جامعة بغداد

كلية التربية ابن الهيثم للعلوم الصرفة

قسم الرياضيات

FOUNDATIONS OF MATHEMATICS

أسس الرياضيات

المرحلة الأولى

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Foundations of mathematics

أسس الرياضيات

Foundations of mathematics is the study of the basic mathematical concepts (logic statements العبارات المنطقية, numbers, relations, sets, functions...).

أسس الرياضيات هو علم دراسة المفاهيم الرياضية الأساسية كالعبارات المنطقية ، أنظمة الأعداد ، العلاقات ، المجاميع والدوال.

Set of Numbers (Subsets of the set of real numbers R)

1. Set of Natural numbers $N = \{1,2,3, \dots\}$

2. Set of Prime numbers $P = \{2,3,5,7,11, \dots\}$

3. Set of Integer numbers $Z = I = \{\dots, -2, -1,0,1,2, \dots\}$

4. Set of Even numbers $E = \{\dots, -4, -2,0,2,4, \dots\}$

5. Set of Odd numbers $O = \{\dots, -3, -1,1,3, \dots\}$

6. Set of Rational numbers $Q = \{\frac{a}{b} : a, b \in Z, b \neq 0\}$

Example: $\frac{2}{3}, -\frac{1}{5}, 3, 0.5, 0.3333$ are examples of rational numbers

7. Set of Irrational numbers $H = \{x : x \notin \mathrm{Q}\}$

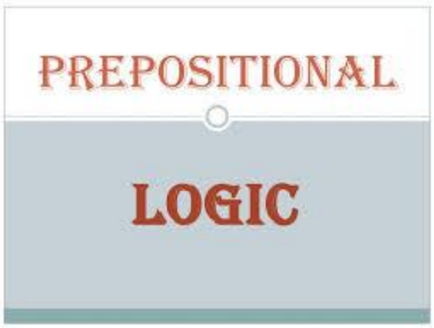
Example: $\pi = 3.1415$ .... is irrational number

$e = 2.71828 \dots$ is irrational number

$\sqrt{2}, \sqrt{5}$ are irrational numbers

CHAPTER ONE: **Mathematical Logic and proof Using Propositional Calculus**

مفهوم المنطق والبرهان الرياضي باستخدام العبارات الخبرية



Chapter 1 Contents:

1. **Propositions (statements) العبارات**
2. **Compound propositions العبارات المركبة**
3. **Mathematical proof البرهان الرياضي**
4. **Quantifiers المسورات**

Definition1.1: **Mathematical Logic المنطق الرياضي** is a subfield of mathematics exploring the applications of formal logic to mathematics. Mathematical logic are widely used in theoretical computer science and other sciences.

المنطق الرياضي هو احد الحقول الرياضية التي تدرس تطبيقات المنطق في الرياضيات. المنطق الرياضي يستخدم بشكل واسع في علوم الحاسبات وعلوم أخرى.

Propositions or Statements العبارات

Definition1.2: A **proposition** is a declarative sentence which is either '**true: T**' or '**false: F**', but not both. We use the letters $p, q, r, s, ..etc$ to denote a proposition.

العبارة هي **جملة خبرية** والتي قد تكون صادقة أو كاذبة ومن غير الممكن أن تكون العبارة صادقة وكاذبة بنفس الوقت.

Example1.3: Which of the following sentences are called propositions (statements), and which ones are not propositions.

1) p: $\sqrt{4} = 2$ is a **true proposition**

2) q: $\sum_{x=1}^{3} (x+2) = 13$ is a **false proposition**

Because $\sum_{x=1}^{3} (x+2) = (1+2) + (2+2) + (3+2) = 3+4+5 = 12 \neq 13$.

3) r: Baghdad isn't in Iraq is a **false statement**

4) s: What time is it ? is **not a proposition**

لأنها جملة استفهامية وليست جملة خبرية

5) w: Study hard is **not a proposition**

Because it is not a declaration sentence ليست جملة خبرية

6) v: $x + y = 0$ is **not a proposition**

لان الجملة ليست صادقة ولا كاذبة

Example1.4: (**H. W**) Which of the following sentences is called a proposition (statement), and which one is not a proposition.

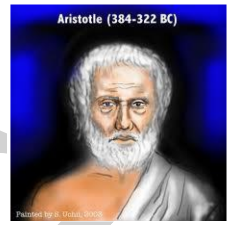
أي من الجمل أدناه تمثل عبارة وأيها لاتمثل عبارة؟

i) p: $x + 1 = 3$

ii) q: $x + y = z$

iii) r: $\frac{3}{4}$ is an even number (عدد زوجي)

The area of logic that deals with propositions is called **propositional logic** or **propositional calculus.** It was first developed by the Greek philosopher **Aristotle** أرسطو more than 2300 years ago.



Definition1.5: Negation of a proposition نفي العبارة

Let p be a proposition. The ***negation*** of p is called ''not p'' and is denoted by ($\sim p$).

لتكن 'p' عبارة. يقال للعبارة 'not p' أو 'ليس p' أنها نفي العبارة p ويرمز لها بالرمز $\sim p$

Example1.6: re-write the following expressions without using the negation

$\sim(3 < 5)$

$\sim(x > y)$

$\sim(x \geq 5)$

$\sim(2 = 10)$

Example1.7: Find the truth value of each of the following statements. Find $(\sim p)$ negations for the statements q and r.

أوجد قيم صدق ونفي كل من العبارات التالية:

1. p : Today is Saturday (F) , $\sim p$: Today is **not** Saturday

2. q : 2+2=4 **(T)**

$\sim q: 2+2 \neq 4$

3. r: : The square has four sides **(H. W)**

Remark1.8: If a proposition p is true, then ~ p is false; and if p is false, then ~ p is true.

أدناه جدول صدق العبارة p ونفيها

The truth table of the negation of a proposition p	
P	~ p
T F	F T

Double Negation Law: If p is a proposition, then ~ ~ p=p.

p	~ p	~ ~ p
T **F**	F T	**T** **F**

Compound propositions

Propositions are divided into two types:

1. **Primitive proposition** **عبارة بدائية أو بسيطة :** A proposition is said to be **primitive,** if it cannot be divided into simpler propositions.

   العبارة تسمى بدائية إذا لم يمكن تحليلها إلى عبارات أبسط .

2. **Composite proposition** **عبارة مركبة:** A proposition is called **composite**, if it is compound of more than one primitive propositions using logical connective operators.

العبارة تسمى مركبة إذا كانت تتكون من عبارتين بسيطتين أو أكثر تربطها أداة ربط واحدة او اكثر.

ادوات الربط التي تكون العبارة المركبة هي: and $\wedge$

or $\vee$

if then $\Rightarrow$

if and only if $\Leftrightarrow$

Example1.9: Propositions **(1)-(3)** in Example (1.3) are primitive.

Example1.10: The following propositions are composite:

1. ‘‘ 2+3 = 5 **and** 6-4 = 1’’

   عبارة مركبة مكونة من عبارتين بسيطة مربوطة بأداة الربط **and**

2. ‘‘Ali is clever **or** he studies every day’’

   عبارة مركبة مكونة من عبارتين بسيطة مربوطة بأداة الربط **or**

ملاحظة: قيمة صدق العبارة المركبة تعتمد على:

1. قيم صدق العبارات البسيطة المكونة لها
2. أدوات الربط المستخدمة لربط العبارات البسيطة

Basic Logical connective Operators أدوات الربط المنطقية الأساسية

There are some basic logical operators that connect simple propositions to produce composite proposition. These operators are:

1. **Conjunction operator (و) أداة الوصل**--English word (**and**), symbol (∧).

Let p and q are two primitive propositions. The conjunction of p and q is denoted by ‘‘p ∧ q’’ and read as ‘‘p and q’’.

If both p and q are true, then p ∧ q is true, otherwise p ∧ q is false.

أي عبارتين بسيطتين p و q يمكن ربطهما بأداة الربط (و) لتكوين العبارة المركبة ‘‘p ∧q’’. إذا كانت كل من p و q صادقة فان p ∧ q تكون صادقة. إذا كانت إحدى العبارتين على الأقل كاذبة فان p ∧ q تكون كاذبة.

Below is the truth table for the conjunction of two propositions:

Conjunction		
p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Example1.11: Find the truth value of the following statements:

أوجد قيم صدق العبارات التالية

1. $2+2=4$ and $2+3=5$

   T ∧ T = T

2. $\frac{x}{x}=1$ such that $x \neq 0$ ∧ Baghdad is not in Iraq

   T ∧ F= F

3. -5 is a prime number ∧ π is a rational number

   F ∧ F= F

Example1.12: Let p: $\mathrm{x}+\mathrm{y}=\mathrm{y}+\mathrm{x}$ such that $x, y \in N$

q: $2 > 10$

r: There are three seasons in Iraq

Find the truth value of:

i) $(q \wedge \sim r) \wedge r$,

ii) $(q \wedge \sim\sim q) \wedge (\sim \mathrm{p} \wedge \mathrm{r})$

Solution of (i): ~r : The seasons in Iraq are not three.

$$(q \wedge \sim \mathrm{r}) \wedge r = (F \wedge T) \wedge F = F \wedge F = F$$

Example1.13: Let p and q are two propositions such that

p: Fouad is poor (T)

q: Fouad is clever (T)

Find the conjunction of p and q. Discuss the truth values of ‘‘p **and** q’’.

اوجد عبارة الوصل بين p و q. ناقش قيم صدق العبارة ‘p and q’ .

<u>Solution</u>: The conjunction ‘‘p **and** q’’ is

‘‘Fouad is poor **and** Fouad is clever’’

The compound proposition ‘‘p **and** q’’ is **true** if

‘‘Fouad is poor and Fouad is clever’’

The compound proposition (p ∧ q) is **false** if:

‘‘Fouad is rich ∧ Fouad is clever’’

‘‘Fouad is poor ∧ Fouad is not clever’’

‘‘Fouad is not rich ∧ Fouad is not clever’’

<u>Properties of the conjunction operators:</u> **(∧) خواص أداة الوصل**

Let p, q and r are three propositions. Using the truth table, show that:

1. $p \wedge q = q \wedge p$ (commutative خاصية الإبدال)
2. $(p \wedge q) \wedge r = p \wedge (q \wedge r)$ (associative خاصية التجميع) **(H.W.)**
3. $p \wedge p = p$ (Idempotent law قانون تساوي القوى)
4. $p \wedge T = p$ (Identity law)
5. $p \wedge F = F$ (Domination Law)
6. $p \wedge \sim p = F$

Solution:

1. $p \wedge q = q \wedge p$ 3. $p \wedge p = p$ 4. $p \wedge F = F$

p	q	$p \wedge q$	$q \wedge p$
T	F	**F**	**F**
T	T	**T**	**T**
F	T	**F**	**F**
F	F	**F**	**F**

p	p	$p \wedge p$
T	T	**T**
F	F	**F**

p	F	$p \wedge F$
T	F	**F**
F	F	**F**

2. Disjunction operator (أو) أداة الفصل--English word (**or**), symbol (∨).

Let p and q be two propositions. The disjunction of p and q is denoted by ''p ∨ q'' and read ''p or q''.

We say that ''p ∨ q'' is true when p is true **or** q is true **or both** are true. If both p and q are false, then p ∨ q is false.

إذا كانت كل من p و qعبارتين بسيطة. تكون العبارة المركبة ''p ∨ q'' المرتبطة بأداة الفصل (∨) كاذبة إذا كانت كل من العبارة p و qعبارة كاذبة. وتكون العبارة ''p ∨ q'' صادقة فيما عدا ذلك (أي إذا كانت إحدى العبارتين البسيطتين على الأقل صادقة).

Disjunction		
p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

The truth table for the disjunction of two propositions

Example1.14: (**H. W**) Let p, q and r are three propositions such that

p: dogs can fly

$q: x - x = 0,\ x \in R$

$r: -3 \in N$

Find the truth value of the following statements:

a) (p ∨q) ∨ r

b) ~q ∨ r

c) ~ (~ p ∨ q)

d) (p ∧ q) ∨ (q ∨ r)

Solution of (c):

~ (~ p ∨ q)= ~ (T ∨ T)= ~ T= F

Example1.15: Let p and q are two primitive propositions such that

p: Today is Friday (T)

q: It is raining today (T)

What is the disjunction of the propositions p and q? Discuss the truth value of ‘‘p ∨ q’’.

Solution: The disjunction ‘‘p ∨ q’’ is

‘‘Today is Friday or it is raining today’’

‘‘p ∨ q’’ means that today is **either** Friday **or** raining **or** both.

The compound proposition (p ∨ q) is **false** if:

''Today is not Friday or it is not raining today''

The compound proposition (p ∨ q) is **true** if:

''Today is Friday or it is raining today''

''Today is not Friday or it is raining today''

''Today is Friday or it is not raining today''

Properties of the disjunction operator: (∨) **خواص أداة الفصل**

Let p, q and r are three propositions. Using the **truth table**, show that:

1. p ∨ q = q ∨ p (خاصية الإبدال) (**H. W**)
2. (p ∨ q) ∨ r = p ∨ (q ∨ r) (خاصية التجميع)
3. p ∨ p = p (قانون تساوي القوى) (**H. W**)
4. p ∨ T=T (Domination Law) (**H. W**)
5. p ∨ F= F (Identity Law) (**H. W**)
6. p ∨ ~ p= T (**H. W**)

Solution2: (p ∨ q) ∨ r = p ∨ (q ∨ r)

p	q	r	p ∨ q	q ∨ r	(p ∨ q) ∨ r	p ∨ (q ∨ r)
T	T	T	T	T	**T**	**T**
F	F	F	F	F	**F**	**F**
F	T	T	T	T	**T**	**T**
F	T	F	T	T	**T**	**T**
F	F	T	F	T	**T**	**T**
T	F	F	T	F	**T**	**T**
T	T	F	T	T	**T**	**T**
T	F	T	T	T	**T**	**T**

3. <u>Conditional operator</u> أداة الشرط: English word (**if…then**), Arabic word (**إذا كان فان**), symbol (→).

Let p and q be two propositions. The conditional statement ''$p \to q$'' is the proposition ''if p then q''. The conditional statement ''$p \to q$'' is **false** when p is true and q is false, otherwise ''$p \to q$'' is **true**.

إذا كانت كل من p و qعبارة بسيطة فان العبارة المركبة (if…then) يرمز لها بالرمز

(→). تكون العبارة (if p then q) **كاذبة** في حالة واحدة فقط عندما تكون p عبارة صادقة و q عبارة كاذبة. تكون العبارة (if p then q) **صادقة** فيما عدا ذلك.

ملاحظة: إذا كانت p عبارة خاطئة فان العبارة 'if p then q' تكون غير قابلة للاختبار وبالتالي فأنها لايمكن أن تكون خاطئة.

The following is the truth table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark1.16: In the conditional statement ''p → q'', p is called the **hypothesis** فرضية and q is called the **conclusion** **نتيجة** .

Remark1.17: The conditional statement can be expressed in the following equivalent ways:

a) ''p implies q''

b) ''q if p'' ,

c) ''q only if p'',

d) ''p is sufficient condition for q'',

e) ''q is a necessary condition for p''.

Example1.18: Find the truth value of the following statements:

1. If fish fly, then $3 + 2 = 5$

$F \rightarrow T = T$

2. If fish walk, then $3 + 2 = 6$

$$F \to F = T$$

Example1.19: Let p, q, and r are three propositions such that

p: 3 is an odd number

$q: x + y = y + x,\ x, y \in R$

r: Winter is hot

Find the truth value of the following statements:

1) $(p \to q) \vee (r \to q)$ **(H. W)**
2) if $(p \wedge q)$ then $(q \vee \sim r)$
3) $(p \wedge r) \vee (q \to p)$ **(H. W)**

Solution2:

if $(p \wedge q)$ then $(q \vee \sim r)$ = if $(T \wedge T)$ then $(T \vee T)$= $T \to T = T$

Example1.20: Find the truth value of the following statements:

1. The statement: ''**If** *x* is negative **then** − *5x* is positive''

$$T \Rightarrow T = T$$

2. The statement: "**If** 9 > 5 **then** dogs don't fly"

$$T \Rightarrow T = T$$

3. The statement: ''**If** ($x > 0$ and $x^2 < 0$) **then** $x \geq 10$''

If (T and F**) then** $F(or\ T)$

If F **then** $F(or\ T) = T$

4. The statement: ''**If** x > 0 **then** (x^2 < 0 or 2x < 0)''

$$T \Rightarrow (F\ or\ F) = T \Rightarrow F = F$$

Definition1.21: Let p and q are two propositions, then

1. The proposition ''$q \rightarrow p$'' is called the **converse** of ''$p \rightarrow\ q$''.
2. The proposition ''$\sim p \rightarrow \sim q$'' is called the **inverse** of ''$p \rightarrow\ q$''.

Example1.22: What is the converse and the inverse of the conditional statement:

''**if** $x > 5, x \in N$ **then** $x > 3$''?

What is the truth value of the statement and its inverse and converse?

Solution: The statement ''**if** $x > 5$ **then** $x > 3$'' Type equation here.

The truth value: T → T=T

The **converse** is ''**if** $x > 3$ **then** $x > 5$''

The truth value: for $x = \{4,5\}$; T → F= F

For $x = \{6,7, \dots..\}$, }; T → T= T

The **inverse** is **if** $x \leq 5$ **then** $x \leq 3$''

The truth value: for $x = \{1,2,3\}$; T → T= T

For $x = \{4,5\}$, }; T → F= F

Properties of the conditional operator: **(→) خواص أداة الشرط**

Let p, q and r are three propositions. Using the **truth table** show that: (**H. W**)

1. $p \to q \neq q \to p$
2. $(p \to q) \to r \neq p \to (q \to r)$
3. **Find** the truth value of: $p \to T,\ p \to F,\ p \to \sim p,\ p \to p$

4. Bi-conditional operator أداة الشرط المزدوج : English word (if and only if), Arabic word (إذا وفقط إذا), symbol **(↔)**

Let p and q be propositions. The *bi-conditional* statement "$p \leftrightarrow q$" is the proposition "p if and only if q". The bi-conditional statement is **true** when p and q have the same true value, and is **false** otherwise.

لتكن كل من **p** و **q** عبارة بسيطة. تكون العبارة المركبة "**p** إذا وفقط إذا **q**" والتي يرمز لها بالرمز (↔) صادقة في حالة تشابه قيم صدق العبارتين وكاذبة فيما عدا ذلك.

P	Q	P if and only if Q
T	T	T
T	F	F
F	T	F
F	F	T

The truth table for the bi-conditional of two propositions

Remark1.23: There are some other ways to express "$p \leftrightarrow q$":

"p iff q"

"if p then q , and if q then p"

"p is necessary and sufficient for q".

<u>Example1.24</u>: Find the truth value of ''$x > 0 \leftrightarrow 2x > 0$''

<u>Solution</u>: The statement is true because

If $x > 0$ then $2x > 0$ and if $2x > 0$ then $x > 0$

<u>Example1.25</u>: Find the truth value of ''$x > 0 \leftrightarrow x^2 > 0$'' (**H. W.**)

<u>Example1.26</u>: Let p: you can take the flight (True)

q: you can buy a ticket (True)

Then the bi-conditional statement $p \leftrightarrow q$ is

''you can take the flight if and only if you can buy a ticket''

Discuss the truth values of the bi-conditional statement.

<u>Solution</u>: The statement $p \leftrightarrow q$ is **true**

'' you buy a ticket ↔ can take the flight''

or

''you do not buy a ticket ↔ cannot take the flight''.

The statement $p \leftrightarrow q$ is **false** when p and q have opposite truth values.

''you do not buy a ticket ↔ you can take the flight''

or

''you buy a ticket ↔ cannot take the flight''.

Properties of the biconditional operator خواص أداة الشرط المزدوج

Let p, q and r are three propositions. Using the **truth table** show that: (**H. W**)

1. $p \leftrightarrow q = q \leftrightarrow p$
2. $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$
3. **Find** the truth value of: $p \leftrightarrow T, p \leftrightarrow F, p \leftrightarrow \sim p, p \leftrightarrow p.$

Exercise1.27:

1. Find the truth value of the following statements:

[(if $2+3=4$ then $x+4=4+x$) and 8 is an even number] iff ($2 \leq -10$ or $2 \geq -10$).

Solution: $[(F \rightarrow T) \wedge T] \leftrightarrow (F \vee T) = [T \wedge T] \leftrightarrow T = T \leftrightarrow T = T$

2. Let p: horse can swim

q: Conjunction operator is useful

r: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ $for\ x, y \in N$

Find the truth value of the following statements:

1. $p \leftrightarrow r$
2. $(p \rightarrow r) \wedge q$
3. $[(p \rightarrow r) \vee (q \rightarrow \sim p)]$

3. Write the truth table of the following statements:

i) $\sim p \wedge q$

ii) $(p \wedge q) \to (p \vee q)$

iii) $(p \to q) \vee \sim (q \leftrightarrow p)$

4. Write the following statements using the connections operators $\to, \leftrightarrow, \wedge, \vee$

i) If p and q integer numbers and $q \neq 0$ then $\frac{p}{q}$ is a rational number

ii) If $x^2$is integer number then x is even or odd number

iii) $xy > 0$ if and only if $(x > 0$ and $y > 0)$ or $(x < 0$ and $y < 0)$

<u>Definition1.28:</u> A compound proposition that is always true is called a ***tautology*** or ***lemma*** or ***theorem*.**

يقال للعبارة المركبة التي تكون صادقة دائما بأنها **نظرية** أو **تحصيل حاصل**

A compound proposition that is always false is called a ***contradiction***.

يقال للعبارة المركبة والتي تكون خاطئة دائما بأنها **تناقض**.

<u>Example1.29</u>: Show that (p v ~ p) is tautology and $(p \wedge \sim p)$ is contradiction.

<u>Solution</u>:

p	~ p	p ∨ ~p tautology	p ∧ ~ p contradiction
T	F	T	F
F	T	T	F

Example1.30: (**H. W**) which of the following compound statements is theorem (tautology) and which one is contradiction

$p \wedge F$, $p \vee T$, $p \leftrightarrow \sim p$, $[(p \to q) \wedge p] \wedge \sim q$

Definition1.31: Logical Equivalence التكافؤ المنطقي

Two statements (propositions) that have **same** truth values are called **logically equivalent**. The notation $p \equiv q$ or $p = q$ denotes that p and q are logically equivalent.

تكون عبارتين متكافئة منطقيا إذا كان لهما نفس قيمة الصدق ويرمز للتكافؤ المنطقي بالرمز ≡ أو =

Example1.32: show that $\sim (p \vee q) = \sim p \wedge \sim q$ (logically equivalent).

