الاحصاء

قسم الرياضيات - المرحله الرابعه

للدراسه الصباحيه و المسائيه

د.حازم منصور

د.مي محمد هلال

د.ناديه فائق

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الفصل الاول

Chapter one

1- Discrete probability distributions

1- Bernoulli distribution

If the random experiment being repeated has only two outcomes such as (success, failure) for example: (Male, female), (yes, no), (head, tail) and so on. the we have a particularly important case of repeated trials, known as Bernoulli trials.

<u>Def.</u> The discrete r.v X is said to have a Bernoulli distribution with parameter p denoted as $X \sim Ber(1,p)$ if its probability mass function (p.m.f) is given as:

$$f(x) = \begin{cases} p^{x}(1-p)^{1-x} & , \quad x = 0,1 \\ 0 & \quad o.w \end{cases}$$

<u>Properties:</u> ① The mean $\mu_x = E(x) = p$

Proof: $E(X)=\sum_{x=0}^{1} x\, f(x) = 0\, f(0) + 1.f(1)$

$$= 0 + p(1-p)^0 = p$$

(2) The variance $\sigma_X^2 = p(1-p)$

Proof: $E(X^2)=\sum_{x=0}^{1} x^2 f(x) = (0)^2 f(0) + (1)^2 f(1) = p(1-p)^0 = p$

$\therefore \sigma_X^2 = E(X^2) - (E(X))^2 = p - p^2 = p(1-p)$

(3) The m.g.f of X is $M_X(t) = (1-p+pe^t)$

Proof: $M_X(t) = E(e^{tX}) = \sum_{x=0}^{1} e^{tx} f(x) = e^0 f(0) + e^t f(1)$

$$= 1.(1-p) + e^t p = (1-p+pe^t)$$

(2) binomial distribution

A random variable X that has a p.m.f.

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \dots, n \\ 0 & o.w \end{cases}$$

is said to have a binomial distribution denoted as

$X \sim b(n, p)$, where n is positive integer and

$0 < p < 1$ are the parameters of the distribution.

Ex: verify that $f(x)$ given above is a p.m.f

Sol: Two conditions must be satisfied (1) $f(x) > 0$ (2) $\sum f(x) = 1$

(1) $f(x) > 0$ and (2) $\sum f(x) = 1$

It is clear that the first condition is satisfied since $0<p<1$ and n is positive integer

For the second condition we have:

$$\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} C_x^n\, p^x (1-p)^{n-x} = [p+(1-p)]^n = 1 \quad \left((a+b)^n = \sum C_x^n\, a^x\, b^{n-x}\right)$$

properties

① $\mu_x = E(x) = np$

Proof:
$$\mu_x = E(x) = \sum_{x=0}^{n} x\, f(x) = \sum_{x=0}^{n} x\, C_x^n\, p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!}\, p^x (1-p)^{n-x}$$

$$= \sum \cancel{x} \frac{n(n-1)!}{\cancel{x}(x-1)!\,(n-x)!}\, p\, p^{x-1} (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n-1} \frac{(n-1)!}{(x-1)!\,(n-x)!}\, p^{x-1}\, (1-p)^{n-x}$$

putting $m=n-1$, $y=x-1 \Rightarrow$ then $m-y = n-x$

$$\mu_x = E(x) = np \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!}\, p^y (1-p)^{m-y}$$

$$= np \underbrace{\sum_{y=0}^{m} C_y^m\, p^y (1-p)^{m-y}}_{=1} = np(1) = np$$

② $var(x) = \sigma_x^2 = np(1-p)$

Proof: $var(x) = E(x^2) - [E(x)]^2$

writing $E(x^2)$ as $E(x(x-1)) + E(x)$

$$E\,x(x-1) = \sum x(x-1) f(x) = \sum x(x-1)\, C_x^n p^x (1-p)^{n-x}$$

$$= \sum x(x-1) \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum \cancel{x(x-1)} \frac{n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!\,(n-x)!} p^2\, p^{x-2} (1-p)^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n-2} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} (1-p)^{n-x}$$

putting $y = x-2$, $m = n-2$ then $m-y = n-x$, if $x=2$ then $y=0$ and

$$E\,x(x-1) = n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y (1-p)^{m-y}$$

$$= n(n-1)p^2 \underbrace{\left(\sum_{y=0}^{m} C_y^m p^y (1-p)^{m-y}\right)}_{=1\ \text{(binomial formula)}}$$

$$= n(n-1)p^2 (1)$$

$$= n(n-1)p^2$$

$$E(x^2) = n(n-1)p^2 + E(x) = n(n-1)p^2 + np = n^2p^2 - np^2 + np$$

$$var(x) = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np - np^2$$

$$\therefore var(x) = \sigma_x^2 = np(1-p)$$

(3) the moment generating function is

$$M_x(t) = [1-p+pe^t]^n$$

<u>Proof.</u> $M_x(t) = E(e^{tx}) = \sum e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} C_x^n p^x (1-p)^{n-x}$

$$= \sum C_x^n (pe^t)^x (1-p)^{n-x} = [1-p+pe^t]^n \quad \left[[a+b]^n = \sum C_x^n a^x b^{n-x}\right]$$

Ex: Find $E(x)$ and $var(x)$ by using the m.g.f

Hint: $E(x) = M'_x(0)$

$$\sigma^2_x = M''_x(0) - [M'_x(0)]^2$$

Proof: we have $M_x(t) = [1-p+pe^t]^n$

$M'_x(t) = n[1-p+pe^t]^{n-1}(pe^t)$

$M'_x(0) = np$

$M''_x(t) = n(n-1)[1-p+pe^t]^{n-2}(pe^t)^2 + n[1-p+pe^t]^{n-1} pe^t$

$M''_x(0) = n(n-1)p^2 + np = n^2p^2 - np^2 + np$

$\sigma^2_x = M''_x(0) - [M'_x(0)]^2 = n^2\cancel{p^2} - np^2 + np - n^2\cancel{p^2}$

$= np - np^2 = np(1-p)$

Ex: If $x \sim b(n,p)$, show that : $E(\frac{x}{n}) = p$ and

$E(\frac{x}{n} - p)^2 = \frac{p(1-p)}{n}$

Sol: $E(\frac{x}{n}) = \frac{1}{n}E(x) = \frac{1}{n}(np) = p$ [since $x \sim b(n,p)$, $E(x) = np$]

let $\frac{x}{n} - p = y$ then $E(\frac{x}{n} - p)^2 = E(y^2)$

but $E(y^2) = var(y) + [E(y)]^2$

$$E\left(\frac{X}{n}-p\right)^2 = \operatorname{Var}\left(\frac{X}{n}-p\right) + \left[E\left(\frac{X}{n}-p\right)\right]^2$$

$$= \operatorname{Var}\left(\frac{X}{n}\right) + \left[\frac{1}{n}E(X)-p\right]^2$$

$$= \frac{1}{n^2}\operatorname{Var}(X) + \left[\frac{1}{n}np-p\right]^2 = \frac{1}{n^2}np(1-p)+0 = \frac{p(1-p)}{n}$$

Ex: let the independent r'vs X_1, X_2, X_3 have the same p.d.f. $f(x)=3x^2$, $0<x<1$, Find the probability that exactly two of these three variable exceed $\frac{1}{2}$

Solution At the first we have to find the probability that any one of these three variables exceed $\frac{1}{2}$ as follows:

$$P = \int_{1/2}^{1} 3x^2\,dx = x^3\Big|_{1/2}^{1} = 1-\frac{1}{8} = \frac{7}{8}$$

The Probability of exactly two of these three variables exceed $\frac{1}{2}$ is

$$f(2) = Pr(X=2) = \binom{3}{2}\left(\frac{7}{8}\right)^2\left(\frac{1}{8}\right) = \frac{147}{512}$$

Ex: let $x_1, x_2, \dots, x_k$ be independent r.vs such that

$x_i \sim b(n_i, p)$, $i = 1, 2, \dots, k$

show that $\sum_{i=1}^{k} x_i \sim b(\sum_{i=1}^{k} n_i, p)$

proof: let $y = \sum_{i=1}^{k} x_i$ by using the m.g.f

$M_y(t) = E(e^{ty}) = E(e^{t\sum x_i}) = E(e^{t(x_1 + x_2 + \dots + x_k)})$

$= E(e^{tx_1} e^{tx_2} \dots e^{tx_k})$

Since the variables are independent, then

$M_y(t) = E(e^{tx_1}) E(e^{tx_2}) \dots E(e^{tx_k})$

$= M_{x_1}(t) M_{x_2}(t) \dots M_{x_k}(t)$

$= [1-p+pe^t]^{n_1} [1-p+pe^t]^{n_2} \dots [1-p+pe^t]^{n_k}$ [since $x_i \sim b(n_i, p)$]

$= (1-p+pe^t)^{\sum n_i}$

$y = \sum_{i=1}^{k} x_i \sim b(\sum_{i=1}^{k} n_i, p)$.

Ex: let $x \sim b(n, p)$ show that

$f(x+1) = \left[\frac{n-x}{x+1} \cdot \frac{p}{1-p}\right] f(x)$

solution: $f(x) = C_x^n p^x (1-p)^{n-x}$ [since $x \sim b(n,p)$]

$$f(x+1) = \binom{n}{x+1} p^{x+1} (1-p)^{n-x-1}$$

$$\frac{f(x+1)}{f(x)} = \frac{\binom{n}{x+1} p^{x+1} (1-p)^{n-x-1}}{\binom{n}{x} p^{x} (1-p)^{n-x}}$$

$$= \frac{\frac{n!}{(x+1)!\,(n-x-1)!}\, p\, p^{x} (1-p)^{n-x-1}}{\frac{n!}{x!\,(n-x)!}\, p^{x} (1-p)^{n-x}}$$

$$= \frac{\cancel{n!}}{(x+1)!\,(n-x-1)!}\, p\,(1-p)^{n-x-1} \times \frac{x!\,(n-x)!}{\cancel{n!}\, p^{x}(1-p)^{n-x}}$$

$$= \frac{x!\,(n-x)(n-x-1)!}{(x+1)\,\cancel{x!}\,(n-x-1)!} \frac{p}{1-p} = \frac{n-x}{x+1} \cdot \frac{p}{1-p}$$

$$f(x+1) = \left[\frac{n-x}{x+1} \cdot \frac{p}{1-p}\right] f(x)$$

(3) Poisson distribution

let X be a discrete r.v which can take on the values $0, 1, 2, \ldots$ such that the p.m.f. of X is given by

$$f(x) = \begin{cases} \frac{e^{-\lambda} \lambda^{x}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{o.w} \end{cases}$$

the distribution is called Poisson distribution

denoted as $X \sim P(\lambda)$ where the positive constant λ represent the parameter of the distribution.

Properties

① the m.g.f of the distribution is

$$M_X(t) = e^{\lambda(e^t-1)}$$

Proof: $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda e^t - \lambda} = e^{\lambda(e^t-1)}$$

$$\therefore M_X(t) = e^{\lambda(e^t-1)}$$

② $\mu_X = \sigma_X^2 = \lambda$

Proof: By using the m.g.f ($M_X(t) = e^{\lambda(e^t-1)}$)

$$M_X(0) = e^{\lambda(e^0-1)} = 1$$

$$M_X'(t) = \lambda e^t e^{\lambda(e^t-1)} = \lambda e^t M_X(t)$$

$$\mu_X = E(X) = M_X'(0) = \lambda e^0 M_X(0) = \lambda(1)(1) = \lambda$$

$$M_X''(t) = \lambda e^t M_X'(t) + \lambda e^t M_X(t)$$

$$M_X''(0) = \lambda e^0 \underbrace{M_X'(0)}_{\lambda} + \lambda e^0 M_X(0) = \lambda^2 + \lambda$$

$var(x) = \sigma_x^2 = M_x''(0) - [M_x'(0)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

(3) The Poisson distribution is an approximation of binomial distribution as $\lambda = np$ and n approaches to infinity:

Proof: the m.g.f of the binomial distribution is

$$M_x(t) = (1 - p + pe^t)^n = [1 + p(e^t - 1)]^n$$

putting $p = \frac{\lambda}{n}$, then

$$M_x(t) = [1 + \frac{\lambda(e^t - 1)}{n}]^n$$

using the well known result from calculus that

$$\lim_{n \to \infty} (1 + \frac{x}{n})^n = e^x$$

$$\lim_{n \to \infty} [1 + \frac{\lambda(e^t - 1)}{n}]^n = e^{\lambda(e^t - 1)}$$

which is the m.g.f of the Poisson dist. with parameter λ.

Ex: verify that the function $f(x) = \frac{e^{-\lambda} \lambda^x}{x!}$, $x = 0, 1, 2, \ldots$ is actually a probability function $< \begin{matrix} f(x) > 0 \\ \Sigma f(x) = 1 \end{matrix}$

solution First, we see that $f(x) > 0$ for $x = 0, 1, 2, \ldots$ given that $\lambda > 0$

second, we have

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!} = e^{-\lambda} e^{\lambda} = 1$$

Ex: let $x_1, x_2, \ldots, x_n$ be independent r.vs such that

$x_i \sim P(\lambda_i)$, $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} x_i \sim P(\sum_{i=1}^{n} \lambda_i)$

proof. let $y = \sum_{i=1}^{n} x_i$, then $M_y(t) = E(e^{ty})$

$$M_y(t) = E(e^{t\sum x_i}) = E(e^{t(x_1 + x_2 + \cdots + x_n)}) = E(e^{tx_1 + tx_2 + \cdots + tx_n})$$

since $x_1, x_2, \ldots, x_n$ are independent then

$$M_y(t) = E(e^{tx_1})\, E(e^{tx_2}) \cdots E(e^{tx_n})$$

$$= M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t) = e^{\lambda_1(e^t - 1)} \cdot e^{\lambda_2(e^t-1)} \cdots e^{\lambda_n(e^t-1)}$$

$$= e^{\sum \lambda_i (e^t - 1)}$$

$$y = \sum_{i=1}^{n} x_i \sim P(\sum_{i=1}^{n} \lambda_i)$$

(4) Negative binomial distribution :-

Consider an experiment of independent Bernoulli trials performed until we get a total of (r) successes and then stops. The probability of each individual trial resulting in a success is (p) where $0<p<1$. let x denote the number of failures encountered before we get

the first r successes, then the p.m.f of X is given by

$$f(x)=\begin{cases} C_x^{x+r-1}\, p^r (1-p)^x &, x=0,1,2,\dots,\ r=1,2,\dots \\ 0 & \text{o.w} \end{cases}$$

and we write $X \sim Nb(r,p)$ where the constants r, p are the parameters of dist.

<u>Ex</u>: show that $f(x)$ is exactly a p.m.f.

<u>Solution</u>: ① It is clear that $f(x)>0$ since each x, r are positive and $0<p<1$

② Applying the rule $\sum_{j=0}^{\infty} C_j^{n+j-1} z^j = (1-z)^{-n}$

then

$$\sum f(x) = \sum_{x=0}^{\infty} C_x^{x+r-1} p^r (1-p)^x = p^r \sum_{x=0}^{\infty} C_x^{x+r-1} (1-p)^x$$

$$= p^r [1-(1-p)]^{-r} = p^r p^{-r} = 1$$

<u>Properties</u>: ① the moment generating function

$$M_x(t) = \left[\frac{p}{1-(1-p)e^t}\right]^n$$

Proof: $M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f(x)$

$$= \sum e^{tx} C_x^{x+r-1} p^r (1-p)^x = p^r \sum C_x^{x+r-1} [(1-p)e^t]^x$$

$$= p^r [1-(1-p)e^t]^{-r} = \left[\frac{p}{1-(1-p)e^t}\right]^r$$

by $\sum C_j^{r+j-1} z^j = (1-z)^{-r}$

(2) The mean of the distribution is given by

$$\mu_x = \frac{r(1-p)}{p}$$

Proof: we have $M_x(t) = \left[\frac{p}{1-(1-p)e^t}\right]^r$

$$M_x'(t) = r\left[\frac{p}{1-(1-p)e^t}\right]^{r-1} \frac{p(1-p)e^t}{[1-(1-p)e^t]^2}$$

$$\mu_x = E(x) = M_x'(0) = r\left[\frac{p}{1-(1-p)}\right]^{r-1} \frac{p(1-p)}{[1-(1-p)]^2}$$

$$= r\left[\frac{p}{p}\right]^{r-1} \frac{p(1-p)}{p^2} = \frac{r(1-p)}{p}$$

(3) The variance of the distribution is

$$\sigma_x^2 = \frac{r(1-p)}{p^2}$$

Proof: we have $M_x(t) = \left[\frac{p}{1-(1-p)e^t}\right]^r \Rightarrow M_x(0) = 1$

$$M_x'(t) = r\left[\frac{p}{1-(1-p)e^t}\right]^{r-1} \frac{p(1-p)e^t}{[1-(1-p)e^t]^2}$$

$$= r\left[\frac{p}{1-(1-p)e^t}\right]^r \left[\frac{p}{1-(1-p)e^t}\right]^{-1} \frac{p(1-p)e^t}{[1-(1-p)e^t]^2}$$

De It can be written as

$$\therefore M_x'(t) = r\, M_x(t)\, \frac{(1-p)e^t}{1-(1-p)e^t}$$

$$\therefore M_x'(0) = r M_x(0)\, \frac{1-p}{p} = \frac{r(1-p)}{p}$$

putting $u = (1-p)e^t$, $\frac{du}{dt} = (1-p)e^t = u$

$$M_x'(t) = r M_x(t)\, \frac{u}{1-u}$$

$$M_x''(t) = r M_x(t)\, \frac{1-u+u}{(1-u)^2}\, \frac{du}{dt} + \frac{u}{1-u}\, r M_x'(t)$$

$$= r M_x(t)\, \frac{1}{(1-u)^2}\, u + \frac{u}{1-u}\, r M_x'(t)$$

$$= \frac{ru}{1-u}\left[M_x(t)\, \frac{1}{1-u} + M_x'(t) \right]$$

$$M_x''(0) = r\, \frac{1-p}{p}\left[\frac{1}{p} + r\frac{(1-p)}{p} \right]$$

$$\sigma_x^2 = M_x''(0) - [M_x'(0)]^2 = \frac{r(1-p)}{p^2} + \frac{r^2(1-p)^2}{p^2} - \cancel{\frac{r^2(1-p)^2}{p^2}}$$

$$\therefore \sigma_x^2 = Var(x) = \frac{r(1-p)}{p^2}$$

5 Geometric distribution

Z the geometric distribution is special case of a negative binomial distribution when $r=1$, hence

$$f(x) = \begin{cases} p(1-p)^x, & x = 0, 1, 2, \dots \\ 0 & o.w \end{cases}$$

The properties of geometric distribution can be obtained from the corresponding properties of negative binomial dist. by putting $r=1$. It follows that:

$$M_X(t) = \frac{p}{1-(1-p)e^t}, \quad \mu_X = \frac{1-p}{p}, \quad \sigma_X^2 = \frac{1-p}{p^2}$$

Ex. A fair die is thrown is successive independent trials until the second three is observed. Let X be a r.v that denotes the number of failures before the second three is observed.

