1}^{k+1} \alpha_i x_i, y \right\rangle = \sum_{i=1}^{k+1} \alpha_i \left\langle x_i, y \right\rangle$

$$\begin{aligned}
\left\langle \sum_{i=1}^{k+1} \alpha_i x_i, y \right\rangle &= \left\langle \sum_{i=1}^{k} \alpha_i x_i + \alpha_{k+1} x_{k+1}, y \right\rangle \\
&= \left\langle \sum_{i=1}^{k} \alpha_i x_i, y \right\rangle + \left\langle \alpha_{k+1} x_{k+1}, y \right\rangle \\
&= \sum_{i=1}^{k} \alpha_i \left\langle x_i, y \right\rangle + \alpha_{k+1} \left\langle x_{k+1}, y \right\rangle \\
&= \sum_{i=1}^{k+1} \alpha_i \left\langle x_i, y \right\rangle
\end{aligned}$$

(ii) The proof is similar to the proof of (i).

(iii) Let $z = \sum_{j=1}^{m} \beta_j y_j$

$$\begin{aligned}
\left\langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j \right\rangle &= \left\langle \sum_{i=1}^{n} \alpha_i x_i, z \right\rangle \\
&= \sum_{i=1}^{n} \alpha_i \left\langle x_i, z \right\rangle \qquad \text{(by part (i))} \\
&= \sum_{i=1}^{n} \alpha_i \left\langle x_i, \sum_{j=1}^{m} \beta_j y_j \right\rangle \\
&= \sum_{i=1}^{n} \alpha_i \Big(\sum_{j=1}^{m} \overline{\beta}_j \left\langle x_i, y_j \right\rangle \Big) \qquad \text{(by part (ii))}
\end{aligned}$$

□

Theorem 4.10.
Let $(X, \langle\ ,\ \rangle)$ is an I.P.S. such that $\langle v_1, w \rangle = \langle v_2, w \rangle \quad \forall w \in X$. Then $v_1 = v_2$. Also, if $\langle v_1, w \rangle = 0 \quad \forall w \in X$ then $v_1 = \mathbf{0}_X$.

Proof. By assumption, $\langle v_1 - v_2, w \rangle = \langle v_1, w \rangle - \langle v_2, w \rangle = 0 \quad \forall w \in X$.

Put $w = v_1 - v_2$, then $\langle v_1 - v_2, v_1 - v_2 \rangle = 0 \implies v_1 - v_2 = 0 \implies v_1 = v_2$.

Now, $\langle v_1, w \rangle = 0 \quad \forall w \in X \implies \langle v_1, v_1 \rangle = 0 \implies v_1 = 0$. □

Theorem 4.11. ***General Cauchy Schwarz's Inequality***
Let $(X, \langle\ ,\ \rangle)$ is an I.P.S. and let $\|\ \| : X \to \mathbb{R}$ is defined by $\|x\| = \sqrt{\langle x, x \rangle} \quad \forall x \in X$. Then

$$\left|\left\langle x, y \right\rangle\right| \leq \|x\|\, \|y\| \quad \forall x, y \in X.$$

Proof. If $x = 0$ or $y = 0$ then $\langle x, y \rangle = 0$, and hence $\langle x, y \rangle = 0 \leq \|x\|\, \|y\|$

If $y \neq 0$, put $z = \dfrac{y}{\|y\|}$ $\qquad$ (**I**)

$$\|z\|^2 = \langle z, z\rangle = \langle \frac{y}{\|y\|}, \frac{y}{\|y\|}\rangle$$
$$= \frac{1}{\|y\|^2}\langle y, y\rangle = \frac{1}{\|y\|^2}\|y\|^2 = 1 \qquad \textbf{(II)}$$

Next, it is enough to show that $|\langle x, z\rangle| \le \|x\|$

because if $|\langle x, z\rangle| \le \|x\|$ then from (**I**)

$$|\langle x, z\rangle| = \left|\langle x, \frac{y}{\|y\|}\rangle\right| = \frac{1}{\|y\|}|\langle x, y\rangle| \le \|x\|$$
$$|\langle x, y\rangle| \le \|x\|\,\|y\|$$