Hint: make truth table

Solution: The truth table for $\sim(p \vee q)$ and $\sim p \wedge \sim q$ is

P	Q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

<u>Definition</u>1.33: Let p, q and r three propositions, then define the following logical equivalence:

1. $p \wedge q = \sim(\sim p \vee \sim q)$

2. $p \rightarrow q = \sim p \vee q$

3. $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

ملاحظة: التعريف أعلاه مهم جدا ويجب أن يحفظ

<u>De Morgan's Theorem:</u> Let p and q are two propositions. Then

1. $\sim(p \wedge q) = \sim p \vee \sim q$ (**H. W**)

2. $\sim(p \vee q) = \sim p \wedge \sim q$

<u>Proof (2)</u>: Take the **right hand side** (**R. H. S**)

$\sim p \wedge \sim q = \sim(\sim\sim p \vee \sim\sim q)$ [definition of $\wedge$]

$= \sim (p \vee q)$ [double negation law: $\sim\sim p = p$]

= **Left hand side (L. H. S)**.

Exercise1.34: Simplify the following statements:

1. $\sim(p \vee \sim q)$

2. $\sim(\sim p \to q)$

3. $\sim(\sim p \leftrightarrow q)$

Solution(1): $\sim(p \vee \sim q) = \sim p \wedge \sim\sim q$ [De Morgan's law]

$$= \sim p \wedge q \qquad [\sim\sim q = q]$$

Solution(2): $\sim(\sim p \to q) = \sim(\sim\sim p \vee q)$

$$= \sim(p \vee q) \quad [\sim\sim p = p]$$

$$= \sim p \wedge \sim q \text{ [De Morgan's law]}$$

Laws of Logical Equivalence قوانين التطابق المنطقي

Let p, q and r are propositions. The following are some of the common logical equivalence rules:

1. Commutative Law قانون الإبدال: $p \wedge q = q \wedge p$

$$p \vee q = q \vee p$$

$$p \leftrightarrow q = q \leftrightarrow p$$

2. Associative Law قانون التجميع: $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$$(p \vee q) \vee r = p \vee (q \vee r)$$

$$(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$$

3. Distributive Law (from left) قانون التوزيع من اليسار:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$$p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$$p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$$

$$p \vee (q \rightarrow r) = (p \vee q) \rightarrow (p \vee r)$$

$$p \vee (q \leftrightarrow r) = (p \vee q) \leftrightarrow (p \vee r)$$

4. Distributive Law (from right) قانون التوزيع من اليمين:

$$(q \vee r) \wedge p = (q \wedge p) \vee (r \wedge p)$$

$$(q \wedge r) \wedge p = (q \wedge p) \wedge (r \wedge p)$$

$$(q \wedge r) \vee p = (q \vee p) \wedge (r \vee p)$$

$$(q \vee r) \vee p = (q \vee p) \vee (r \vee p)$$

$$(q \rightarrow r) \vee p = (q \vee p) \rightarrow (r \vee p)$$

$$(q \leftrightarrow r) \vee p = (q \vee p) \leftrightarrow (r \vee p)$$

5. Idempotent Law قانون تساوي القوى : $p \wedge p = p$; $p \vee p = p$

6. Identity Law: $p \wedge T = p$; $p \vee F = p$

7. Domination Law: $p \wedge F = F$; $p \vee T = T$

<u>Exercise1.35:</u> Simplify the following statements using laws of logical equivalence:

بسط العبارات التالية باستخام قوانين التطابق المنطقي

1. $(p \vee q) \wedge \sim p$

2. $(p \vee q) \vee (\sim p \wedge q)$

<u>Exercise1.36:</u> Prove (**without using the truth table**) that

برهن بدون استخدام جداول الصدق

$$\sim (p \vee (\sim p \wedge q)) = \sim p \wedge \sim q$$

<u>Solution</u>: Take the **L. H. S**

$\sim (p \vee (\sim p \wedge q)) = \sim p \wedge \sim (\sim p \wedge q)$ [De Morgan's law]

$= \sim p \wedge (\sim\sim p \vee \sim q)$ [De Morgan's law]

$= \sim p \wedge (p \vee \sim q)$ [by double negation law]

$= (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ [by distributive law]

$= F \vee (\sim p \wedge \sim q)$ $[\sim p \wedge p = F]$

$= (\sim p \wedge \sim q) \vee F$ [by commutative law]

$= \sim p \wedge \sim q$ **R. H. S**

Theorem1.37: (Properties of $\rightarrow$)

Let p, q and r are three propositions. Prove the following properties **without using truth tables**:

1. $p \rightarrow p = T$

2. $\sim p \rightarrow p = p$

3. $p \rightarrow T = T$

4. $T \rightarrow p = p$

5. $p \rightarrow F = \sim p$

6. $F \rightarrow p = T$

7. $p \rightarrow q = \sim q \rightarrow \sim p$

8. $p \rightarrow q = (p \wedge \sim \mathrm{q}) \rightarrow \sim p$

9. $p \rightarrow q = (p \wedge \sim \mathrm{q}) \rightarrow (r \wedge \sim r)$

10. $\sim (p \rightarrow q) = p \wedge \sim \mathrm{q}$

Proof 1: To prove $p \rightarrow p = T$

$$p \rightarrow p = \sim p \vee p \quad \text{[def. of } \rightarrow]$$

$$= T$$

Proof 4: To prove $T \rightarrow p = p$

$$T \rightarrow p = \sim T \vee p \quad \text{[def. of } \rightarrow]$$

$$= F \vee p \quad [\sim T = F]$$

$$= p$$

Proof 7: To prove $p \to q = \sim q \to \sim p$

$p \to q = \sim p \vee q$ [def. of $\to$]

$= q \vee \sim p$ [v is commutative]

$= \sim q \to \sim p$

Proof 8: To prove $p \to q = (p \wedge \sim q) \to \sim p$

Take the R. H. S: $(p \wedge \sim q) \to \sim p$

$= \sim (p \wedge \sim q) \vee \sim p$ [def. of $\to$]

$= (\sim p \vee \sim \sim q) \vee \sim p$ [De Morgan]

$= (\sim p \vee q) \vee \sim p$ [$\sim \sim q = q$]

$= \sim p \vee (q \vee \sim p)$ [v is associative]

$= \sim p \vee (\sim p \vee q)$ [v is comm.]

$= (\sim p \vee \sim p) \vee q$ [v is asso.]

$= \sim p \vee q$ [p v p= p]

$= p \to q$ [def. of $\to$]

= L. H. S

Theorem1.38: (Properties of $\leftrightarrow$)

Let p and q are two propositions. Prove the following properties without using truth tables:

1. $p \leftrightarrow p = T, \; p \leftrightarrow T = p, \; p \leftrightarrow F = \sim p$

2. $p \leftrightarrow \sim p = F$

3. $p \leftrightarrow q = q \leftrightarrow p$

4. $p \leftrightarrow q = \sim p \leftrightarrow \sim q$

5. $\sim p \leftrightarrow q = p \leftrightarrow \sim q$

6. $\sim (p \leftrightarrow q) = \sim p \leftrightarrow q$

7. $\sim (p \leftrightarrow q) = p \leftrightarrow \sim q$

Proof 1: To prove $p \leftrightarrow T = p$

$$p \leftrightarrow T = (p \rightarrow T) \wedge (T \rightarrow p) \quad [\text{ def. of } \leftrightarrow]$$

$$= (\sim p \vee T) \wedge (\sim T \vee p) \quad [\text{ def. of } \rightarrow]$$

$$= (\sim p \vee T) \wedge (F \vee p) \quad [\sim T = F]$$

$$= T \wedge p \quad [\sim p \vee T = T]$$

$$= p$$

Proof 6: $\sim (p \leftrightarrow q) = \sim p \leftrightarrow q$

Take **L. H. S** : $\sim (p \leftrightarrow q) = \sim [(p \rightarrow q) \wedge (q \rightarrow p)] \quad [\text{ def. of } \leftrightarrow]$

$$= \sim (p \rightarrow q) \vee \sim (q \rightarrow p) \quad [\text{ De Morgan}]$$

$= \sim (\sim p \vee q) \vee \sim (\sim q \vee p)$ [def. of $\rightarrow$]

$= (p \wedge \sim q) \vee (q \wedge \sim p)$ [De Morgan]

$= [(p \wedge \sim q) \vee q] \wedge [(p \wedge \sim q) \vee \sim p)]$ [distributive (v on ∧)]

$= [(p \vee q) \wedge (\sim q \vee q)] \wedge [(p \vee \sim p) \wedge (\sim q \vee \sim p)]$ [dist. (v on ∧)]

$= [(p \vee q) \wedge T] \wedge [T \wedge (\sim q \vee \sim p)]$

$= (p \vee q) \wedge (\sim q \vee \sim p)$

$= (\sim p \rightarrow q) \wedge (q \rightarrow \sim p)$ [def. of $\rightarrow$] $= \sim p \leftrightarrow q$ [def. of $\leftrightarrow$]

Mathematical Proof البرهان الرياضي

A mathematical proof is a valid argument that establishes the truth of a mathematical statement.

البرهان الرياضي هو إثبات صحة عبارة رياضية من خلال حجة أو تعليل منطقي.

Methods of Proving Mathematical Statements (or Theorems)

1. Direct Proof of a conditional statement $p \rightarrow q$

Direct proofs lead from the hypothesis of a theorem to the conclusion.

Definition1.39: The integer number x is called **even** if there exist $k \in Z$ such that $x = 2k$.

Definition1.40: The integer number x is called **odd** if there exist $k \in Z$ such that $x = 2k + 1$.

Theorem1.41: If x is an odd natural number ($x \in$ O) then x^2 is odd

Proof: Assume that x is an odd natural number. We must prove x^2 is odd

Since x is odd, then $x = 2k + 1$ for some $k \in N$.

$x^2 = x.x = (2k+1)(2k+1) = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Let $s = 2k^2 + 2k \in N$, then $x^2 = 2s + 1$

Hence, x^2 is an odd number.

<u>Theorem1.42</u>: (**H. W.**) If x is an even natural number ($x \in$ E) then x^2 is even.

<u>Theorem1.43</u>: The sum of two even natural numbers is even

<u>The theorem can be written as follows</u>: If x, y$\in E^+$ then $x + y \in E^+$ where $E^+ =$ set of positive even numbers.

<u>Proof</u>: Let p: $x \text{ and } y$ are even positive numbers,

q: $x + y$ is an even positive number

Let x = 2r and y = 2s ($r, s \in N$). Then x + y = 2r + 2s = 2(r + s) such that $r + s \in N$

x + y= $2k$ where $k = r + s$. Therefore x + y is a positive even number.

<u>Theorem1.44</u>: (H. W.)

i) If $x \in E$ and $y \in O$ then $x + y \in O$

ii) If $x \in E$ and $y \in O$ then $xy \in E$

iii) If $x, y \in E$ then $x + y \in E$

2. <u>Direct Proof of a conditional statement p ↔ q</u>

To prove a proposition in the form p ↔ q, we prove its equivalence. i.e.,

$$\boldsymbol{p} \leftrightarrow \boldsymbol{q} = (\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\mathrm{q} \rightarrow \boldsymbol{p})$$

Theorem1.45: x is odd number ↔ x+1 is an even number

Proof: Let p: x is odd number

q: x+1 is an even number

من التعريف أعلاه يجب أن نبرهن بان **p →q** و **q →p**

1. **Prove p →q:** Let $x \in O$, $x = 2k + 1$; $k \in Z$

x+1=$2k + 2 = 2(k + 1)$; $(k + 1) \in Z$

$x + 1 = 2r$; $r = k + 1 \in Z$

$x + 1 \in E$

2. **Prove q → p:** Let $x + 1 \in E$ To prove $x \in O$

$x + 1 = 2k$; k$\in Z$

$x = 2k - 1; k \in Z$ ……(1)

Since $k \in Z,$ then $r = k - 1 \in Z$

$k = r + 1$……(2)

Substitute (2) in (1), $x = 2(r + 1) - 1 = 2r + 1;$ $r \in Z$

$x = 2r + 1 \in O$

Theorem1.46: x is even ↔ x^2 is even

Proof: Let p: x is even number

q: x^2 is even number

من التعريف أعلاه يجب أن نبرهن بان $\boldsymbol{p} \rightarrow \boldsymbol{q}$ و $\boldsymbol{q} \rightarrow \boldsymbol{p}$

1. **Prove p →q:** Let $x \in E$, $x = 2k$; $k \in Z$

Prove $x^2 \in E$ (Theorem (1.44) مشابه لبرهان)

2. **Prove q → p:** Let $x^2 \in E$ To prove $x \in E$

Take $\mathrm{x}^2 + \mathrm{x} = \mathrm{x}(\mathrm{x}+1) \in \mathrm{E}$ [from Theorem 1.46(ii)]

$\Longrightarrow x = \mathrm{x}(\mathrm{x}+1) - \mathrm{x}^2 \in \mathrm{E}$ [Theorem 1.46(iii)]

$\Longrightarrow x \in E$

<u>Theorem1.47:</u> (H. W.) $\boldsymbol{x}$ is odd number if and only if $\boldsymbol{x^2}$ is odd number.

3. <u>Proof by Contradiction</u>

البرهان بالتناقض هو أن نفرض عكس المطلوب إثباته ثم نحصل على تناقض مع الفرض

<u>Theorem1.48:</u> Prove that: If $x^2 \in O$ then $x \in O$

<u>Proof</u>: Assume that $x^2 \in O$. To prove $x \in O$

By contradiction, assume that $x \in E$

$x = 2k \,;\, k \in Z$

$x^2 = 4k^2 \in E$

تناقض مع الفرض لأنه في الفرض $x^2 \in O$

$\therefore x \in O$.

<u>Theorem1.49:</u> If x^2 is even then x is even

<u>Proof</u>: Assume that $x^2 \in E$. To prove $x \in E$

By contradiction, assume that $x \in O$

$x = 2k + 1 \,;\, k \in Z$

$x^2 = 4k^2 + 4K + 1 \in O$ تناقض مع الفرض

$\Longrightarrow x^2 \in E$. Hence, $x \in E$.

Theorem1.50: Prove that: If $n = ab$ where a and b are positive integers, then $a \leq \sqrt{n} \vee b \leq \sqrt{n}$.

Proof: Let p: $n = ab$ where a and b are positive integer **hypothesis**

q: $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ **conclusion**

The first step is to assume that the **conclusion** is false as follows:

Assume that $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ is false (F). Hence, $\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n})$ is true (T).

$\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n})$= $\sim(a \leq \sqrt{n})$ and $\sim(b \leq \sqrt{n})$ [De Morgan's law]

= $a > \sqrt{n}$ and $b > \sqrt{n}$

Multiply the two inequalities together, $ab > n$ هذه المتراجحة تناقض الفرض

This shows that $ab \neq n$ **contradiction with the hypothesis**

Thus, $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ is true.

Definition1.51: Variable المتغير

An alphabetic letter x, $y, z, \ldots$ which represents a number that is either arbitrary or unknown.

Example1.52: ''$4x - 7 = 5$'': x is a variable

'' $\sqrt[3]{z}$ = 3'': z is a variable

Definition1.53: Open Sentence الجملة المفتوحة

A sentence is called **open sentence** (or propositional function), if it contains one or more variables. Open sentence is denoted by p(x), q(x), g(x)…etc.

Example1.54: The following are open sentences:

$p(x)$: x is an odd number

$q(x, y)$: $x + y = 5$ such that $x, y \in N$

$r(z)$: $\sqrt[3]{z} = 3$ such that $z \in R$

$s(y)$: computer y is working properly

Example1.55: Let the open sentence `` $\boldsymbol{p(x)}$: $x > 3$''.

What are the truth value of $p(5)$ and $p(-1)$? Which values $x \in N$ that make $p(x)$ true?

Solution: $p(5)$: $5 > 3$ is a true proposition

$p(-1)$: $-1 > 3$ is a false proposition

$p(x)$ is a true statement for $x \in \{4, 5, 6, \dots\}$.

Example1.56: Let the propositional function $\boldsymbol{q(x, y)}$: $x = y + 3$. What are the truth values of $q(1,2)$ and $q(3,0)$?

Solution: $q(1,2)$: $1 = 2 + 3$. This means $1 = 5$ which is false. Thus, $q(1,2)$ is false statement

$q(3,0)$: $3 = 0 + 3 = 3$. Hence $q(3,0)$ is true proposition

Example1.57: (H. W) Let the open sentence ``$\boldsymbol{r(x, y, z)}$: $x + y = z''$.

What are the truth values of $r(1,2,3)$ and $r(0,0,1)$?

<u>Definition</u>1.58: Solution Set (Truth set)

Let $p(x)$ be an open sentence and let A be a set. The solution set denoted by T_p is the set of all elements x of A for which $p(x)$ is true. In other words

T_p ={$x \in A$: $p(x)$ is true }

مجموعة الحل أو مجموعة الصدق: هي مجموعة العناصر التي تجعل التعبير المفتوح p(x) عبارة صادقة.

<u>Example1.59</u>: Find the solution set for each of the following open sentences:

1) Let $p(x)$ be "$x+2>7$" and $A=N$. Then

   $T_p=\{x \in N: x+2>7\}=\{x \in N: x>5\}=\{6,7,\dots\}$

   ماهي الأعداد الطبيعية الأكبر من 5

2) Let $q(x)$ be "$x+5<3$" and $A=N$. Then

   $T_q=\{x \in N: x+5<3\}=\{x \in N: x<-2\}=\emptyset$

   ماهي الأعداد الطبيعية الاقل من2-.

3) Let $p(x)$ be "$x+5>1$" and $A=N$. Then

   $T_p=\{x \in N: x+5>1\}=\{x \in N: x>-4\}=N$

   ماهي الأعداد الطبيعية الأكبر من 4-

<u>Example1.60</u>: (H. W.) Find the following solution sets. Also determine $p(x)$ and A for each solution set

1) $T_P=\{x \in N: -2<x<2\}=\{1\}$

   p(x) : $-2<x<2$ $\qquad A=N$

2) $T_P=\{x \in Z: -2<x<2\}$ **(H. W.)**

3) $T_P=\{x \in Z: -1<x<1\}$ **(H.W.)**

Example1.61: Assume we have the following statement:

''x > 2 and x < 5''

Which values of $x \in N$ that make the statement true? Which values of x that make the statement false? Discuss all the possible cases.

ماهي قيم x التي تجعل العبارة أعلاه صحيحة؟ وماهي القيم التي تجعلها خاطئة؟

Solution: For $A = \{3,4\}$, we have ''x > 2 and x < 5'' is **true** because the values in A are greater than 2 and less than 5.

For $A^c = N - A =$\{1,2,5, 6, 7, 8, …\}, then ''x > 2 and x < 5'' is **false**

Example1.62: Assume we have the following statement:

$$''x \leq -3 \; or \; x \geq 6''$$

Which values of x $\in N$ that make the statement true? Which values of x that make the statement false?

For $A = \{x \in N: x = 6, 7, \dots..\}$, the statement above is true
The statement is **false** for $A^c = N - A = \{1,2, \dots ,5\}$

Quantifiers المسورات

Quantifiers are open sentences written in a special way.

المسورات هي جمل مفتوحة مكتوبة بطريقة معينة

There are two types of quantifiers:

1. **Universal quantifiers العبارة المسورة كلياً**

2. **Existential quantifiers العبارة المسورة جزئيا**

<u>Universal quantifiers</u>:

Let p(x) be an open sentence on a set A. The notation

$$``\forall\, x \in A, p(x)''$$

Denote the **universal quantification تسوير كلي** of $p(x)$ and it reads as: ''for all $x, p(x)$'' or ''for every $x, p(x)$'' or ''for each $x, p(x)$''.

The symbol ∀ is called **universal quantifier** مسوراً كلياً.

The set A is called **domain المجال**

<u>Example</u>1.63: $\forall\, x\ \in N, x > 0$

All seasons in Iraq have rain

<u>Remark1.64</u>: 1. The universal quantifier $p(x)$ on a domain A is **true** if and only if $T_p = A$.

2. universal quantifier $p(x)$ on a domain A is **false** if and only if there exist $x \in A$ such that $p(x)$ is false.

<u>Example</u>1.65: Find the truth value of the following open sentences:

1. $\forall\, x \in R,\ x + 1 > x$

Let $A = R \ and\ p(x): x + 1 > x$

Because p(x) is true for all $x \in R$, the solution set $T_p = R$

$\Longrightarrow$ the quantification $\forall\, x \in R,\ x + 1 > x$ is **true.**

2. $\forall\, x \in N, x < 2$

Let $A = N\ and\ p(x): x < 2$

$p(x)$ is not true for all $x \in N$. Take $x = 3, p(3)$ is false.

$\Longrightarrow T_p \neq N$

3. $\forall\, x \in N, (x > 0\ and\ x = 0)$

The statement is **false**, there exists $x = 4 \in N$ such that $4 > 0\ and\ 4 \neq 0$.

4. $\forall\, x \in Z, |\mathrm{x}| > 0$ **(H. W.)**
5. For all $x \in \{1, -1\}$, $x^2 - 1 = 0$ **(H. W.)**

Existential quantifiers:

Let p(x) be an open sentence on a set A. The notation

$$“\exists\, x \in A, p(x)”$$

Denote the **existential quantification تسوير جزئي** of $\mathrm{p(x)}$ and it read as: ‘‘there exists $\mathrm{x, p(x)}$’’ or ‘‘there is $x, p(x)$’’ or ‘‘some $x, p(x)$’’.

The symbol ∃ is called **existential quantifier مسوراً جزئياً**.

The set A is called **domain المجال**

Example1.66: $\exists\, x \in N, x < 0$

There exists seasons in Iraq do not have rain

Remark1.67:

The existential quantifier $p(x)$ on a domain A is **true** if and only if $T_p \neq \emptyset$.

العبارة المسورة جزئياً تكون صادقة إذا وجد على الأقل عنصر واحد يحقق العبارة $p(x)$)

The existential quantifier $p(x)$ on a domain A is **false** if and only if $T_p = \emptyset$.

العبارة المسورة جزئياً تكون كاذبة إذا لم يكن هناك عنصر في المجموعة A يحقق العبارة $p(x)$).

Example1.68: Find the truth value of the following open sentences:

1. $\exists x \in R, x^2 = x$

A= R and p(x): $x^2 = x$

$T_p = \{0, 1\}$

$\Longrightarrow$ the existential quantifier $\exists x \in R, x^2 = x$ is true

2. $\exists x \in N,\ 3x + 5 = 1$

$x = \frac{-4}{3} \notin$ N

$\Longrightarrow T_p = \emptyset$

$\Longrightarrow \exists x \in N,\ 3x + 5 = 1$ is false

3. $\exists x \in Z,\ [(x+1)^2 = 0 \ and \ x^2 - 1 = 0]$

$(x+1)^2 = 0 \Longrightarrow$ x=-1

And $x^2 - 1 = 0 \Longrightarrow$ x=-1, 1

$T_p = \{-1\} \subset Z$

$\exists x \in Z,\ [(x+1)^2 = 0 \ and \ x^2 - 1 = 0]$ is true

De Morgan's law for the existential quantifier

$\sim [\exists\, x \in A, \sim p(x)] = \forall x \in A,\ p(x)$

قانون دي موركان للعلاقة بين التسوير الكلي والجزئي

Example1.69:

1. $\sim [\exists\, x \in E, x + 2 \notin E] = \forall x \in E, x + 2 \in E$

2. $\forall x \in N,\ \sqrt{3x} = \sqrt{3}\sqrt{x} = \sim[\exists x \in N, \sqrt{3x} \neq \sqrt{3}\sqrt{x}]$

<u>Theorem</u>1.70: Let p(x) be an open sentence and A is the domain. Then

1. $\sim[\forall x \in A,\ p(x)] = \exists\, x \in A, \sim p(x)$

2. $\sim[\forall x \in A,\ \sim p(x)] = \exists\, x \in A,\ p(x)$ **(H. W.)**

3. $\sim[\ \exists x \in A,\ p(x)] = \forall x \in A, \sim p(x)$ **(H. W.)**

<u>Proof1</u>: $\sim[\forall x \in A,\ p(x)] = \sim[\sim [\exists\, x \in A, \sim p(x)]]$ {from De Morgan}

$= \sim\sim[\exists\, x \in A, \sim p(x)]$

$= \exists\, x \in A, \sim p(x)$ $[\sim\sim p = p]$

<u>Definition1.71:</u> Nested Quantifiers المسورات المتداخلة

Two quantifiers are nested if one is within the area of the other.