(i) Find the distribution of X

(ii) Find the probability of observing 10 no threes before the second three is observed

(iii) Find the mean, variance, and m.g.f of the distribution

Solution: $X \sim Nb(2, \frac{1}{6})$, that is $r=2$, $p=\frac{1}{6}$

$$f(x) = C^{x+2-1}_{x} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^x$$

(ii) $Pr(X=10) = f(10) = C^{11}_{10} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10}$

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De (iii) $\mu_x = E(x) = r\frac{(1-p)}{p} = 2\frac{5/6}{1/6} = 10$

$var(x) = \sigma_x^2 = r\frac{(1-p)}{p^2} = 2\frac{5/6}{(1/6)^2}$

$M_x(t) = \left[\frac{p}{1-(1-p)e^t}\right]^r = \left[\frac{1/6}{1-\frac{5}{6}e^t}\right]^2$

Ex: Suppose we flip a fair coin until we get a head. Let x be the number of tails before we get a head.

(i) Find the p.m.f of x.

(ii) Find the mean, variance and m.g.f of x.

Solution: Since r=1 (first head) then we have a geometric distribution with $p=\frac{1}{2}$ and hence

$$f(x) = p(1-p)^x = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^x$$

(ii) $E(x) = \mu_x = \frac{1-p}{p} = \frac{1/2}{1/2} = 1$

$var(x) = \sigma_x^2 = \frac{1-p}{p^2} = \frac{1/2}{1/4} = 2$

$M_x(t) = \frac{p}{1-(1-p)e^t} = \frac{1/2}{1-\frac{1}{2}e^t}$

2- Continuous probability Distributions

① The uniform distribution :-

A continuous r.v X is said to follow a uniform distribution denoted as $X \sim U(a,b)$ if the p.d.f of x is

$$f(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & o.w \end{cases}$$

The real numbers a, b are the parameters of the distribution

It can be shown that $f(x)$ is actually a p.d.f

Since $f(x) = \frac{1}{b-a} > 0$ (because $a < b$)

and $\int_a^b f(x)\,dx = \frac{1}{b-a} \int_a^b f(x) = 1$

properties :- ① the mean $\mu_x = \frac{b+a}{2}$

proof: $\mu_x = E(x) = \int_a^b x\, f(x)\, dx = \frac{1}{b-a} \int_a^b x\, dx$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{1}{2(b-a)} (b^2 - a^2) = \frac{b+a}{2}$$

De ② $Var(x) = \sigma_x^2 = \frac{(b-a)^2}{12}$

proof: $\sigma_x^2 = E(x^2) - [E(x)]^2$

$$E(x^2) = \int_a^b x^2 f(x)\, dx = \int_a^b \frac{1}{b-a} x^2 \, dx = \frac{1}{b-a} \frac{x^3}{3} \Big|_a^b$$

$$= \frac{1}{3(b-a)} (b^3 - a^3) = \frac{1}{3(b-a)} (b-a)(b^2+ab+a^2)$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\sigma_x^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{b^2+ab+a^2}{3} - \frac{\overset{b^2+2ab+a^2}{(b+a)^2}}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

③ the M.g.f of the distribution is

$$M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}, \quad t > 0 \qquad \text{(prove it)}$$

④ the k^{th} moment about origin is:

$$E(x^k) = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$$

②

proof: $E(x^k) = \int_a^b x^k f(x)\, dx = \frac{1}{b-a} \int_a^b x^k \, dx$

$$= \frac{1}{(k+1)(b-a)} x^{k+1} \Big|_a^b = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$$

Ex: let $x \sim U(-a,a)$, $a>0$ find the value of a if it is known that $Pr(x>1) = \frac{1}{3}$

solution: $f(x) = \frac{1}{a-(-a)} = \frac{1}{2a}$

$$Pr(x>1) = \int_1^a \frac{1}{2a}\,dx = \frac{1}{2a}\int_1^a dx = \frac{1}{2a}\,x\Big|_1^a = \frac{a-1}{2a} = \frac{1}{3}$$

$$2a = 3a-3 \Rightarrow a = 3$$

(2) Gamma distribution

Def: If $\alpha>0$, we define the gamma function

$$\Gamma\alpha = \int_0^\infty y^{\alpha-1} e^{-y}\,dy$$

properties of gamma function

(1) $\Gamma(\alpha+1) = \alpha\,\Gamma\alpha$

(2) If α is positive integer then $\Gamma(\alpha+1) = \alpha!$

(3) $\Gamma\alpha = 2\int_0^\infty x^{2\alpha-1} e^{-x^2}\,dx$

(4) $\Gamma\frac{1}{2} = \sqrt{\pi}$

Def: the continuous r.v X is said to have a gamma distribution with parameters $\alpha, \beta > 0$

denoted as $X \sim G(\alpha, \beta)$ if the p.d.f of X is

$$f(x) = \begin{cases} \frac{1}{\Gamma\alpha \, \beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0 & o.w \end{cases}$$

properties of gamma distribution

① The m.g.f of the distribution is

$$M_X(t) = (1-\beta t)^{-\alpha}, \quad t < \frac{1}{\beta}$$

Proof: $M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx} \frac{1}{\Gamma\alpha\,\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}\,dx$

$$= \frac{1}{\Gamma\alpha\,\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{tx - \frac{x}{\beta}}\,dx = \frac{1}{\Gamma\alpha\,\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{\frac{\beta t x - x}{\beta}}\,dx$$

$$= \frac{1}{\Gamma\alpha\,\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1-\beta t}{\beta}\right)}\,dx$$

putting $y = x\left(\frac{1-\beta t}{\beta}\right) \Rightarrow x = \frac{\beta}{1-\beta t} y, \; dx = \frac{\beta}{1-\beta t} dy$

$$M_X(t) = \frac{1}{\Gamma\alpha\,\beta^\alpha} \int_0^\infty \left[\frac{\beta y}{1-\beta t}\right]^{\alpha-1} e^{-y} \frac{\beta}{1-\beta t}\,dy$$

$$= \frac{1}{\Gamma\alpha\,\beta^\alpha} \frac{\beta^\alpha}{(1-\beta t)^\alpha} \int_0^\infty e^{-y} y^{\alpha-1}\,dy$$

$$= \frac{1}{\Gamma_\alpha \beta^\alpha} \left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma_\alpha = \frac{1}{(1-\beta t)^\alpha} = (1-\beta t)^{-\alpha}$$

② $\mu_x = E(x) = \alpha\beta$

prove it

③ $\sigma_x^2 = Var(x) = \alpha\beta^2$

④ the k^{th} moment about origin is

$$E(x^k) = \frac{\beta^k \Gamma_{\alpha+k}}{\Gamma_\alpha}, k = 1, 2, 3, \dots$$

proof. $E(x^k) = \int_0^\infty x^k f(x)\,dx = \int_0^\infty x^k \frac{1}{\Gamma_\alpha \beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}}\,dx$

$$= \frac{1}{\Gamma_\alpha \beta^\alpha} \int_0^\infty x^{k+\alpha-1} e^{-x/\beta}\,dx$$

let $y = \frac{x}{\beta} \Rightarrow x = \beta y \Rightarrow dx = \beta\,dy$

$$E(x^k) = \frac{1}{\Gamma_\alpha \beta^\alpha} \int_0^\infty (\beta y)^{\alpha+k-1} e^{-y} (\beta\,dy)$$

$$= \frac{1}{\Gamma_\alpha \beta^\alpha} \beta^{\alpha+k} \int_0^\infty y^{\alpha+k-1} e^{-y}\,dy$$

$$= \frac{1}{\Gamma_\alpha \beta^\alpha} \beta^\alpha \beta^k \Gamma_{(\alpha+k)} = \frac{\beta^k \Gamma_{\alpha+k}}{\Gamma_\alpha}, k = 1, 2, \dots$$

Ex: Use the formula of $E(x^k)$, to find μ_x, σ_x^2

Solution: putting $k=1$ then

$$E(x) = \mu_x = \frac{\beta \Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta \alpha \Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta$$

putting $k=2$, we get

$$E(x^2) = \frac{\beta^2 \Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\beta^2 (\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$= \frac{\beta^2(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \beta^2 \alpha(\alpha+1)$$

$$\sigma_x^2 = var(x) = E(x^2) - [E(x)]^2$$

$$= \beta^2(\alpha+1)\alpha - \beta^2\alpha^2$$

$$= \beta^2\alpha^2 + \beta^2\alpha - \beta^2\alpha^2 = \alpha\beta^2$$

(3) The Chi square distribution

The chi square distribution is a special case of gamma distribution which $\alpha = \frac{r}{2}$ and $\beta = 2$ where r is positive integer. Hence the p.d.f of the r.v. X is

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\frac{r}{2})\, 2^{r/2}}\, x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, & x \geq 0 \\ 0 & o.w \end{cases}$$

and we write $X \sim \chi^2(r)$ where r is the number of degrees freedom representing the parameter of the distribution

Properties

: the properties of chi square dist. are the same of properties of gamma dist. when $\alpha = \frac{r}{2}$, $\beta = 2$

that is:

(1) $M_X(t) = (1-2t)^{-\frac{r}{2}}$, (2) $\mu_X = E(X) = \frac{r}{2} \cdot 2 = r$

(3) $var(X) = \sigma_X^2 = \left(\frac{r}{2}\right) 2^2 = 2r$

(4) $E(X^k) = \dfrac{2^k \, \Gamma\left(\frac{r}{2}+k\right)}{\Gamma\left(\frac{r}{2}\right)}$, $k = 1, 2, \dots$

(4) Beta distribution

Def: If $\alpha > 0$, $\beta > 0$, we define the Beta function as

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, dx$$

It can be show that

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} , \quad \alpha, \beta > 0$$

Def. The continuous r.v. X is said to have a Beta distribution denoted as $X \sim B(\alpha, \beta)$. If the p.d.f of X is

$$f(x) = \begin{cases} \dfrac{\Gamma_{\alpha+\beta}}{\Gamma_\alpha \Gamma_\beta} x^{\alpha-1} (1-x)^{\beta-1} & , 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

Properties

(1) The k^{th} moment about origin is

$$E(x^k) = \frac{\Gamma(k+\alpha)\,\Gamma(\alpha+\beta)}{\Gamma_\alpha\,\Gamma(k+\alpha+\beta)} \quad , k = 1, 2, \ldots$$

Proof. $$E(x^k) = \int_0^1 x^k f(x)\,dx = \frac{\Gamma_{\alpha+\beta}}{\Gamma_\alpha \Gamma_\beta} \underbrace{\int_0^1 x^{\alpha+k-1} (1-x)^{\beta-1}\,dx}_{B(\alpha+k,\beta)}$$

$$= \frac{\Gamma_{\alpha+\beta}}{\Gamma_\alpha \cancel{\Gamma_\beta}} \cdot \frac{\Gamma_{\alpha+k}\,\cancel{\Gamma_\beta}}{\Gamma_{\alpha+k+\beta}} \qquad \left[B(\alpha,\beta) = \frac{\Gamma_\alpha \Gamma_\beta}{\Gamma_{\alpha+\beta}} \right]$$

$$= \frac{\Gamma_{\alpha+\beta}\,\Gamma_{\alpha+k}}{\Gamma_\alpha\,\Gamma_{k+\alpha+\beta}}$$

(2) From the above formula, the mean and variance of the distribution can be derived as follows:-

putting $k=1$ we obtain

$$E(x) = \mu_x = \frac{\Gamma\alpha+1 \; \Gamma\alpha+\beta}{\Gamma\alpha \; \Gamma\alpha+\beta+1} = \frac{\alpha\,\Gamma\alpha \; \Gamma\alpha+\beta}{\Gamma\alpha \; (\alpha+\beta)\,\Gamma\alpha+\beta} \quad \left[\text{Since } \Gamma\alpha+1 = \alpha\,\Gamma\alpha\right]$$

$$\boxed{\mu_x = \frac{\alpha}{\alpha+\beta}}$$

Putting $k=2$ we get

$$E(x^2) = \frac{\Gamma\alpha+2 \; \Gamma\alpha+\beta}{\Gamma\alpha \; \Gamma\alpha+\beta+2} = \frac{(\alpha+1)\alpha\,\cancel{\Gamma\alpha}\;\cancel{\Gamma\alpha+\beta}}{\cancel{\Gamma\alpha}\,(\alpha+\beta+1)(\alpha+\beta)\,\cancel{\Gamma\alpha+\beta}}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$Var(x) = \sigma_x^2 = E(x^2) - [E(x)]^2$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\cancel{\alpha^3}+\cancel{\alpha^2\beta}+\cancel{\alpha^2}+\alpha\beta-\cancel{\alpha^3}-\cancel{\alpha^2\beta}-\cancel{\alpha^2}}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\boxed{\therefore \sigma_x^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

(5) The normal distribution

A Continuous r.v X is said to have normal distribution with parameters μ, σ^2 denoted as $X \sim N(\mu, \sigma^2)$ if the p.d.f of X is:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty}$$

Properties:

(1) The m.g.f of normal dist. is

$$\boxed{M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

Proof:

$$M_X(t) = E(e^{tX}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $y = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma y \Rightarrow dx = \sigma dy$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma y)} e^{\frac{-y^2}{2}} (\sigma dy)$$

$$= e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma y} e^{\frac{-y^2}{2}} dy$$

$$= e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y^2 - 2t\sigma y)}{2}} \, dy$$

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$$M_x(t) = e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{[(y^2 - 2\sigma t y + \sigma^2 t^2) - \sigma^2 t^2]}{2}} \, dy$$

$$= e^{\mu t} e^{\frac{\sigma^2 t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y - \sigma t)^2}{2}} \, dy$$

let $z = y - \sigma t \Rightarrow y = z + \sigma t \Rightarrow dy = dz$

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz \right]}_{=1 \text{ normal dist.}}$$

$$\therefore M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

(2) the mean of the normal dist. is :-

$$\boxed{\mu_x = E(x) = \mu}$$

proof:

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$M_x'(t) = \left(\mu + \frac{2\sigma^2 t}{2}\right) e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$= (\mu + \sigma^2 t) \, M_x(t)$$

$$\mu_X = E(X) = M_X'(0) = (\mu + 0)\, M_X(0) = \mu \cdot 1 = \mu$$

(3) the variance of the distribution is

$$\boxed{\sigma_X^2 = Var(X) = \sigma^2}$$

Proof:

$$M_X''(t) = (\mu + \sigma^2 t) M_X'(t) + \sigma^2 M_X(t) \quad [\text{since } M_X'(t) = (\mu + \sigma^2 t) M_X(t)]$$

$$M_X''(0) = (\mu + 0) M_X'(0) + \sigma^2 M_X(0)$$
$$= (\mu + 0)\mu + \sigma^2 \cdot 1 = \mu^2 + \sigma^2$$

$$\sigma^2 = Var(X) = M_X''(0) - [M_X'(0)]^2$$
$$= \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

Def: If the r.v $Z \sim N(0,1)$, then we say that Z distributed as standard normal distribution with p.d.f.

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty}$$

the mean, variance and moment generating functio

of the r.v Z is

$$\boxed{\mu_Z = 0 , \quad \sigma_Z^2 = 1 , \quad M_Z(t) = e^{t^2/2}}$$

theorem 1 If the r.v $X \sim N(\mu, \sigma^2)$ then

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

proof: By using the transformation method,

we have $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} , \quad -\infty < x < \infty$

The space of X denoted by A and the space of Z denoted by B are defined as:

$A = \{x : -\infty < x < \infty\}$, $B = \{z : -\infty < z < \infty\}$

$Z = u(x) = \frac{x-\mu}{\sigma}$ is (1-1) transformation maps A onto B

$x = u^{-1}(z) = \mu + \sigma z$ is (1-1) transformation maps B onto A

$$|J| = \left|\frac{dx}{dz}\right| = \sigma \Rightarrow g(z) = f\left[u^{-1}(z)\right] |J|$$

$$g(z) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}z^2} (\sigma) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

$$\therefore Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

Calculating the probabilities

The probabilities concerning the r.v. X which distributed as $N(\mu, \sigma^2)$ can be expressed in terms of probabilities concerning $Z = \frac{X-\mu}{\sigma}$ which distributed as $N(0,1)$. However an integral like $\int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$ cannot be evaluated. Insted, we use tables which approximate the value of this integral for different values of k. In general, the following rules are important.

① $Pr(Z<0) = Pr(Z>0) = 0.5$

② $Pr(Z<-z_1) = 1 - Pr(Z<z_1)$, $z_1 > 0$

③ $Pr(z_1 < Z < z_2) = Pr(Z<z_2) - Pr(Z<z_1)$

$= N(z_2) - N(z_1)$

Ex: Given that $x \sim N(2, 25)$, find $P_r(0 < x < 10)$

solution

$\therefore P_r(0 < x < 10) = P_r\left(\frac{0-2}{5} < z < \frac{10-2}{5}\right)$

$= P_r(-0.4 < z < 1.6)$

$= P_r(z < 1.6) - P_r(z < -0.4)$

$= P_r(z < 1.6) - [1 - P_r(z < 0.4)]$

$= N(1.6) - [1 - N(0.4)]$

$= 0.945 - [1 - 0.655] = 0.6$ [from tables]

theorem ②: If the r.v $x \sim N(\mu, \sigma^2)$ then the r.v $y = \left(\frac{x-\mu}{\sigma}\right)^2 \sim \chi^2(1)$

proof.