Let $\alpha \in F$ then $\langle x - \alpha z, x - \alpha z\rangle \ge 0$

$$\langle x, x\rangle - \alpha\langle z, x\rangle - \overline{\alpha}\langle x, z\rangle + \alpha\overline{\alpha}\langle z, z\rangle \ge 0$$

$$\|x\|^2 - \overline{\alpha}\langle x, z\rangle - \alpha\langle z, x\rangle + \alpha\overline{\alpha}\underbrace{\|z\|^2}_{=1 \text{ from (I)}} \ge 0$$

$$\|x\|^2 - \langle x, z\rangle\overline{\langle x, z\rangle} + \langle x, z\rangle\overline{\langle x, z\rangle} - \overline{\alpha}\langle x, z\rangle - \alpha\langle z, x\rangle + \alpha\overline{\alpha} \ge 0$$

$$\|x\|^2 - |\langle x, z\rangle|^2 + \langle x, z\rangle(\overline{\langle x, z\rangle} - \overline{\alpha}) - \alpha(\langle z, x\rangle - \overline{\alpha})) \ge 0$$

$$\|x\|^2 - |\langle x, z\rangle|^2 + \langle x, z\rangle(\overline{\langle x, z\rangle - \alpha}) - \alpha(\overline{\langle x, z\rangle - \alpha})) \ge 0$$

$$\|x\|^2 - |\langle x, z\rangle|^2 + (\langle x, z\rangle - \alpha)(\overline{\langle x, z\rangle - \alpha}) \ge 0$$

$$\|x\|^2 - |\langle x, z\rangle|^2 + |\langle x, z\rangle - \alpha|^2 \ge 0 \quad \forall \alpha \in F \qquad \textbf{(III)}$$

Put $\alpha = \langle x, z\rangle$, then (**III**) becomes

$$\|x\|^2 - |\langle x, z\rangle|^2 \ge 0 \implies |\langle x, z\rangle|^2 \le \|x\|^2$$
$$|\langle x, z\rangle| \le \|x\|$$
$$\left|\langle x, \frac{y}{\|y\|}\rangle\right| \le \|x\| \quad \text{(using } \textbf{(I)}\text{)}$$
$$|\langle x, y\rangle|\frac{1}{\|y\|} \le \|x\|$$
$$|\langle x, y\rangle| \le \|x\|\,\|y\|. \qquad \square$$

As an application to Theorem 4.11:

If $X = \mathbb{R}^n$ and $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$ for any $X = (x_1, ..., x_n), Y = (y_1, ..., y_n)$. Apply Cauchy Schwarz inequality.

Sloution: We have , $\|X\| = [\langle X, X \rangle]^{\frac{1}{2}} = [\sum_{i=1}^{n} x_i^2]^{\frac{1}{2}}$ and $\|Y\| = [\langle Y, Y \rangle]^{\frac{1}{2}} = [\sum_{i=1}^{n} y_i^2]^{\frac{1}{2}}$

From Theorem 4.11, $|\langle X, Y \rangle| \leq \|X\| \|Y\|$; that is

$$\left| \sum_{i=1}^{n} x_i y_i \right| \leq [\sum_{i=1}^{n} x_i^2]^{\frac{1}{2}} [\sum_{i=1}^{n} y_i^2]^{\frac{1}{2}}$$

Theorem 4.12.

Every inner product space is a normed space and hence a metric space.

Proof. Let $(X, \langle \ , \ \rangle)$ is an I.P.S. and let the function $\| \ \| : X \to \mathbb{R}$ is defined by $\|x\| = \sqrt{\langle x, x \rangle} \quad \forall x \in X$. To prove $\| \ \|$ is a norm on X

(1) Since $\langle x, x \rangle \geq 0 \quad \forall x \in X \implies \|x\| = \sqrt{\langle x, x \rangle} \geq 0 \quad \forall x \in X$

(2) $\|x\| = 0 \iff \sqrt{\langle x, x \rangle} = 0 \iff \langle x, x \rangle = 0 \iff x = \mathbf{0_X}$

(3) Let $\alpha \in F$ and $x \in X$

$\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle = \alpha \overline{\alpha} \langle x, x \rangle = |\alpha|^2 \|x\|^2$

Thus, $\|\alpha x\| = |\alpha| \|x\|$

(4) T.P. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

$\|x + y\|^2 = \langle x + y, x + y \rangle$

$= \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle$

$= \|x\|^2 + \overline{\langle x, y \rangle} + \langle x, y \rangle + \|y\|^2$

$= \|x\|^2 + 2Re\langle x, y \rangle + \|y\|^2$

$\leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2$

$\leq \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2$ (by Cauchy Schwarz)

$= (\|x\| + \|y\|)^2$

Thus, $\|x + y\| \leq \|x\| + \|y\|$ □

Theorem 4.13.