في حالة وجود أكثر من متغير واحد في الجملة المفتوحة فان ذلك يستلزم وجود أكثر من مسور.

وهناك ثمانية طرق للتعبير عن المسورات المتداخلة وهي كالأتي:

Let $p(x, y)$ be an open sentence defined on the domain sets A and B. Then, the quantifiers can be expressed as follows:

1. $\forall x \in A,\ \forall y \in B,\ p(x, y)$

2. $\forall y \in B, \forall x \in A, p(x, y)$

3. $\exists x \in A,\ \exists y \in B,\ p(x, y)$

4. $\exists y \in B,\ \exists x \in A,\ p(x, y)$

5. $\forall x \in A,\ \exists y \in B,\ p(x, y)$

6. $\exists\ y \in B, \forall x \in A,\ p(x, y)$

7. $\exists\ x \in A, \forall y \in B,\ p(x, y)$

8. $\forall y \in B, \exists\ x \in A,\ p(x, y)$

Remark1.72: In the above definition, the quantifiers (1) and (2) are logically equivalent. i.e.,

$$\forall x \in A, \quad \forall y \in B, \quad p(x,y) = \forall y \in B, \quad \forall x \in A, \quad p(x,y)$$

Similarly, the quantifiers (3) and (4) are logically equivalent. i.e.,

$$\exists x \in A, \quad \exists y \in B, \quad p(x,y) = \exists y \in B, \quad \exists x \in A, \quad p(x,y)$$

Example1.73:

1. $\forall x \in R,\ \forall y \in N, x^2 + y^2 \ \geq 0\ (\boldsymbol{True}) = \ \forall y \in N,\ \forall x \in R,\ x^2 + y^2 \ \geq 0\ (\boldsymbol{True})$

2. $\exists x \in N,\ \exists y \in N,\ x + 2y < 0$ **(F)** $\equiv \exists y \in N,\ \exists x \in N,\ x + 2y < 0$ **(F)**

Remark1.74: In the above definition, the quantifiers (5) and (6) are not logically equivalent. i.e.,

$$\forall x \in A, \quad \exists y \in B, \quad p(x,y) \neq \exists y \in B, \quad \forall x \in A, \quad p(x,y)$$

Similarly, the quantifiers (7) and (8) are not logically equivalent. i.e.,

$$\exists x \in A, \quad \forall y \in B, \quad p(x,y) \neq \forall y \in B, \quad \exists x \in A, \quad p(x,y)$$

Example1.75:

$\exists x \in R,\ \forall y \in N, x + y = 0$ **(False)**

المسورة أعلاه تعني بأنه يوجد عدد حقيقي x بحيث أن حاصل جمع x و y يساوي صفر لكل عدد طبيعي y.

$\forall y \in N,\ \exists x \in R,\ x + y = 0$ **(True)**

العبارة تؤكد بأنه لكل عدد طبيعي y يوجد عدد حقيقي x بحيث $x + y = 0$.

$\Longrightarrow \exists x \in R,\ \forall y \in N, x + y = 0 \ \neq \forall y \in N,\ \exists x \in R,\ x + y = 0$.

Example1.76: Let x= computer, y= student,

p(x,y)= student uses the computer

Show that $\exists x,\ \forall y, p(x,y)\ \neq \forall y,\ \exists x, p(x,y)$

Solution:

$\exists x,\ \forall y,\ p(x,y)$

العبارة تعني بأنه يوجد كومبيوتر كل الطلاب تستخدمه

$\forall y, \quad \exists x, \quad p(x,y)$

العبارة تعني بأنه لكل طالب يوجد كومبيوتر يستخدمه

نلاحظ أن المعنى مختلف للعبارتين المسورتين

De Morgan's laws for nested quantifiers

Let x *and* y are two variables defined on the sets A and B, respectively and $p(x,y)$ an open sentence. Then:

1. $\sim[\forall\ x \in A, \forall\ y \in B,\ p(x,y)] = \exists x, \exists y,\ \sim p(x,y)$ (**H. W.**)
2. $\sim[\ \exists\ x \in A, \exists\ y \in B,\ p(x,y)] =\ \forall x, \forall\ y,\ \sim p(x,y)$
3. $\sim[\ \forall\ x \in A, \exists\ y \in B,\ p(x,y)] = \exists x, \forall y,\ \sim p(x,y)$ (**H. W.**)
4. $\sim[\ \exists\ x \in A, \forall\ y \in B,\ p(x,y)] = \forall x, \exists y,\ \sim p(x,y)$ (**H. W.**)

Proof 2:

Take the L. H. S

$\sim[\ \exists\ x \in A, \exists\ y \in B,\ p(x,y)] = \forall x \in A \sim [\exists\ y \in B,\ p(x,y)]$

$= \forall x \in A, \forall\ y \in B,\ \sim p(x,y)$

= R. H. S

Example1.77: Find the truth values of the following statements and of their negations:

1. $\forall\ x \in R\ (x \neq 0), \exists\ y \in R,\ xy = 1$

The statement is **true** because$\forall\ x \in R\ (x \neq 0),\ \exists\ y = \frac{1}{x} \in R,\quad x\ \ \frac{1}{x} = 1$**Negation:**

$\sim [\forall\ x \in R\ (x \neq 0), \exists\ y \in R, \qquad xy = 1]$

$= \ \ \exists x \in R\ (x \neq 0),\ \forall y \in R,\ xy \neq 1$

The statement is **false**

Let x=2 and y= $\frac{1}{2}$ then $xy = 1$

2. $\exists x \in R, \exists y \in R, x^2 + y^2 \geq 0$ is true

يوجد عددين حقيقيين حاصل جمع مربعيهما عدد غير سالب

Negation:

$\sim[\ \exists x \in R, \exists y \in R, x^2 + y^2 \geq 0]$

$= \forall\ x \in R, \forall y \in R, x^2 + y^2 < 0$ is **false**

3. $\forall\ x \in N, \forall y \in N, x + y \in N$ (**H. W.**)

4. $\forall\ x \in N, \exists y \in Z, x + y \in N$ (**H. W.**)

<u>Exercise1.78</u>:

1. Express the following using connective operators and/or quantifiers

عبر عما يلي باستخدام ادوات الربط او المسورات

i) there exists p, and there exist q such that $pq = 32$

ii) for each x, there exists y such that $x < y$

iii) each even number is not odd number

iv) for each x, if x is natural number then x is an integer number

v) for each natural number x, x is even number or x is odd number

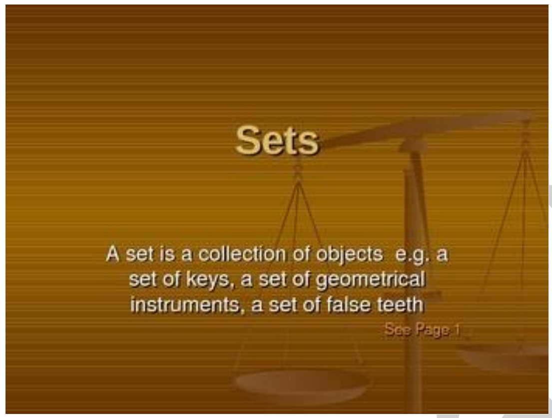
2. Find the negation of the following sentences:

i) $\forall x, \forall\, y, \exists z, x + y + z = 18$

ii) there exists y such for each $x, xy \leq 2$

iii) $\exists x, [p(x) \rightarrow Q(x)]$

CHAPTER Two: **Set Theory نظرية المجموعات**



Chapter Two Contents:

1. Basic notion of sets مفهوم المجموعات

2. Subsetsالمجموعات الجزئية

3. Algebra of sets (union, intersection, difference, complement, symmetric difference) جبر المجموعات أو العمليات على المجموعات

Definition 2.1: Set

A set is an unordered collection of objects. The objects are called **the elements** or **members** of the set.

المجموعة هي تجمع من الأشياء المعرفة بدون ترتيب والتي تسمى بالعناصر أو أعضاء تنتمي للمجموعة.

<u>Remark</u> 2.2:

1. The capital letters usually used to represents sets such as A, B, C,…etc.
2. The small letters such as a, b, c, d,…etc are used to represents the members or the elements of the set.

3. Membership in a set is denoted as follows: انتماء عنصر لمجموعة يعبر عنه بالشكل التالي

$\boldsymbol{a} \in \boldsymbol{A}$ denotes that ***a*** **belongs to** a set *A*

4. Non-membership to a set is denoted as follows: عدم انتماء عنصر لمجموعة يعبر عنه بالشكل التالي

$\boldsymbol{a} \notin \boldsymbol{A}$ denotes that ***a*** **does not belong to** a set *A*

<u>Specifying a Set</u>: طرق التعبير عن المجموعة

1. Listing members of a set: الطريقة الجدولية

In this way, we list all non-repeated members of a set separated by commas and contained in braces { }. The members are not in an order.

الطريقة الجدولية أو طريقة القائمة: في هذه الطريقة نضع جميع العناصر الغير المعادة بين قوسي مجموعة وبفواصل تفصل بينها. عناصر المجموعة لايشترط أن تكون مرتبة بطريقة معينة

<u>Example2.3</u>:

1. $\mathrm{A} = \{1,2,-5,0,9\}$, $B = \{x, y$, Ali, fish$\}$, $C = \{y_1, y_2, y_3\}$ are sets

2. The set of vowel letters in English: $V = \{a, e, i, o, u\}$

3. The set of even positive numbers less than 8 is: $W = \{0, 2, 4, 6\}$.

4. The set of positive numbers less than 50 is: $K = \{1, 2, \ldots, 49\}$

2. Listing a set property: استخدام الصفة المميزة للمجموعة

In this way, we state the property that characterize the elements in a set in as follows: {x: p(x)}, where x is a variable and p(x) is an open sentence.

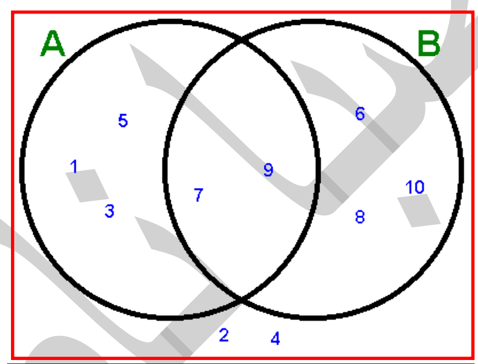
Example2.4: A=$\{x: x \in Q\}$

B=$\{x: x \text{ is positive odd and } x < 10\}$= $\{1,3,5,7,9\}$

C=$\{x \in N: -3 \le x \le 5\} = \{1,2,3,4,5\}$

3. Venn Diagrams:مخططات فن

في هذه الطريقة توضع عناصر المجموعة داخل منحني مغلق يمثل المجموعة وتستخدم هذه الطريقة لأغراض توضيحية فقط.



Definition2.5: Empty Set المجموعة الخالية

The set that contains no elements is called an empty set and is denoted by { } or Ø.

تسمى المجموعة التي لاتحوي أي عنصر بالمجموعة الخالية.

Example2.6: $A = \{x \in N: 2 < x < 3\} = \emptyset$

$B = \{x \in E: \sqrt{x}=1\} = \emptyset$

$C = \{x \in N: x < 0\} = \{\ \}$

Subsets: المجموعات الجزئية

The set A is a subset of a set B ($A \subseteq B$) if and only if every element of A is an element of B. In other words,

$$A \subseteq B \text{ iff } \forall\, x,\ x \in A \ \Rightarrow x \in B$$

Remark2.7: A is **not a subset** of B is denoted by$A \nsubseteq B$.

$$A \nsubseteq B \text{ if and only if } \sim[\ \forall\, x, x \in A \ \Rightarrow x \in B]$$
$$\text{if and only if } \exists x; x \in A \ \wedge x \notin B$$

Example2.8: Consider the sets A= {2}, B= {1, 2, 3} and C = {4, 5} and D= {-2, 1, 2, 3, 4, 5}. Then $A \subseteq B, .\, A \subseteq D,\ B \subseteq D \ and\ \ C \subseteq D$.
It is true that $A \subseteq A, .\, B \subseteq B,\ C \subseteq C \ and\ \ D \subseteq D$.

Example2.9: Let A= {4, 9} and $B = \{x \in N{:}\, 1 < x < 10\}$. Determine whether $A \subseteq B$ or$B \subseteq A$.

Solution: The set B be can be written as B={2,…,9}. Then
$\forall x,\ x \in A \ \Rightarrow x \in B$
Hence, $A \subseteq B$.
But $B \nsubseteq A$ because, for example, $\exists x = 5 \in B \ \wedge x \notin A$.

Example2.10: Let A= $\{x \in N{:}\, x > 3\}$ and $B = \{x \in N{:}\, x^2 > 4\}$. Is $A \subseteq B$? Is $B \subseteq A$?

Solution: Let $x \in A \Rightarrow\ x \in N \ and \ x > 3$
$\Rightarrow\ x^2 > 9$
$\Rightarrow\ x^2 > 4 \Rightarrow x \in B.$
$\Rightarrow A \subseteq B.$
Is $B \subseteq A$?

Example2.11: (H. W.) Let $A = \{-2, 3\}, B = \{x \in Z: x^3 - x^2 - 6x = 0\}$.
Determine whether $A \subseteq B$ or $B \subseteq A$?

Example2.12: (H. W.) Let $A = \{x \in N : x \geq 4\}$ and $B = \{x \in N : x < 9\}$.
Determine whether $A \subseteq B$ or $B \subseteq A$?

Theorem2.13: Let A, B and C be any sets, then

1. $\emptyset \subseteq A$
2. $A \subseteq A$
3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Proof 1: T. P $\emptyset \subseteq A$, $i.e.$, T. P $\forall x \in \emptyset \Rightarrow x \in A$

$$F \Rightarrow (\text{T or F}) = T$$

$$\Rightarrow \emptyset \subseteq A.$$

Proof 2: T. P $A \subseteq A$, $i.e.$, T. P $\forall x \in A \Rightarrow x \in A$

$$T \Rightarrow T = T$$

$$\Rightarrow A \subseteq A.$$

Proof 3: T. P If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

T. P $\forall x, x \in A \Rightarrow x \in C$

$\because A \subseteq B \Rightarrow \forall x, x \in A \Rightarrow x \in B$

$\because B \subseteq C \Rightarrow \forall x, x \in B \Rightarrow x \in C$

$\therefore \forall x, x \in A \Rightarrow x \in B \Rightarrow x \in C$

$\therefore \forall x, x \in A \Rightarrow x \in C$

$\therefore A \subseteq C$

Definition2.14: Proper Subset المجموعة الجزئية الفعلية

A set A is called a **proper** subset of B and denoted by $(A \subset B)$ if and only if $A \subseteq B$ and there exist an element $x \in B$ that is $x \notin A$.

i.e., $A \subset B$ iff $\{\forall x, x \in A \Rightarrow x \in B\} \land \{\exists y, y \in B \land y \notin A\}$

<u>Example2.15</u>: Let $A = \{x \in N \text{ v } x^2 - 16 = 0\}$
$B = \{x \in N\text{:}\ x^2 - 16 = 0\}$
Determine if $A \subset B\ \ or\ B \subset A$.

<u>Solution</u>: $A = \{1, 2, 3, \dots\} \cup \{4, -4\}$ and B ={4}
It is clear that $B \subset A$ because $B \subseteq A$ and
$\exists \text{y} = \{1{,}2{,}3{,}5, \dots\} \in \text{A}\ \wedge \text{y} \notin\ B$.

<u>Example2.16</u>: (H. W.) Let A ={fish, dog, bird}, $B = \{x, y, z, w\}$. Determine if $A \subset B\ \ or\ B \subset A$.

<u>Example2.17</u>: (H. W.) Let $A = \{x \in Z\text{:}\ -2 \leq x \leq 10\ \}$
$B = \{x \in Z\ \text{ v }\ x^2 + 9 = 0\}$
Determine if $A \subset B\ \ or\ B \subset A$.

<u>Solution</u>: $A = \{-2, -1{,}0{,}1, \dots, 10\}$ and $B = \{0, \pm 1, \pm 2, \dots\}\ \cup \{3i, -3i\}$

Complete the solution!

<u>Definition2.18:</u> Equal Sets المجموعات المتساوية

Two sets A and B are equal if they both have the same elements or, equivalently, if each is contained in the other.
يقال أن المجموعة A تساوي المجموعة B إذا كانت لهما نفس العناصر أو إذا كانت كل مجموعة محتواة في الأخرى.

$$\boxed{\begin{aligned} A = B\ iff\ A &\subseteq B\ \wedge \text{B} \subseteq A \\ &\leftrightarrow \{\ \forall x, x \in A \rightarrow x \in B\}\ \wedge\ \{\forall x, x \in B \rightarrow x \in A\} \\ &\leftrightarrow \{\ \forall x, x \in A \leftrightarrow x \in B\} \end{aligned}}$$

Example2.19: Let $A = \{x \in Z \ \wedge 5x^2 + 2 = 0\}$
$B = \{x \in N: 2x + 3 = 0\}$
Is $A = B$?

Solution: $A = Z \ \cap \{\mp\sqrt{\frac{2}{5}}\ i\} = \emptyset$

$B = \emptyset$

$\Longrightarrow A = B$

Lemma2.20: (**H. W.**) Prove that: $A = A, \ \ \text{for any set } A.$

Definition2.21: Universal Set المجموعة الشاملة

Universal set U is the set that contains all the elements or the sets we have under discussion.
المجموعة الشاملة: هي المجموعة التي تحوي جميع العناصر أو المجموعات قيد المناقشة ويرمز لها بالرمز U.

Example2.22: Let $A = \{x, y, 3\}, B = \{2, -5, 100\}, C = \{2, 3, 1\}$
Find a universal set U.

Example2.23: Let $A = \{x \in R: 2 \le x \le 5\}\ and \ \ B = \{x \in R: \ -1 \le x \le 2\}$
Find a universal set U.

<u>Definition2.24</u>:Family of Sets عائلة المجموعات

Family of sets is a set that have other sets as members.

يقال للمجموعة التي يكون كل عنصر من عناصرها مجموعة أنها **عائلة مجموعات**

<u>Example2.25</u>:

1. $A = \{\{1\}, \{2\}\}$ is a family of sets

2. $B = \{\emptyset\}$ is a family of a set

3. $X = \{X\}$ is a family of a set

4. $A = \{x, \{y, z\}, \{1, \dots, 5\}\}$

5. $H = \{\, A: A \; is \; a \; subset \; of \; \{1,2,3\}\}$

6. $K = \{A_i: A_i = \{2, \frac{2}{i}\}, i = 1, 2, 3\}$

<u>Definition2.26</u>: Power Set مجموعة القوى أو مجموعة الأجزاء

Given a set X, the **power set of X** is the set of all subsets of X. The power set of X is denoted by $P(X)$.

لتكن X مجموعة يقال لمجموعة كل المجموعات الجزئية من X أنها مجموعة القوى ل X ويرمز لها بالرمز $P(X)$.

$$\boxed{P(X) = \{A: A \subseteq X\}, \qquad A \in P(X) \Leftrightarrow A \subseteq X}$$

<u>Example2.27</u>: Find $P(X)$ for the following sets X:

1. X=$\{1, 2, a\}$, $\quad P(X) = \{\emptyset, X, \{1\}, \{2\}, \{a\}, \{1,2\}, \{1, a\}, \{2, a\}\}$

2. $X = \{\emptyset\}$, $\qquad P(X) = \{\emptyset, X\}$

3. $X = \{\{-2\}, 3\}$, $\quad P(X) = \{\emptyset, X, \{\{-2\}\}, \{3\}\}$

Remark2.28: 1. Since $X \subseteq X, then\ P(X) \neq \emptyset$.

2. If X is finite and has n elements, then $P(X)$ has 2^n elements.

Theorem2.29: Let X, Y be any sets, then $X \subseteq Y \iff P(X) \subseteq P(Y)$.

Proof: ($\Longrightarrow$) Let $X \subseteq Y$ T.P $P(X) \subseteq P(Y)$

Let $A \in P(X) \Longrightarrow A \subseteq X$ (By def. of $P(X)$)

$\Longrightarrow A \subseteq Y$ $(X \subseteq Y)$

$\Longrightarrow A \in P(Y)$

$\therefore P(X) \subseteq P(Y)$

($\Longleftarrow$) Let $P(X) \subseteq P(Y)$ To Prove $X \subseteq Y$

Let $x \in X \Longrightarrow \{x\} \subseteq X$

$\Longrightarrow \{x\} \in P(X)$

$\Longrightarrow \{x\} \in P(Y)$ $\{P(X) \subseteq P(Y)\}$

$\Longrightarrow \{x\} \subseteq Y$

$\Longrightarrow x \in Y$

$\therefore X \subseteq Y$

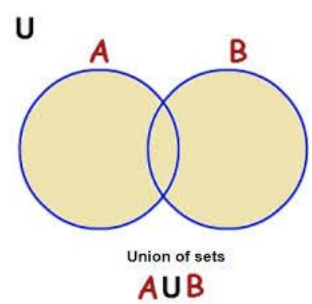
Algebra of sets:

1. Union الاتحاد

The **union** of the sets A and B, denoted by $\boldsymbol{A \cup B}$, is the set of elements which belong to A **or** to B.