By using the m.g.f. method

$M_y(t) = E(e^{ty}) = E\left[e^{t\left(\frac{x-\mu}{\sigma}\right)^2}\right]$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{t\left(\frac{x-\mu}{\sigma}\right)^2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

putting $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma\, dz$

$M_x(t) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tz^2 - \frac{1}{2}z^2} (\sigma\, dz)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z^2\left(t-\frac{1}{2}\right)}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}(1-2t)}\, dz$$

let $w = z\sqrt{1-2t} \Rightarrow z = \frac{1}{\sqrt{1-2t}}\, w$, $dz = \frac{1}{\sqrt{1-2t}}\, dw$

$$\Rightarrow M_y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} \frac{1}{\sqrt{1-2t}}\, dw$$

$$= \frac{1}{\sqrt{1-2t}} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}}\, dw\right]}_{1} = \frac{1}{\sqrt{1-2t}}\,(1)$$

$M_y(t) = (1-2t)^{-1/2}$ which is the m.g.f. of

chisquare dist. with 1 degree of freedom

$$y = \left(\frac{x-\mu}{\sigma}\right)^2 \sim \chi^2(1)$$

theorem (3): If X_i, $i = 1, 2, \dots, n$ distributed as $N(0,1)$ then $\sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$

Proof: let $y = \sum_{i=1}^{n} X_i^2$, then $M_y(t) = E(e^{ty})$

$$M_y(t) = E\left(e^{t(X_1^2 + X_2^2 + \dots + X_n^2)}\right)$$

$$= E(e^{tX_1^2})\, E(e^{tX_2^2}) \dots E(e^{tX_n^2})$$

since each of $X_1, X_2, \dots, X_n \sim N(0,1)$ then each of $X_1^2, X_2^2, \dots, X_n^2 \sim \chi^2(1)$ (theorem 2)

$$M_y(t) = \underbrace{(1-2t)^{-1/2} (1-2t)^{-1/2} \dots (1-2t)^{-1/2}}_{n \text{ terms}}$$

$$M_y(t) = \left[(1-2t)^{-1/2}\right]^n = (1-2t)^{-n/2}$$

$$y = \sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$$

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(6) The student's t distribution

let the r.v. $W \sim N(0,1)$ and the r.v. $V \sim \chi^2(r)$ where W and V are stochastically independent. then $T = \frac{W}{\sqrt{\frac{V}{r}}}$ has a student's t distribution with p.d.f given by

$$g(t) = \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)} \frac{1}{\left(1+\frac{t^2}{r}\right)^{(r+1)/2}} \quad , -\infty < t < \infty$$

Proof: the joint p.d.f of W and V is

$$\phi(w,v) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}} (v)^{\frac{r}{2}-1} e^{-\frac{v}{2}}$$

let $t = \frac{w}{\sqrt{\frac{v}{r}}}$ and $u = v$ $\quad -\infty < w < \infty, 0 < v < \infty$ define a 1-1

(1) transformation mapping the space

(2) $\{(w,v): -\infty < w < \infty, 0 < v < \infty\}$ onto the

(3) space $\{(t,u): -\infty < t < \infty, 0 < u < \infty\}$

$$w = t\frac{\sqrt{u}}{\sqrt{r}} , \quad v = u , \quad J = \begin{vmatrix} \frac{\partial w}{\partial t} & \frac{\partial w}{\partial u} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial u} \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{u}}{\sqrt{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{\sqrt{u}}{\sqrt{r}}$$

$$g(t,u) = \cancel{\phi}\left[\frac{t\sqrt{u}}{\sqrt{r}}, u\right] |J|$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma\frac{r}{2}\,2^{r/2}} u^{\frac{r}{2}-1} e^{-\frac{u}{2}} e^{-\frac{t^2 u}{2r}} \frac{\sqrt{u}}{\sqrt{r}}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma\frac{r}{2}\,2^{r/2}} u^{\frac{r}{2}-1} e^{-\frac{u}{2}\left[1+\frac{t^2}{r}\right]} \frac{\sqrt{u}}{\sqrt{r}},$$

$$-\infty < t < \infty, \quad 0 < u < \infty$$

Now $g(t) = \int_0^\infty g(t,u)\,du$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\,\Gamma\frac{r}{2}\,2^{r/2}} u^{\frac{r}{2}-1} e^{-\frac{u}{2}\left[1+\frac{t^2}{r}\right]} \frac{\sqrt{u}}{\sqrt{r}}\,du$$

Putting $z = \frac{u}{2}\left[1+\frac{t^2}{r}\right] \Rightarrow u\left[1+\frac{t^2}{r}\right] = 2z$

$$u = \frac{2z}{1+\frac{t^2}{r}} \Rightarrow du = \frac{2}{1+\frac{t^2}{r}}\,dz$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\,\Gamma\frac{r}{2}\,2^{r/2}\,r^{1/2}} \left(\frac{2z}{1+\frac{t^2}{r}}\right)^{\frac{r}{2}-1} e^{-z} \frac{\sqrt{2z}}{\sqrt{1+\frac{t^2}{r}}} \frac{2\,dz}{1+\frac{t^2}{r}}$$

$$= \frac{1}{\sqrt{2\pi r}\,\Gamma\frac{r}{2}\,2^{r/2}} \frac{2^{\frac{r}{2}-1}}{\left(1+\frac{t^2}{r}\right)^{\frac{r}{2}-1}} \frac{\sqrt{2}}{\left(1+\frac{t^2}{r}\right)^{1/2}} \frac{2}{\left(1+\frac{t^2}{r}\right)} \int_0^\infty z^{\frac{r}{2}-1} z^{1/2} e^{-z}\,dz$$

$$= \frac{1}{\sqrt{2\pi r}\,\Gamma\frac{r}{2}} \frac{\sqrt{2}}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} \int_0^\infty z^{\frac{r}{2}+\frac{1}{2}-1} e^{-z}\, dz$$

$$\boxed{g(t) = \frac{\Gamma\frac{r+1}{2}}{\sqrt{\pi r}\,\Gamma\frac{r}{2}} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} \Gamma\frac{r+1}{2}, \quad -\infty < t < \infty)}$$

the mean and variance of distribution

the mean of student's t distribution is:

$E(t) = E\left(\frac{w}{\sqrt{\frac{v}{r}}}\right) = E(w)\, E\left(\frac{1}{\sqrt{\frac{v}{r}}}\right)$ [since w, v are indep.]

but $w \sim N(0,1)$, $E(w) = 0$

hence $E(t) = 0 \cdot E\left(\frac{1}{\sqrt{\frac{v}{r}}}\right) = 0$

- the variance of distribution is derived as follows

$var(t) = E(t^2) - [\underbrace{E(t)}_{0}]^2 = E(t^2)$

$var(t) = E\left[\frac{w}{\sqrt{\frac{v}{r}}}\right]^2 = E\left[\frac{w^2}{\frac{v}{r}}\right] = E(w^2)\, E\left(\frac{1}{\frac{v}{r}}\right)$

Since $w \sim N(0,1)$ then $E(w^2) = var(w) + [\underbrace{E(w)}_{0}]^2$

$= 1$

$$var(t) = 1 \cdot E\left(\frac{1}{\frac{V}{r}}\right) = r\,E\left(\frac{1}{V}\right)$$

but $V \sim \chi^2(r)$ then $E\left(\frac{1}{V}\right) = \int_0^\infty \frac{1}{V} \frac{1}{\Gamma\frac{r}{2}\, 2^{r/2}} V^{\frac{r}{2}-1} e^{-\frac{V}{2}} dV$

$$= \frac{1}{\Gamma\frac{r}{2}\, 2^{r/2}} \int_0^\infty V^{-1} V^{\frac{r}{2}-1} e^{-\frac{V}{2}} dV = \frac{1}{\Gamma\frac{r}{2}\, 2^{r/2}} \int_0^\infty V^{\frac{r}{2}-2} e^{-\frac{V}{2}} dV$$

Putting $z = \frac{V}{2} \Rightarrow V = 2z \Rightarrow dV = 2\,dz$

$$E\left(\frac{1}{V}\right) = \frac{1}{\Gamma\frac{r}{2}\, 2^{r/2}} \int_0^\infty (2z)^{\frac{r}{2}-2} e^{-z}\, 2\,dz$$

$$= \frac{2^{\frac{r}{2}-2}\cdot 2}{\Gamma\frac{r}{2}\, 2^{r/2}} \underbrace{\int_0^\infty z^{\frac{r}{2}-1-1} e^{-z}\, dz}_{\Gamma\frac{r}{2}}$$

$$= \frac{1}{2\,\Gamma\frac{r}{2}} \Gamma\left(\frac{r}{2}-1\right) = \frac{1}{2\left(\frac{r}{2}-1\right)\cancel{\Gamma\left(\frac{r}{2}-1\right)}} \cancel{\Gamma\left(\frac{r}{2}-1\right)}$$

$$= \frac{1}{2\left(\frac{r}{2}-1\right)} = \frac{1}{r-2}$$

$$var(t) = r\,E\left(\frac{1}{V}\right) = \frac{r}{r-2}$$

(7) The F distribution

let $X_1 \sim \chi^2(n_1)$ is independent from $X_2 \sim \chi^2(n_2)$

then the ration $f = \frac{X_1/n_1}{X_2/n_2}$ is said to have an F distribution with n_1, n_2 degrees of freedom

the p.d.f of the distribution is:-

$$g(f) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \frac{f^{\frac{n_1}{2}-1}}{\left[1+\frac{n_1}{n_2}f\right]^{\frac{n_1+n_2}{2}}}, f > 0$$

and we say that $f \sim F(n_1, n_2)$

The mean of the distribution is given by

$$\mu_f = E(f) = \frac{n_2}{n_2 - 2}, n_2 > 2$$

<u>proof</u>:- $E(f) = E\left(\frac{X_1/n_1}{X_2/n_2}\right) = \frac{\frac{1}{n_1}}{\frac{1}{n_2}} E\left(\frac{X_1}{X_2}\right) =$

$= \frac{n_2}{n_1} E(X_1) \cdot E\left(\frac{1}{X_2}\right)$

since $X_1 \sim \chi^2(n_1)$, $X_2 \sim \chi^2(n_2)$, then

$E(X_1) = n_1$

$$E(F) = \frac{n_2}{n_1} n_1 E\left(\frac{1}{X_2}\right) = n_2 E\left(\frac{1}{X_2}\right)$$

$$E\left(\frac{1}{X_2}\right) = \int_0^\infty \frac{1}{\Gamma\frac{n_2}{2} 2^{n_2/2}} \cdot \frac{1}{X_2} X_2^{\frac{n_2}{2}-1} e^{-\frac{X_2}{2}} dX_2$$

$$= \frac{1}{\Gamma\frac{n_2}{2} 2^{n_2/2}} \int_0^\infty X_2^{\frac{n_2}{2}-1-1} e^{-\frac{X_2}{2}} dX_2$$

$$z = \frac{X_2}{2} \Rightarrow 2z = X_2 \Rightarrow 2dz = dX_2$$

$$= \frac{1}{\Gamma\frac{n_2}{2} 2^{n_2/2}} \int_0^\infty (2z)^{\frac{n_2}{2}-2} e^{-z} 2dz$$

$$= \frac{1}{\Gamma\frac{n_2}{2} 2^{n_2/2}} 2^{\frac{n_2}{2}-2} \cdot 2 \int_0^\infty z^{\frac{n_2}{2}-2} e^{-z} dz$$

$$= \frac{1}{2\Gamma\frac{n_2}{2}} \Gamma\left(\frac{n_2}{2}-1\right) \qquad \Gamma\left(\frac{n_2}{2}-1\right)$$

$$= \frac{1}{2\left(\frac{n_2}{2}-1\right)\Gamma\left(\frac{n_2}{2}-1\right)} \Gamma\left(\frac{n_2}{2}-1\right) = \frac{1}{n_2-2}$$

$$E(F) = n_2 E\left(\frac{1}{X_2}\right) = \frac{n_2}{n_2-2}, \quad n_2 > 2$$

(7) the variance of the distribution is

$$var(f) = 2\mu_f^2 \frac{n_1+n_2-2}{n_1(n_2-4)}, \quad n_2 > 4$$

Problems:

(1) let t be a r.v with p.d.f

$$g(t) = c\left[1 + \frac{1}{5}t^2\right]^{-3}$$

find the value of C such that the r.v. follows t distribution

(2) let the r.v. $t \sim t(1)$ show that the C.D.F of t is $F(t) = 0.5 + \frac{1}{\pi}\tan^{-1} t$

(3) let $T = \frac{w}{\sqrt{v/r}}$, where $w \sim N(0,1)$ and $v \sim \chi^2(r)$ show that T^2 has an F distribution with parameters $r_1 = 1$, $r_2 = r$

(4) let f has F distribution with parameters r_1, r_2 prove that $\frac{1}{f}$ has an F distribution with parameters r_2 and r_1

③ Distribution of sample mean and Sample variance

① the distribution of $\overline{X}$

let $X_i, i=1,2,\dots,n$ be a r.s from $N(\mu,\sigma^2)$

the distribution $\overline{X}$ is derived by using the m.g.f as follows:

$$M_{\overline{X}}(t) = E(e^{t\overline{X}}) = E(e^{t\frac{\Sigma X_i}{n}}) = E(e^{\frac{t}{n}(X_1+X_2+\dots+X_n)})$$

$$= E(e^{\frac{t}{n}X_1})\, E(e^{\frac{t}{n}X_2}) \dots E(e^{\frac{t}{n}X_n})$$

$$= M_{X_1}(\tfrac{t}{n})\, M_{X_2}(\tfrac{t}{n}) \dots M_{X_n}(\tfrac{t}{n})$$

Since each of $X_1, X_2, \dots, X_n$ distributed $N(\mu,\sigma^2)$

then $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$M_{\overline{X}}(t) = \left[M_X(\tfrac{t}{n})\right]^n = \left[e^{\mu\frac{t}{n} + \frac{\sigma^2(\frac{t}{n})^2}{2}}\right]^n = e^{\mu t + \frac{\frac{\sigma^2}{n}t^2}{2}}$$

which is similar to the m.g.f of normal dist. with mean μ and Variance $\frac{\sigma^2}{n}$

$$\overline{X} \sim N(\mu, \tfrac{\sigma^2}{n})$$

$$f(\bar{x}) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{n}}} e^{-1/2 \frac{(\bar{x}-\mu)^2}{\sigma^2/n}} , \quad -\infty < \bar{x} < \infty$$

for example if $x_1, x_2, \ldots, x_n$ be a r.s from $N(1, 2)$

then $\bar{X} \sim N(1, \frac{2}{n})$ and $f(\bar{x}) = \frac{1}{\sqrt{2\pi \frac{2}{n}}} e^{\frac{-(\bar{x}-1)^2}{2 \frac{2}{n}}} , \quad -\infty < \bar{x} < \infty$

Properties :- According to theorems (1), (2), (3) if

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, then:-

① $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

② $\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$

③ $\sum_{i=1}^{k} \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(k)$

② Distribution of Sample variance S^2

The sample variance S^2 is defined as:-

$S^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$

consider the sum of squares

$$\Sigma (x_i - \mu)^2 = \Sigma [(x_i - \bar{x}) + (\bar{x} - \mu)]^2$$

$$= \Sigma (x_i - \bar{x})^2 + 2(\bar{x} - \mu)\underbrace{\Sigma (x_i - \bar{x})}_{=0} + n(\bar{x} - \mu)^2$$

$$\Sigma (x_i - \mu)^2 = \Sigma (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

dividing both sides by σ^2 we get,

$$\Sigma \left(\frac{x_i - \mu}{\sigma}\right)^2 = \frac{ns^2}{\sigma^2} + \left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)^2$$

Since $\Sigma\left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2(n)$ [theorem 3]

Also $\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$ [property 2]

it follows that $\frac{ns^2}{\sigma^2} \sim \chi^2(n-1)$ (additive property of χ^2 dist.)

Hence $E\left(\frac{ns^2}{\sigma^2}\right) = n-1 \Rightarrow \frac{n}{\sigma^2} E(s^2) = n-1$

$$E(s^2) = \sigma^2 \frac{n-1}{n}$$

$$Var\left(\frac{ns^2}{\sigma^2}\right) = 2(n-1) \Rightarrow \frac{n^2}{\sigma^4} Var(s^2) = 2(n-1)$$

$$Var(s^2) = \frac{2(n-1)}{n^2} \sigma^4$$

Ex: If $\bar{X}$ is the mean of r.v of size n from $N(M, 100)$. find n such that

$$Pr(M-5 < \bar{X} < M+5) = 0.954$$

solution:

let $x_1 = M-5$, $x_2 = M+5$

$$Pr[M-5 < \bar{X} < M+5] = Pr[x_1 < \bar{X} < x_2]$$

$$= Pr\left[\frac{x_1 - M}{\sigma/\sqrt{n}} < \frac{\bar{X} - M}{\sigma/\sqrt{n}} < \frac{x_2 - M}{\sigma/\sqrt{n}}\right]$$

$$= Pr\left[\frac{-5}{10/\sqrt{n}} < Z < \frac{5}{10/\sqrt{n}}\right] = Pr\left(\frac{-1}{2}\sqrt{n} < Z < \frac{1}{2}\sqrt{n}\right)$$

$$= Pr\left[Z < \frac{1}{2}\sqrt{n}\right) - \left[1 - Pr\left(Z < \frac{1}{2}\sqrt{n}\right)\right]$$

$$= 2Pr\left[Z < \frac{1}{2}\sqrt{n}\right] - 1 = 0.954$$

$$Pr\left(Z < \frac{1}{2}\sqrt{n}\right) = \frac{1.954}{2} = 0.977$$

$$N\left(\frac{1}{2}\sqrt{n}\right) = 0.977$$

$\frac{1}{2}\sqrt{n} = 2$ (approximatly) from tables

$$\sqrt{n} = 4 \Rightarrow n = 16$$

Ex: let S^2 be the variance of a r.s of size six from $N(\mu, 12)$ find $Pr(2.30 < S^2 < 22.2)$

solution

Since $\frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$, then

$\frac{6S^2}{12} = \frac{S^2}{2} \sim \chi^2(5)$

$Pr(2.30 < S^2 < 22.2) = Pr(1.15 < \frac{S^2}{2} < 11.1)$

$= Pr(\frac{S^2}{2} < 11.1) - Pr(\frac{S^2}{2} < 1.15)$

$= 0.950 - 0.050 = 0.90$ (from χ^2 tables)

Ex: let $X_1, X_2, \ldots X_n$ be a r.s. from $N(\mu, \sigma^2)$ let $\bar{X}$ and S^2 denote the mean and variance of the sample where $\bar{X}$ has S^2 are independent and $E(S^2) = \sigma^2$. show that $t = \frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t(n-1)$