Let $(X, \langle\ ,\ \rangle)$ is an I.P.S. and $x, y \in X$. Then

(1) $\|x+y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$ (Polarization Identity)

(2) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Law of Parallelogram)

(3) $\langle x, y\rangle = \frac{1}{4}\left[\, \|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 \,\right]$

Proof. (1)
$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \overline{\langle x, y\rangle} + \langle x, y\rangle + \|y\|^2 \\ &= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2\end{aligned}$$

(2) T.P. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

By part (1), $\|x+y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2$ (**I**)

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle \\ &= \langle x, x\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|^2 - \overline{\langle x, y\rangle} - \langle x, y\rangle + \|y\|^2 \\ &= \|x\|^2 - 2Re\langle x, y\rangle + \|y\|^2 \qquad (\mathbf{II})\end{aligned}$$

By summing up (**I**) and (**II**) we get $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(3) By parts (1) and (2), we have

$$\begin{aligned}\|x+y\|^2 - \|x-y\|^2 &= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2 - \left(\|x\|^2 - 2Re\langle x, y\rangle + \|y\|^2\right) \\ &= 2Re\langle x, y\rangle + 2Re\langle x, y\rangle \\ &= \langle \overline{x, y}\rangle + \langle x, y\rangle + \langle \overline{x, y}\rangle + \langle x, y\rangle \\ &= 2\langle y, x\rangle + 2\langle x, y\rangle \qquad (\mathbf{I})\end{aligned}$$

$$\|x+iy\|^2 = \langle x+iy, x+iy\rangle$$

$$= \langle x,x\rangle + i\langle y,x\rangle + \bar{i}\langle x,y\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + i\langle y,x\rangle - i\langle x,y\rangle + \|y\|^2$$

$$\|x-iy\|^2 = \langle x-iy, x-iy\rangle$$

$$= \langle x,x\rangle - i\langle y,x\rangle - \bar{i}\langle x,y\rangle + \langle y,y\rangle$$

$$= \|x\|^2 - i\langle y,x\rangle + i\langle x,y\rangle + \|y\|^2$$

Hence we get,

$$i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = i\big[\,\|x\|^2 + i\langle y,x\rangle - i\langle x,y\rangle + \|y\|^2\,\big] - i\big[\,\|x\|^2 - i\langle y,x\rangle + i\langle x,y\rangle + \|y\|^2\,\big]$$

$$= i\,\|x\|^2 - \langle y,x\rangle + \langle x,y\rangle + i\,\|y\|^2 - i\,\|x\|^2 - \langle y,x\rangle + \langle x,y\rangle - i\,\|y\|^2$$

$$= 2\langle x,y\rangle - 2\langle y,x\rangle \qquad \textbf{(II)}$$

By (**I**) and (**II**), we have

$$\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = 2\langle y,x\rangle + 2\langle x,y\rangle + 2\langle x,y\rangle - 2\langle y,x\rangle$$

$$\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = 4\langle x,y\rangle$$

$$\frac{1}{4}\,\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = \langle x,y\rangle \qquad \square$$

Remark 4.14.

Any normed linear space generated from inner product space must satisfies the three laws of Theorem 4.13.

Example 4.15.

Let $X = C[a,b]$ and let $\|f\| = \max\{|f(x)| : x \in [a,b]\}$. Then the converse of Theorem 4.12. i.e.,

(1) Show that $(X, \|\ \|)$ is a normed linear space (**H.W.**)

(2) Show that X is not generated by I.P.S (i.e, X is not I.P.S)

Solution: (2) To show that X is not I.P.S, we shall show that parallelogram law does not hold. i.e., $\|f+g\|^2 + \|f-g\|^2 \neq 2\|f\|^2 + 2\|g\|^2$ for some $f, g \in C[a,b]$

Let $f(x) = 1$ and $g(x) = \dfrac{x-a}{b-a}$ $\forall x \in [a,b]$

Note that f, g are continuous on $[a,b]$. Thus, $f, g \in C[a,b]$.