إتحاد مجموعتين B و A هي مجموعة العناصر التي تنتمي للمجموعة A أو المجموعة B.

$$A \cup B = \{x, x \in A \vee x \in B\}$$
$$x \in A \cup B \iff x \in A \vee x \in B$$
$$x \notin A \cup B \iff x \notin A \wedge x \notin B$$



<u>Example2.30</u>: Let $A = \{x \in N: 1 \leq x \leq 5\} = \{1,2,3,4,5\}$
$B = \{x \in N: 8 \leq x \leq 12\} = \{8,9,10,11,12\}$
Find $A \cup B,\ B \cup A, A \cup A,\ and\ B \cup \emptyset$
<u>Solution</u>: $A \cup B = B \cup A = \{1, \dots, 5, 8, \dots, 12\}$

$A \cup A = A$
$B \cup \emptyset = B$

<u>Example2.31</u>: Let $A = \{x \in R: -2 \leq x \leq 5\} = [-2,5]$,
$B = \{x \in E: x^2 - 16 = 0\} = \{4, -4\}$
$C = \{1, 4\}$
Find $A \cup (B \cup C), (A \cup B) \cup C, P(B), P(C), P(B \cup C)$

<u>Definition</u>2.32: Generalization of the union تعميم الاتحاد
Let $A_1, A_2, \dots, A_n, A_{n+1}, \dots$ be any sets. Then:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

اتحاد عدد منتهي من المجموعات

In general,

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup \ldots \cup A_n \cup A_{n+1} \cup \ldots$$

اتحاد عدد غير منتهي من المجموعات

Example2.33: Let $H = \{A_i;\ A_i = \{2i+3\},\ i \in Z\}$
Find

$\bigcup_{i=1}^{4} A_i$, and $\bigcup_{i=-1}^{3} A_i$

Solution: $\bigcup_{i=1}^{4} A_i = A_1 \cup A_2 \cup A_3 \cup A_4 = \{5\} \cup \{7\} \cup \{9\} \cup \{11\}$
$= \{5,7,9,11\}$

$\bigcup_{i=-1}^{3} A_i = A_{-1} \cup A_0 \cup A_1 \cup A_2 \cup A_3 = \{1\} \cup \{3\} \cup \{5\} \cup \{7\} \cup \{9\}$
$= \{1,3,5,7,9\}$

Example2.34: Let $K = \{A_n;\ A_n = (-n, n),\ n \in N\}$
Find $A_1 \cup \bigcup_{n=2}^{\infty} A_n$

Solution: $A_1 \cup \bigcup_{n=2}^{\infty} A_n = (-1,1) \cup (-2,2) \cup \ldots \cup (-k, k) \cup \ldots$

Example2.35: (H. W.) Let $K = \{A_n;\ A_n = [n, n+10),\ n \in Z\}$
Find $\bigcup_{n=-2}^{2} A_n$

Example2.36: (H. W.) Let $K = \{A_j; A_j = \{j+1, j+2\},\ j \in N\}$
Find $\bigcup_{j=3}^{10} A_j$

Theorem2.37: Let A, B and C any three sets. Then:
1. $A \cup \emptyset = A$ (Identity law)
2. $A \cup A = A$ (Idempotent law)
3. $A \cup U = U$ (Domination law)
4. $(A \cup B) \cup C = A \cup (B \cup C)$ (**H.W.**)
5. $A \cup C = C \cup A$
6. $A \subseteq B \iff A \cup B = B$
   $B \subseteq A \iff A \cup B = A$
7. $A \subseteq A \cup B$ (**H.W.**)
   $B \subseteq A \cup B$
8. $P(A) \cup P(B) \subseteq P(A \cup B)$

Proof 1: To prove $A \cup \emptyset \subseteq A \wedge A \subseteq A \cup \emptyset$

T. P $A \cup \emptyset \subseteq A$ (T. P $\forall x \in A \cup \emptyset \Longrightarrow x \in A$)
Let $x \in A \cup \emptyset \Longrightarrow x \in A \vee x \in \emptyset$ (def. of ∪)
$\Longrightarrow x \in A \vee$ F
$\Longrightarrow x \in A$ (p v F=p)
$\therefore A \cup \emptyset \subseteq A$ **……………(1)**

Let $x \in A \Longrightarrow x \in A \vee$ F (p ∨ F=p)
$\Longrightarrow x \in A \vee x \in \emptyset$
$\Longrightarrow x \in A \cup \emptyset$ (def. of ∪)
$\therefore A \subseteq A \cup \emptyset$ **……………(2)**

From (1) & (2), $A \cup \emptyset = A$

Proof 2: To prove $A \cup A \subseteq A \wedge A \subseteq A \cup A$

T. P $A \cup A \subseteq A$ (T. P $\forall x \in A \cup A \Longrightarrow x \in A$)
Let $x \in A \cup A \Longrightarrow x \in A \vee x \in A$ (def. of ∪)
$\Longrightarrow x \in A$ (p ∨ p=p)
$\therefore A \cup A \subseteq A$ **……………(1)**

Let $x \in A \Longrightarrow x \in A \vee x \in A$ (p ∨ p=p)
$\Longrightarrow x \in A \cup A$ (def. of ∪)
$\therefore A \subseteq A \cup A$ **……………(2)**

From (1) & (2), $A \cup A = A$

<u>Proof 3</u>: To prove $A \cup U \subseteq U \wedge \mathrm{U} \subseteq A \cup U$

T. P $A \cup U \subseteq U$ (T. P $\forall x \in A \cup U \Longrightarrow x \in U$)
Let $x \in A \cup U \Longrightarrow x \in A \vee x \in U$ (def. of ∪)
$\Longrightarrow x \in U \vee x \in U$ $(A \subseteq U)$
$\Longrightarrow x \in U$
$\therefore A \cup U \subseteq U$ **……………(1)**

Let $x \in U \Longrightarrow x \in U \vee x \in A$ (T ∨ p=T)
$\Longrightarrow x \in A \cup U$ (def. of ∪)
$\therefore U \subseteq A \cup U$ **……………(2)**
From (1) & (2), $A \cup U = U$
<u>Proof 5</u>: To prove $A \cup C \subseteq C \cup A \wedge C \cup A \subseteq A \cup C$

T. P $A \cup C \subseteq C \cup A$
Let $x \in A \cup C \Longrightarrow x \in A \vee x \in C$ (def. of ∪)
$\Longrightarrow x \in C \vee x \in A$ (∨ is commutative)
$x \in C \cup A$ (def. of ∪)

$\therefore A \cup C \subseteq C \cup A$ **……………(1)**

Similarly, show that $C \cup A \subseteq A \cup C$ (**H. W.**) **……………(2)**

From (1) & (2), $A \cup C = C \cup A$

<u>Proof 6</u>: To prove $A \subseteq B \Leftrightarrow A \cup B = B$

(⟹) if $A \subseteq B$ T. P $A \cup B = B$
T. P $A \cup B \subseteq B \wedge B \subseteq A \cup B$
Let $x \in A \cup B \Longrightarrow x \in A \vee x \in B$ (def. of ∪)
$\Longrightarrow x \in B \vee x \in B$ (by hypo. $A \subseteq B$)
$\Longrightarrow x \in B$ (p ∨ p=p)
$\therefore A \cup B \subseteq B$ **……………(1)**
Let $x \in B \Longrightarrow x \in B \vee x \in A$ (T ∨ p=T)
$\Longrightarrow x \in A \cup B$ (def. of ∪)

$\therefore B \subseteq A \cup B$ **..............(2)**

From (1) & (2), $A \cup B = B$

($\Longleftarrow$) If $A \cup B = B$ To prove $A \subseteq B$

Let $x \in A \Longrightarrow x \in A \ \vee \ x \in B$ (T ∨ p=T)

$\Longrightarrow \quad x \in A \cup B$ (def. of ∪)

$\Longrightarrow \quad x \in B$ (by hypo. $A \cup B = B$)

$\therefore A \subseteq B$

<u>Proof 8</u>: T. P $P(A) \cup P(B) \subseteq P(A \cup B)$

Let $X \in P(A) \cup P(B)$ To prove $X \in P(A \cup B)$

$X \in P(A) \cup P(B) \Longrightarrow X \in P(A) \ \vee \ X \in P(B)$ (def. of ∪)

$\Longrightarrow X \subseteq A \ \vee \ X \subseteq B$ (def. of $P(A)$)

$\Longrightarrow X \subseteq A \cup B$ (def. of ∪)

$\Longrightarrow X \in P(A \cup B)$ (def. of $P(A \cup B)$)

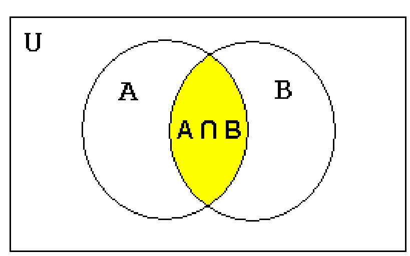
$\therefore P(A) \cup P(B) \subseteq P(A \cup B)$

2. <u>Intersection</u> التقاطع

The **intersection** of the sets A and B, denoted by $\boldsymbol{A \cap B}$, is the set of elements which belong to both A **and** to B.

تقاطع مجموعتين B و A هي مجموعة العناصر التي تنتمي لكلا المجموعتين معاً.

$$A \cap B = \{x; x \in A \ \wedge \mathrm{x} \in \mathrm{B}\}$$
$$x \in A \cap B \Leftrightarrow x \in A \ \wedge \mathrm{x} \in \mathrm{B}$$
$$x \notin A \cap B \Leftrightarrow x \notin A \ \vee \ \mathrm{x} \notin \mathrm{B}$$



Example2.38: Let $A = [2,9], B = (5, 14], C = (8, 12)$
Find $A \cap B, A \cap \emptyset, B \cap B, B \cap C, A \cap C, (A \cap B) \cup (B \cap C)$

Definition2.39: Generalization of the intersection تعميم التقاطع
Let $A_1, A_2, \dots, A_n, A_{n+1}, \dots$ be any sets. Then:

$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i$=$\{x, x \in A_i \;\; \forall i = 1,2, \dots, n\}$
تقاطع عدد منتهي من المجموعات

In general,

$A_1 \cap A_2 \cap \dots \cap A_n \cap A_{n+1} \cap \dots = \bigcap_{i=1}^{\infty} A_i$=$\{x, x \in A_i, \;\forall\, i\}$
تقاطع عدد غير منتهي من المجموعات

Example2.40: Let $X = \{A_i; A_i = \{1, 2, 3, \dots, i\}; i \in N\}$
Find

$$(\bigcup_{i=1}^{5} A_i) \cap (\bigcap_{i=1}^{5} A_i)$$

Solution: $A_1 = \{1\}, A_2 = \{1,2\}, \dots\ A_5 = \{1, 2, 3, 4, 5\}$

$$\bigcup_{i=1}^{5} A_i = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = \{1,2,3,4,5\}$$

$$\bigcap_{i=1}^{5} A_i = A_1 \cap \ldots \cap A_5 = \{1\}$$

$(\bigcup_{i=1}^{5} A_i) \cap (\bigcap_{i=1}^{5} A_i) = \{1\}$

<u>Example2.41</u>: Let $X = \left\{A_j; A_j = \left(\frac{-1}{j}, \frac{1}{j}\right); j \in N\right\}$

Find

$$\bigcap_{j=2}^{4} A_j \quad and \quad \bigcup_{j=2}^{4} A_j$$

<u>Solution</u>:

$$\bigcap_{j=2}^{4} A_j = A_2 \cap A_3 \cap A_4 = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{1}{3}, \frac{1}{3}\right) \cap \left(-\frac{1}{4}, \frac{1}{4}\right) = \left(-\frac{1}{4}, \frac{1}{4}\right)$$

$$\bigcup_{j=2}^{4} A_j = A_2 \cup A_3 \cup A_4 = \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{1}{3}\right) \cup \left(-\frac{1}{4}, \frac{1}{4}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

<u>Example2.42</u>: **(H. W.)** Let $X = \{A_n; A_n = \{n^3 + 1\}; n \in Z\}$

Find

$\bigcap_{n=-3}^{0} A_n$, $\bigcap_{n=1}^{\infty} A_n$, $P(\bigcup_{n=1}^{3} A_n)$

Example2.43: (**H. W.**) Let $X = \{A_n;\ A_n = \{n-2, n-1, n\}; n \in N\}$
Find

$\bigcap_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} A_n$

Theorem2.44: Let $A, B\ and\ C$ any three sets. Then:

1. $A \cap \emptyset = \emptyset$ (**H. W.**) (Domination law)
2. $A \cap A = A$ (**H. W.**) (Idempotent law)
3. $A \cap U = A$ (Identity law)
4. $(A \cap B) \cap C = A \cap (B \cap C)$
5. $A \cap C = C \cap A$ (**H. W.**)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (**H. W.**)
8. $A \cap B \subseteq A$ (**H. W.**)
   $A \cap B \subseteq B$
9. $A \subseteq B \iff A \cap B = A$
10. $P(A) \cap P(B) = P(A \cap B)$ (**H. W.**)

Proof 3: **T. P** $A \cap U \subseteq A \wedge\ A \subseteq A \cap U$
Assume that $x \in A \cap U \Longrightarrow \mathrm{x} \in A \wedge \mathrm{x} \in U$
$\Longrightarrow \mathrm{x} \in A$
$\therefore A \cap U \subseteq A$ **.............(1)**

Let $x \in A$ T.P $x \in A \cap U$
$x \in A \Longrightarrow x \in A \wedge \mathrm{x} \in U \quad [A \subseteq U]$
$\Longrightarrow x \in A \cap U$ **.............(2)**

From (1) & (2), $A \cap U = A$

Proof 4:

Let $x \in (A \cap B) \cap C \Leftrightarrow x \in (A \cap B) \wedge x \in C$ (def. of ∩)
$\Leftrightarrow (x \in A \wedge x \in B) \wedge x \in C$ (def. of ∩)
$\Leftrightarrow x \in A \wedge (x \in B \wedge x \in C)$ (∧ is assoc.)
$\Leftrightarrow x \in A \wedge x \in B \cap C$ (def. of ∩)
$\Leftrightarrow x \in A \cap (B \cap C)$ (def. of ∩)
$\therefore (A \cap B) \cap C = A \cap (B \cap C)$

<u>Proof 6</u>:

Let $x \in A \cap (B \cup C) \Leftrightarrow x \in A \wedge x \in B \cup C$ (def. of ∩)

$\Leftrightarrow x \in A \wedge (x \in B \vee x \in C)$ (def. of ∪)

$\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ (distribute ∧ on v)

$\Leftrightarrow x \in A \cap B \vee x \in A \cap C$ (def. of ∩)

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$ (def. of ∪)

<u>Proof 9</u>: T. P $A \subseteq B \Leftrightarrow A \cap B = A$

($\Longrightarrow$) Suppose $A \subseteq B$ T.P $A \cap B \subseteq A \wedge A \subseteq A \cap B$

Let $x \in A \cap B \Longrightarrow x \in A \wedge x \in B$ (def. of ∩)

$\Longrightarrow x \in A$

$\therefore A \cap B \subseteq A$**............(1)**

Let $x \in A \Longrightarrow x \in A \wedge x \in B$ $[A \subseteq U]$

$\Longrightarrow x \in A \cap B$

$\Longrightarrow A \subseteq A \cap B$**.......(2)**

From (1) & (2), $A \cap B = A$

($\Longleftarrow$) Let $A \cap B = A$ **T. P** $A \subseteq B$ (**H. W.**)

3. <u>The Complement</u> المتممة أو المكملة

Let U be a universal set and A be any subset of U. The **complement** of a set A, denoted by A^c, is the set of elements which belong to U but do not belong to A.

متممة المجموعة A هي مجموعة العناصر التي تنتمي للمجموعة الشاملة ولا تنتمي للمجموعة A.

$$A^c = \{x, x \in U \wedge x \notin A\} = U \backslash A$$
$$x \in A^c \Leftrightarrow x \notin A$$
$$x \in \mathrm{A} \Leftrightarrow \mathrm{x} \notin \mathrm{A}^{\mathrm{c}}$$

<u>Example2.45</u>: Let $U = Z, A = \{-1, 0, 1\}$. Find A^c.

<u>Solution</u>: $A^c = Z \backslash A$

Example2.46: Let $U = [0, 8), A = \{1, 2\}$. Find A^c.

$A^c = [0,1) \cup (1,2) \cup (2,8)$

Example2.47: Let $U = \{1, 2, \dots, 10\}$,

$A = \{x \in N: 1 \leq x \leq 3\} = \{1,2,3\}$

$B = \{x \in N: 8 \leq x \leq 10\} = \{8,9,10\}$

$C = \{x \in N: 1 \leq x \leq 2\} = \{1,2\}$

Find $A^c, B^c, C^c, (A \cup B)^c, (A \cap C)^c, (C \cup B)^c$

Solution: $A^c = \{4,5, \dots, 10\}$

$B^c = \{1,2, \dots, 7\}$

$C^c =$

$(A \cup B)^c = \{1,2,3,8,9,10\}^c = \{4,5,6,7\}$

$(A \cap C)^c = \{1,2\}^c = \{3,4, \dots, 10\}$

Theorem2.48: Let A and B any two sets. Then:

1. $\emptyset^c = U$, $U^c = \emptyset$
2. $A^c \cap A = \emptyset$ (**H. W.**), $A^c \cup A = U$, $(A^c)^c = A$ (**H. W.**)
3. $(A \cup B)^c = A^c \cap B^c$
4. $(A \cap B)^c = A^c \cup B^c$ (**H. W.**)
5. $A \subseteq B \Leftrightarrow B^c \subseteq A^c$ (**H. W.**)
6. $A \subseteq B \Leftrightarrow A \cap B^c = \emptyset$

Proof 1: T. P $\emptyset^c = U$

Assume that $\emptyset^c \neq U$ برهان غير مباشر

$\exists x, x \in \emptyset^c \wedge x \notin \mathrm{U}$

$x \in \emptyset^c \Longrightarrow x \in U \wedge x \notin \emptyset$ (def. of $\emptyset^c$)

$\Longrightarrow x \in U$ تناقض مع الفرض

$\therefore \emptyset^c = U$

Proof 1: T. P $U^c = \emptyset$

Assume that $U^c \neq \emptyset$ برهان غير مباشر

$\exists x, x \in U^c \wedge x \notin \emptyset$

$x \in U^c \Longrightarrow x \in U \wedge x \notin \mathrm{U}$ تناقض
$\therefore U^c = \emptyset$

<u>Proof 2</u>: T. P $A^c \cup A = U$
Assume that $A^c \cup A \neq U$ برهان غير مباشر
$\exists x, x \in A^c \cup A \wedge x \notin \mathrm{U} \ldots\ldots (*)$
$x \in A^c \cup A \Longrightarrow x \in A^c \vee x \in A$ (def. of ∪)
If $x \in A^c \Longrightarrow \boldsymbol{x} \in \boldsymbol{U} \wedge x \notin \mathrm{A}$ (contradiction with $x \notin \mathrm{U}$ in *)
If $x \in A \Longrightarrow \boldsymbol{x} \in \boldsymbol{U} \wedge x \notin A^c$ (contradiction with $x \notin \mathrm{U}$ in *)
$\therefore A^c \cup A = U$

<u>Proof 3</u>: T. P $(A \cup B)^c = A^c \cap B^c$
$x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B$ (def of complement)
$\Leftrightarrow x \notin A \wedge x \notin B$ (def of ∪)
$\Leftrightarrow x \in A^c \wedge x \in B^c$ (def. of A^c)
$\Leftrightarrow x \in A^c \cap B^c$ (def. of ∩)

<u>Proof 6</u>: (⟹) Let $A \subseteq B$ **T. P** $A \cap B^c = \emptyset$
Assume that $A \cap B^c \neq \emptyset$ برهان غير مباشر بالتناقض
$\exists x, x \in A \cap B^c \Longrightarrow x \in A \wedge x \in B^c$
$\Longrightarrow x \in B \wedge x \in B^c$ (by hypo. $A \subseteq B$)
$\Longrightarrow$ تناقض
$\therefore A \cap B^c = \emptyset$

(⟸) Let $A \cap B^c = \emptyset$ T. P $A \subseteq B$
Let $x \in A \Longrightarrow x \notin B^c$ (by hypo. $A \cap B^c = \emptyset$)
$\Longrightarrow x \in B$ (def. of B^c)
$\therefore A \subseteq B$

<u>4. Difference or relative complement</u> : الفضلة أو الفرق

Let A and B are two sets. The **difference** between A and B, denoted as A-B or $A\backslash B$, is the set of elements which belong to A but do not belong to B.

يقال لمجموعة العناصر المنتمية إلى A وغير منتمية إلى B بأنها فضلة A على B.

$$A - B = \{x, x \in A \wedge x \notin B\} = A\backslash B$$
$$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$$
$$x \notin A - B \Leftrightarrow x \notin A \vee x \in B$$

Example2.49: Let $A = \{x \in Z: x \geq 1\} = N$

$$B = \{x \in R: x^2 + 3 = 0\} = \emptyset$$

$$C = \{x \in Z: -3 < x \leq 4\} = \{-2, -1, \dots, 4\}$$

$$D = \{x \in O: x^2 - 9 = 0\} = \{3, -3\}$$

Find $A \backslash D$, $B - B$, $C \backslash (A \cap D)$, $(B \cup A) \backslash C$, $(C \cup A) \backslash (B \cap D)$

Solution: $A \backslash D = N \backslash \{\mathbf{3}, -\mathbf{3}\}$

$$(C \cup A) \backslash (B \cap D) = (N \cup \{-2, -1, 0\}) \backslash \emptyset = N \cup \{-2, -1, 0\}$$

$$B - B =$$

$$C \backslash (A \cap D) =$$

$$(B \cup A) \backslash C$$

Theorem2.50: Let A, B *and* C any three sets. Then:

1. $A \backslash A = \emptyset$, $A \backslash U = \emptyset$ (**H. W.**)
2. $A \backslash \emptyset = A$, $\emptyset \backslash A = \emptyset$ (**H. W.**)
3. $A \backslash B \subseteq A$, $B \backslash A \subseteq B$ (**H. W.**)
4. $A \subseteq B \Leftrightarrow A \backslash B = \emptyset$ (**H. W.**)
5. $A \backslash B \cap B = \emptyset$
6. $A \cap B = \emptyset \Leftrightarrow A \backslash B = A \wedge B \backslash A = B$
7. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$ (**H. W.**)
8. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$
9. $A \backslash A^c = A$, $A^c \backslash A = A^c$ (**H. W.**)

Proof 5: Assume that $A \backslash B \cap B \neq \emptyset$ برهان غير مباشر

$$\exists x, x \in A \backslash B \cap B \Longrightarrow x \in A \backslash B \wedge x \in B$$

$$\Longrightarrow (x \in A \wedge x \notin B) \wedge x \in B \quad \text{(def. of difference)}$$

$\Longrightarrow x \in A \wedge (x \notin B \wedge x \in B)$ ($\wedge$ is asso.)

$\Longrightarrow x \in A \wedge F$

$\Longrightarrow x \in F$ تناقض ($p \wedge \mathrm{F} = F$)

$\therefore A \backslash B \cap B = \emptyset$

Proof 6: ($\Longrightarrow$) Let $A \cap B = \emptyset$ T. P $A \backslash B = A \wedge B \backslash A = B$

Let $x \in A \backslash B \Leftrightarrow x \in A \wedge x \notin B$ (def. of \)

$\Leftrightarrow x \in A \wedge T$ (by hypo.)