توزيع شكل ترمينه

proof, Since $X \sim N(\mu, \sigma^2) \Rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

let $Z_1 = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \Rightarrow Z_1 \sim N(0,1)$

let $Z_2 = \frac{(n-1)S^2}{\sigma^2} \Rightarrow Z_2 \sim \chi^2(n-1)$

$$\frac{Z_1}{\sqrt{\frac{Z_2}{n-1}}} \sim t(n-1)$$

but $$\frac{Z_1}{\sqrt{\frac{Z_2}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\frac{\sqrt{S^2}}{\sqrt{\sigma^2}}} = \sqrt{n}\left(\frac{\bar{X}-\mu}{\cancel{\sigma}}\right)\frac{\cancel{\sigma}}{S}$$

$$t = \sqrt{n}\frac{(\bar{X}-\mu)}{S} \sim t(n-1)$$

تمارين التوزيعات

Problems :-

(1) Given that $X \sim Ber(1, \frac{1}{3})$ find

(i) the p.m.f of X ; (ii) μ_X, σ_X^2 and $M_X(t)$

(2) The m.g.f of a r.v X is $M_X(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^9$

(i) find the p.m.f of X. (ii) find the mean μ_X and variance σ_X^2, (iii) show that

$$Pr(\mu_X - 2\sigma_X < X < \mu_X + 2\sigma_X) = \sum_{x=1}^{5} C_x^9 \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

(3) let $X \sim b(2, p)$ and $Y \sim b(4, p)$ if $Pr(X \geq 1) = \frac{5}{9}$

find $Pr(Y \geq 1)$

(4) let X_1, X_2 be independent r.v.'s such that

$X_1 \sim P(4)$ and $X_2 \sim P(6)$ let $Y = X_1 + X_2$

(i) Find the p.m.f of Y, (ii) find μ_Y, σ_Y^2

(iii) find $Pr(Y \leq 1)$

5) Given that $X \sim P(\lambda)$, find the value of λ it is known that $f(x) = \frac{4}{x} f(x-1)$, $x = 1, 2, \dots$

6) let $X_i \sim Nb(r_i, p)$, $i = 1, 2, \dots, n$

show that $\sum_{i=1}^{n} X_i \sim Nb(\sum_{i=1}^{n} r_i, p)$

7) let $X \sim Nb(4, 0.3)$, (i) Find the p.m.f of X, (ii) find μ_x, σ_x^2, $M_x(t)$, (iii) let $y = 4 + 5X$

find μ_y, σ_y^2

Problems: تمارين التوزيعات

(1) let $X \sim U(0,1)$, use the transformation method to find the distribution of $y = -2\ln X$ then find M_y, d_y

(2) Given that $X_1, X_2, \dots, X_n$ are independent random variables where $X_i \sim G(\alpha_i, \beta)$, $i = 1, 2, \dots, n$ show that $y = \sum_{i=1}^{n} X_i \sim G\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$

(3) If $X_1 \sim G(2,3)$ and $X_2 \sim G(1,3)$ are two independent r.vs

i) find the distribution of $y = X_1 + X_2$

ii) find the mean and varinace of y

(4) let $X \sim B(\alpha, \beta)$, let $y = \ln \dfrac{X}{1-X}$ find $M_y(t)$

(5) In each case the r.v. X follow Beta distribution, find the value of the constant C

i) $f(x) = c\,x^2(1-x)^5$

ii) $f(x) = c\,(x - x^2)^{0.5}$

6) If $x \sim B(\alpha, \beta)$. show that $y = (1-x) \sim B(\beta, \alpha)$

7) If $x \sim N(0,1)$ find $E(x^{2k})$, $k \in I^{+}$, then find $E(x^2)$ and $E(x^4)$

8) If $x_1, x_2, \ldots, x_n$ are independent r.v.s where $x_i \sim N(\mu_i, \sigma_i^2)$; $i = 1, 2, \ldots n$ show that

$$y = \sum_{i=1}^{n} x_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

9) If $x_1 \sim N(\mu_1, \sigma_1^2)$, $x_2 \sim N(\mu_2, \sigma_2^2)$, where x_1, x_2 are independent r.v.s show that

$$y = x_1 - x_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$

10) If $X \sim \chi^2(n)$ and $(X+Y) \sim \chi^2(n+m)$ where X, Y are independent r.v.s. Use the m.g.f to find the distribution of Y

11) If $X \sim N(0,2)$, find $E(X^{k/2})$, where k is even positive number, then find $E(X^2)$

12) If the m.g.f of the r.v X is

$$M_X(t) = e^{3t + 8t^2}$$

i) find the distribution of X

ii) find the mean and variance of X

$S^2, \overline{X}$ توزيع

problems

1) let $\overline{X}$ be the mean of r.v of size (5) from $N(0, 125)$. find the value of c if $Pr(\overline{X} < c) = 0.90$, and from tables it is known that $N(1.282) = 0.90$

2) let $X_1, X_2, \ldots, X_n$ be a r.s from $G(\alpha, \beta)$. show that $\overline{X} \sim G(\alpha n, \frac{\beta}{n})$ then show that

$$E(\overline{X}) = \alpha\beta, \quad var(\overline{X}) = \frac{\alpha\beta^2}{n}$$

<u>Hint</u>: use $M_{\overline{X}}(t) = E(e^{t\overline{X}}) = E(e^{\frac{t}{n}(X_1 + X_2 \ldots + X_n)})$

3) let $X_1, X_2, \ldots X_n$ be a r.s from $G(3, 2)$. find the p.d.f of $\overline{X}$, then find $E(\overline{X})$ and $var(\overline{X})$

(4) Distribution of order statistics

let $x_1, x_2, \dots, x_n$ denote a r.s from a dist. of continuous type having p.d.f $f(x)$ which is positive provided $a < x < b$. let y_1 be the smallest of these x_i, y_2 the next x_i in order of magnitude, ... and y_n the Largest x_i, that is $y_1 < y_2 < \dots < y_n$ represent $x_1, x_2, \dots, x_n$ when the latter are arranged in ascending order of magnitude, then y_i, $i = 1, 2, \dots, n$ is called the ith order statistic of the r.s $x_1, x_2, \dots, x_n$

It can be shown that the joint p.d.f of $y_1, y_2, \dots, y_n$ is

$$g(y_1, y_2, \dots, y_n) = n!\, f(y_1)\, f(y_2) \dots f(y_n) \quad \text{--- (1)}$$

the marginal distribution of y_k is

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!} [F(y_k)]^{k-1} [1 - F(y_k)]^{n-k} f(y_k) \quad \text{--- (2)}$$

the joint p.d.f for any two order statistics y_i, y_j is

$$g(y_i, y_j) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} [F(y_i)]^{i-1} [F(y_j) - F(y_i)]^{j-i-1}$$

$$[1 - F(y_j)]^{n-j} f(y_i) f(y_j) \quad \ldots \; (3)$$

Ex: let $y_1 < y_2 < y_3 < y_4$ denote the order statistics of a r.s. of size 4 from a distribution having p.d.f

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & o.w \end{cases}$$

Find the p.d.f of y_3 then find $Pr(\frac{1}{2} < y_3)$

Solution: Applying formula (2) with $n = 4$ and $k = 3$ we get

$$g(y_3) = \frac{4!}{2!\;1!} [F(y_3)]^2 [1 - F(y_3)]^1 f(y_3)$$

since $F(x) = \int_0^x f(u)\,du = \int_0^x 2u\,du = \frac{2u^2}{2}\Big|_0^x = x^2$

then $F(y_3) = y_3^{\,2}$

$$g(y_3) = 12\left[y_3^2\right]^2\left[1-y_3^2\right]2y_3$$

$$= 24\, y_3^4 (1-y_3^2)\, y_3 = 24\, y_3^5 (1-y_3^2) \qquad 0<y_3<1$$

$$Pr\left(\tfrac{1}{2}<y_3\right) = \int_{1/2}^{1} 24\, y_3^5 (1-y_3^2)\, dy_3$$

$$= \int_{1/2}^{1} 24\,(y_3^5 - y_3^7)\, dy_3 = 24\left[\left(\frac{y_3^6}{6} - \frac{y_3^8}{8}\right)\right|_{1/2}^{1}$$

Distribution of functions of order statistic

① **The median** الوسيط

when the observations are arranged in ascending order of magnitude, then

ⓐ if n is odd, the median is the observation of order $\frac{n+1}{2}$

ⓑ If n is even, the median is the average observations of order $\frac{n}{2}$, $\frac{n}{2}+1$

If X is a r.v with p.d.f $f(x)$ and distribution function $F(x)$, then the value of the median is the value of x that satisfies the equation $F(x)=\frac{1}{2}$

Ex: let X_1, X_2, X_3 be a r.s from the dist.

$$f(x) = e^{-x}, \quad x>0$$

(i) Find the p.d.f of the smallest value of the sample

(ii) Find the joint p.d.f of the largest and smallest value of the sample.

(iii) find the p.d.f of the median and the value of the median.

Solution: let y_1 is the smallest value of the sample followed by y_2 then y_3

$$f(x) = e^{-x} \Rightarrow F(x) = \int_0^x f(u)\,du = \int_0^x e^{-u}\,du = -e^{-u}\Big|_0^x$$

$$= -e^{-x} + e^{0} = 1 - e^{-x}$$

From formula (2) with $k=1$ and $n=3$ we get

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!}\,[F(y_k)]^{k-1}\,(1-F(y_k))^{n-k}\,f(y_k)$$

$$g(y_1) = \frac{3!}{0!\,2!}\,[F(y_1)]^0\,[1-F(y_1)]^2\,f(y_1)$$

$$g(y_1) = 3\,(1-1+e^{-y_1})^2\,e^{-y_1} = 3\,e^{-2y_1}\,e^{-y_1} = 3e^{-3y_1}$$

(ii) From formula (3) with $i=1, j=3, n=3$

$$g(y_1, y_3) = \frac{3!}{0!\,1!\,0!}\,[F(y_1)]^0\,[F(y_3) - F(y_1)]^{3-1-1}$$

$$[1-F(y_3)]^0\,f(y_1)\,f(y_3)$$

$$= 6\,[1-e^{-y_3} - (1-e^{-y_1})]\,e^{-y_1}\,e^{-y_3}$$

$$= 6\,[\cancel{1} - e^{-y_3} - \cancel{1} + e^{-y_1}]\,e^{-y_1}\,e^{-y_3}$$

$$= 6\,(-e^{-y_3} + e^{-y_1})\,e^{-y_1}\,e^{-y_3} = 6\,(e^{-y_1} - e^{-y_3})\,e^{-(y_1+y_3)}$$

$0 < y_1 < y_3 < \infty$

(iii) the median is the observation of order $\frac{n+1}{2} = \frac{3+1}{2} = 2$

which is y_2

$$g(y_2) = \frac{3!}{1!\,1!}\, \overset{(1-e^{-y_2})}{F(y_2)}\, [1 - F(y_2)]\, f(y_2)$$

$$= 6\,(1-e^{-y_2})\, e^{-y_2}\, e^{-y_2} = 6\, e^{-2y_2}(1-e^{-y_2}) \qquad 0 < y_2 < \infty$$

To find the value of y_2 we get

$$F(y_2) = \frac{1}{2} \Rightarrow 1 - e^{-y_2} = \frac{1}{2} \Rightarrow e^{-y_2} = \frac{1}{2}$$

$$y_2 = \ln 2$$

(2) The Range

The range of the sample is the difference between the largest and smallest value of the sample that is $\boxed{R = y_n - y_1}$

Ex: let x_1, x_2, x_3 be a r.s from $B(2,1)$ and let $y_1 < y_2 < y_3$ be the order statistics of the sample

(a) Find the probability distribution of R

(b) Find the mean and variance of the distribution

Solution: $X \sim B(\overset{\alpha}{2}, \overset{B}{1})$

$$f(x;\alpha,B) = \frac{\Gamma_{\alpha+B}}{\Gamma_\alpha \Gamma_B} x^{\alpha-1}(1-x)^{B-1} \qquad 0<x<1$$

if $\alpha = 2$, $B = 1$ then $f(x) = \dfrac{\Gamma_3}{\Gamma_1 \Gamma_2} x (1-x)^0$

$$= 2x \quad , 0<x<1$$

$$F(x) = \int_0^x f(u)\,du = \int_0^x 2u\,du = \frac{2u^2}{2}\Big|_0^x = x^2$$

let $R = y_3 - y_1 = u_1(y_1, y_3)$, $Z = y_3 = u_2(y_1, y_3)$ } (1-1) transformation from space of y_1, y_3 to space of R, Z

$y_1 = Z - R = u_1^{-1}(R, Z)$, $y_3 = Z = u_2^{-1}(R, Z)$ } (1-1) transformation from space of R, Z to space of y_1, y_3

Applying formula (3) we get.

$$g(y_1, y_3) = 6\,[F(y_3) - F(y_1)]\,f(y_1)\,f(y_3)$$

$$= 6\,[y_3^2 - y_1^2]\,(2y_1)(2y_3)$$

$$= 24\,y_1 y_3\,[y_3^2 - y_1^2] \qquad 0<y_1<y_3<1$$

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial Z} & \frac{\partial y_1}{\partial R} \\ \frac{\partial y_3}{\partial Z} & \frac{\partial y_3}{\partial R} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$$g(R, Z) = g[u_1^{-1}(R, Z), u_2^{-1}(R, Z)]\,|J|$$

$= 24\left[z^2 - (z-R)^2\right](z-R)z \qquad 0<R<z<1$

$= 24\left[z^2 - \cancel{z^2} + 2Rz - R^2\right](z-R)z$

$= 24\left[2z^2R - R^2z\right](z-R)$

$= 24\left[2z^3R - R^2z^2 - 2z^2R^2 + R^3z\right]$

$= 24R\left[2z^3 - R^2z^2 - 2z^2R + R^2z\right]$

$= 24R\left[2z^3 - 3z^2R + R^2z\right]$

$$h(R) = \int_R^1 g(R,z)\,dz = \int_R^1 24R\left[2z^3 - 3z^2R + R^2z\right]dz$$

$$= 24R\left(\frac{2z^4}{4} - \frac{3z^3R}{3} + \frac{R^2z^2}{2}\right)\Big|_R^1$$

$$= 24R\left(\frac{1}{2} - R + \frac{R^2}{2} - \cancel{\frac{1}{2}R^4} + \cancel{R^4} - \cancel{\frac{R^4}{2}}\right)$$

$$= \frac{24}{2}R\left[1 - 2R + R^2\right] = 12R(R-1)^2$$

$(R^2 - 2R + 1) \qquad = 12(1-R)^2R$

$R \sim B(2,3)$

$$E(R) = \frac{\alpha}{\alpha+\beta} = \frac{2}{5}$$

$$\text{var}(R) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{6}{25(6)} = \frac{1}{25}$$

Ex: let $y_1 < y_2 < \dots < y_5$ denote the order statistics of a r.s. of size 5 from a distribution having p.d.f. $f(x) = e^{-x}$, $0 < x < \infty$

show that the statistics $z_1 = y_2$, $z_2 = y_4 - y_2$ are stochastically independent.

solution:

$F(x) = 1 - e^{-x}$, $0 < x < \infty$ and according to formula (2) we get

$$g(y_2, y_4) = \frac{5!}{1!\,1!\,1!}\left(1 - e^{-y_2}\right)\left[1 - e^{-y_4} - (1 - e^{-y_2})\right]$$
$$\left[1 - (1 - e^{-y_4})\right] e^{-y_2 - y_4}$$

$$= 5!\,(1 - e^{-y_2})\left[e^{-y_2} - e^{-y_4}\right] e^{-y_4}\, e^{-y_2 - y_4}$$

$0 < y_2 < y_4 < \infty$

let the space of y_2, y_4 is

$A = \{(y_2, y_4) : 0 < y_2 < y_4 < \infty\}$

the space of z_1, z_2 is

$$B = \{(z_1, z_2) : 0 < z_1 < \infty,\ 0 < z_2 < \infty\}$$

$$\left.\begin{aligned} z_1 &= y_2 = u_1(y_1, y_2) \\ z_2 &= y_4 - y_2 = u_2(y_1, y_2) \end{aligned}\right\} \text{(1-1) transformation maps A onto B}$$

$$\left.\begin{aligned} y_2 &= z_1 = u_1^{-1}(z_1, z_2) \\ y_4 &= z_2 + z_1 = u_2^{-1}(z_1, z_2) \end{aligned}\right\} \text{(1-1) transformation maps B onto A}$$

$$J = \begin{vmatrix} \frac{\partial y_2}{\partial z_1} & \frac{\partial y_2}{\partial z_2} \\ \frac{\partial y_4}{\partial z_1} & \frac{\partial y_4}{\partial z_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$g(z_1, z_2) = g\left(u_1^{-1}(z_1, z_2), u_2^{-1}(z_1, z_2)\right) |J|$$

$$= 120\ (1 - e^{-z_1})\left(e^{-z_1} - e^{-(z_2+z_1)}\right) e^{-(z_2+z_1)} \left(e^{-z_1-z_2-z_1}\right.$$

$$= 120\ (1 - e^{-z_1})\ e^{-z_1}\left(1 - e^{-z_2}\right) e^{-z_2}\ e^{-z_1}\ e^{-2z_1}\ e^{-z_2}$$

$$= 120\ e^{-4z_1}\left(1 - e^{-z_1}\right) e^{-2z_2}\left(1 - e^{-z_2}\right), \quad 0 < z_1 < \infty, \quad 0 < z_2 < \infty$$

the marginal distribution for each of z_1, z_2 are as follows :-

$$h(z_1) = \int_0^\infty g(z_1, z_2)\, dz_2 = \int_0^\infty 120\, e^{-4z_1}(1 - e^{-z_1})\, e^{-2z_2}(1 - e^{-z_2})\, dz_2$$

$$= 120\, e^{-4z_1}(1 - e^{-z_1}) \int_0^\infty e^{-2z_2}(1 - e^{-z_2})\, dz_2$$

$$= 120\, e^{-4z_1}\left(1-e^{-z_1}\right)\int_0^{\infty}\left(e^{-2z_2}-e^{-3z_2}\right)dz_2$$

$$= 120\, e^{-4z_1}\left(1-e^{-z_1}\right)\left(-\frac{1}{2}e^{-2z_2}+\frac{1}{3}e^{-3z_2}\Big|_0^{\infty}\right.$$

$$= 120\, e^{-4z_1}\left(1-e^{-z_1}\right)\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$= 120\, e^{-4z_1}\left(1-e^{-z_1}\right)\frac{3-2}{6}$$

$$= 20\, e^{-4z_1}\left(1-e^{-z_1}\right)$$

$$h(z_2) = \int_0^{\infty} g(z_1, z_2)\, dz_1$$

$$= 120\, e^{-2z_2}\left(1-e^{-z_2}\right)\int_0^{\infty} e^{-4z_1}\left(1-e^{-z_1}\right)dz_1$$

$$= 120\, e^{-2z_2}\left(1-e^{-z_2}\right)\int_0^{\infty} e^{-4z_1}-e^{-5z_1}\, dz_1$$

$$= 120\, e^{-2z_2}\left(1-e^{-z_2}\right)\left(-\frac{1}{4}e^{-4z_1}+\frac{1}{5}e^{-5z_1}\Big|_0^{\infty}\right.$$

$$= 120\, e^{-2z_2}\left(1-e^{-z_2}\right)\left(\frac{1}{4}-\frac{1}{5}\right)$$

$$= 120\, e^{-2z_2}\left(1-e^{-z_2}\right)\frac{5-4}{20}$$

$$= 6\, e^{-2z_2}\left(1-e^{-z_2}\right)$$

Since $g(z_1, z_2) = h(z_1)\, h(z_2)$ then z_1, z_2 are stochastially independent.