$\|f\| = 1$ and $\|g\| = 1$

$$\|f+g\| = \left\|1 + \frac{x-a}{b-a}\right\| = \max\left\{\left|1 + \frac{x-a}{b-a}\right| : x \in [a,b]\right\} = 2$$

$$\|f-g\| = \left\|1 - \frac{x-a}{b-a}\right\| = \max\left\{\left|1 - \frac{x-a}{b-a}\right| : x \in [a,b]\right\} = 1$$

$$\|f+g\|^2 + \|f-g\|^2 = 4 + 1 = 5 \qquad \textbf{(I)}$$

$$2\|f\|^2 + 2\|g\|^2 = 2.1^2 + 2.1^2 = 4 \qquad \textbf{(II)}$$

By (**I**) and (**II**), we get $\|f+g\|^2 + \|f-g\|^2 \neq 2\|f\|^2 + 2\|g\|^2$

i.e., $5 \neq 4$

Example 4.16.

Let $L = \mathbb{R}^2$ and let $\|X\| = |x| + |y|$ $\forall X = (x,y) \in \mathbb{R}^2$. Then the converse of Theorem 4.12. i.e.,

(1) Show that $(\mathbb{R}^2, \|\ \|)$ is a normed linear space (**H.W.**)

(2) Show that $\mathbb{R}^2$ is not generated by I.P.S (i.e, $\mathbb{R}^2$ is not I.P.S)

Solution: (2) To show that X is not I.P.S, we shall show that parallelogram law does not hold. i.e., $\|X+Y\|^2 + \|X-Y\|^2 \neq 2\|X\|^2 + 2\|Y\|^2$ for some $X, Y \in \mathbb{R}^2$

Let $X = (2,3)$ and $Y = (-6,1)$

$\|X\| = |2| + |3| = 5 \implies 2\|X\|^2 = 50$

$\|Y\| = |-6| + |1| = 7 \implies 2\|Y\|^2 = 98$

$\|X+Y\| = \|(-4,4)\| = |-4| + |4| = 8$

$\|X+Y\|^2 = 64$

$\|X-Y\| = \|(8,2)\| = |8| + |2| = 10$

$\|X-Y\|^2 = 100$

Thus, $\|X+Y\|^2 + \|X-Y\|^2 = 64 + 100 = 164$

and $2\|X\|^2 + 2\|Y\|^2 = 50 + 98 = 148$

Hence, $\|X+Y\|^2 + \|X-Y\|^2 \neq 2\|X\|^2 + 2\|Y\|^2$

i.e., $\| \ \|$ does not satisfy paralleogram law.

Example 4.17.

Let $L = \mathbb{R}^2$ and let $\|X\| = \max\{|x|, |y|\} \quad \forall (x,y) \in \mathbb{R}^2$. Then

(1) Show that $(\mathbb{R}^2, \| \ \|)$ is a normed linear space (**H.W.**)

(2) Is $\mathbb{R}^2$ generated by I.P.S? (**H.W.**)

Theorem 4.18.

Let $(X, \langle \, , \, \rangle)$ is an I.P.S. Then

(1) If $(x_n) \to x$ and $(y_n) \to y$ then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

(2) If (x_n) and (y_n) are Cauchy sequences in X then $\langle x_n, y_n \rangle$ is a Cauchy sequence in F.

Proof. (1) $\langle x_n, y_n \rangle = \langle x + (x_n - x), y + (y_n - y) \rangle$

$$= \langle x, y \rangle + \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle$$

$$\langle x_n, y_n \rangle - \langle x, y \rangle = \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle$$

$$\left| \langle x_n, y_n \rangle - \langle x, y \rangle \right| = \left| \langle x, y_n - y \rangle + \langle x_n - x, y \rangle + \langle x_n - x, y_n - y \rangle \right|$$

$$\leq \left| \langle x, y_n - y \rangle \right| + \left| \langle x_n - x, y \rangle \right| + \left| \langle x_n - x, y_n - y \rangle \right|$$

$$\leq \|x\| \, \|y_n - y\| + \|x_n - x\| \, \|y\| + \|x_n - x\| \, \|y_n - y\| \quad \text{(By Cauchy Schwarz)}$$

But $(x_n) \to x$ and $(y_n) \to y$ then $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$

Hence, $\left| \langle x_n, y_n \rangle - \langle x, y \rangle \right| \to 0$, and hence, $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