$\Leftrightarrow x \in A$ ($\mathrm{p} \wedge T = p$)

$\therefore A \backslash B = A$

Similarly, one can prove that $B \backslash A = B$

($\Longleftarrow$) Let $A \backslash B = A \wedge B \backslash A = B$ T. P $A \cap B = \emptyset$

Assume that $A \cap B \neq \emptyset$ برهان بالتناقض

$\exists x, x \in A \cap B \Longrightarrow x \in A \wedge x \in B$ (def. of $\cap$)

$\Longrightarrow \exists x, x \in A \backslash B \wedge x \in B$ ($A \backslash B = A$)

$\Longrightarrow \exists x, x \in A \backslash B \wedge x \in B$

$\Longrightarrow \exists x, (x \in A \wedge x \notin B) \wedge x \in B$

$\Longrightarrow \exists x, x \in A \wedge (x \notin B \wedge x \in B)$ ($\wedge$ is asso.)

$\Longrightarrow \exists x, x \in A \wedge F = F$ تناقض ($p \wedge F = F$)

$\therefore A \cap B = \emptyset$

Proof 8: T. P $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

let $x \in A \backslash (B \cap C) \Leftrightarrow x \in A \wedge x \notin B \cap C$ (def. of \)

$\Leftrightarrow x \in A \wedge (x \notin B \vee x \notin C)$ (def. of $\cap$)

$\Leftrightarrow (x \in A \wedge x \notin B)$ v $(x \in A \wedge x \notin C)$ (dist. ∧ over v)

$\Leftrightarrow x \in A \backslash B$ v $x \in A \backslash C$ (def. of \)

$\Leftrightarrow x \in A \backslash B \cup A \backslash C$

$\therefore A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

5. Symmetric Difference الفرق التناظري

The **symmetric difference** between two sets A and B is denoted by $A \Delta B$ and is defined as:

$$\boxed{\begin{aligned} A \,\Delta\, B &= (A \backslash B) \cup (B \backslash A) \\ &= (A \cup B) \backslash (A \cap B) \end{aligned}}$$

<u>Example2.51</u>: Let $A = \{x \in E : -8 \leq x < 9\} = \{-8, -6, \ldots, 0, 2, \ldots 8\}$

$B = \{1, 2, 4, 6\}$

Find $A \Delta B$

<u>Solution</u>: $A \,\Delta\, B = (A \backslash B) \cup (B \backslash A)$

$= \{-8, -6, \ldots 0, 6, 8\} \cup \{1\}$

<u>Theorem2.52</u>: Let A, B *and* C any three sets. Then:

1. $A \Delta A = \emptyset$ $(\boldsymbol{H}.\boldsymbol{W}.)$, $A \,\Delta\, \emptyset = A$
2. $A \,\Delta\, B = B \,\Delta\, A$
3. $A \,\Delta\, B = \emptyset \Leftrightarrow A = B$
4. $A \,\Delta\, (B \,\Delta\, C) = (A \Delta B) \Delta\, C$ **(بدون برهان)**
5. $A \cap (B \,\Delta\, C) = (A \cap B) \Delta (A \cap C)$ **(بدون برهان)**

Proof 1: T. P $A \, \Delta \, \emptyset = A$

Let $x \in A \, \Delta \, \emptyset \Leftrightarrow x \in (A \backslash \emptyset \cup \emptyset \backslash A)$ (def. of Δ)

$\Leftrightarrow x \in A \backslash \emptyset \vee x \in \emptyset \backslash A$

$\Leftrightarrow x \in A \cup \emptyset$ $(A \backslash \emptyset = A \, , \emptyset \backslash A = \emptyset)$

$\Leftrightarrow x \in A$

$\therefore \; A \, \Delta \, \emptyset = A$

Proof 2: T. P $A \, \Delta \, B = B \, \Delta \, A$

Let $x \in A \, \Delta \, B \Leftrightarrow x \in (A \backslash B \cup B \backslash A)$ (def. of Δ)

$\Leftrightarrow x \in (B \backslash A \cup A \backslash B)$ (∪ is comm.)

$\Leftrightarrow x \in B \, \Delta \, A$

$\therefore A \, \Delta \, B = B \, \Delta \, A$

Proof 3: $A \, \Delta \, B = \emptyset \Leftrightarrow A = B$

(⟹) let $A \, \Delta \, B = \emptyset$ T. P $A = B$

Suppose $A \neq B \Longrightarrow \exists x, x \in A \wedge x \notin B$

$\Longrightarrow \exists x, x \in A \backslash B$ (def. of \)

$\Longrightarrow \exists x, x \in A \backslash B \vee x \in B \backslash A$ (T ∨ p=T)

$\Longrightarrow \exists x, x \in A \backslash B \cup B \backslash A$ (def. of ∪)

$\Longrightarrow \exists x, x \in A \, \Delta \, B = \emptyset$

$\Longrightarrow \exists x, x \in \emptyset$ contradiction

$\therefore \; A = B$

(⟸) suppose $A = B$ T. P $A \, \Delta \, B = \emptyset$

$A \, \Delta \, B = A \backslash B \cup B \backslash A$ (def. of \)

$= A \backslash A \cup A \backslash A$ $(A = B)$

= ∅ ∪ ∅=∅

Chapter Three: Relations العلاقات

Chapter Three Contents:

1. Cartesian Product الضرب الديكارتي

2. Relations العلاقات

3. Properties of Relationsأنواع العلاقات

4. Ordering الترتيب

Definition3.1: Ordered Pair الزوج المرتب

An **ordered pair** of elements *a* and *b is* denoted by *(a, b)* where *a* is called the first element and *b* is the second element.

ليكن a وb عنصراً . الزوج المرتب المتكون من a وb يرمز له بالرمز (a, b) حيث أن aيسمى مسقط أول للزوج المرتب والعنصر b يسمى المسقط الثاني.

Remark3.2: Let a, b, c and d be four elements. Then:

1. $(a, b) \neq (c, d)$ in general
2. $(a, b) = (c, d) \Leftrightarrow a = c \ \wedge \mathrm{b} = \mathrm{d}$
3. $(a, b) = (b, a) \Leftrightarrow a = b$

Cartesian Product الضرب الديكارتي

Consider two arbitrary sets *A* and *B*. The set of all ordered pairs *(a, b)* where $a \in A$ and $b \in B$ is called the **product**, or **Cartesian product**, of *A* and *B* and denoted by $\boldsymbol{A} \times \boldsymbol{B}$.

$$\boldsymbol{A} \times \boldsymbol{B} = \{(\mathrm{a}, \mathrm{b}): \mathrm{a} \in \mathrm{A} \ \wedge \mathrm{b} \in \mathrm{B}\}$$
$$(\mathrm{a}, \mathrm{b}) \in A \times B \Leftrightarrow \mathrm{a} \in \mathrm{A} \ \wedge \mathrm{b} \in \mathrm{B}$$
$$(\mathrm{a}, \mathrm{b}) \notin A \times B \Leftrightarrow \mathrm{a} \notin \mathrm{A} \ \vee \ \mathrm{b} \notin \mathrm{B}$$

<u>Example</u>3.3: Let $A=\{a,b,c\}$, $B=\{5,4\}$. Find

$A\times B$=

$B\times A$=

$A\times A$ =

$B\times B$=

<u>Remark</u>3.4: If the number of a set A equals n and the number of a set B equals m. Then the number of the elements of $A\times B$ is nm.

<u>Example</u>3.5: Let $A=\{x\in N: x\leq 3\}=\{1,2,3\}$
$B=\{0,3\}$, $C=\{1\}$. Find

$A\times A$=

$B\times B$ $=\{$

$C\times C$=

$B\times C$=

$(B\cap C)\times A$=

$(B\cup C)\times B$=

Is , $A\times B=B\times A$?

<u>Theorem</u>3.6: Let A, B, C and D be non empty sets. Then:

1. $A \times \emptyset = \emptyset$ and $\emptyset \times A = \emptyset$
2. $A \times B = B \times A \Leftrightarrow A = B$
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (**H. W.**)
4. $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (**H. W.**)
5. $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$
6. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ (**H. W.**)

<u>Proof 1</u>: T. P. $\emptyset \times A = \emptyset$
Suppose that $\emptyset \times A \neq \emptyset$ نفرض العكس
$\exists\ (x, y) \in \emptyset \times A \Rightarrow x \in \emptyset\ \wedge\ y \in A$ (def. of $A \times B$)
$\Rightarrow F \wedge y \in A$
$\Rightarrow F$ ($F \wedge p = F$) تناقض
$\therefore \emptyset \times A = \emptyset$

In the same way, prove that $A \times \emptyset = \emptyset$ (**H. W**)

<u>Proof 2</u>: ($\Rightarrow$) Suppose $A \times B = B \times A$ T. P. $A = B$
Let $(x, y) \in A \times B \Rightarrow x \in A\ \wedge\ y \in B$ (def. of $A \times B$)
$(x, y) \in B \times A \Rightarrow x \in B\ \wedge\ y \in A$ ($A \times B = B \times A$)
$\Rightarrow A \subseteq B\ \wedge\ B \subseteq A$
$\Rightarrow A = B$ (def. of equal sets)

($\Leftarrow$) Suppose $A = B$ T. P. $A \times B = B \times A$
Let $(x, y) \in A \times B \Leftrightarrow x \in A\ \wedge\ y \in B$ (def. of $A \times B$)
$\Leftrightarrow x \in B\ \wedge\ y \in A$ ($A = B$)
$\Leftrightarrow (x, y) \in B \times A$

$\therefore A \times B = B \times A$

<u>Proof 5</u>: T. P. $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$
Let $(x, y) \in A \times (B \backslash C) \Leftrightarrow x \in A\ \wedge\ y \in (B \backslash C)$ (def. of $A \times B$)
$\Leftrightarrow x \in A\ \wedge (y \in B\ \wedge\ y \notin C)$ (def. of $\backslash$)
$\Leftrightarrow (x \in A\ \wedge y \in B) \wedge (x \in A\ \wedge y \notin C)$ (dist. $\wedge$ on $\wedge$)
$\Leftrightarrow (x, y) \in A \times B\ \wedge\ (x, y) \notin A \times C$

$\Leftrightarrow (x, y) \in (A \times B) \backslash (A \times C)$
$\therefore A \times (B \backslash C) = (A \times B) \backslash (A \times C)$

<u>Definition</u>3.7: Generalization of the Cartesian product تعميم الضرب الديكارتي

Let $A_1, A_2, \dots, A_n$ be any sets. Then

$$\prod_{i=1}^{n} A_i = A_1 \times \dots \times A_n = \{(x_1, \dots, x_n) : x_i \in A_i, i = 1, \dots, n\}$$

<u>Example</u>3.8: What is the Cartesian product $A \times B \times C$, where $A = \{0,1\}, B = \{1,2\}, C = \{2\}$?

<u>Solution</u>:

$A \times B \times C = \{(0,1,2), (0,2,2), (1,1,2), (1,2,2)\}$

<u>Remark</u>3.9: Let A be a set, then

$$A \times A = A^2$$
$$A \times A \times A = A^3$$

In general, $A \times \dots \times A = A^n$

<u>Example3.10</u>: $R \times R = R^2$

$R \times R \times R = R^3$

<u>Relation</u>: العلاقة

Let A and B are two sets. Any subset R of $A \times B$ is called a **relation** from A to B. In other words,

R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

$(x, y) \in R$ can be written as xRy or $x \sim y$ $(x, y) \notin R$ can be written as xRy or $x \nsim y$

Remark3.11: The relations are denoted by $R, S, T, W, \ldots$

Definition3.12: If R is a relation from $\boldsymbol{A\ to\ A}$ $(R \subseteq A \times A)$ then R is called a **relation on A**

Example3.13 Let $A = \{1, 4, 5\}$, $B = \{1, a\}$

$A \times B = \{(1,1), (1, a), (4,1), (4, a), (5,1), (5, a)\}$. Write three relations from A to B.

Solution:

R=

S=

T=

Remark3.14: The empty set ∅ is the **smallest relation** from A to B

(∅ $\subseteq A \times B$). And, $A \times B$ is the **largest relation** from A to B

$(A \times B \subseteq A \times B)$.

Example3.15: (H. W.) Let $A = \{1, 4, 5\}$, $B = \{1, a\}$. Find $B \times A$ and write three relations from B to A.

Example3.16: Let $A = \{x, y, -1\}$. Find a relation from A to A.

Solution: $A \times A = \{(a, , b): a, b \in A\}$

$A \times A$ =

$\{(x, x), (x, y), (x, -1), (y, x), (y, y), (y, -1), -1, x), (-1, y), (-1, -1)\}$

Specifying a relation: طرق التعبير عن العلاقة

1. Listing members of a relation: الطريقة الجدولية

List the members of the relation separated by commas and contained in braces { }.

Example3.17: Let $A = \{1,2\}$, and B={x, y} are two sets and R is a relation from $A\ to\ B$.

R=$\{(1,x),(1,y),(2,x),(2,y)\}$

2. Listing a relation property: استخدام الصفة المميزة للعلاقة

State the property that characterizes the elements in a relation

Example3.18: A=$\{-1,5,0\}$
B=$\{-3,0,-1\}$
Let $R_1 = \{(a,b) \in A \times B : a \geq b\} =$
$\{(-1,-3),(-1,-1),(5,-3),(5,0),(5,-1),$ (0,-3),(0,0),(0,-1)}

$$R_2 = \{(x,y) \in A \times B : x = y\} = \{(-1,-1),(0,0)\}$$

Definition3.19: Let $x\ and\ y$ are integers with $x \neq 0$ $(x, y \in Z)$. Then ''x divides يقسمy'' is denoted by $x|$y and is defined as:

$$x|\text{y} \Leftrightarrow \exists k \in Z\ such\ that\ y = kx$$
$$x|\text{v} \Leftrightarrow \forall k \in Z :\ v \neq kx$$

When ''x divides يقسمy'' we say that ''x is a **factor** of y'' or ''y is a **multiple** of x''

Example3.20: Are $3|6, -4|8, \ -5|1, \ 9|30$?

Solution:

$\exists k = 2 \ such \ that \ 6 = 3k = 3(2) \ \Longrightarrow 3|6$

$\exists k = -2 \ such \ that \ 8 = -4k = -4(-2) \ \Longrightarrow -4|8$

$\forall k \in Z, 30 \neq 9k \ \ \Longrightarrow 9|30$

$-5|1$ (**H. W.**)

Example3.21: A=$\{x \in Z: 0 \leq x \leq 4\}$. Write a relation R on A such that $R = \{$(a,b)$\in A \times A$: $a|$b$\}$

Solution: $A = \{0,1,2,3,4\}$

$R = \{(1,0),(1,1),(1,2),(1,3),(1,4),(2,0),(2,2),(2,4),(3,0),(4,0),(4,4)\}$

Definition3.22: Let R_1 and R_2 are two relations from A to B, then $R_1 \cap R_2, R_1 \cup R_2, and \ \ R_1 \backslash R_2$ are also relations from A to B.

Example3.23: A=$\{x, y, z\}$
B=$\{x \in Z: -2 \leq x \leq 3\}$
Write two relations R_1 and R_2 from A to B.

R_1=

R_2=

Then find

$R_1 \cap R_2$=

$R_1 \cup R_2$=

$R_1 \backslash R_2$=

$R_2 \backslash R_1$=

<u>Definition3.24</u>: Let R a relation from A to B then R^{-1} is a relation from $\boldsymbol{B}$ **to** $\boldsymbol{A}$. R^{-1} is called **the inverse** of R.

$$R^{-1} = \{(b, a): (a, b) \in R\}$$

<u>Remark3.25</u>: $(R^{-1})^{-1} = R$

<u>Example3.26</u>: Let $A = \{0,3,8,-10\} and\ B = \{0,1,2\}$. A relation R from A to B is defined as:

$$R = \{(a, b) \in A \times B: a + b = 2k, k \in Z\}$$

Write the elements of the relation R then find R^{-1}?

<u>Definition</u>3.27: Domain of a relation مجال العلاقة

The domain of a relation $R \subseteq A \times B$ is the set of the first coordinates of each pair**.** In other words:

dom $R = \{x \in A;\ \exists y \in B: (x, y) \in R\}$

It is clear that dom $R \subseteq A$

منطلق العلاقة هو مجموعة المساقط الأولى للعلاقة

<u>Definition3.28</u>: Range of a relation مدى العلاقة

The range of a relation $R \subseteq A \times B$ is the set of the second coordinates of each pair**.** In other words:

range $R = \{y \in B;\ \exists x \in A: (x, y) \in R\}$

It is clear that range $R \subseteq B$

مدى العلاقة هو مجموعة المساقط الثانية

Example3.29: Let $A = N$ and $R = \{(a,b) \in N \times N: b = 2a\}$

Find dom R and $range$ R.

Solution:

The relation R can be written as follows

$$R = \{(1,2), (2,4), (3,6), (4,8), \dots.\}$$

DomR={1,2,3,4,5,....}=N

Range R={2,4,6,8,....}= even positive numbers

Example3.30: (H. W.) Let $A = Z$ and $R = \{(a,b) \in Z \times Z: b = 1 - a\}$

Find dom R and $range$ R.

Lemma3.31: Let R be a relation on $A \times B$ then:

1. dom $R = range\ R^{-1}$ **(H. W.)**

2. range R=dom R^{-1}

Proof 2: $A \times B = \{(a,b): a \in A, b \in B\}$

$$B \times A = \{(b,a): (a,b) \in A \times B\}$$

range$R = \{b \in B; \exists a \in A\ such\ that\ (a,b) \in R\}$

dom$R^{-1} = \{b \in B; \exists a \in A\ such\ that\ (b,a) \in R^{-1}\}$

T. P range R= $d$om R^{-1}

Let $b \in rangeR \quad \Leftrightarrow \exists a \in A\ such\ that\ (a,b) \in R$

$$\Leftrightarrow \exists a \in A\ such\ that\ (b,a) \in R^{-1}$$

$$\Leftrightarrow b \in dom\ R^{-1}$$

$\therefore$ range R= $d$om R^{-1}

Properties of Relations: خواص العلاقات

1. Reflexive relation: العلاقة الانعكاسية

A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every $a \in A$

العلاقة Rعلى المجموعة A تسمى انعكاسية اذا كان كل عنصر في المجموعة A مرتبط بنفسه وفق العلاقة R.

R **reflexive** on $A \Leftrightarrow a\ R\ a \quad \forall a \in A$

$\Leftrightarrow (a,a) \in R \quad \forall a \in A$

R **not reflexive** on $A \Leftrightarrow \exists a \in A,\ a \not\sim a$

$\Leftrightarrow \exists a \in A,\ (a,a) \notin R$

Example3.32: Let $A = \{1,2,3,4\}$. Which of these relations are reflexive?

Solution:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,1),(4,4)\}$

R_1 is reflexive on A because

$(1,1) \in R_1, (2,2) \in R_1, (3,3) \in R_1, (4,4) \in R_1$

$\Longrightarrow\ (a,a) \in R_1 \quad \forall a \in A$

$R_2 = \{(1,1),(1,2),(2,1)\}$ is not reflexive because $\exists 2 \in A\ such\ that$

$(2,2) \notin R_2$

$\exists 3 \in A\ such\ that\ (3,3) \notin R_2$

$\exists 4 \in A\ such\ that\ (4,4) \notin R_2$

$R_3 = \{(1,1),(2,1),(3,4),(2,2),(2,3),(3,3),(3,1),(3,1),(4,3)\}$ is not reflexive because $(4,4) \notin R_3$

$R_4 = \{(3,4)\}$ is not reflexive

Example3.33: Let $A = \{-2, \frac{1}{2}, 0, 3\}$. Let R_1 and R_2 be two relations on A such that

$R_1 = \{(a,b): a \leq b\}$

$R_2 = \{(a,b): a = b\}$

Are R_1, R_2 reflexive? Is $A \times A$ reflexive on A?

Solution: $R_1 = \{(a,b): a \leq b\} = \{(-2,-2), \left(-2,\frac{1}{2}\right), (-2,0), (-2,3),$

$\left(0,\frac{1}{2}\right), (0,3), \left(\frac{1}{2},3\right), \left(\frac{1}{2},\frac{1}{2}\right), (0,0), (3,3)\}$

$(a,a) \in R_1 \quad \forall a \in A \Longrightarrow R_1$ is reflexive

Similarly, $(a,a) \in R_2 \quad \forall a \in A$ and $(a,a) \in A \times A \; \forall a \in A$

$\Longrightarrow R_2$ and $A \times A$ are reflexive relations

Example3.34: Let $A = Z$. Let R be relations on A such that

$R = \{(a,b): a = b \text{ or } a = -b\}$.

Is R reflexive? Is $Z \times Z$ reflexive on Z?

Solution:

$R = \{(-1,-1), (-1,1), (1,-1), (0,0), (2,-2), \ldots.\}$

Since $(a,a) \in R \quad \forall a \in Z \Longrightarrow R$ is reflexive

Similarly, $(a,a) \in Z \times Z \quad \forall a \in Z \Longrightarrow Z \times Z$ is reflexive

Remark3.35:

1. $A \times A$ is reflexive on A

2. $\emptyset$ is not reflexive on A

Example3.36: Let $A = N$. Let R be relations on A such that

$R = \{(a, b) \in N \times N: a|b\}$. Is R reflexive?

Solution:

a R a $\forall$a $\in$ N? Is a|a $\forall$a $\in$ N?

a|a$\Longrightarrow \exists k = 1$ s.t. a=1(a)

$\Longrightarrow$ a R a $\forall$a $\in$ N

R is reflexive

Example3.37: Let $A = \{-2, -3, 2, 4\}$. Let R be relations on A such that

$R = \{(a, b): a + b \leq 3\}$. Is R reflexive?

Solution: Let $a = 2$, $b = 2$

$a + b = 4 > 3$

$\Longrightarrow R$ is not reflexive

Identity Relation: العلاقة الذاتية او علاقة الوحدة

Let A be a set. The **identity** relation on A is denoted by I_A and is defined as:

$$I_A = \{(a, b) \in A \times A: a = b\}$$

Remark3.38: Let A be a set. The identity relation I_A is a subset of the reflexive relation on A.

Example3.39: Let $A = \{-2, -3, 2, 4\}$. The following relation is Identity relation

$R = \{(-2,-2), (-3,-3), (2,2), (4,4)\} = I_A$

Example3.40: (**H. W.**) Which relations from **examples 3.32-3.34, 3.36, 3.37** are identity relations?

Symmetric Relation: العلاقة التناظرية

A relation R on a set A is called **symmetric** if the following condition satisfied:

If $(a, b) \in R \; then \; (b, a) \in R \quad \forall a, b \in A$

The relation R is **not symmetric** غير تناظرية if

$\exists (a, b) \in A \times A$ such that $(a, b) \in R \; but \; (b, a) \notin \mathrm{R}$

Example3.41: Let $A = \{1,2,3,4\}$. Which of these relations are symmetric?