Problems :- ① let $y_1 < y_2 < y_3 < y_4$ be the order statistics of r.s of size 4 from the dist. having p.d.f $f(x) = e^{-x}$, $0 < x < \infty$, find $Pr(3 \le y_4)$

② let X_1, X_2, X_3 be a r.s from $f(x) = 2x$, $0 < x < 1$, compute the probability that the smallest of these X_i exceeds the median of the distribution.

Chapter two

1- Estimation Theory

let $x_1, x_2, \dots, x_n$ be a r.s. from a distribution having p.d.f $f(x,\theta)$, where $f(x,\theta)$ is of known from with unknown parameter θ, therefore it have to be estimator from the sample date. Two types of estimatier can be done, namely the point estimatior and the interval estimation.

Def. The point estimaton of θ is a rule (function) that assignes each element of the sample a value (estimate) of θ denoted as $\hat{\theta} = t(x_1, x_2, \dots, x_n)$

properties of Good Estimator :-

① unbiasedness: An estimator $\hat{\theta}$ is said to be unbiased estimator of θ iff $\boxed{E(\hat{\theta}) = \theta}$

otherwise the estimator is said to be biased

The value of biase $b(\theta)$ is defined as

$$\boxed{b(\theta) = E(\hat{\theta} - \theta) = E(\hat{\theta}) - \theta}$$

<u>Ex:</u> let $x_1, x_2, \dots, x_n$ be a r.s from $N(\mu, 1)$ show that

$\hat{\theta} = \bar{x}$ is unbiased estimator of μ

<u>solution:</u> we have to show that $E(\bar{x}) = \mu$

$E(\bar{x}) = E\left(\frac{\sum x_i}{n}\right) = \frac{1}{n} E(\sum x_i) = \frac{1}{n} \sum E(x_i)$.

Since $x_i \sim N(\mu, 1)$, then $E(x_i) = \mu$

$E(\bar{x}) = \frac{1}{n} \sum_{i=1}^{n} \mu = \frac{1}{n} n\mu = \mu$

$\hat{\theta} = \bar{x}$ is unbiased estimator of μ.

<u>Ex:</u> let $x_1, x_2, \dots, x_n$ be a r.s from $N(\mu, \sigma^2)$, show that

$\frac{1}{n-1} \sum (x_i - \bar{x})^2$ is unbiased est. of σ^2

<u>solution:</u> Recalling that $S^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$

$\sum (x_i - \bar{x})^2 = nS^2 \Rightarrow E\left[\frac{1}{n-1} \sum (x_i - \bar{x})^2\right] = \frac{1}{n-1} E(nS^2)$

$= \frac{n}{n-1} E(S^2) = \frac{n}{n-1} \left[\frac{n-1}{n} \sigma^2\right] = \sigma^2$

$\Rightarrow \frac{1}{n-1} \sum (x_i - \bar{x})^2$ is unbiased est. of σ^2

(2) Mean Square Error متوسط مربعات الخطأ

The mean square error (MSE) of an est. $\hat{\theta}$ is defined as

$$\boxed{MSE(\hat{\theta}) = E(\hat{\theta}-\theta)^2 = var(\hat{\theta}) + b^2(\theta)}$$

if $\hat{\theta}$ is unbiased then $b(\theta)=0$ and $MSE(\hat{\theta}) = var(\hat{\theta})$

The good estimator has MSE as small as possible

Ex: let $x_1, x_2, \dots, x_n$ be a r.s from $f(x,\theta) = \theta^x (1-\theta)^{1-x}, x=0,1$

Use MSE to compare between the two statistics (estimators) $\bar{X}, X_1$.

Solution: Since $X_i \sim$ Bernoulli $(1,\theta)$ then $E(X_i)=\theta$, $i=1,2,\dots,n$, and hence $E(X_1)=\theta$

Also, $E(\bar{X}) = E\left(\frac{\sum X_i}{n}\right) = \frac{1}{n} E(\sum X_i) = \frac{1}{n} \sum E(X_i) = \frac{1}{n}\sum\theta$

$= \frac{1}{n} n\theta = \theta$

Each of $X_1, \bar{X}$ are unbiased est. of θ.

$MSE(X_1) = var(X_1) = \theta(1-\theta)$

$MSE(\bar{X}) = var(\bar{X}) = var\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} var(\sum X_i) = \frac{1}{n^2}\sum var(X_i)$

$= \frac{1}{n^2} \sum \theta(1-\theta) = \frac{1}{n^2} n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}$

$MSE(\bar{X}) < MSE(X_1)$, This means that $\bar{X}$ is better than X_1

③ Consistency الاتساق

$\hat{\theta}$ is Consistent est. of θ if

$$\boxed{\begin{array}{l} \text{① } \hat{\theta} \text{ is unbiased} \\ \text{② } \lim_{n \to \infty} var(\hat{\theta}) = 0 \end{array}}$$

Ex. let $X_1, X_2, \ldots, X_n$ be a r.s from $P(\theta)$. Show that $\hat{\theta} = \bar{X}$ is Consistent est. of θ

Solution: Since $X \sim P(\theta)$, then $f(x,\theta) = \frac{e^{-\theta} \theta^{x}}{x!}$

$x = 0, 1, 2, \ldots$

$E(\hat{\theta}) = E(\bar{X}) = E\left(\frac{\sum X_i}{n}\right) = \frac{1}{n} \sum E(X_i) = \frac{1}{n} \sum \theta = \frac{1}{n} n\theta$

$= \theta \Rightarrow \hat{\theta} = \bar{X}$ is unbiased est. of θ

$var(\hat{\theta}) = var\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} \sum var(X_i) = \frac{1}{n^2} \sum \theta = \frac{n\theta}{n^2} = \frac{\theta}{n}$

$\lim_{n \to \infty} var(\hat{\theta}) = \lim_{n \to \infty} \frac{\theta}{n} = 0$, the two Conditions Consistency are satisfied $\Rightarrow \hat{\theta} = \bar{X}$ is Consistent

est. of θ

④ Minimum Variance Unbiased Estimate

If a statistic $T = t(x_1, x_2, \ldots, x_n)$ is such that

① T is unbiased statistic of θ.

② It has smallest variance among all the unbiased statistics of θ, then T is called a minimum variance unbiased estimate (MVUE) of θ.

Ex: let y_1 and y_2 be two stochastically independent unbiased statistics for θ. Say the variance of y_1 is twice the variance of y_2. Find the constants k_1 and k_2 so that $k_1 y_1 + k_2 y_2$ is an unbiased statistic with smallest possible variance for such a linear combination

Solution: Since each of y_1, y_2 and $k_1 y_1 + k_2 y_2$ are unbiased then $E(y_1) = \theta$, $E(y_2) = \theta$

$E(k_1 y_1 + k_2 y_2) = k_1 E(y_1) + k_2 E(y_2) = \theta$

$k_1\theta + k_2\theta = \theta \Rightarrow (k_1 + k_2)\theta = \theta$

$k_1 + k_2 = 1 \Rightarrow \boxed{k_2 = 1 - k_1}$

let $\sigma^2 = var(y_2) \Rightarrow var(y_1) = 2\sigma^2$

putting $Q = var(k_1 y_1 + k_2 y_2)$ then

$Q = k_1^2 var(y_1) + k_2^2 var(y_2)$

$= 2k_1^2 \sigma^2 + (1-k_1)^2 \sigma^2$

$\frac{\partial Q}{\partial k_1} = 4k_1 \sigma^2 - 2(1-k_1)\sigma^2 = 0$

$4k_1 - 2 + 2k_1 = 0 \Rightarrow 6k_1 = 2 \Rightarrow k_1 = \frac{1}{3}$

$k_2 = 1 - k_1 = 1 - \frac{1}{3} = \frac{2}{3}$

5) Efficiency ﺗﻌﺮﻳﻒ let T be unbiased est. for a Parameter θ. then T is called an efficient estimator of θ iff the variance of T attains the Rao-Cramer lower bound given by

$$\boxed{var(T) \geq \frac{1}{nE\left(\frac{\partial \ln f(x,\theta)}{\partial \theta}\right)^2}}$$

it can be shown that:-

$$\boxed{E\left[\frac{\partial \ln f(x,\theta)}{\partial \theta}\right]^2 = -E\left[\frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2}\right]}$$

__Ex:__ let $X_1, X_2, \ldots, X_n$ be a r.s from $P(\theta)$, show that $\bar{X}$ is an efficient statistic for θ

__solution:__ $f(x,\theta) = \dfrac{e^{-\theta}\theta^x}{x!}$, $x = 0,1,2,\ldots$

$$\ln f(x,\theta) = \ln \frac{e^{-\theta}\theta^x}{x!} = \ln e^{-\theta}\theta^x - \ln x!$$

$$= \ln e^{-\theta} + \ln \theta^x - \ln x! = -\theta + x\ln\theta - \ln x!$$

$$\frac{\partial \ln f(x,\theta)}{\partial\theta} = -1 + \frac{x}{\theta} = \frac{x-\theta}{\theta}$$

$$E\left(\frac{\partial \ln f(x,\theta)}{\partial\theta}\right)^2 = E\left(\frac{x-\theta}{\theta}\right)^2 = \frac{1}{\theta^2}\underbrace{E(x-\underset{E(x)}{\theta})^2}_{var(x)}$$

$= \dfrac{1}{\theta^2}E[x-E(x)]^2$ [because $X \sim P(\theta)$ and $E(x) = \theta$]

$$= \frac{1}{\theta^2}var(x) = \frac{1}{\theta^2}\theta = \frac{1}{\theta}$$

$$R.C.L.B = \frac{1}{n\frac{1}{\theta}} = \frac{\theta}{n}$$

Rao-Cramer lower bound

on the other hand we have

$$var(\bar{x}) = var\frac{\sum x_i}{n} = \frac{1}{n^2}\sum var X_i = \frac{1}{n^2}\sum\theta = \frac{1}{n^2}n\theta = \frac{\theta}{n}$$

$var(\bar{x}) = R.C.L.B$

$\bar{X}$ is an efficient statistic for θ.

<u>Ex</u>: let $x_1, x_2, \dots, x_n$ be a r.s from $N(0,\theta)$, show that $\hat{\theta} = \frac{\sum x_i^2}{n}$ is :

① efficient statistic for θ

② consistent statistic for θ.

<u>Solution</u>: ① $E(\hat{\theta}) = E\left(\frac{\sum x_i^2}{n}\right) = \frac{1}{n} \sum E(x_i^2)$

$= \frac{1}{n} \sum \left[var(x_i) + (E(x_i))^2 \right] = \frac{1}{n} \sum (\theta + 0)$

$= \frac{1}{n} n\theta = \theta$

$\hat{\theta} = \frac{\sum x_i^2}{n}$ is unbiased statistic for θ

$$f(x,\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}$$

$$\ln f(x,\theta) = -\frac{1}{2} \ln 2\pi\theta + \ln e^{-\frac{x^2}{2\theta}}$$

$$= -\frac{1}{2} \ln 2\pi - \frac{1}{2} \ln \theta - \frac{x^2}{2\theta}$$

$$\frac{\partial \ln f(x,\theta)}{\partial \theta} = \frac{-1}{2\theta} + \frac{x^2}{2\theta^2}$$

$$\frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}$$

$$E\left[\frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2}\right] = E\left[\frac{1}{2\theta^2} - \frac{x^2}{\theta^3}\right]$$

$$= \frac{1}{2\theta^2} - \frac{1}{\theta^3} E(x^2) = \frac{1}{2\theta^2} - \frac{1}{\theta^3} \left[var(x) + [E(x)]^2\right]$$

$$= \frac{1}{2\theta^2} - \frac{1}{\theta^3} (\theta + 0) = \frac{1}{2\theta^2} - \frac{1}{\theta^2} = \frac{-1}{2\theta^2}$$

$$R.C.L.B = \frac{1}{-n E\left[\frac{\partial^2 \ln f(x,\theta)}{\partial \theta^2}\right]} = \frac{1}{-n\left(-\frac{1}{2\theta^2}\right)} = \frac{2\theta^2}{n}$$

The derive $var(\hat{\theta}) = var(\Sigma x_i^2/n)$ it is known that since $x_i \sim N(0, \theta)$ then

$$\frac{x_i}{\sqrt{\theta}} \sim N(0,1) \quad \text{and} \quad \frac{x_i^2}{\theta} \sim \chi^2(1)$$

$$\Rightarrow \frac{\Sigma x_i^2}{\theta} \sim \chi^2(n)$$

$$E\left(\frac{\Sigma x_i^2}{\theta}\right) = n \quad , \quad var\left(\frac{\Sigma x_i^2}{\theta}\right) = 2n$$

$$\frac{1}{\theta^2} var(\Sigma x_i^2) = 2n \Rightarrow var \Sigma x_i^2 = 2n\theta^2$$

$$var(\hat{\theta}) = var\left(\frac{\Sigma x_i^2}{n}\right) = \frac{1}{n^2} var \Sigma x_i^2 = \frac{1}{n^2} 2n\theta^2$$

$$var(\hat{\theta}) = \frac{2\theta^2}{n^2}$$

$$var(\hat{\theta}) = R.C.L.B$$

$\hat{\theta} = \frac{\sum x_i^2}{n}$ is an eff. stat. for θ

(2) We proved the first condition of consistency

unbiased $\lim_{n \to \infty} Var(\hat{\theta})$

that is $\hat{\theta}$ is unbiased

$\lim_{n \to \infty} var(\hat{\theta}) = \lim_{n \to \infty} \frac{2\theta^2}{n} = 0$, thus the second condition of consistency is satisfied

$\therefore \hat{\theta} = \frac{\sum x_i^2}{n}$ is consistent est. of θ

(6) sufficiency الكفاية

(1) The Fisher Neyman theorem

let $x_1, x_2, \dots, x_n$ denote a r.s from a dist. that has p.d.f $f(x, \theta)$. let $y = u(x_1, x_2 \dots x_n)$ be a statistic whose p.d.f is $g(y, \theta)$.

Define $L(x_1, x_2, \dots, x_n, \theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdots f(x_n, \theta)$

$$= \prod_{i=1}^{n} f(x_i, \theta)$$

Then $y = u(x_1, x_2 \dots x_n)$ is a sufficient statistic for θ iff:-

$$\frac{L(x_1, x_2, \ldots, x_n, \theta)}{g(y, \theta)} = H(x_1, x_2, \ldots, x_n) \text{ does not depend upon } \theta.$$

Ex: let $x_1, x_2, \ldots, x_n$ denote ar.s from a distribution that has a p.d.f.

$$f(x,\theta) = \begin{cases} \theta^{x}(1-\theta)^{1-x} & , x = 0, 1, \quad 0 < \theta < 1 \\ 0 & \text{o.w} \end{cases}$$

Show that $y = \sum_{i=1}^{n} x_i$ is a suff. stat. for θ.

Solution:

$$L(x_1, x_2, \ldots, x_n, \theta) = \theta^{x_1}(1-\theta)^{1-x_1} \theta^{x_2}(1-\theta)^{1-x_2} \cdots \theta^{x_n}(1-\theta)^{1-x_n}$$

$$L = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

Since $x_i \sim$ Bernouli $(1,\theta) \Rightarrow y = \sum x_i \sim$ binomial (n,θ) since

$$M_y(t) = E(e^{ty}) = E(e^{t\sum x_i}) = E(e^{tx_1} e^{tx_2} \cdots e^{tx_n})$$

$$= E(e^{tx_1}) E(e^{tx_2}) \cdots E(e^{tx_n}) = M_{x_1}(t) M_{x_2}(t) \cdots M_{x_n}(t)$$

$$= [1-\theta+\theta e^{t}][1-\theta+\theta e^{t}] \cdots [1-\theta+\theta e^{t}] = [1-\theta+\theta e^{t}]^{n}$$

$\therefore g(y,\theta) = \binom{n}{y} \theta^{y} (1-\theta)^{n-y}, \quad y = 0, 1, \ldots, n$

$$\frac{L}{g} = \frac{\theta^{\sum x_i} (1-\theta)^{n-\sum x_i}}{\binom{n}{y} \theta^{y} (1-\theta)^{n-y}} = \frac{\cancel{\theta^{\sum x_i} (1-\theta)^{n-\sum x_i}}}{\binom{n}{\sum x_i} \cancel{\theta^{\sum x_i} (1-\theta)^{n-\sum x_i}}}$$

$$= \frac{1}{\binom{n}{\sum x_i}} = H(x_1, x_2, \ldots, x_n) \text{ does not contain } \theta.$$

$\therefore y = \sum x_i$ is a suff. stat. for θ.