(2) for any $n, m \in Z_+$

$$\langle x_n, y_n \rangle = \langle (x_n - x_m) + x_m, (y_n - y_m) + y_m \rangle$$

$$= \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_n - x_m, y_m \rangle$$

$$\langle x_n, y_n \rangle - \langle x_m, y_m \rangle = \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle$$

$$\left| \langle x_n, y_n \rangle - \langle x_m, y_m \rangle \right| = \left| \langle x_n - x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle + \langle x_m, y_n - y_m \rangle \right|$$

$$\leq \left| \langle x_n - x_m, y_n - y_m \rangle \right| + \left| \langle x_m, y_n - y_m \rangle \right| + \left| \langle x_m, y_n - y_m \rangle \right|$$

$$\leq \|x_n - x_m\| \, \|y_n - y_m\| + \|x_m\| \, \|y_n - y_m\| + \|x_n - x_m\| \, \|y_m\| \quad \text{(By Cauchy Schwarz)}$$

But (x_n) and (y_n) are Cauchy sequences, then $\|x_n - x_m\| \to 0$ and $\|y_n - y_m\| \to 0$ as $n \to \infty$. Also, (x_n) and (y_n) are bounded sequences, then as $n \to \infty$

$$\left| \langle x_n, y_n \rangle - \langle x_m, y_m \rangle \right| \to 0$$

□

Corollary 4.19.

Let $(X, \langle\ ,\ \rangle)$ is an I.P.S. Then

(1) If $(x_n) \to x$ then $\|x_n\| \to \|x\|$

(2) If (x_n) is a Cauchy sequences in X then $\left\langle\ \|x_n\|\ \right\rangle$ is a convergent sequence in $\mathbb{R}$.

Proof. (1) Since $(x_n) \to x$ then $\left\langle x_n, x_n \right\rangle \to \left\langle x, x \right\rangle$ (By Theorem 4.18)

Hence, $\|x_n\|^2 \to \|x\|^2$. i.e., $\|x_n\| \to \|x\|$

(2) Since (x_n) is a Cauchy sequences in X, then by Theorem 4.18(2), $\left\langle x_n, x_n \right\rangle$ is a Cauchy sequence in F. Since $F = \mathbb{R}$ or C then F is complete. Thus, $\left\langle\ \|x_n\|^2\ \right\rangle$ is a convergent sequence in F. Thus, $\left\langle\ \|x_n\|\ \right\rangle$ is a convergent sequence in F □

Definition 4.20. ***Hilbert Space***

Hilbert space is an I.P.S. $(X, \langle\ ,\ \rangle)$ which is a Banach space with respect to $\|x\| = \sqrt{\langle x, x\rangle}$.

Example 4.21.

Consider the I.P.S. $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ such that $\langle X, Y\rangle = \sum_{i=1}^n x_i\overline{y}_i$ where $X = (x_1, ..., x_n), Y = (y_1, ..., y_n) \in \mathbb{R}^n$ (or C^n). (see Example 4.5)

Show that $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ is Hilbert space.

Solution: Since $\sqrt{\langle X, X\rangle} = \left[\sum_{i=1}^n x_i\overline{x}_i\right]^{\frac{1}{2}} = \left[\sum_{i=1}^n |x_i|^2\right]^{\frac{1}{2}} = \|X\|$

From Example 3.2, $\mathbb{R}^n$ (or C^n) is a Banach space w.r.t. $\|X\| = \sqrt{\langle X, X\rangle}$, and thus, $(\mathbb{R}^n, \langle\ ,\ \rangle)$ (or $(C^n, \langle\ ,\ \rangle)$ is a Hilbert space.

Example 4.22.

The space $C[-1, 1]$ with the inner product defined by $\langle f, g\rangle = \displaystyle\int_{-1}^{1} f(x)\ g(x)\ dx$ is not a Hilbert space.