$R_1 = \{(1,2), (2,1)\}$ is symmetric because $(1,2) \in R_1 \wedge (2,1) \in R_1$

$R_2 = \{(1,1), (1,2), (2,1)\}$ is symmetric

$R_3 = \{(1,1), (3,4), (2,2), (2,3), (3,3), (3,1), (1,3), (4,3)\}$ is not symmetric because $(2,3) \in R_3$ but $(3,2) \notin \mathrm{R}_3$

$R_4 = \{(3,4)\}$ is not symmetric

<u>Example</u>3.42: Let $A = \{-2, -3, 2, 4\}$. Let R be relations on A such that

$R = \{(a, b): a + b \leq 3\}$. Is R symmetric?

<u>Solution</u>:

$R =$
$\{(-2,-2), (-2,-3), (-2,2), (-2,4), (-3,-2), (-3,2), (-3,4), (2,-2), (2,-3), (4,-2), (4,-3)\}$

R is symmetric , $(a, b) \in R \Leftrightarrow (b, a) \in R$

<u>Example</u>3.43: Let $A = Z$. Let R be relations on A such that

$R = \{(a, b) \in Z \times Z: a + b \leq 3\}$. Is R symmetric?

<u>Solution</u>: $R = \{(-1,-1), (-2,-2), (2,1), (1,2), (1,1), \dots.\}$

نلاحظ في هذا المثال ان العلاقة Rمعرفة على مجموعة الاعداد الصحيحة الغير منتهية وانه من غير الممكن كتابة جميع عناصر العلاقة R. لذلك فانه من الصعب اختبار خاصية التناظر من خلال كتابة كل عناصر العلاقة. لذلك سنلجأ الى طريقة الاختبار عن طريق البرهان العام كالتالي:

$(a, b) \in R \Leftrightarrow a + b \leq 3$ **or** $a\ R\ b \Leftrightarrow a + b \leq 3$

Let $(a, b) \in R \Longrightarrow (b, a) \in R$?

$(a, b) \in R \Longrightarrow a + b \leq 3$

$\Longrightarrow b + a \leq 3$

$\Longrightarrow (b, a) \in R$

$\therefore$ R is symmetric relation

<u>Example3.44</u>: Let $A = Z$ and define $a\ R\ b \Leftrightarrow ab \geq 0 \quad \forall a, b \in Z$

Is R reflexive? Symmetric?

Solution:

reflexive: Is $aRa \quad \forall a \in Z$?

Let $a \in Z \Longrightarrow \mathrm{a.a} = \mathrm{a}^2 \geq 0$

$\therefore R$ is reflexive

symmetric? Let $(a, b) \in R, Is\ (b, a) \in R$?

$$(a, b) \in R \Longrightarrow ab \geq 0$$

$$\Longrightarrow ba \geq 0$$

$$\Longrightarrow (b, a) \in R$$

$\therefore R$ is symmetric

Example3.45: Let $A = R$ and define $a\ S\ b \Leftrightarrow a - b > 0 \quad \forall a, b \in S$

Is S reflexive? Symmetric?

Solution:

1. reflexive: Is $aSa \quad \forall a \in R$?

Let $a \in R \Longrightarrow \mathrm{a} - \mathrm{a} = 0$

$\therefore S$ is not reflexive

2. symmetric? Let $(a, b) \in S, Is\ (b, a) \in S$?

$$(a, b) \in S \Longrightarrow a - b > 0$$

$$\Longrightarrow b - a < 0$$

$$\Longrightarrow (b, a) \notin \mathrm{S}$$

$\therefore S$ is not symmetric

<u>Example3.46</u>: (H. W.) Let $A = Z$ and define $a\ R\ b \Leftrightarrow |a| = |\mathrm{b}|\quad \forall a, b \in Z$

Is R reflexive? Symmetric?

<u>Example3.47</u>: (H. W.) Let $A = Z$ and define $a\ R\ b \Leftrightarrow \mathrm{a} = 1\ \ \forall a \in Z$

$R = \{(1, b) \colon b \in Z\}$

Is R reflexive? Symmetric?

<u>Theorem</u>3.48: A relation R on a set A is symmetric iff $R = R^{-1}$

<u>Proof</u>: $\Longrightarrow$) Suppose R is symmetric T. P. $R = R^{-1}$

Let $(a, b) \in R \Leftrightarrow (b, a) \in R$ (R is symmetric)

$\Leftrightarrow (a, b) \in R^{-1}$ (def. of R^{-1})

$\therefore R = R^{-1}$

$\Longleftarrow$) Suppose $R = R^{-1}$ T. P. R is symmetric

Let $(a, b) \in R \Longrightarrow (a, b) \in R^{-1}$ $(R = R^{-1})$

$\Longrightarrow (b, a) \in (R^{-1})^{-1} = R$

$\therefore R$ is symmetric

<u>Anti Symmetric Relation</u>: علاقة ضد التناظرية

A relation R on a set A is called **anti symmetric** if

$$(a\ R\ b\ \wedge \mathrm{b}\ Ra) \Longrightarrow a = b\ \ \forall a, b \in A$$

R is not **anti symmetric** if $\exists a, b \in A\ such\ that\ (a\ Rb\ \wedge \mathrm{b}\ Ra) \wedge \mathrm{a} \neq b$

Example3.49: Let $A = \{1,2,3,4\}$ and R_1, R_2 are two relations on A such that

$R_1 = \{(2,1),(3,1),(3,2),(1,1)\}$ and $R_2 = \{(2,1),(3,1),(1,2),(1,1)\}$

Are R_1, R_2 anti symmetric?

Solution: R_1 is anti symmetric because $(1\ R\ 1\ \wedge 1\ R 1) \Longrightarrow 1 = 1$

R_2 is not anti symmetric because $\exists (2,1) \in R \wedge (1,2) \in R$ but $1 \neq 2$

Example3.50: (H. W.) Let $A = \{1,2,3,4\}$ and R is a relation on A such that

$R = \{(4,2)\}$. Is R anti symmetric?

Example3.51: Let $A = Z$ and R is a relation on Z such that $a\ R\ b \iff a = b + 1$

Is R reflexive? symmetric? Anti symmetric?

Solution:

$R = \{(2,1),(1,0),(0,-1),\ldots.\}$

Reflexive? Let $(a,a) \in R \Longrightarrow a \neq a + 1$

$\therefore R$ is not reflexive

symmetric? Let $(a,b) \in R \Longrightarrow a = b + 1$

but $b \neq a + 1$

Take $(3,2) \in R$ $(3 = 2 + 1)$

but $(2,3) \notin R$ $(2 \neq 3 + 1)$

$\therefore$ R is not symmetric

anti symmetric? Suppose $a\ R\ b\ \wedge b\ R\ a\ \Longrightarrow a = b$?

if $a\ R\ b$ then $a = b + 1$

if $b\ R\ a$ then $b = a + 1$

$a = b + 1$ and $b = a + 1$ iff $a = b$

$\therefore R$ is anti symetric

<u>Example3.52</u>: Let $A = Z$ and R is a relation on Z such that $a\ R\ b \iff a|b$

Is R anti symmetric?

<u>Solution</u>:

Let $a\ R\ b\ \wedge b\ R\ a\ \Longrightarrow a = b$?

$$a\ R\ b\ \wedge b\ R\ a \Longrightarrow a|b \wedge b|a$$

$$\Longrightarrow b = k_1 a \wedge a = k_2 b\ ,\ \ k_1, k_2 \in Z \ldots.(*)$$

$$\Longrightarrow b = k_1(k_2 b)$$

$$\Longrightarrow b = (k_1 k_2) b$$

$$\Longrightarrow k_1 k_2 = 1$$

$\Longrightarrow \mathrm{k}_1 = k_2 = 1\ or\ k_1 = k_2 = -1$

If $\mathrm{k}_1 = k_2 = 1\ then\ b = 1.a$ (from *)

If $\mathrm{k}_1 = k_2 = -1\ then\ b = -1.a$ (from *)

$\Longrightarrow b = a\ or\ b = -a$

$\therefore R$ is not anti symmetric

<u>Example3.53</u>: (H. W.) Let $A = Z$ and R is a relation on Z such that

$$a\ R\ b \iff a + b = 2k, \qquad k \in Z$$

Is R anti symmetric?

Theorem3.54: Let R be a relation on A, then R is anti symmetric iff $R \cap R^{-1} \subseteq I_A$

Proof: $\Longrightarrow$)Let R is anti symmetric **T. P.** $R \cap R^{-1} \subseteq I_A$

Let $(a,b) \in R \cap R^{-1} \Longrightarrow (a,b) \in R \wedge (a,b) \in R^{-1}$ (def. of ∩)

$\Longrightarrow (a,b) \in R \wedge (b,a) \in R$

$\Longrightarrow a = b$ (R is anti symmetric)

$\Longrightarrow (a,b) \in I_A$

$\therefore R \cap R^{-1} \subseteq I_A$

$\Longleftarrow$) Let $R \cap R^{-1} \subseteq I_A$ **T. P.** R is anti symmetric

Let $aRb \wedge b\,Ra \Longrightarrow (a,b) \in R \wedge (b,a) \in R$

$\Longrightarrow (a,b) \in R \wedge (a,b) \in R^{-1}$

$\Longrightarrow (a,b) \in R \cap R^{-1}$ (def. of ∩)

$\Longrightarrow (a,b) \in I_A$

$\Longrightarrow a = b$

Transitive Relation: العلاقة المتعدية

A relation R on a set A is **transitive**

If $(a,b) \in A \wedge (b,c) \in A$ then $(a,c) \in A \ \forall a,b,c \in A$

or

If $aR\,b \wedge b\,R\,c$ then $a\,R\,c \ \forall a,b,c \in A$

A relation R on a set A is **not transitive** if

$$\boxed{\exists a, b, c \in A \text{ such that } (a,b) \in R \wedge (b,c) \in R \ \wedge (a,c) \notin R}$$

Example3.55: Let $A = \{1,2,3,4\}$. Which of these relations are transitive?

$R_1 = \{(1,1),(1,2),(2,3),(1,3)\}$

R_1 is transitive on A because

$(1,2) \in R_1 \wedge (2,3) \in R_1 \Longrightarrow \ (1,3) \in R_1$

$(1,1) \in R_1 \wedge (1,2) \in R_1 \Longrightarrow \ (1,2) \in R_1$

$R_2 = \{(1,2),(2,3)\}$

R_2 is not transitive on A because

$(1,2) \in R_2 \wedge (2,3) \in R_2$ but $(1,3) \notin R_2$

Example3.56: Let $A = N$. Define a relation R on A such that

$R = \{(a,b): a \leq b\}$. Is R transitive on A? symmetric on A?

Solution:

Transitive? Let $(a,b) \in R \wedge (b,c) \in R$, Is $(a,c) \in R$?

$(a,b) \in R \wedge (b,c) \in R \Longrightarrow \ a \leq b \ \wedge b \leq c$

$\Longrightarrow \ a \leq c$

$\Longrightarrow (a,c) \in R$

$\therefore$ R is transitive on A

symmetric? Let $(a, b) \in R$, Is $(b, a) \in R$?

Let $(a, b) \in R \implies a \leq b$

but this does not mean that $b \leq a$

for example, let a=2, b=5

$2 \leq 5$ but $5 > 2$

$\therefore$ R is not symmetric on A

<u>Example</u>3.57: Let $A = N$. Define $a\ R\ b \iff a|$b

Prove that R **transitive** and **not symmetric** on A

<u>Solution</u>:

transitive: let $a, b, c \in N$ such that $a\ R\ b \wedge b\ R\ c$ To Prove $a\ R\ c$

$a\ R\ b \implies a|$b

$\implies \exists k_1 \in Z\ s.t. b = k_1 a$ **.....(1)**

$b\ R\ c \implies b|$c

$\implies \exists k_2 \in Z\ s.t. c = k_2 b$ **.....(2)**

substitute (1) into (2), $c = k_2 k_1 a$

$\implies c = k_3 a, \quad k_3 = k_1 k_2 \in Z$

$\implies a|$c

$\implies a\ R\ c$

$\therefore R$ is transitive

R is not symmetric. Take a=2 and b=4

It is clear that $2|4$ (4=2(2))

but ''4 does not divides 2'' $\implies \forall k \in Z, 2 \neq 4k$

Example3.58: (H. W.) Let $A = Z$ and define $a\ R\ b \Leftrightarrow ab \geq 0 \quad \forall a, b \in Z$

Show that R is transitive.

Example3.59: (H. W.) Let $A = R$ and define

$a\ S\ b \Leftrightarrow a - b > 0\ \forall a, b \in R$. Is S transitive?

Equivalence Relation: علاقة التكافؤ

A relation R on a set A is called equivalence relation if and only if R is reflexive, symmetric and transitive

العلاقة التي تكون انعكاسية وتناظرية ومتعدية تسمى علاقة تكافؤ

Example3.60: Let $A = Z$ and R is a relation on Z such that

$$a\ R\ b \Leftrightarrow a + b = 2k, \qquad k \in Z$$

Show that R is equivalence relation?

Solution:

reflexive: T. P. $a\ R\ a \quad \forall a \in Z$

let $a \in Z \Longrightarrow a + a = 2a \Longrightarrow (\mathrm{a}, \mathrm{a}) \in \mathrm{R}$

Symmetric? Let $(a, b) \in R$ T. P. $(b, a) \in R$

$(a, b) \in R \Longrightarrow a + b = 2k$

$\Longrightarrow \mathrm{b} + a = 2k$

$\Longrightarrow (b, a) \in R$

$\therefore R$ is Symmetric

Transitive: Let $(a, b) \in R \wedge (b, c) \in R$ To Prove $(a, c) \in R$

$(a,b) \in R \Longrightarrow a+b=2k_1,\ \ k_1 \in Z$**……(1)**

$(b,c) \in R \Longrightarrow b+c=2k_2,\ \ k_2 \in Z$**……(2)**

By summing up equations (1) and (2)

$$a+b+b+c=2k_1+2k_2$$

$$a+c=2(k_1+k_2-b)$$

$$a+c=2s \quad s=k_1+k_2-b \in Z$$

$$(a,c) \in R$$

$\therefore R$ is transitive

<u>Example3.61</u>:(**H.W.**) Which relations from previous examples are equivalence relations?

<u>Equivalence Classes</u>: صفوف التكافؤ

Let R be an equivalence relation on A. The set of all elements that are related to an element $a \in A$ is called an **equivalence class** of a. The equivalence class of a is denoted by $[a]$.

لتكن R علاقة تكافؤ معرفة على A وليكن a عنصرا في A. فان مجموعة كل العناصر التي ترتبط وفق العلاقة R مع a تسمى صف تكافؤ ال a ويرمز لها ب $[a]$.

$[a]=\{x \in A: x \sim a\}$

$x \in [a] \Leftrightarrow x \sim a$

$x \notin [a] \Leftrightarrow x \nsim a$

<u>Example</u>3.62: Let $A = \{1,2,3,4\}$. $R = \{(1,1),(2,2),(1,2),(2,1),(3,3),(4,4)\}$ be an equivalence relation on A. Find all equivalence classes on A.

Solution:

$[1] = \{x \in A: x \sim 1\} = \{1,2\}$

$[2] = \{x \in A: x \sim 2\} = \{1,2\}$

$[3] = \{x \in A: x \sim 3\} = \{3\}$

$[4] = \{x \in A: x \sim 4\} = \{4\}$

<u>Example</u>3.63: Let $A = \{-1,1,0\}$ and $R = \left\{(a,b) \in A \times A: \sqrt[3]{a} = \sqrt[3]{b}\right\}$. Show that R is an equivalence relation. Find all equivalence classes on A.

<u>Solution</u>: $R = \{(-1,-1),(1,1),(0,0)\}$

$[-1] = \{x \in A: x \sim -1\} = \{-1\}$

$[1] = \{x \in A: x \sim 1\} = \{1\}$

$[0] = \{x \in A: x \sim 0\} = \{0\}$

<u>Example</u>3.64: Let $A = Z$ and $R = \{(a,b) \in Z \times Z: a - b = 3k,\ k \in Z\}$. Show that R is an equivalence relation. Find all different equivalence classes on Z.

<u>Solution</u>:

reflex.: Let $a \in Z$ T.P. $(a,a) \in R$

$$a \in Z \Longrightarrow a - a = 3(0), \quad k = 0 \in Z$$

Symm.: Let $(a,b) \in R \Longrightarrow a - b = 3k, \quad k \in Z$

$$\Longrightarrow b - a = -3k = 3(-k), \ -k \in Z$$

$$\Longrightarrow (b,a) \in R$$

Trans.: Let $(a,b) \in R \wedge (b,c) \in R$ To Prove $(a,c) \in R$

$(a,b) \in R \Longrightarrow a-b=3k_1,\ \ k_1 \in Z$**……(1)**

$(b,c) \in R \Longrightarrow b-c=3k_2,\ \ k_2 \in Z$**……(2)**

By summing up equations (1) and (2)

$$a-b+b-c=3k_1+3k_2$$

$$a-c=3(k_1+k_2)$$

$$a-c=3s \quad s=k_1+k_2 \in Z$$

$$(a,c) \in R$$

$\therefore R$ is reflexive, symm. and trans.

$\therefore R$ is equivalence relation on Z

To find all equivalence classes, we start with

$[0]=\{x \in Z: x\sim 0\}=\{x \in Z: x-0=3k,\ k \in Z\}$

$=\{x \in Z: x=3k,\ k \in Z\}$

={0,3,-3,6,-6,…..}

$[1]=\{x \in Z: x\sim 1\}=\{x \in Z: x-1=3k,\ k \in Z\}$

$=\{x \in Z: x=3k+1,\ k \in Z\}$

= {1,4,-2,7,-5,…..}

$[2]=\{x \in Z: x\sim 2\}=\{x \in Z: x-2=3k,\ k \in Z\}$

$=\{x \in Z: x=3k+2,\ k \in Z\}$

= {2,5,-1,8,-4,…..}

$[3] = \{x \in Z: x{\sim}1\} = \{x \in Z: x - 3 = 3k,\ k \in Z\}$

$= \{x \in Z: x = 3k + 3,\ k \in Z\}$

= {3,6,0,9,-3,.....}=$[0]$

$[4] = [1]$

$[5] = [2]$

<u>Example</u>3.65: **(H. W.)** Let $A = Z$ and $R = \{(a,b) \in Z \times Z: a - b = 5k,\ k \in Z\}$. Show that R is an equivalence relation. Find all different equivalence classes on Z.

<u>Example</u>3.66: Let $A = N$. Define a relation R on A such that

$R = \{(a,b): a = b\}$. Show that R is an equivalence relation. Find all equivalence classes on N.

<u>Solution</u>:

R is an equivalence relation (**H. W.**)

$[1] = \{x \in N: x{\sim}1\} = \{x \in N: x = 1\} = \{1\}$

$[2] = \{x \in N: x{\sim}2\} = \{x \in N: x = 2\} = \{2\}$

$[3] = \{x \in N: x{\sim}3\} = \{x \in N: x = 3\} = \{3\}$

.... etc

<u>Theorem3.67</u>: Let R be an equivalence relation on A, then:

1. $[a] \neq \emptyset \quad \forall a \in A$

2. $a \sim b$ if and only if $[a] = [b]$

3. $a \nsim b \iff [a] \cap [b] = \emptyset$

4. $[a] \neq [b] \iff [a] \cap [b] = \emptyset$

5. $a \in [b] \iff [a] = [b]$

Proof1: $[a] = \{x \in A: x \sim a\}$

Since R is an equivalence relation $\Longrightarrow R$ is reflexive

$\Longrightarrow$ a~a $\quad \forall a \in A$

$\Longrightarrow a \in [a]$ (def. of equi. classes)

$\Longrightarrow [a] \neq \emptyset$

Proof2: $\Longrightarrow$) Suppose $a \sim b$ T. P [a] $\subseteq [b] \land [b] \subseteq [a]$

let $x \in [a] \Longrightarrow x \sim a \land a \sim b$

$\Longrightarrow$ x$\sim b$ (R is trans.)

$\Longrightarrow x \in [b]$

$\therefore$ [a] $\subseteq [b]$

Similarly, prove that $[b] \subseteq [a]$ (H. W.)

($\Longleftarrow$) Let $[a] = [b]$ T. P. $a \sim b$

since R is reflexive $\Longrightarrow a \sim a \Longrightarrow$ a $\in$ [a] = [b] (from hypo.)

$\Longrightarrow$ a $\in$ [b] $\Longrightarrow$ a$\sim b$

Proof3: $\Longrightarrow$) Suppose $a \nsim b$ T. P $[a] \cap [b] = \emptyset$

suppose $[a] \cap [b] \neq \emptyset$

$\exists x \in [a] \cap [b] \Longrightarrow x \in [a] \land x \in [b]$

$\Longrightarrow x \sim a \land x \sim b$ (def. of equi. classes)

$\Longrightarrow a \sim x \land x \sim b$ (R is symm.)

$\Longrightarrow a \sim b$ (R is trans.) contradiction

$\therefore [a] \cap [b] = \emptyset$

($\Leftarrow$) suppose $[a] \cap [b] = \emptyset$ T. P. $a \nsim b$

suppose $a \sim b \Longrightarrow [a] = [b]$ (from 2)

$\Longrightarrow [a] \cap [b] \neq \emptyset$ contradiction

$\therefore\ a \nsim b$

Proof4: $\Longrightarrow$) suppose $[a] \neq [b]$ and $[a] \cap [b] \neq \emptyset$

$\Longrightarrow a \sim b$ (from 3)

$\Longrightarrow [a] = [b]$ تناقض مع الفرض

($\Leftarrow$) suppose $[a] \cap [b] = \emptyset$ T. P. $[a] \neq [b]$

suppose $[a] = [b] \Longrightarrow [a] \cap [b] \neq \emptyset$ تناقض

$\therefore\ [a] \neq [b]$

Definition: Partition of a Set تجزئة المجموعة

A collection of subsets $\{A_i : i \in I \subseteq N\}$ of A is called **partition** of A if

تجمع من المجموعات الجزئية الغير خالية$\{A_i : i \in N\}$ من A تسمى **تجزئة** ل A اذا حققت الشروط التالية

1. $A_i \neq \emptyset \quad \forall i \in I$

2. $A_i \cap A_j = \emptyset \quad \forall\ i \neq j$

3. $\bigcup_{i \in I} A_i = A$

Example3.68: Let $A = \{0,1,2,3,5,-2\}$. Find two partitions of A.

Solution: First partition

The collection $A_1 = \{0,1\}, A_2 = \{2,3,5\}, A_3 = \{-2\}$ form a partition of A

because

1. $A_1 \neq \emptyset, A_2 \neq \emptyset$ and $A_3 \neq \emptyset$

2. $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$

3. $A_1 \cup A_2 \cup A_3 = A$

Second Partition (H.W.)

Example3.69: Let $X = [-2,5)$. Find two partitions of X.

Solution: First partition

The collection $A_1 = [-2,3], A_2 = (3,4), A_3 = [4,5)$ form a partition of X

Second Partition (H.W.)

Theorem3.70: Let R be an equivalence relation on a nonempty set A. Then the set of all different equivalence classes forms a partition for A.