Ex: let $y_1 < y_2 < \cdots < y_n$ denote the order statistics of a r.s $x_1, x_2, \ldots, x_n$ from the dist. that has p.d.f $f(x,\theta) = e^{-(x-\theta)}, \quad \theta < x < \infty$

$-\infty < \theta < \infty$

show that y_1 is a suff. stat. for θ

Solution:

$$L(x_1, x_2, \ldots, x_n, \theta) = e^{-(x_1-\theta)} e^{-(x_2-\theta)} \cdots e^{-(x_n-\theta)}$$

$$= e^{-\sum(x_i-\theta)} = e^{-\sum x_i + \sum \theta} = e^{-\sum x_i + n\theta}$$

$$g(y_1,\theta) = \frac{n!}{(1-1)!(n-1)!} [F(y_1)]^{1-1} [1-F(y_1)]^{n-1} f(y_1)$$

$$F(x) = \int_{\theta}^{x} e^{-(u-\theta)} du = -e^{-(u-\theta)} \Big|_{\theta}^{x} = -e^{-(x-\theta)} + e^{-(\theta-\theta)}$$

$$= -e^{-(x-\theta)} + 1 = 1 - e^{-(x-\theta)}$$

$$F(y_1) = 1 - e^{-(y_1-\theta)}$$

$$g(y_1,\theta) = \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} [1 - \underbrace{(1 - e^{-(y_1-\theta)})}_{-1+e^{-(y_1-\theta)}}]^{n-1} e^{-(y_1-\theta)} = n\, e^{-n(y_1-\theta)}$$

$$g(y_1,\theta) = n\, e^{-n(y_1-\theta)} \qquad \theta < y_1 < \infty$$

(from formula 2 of order stat.)

$$g(y_1,\theta) = n\, e^{-n(\min x_i)+n\theta}$$

$$\frac{L}{g} = \frac{e^{-\sum x_i + n\theta}}{n\, e^{-n(\min x_i)+n\theta}} = \frac{e^{-\sum x_i}}{n\, e^{-n(\min x_i)}} = H(x_1, x_2, \dots, x_n)$$

(does not contain θ)

$y_1 = \min(x_1, x_2, \dots, x_n)$ is a suff. stat for θ.

② The Factorization theorem

Let $X_1, X_2, \dots, X_n$ denote a r.s from a distribution that has p.d.f. $f(x,\theta)$. The statistic $T = t(x_1, x_2, \dots, x_n)$ is a sufficient statistic for θ iff we can find two non negative function k_1 and k_2 such that.

$$\boxed{L(x_1, x_2, \dots, x_n, \theta) = k_1(T,\theta)\, k_2(x_1, x_2, \dots, x_n)}$$

where $k_2(x_1, x_2, \dots, x_n)$ does not depend on θ

يقال عن الاحصاء T انه suff. للمعلمة θ اذا كان بالامكان تجزئة L
الى حاصل ضرب دالتين احداهما تحوي التقدير والمعلمة $k_1(T,\theta)$
والاخرى خالية من المعلمة θ تماما $k_2(x_1, x_2, \dots, x_n)$

<u>Ex:</u> let $X_1, X_2, \dots, X_n$ denote a r.s from a dist. which is $N(\theta, \sigma^2)$, $-\infty < \theta < \infty$, where the variance σ^2 is known. show that $\bar{X} = \frac{\sum X_i}{n}$ is suff. stat. for θ.

<u>Solution:</u> $X \sim N(\theta, \sigma^2)$

$$\therefore f(x;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$$L(x_1, x_2, \dots, x_n, \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-\theta)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_2-\theta)^2}{2\sigma^2}} \cdots \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_n-\theta)^2}{2\sigma^2}}$$

$$L(x_1, x_2, \dots, x_n, \theta) = (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum (x_i-\theta)^2}{2\sigma^2}}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum[(x_i-\bar{x})+(\bar{x}-\theta)]^2}{2\sigma^2}}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum[(x_i-\bar{x})^2+(\bar{x}-\theta)^2+\overset{=0}{2(x_i-\bar{x})(\bar{x}-\theta)}]}{2\sigma^2}}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum (x_i-\bar{x})^2}{2\sigma^2} + \frac{n(\bar{x}-\theta)^2}{2\sigma^2}}$$

$$= \underbrace{(2\pi\sigma^2)^{-n/2} e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2}}_{K_1} \quad \underbrace{e^{-\frac{\sum (x_i-\bar{x})^2}{2\sigma^2}}}_{K_2}$$

تحوي التقدير $\bar{X}$ والمعلمة θ — لا تحوي المعلمة

$\therefore \bar{X}$ is a suff. stat. for θ

EX: let $X_1, X_2, \dots, X_n$ denote a r.s from a dist.

$f(x,\theta) = \theta x^{\theta-1}, \quad 0 < x < 1$

Show that the product $T(X_1, X_2, \dots, X_n) = X_1 X_2 \dots X_n$ is a suff. stat. for θ.

Solution :-

$$L(x_1, x_2, \dots, x_n, \theta) = \underbrace{(\theta x_1^{\theta-1})}_{f(x_1,\theta)} \underbrace{(\theta x_2^{\theta-1})}_{f(x_2,\theta)} \dots \underbrace{(\theta x_n^{\theta-1})}_{f(x_n,\theta)}$$

$$= \theta^n (x_1 \cdot x_2 \dots x_n)^{\theta-1} = \underbrace{\theta^n (x_1 \cdot x_2 \dots x_n)^{\theta}}_{K_1} \underbrace{\frac{1}{x_1 \cdot x_2 \dots x_n}}_{K_2}$$

the product $x_1 \cdot x_2 \dots x_n$ is a suff. stat. for θ.

Problems :-

① let $X_1, X_2, \dots, X_n$ be a r.s. from

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & 0 < x < \infty, \quad 0 < \theta < \infty \\ 0 & \text{o.w} \end{cases}$$

show that $\bar{X}$ is unbiased statistic for θ.

② let $y_1 < y_2 < y_3$ be the order statistics of a r.s. of size 3 from the uniform dist. having p.d.f. $f(x,\theta) = \frac{1}{\theta}, \quad 0 < x < \theta$, $0 < \theta < \infty$

show that $4y_1$, $2y_2$ and $\frac{4}{3}y_3$ are all unbiased statistics for θ. find the variance of each of these unbiased statistics.

3.) let $x_1, x_2, \dots, x_n$ be a r.s from $P(\theta)$. show that $\sum x_i$ is a suff. stat. for θ.

4) show that the nth order statistic of a r.s. of size n from the uniform dist. having p.d.f $f(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$, $0 < \theta < \infty$ is a suff. statistic for θ.

② Methods of Estimator

2.1 The maximum liklihood Method

Def. Let $x_1, x_2, \dots x_n$ be a r.s. from a distribution having p.d.f $f(x, \theta)$ then:-

① the joint function $f(x_1, \theta) \cdot f(x_2, \theta) \dots f(x_n, \theta)$
$$= \prod_{i=1}^{n} f(x_i, \theta)$$ is called the liklihood function denoted as $L(x_1, x_2, \dots x_n, \theta)$.

(ii) let $\hat{\theta}$ be the value of θ that maximize L. thus, $\hat{\theta}$ is the root of the equation $\frac{\partial L}{\partial \theta} = 0$ such that $\frac{\partial^2 L}{\partial \theta^2} < 0$ and it is called the Maximum liklihood estimate (MLE) for θ.

(iii) the value of θ that maximize L, maximize $\ln L$ also, thus $\hat{\theta}$ may be regard as a solution of $\frac{\partial \ln L}{\partial \theta} = 0$, such that $\frac{\partial^2 \ln L}{\partial \theta^2} < 0$

The following assumptions have to be done:

① the first and second partial derivatives are continuous function of θ.

② the range of the r.v. X does not depend upon θ

properties of MLE

① MLE are consistent estimators

② If MLE exist then it is the most efficient in the class of such estimators.

③ If $\hat{\theta}$ is MLE for θ and $g(\theta)$ is the single valued function of θ, then $g(\hat{\theta})$ is the MLE for $g(\theta)$. this is called the invariance property [illegible]

Ex: let $X_1, X_2, \ldots, X_n$ be a r.s from the distribution having p.d.f

$$f(x,\theta) = \begin{cases} \theta x^{\theta-1} & 0<x<1 \\ 0 & o.w \end{cases}$$

Find the MLE for θ.

solution:

$$L(x_1, x_2, \ldots, x_n, \theta) = (\theta x_1^{\theta-1})(\theta x_2^{\theta-1}) \ldots (\theta x_n^{\theta-1})$$

$$= \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

$$\ln L = \ln \theta^n \prod_{i=1}^{n} x_i^{\theta-1} = \ln \theta^n + \ln \prod_{i=1}^{n} x_i^{\theta-1}$$

$$= n \ln \theta + (\theta-1) \ln \prod_{i=1}^{n} x_i$$

$$= n \ln \theta + (\theta-1) \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \sum \ln x_i = 0 \Rightarrow \frac{n}{\theta} = -\sum \ln x_i$$

$$\hat{\theta} = \frac{-n}{\sum \ln x_i}$$ is the MLE for θ.

Ex: let $x_1, x_2, \ldots, x_n$ be a r.s from $N(\mu, \sigma^2)$ use the MLE method to estimate μ and σ^2

solution:

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$$L(x_1, x_2, \ldots x_n, \mu, \sigma^2) = \prod_{i=1}^{n} f(x_i, \mu, \sigma^2)$$

$$L = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$$

$$\ln L = \frac{-n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i - \mu)^2$$

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{\sigma^2}\sum(x_i - \mu) = 0 \Rightarrow \sum(x_i - \mu) = 0$$

$$\sum x_i - n\mu = 0 \Rightarrow \hat{\mu} = \frac{\sum x_i}{n} = \bar{x}$$ is the MLE for μ.

$$\frac{\partial \ln L}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i - \hat{\mu})^2 = 0$$

$$\frac{n}{2\sigma^2} = \frac{1}{2\sigma^4}\sum(x_i - \hat{\mu})^2$$

$$\frac{2\sigma^4}{2\sigma^2} = \frac{\sum(x_i - \hat{\mu})^2}{n}$$

$$\hat{\sigma}^2 = \frac{\sum(x_i - \hat{\mu})^2}{n} = \frac{\sum(x_i - \bar{x})^2}{n}$$

Ex: let $x_1, x_2, \dots, x_n$ be a r.s drawn from

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} & 0 < x \le \theta \\ 0 & \text{o.w} \end{cases}, \quad 0 < \theta < \infty$$

Find the MLE for θ.

solution: $L(x_1, x_2, \dots, x_n, \theta) = \left(\frac{1}{\theta}\right)\left(\frac{1}{\theta}\right)\cdots\left(\frac{1}{\theta}\right) = \frac{1}{\theta^n}$

We can't use the differentiation method because

the range of X depend upon θ, but it is clear that L has maximum value at the smallest value of θ. which coincide with the maximum value of X hence, $\hat{\theta} = \max(x_i) =$ the Largest order statistic of the sample.

Ex: let $X_1, X_2, \dots, X_n$ be a r.s from a distribution having p.d.f.

$$f(x, \alpha, \beta) = \begin{cases} \beta e^{-\beta(x-\alpha)} & \alpha \le x, \beta \ge 0 \\ 0 & \text{o.w} \end{cases}$$

Find the MLE for α, β.

Solution:

$$L(x_1, x_2, \dots, x_n, \alpha, \beta) = \beta^n e^{-\beta \sum_{i=1}^{n} (x_i - \alpha)}$$

the MLE for α can't be found by the method of differentiation since the range of X depend upon α.

It is clear that L has maximum value at the largest value of α which coincide with the smallest value of x. Hence $\hat{\alpha} = \min(x_i)$ = the smallest order statistic of the sample.

To find $\hat{\beta}$ we can use the differentiation method as follows.

$$\ln L = \ln\left(\beta^n e^{-\beta\sum_{i=1}^{n}(x_i-\hat{\alpha})}\right)$$

$$= \ln \beta^n - \beta \sum (x_i - \hat{\alpha})$$

$$= n \ln \beta - \beta \sum (x_i - \hat{\alpha})$$

$$= n \ln \beta - \beta \sum (x_i - \min(x_i))$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum (x_i - \min(x_i)) = 0$$

$$\frac{n}{\beta} = \sum (x_i - \min(x_i))$$

$$\hat{\beta} = \frac{n}{\sum (x_i - \min(x_i))} = \frac{n}{\sum x_i - \sum \min(x_i)} = \frac{n}{n\bar{x} - n\min(x_i)}$$

$$= \frac{1}{\bar{x} - \min(x_i)}$$

Ex: let $X_1, X_2, \dots, X_n$ be a r.s from a dist. having p.d.f

$$f(x,\theta) = \begin{cases} \theta^{x}(1-\theta)^{1-x} & , x = 0,1 \\ 0 & \text{o.w} \end{cases}$$

Find the MLE for $w = \frac{\theta}{1-\theta}$

Solution: At the first we find MLE for θ

$$L(x_1, x_2, \dots, x_n, \theta) = f(x_1,\theta) \cdot f(x_2,\theta) \dots f(x_n,\theta)$$

$$= \theta^{x_1}(1-\theta)^{1-x_1} \, \theta^{x_2}(1-\theta)^{1-x_2} \dots \theta^{x_n}(1-\theta)^{1-x_n}$$

$$= \theta^{\sum x_i}(1-\theta)^{\sum(1-x_i)} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$\ln L = \ln\left(\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\right) = \ln \theta^{\sum x_i} + \ln(1-\theta)^{n-\sum x_i}$$

$$= \sum x_i \ln\theta + (n - \sum x_i)\ln(1-\theta)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1-\theta} = 0$$

$$\frac{\sum x_i}{\theta} = \frac{n - \sum x_i}{1-\theta} \Rightarrow \frac{1-\theta}{\theta} = \frac{n - \sum x_i}{\sum x_i}$$

$$\frac{1}{\theta} - \cancel{1} = \frac{n}{\sum x_i} - \cancel{1} \Rightarrow \hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$$

$$\hat{w} = \frac{\hat{\theta}}{1-\hat{\theta}} = \frac{\bar{x}}{1-\bar{x}} \quad \text{is the MLE for } w$$

خاصية

(according to the invariance property)

Ex: Eight trials are conducted of a given system with the following results (S, F, S, F, S, S, S, S) where S denote success and F denote failure find the MLE of p the probability of the successful events.

Solution:
let the r.v. X denote the success event

then $X = \begin{cases} 1 & \text{if the event } S \text{ occur} \\ 0 & \text{if the event } S \text{ does not occur} \end{cases}$

$X \sim Ber(1, p)$ then $f(x) = p^x (1-p)^{1-x}$, $x = 0, 1$

$L = f(x_1, p) \cdots f(x_n, p) = p^{x_1}(1-p)^{1-x_1} \cdots p^{x_n}(1-p)^{1-x_n}$

$= p^{\sum x_i} (1-p)^{n-\sum x_i} = p^6 (1-p)^{8-6} = p^6 (1-p)^2$

$\ln L = \ln(p^6 (1-p)^2) = 6\ln p + 2\ln(1-p)$

$\frac{d \ln L}{dp} = \frac{6}{p} - \frac{2}{1-p} = 0$

$\frac{6}{p} = \frac{2}{1-p} \Rightarrow \frac{1-p}{p} = \frac{2}{6} \Rightarrow \frac{1}{p} - 1 = \frac{2}{6}$

$\frac{1}{p} = 1 + \frac{2}{6} = \frac{8}{6} \Rightarrow \hat{p} = \frac{6}{8} = \frac{3}{4}$ is the MLE of p

2) The moments method:

let $f(x, \theta_1, \theta_2, \dots, \theta_k)$ be the p.d.f of the population with k parameters $\theta_1, \theta_2, \dots, \theta_k$. By this method we equate the population moments $\mu_r' = E(x^r)$ with the sample moments $m_r = \frac{1}{n} \sum x^r$, $r = 1, 2, \dots, k$ then solving for the unknown parameters.

Ex: let $X_1, X_2, \dots, X_n$ be a r.s from $p(\theta)$. Find the moment estimator for θ.

Solution we have the population moment

$\mu_1 = E(X) = \theta$, and the sample moment $m_1 = \frac{1}{n} \sum x_i = \bar{X}$

$\mu_1 = m_1 \Rightarrow \hat{\theta} = \bar{X}$ is the moment est. for θ.

Ex. 2 Let $X_1, X_2, \dots X_n$ be a r.s from $f(x, \theta) = \frac{1}{\theta}$, $0 < x \le \theta$

Find the moment est. for θ

Solution: $\mu_1 = E(x) = \int_0^\theta x f(x)\, dx = \int_0^\theta x \frac{1}{\theta}\, dx = \frac{1}{\theta} \frac{x^2}{2} \Big|_0^\theta$

$= \frac{\theta^2}{2\theta} = \frac{\theta}{2}$

$m_1 = \frac{1}{n} \sum x_i \; \bar{x}$

$\mu_1 = m_1 \Rightarrow \frac{\theta}{2} = \bar{x} \Rightarrow \hat{\theta} = 2\bar{x}$ is the moment est. for θ

<u>Ex:</u> let $x_1, x_2, \ldots x_n$ be a r.s. from $U(\alpha, \beta)$

find the moment est. for α and β

solution

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{\beta - \alpha} & , \alpha < x < \beta \\ 0 & o.w \end{cases}$$

$\mu_1 = E(x) = \frac{\alpha + \beta}{2} \quad , m_1 = \frac{1}{n} \sum x_i = \bar{x}$

$\mu_1 = m_1 \Rightarrow \frac{\alpha + \beta}{2} = \bar{x} \quad \text{---} (1)$

$\mu_2 = E(x^2) = Var(x) + [E(x)]^2 = \frac{(\beta - \alpha)^2}{12} + \left(\frac{\alpha + \beta}{2}\right)^2$

$m_2 = \frac{1}{n} \sum x_i^2$

$\mu_2 = m_2 \Rightarrow \frac{(\beta - \alpha)^2}{12} + \left(\frac{\alpha + \beta}{2}\right)^2 = \frac{1}{n} \sum x_i^2 \quad \text{---} (2)$

by (1) $\Rightarrow \frac{(2\bar{x} - \alpha - \alpha)^2}{12} + \bar{x}^2 = \frac{1}{n} \sum x_i^2$

$\frac{4(\bar{x} - \alpha)^2}{12} + \bar{x}^2 = \frac{1}{n} \sum x_i^2$

$$\frac{(\bar{x}-\alpha)^2}{3} + \bar{x}^2 = \frac{1}{n}\sum x_i^2$$

$$(\bar{x}-\alpha)^2 + 3\bar{x}^2 = \frac{3\sum x_i^2}{n}$$

$$(\bar{x}-\alpha)^2 = \frac{3}{n}\sum x_i^2 - 3\bar{x}^2$$

$$\bar{x}-\alpha = \sqrt{\frac{3}{n}\sum x_i^2 - 3\bar{x}^2}$$

$$\hat{\alpha} = \bar{x} - \sqrt{\frac{3}{n}\sum x_i^2 - 3\bar{x}^2}$$

$$\hat{\beta} = 2\bar{x} - \bar{x} + \sqrt{\frac{3}{n}\sum x_i^2 - 3\bar{x}^2}$$

$$= \bar{x} + \sqrt{\frac{3}{n}\sum x_i^2 - 3\bar{x}^2}$$

the moment est. for α and β

Exercises

(1) let $x_1, x_2, \ldots, x_n$ be a r.s from $P(\theta)$ find the MLE for $Pr(X > 0)$

Ex. H.W

(2) let $x_1, x_2, \ldots, x_n$ be a r.s. from

$$f(x, \theta_1, \theta_2) = \begin{cases} \frac{1}{\theta_2} e^{-\frac{(x-\theta_1)}{\theta_2}} & , \theta_1 \le x < \infty \\ & -\infty < \theta_1 < \infty, 0 < \theta_2 < \infty \\ 0 & \text{o.w} \end{cases}$$

find the MLE for θ_1 and θ_2

3) let $x_1, x_2, \dots, x_n$ be a r.s from $N(\mu, \sigma^2)$

find the moment est. for μ and σ^2

4) let $x_1, x_2, \dots, x_n$ be a r.s from $G(\alpha, \beta)$

find the moment est. for α and β.