Solution: Let

$$f_n(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ nx & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$$

$\|f_n - f_m\|^2 = \langle f_n - f_m, f_n - f_m\rangle$

Suppose $n > m$, then $\dfrac{1}{n} < \dfrac{1}{m}$. We must find $f_n(x) - f_m(x)$

$$f_n(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ nx & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$$

and

$$f_m(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ mx & \text{if} \quad 0 < x < \frac{1}{m} \\ 1 & \text{if} \quad \frac{1}{m} \leq x \leq 1 \end{cases}$$

Then

$$f_n(x) - f_m(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ (n-m)x & \text{if} \quad 0 < x < \frac{1}{n} \\ 1 - mx & \text{if} \quad \frac{1}{n} \leq x \leq \frac{1}{m} \\ 0 & \text{if} \quad \frac{1}{m} \leq x \leq 1 \end{cases}$$

$$\|f_n - f_m\|^2 = \int_{-1}^{1} (f_n(x) - f_m(x))^2 \, dx = \int_0^{\frac{1}{n}} (n-m)^2 x^2 \, dx + \int_{\frac{1}{n}}^{\frac{1}{m}} (1 - mx)^2 \, dx$$

$$= \frac{(n-m)^2 x^3}{3}\Big]_0^{\frac{1}{n}} + (\frac{-1}{m}) \frac{(1-mx)^3}{3}\Big]_{\frac{1}{n}}^{\frac{1}{m}}$$

$$= \frac{(n-m)^2}{3} \frac{1}{n^3} - \frac{1}{m}\Big[0 - \frac{1}{3}(1 - \frac{m}{n})^3\Big]$$

$$= \frac{(n-m)^2}{3n^3} + \frac{1}{3m}(\frac{n-m}{n})^3$$

$$= \frac{(n-m)^2}{3n^2 m}$$

Thus, $\|f_n - f_m\|^2 = \dfrac{(n-m)^2}{3n^2 m}$

Since $n > m$, then $n = m + t$

$$\|f_n - f_m\|^2 = \frac{t^2}{3(m+t)^2 m} \to 0 \quad \text{as } m \to \infty$$

Hence, $\|f_n - f_m\| \to 0$. Thus, $\langle f_n \rangle$ is a Cauchy sequence.

But $f_n \to f$ where

$$f(x) = \begin{cases} 0 & \text{if} \quad -1 \leq x \leq 0 \\ 1 & \text{if} \quad 0 < x \leq 1 \end{cases}$$

Thus, $f \notin C[-1,1]$. Then, $\langle f_n \rangle$ is not convergent in $C[-1,1]$. i.e., The space is not Hilbert space.

Remark 4.23.

Every Hilbert space is a Banach space but the converse is not true. For example, the space $C[a,b]$ with $\|f\| = \max\{|f(x)| : x \in [a,b]\}$ is a Banach space (see Example 3.5). However, $C[a,b]$ is not a Hilbert space since it does not satisfy parallelogram law; that is $\| \ \|$ can not be obtained from inner product (see Example 4.15).

Orthogonality and Orthonormality

Definition 4.24.

Let $(X, \langle \ , \rangle)$ be an I.P.S and $x, y \in X$. Then x is said to be **orthogonal** on y (denoted by $x \perp y$) iff $\langle x, y \rangle = 0$.

Example 4.25.

Let $X = \mathbb{R}^2$ is I.P.S such that $\langle X, Y \rangle) = x_1 y_1 + x_2 y_2 \ \ \forall X = (x_1, x_2), Y = (y_1, y_2) \in \mathbb{R}^2$. Let $X = (-6, 3), Y = (2, -1), Z = (1, 2)$. Show that $x \perp z, y \perp z$ and $y \not\perp x$.

Proposition 4.26.

Let $(X, \langle\ ,\rangle)$ be an I.P.S and $x, y \in X$. Then

(i) If $x \perp y$ then $y \perp x$.

(ii) $\mathbf{0} \perp x\ \ \forall x \in X$.

(iii) if $x \perp x$ then $x = \mathbf{0}$.

Proposition 4.27.

Let $(X, \langle\ ,\rangle)$ be an I.P.S and $x, x_1, ..., x_n \in X$ such that x is orthogonal on $x_1, ..., x_n$. Prove that x is orthogonal on any linear combination of $x_1, ..., x_n$.

Proof. Let w be a linear combination of $x_1, ..., x_n$. i.e., there exists $\alpha_i \in F$ such that $w = \sum_{i=1}^{n} \alpha_i x_i$. We must show $\langle x, w\rangle = 0$.

$$\begin{aligned}\langle x, w\rangle = \langle x, \textstyle\sum_{i=1}^{n} \alpha_i x_i\rangle &= \textstyle\sum_{i=1}^{n} \overline{\alpha_i}\langle x, x_i\rangle \quad \text{(by Corollary 4.9(ii))}\\ &= \textstyle\sum_{i=1}^{n} \overline{\alpha_i}.0 \quad \text{(From the assumption)}\\ &= 0.\end{aligned}$$

□

Example 4.28.