Proof: Let $P = \{[a]: a \in A\}$ the set of all different equivalent classes of A

T. P. P is a partition of A

(1) $[a] \neq \emptyset \quad \forall a \in A$ (from Theorem 3.67(1))

(2) Let $[a], [b]$ are two different equivalence classes **T. P.** $[\boldsymbol{a}] \cap [\boldsymbol{b}] = \emptyset$

Since $[a] \neq [b] \Longrightarrow [\boldsymbol{a}] \cap [\boldsymbol{b}] = \emptyset$ (from Theorem 3.67(3))

(3) T.P. $\bigcup_{a \in A}[a] = A$

T. P. $\bigcup_{a \in A}[a] \subseteq A \wedge A \subseteq \bigcup_{a \in A}[a]$

let $x \in \bigcup_{a \in A}[a] \Longrightarrow x \in [a]$ for some $a \in A$

$\Longrightarrow x \in A \quad ([a] \subseteq A)$

$\therefore \bigcup_{a\in A}[a] \subseteq A$ **.........(1)**

let $x \in A \Longrightarrow x \in [a]$ for some $a \in A$

$\Longrightarrow x \in \bigcup_{a\in A}[a] \quad ([a] \subseteq \bigcup_{a\in A}[a])$

$\therefore A \subseteq \bigcup_{a\in A}[a]$ **.......(2)**

From (1) and (2), $\bigcup_{a\in A}[a] = A$

<u>Example</u>3.71: Let $A = Z$ and define $a\ R\ b \Leftrightarrow |a| = |\mathrm{b}| \quad \forall a, b \in Z$

Prove that R is an equivalence relation. Then find all different equivalence classes (i.e., find a partition set of Z).

<u>Solution</u>: R is an equivalence relation **(H. W.)**

$[0] = \{\mathrm{x} \in \mathrm{Z}: \mathrm{x} \sim 0\} = \{0\}$

$[1] = \{1, -1\}$

$[2] = \{2, -2\}$

.

.

etc

Partition set=P=$\{[a]: a \in Z^+\}$

<u>Example</u>3.72: Let $A = Z$ and $R = \{(a, b) \in Z \times Z: a - b = 3k,\ k \in Z\}$**.** Find the partition set on Z.

<u>Solution</u>: From Example 3.64, the relation R is an equivalence relation

Partition set=The set of all different equivalence classes

P= $\{[0], [1], [2]\}$

Example3.73: Let $A = \{1,2,3,4,5,6\}$ and $A_1 = \{1,2,3\}, A_2 = \{4,5\}, and\ A_3 = \{6\}$ are partition of A. Write the equivalence relation R produced from A_1, A_2 and A_3.

Solution: The subsets A_1, A_2 and A_3 are the equivalence classes of A

1. $A_1 = \{1,2,3\}$ is an equivalence class of A

$\Longrightarrow (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3) \in R$

2. $A_2 = \{4,5\}$ is an equivalence class of A

$\Longrightarrow (4,5), (4,4), (5,4), (5,5) \in R$

3. $A_3 = \{6\}$ is an equivalence class of A

$\Longrightarrow (6,6) \in R$

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,5), (4,4), (5,4), (5,5), (6,6)\}$

Order Relations علاقات الترتيب

1. Partially Ordered Relation: العلاقة المرتبة جزئيا

A relation R on a set A is called **Partially Ordered Relation (P. O. R)** or **partially ordering if** it is reflexive, anti symmetric and transitive. The pair (A, R) is called **partially ordered set**.

العلاقة على المجموعة A تسمى مرتبة جزئيا اذا كانت العلاقة انعكاسية وضد تناظرية ومتعدية كما يسمى الزوج (A, R) بالمجموعة المرتبة جزئيا.

Mathematically,

R is P.O.R $\Longleftrightarrow R$ reflexive $\wedge$ anti symmetric $\wedge$ transitive
R is not P.O.R $\Longleftrightarrow R$ not reflexive $\vee$ not anti symmetric $\vee$ not transitive

Example3.74: (**H.W.**) Let $A = \{1,2,-3\}$ be a set. Let

$R_1 = \{(a,b)) \in A \times A: a \geq b\} =$
$\{(1,1),(2,2),(-3,-3),(1,-3),(2,1),(2,-3)\}$

$R_2 = \{(1,1),(2,2),(-3,-3),(1,2),(2,1)\}$

$R_3 = \{(1,1),(2,2),(-3,-3)\} = I_A$

Are $(A,R_1),(A,R_2)$ and (A,R_3) partially ordered sets?

Example3.75: Show that $(Z,\geq)$ is a partially ordered set

Solution: Let R be a relation such that

$R = \{(a,b) \in Z \times Z: a \geq b\}$

We must show R is reflexive, anti symmetric and transitive

Reflexive: since $a \geq a \implies (\mathrm{a},\mathrm{a}) \in \mathrm{R} \implies \mathrm{R}$ is reflexive

Anti Symmetric: let $(a,b) \in R \wedge (\mathrm{b},\mathrm{a}) \in \mathrm{R}$ T. P. a=b

$$\implies \mathrm{a} \geq \mathrm{b} \implies \mathrm{a} = \mathrm{r} + \mathrm{b}, \quad \mathrm{r} \geq 0 \ldots.(\mathbf{1})$$

And $$\mathrm{b} \geq \mathrm{a} \implies \mathrm{b} = \mathrm{s} + \mathrm{a}, \quad \mathrm{s} \geq 0 \ldots.(\mathbf{2})$$

Substitute (1) in (2) $\implies \mathrm{b} = \mathrm{s} + \mathrm{r} + \mathrm{b}$

$$\implies \mathrm{s} + \mathrm{r} = 0 \overset{\mathrm{r,s} \geq 0}{\implies} \mathrm{r} = \mathrm{s} = 0$$

Substitute $r = 0$ in (1) $\implies \mathrm{a} = \mathrm{b}$

R is anti symmetric

Transitive: let $(a,b) \in R \wedge (\mathrm{b},\mathrm{c}) \in \mathrm{R} \implies \mathrm{a} \geq \mathrm{b} \wedge \mathrm{b} \geq \mathrm{c}$

$$\implies \mathrm{a} \geq \mathrm{b} \implies \mathrm{a} = \mathrm{r} + \mathrm{b}, \quad \mathrm{r} \geq 0 \ldots.(\mathbf{1})$$

And $$\mathrm{b} \geq \mathrm{c} \implies \mathrm{b} = \mathrm{s} + \mathrm{c}, \quad \mathrm{s} \geq 0 \ldots.(\mathbf{2})$$

Substitute (2) in (1) $\Longrightarrow \mathrm{a} = (\mathrm{r}+\mathrm{s}) + \mathrm{c}$ and $r+s \geq 0$

$\Longrightarrow \mathrm{a} \geq \mathrm{c} \Longrightarrow (\mathrm{a},\mathrm{c}) \in \mathrm{R} \Longrightarrow$ R is transitive

Remark3.76: For any nonempty set A, the relation $A \times A$ is not P.O.R

Example3.77: Let $A = Z$ and $R = Z \times Z$. Show that R is not P.O.R

The relation R is an equivalence relation $\Longrightarrow R$ is symmetric

$\Longrightarrow (\mathrm{a},\mathrm{b}) \in \mathrm{R}$ and $(b,a) \in R \quad \forall a,b \in R$

But $a \neq b$ (in general)

$\Longrightarrow$ R is not anti symmetric

$\Longrightarrow$ R is not $P.O.R$

Example3.78: (H .W.) Let $A = Z$ and $R_1 = \{(a,b) \in Z \times Z : a|\mathrm{b}\}$

$R_2 = \{(a,b) \in Z \times Z : a \leq b\}$

$R_3 = \{(a,b) \in Z \times Z : a < b\}$

$R_4 = \{(a,b) \in Z \times Z : a > b\}$

Show that **(**$Z, a|\mathrm{b}$) is not a partially ordered set

Show that R_2 is P.O.R

Show that R_3 and R_4 are not P.O.R

Example3.79: **(H .W.**) Let $X = \{1,2,3\}$ and $R = \{(A,B) \in P(X) \times P(X) : A \subseteq B\}$

Show that R is a partially ordered relation on $P(X)$

<u>Definition3.80</u>: Let R be a P.O.R on a set A and let$a, b \in A$. Then a, b are called **comparable عنصرين قابلين للمقارنة** with respect to R if $(a, b) \in R$ or $(b, a) \in R$.

Mathematically,

a, b are comparable $\Leftrightarrow \quad (a, b) \in R \vee (b, a) \in R$
a, b are not comparable $\Leftrightarrow \quad (a, b) \notin R \wedge (b, a) \notin R$

<u>Example3.81</u>: Let $A = \{1,2,3\}$, let

$R = \{(1,1), (2,2), (3,3), (1,2), (3,1), (3,2)\}$ be a P.O.R

Find the comparable element in A with respect to R

1R 1 $\Longrightarrow$ 1,1 are comparable

2R2 $\Longrightarrow$ 2,2 are comparable

3R3 $\Longrightarrow$ 3,3 are comparable

1R2$\Longrightarrow$ 1,2 are comparable

3R1$\Longrightarrow$ 1,3 are comparable

2R3$\Longrightarrow$ 2, 3 are comparable

<u>Example3.82</u>: (H. W.) Let $A = \{3,4,6,8,10\}$ and $R = \{(a, b) \in A \times A : a \mid b\}$

Find the comparable element in A with respect to R

2. Totally Ordered Relation العلاقة المرتبة كليا

A relation R on a set A is called **totally ordered relation (T. O. R)** or **totally ordering** if

1. R is P.O.R

2. a, b are comparable $\forall a, b \in A$

العلاقة على المجموعة A تسمى مرتبة كليا اذا كانت العلاقة مرتبة جزئيا وكل عنصرين في A قابلين للمقارنة بالنسبة للعلاقة R

Example3.83: Let $A = \{1,2,-3\}$ be a set. Let

$R = \{(a, b)) \in A \times A: a \geq b\} =$
$\{(1,1), (2,2), (-3,-3), (1,-3), (2,1), (2,-3)\}$

Is R T.O.R?

Solution: P.O.R: From Example 3.74, R is P.O.R

Comparable: From reflexive relation, each element is comparable with itself

$(1,-3) \in R \Longrightarrow 1,-3$ are comparable

$(2,1) \in R \Longrightarrow 1,2$ are comparable

$(2,-3) \in R \Longrightarrow 2,-3$ are comparable

$\therefore$ a, b are comparable $\forall a, b \in A$

$\therefore$ R is T.O.R

Example3.84: Show that $(Z, \geq)$ is a totally ordered set

Solution: P.O.R: From Example3.75, R is P.O.R

Comparable: T.P. $a \geq b \vee b \geq a \quad \forall a, b \in Z$

Let $a, b \in Z \Longrightarrow \mathrm{a} = \mathrm{b} \vee a > b \vee b > a$

$\Longrightarrow \mathrm{a} \geq \mathrm{b} \vee a \geq b \vee b \geq a$

$\Longrightarrow \mathrm{a} \geq \mathrm{b} \vee b \geq a$

$\Longrightarrow \mathrm{a}R\mathrm{b} \vee bRa$

$\therefore a, b$ are comparable $\forall a, b \in Z$

$\therefore R$ is T.O.R

Example3.85: (H .W.) Show that **(**$Z, \leq$) is a totally ordered set

Example3.86: (R , $\leq$), (R , $\geq$), $(Q, \geq), (Q, \leq), (N, \leq), (N, \geq)$ are totally ordered sets

Example3.87: Give an example of a P.O.R that is not T.O.R

CHAPTER Four: Mappings التطبيقات

Chapter Four Contents:

1. Mappings التطبيقات

2. Types of mappings أنواع التطبيقات

3. Composition of mappings تركيب التطبيقات

4. Direct image and inverse image الصورة المباشرة والصورة العكسية

Mapping (Function) التطبيق او الدالة

Let A and B be two nonempty sets. A **relation f** from A to B ($f \subseteq A\times B$) is called a **mapping** or **function** if each element in A is related to a unique element in B. This relation is denoted by $f: A \to B$.

الدالة او التطبيق fهي **علاقة خاصة** من A الى B تربط كل عنصر في المجموعة A بعنصر وحيد في المجموعة B ويرمز لهذه العلاقة بالرمز $f: A \to B$.

Mathematically,

$$f: A \to B \text{ is a mapping } \Leftrightarrow \forall x \in A\ \exists!\, y \in B\ s.t.\ f(x) = y$$

Remark 4.1: A mapping is (generally) denoted by f, Fg, G, h, H, ….

<u>Example 4.2</u>: Let $A = \{1,2,3,4\}$ and $B = Z$. Which of the following relations is mapping?

$R_1 = \{(x, y) \in \mathrm{A} \times B: y = 2x\}$

$R_2 = \{(1,1), (1,2), (2,0), (3, -1), (4,1)\}$

$R_3 = \{(1,1), (2,0), (3,3)\}$

<u>Remark</u> 4.3: Every function is a relation but not every relation is a function

Mapping can be defined in another way :

<u>Definition</u> 4.4: Let A and B be two nonempty sets. A **relation $\boldsymbol{f}$** from A to B ($f \subseteq \mathrm{A} \times B$) is called a **mapping** or **function** if it satisfies two conditions:

1) <u>Closure الانغلاق</u> : $\forall \mathrm{x} \in \mathrm{A} \implies f(x) \in B$ كل عنصر في المجال صورته تنتمي للمجال المقابل

2) <u>Well-defined</u> <u>التعريف الجيد</u>: كل عنصر في المجال له صورة وحيدة في المجال المقابل

If $x_1 = x_2$ then $f(x_1) = f(x_2) \quad \forall x_1, x_2 \in A$

<u>Example 4.5</u>: Determine whether $f: Z \to R$ is a function or not

a) $f(x)=\sqrt{x^2 + 1}$

1. closure: Let $\forall x, x \in Z \overset{?}{\Rightarrow} f(x) \in R$

$$\forall x, x \in Z \implies x^2 + 1 \in Z \implies \sqrt{x^2 + 1} \in R$$

$\therefore$ closure is held الانغلاق متحقق

2. well defined: $\forall x_1, x_2 \in Z, x_1 = x_2 \implies x_1^2 = x_2^2 \implies x_1^2 + 1 = x_2^2 + 1$

$$\Rightarrow \sqrt{x_1^2+1} = \sqrt{x_2^2 \ +1}$$

$$\Rightarrow f(x_1) = f(x_2)$$

$\therefore f$ is mapping

b) $f(x) = \begin{cases} -x, & x \leq 1 \\ x, & x \geq 1 \end{cases}$

f is not well defined because $x = 1 \in Z$

but $f(1) = 1$ and $f(1) = -1$ يوجد عنصر له صورتان

3) $f(x) = \frac{1}{x}$

Closure condition is not held شرط الانغلاق غير متحقق

Let $x = 0 \in Z$ but $f(0) = \frac{1}{0} \notin R$

$\therefore f$ is not a mapping

<u>Example 4.6</u>: (H.W.) Is f mapping?

1. Let $f: N \rightarrow N$ s.t. $f(x) = x/(|\mathrm{x}| - 5)$

2. Let $f: R \rightarrow R$ s.t. $f(x) = \frac{\sqrt[3]{x}}{x-1}$

Graph of Mapping رسم الدالة

Let A and B be two non-empty sets and $f: A \rightarrow B$**.** The graph of f is denoted by $Graph\ f$ and is defined as

$$Graph\ f = \{(x, y): x \in A \text{ and } y = f(x)\}$$

<u>Definition</u> 4.7: Let $f: A \to B$ be a mapping. Then

1) The set A is called the **domain of f مجال الدالة** and is denoted by D_f

2) The set B is called the **Codomain of f المجال المقابل للدالة** and is denoted by Cod_f

3) If $f(x) = y$ then y is called the **image** of x and x is called the **preimage** of y

4) The set of all images of the elements of A is called the **range** of f and is denoted by R_f

$$R_f = f(A) = \{y = f(x): x \in A\} \subseteq B = Cod_f$$

ملاحظة 3: يمكن ايجاد المجال للدالة بالاعتماد على نوعها

1. الدالة الخطية مجالها جميع الاعداد الحقيقية $D_f = R$
2. *الدالة الكسرية مجالها جميع الاعداد الحقيقية ماعدا القيم التي تجعل المقام يساوي صفر*
3. *الدالة الجذرية مجالها جميع الاعداد الحقيقية عدا القيم التي تجعل القيمة تحت الجذر سالبة*

لإيجاد مجموعة الصور R_f هناك عدة طرق منها الاعتماد على منحني الدالة. اذا كان مجال الدالة مجموعة جزئية من مجموعة الاعداد الحقيقية فأن من الممكن ايجاد المدى عن طريق ايجاد قيم x بدلالة y

<u>Example 4.8</u>: Write the graph set, the domain and the range of the following functions:

1) Let $f: \{-2, -1, 0, 1, 2\} \to Z$ s.t. $f(x) = x^3$

$$Graph\ f = \{(x, x^3): x \in \{-2, -1, 0, 1, 2\} \text{ and } f(x) = x^3\}$$

$$= \{(-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8)\}$$

$D_f = \{-2, -1, 0, 1, 2\}$

$R_f = \{-8, -1, 0, 1, 8\} \subseteq Z = Cod_f$

2) Let $g: Z \to Z$ s.t. $g(x) = x^2$

$$Graph\ g = \{(x, x^2): x \in Z \text{ and } g(x) = x^2\}$$

$$= \{\dots, (-2,4), (-1,1), (0,0), (1,1), (2,4), \dots\}$$

D_g=Z

$$R_g = \{0,1,4,9,16, \dots\} \subseteq Z = Cod_g$$

<u>Example 4.9</u>: (**H. W.**) Find the domain and the range of the following functions:

1) $f(x) = \frac{x}{x+2}$ 2) $F(x) = \sqrt{1-2x}$ 3) $G(x) = \sqrt{\frac{x}{x+2}}$

<u>Types of Mappings أنواع التطبيقات</u>

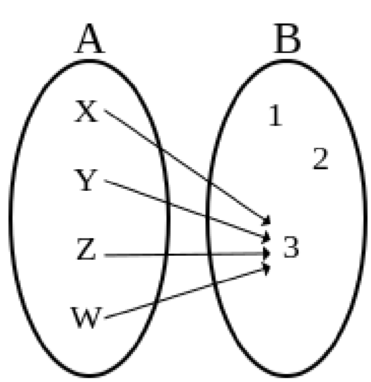
1. <u>Constant Mapping التطبيق الثابت</u>

A mapping $f: A \to B$ is called

constant map دالة ثابتة $\Leftrightarrow$ $\exists!\, c \in B\ s.t.\ f(x) = c \quad \forall x \in A$

or

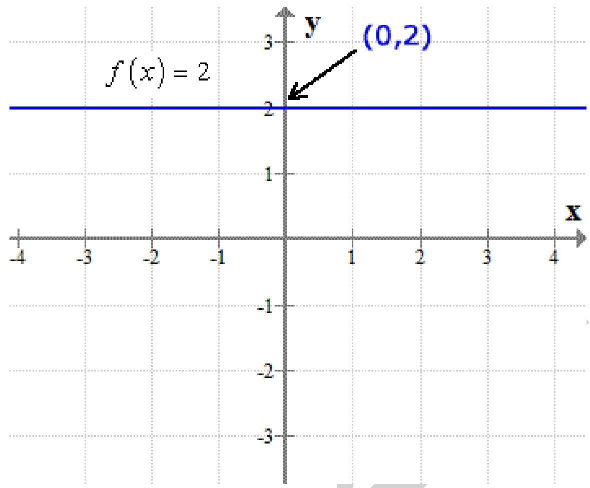
f is called constant $\Leftrightarrow R_f = \{c\}$



Constant Mapping

<u>Example 4.10</u>: Let $f: R \to R$ s.t. $f(x) = 2 \quad \forall x \in R$

f is constant function



<u>Example 4.11</u>: Give two examples of non-constant functions

1. let $f: R \to R$ s.t. $f(x) = \begin{cases} 2, & x \geq 1 \\ -3, & x < 1 \end{cases}$

f is not a constant mapping because $f(1) = 2$ and $f(0) = -3$

Give another example (H. W.)

2. <u>Identity Mapping الدالة الذاتية</u>

A mapping $f: A \to A$ is called **identity map دالة ذاتية** denoted by $i_A \Leftrightarrow$
$f(x) = x \quad \forall x \in A$

<u>Example 4.12</u>:

1. let $f = \{(1,1), (3,3), (0,0), (-6,-6)\}$

f is identity function defined on $A = \{0,1,3,-6\}$

$\therefore f = i_A$

2. let $f: Z \to Z$ s.t. $f(x) = x \quad \forall x \in Z$

$f = i_Z$

3. let $f: Z \to Z$ s.t. $f(x) = |x| \quad \forall x \in Z$

$f(x) = f(-x) = x \quad \forall x \in Z$

$\therefore f$ is not identity function

4. let $f: N \to N$ s.t. $f(x) = |x| \quad \forall x \in N$

$f(x) = x \quad \forall x \in N$

$\therefore f$ is identity function (i.e., $f = i_N$)

3. <u>Injective Mapping التطبيق المتباين</u>

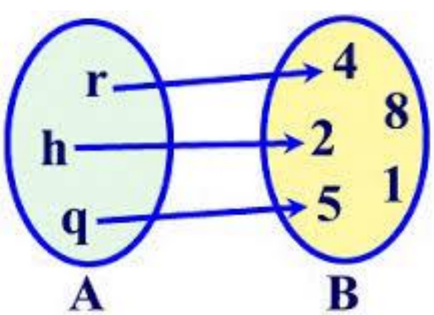
A function $f: A \to B$ is called **one to one** (1-1) or **injective** if different elements in the domain A have different images in B
تسمى الدالة متباينة اذا كانت للعناصر المختلفة في المجال صورا مختلفة في المجال المقابل

$f: A \to B$ is called **1-1** $\Leftrightarrow$ $\forall\, x_1, x_2 \in A;\ if\ x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

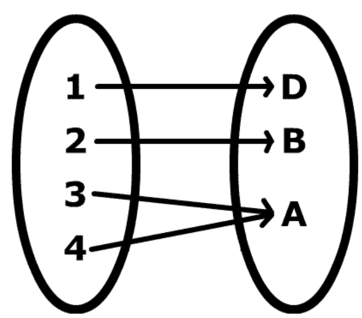
Or

$f: A \to B$ is called **1-1** $\Leftrightarrow$ $\forall\, x_1, x_2 \in A;\ if\ f(x_1) = \mathrm{f}(\mathrm{x}_2)$ then $x_1 = x_2$

$f: A \to B$ is **not 1-1** $\Leftrightarrow$ $\exists\, x_1, x_2 \in A;\ x_1 \neq \mathrm{x}_2 \wedge f(x_1) = \mathrm{f}(\mathrm{x}_2)$



One to one function

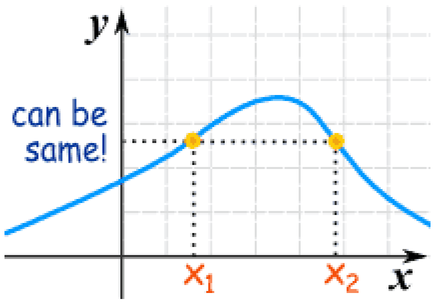


not one to one function

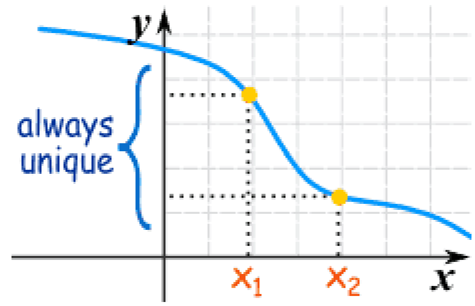
$f(3) = f(4) = A$

f is Many to one

<u>Remark 4.13</u>: In the **graph of Injective** map, a horizontal line should never intersect the curve at 2 or more points**.**



not one to one (many to one)



Injective (one to one) function

4. <u>Surjective Mapping التطبيق الشامل</u> A function $f: A \to B$ is called (onto) or **surjective if** every element in "B" has **at least one** relating element in "A" (maybe more than one).