3-4-3. The method of least squares

Suppose that we can write the observations in the form:

$$y_i = g_i(\theta_1, \theta_2, \cdots \theta_k) + \epsilon_i \ , \quad i = 1, 2, \cdots n$$

where g_i are known functions and the real numbers $\theta_1, \theta_2, \cdots \theta_k$ are the unknown parameters of interest. Suppose that ϵ_i satisfy the conditions:

1) $E(\epsilon_i) = 0$, $var(\epsilon_i) = \sigma^2 > 0$, $cov(\epsilon_i, \epsilon_j) = 0$

$i = 1, 2, \cdots n$, $j = 1, 2, \cdots n$

The method of least squares says that we should find the point $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \cdots \hat{\theta}_k)$ which makes the expected value vector as close as possible to the observed value, that is we should minimize

$$\sum_{i=1}^{n} [y_i - E(y_i)]^2$$

Ex) Let $y_i = \theta_1 + \epsilon_i$, $i = 1, 2, \cdots n$ Estimate θ_1 using LS method.

<u>Solution.</u> We have $E(y_i) = E(\theta_1 + \epsilon_i)$

$$= E(\theta_1) + E(\epsilon_i) = \theta_1 \implies$$

$$\text{Let } Q = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} [y_i - E(y_i)]^2 = \sum_{i=1}^{n} (y_i - \theta_1)^2$$

$$\frac{\partial Q}{\partial \theta_1} = -2\sum (y_i - \theta_1) = 0 \Rightarrow \sum (y_i - \theta_1) = 0$$

$$\Rightarrow \sum y_i - n\theta_1 = 0 \Rightarrow \sum y_i = n\theta \Rightarrow \hat{\theta}_1 = \frac{\sum y_i}{n} = \bar{y}$$

is the LS est. for θ_1

Ex) Let X_1, X_2, X_3 be three random variables with the same variance σ^2. Let $E(X_1) = \theta_1$, $E(X_2) = \theta_1 + \theta_2$, $E(X_3) = 2\theta_1 + \theta_2$. Find the LS estimators for θ_1 and θ_2, then find the mean and variance for each est.

<u>Solution</u> Let $Q = \sum_{i=1}^{3} [X_i - E(X_i)]^2$

$$= [X_1 - E(X_1)]^2 + [X_2 - E(X_2)]^2 + [X_3 - E(X_3)]^2$$

$$= (X_1 - \theta_1)^2 + (X_2 - \theta_1 - \theta_2)^2 + (X_3 - 2\theta_1 - \theta_2)^2$$

$$\frac{\partial Q}{\partial \theta_1} = 0 \Rightarrow -2(X_1 - \theta_1) - 2(X_2 - \theta_1 - \theta_2)$$

$$-4(X_3 - 2\theta_1 - \theta_2) = 0 \Rightarrow$$

$$-2X_1 + 2\theta_1 - 2X_2 + 2\theta_1 + 2\theta_2 - 4X_3 + 8\theta_1 + 4\theta_2 =$$

$$\Rightarrow 6\theta_1 + 3\theta_2 = X_1 + X_2 + 2X_3 \quad \ldots \quad (1)$$

$$\frac{\partial Q}{\partial \theta_2} = 0 \Rightarrow 3\theta_1 + 2\theta_2 = X_2 + X_3 \quad \ldots \quad (2)$$

Solving eq(1), eq(2) for θ_1, θ_2 we get:

$$\hat{\theta}_1 = \frac{1}{3}(2X_1 - X_2 + X_3), \quad \hat{\theta}_2 = X_2 - X_1$$ are the LS est. for θ_1, θ_2.

(Complete the solution)

Ex) Let $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, $i = 1, 2, \dots n$ (simple linear regression model). Estimate β_0, β_1 using LS method.

<u>Solution</u> Let $Q = \sum \epsilon_i^2 = \sum [y_i - E(y_i)]^2$

$$= \sum (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial Q}{\partial \beta_0} = 0 \Longrightarrow -2\sum (y_i - \beta_0 - \beta_1 x_i) = 0 \Longrightarrow$$

$$\sum (y_i - \beta_0 - \beta_1 x_i) = 0 \dots \quad (1)$$

$$\frac{\partial Q}{\partial \beta_1} = 0 \Longrightarrow -2\sum x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \Longrightarrow$$

$$\sum x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \dots \quad (2)$$

From eq(1) we obtain: $\sum y_i = n\beta_0 + \beta_1 \sum x_i \dots (3)$

From eq(2) we obtain $\sum x_i y_i = \beta_0 \sum x_i + \beta_1 \sum x_i^2 \dots (4)$

Solving eq(3), eq(4) for β_0, β_1 we obtain:

$$\hat{\beta}_0 = \frac{\sum y_i \sum x_i^2 - (\sum x_i)(\sum x_i y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

$$\hat{\beta}_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

are the LS estimators for β_0, β_1

Ex) For the simple linear regression model

Show that $\hat{\beta}_1$ can be written as:

$$\hat{\beta}_1 = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2} = \frac{\sum(x_i-\bar{x})y_i}{\sum(x_i-\bar{x})^2}$$

then show that $\hat{\beta}_1$ is unbiaser est. for β

Solution: we have: $\sum(x_i-\bar{x})(y_i-\bar{y}) =$

$$= \sum x_i y_i - \bar{x}\underbrace{\sum y_i}_{n\bar{y}} - \bar{y}\sum x_i + n\bar{x}\bar{y} = \sum x_i y_i - n\bar{x}\bar{y}$$

$$= \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}$$

$$\sum(x_i-\bar{x})^2 = \sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2 = \sum x_i^2 - n\bar{x}^2$$

$$= \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

$$\Rightarrow \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2} = \frac{\sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}}{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}$$

$$= \frac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2} = \hat{\beta}_1$$

Also we have $\sum(x_i-\bar{x})y_i = \sum x_i y_i - \bar{x}\sum y_i$

$$= \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum(x_i-\bar{x})y_i}{\sum(x_i-\bar{x})^2}$$

$$E(\hat{\beta}_1) = E\,\frac{\sum(x_i-\bar{x})y_i}{\sum(x_i-\bar{x})^2} = \frac{\sum(x_i-\bar{x})E y_i}{\sum(x_i-\bar{x})^2}$$

$$= \frac{\sum(x_i-\bar{x})(\beta_0+\beta_1 x_i)}{\sum(x_i-\bar{x})^2} = \beta_0\underbrace{\frac{\sum(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}}_{=0} + \beta_1\frac{\sum x_i(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}$$

$$= \beta_1\frac{\sum x_i^2 - n\bar{x}^2}{\sum x_i^2 - n\bar{x}^2} = \beta_1$$

8-2 Interval Estimation

Def: An $(1-\alpha)\%$ confidence interval (C.I.) estimator is an interval whose end points are functions of the sample statistics such that if we could generate indefinitely samples, the interval should contain the true parameters $(1-\alpha)\%$ of the times.

Constructing C.Is.:-

The following steps are necessary to construct the C.I.

step (1): Obtain the probability distribution of the point estimator for the unknown parameter

step (2) Standardize the estimator such that we get a r.v with completely known distribution.

step (3): Construct C.I for standardized r.v. then solve for the unknown parameter.

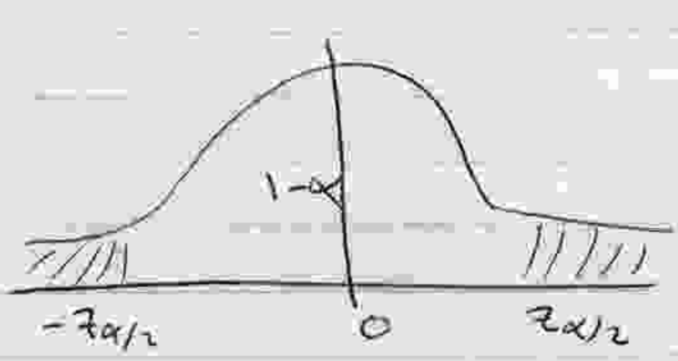
8-2-1 C.I. for means of normal population

(i) if σ^2 is known :- let $X_1, X_2, \dots, X_n$ be a r.s. from normal population with unknown mean μ and known variance σ^2 Applying the above steps:-

① the sample mean $\bar{X}$ is a point estimate of μ with probability distribution $N(\mu, \frac{\sigma^2}{n})$

② standardizing $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

③ the values $-Z_{\alpha/2}, Z_{\alpha/2}$ place $\frac{1}{2}\alpha$ in each tail of normal distribution



Ex

therefore

$$Pr\left[-Z_{\alpha/2} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < Z_{\alpha/2}\right] = 1-\alpha$$

solving for μ we obtain

$$\boxed{Pr\left[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1-\alpha}$$

where $0<\alpha<1$ and selected often to be $0.1, 0.01$ or 0.05

<u>Ex:</u> Find 95% C.I for the mean of normal population $N(\mu, \overset{\sigma^2}{25})$ if it is known that $\bar{X} = 10$, $n = 100$

<u>Solution:</u> we have $1-\alpha = 0.95 \Rightarrow \alpha = 0.05$

$\frac{\alpha}{2} = \frac{0.05}{2} = 0.025$ From tables of standard normal distribution we get $z_{\alpha/2} = z_{0.025} = 1.96$

$$Pr\left[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1-\alpha$$

$$Pr\left[10 - (1.96)\frac{5}{\sqrt{100}} < \mu < 10 + (1.96)\frac{5}{\sqrt{100}}\right] = 0.95$$

$$Pr\left[10 - (1.96)\frac{5}{10} < \mu \; 10 + (1.96)\frac{5}{10}\right] = 0.95$$

$$Pr\left[9.02 < \mu < 10.98\right] = 0.95$$

lower bound $C_L = 9.02$
upper bound $C_u = 10.98$

ii) If σ^2 is unknown

(a) For small samples ($n < 30$)

In this case the r.v $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$

Applying the steps stated earlier we get

$$\boxed{Pr\left[\bar{X} - t_{\alpha/2} \frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2} \frac{S}{\sqrt{n}}\right] = 1 - \alpha}$$

Ex: let $\bar{X} = 20$, $S^2 = 9$ denote the mean and variance of a r.s of size 15 from $N(\mu, \sigma^2)$

Find 95% C.I. for μ.

Solution: we have $1 - \alpha = 0.95 \Rightarrow \alpha = 0.05$, $\frac{\alpha}{2} = 0.025$

From tables of t distribution we get

$t_{\alpha/2}(n-1) = t_{0.025}(14) = 2.145$

As another way to represent C.I we write

$$\text{C.I. for } \mu = \bar{X} \pm t_{\alpha/2} \frac{S}{\sqrt{n}} = 20 \pm (2.145) \frac{\sqrt{9}}{\sqrt{15}}$$

$$= 20 \pm (2.145) \frac{3}{\sqrt{15}} \quad ,$$

$C_L = 18.338$, $C_u = 21.661$

b) For Large Samples ($n > 30$)

In this case and form statistical inference theory the distribution of the r.v.

$t = \frac{\sqrt{n}(\bar{X} - \mu)}{S}$ will converge to $N(0,1)$

which means that we can use the standard normal tables instead of t distribution table and hence

$$\boxed{C.I \text{ for } \mu = \bar{X} \pm Z_{\alpha/2} \frac{S}{\sqrt{n}}}$$

Ex: let $\bar{X} = 20$, $S^2 = 16$ denote the mean and variance of a r.s of size 100. Find 99% c.I for μ.

Solution: we have $1 - \alpha = 0.99 \Rightarrow \alpha = 0.01 \Rightarrow \frac{\alpha}{2} = 0.005$.

$Z_{\alpha/2} = Z_{0.005} = 2.58$ (from table)

$C.I. = \bar{X} \pm Z_{\alpha/2} \frac{S}{\sqrt{n}} = 20 \pm (2.58) \frac{\sqrt{16}}{\sqrt{100}} = 20 \pm (2.58) \frac{4}{10}$

$= (18.968, 21.032)$

9-2-2: C.I. for difference between two means

(i) If σ_1^2, σ_2^2 are known

let $\overline{X_1}, \overline{X_2}$ denote the means of two independent random samples of size n_1, n_2 from normal populations with variances σ_1^2, σ_2^2 respectively

A $(1-\alpha)\%$ C.I. for $\mu_1 - \mu_2$ is

$$\boxed{\text{C.I. for } \mu_1 - \mu_2 = (\overline{X_1} - \overline{X_2}) \pm Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

(ii) If σ_1^2, σ_2^2 are unknown

(a) For Large samples ($n_1, n_2 > 30$)

A $(1-\alpha)\%$ C.I. for $\mu_1 - \mu_2$ is given by

$$\boxed{\text{C.I. for } \mu_1 - \mu_2 = (\overline{X_1} - \overline{X_2}) \pm Z_{\alpha/2} \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

where S_1^2, S_2^2 denote the variances of the two samples.

Ex: Construct 96% C.I. for $\mu_1-\mu_2$ if it is known that $n_2=50$, $\bar{X}_2=76$, $S_2=6$, $n_1=75$, $\bar{X}_1=82$ and $S_1=8$

Solution: we have $1-\alpha=0.96 \Rightarrow \alpha=0.04$, $\frac{\alpha}{2}=0.02$

From tables of standard normal distribution

$Z_{\alpha/2}=Z_{0.02}=2.054$, then 96% C.I. for $\mu_1-\mu_2$

$$= (\bar{X}_1-\bar{X}_2) \pm Z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

$$= (82-76) \pm (2.054)\sqrt{\frac{64}{75}+\frac{36}{50}}$$

$$= 6 \pm (2.054)\sqrt{\frac{64}{75}+\frac{36}{50}}$$

(b) For Small samples ($n_1, n_2 < 30$)

A $(1-\alpha)\%$ C.I. for $\mu_1-\mu_2$ is given by

$$\boxed{\text{C.I. for } \mu_1-\mu_2 = (\bar{X}_1-\bar{X}_2 \pm t_{\alpha/2}\, S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

where S_p is the pooled variance obtained from the sample variances S_1^2, S_2^2 as

$$\boxed{S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}}$$

Ex: Given that $n_1 = 12$, $\bar{X}_1 = 85$, $S_1 = 4$, $n_2 = 10$, $\bar{X}_2 = 81$

$S_2 = 5$, find 90% C.I. for $\mu_1 - \mu_2$

Solution: we have $1-\alpha = 0.90 \Rightarrow \alpha = 0.10$, $\frac{\alpha}{2} = 0.05$

$t_{\alpha/2}(n_1+n_2-2) = t_{0.05}(20) = 1.725$ from tables

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{(11)(16)+(9)25}{12+10-2} = 20.05$$

$S_p = 4.478$

90% C.I. for $\mu_1 - \mu_2$ is $\left(\bar{X}_1 - \bar{X}_2 \pm t_{\alpha/2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)$

$$\left(85-81 \pm (1.725)(4.478)\sqrt{\frac{1}{12} + \frac{1}{10}}\right)$$

$= (0.69, 7.31)$

2-2-3 C.I. for variances

(i) μ is known

$(1-\alpha)\%$ C.I. for σ^2 is given by

$$\boxed{Pr\left[\frac{nS^2}{\chi^2_{1-\frac{\alpha}{2}}} < \sigma^2 < \frac{nS^2}{\chi^2_{\alpha/2}}\right] = 1-\alpha}$$

where $\chi^2_{\alpha/2}$, $\chi^2_{1-\alpha/2}$ are the χ^2 values obtained from χ^2 distribution table with n degrees of freedom and level of significant $\frac{\alpha}{2}$, $1-\frac{\alpha}{2}$ respectively.

<u>Ex:</u> let $S^2 = 9$ denote the variance of r.s of size 25 from $N(10, \sigma^2)$, find 95% C.I for σ^2

<u>Solution:</u> we have $1-\alpha = 0.95 \Rightarrow \alpha = 0.05$, $\frac{\alpha}{2} = 0.025$

$1-\frac{\alpha}{2} = 1-0.025 = 0.975$

From χ^2 table we get

$$\chi^2_{\alpha/2}(n) = \chi^2_{0.025}(25) = 13.1197$$

$$\chi^2_{1-\frac{\alpha}{2}}(n) = \chi^2_{0.975}(25) = 40.6465$$

$$Pr\left[\frac{nS^2}{\chi^2_{1-\frac{\alpha}{2}}} < \sigma^2 < \frac{nS^2}{\chi^2_{\alpha/2}}\right] = 1-\alpha$$

$$Pr\left[\frac{25(9)}{40.6465} < \sigma^2 < \frac{25(9)}{13.1197}\right] = 0.95$$

$C_L = 5.5355$, $C_u = 17.1498$

(ii) If μ is unknown

A $(1-\alpha)\%$ C.I. for σ^2 is given by

$$\boxed{P_r\left[\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{\alpha/2}}\right] = 1-\alpha}$$

where $\chi^2_{\alpha/2}$, $\chi^2_{1-\frac{\alpha}{2}}$ are the χ^2 values obtained from χ^2 distribution table with $(n-1)$ degrees of freedom and level of significant $\frac{\alpha}{2}$, $1-\frac{\alpha}{2}$ respectively.