(1) Find the value of a that makes the vectors $X = (a, 2, -1), y = (3, -5, 2)$ orthogonal vectors in $\mathbb{R}^3$. (**H.W.**)

(2) Let $(X, \langle\ ,\rangle)$ be an I.P.S over $\mathbb{R}$ and let $x, y \in X$ such that $\|x\| = \|y\| = 1$. Prove that $x + y \perp x - y$.

Answer: $\langle x+y, x-y\rangle = \langle x, x\rangle - \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle = \|x\|^2 - \langle x, y\rangle + \langle x, y\rangle - \|y\|^2 = 0$. Hence, $x + y \perp x - y$.

(3) Let $(X, \langle\ ,\rangle)$ be an I.P.S and let $x, y \in X$ such that $x \perp y$. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2 = \|x - y\|^2$.

Answer:
$$\begin{aligned}\|x + y\|^2 = \langle x + y, x + y\rangle &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle\\ &= \|x\|^2 + 0 + 0 + \|y\|^2 = \|x\|^2 + \|y\|^2\end{aligned}$$

Similarly, $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

(4) Let $(X, \langle \ , \rangle)$ be an I.P.S and let $x, y \in X$ such that $x \perp y$. Prove that $\|x + \lambda y\| = \|x - \lambda y\|$.

Answer: (H.W.)

(5) Let $(X, \langle \ , \rangle)$ be an I.P.S and let $x_1, x_2, ..., x_n \in X$ such that $x_i \perp x_j \ \ \forall i \neq j$. Prove that $\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$.

Answer: We prove using induction. If $n = 1$, the statement is true.

If $n = 2$. Since $x_1 \perp x_2$ then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ (by part (3)).

Suppose the statement is true for $n = k$. i.e., $\left\|\sum_{i=1}^{k} x_i\right\|^2 = \sum_{i=1}^{k} \|x_i\|^2$

To prove the statement is true when $n = k + 1$. i.e., T.P. $\left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \sum_{i=1}^{k+1} \|x_i\|^2$

$$\begin{aligned}
\left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \left\|\sum_{i=1}^{k} x_i + x_{k+1}\right\|^2 &= \left\|\sum_{i=1}^{k} x_i\right\|^2 + \|x_{k+1}\|^2 \\
&= \sum_{i=1}^{k} \|x_i\|^2 + \|x_{k+1}\|^2 \ \ \text{(by induction } n = k\text{)} \\
&= \sum_{i=1}^{k+1} \|x_i\|^2.
\end{aligned}$$

Definition 4.29.

Let $(X, \langle \ , \rangle)$ be an I.P.S, $x \in X$, and $A \subseteq X$. Then x is said to be orthogonal on A $(x \perp A)$ if $x \perp a \ \ \forall a \in A$.

Example 4.30.

Consider the space $\mathbb{R}^2$ and $A = \{(0, y) : y \in \mathbb{R}\}$. Then $(2, 0) \perp A$ because $\langle (2, 0), (0, y) \rangle = 2.0 + 0.y = 0$.

Definition 4.31.

Let $(X, \langle \ , \rangle)$ be an I.P.S. and $A \subseteq X$. Then

(1) A is said to be **orthogonal** if $x \perp y \ \ \forall x, y \in A, \ \ x \neq y$.

(2) A is said to be **orthonormal** if A is orthogonal and $\langle x, x \rangle^{\frac{1}{2}} = \|x\| = 1 \ \ \forall x \in A$. In other words, A is orthonormal if $\forall x, y \in A$

$$\langle x, y \rangle = \begin{cases} 0 & \text{if } \ x \neq y \\ 1 & \text{if } x = y \ . \end{cases}$$

Definition 4.32.

Let $(X, \langle \ , \rangle)$ be an I.P.S. and (x_n) is a sequence in X. Then

(1) (x_n) is said to be **orthogonal** if $x_n \perp x_m \ \ \forall n \neq m$.

(2) (x_n) is said to be **orthonormal** if (x_n) is orthogonal and $\langle x_n, x_n\rangle^{\frac{1}{2}} = \|x_n\| = 1 \ \ \forall n \in N$. In other words, (x_n) is orthonormal if $\forall n, m \in N$

$$\langle x_n, x_m\rangle = \begin{cases} 0 & \text{if } \ n \neq m \\ 1 & \text{if } n = m \ . \end{cases}$$

Remark 4.33.