Ex: let $x_1, x_2, \ldots, x_{10}$ be a r.s. drawn from $N(\mu, \sigma^2)$ where both are unknown. Suppose $\sum x_i = 159$, and $\sum x_i^2 = 2531$. Compute 95% C.I. for σ^2 if it is known that $\chi^2_{0.025}(9) = 2.70$ and $\chi^2_{0.975}(9) = 19$

Solution: Since $S^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$

$$(n-1)S^2 = \sum(x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2$$

$$(n-1)S^2 = 2531 - 10\left(\frac{159}{10}\right)^2 = 2531 - 10(15.9)^2 = 2.90$$

we have $1-\alpha=0.95 \Rightarrow \alpha=0.05 \Rightarrow \frac{\alpha}{2}=0.025, 1-\frac{\alpha}{2}=0.975$

then 95% C.I. for σ^2 is

$$pr\left[\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{\alpha/2}}\right]=1-\alpha$$

$$pr\left[\frac{2.90}{19} < \sigma^2 < \frac{2.90}{2.70}\right]=0.95$$

$$pr\left[0.15 < \sigma^2 < 1.07\right]=0.95$$

(iii) C.I. for the ratio of two variances

let S_1^2, S_2^2 be the variances of two independent random samples of size n_1, n_2 respectively

let $v_1=n_1-1$, $v_2=n_2-1$ be the degrees of freedom

then $(1-\alpha)$% C.I. for the ratio $\frac{\sigma_1^2}{\sigma_2^2}$ is given by.

$$\boxed{Pr\left[\frac{S_1^2}{S_2^2}\,\frac{1}{f_{\alpha/2}(v_1,v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2}\, f_{\alpha/2}(v_2,v_1)\right]=1-\alpha}$$

The values of $f_{\alpha/2}(v_1,v_2)$ and $f_{\alpha/2}(v_2,v_1)$ obtained from the F distribution table.

Ex: Find 98% C.I. for σ_1^2/σ_2^2 if it is known that $n_1=25$, $n_2=16$, $S_1=8$, $S_2=7$

solution: we have $1-\alpha=0.98 \Rightarrow \alpha=0.02$, $\frac{\alpha}{2}=0.01$

$$f_{\alpha/2}(\underset{n_1-1}{\underset{\parallel}{v_1}},\underset{n_2-1}{\underset{\parallel}{v_2}})=f_{0.01}(24,15)=3.29$$

$$f_{\alpha/2}(v_2,v_1)=f_{0.01}(15,24)=2.89$$

$$Pr\left[\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}(v_1,v_2)}<\frac{\sigma_1^2}{\sigma_2^2}<\frac{S_1^2}{S_2^2}f_{\alpha/2}(v_2,v_1)\right]=1-\alpha$$

$$Pr\left[\frac{64}{49}\frac{1}{3.29}<\frac{\sigma_1^2}{\sigma_2^2}<\frac{64}{49}(2.89)\right]=0.98$$

C.I $=(0.397, 3.775)$

Exercises :

(1) If it is known that $n = 17$ is the size of r.s. from $N(\mu, \sigma^2)$ with $\bar{x} = 5.3$, $s^2 = 6.2$ find 95 % C.I for both μ and σ^2

the tabulated values are

$t_{0.025}(16) = 2.120$, $\chi^2_{0.025}(16) = 6.91$,

$\chi^2_{0.975}(16) = 28.8$

(2) Given $\bar{x} = 18$, is the mean of a r.s. of size 20 from $N(\mu, 25)$. find 99 % C.I for μ if it is known that $Z_{0.005} = 2.58$

(3) A r.s. of size 10 is drawn from $N(\mu, \sigma^2)$ the values of individuals are 10.7, 12.6, 9.3, 9.5, 11.3, 12.2, 11.5, 11.1, 10.4 and 10.2. find 95 % C.I for μ and σ^2,

$t_{0.025}(9) = 2.262$

4) Two random samples each of size 10 from $N(\mu_1, \sigma^2)$, $N(\mu_2, \sigma^2)$ yield

$\overline{x_1} = 4.8$, $S_1^2 = 8.64$, $\overline{x_2} = 5.6$, $S_2^2 = 7.88$.

Find 95% C.I. for $\mu_1 - \mu_2$ if it known that $t_{0.025}(18) = 2.101$

5) let $x_1, x_2, \dots, x_n$ be a r.s from $N(\mu, \sigma^2)$ let $0 < a < b$. show that the mathematical expectation of the length of random interval

$$\left[\frac{\sum(x_i - \mu)^2}{b}, \frac{\sum(x_i - \mu)^2}{a}\right] \text{ is } (b-a)\left(\frac{n\sigma^2}{ab}\right)$$

Chapter three الفصل الثالث

Testing of Hypothesis اختبار الفرضيات

Basic concepts مفاهيم أساسية

Statistical hypothesis الفرضية الإحصائية :

هي عبارة عن ادعاء قد يكون صائبا (true) أو خاطئا (false) حول معلمة أو أكثر لمجتمع (أو عدة مجتمعات) ان الفرضية التي نضعها ابتداءً تدعى بالفرضية الصفرية أو فرضية العدم null hypothesis ويرمز لها H_0 . ان نقيض فرضية العدم يقودنا الى قبول فرضية أخرى بديلة عنها تدعى بالفرضية البديلة alternative hypothesis ويرمز لها H_1 .

Simple and Composite hypothesis الفرضية البسيطة والمركبة

يقال عن الفرضية الإحصائية انها بسيطة اذا افترضت ان معلمة (المعالم) المجتمع تساوي قيمة محددة ، ويقال عنها انها مركبة اذا افترضت ان معلمة (المعالم) المجتمع تأخذ أكثر (أو احدى) من قيمة محددة .

For ex) $H_0: \mu \le 2, \sigma^2 > 0$ Composite

$H_0: \mu = 3, \sigma^2 = 10$ Simple

$H_0: \mu = 3 \;,\; 0 < \sigma^2 < 10$ composite

الاحصائية الاختبار :- Test statistic

عبارة عن متغير عشوائي بدلالة مفردات العينة وله توزيع احتمالي معلوم ويصف العلاقة بين القيم النظرية للمجتمع مع القيم المحسوبة من العينة (مثلاً $\bar{X}$ او $\sum x_i^2$ ... الخ)

منطقة الرفض او المنطقة الحرجة: critical region

هي تلك المنطقة التي اذا وقعت فيها قيمة احصائية الاختبار رفضنا فرضية العدم H_0 ، ويرمز لها بالرمز C .

الخطأ من النوع الاول والثاني :- Type I and II errors

هناك نوعان من الاخطاء قد يقع فيها الباحث عند اختبار الفرضيات وهما الخطأ من النوع الاول ويتمثل برفض فرضية العدم H_0 عندما تكون هي الصحيحة .

والخطأ من النوع الثاني ويتمثل بقبول فرضية العدم H_0 عندما تكون خاطئة .

حجم الاختبار او مستوى المعنوية :-

The size of the test or level of significant

وهو احتمال الوقوع في الخطأ من النوع الاول ويرمز له α
اي انه :-

$$\alpha = P_r[\text{type I error}] = P_r[\text{reject } H_0 \mid H_0 \text{ is true}]$$

$$= P_r[x_1, x_2, \ldots, x_n \in C \mid \underset{\text{صحيحة}}{H_0}\ \underset{\text{true}}{}]$$

The Size of type II error — حجم الخطأ من النوع الثاني

ويعرف كالآتي

$\beta = P_r[\text{type II error}] = P_r[\text{accept } H_0 \mid H_1]$

$= P_r[x_1, x_2, \ldots x_n \in C^c \mid H_1]$

حيث ان C^c تمثل المنطقة المكملة لـ C (C complement)

وهي تمثل منطقة قبول فرض العدم H_0

The power of the test — قوة الاختبار

هو احتمال رفض فرض العدم H_0 إذا كانت خاطئة (الفرض البديل H_1 هو الصحيح) ويرمز له $K(\theta)$ حيث ان:

$K(\theta) = P_r[\text{reject } H_0 \mid H_1]$

$= P_r[x_1, x_2, \ldots x_n \in C \mid H_1] = 1 - P_r[x_1, x_2, \ldots, x_n \in C^c \mid H_1]$

$$\boxed{K(\theta) = 1 - \beta}$$

Ex. let the r.v X has the p.d.f

$$f(x,\theta) = \begin{cases} \theta^x (1-\theta)^{1-x} & , x = 0,1 \\ 0 & \text{o.w} \end{cases}$$

To test the simple hypothesis $H_0: \theta = \frac{1}{4}$

against the alternative hypothesis $H_1: \theta < \frac{1}{4}$

Suppose that critical region is

$C = \{x_1, x_2, \dots, x_{10};\ \sum_{i=1}^{10} x_i \le 1\}$ find:

a: The power of the test $k(\theta)$

b: the power of the test at $\theta = \frac{1}{16}$

c: P_r [type II error] at $\theta = \frac{1}{16}$

d: the significant level α.

Solution Since $X \sim$ Bernoulli $(1, \theta)$ then

$y = \sum_{i=1}^{10} x_i \sim$ binomial $(10, \theta)$

$$f(y,\theta) = \begin{cases} C_y^{10}\, \theta^y (1-\theta)^{10-y} & y = 0, \dots, 10 \\ 0 & o.w \end{cases}$$

a) $k(\theta) = P_r[y \le 1 \mid H_1]$

أي انه يجب علينا ايجاد $P(y \le 1)$ تحت فرض H_1

$P_r(y \le 1) = P_r(y=0) + P_r(y=1)$

$= C_0^{10}\, \theta^0 (1-\theta)^{10} + C_1^{10}\, \theta (1-\theta)^9 = (1-\theta)^{10} + 10\theta(1-\theta)^9$

$= (1-\theta)^9 [1 - \theta + 10\theta] = (1-\theta)^9 (1 + 9\theta)$, $0 < \theta < \frac{1}{4}$

b) $k(\frac{1}{16}) = (1 - \frac{1}{16})^9 (1 + \frac{9}{16}) = (\frac{15}{16})^9 (\frac{25}{16})$

$c:\ \beta\left(\frac{1}{16}\right) = 1 - k\left(\frac{1}{16}\right) = 1 - \left(\frac{15}{16}\right)^9 \left(\frac{25}{16}\right)$

$d:\ \alpha = P_r[\text{type I error}] = P_r[\text{reject } H_0 \mid H_0]$

$= P_r[(y \le 1) \mid H_0]$

المطلوب ايجاد $P(y \le 1)$ تحت فرض H_0 (عندما تكون H_0 صحيحة)

$P[y \le 1 \mid H_0] = P(y=0) + P(y=1)$ عند $\theta = \frac{1}{4}$

$= (1-\theta)^9 (1+9\theta)\Big]_{\theta=\frac{1}{4}} = \left(\frac{3}{4}\right)^9 \left(\frac{13}{4}\right)$

<u>**Ex:**</u> Suppose that $x_1, x_2, \dots, x_n$ is a random sample from $N(\theta, 1)$, to test the simple hypothesis $H_0: \theta = 0$ against the simple alternative hypothesis $H_1: \theta = 1$, the critical region is:

$C = \{x_1, x_2, \dots, x_n : \bar{x} \ge k\}$ find n and k such that $\alpha = \beta = 0.01$

<u>Solution:</u> $\alpha = P_r[\text{reject } H_0 \mid H_0 \text{ is true}]$

$\alpha = P_r[\bar{x} \ge k \mid \theta = 0]$

$\bar{x} \sim N\left(\theta, \frac{1}{n}\right) \Rightarrow \bar{x} / H_0 \sim N\left(0, \frac{1}{n}\right)$

$\alpha = P_r[\bar{x} \ge k \mid \theta = 0] = P_r\left[\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \ge \frac{k - \mu}{\sigma/\sqrt{n}}\right] = P_r\left[Z \ge \frac{k - 0}{1/\sqrt{n}}\right]$

$= Pr[Z \geq \sqrt{n}\, k]$

$0.01 = Pr[Z \geq \sqrt{n}\, k] = 1 - Pr[Z < \sqrt{n}\, k]$

$Pr[Z < \sqrt{n}\, k] = 1 - 0.01 = 0.99$

$\boxed{\sqrt{n}\, k = 2.33}$ ---(1) from tables of standard normal dist.

Also

$\beta = Pr[\text{accept } H_0 \mid H_1] = Pr[\bar{X} < k \mid \theta = 1]$

$\bar{X} / H_1 \sim N(1, \frac{1}{n})$

$\beta = Pr\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{k - \mu}{\sigma/\sqrt{n}}\right] = Pr\left[Z < \frac{k-1}{1/\sqrt{n}}\right] = Pr[Z < \sqrt{n}(k-1)]$

$0.01 = Pr[Z < \sqrt{n}(k-1)]$

$\boxed{\sqrt{n}\,(k-1) = -2.33}$ ---(2) from tables

Solving equation (1) and (2)

by (2) $\Rightarrow \sqrt{n} = \frac{-2.33}{k-1}$

by (1) $\Rightarrow \frac{-2.33}{k-1}\, k = 2.33$

$-2.33k = 2.33k - 2.33$

$4.66k = 2.33$

$k = \frac{2.33}{4.66} = 0.5$

Now: by (2) $\Rightarrow \sqrt{n}\,(0.5 - 1) = -2.33$

$\sqrt{n}\,(-0.5) = -2.33 \Rightarrow \sqrt{n} = 4.66$

$n = 21.7156 \simeq 22$ (approximately)

Ex: The Consumption of electricity in a small township is assumed to be exponentially distributed with parameter θ. Determine the size of type

I and type II errors if: $H_0: \theta = 1000$ k.w. is tested against $H_1: \theta = 2000$ k.w. and if the test criterion is as follows

Select any day at random. If the consumption of that day is 4000 k.w. or more, reject H_0, otherwise accept H_0.

Solution: the critical region is

$C = \{x_1, x_2, \dots, x_n : x_i \geq 4000\}$

$\alpha = P_r[\text{reject } H_0 \mid H_0 \text{ is true}] = P_r[x \geq 4000 \mid \theta = 1000]$

we have $f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}, \; x > 0, \theta > 1$

$$\therefore \alpha = \int_{4000}^{\infty} \frac{1}{1000} e^{-x/1000}\, dx = -e^{-x/1000}\Big|_{4000}^{\infty} = -e^{-\infty} + e^{-\frac{4000}{1000}} = e^{-4}$$

$\beta = P_r[\text{accept } H_0 \mid H_1 \text{ is true}] = P_r[x < 4000 \mid \theta = 2000]$

$$\beta = \int_{0}^{4000} \frac{1}{2000} e^{-x/2000}\, dx = -e^{-x/2000}\Big|_{0}^{4000} = -e^{-\frac{4000}{2000}} + 1$$

$$= -e^{-2} + 1 = 1 - e^{-2}$$

Ex: let $X_1, X_2, \dots, X_n$ be a r.s of size 25 from $N(\mu, 36)$

we shall reject $H_0: \mu = 75$ and accept $H_1: \mu = 80$

iff $\bar{X} > c$. where c is constant. find the value

of c at $\alpha = 0.01$ if it is known that $Z_{0.99} = 2.33$

(Solve it) Ans: $c = 77.79$

Exercises

① let $X_1, X_2, \dots, X_n$ be a random sample from Poisson (θ)

The critical region for testing $H_0: \theta = \frac{1}{10}$ against

$H_1: \theta > \frac{1}{10}$ is: $C = \{X_1, X_2, \dots, X_n, \sum_{i=1}^{10} X_i \geq 1\}$

a) find the power of the test $K(\theta)$.

b) find the level of significant α.

2) Consider a normal distribution $N(\theta, 4)$

the simple hypothesis $H_0: \theta = 0$ is rejected

and the alternative composite hypothesis

$H_1: \theta > 0$ is accepted if and only if

the observed mean $\bar{X}$ of a random

sample of size 25 is greater than or equal to $\frac{3}{5}$, find the power of the test.

3) let $X_1, X_2, \ldots X_{25}$ denote a random sample of size 25 from a normal population with variance $\sigma^2 = 4$. to test $H_0: X \sim N(0,4)$ against $H_1: X \sim N(1,4)$

the critical region is

$C = \{X_1, X_2, \ldots X_{25} : \bar{X} \geq 0.04\}$ find the error size of this test.

Best Critical Region أفضل منطقة حرجة

يقال عن المنطقة الحرجة C انها أفضل منطقة حرجة (BCR)

بحجم α عند اختبار الفرض العدم البسيط $H_0: \theta = \theta_0$

مقابل الفرض البديل البسيط $H_1: \theta = \theta_1$ اذا وجدت

منطقة حرجة اخرى مثل A بحيث ان:

1) $\Pr[X_1, X_2, \dots, X_n \in A \mid H_0] = \Pr[X_1, X_2, \dots, X_n \in C \mid H_0] = \alpha$

2) $\Pr[X_1, X_2, \dots, X_n \in A \mid H_1] \leq \Pr[X_1, X_2, \dots, X_n \in C \mid H_1]$

Ex: Consider the r.v. $X \sim$ binomial $(5, \theta)$ under $H_0: \theta = \frac{1}{2}$, $H_1: \theta = \frac{3}{4}$, the following table gives the density values of X under H_0 and H_1

x	0	1	2	3	4	5
$f(x, \frac{1}{2})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$f(x, \frac{3}{4})$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{1024}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$

Assuming that $\alpha = \frac{1}{32} = \Pr[\text{reject } H_0 \mid H_0]$

we have two critical regions:

$A = \{x : x = 0\}$, $C = \{x : x = 5\}$

$Pr\{x \in A \mid H_0\} = Pr\{x \in C \mid H_0\} = \frac{1}{32}$

Also

$Pr(x \in A \mid H_1) = \frac{1}{1024}$

$Pr(x \in C \mid H_1) = \frac{243}{1024}$

$\Rightarrow Pr(x \in A \mid H_1) < Pr(x \in C \mid H_1)$

$\Rightarrow$ C is the best critical region BCR

<u>Ex:</u> For the above example let us use the r.v. X to test $H_0 : \theta = \frac{1}{2}$ against $H_1 : \theta = \frac{3}{4}$ but by assuming $\alpha = \frac{6}{32}$

In this case we will have four critical regions of size $\alpha = \frac{6}{32}$ these regions are

$C_1 = \{x : x = 0, 1\}$, $C_2 = \{x : x = 0, 4\}$

$C_3 = \{x : x = 1, 5\}$, $C_4 = \{x : x = 4, 5\}$

one of these regions is BCR

$$Pr\{X \in C_1 \mid H_1\} = Pr(X = 0, 1 \mid \theta = \tfrac{3}{4}] = \frac{1}{1024} + \frac{15}{1024} = \frac{16}{1024}$$

$$Pr\{X \in C_2 \mid H_1\} = \frac{406}{1024}$$

$$Pr\{X \in C_3 \mid H_1\} = \frac{258}{1024}$$

$$Pr\{X \in C_4 \mid H_1\} = \frac{648}{1024}$$

$\Rightarrow C_4$ is BCR of size $\alpha = \frac{6}{56}$