Orthonarmal set has no zero vector ($\|0\| \neq 1$).

Example 4.34.

Let $X = \mathbb{R}^3$ and $A = \{(1, 2, 2), (2, 1, -2), (2, -2, 1)\}$. Show that A is orthogonal but not orthonormal. **(H.W.)**

Example 4.35.

Let $X = C[-\pi, \pi]$ and $f_n(x) = sin(nx)$ be a sequence in X. Then (f_n) is orthogonal sequence because for $n \neq m$, $\ \langle f_n, f_m\rangle = \int_{-\pi}^{\pi} sin(nx)sin(mx)\, dx = 0$

Theorem 4.36.

Let X be an I.P.S. and $x_1, ..., x_n$ be orthonormal vectors in X. Then

$$\sum_{i=1}^{n} |\langle x, x_i\rangle|^2 \leq \|x\|^2 \ \ \forall x \in X$$

.

Example 4.37.

Let $X = \mathbb{R}^3$ and $x_1 = \dfrac{1}{3}(1, 2, 2), x_2 = \dfrac{1}{3}(2, 1, -2), x_3 = \dfrac{1}{3}(2, -2, 1)$.

$Let x = (2, 1, 3)$. Then

$|\langle x, x_1\rangle|^2 = [\frac{1}{3}(2 + 2 + 6)]^2 = \dfrac{100}{9}$

$|\langle x, x_2\rangle|^2 = [\frac{1}{3}(4 + 1 - 6)]^2 = \dfrac{1}{9}$

$|\langle x, x_3\rangle|^2 = [\frac{1}{3}(4 - 2 + 3)]^2 = \dfrac{25}{9}$

$\sum_{i=1}^{3} |\langle x, x_i\rangle|^2 = \dfrac{100}{9} + \dfrac{1}{9} + \dfrac{25}{9} = 14.$

on the other hand, $\|x\|^2 = \langle x, x\rangle = 4 + 1 + 9 = 14$.

As in Theorem 4.36, $\sum\limits_{i=1}^{3} |\langle x, x_i\rangle|^2 = \|x\|^2$

Take $x = (1, 1, 1)$ and apply Theorem 4.36. **(H.W.)**

Theorem 4.38.

Let $(X, \langle\ ,\ \rangle)$ be an I.P.S. Let (x_n) be an orthonormal sequence in X and (λ_n) be a sequence in F such that $\sum_{i=1}^{+\infty} |\lambda_i|^2 < +\infty$. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$. Then (y_n) is a Cauchy sequence.

Proof. Let $y_n = \sum_{i=1}^{n} \lambda_i x_i$, $\;y_m = \sum_{i=1}^{m} \lambda_i x_i$. Assume that $n < m$ then $m = n + k$ for some $k \in N$. We must prove $\|y_m - y_n\| \to 0$.

$y_m - y_n = \sum_{i=1}^{m} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n+k} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=n+1}^{n+k} \lambda_i x_i$.

$$\begin{aligned}\|y_m - y_n\|^2 = \left\|\sum_{i=n+1}^{n+k} \lambda_i x_i\right\|^2 &= \langle \sum_{i=n+1}^{n+k} \lambda_i x_i,\ \sum_{i=n+1}^{n+k} \lambda_i x_i\rangle \\ &= \sum_{i=n+1}^{n+k} \lambda_i \sum_{i=n+1}^{n+k} \overline{\lambda_i} \langle x_i, x_i\rangle \\ &= \sum_{i=n+1}^{n+k} \lambda_i \overline{\lambda_i} \langle x_i, x_i\rangle \\ &= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \|x_i\|^2 \\ &= \sum_{i=n+1}^{n+k} |\lambda_i|^2 \qquad (\|x_i\|^2 = 1\ \ \forall i)\end{aligned}$$

As $n \to +\infty$, $\sum_{i=n+1}^{n+k} |\lambda_i|^2 \to 0 \qquad (\sum_{i=1}^{+\infty} |\lambda_i|^2$ convergent)

Thus, $\|y_m - y_n\|^2 \to 0$ which means $\|y_m - y_n\| \to 0$. Hence, (y_n) is a Cauchy sequence.

$\square$