الدالة f تكون شاملة اذا كان كل عنصر في المجال المقابل هو صورة لعنصر واحد او اكثر في المجال

Mathematically,

A function $f: A \to B$ is called (onto) or **surjective** $\Leftrightarrow$ $R_f = Cod_f$

Or

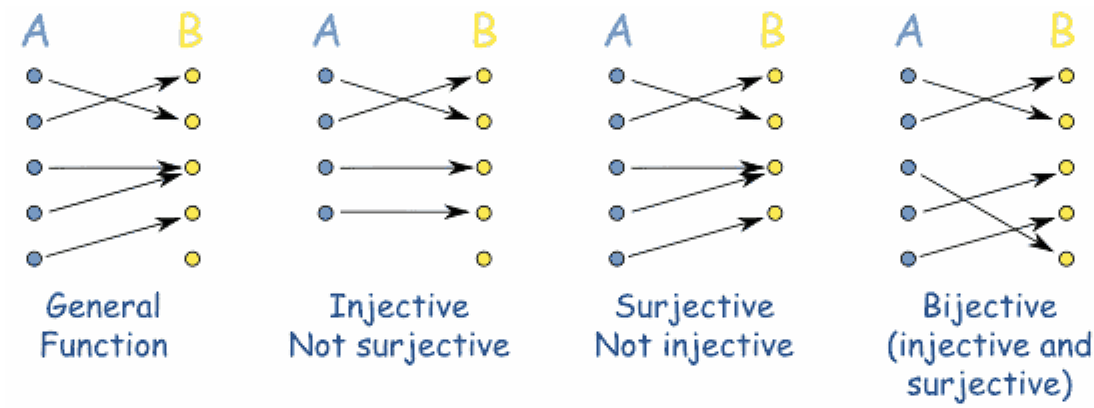
A function $f: A \to B$ is called (onto) or **surjective** $\Leftrightarrow$ $\forall y \in B \ \exists x \in A \ s.t. f(x) = y$

A function $f: A \to B$ is not (onto) or not **surjective** $\Leftrightarrow$ $R_f \neq Cod_f$

5. <u>bijective Mapping التطبيق المتقابل</u>

A function $f: A \to B$ is called **bijective** $\Leftrightarrow$ f is 1-1 and onto

A function $f: A \to B$ is called **not bijective** $\Leftrightarrow$ f is not 1-1 or f is not onto



Example 4.14: Which of the following functions are injective? Surjective? Bijective?

1. $f: R \to R$ s.t. $f(x) = x$ **(H. W.)**

2. $f: R \to R$ s.t. $f(x) = 3$ **(H. W.)**

3. $f: R^+ \to R$ s.t. $f(x) = x^2$ **(H. W.)**

4. $f: R \to R$ s.t. $f(x) = \sqrt{x^2 + 9}$

5. $f: R \setminus \{\frac{5}{2}\} \to R$ s.t. $f(x) = \frac{x+4}{2x-5}$ **(H. W.)**

6. $f: R \to [1, \infty)$ s.t. $f(x) = |x - 4| + 1$

7. $f: R \to [-4, \infty)$ s.t. $f(x) = -4 + (x-4)^2$ **(H. W.)**

8. $f: R \to R$ s.t. $f(x) = \begin{cases} x^3, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

Solution4: $f: R \to R$ s.t. $f(x) = \sqrt{x^2 + 9}$

$D_f = R$ and $cod_f = R$

Surjective? We need to find R_f

When $x \in R \Longrightarrow f(x) = y \geq 3$

$$R_f = \{y : y \geq 3\} = [3, \infty) \neq R = cod_f$$

$\therefore f$ is not surjective (not onto)

Injective? Let $f(x_1) = f(x_2) \overset{?}{\rightarrow} x_1 = x_2$

$$f(x_1) = f(x_2) \Longrightarrow \sqrt{x_1^{\,2} + 9} = \sqrt{x_2^{\,2} + 9}$$

$$\Longrightarrow x_1^{\,2} + 9 = x_2^{\,2} + 9$$

$$\Longrightarrow x_1^{\,2} = x_2^{\,2}$$

$$\Longrightarrow \quad x_1 = \mp x_2$$

$\Longrightarrow x_1 = x_2$ **....(1)**

$Or, \quad x_1 = -x_2 \Longrightarrow x_1 \neq x_2$ **.....(2)**

From (2), f is not injective

$\therefore f$ is not bijective

<u>**Solution6**</u>**:** $f : R \rightarrow [1, \infty)$ s.t. $f(x) = |x - 4| + 1 = \begin{cases} x - 3, & x \geq 4 \\ -x + 5, & x < 4 \end{cases}$

$D_f = R$ and $cod_f = [1, \infty)$

Injective? Let $f(x_1) = f(x_2) \overset{?}{\rightarrow} x_1 = x_2$

$$f(x_1) = f(x_2) \Longrightarrow |x_1 - 4| + 1 = |x_2 - 4| + 1$$

$$\Longrightarrow |x_1 - 4| = |x_2 - 4|$$

Either $x_1 - 4 = x_2 - 4$

$\Longrightarrow x_1 = x_2$ **....(1)**

$Or, x_1 - 4 = -x_2 + 4$

$\Longrightarrow x_1 = 8 - x_2 \Longrightarrow x_1 \neq x_2$ **.....(2)**

From (2), f is not injective (1-1)

Surjective? We need to find R_f

$$f(x) = \begin{cases} x-3, & x \geq 4 \\ -x+5, & x < 4 \end{cases}$$

If $x \geq 4 \implies x-3 \geq 1 \implies y \geq 1$ **……(1)**

If $x < 4 \implies -x > -4 \implies 5-x > 1 \implies y > 1$ **…….(2)**

$\therefore$ From (1) and (2), $R_f = \{y : y \geq 1 \text{ or } y > 1\} = [1, \infty) = cod_f$

$\therefore f$ is surjective (onto)

طريقة اخرى لاختبار ان الدالة شاملة وهي ايجاد x بدلالة y

If $x \geq 4 \implies \mathrm{y} = x-3 \implies x = y+3$

لكي تكون $x \geq 4$ يجب ان تكون قيم y اكبر او تساوي ال 1

$$\implies y \geq 1 \quad \ldots\ldots(\mathbf{1})$$

If $x < 4$, $y = -x+5 \implies x = 5-y$

لكي تكون $x < 4$ يجب انتكون قيم y اكبر من ال 1

$\implies y > 1$ **…….(2)**

$\therefore$ From (1) and (2), $R_f = \{y : y \geq 1 \text{ or } y > 1\} = [1, \infty) = cod_f$

$\therefore f$ is onto

$\therefore f$ is not bijective

Solution8: $f: R \to R$ s.t. $f(x) = \begin{cases} x^3, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

$D_f = R$ and $cod_f = R$

Surjective (onto)? We need to find R_f

When $x < 0 \Longrightarrow y = f(x) = x^3 < 0 \Longrightarrow y < 0$

When $x \geq 0 \Longrightarrow y = f(x) = x^2 \geq 0 \Longrightarrow y \geq 0$

$$R_f = \{y : y < 0 \ \ or\ y \geq 0\} = R = cod_f$$

$\therefore f$ is onto

Injective (1-1)? Let $f(x_1) = f(x_2) \overset{?}{\to} x_1 = x_2$

$$f(x_1) = f(x_2) \Longrightarrow \begin{cases} x_1^3 = x_2^3, & x_1 < 0, x_2 < 0 \\ x_1^2 = x_2^2, & x_1 \geq 0,\ x_2 \geq 0 \\ x_1^3 = x_2^2, & x_1 < 0,\ x_2 \geq 0 \\ x_1^2 = x_2^3, & x_1 \geq 0, x_2 < 0 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = x_2, & x_1 < 0, x_2 < 0 \\ x_1 = x_2, & x_1 \geq 0, x_2 \geq 0 \end{cases}$$

$\therefore x_1 = x_2 \Longrightarrow f$ is injective

$\therefore f$ is bijective

Inverse Mapping: الدالة العكسية

Let f be a bijective mapping from A to B then f^{-1} is a mapping from B to A such that $f^{-1}(y) = x$.

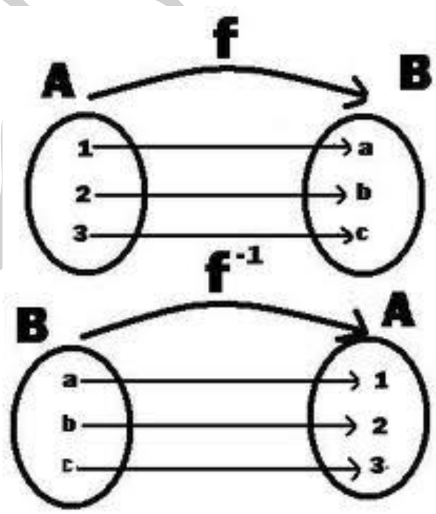
Mathematically,

$f: A \to B$ s.t. $f(x) = y$ is a bijective mapping $\Leftrightarrow f^{-1}: B \to A$ is a map. s.t. $f^{-1}(y) = x$

Example 4.15: Let $f: \{1,2,3\} \to \{a, b, c\}$ s.t.

$f(1) = a,\ \ f(2) = b,\ \ f(3) = c$

Or $f = \{(1, a), (2, b), (3, c)\}$



Since f is 1-1 and onto (bijective) $\Longrightarrow f^{-1}: \{a, b, c\} \to \{1,2,3\}$ s.t.

$f^{-1}(a) = 1,\ f^{-1}(b) = 2,\ \ f^{-1}(c) = 3$

or $f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$

Example 4.16: Let $f: R^+ \to R^+$ s.t. $f(x) = \frac{x}{2}$. Find f^{-1} (if exist)?

Solution: f^{-1}exist $\Leftrightarrow$ f is bijective

1-1? Let $f(x_1) = f(x_2) \Longrightarrow \frac{x_1}{2} = \frac{x_2}{2} \Longrightarrow \mathrm{x}_1 = \mathrm{x}_2 \Longrightarrow$ f is 1-1

Onto? $y = \frac{x}{2} \Longrightarrow \mathrm{x} = 2\mathrm{y}$

$$\Longrightarrow \mathrm{R_f} = \mathrm{R}^+ = \mathrm{cod_f}$$

$\therefore$ f is bijective $\Longrightarrow$ f^{-1} is a mapping $\Longrightarrow$ f^{-1} exist

لايجاد f^{-1} نجد x بدلالة y

$y = \frac{x}{2} \Longrightarrow \mathrm{x} = 2\mathrm{y}$

$\therefore$ $f^{-1}: R \to R$ s.t. $f^{-1}(\mathrm{y}) = 2\mathrm{y}$

Remark 4.17:

1. If a function f is **not injective** then f^{-1} is **not a mapping** (f^{-1}does not exist)

2. If a function f is **not surjective** then f^{-1} is **not a mapping** (f^{-1}does not exist)

3. If a function f is **bijective** then f^{-1} is **a mapping** (f^{-1} exist)

Example 4.18: $f: R \to R$ s.t. $f(x) = 4 + (x-4)^2$. Find f^{-1} (if exist)

Solution:

f^{-1} exist if and only if f is 1-1 and onto

1-1? Let $f(x_1) = f(x_2) \Longrightarrow\ 4 + (x_1 - 4)^2 = 4 + (x_2 - 4)^2$

$$\Longrightarrow\ (x_1 - 4)^2 = (x_2 - 4)^2$$

بجذر الطرفين

$$\Longrightarrow x_1 - 4 = \mp(x_2 - 4)$$

Either, $x_1 - 4 = x_2 - 4 \implies x_1 = x_2$

Or, $x_1 - 4 = -(x_2 - 4) \implies x_1 \neq x_2$

$\therefore$ f is not 1-1

$\therefore$ f^{-1} is not defined

Example 4.19: $f: R \to R$ s.t. $f(x) = \begin{cases} x^3, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

Find f^{-1} (if exist)?

Solution:

f is bijective (see Example 4.14(8))

$\therefore$ f^{-1} is defined

$\therefore$ $f^{-1}: R \to R$ s.t. $f^{-1}(\mathrm{y}) = \begin{cases} \sqrt[3]{y}, & y < 0 \\ \sqrt{y}, & y \geq 0 \end{cases}$

Example 4.20: $f: [3, \infty) \to [0, \infty)$ s.t. $f(x) = \sqrt{x-3}$. Find f^{-1} (if exist)

Let $f(x_1) = f(x_2)$

$\Longrightarrow x_1 - 3 = x_2 - 3$

$\Longrightarrow \mathrm{x}_1 = x_2 \Longrightarrow$ f is 1-1

Onto? Is $R_f = cod_f$?

نجد x بدلالة y

(because $x \geq 3$)$y = \sqrt{x-3} \Longrightarrow x = y^2 + 3 \Longrightarrow \mathrm{y} \geq 0$

يمكن ايجاد المدى بطريقة اخرى

$x \geq 3 \Rightarrow \mathrm{x} - 3 \geq 0$

$\Rightarrow \sqrt{x-3} \geq 0 \Rightarrow y \geq 0$

$\Rightarrow \mathrm{R_f} = [0, \infty) = \mathrm{cod_f}$

f is bijective

$\therefore\ f^{-1}$ is defined

لايجاد f^{-1} نجد x بدلالة y

$y = \sqrt{x-3} \Rightarrow x = y^2 + 3$

$$\therefore\ f^{-1}: [0, \infty) \to [3, \infty)\ \ s.t. f^{-1}(y) = y^2 + 3$$

Remark 4.21:

1. $f^{-1} \neq \frac{1}{f}$

2. $(f^{-1})^{-1} = f$

3. $f = f^{-1} \Leftrightarrow f = i_A$

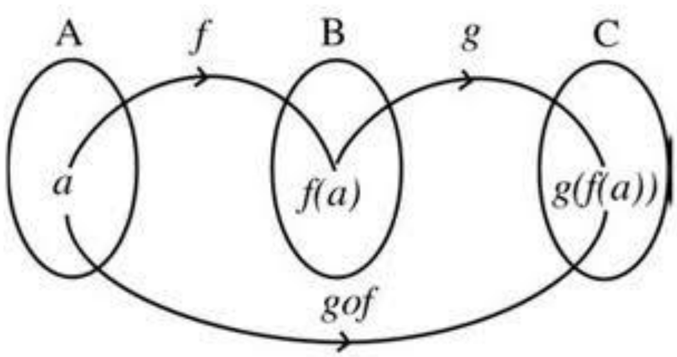
Composition of Mappings تركيب التطبيقات

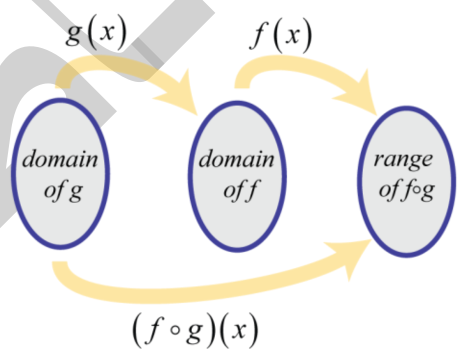
Let $f: A \to B$ be a mapping and $g: B \to C$ be a mapping. The **composition** of g and f is a mapping denoted by gof and is defined as

$$(gof)(a) = g\big(f(a)\big) \quad \forall a \in A$$

Mathematically,

> If $f: A \rightarrow B$ and $g: B \rightarrow C$ then $gof: A \rightarrow C$ is a map. $\Leftrightarrow \forall a \in A$, $\exists!\, g(f(a)) \in C$ s.t. $(gof)(a) = g(f(a))$





<u>Remark</u> 4.22: Let $f: A \rightarrow B$ and $g: C \rightarrow D$. Then

1. gof is defined (exist) if and only if $R_f \subseteq D_g$

2. fog is defined (exist) if and only if $R_g \subseteq D_f$

3. $fog \neq gof$ (in general)

Example 4.23: Let $f: R \to [2, \infty)\ s.t. f(x) = x^4 + x^2 + 2$

$$g: R^+ \to [-4, \infty)\ s.t. g(x) = \sqrt{x} - 4$$

Find fog and gof (if exist)

Solution: fog exist when $R_g \subseteq D_f$

Rg= $[-4, \infty)$ and $D_f = R \Longrightarrow R_g \subseteq D_f$

$\therefore fog$ is defined

$$(fog)(x) = f\big(g(x)\big) = f(\sqrt{x} - 4) = \big(\sqrt{x} - 4\big)^4 + \big(\sqrt{x} - 4\big)^2 + 2$$

gof is defined when $R_f \subseteq D_g$

$R_f = [2, \infty)$ and $D_g = R^+ \Longrightarrow R_f \subseteq D_g$

$\therefore gof$ is defined

$$(gof)(x) = g\big(f(x)\big) = g(x^4 + x^2 + 2) = \sqrt{x^4 + x^2 + 2} - 4$$

Example 4.24: Let $f: R \to R\ s.t. f(x) = \sin(x)$

$$g: [0, \infty) \to (-\infty, 0]\ s.t. g(x) = -\sqrt{x}$$

Find fog and gof (if exist)

Solution: fog exist when $R_g \subseteq D_f$

Find R_g? $g(x) = -\sqrt{x}$

$\sqrt{x} \geq 0 \Longrightarrow \mathrm{g(x)} = -\sqrt{x} \leq 0$

$\therefore R_g = (-\infty, 0] \subseteq D_f = R$

$\therefore fog$ is defined

$$(fog)(x) = f\big(g(x)\big) = f\big(-\sqrt{x}\big) = \sin(-\sqrt{x})$$

To find gof, we need to check if $R_f \subseteq D_g$

$$f(x) = \sin(x) \in [-1,1]$$

$$\therefore R_f = [-1,1] \subseteq D_g = [0,\infty)$$

$\therefore$ gof is not a map (does not exist)

Theorem 4.25: Let $f: A \to B$ and $g: B \to C$ be two mappings, then

1. If f and g are 1-1 then gof is 1-1

2. If gof is 1-1 then f is 1-1

3. . If f and g are onto then gof is onto

4. If gof is onto then g is onto

Proof 1: T.P. $gof: A \to C$ is 1-1 (injective)

$\forall x_1, x_2 \in A, (gof)(x_1) = (gof)(x_2)$ T.P. $x_1 = x_2$

$(gof)(x_1) = (gof)(x_2) \Longrightarrow g\big(f(x_1)\big) = g(f(x_2))$ (def. of gof)

$$\Longrightarrow \mathrm{f(x_1) = f(x_2)} \qquad (g \text{ is 1-1})$$

$$\Longrightarrow \mathrm{x_1} = x_2 \qquad (f \text{ is 1-1})$$

$\therefore$ gof is 1-1

Proof 2: Let gof is 1-1 **T.P.** f is 1-1

$\forall x_1, x_2 \in A, f(x_1) = \mathrm{f(x_2)}$ T.P. $x_1 = x_2$

$f(x_1), f(x_2) \in \mathrm{D_g}$ and $f(x_1) = \mathrm{f(x_2)}$

$\Longrightarrow g\big(f(x_1)\big) = g(f(x_2))$ (g is well defined)

$\Longrightarrow (gof)(x_1) = (gof)(x_2)$ (def. of gof)

$\Longrightarrow x_1 = x_2$ (gof is 1-1)

$\Longrightarrow f$ is 1-1

Proof 3: Assume that f and g are onto **T.P.** gof is onto

T.P. $\forall z \in C, \exists x \in A\ s.t. (gof)(x) = z$

Since f is onto $\Longrightarrow$ $\forall y \in B, \exists x \in A\ s.t. f(x) = y$**……(1)**

Since g is onto $\Longrightarrow$ $\forall z \in C, \exists y \in B\ s.t. g(y) = z$**……(2)**

Substitute (1) in (2)

$\Longrightarrow \forall z \in C, \exists y = f(x) \in B\ s.t. g(f(x)) = z$

$\Longrightarrow \forall z \in C, \exists x \in A\ s.t. (gof)(x) = z$

$\Longrightarrow gof$ is onto

Proof 4: Let gof is onto **T.P.** g is onto

T.P. $\forall z \in C, \exists y \in B\ s.t. g(y) = z$

Since gof is onto $\Longrightarrow$ $\forall z \in C, \exists x \in A\ s.t. (gof)(x) = z$

$\Longrightarrow \forall z \in C, \exists y = f(x) \in B\ s.t.\ g(f(x)) = z$

$\Longrightarrow \forall z \in C, \exists y \in B\ s.t.\ g(y) = z$

$\Longrightarrow g$ is onto

Equal Mappings تساوي التطبيقات

Let f and g are two mappings, then

$$\boxed{f = g \iff \mathrm{D}_f = \mathrm{D}_g \wedge f(x) = g(x) \quad \forall x \in D_f = D_g}$$

Theorem 4.25: Let $f: A \to A$ be a mapping and $i_A: A \to A$ be the identity mapping, then $i_A of = f$ and $f o i_A = f$

Proof: T.P. $\underbrace{i_A of = f}_{(1)} \wedge \underbrace{f o i_A = f}_{(2)}$

$i_A of: A \to A$ is defined

T.P. $i_A of = f$ لكي نبرهن المساواة يجب تحقق شرطين

1) $D_{i_{Aof}} = D_f$?

$i_A of: A \to A$

$\therefore D_{i_{Aof}} = A = D_f$

2) $(i_A of)(x) = f(x)$

$(i_A of)(x) = i_A(f(x))$ (def. of o)

$= f(x)$ (def. of i_A)

$\therefore (i_A of)(x) = f(x) \quad \forall x \in A$

From (1) and (2) $\Longrightarrow$ $i_A of = f$

Similarly, prove that $f o i_A = f$ (**H.W.**)

Theorem 4.26: Let A, B and C are non empty sets. Then:

1. If $f: A \to B$ is bijective and $f^{-1}: B \to A$ then $f^{-1} of = i_A$, $f o f^{-1} = i_B$

2. If $f: A \to A$ is bijective and $f^{-1}: A \to A$ then $f^{-1} of$ = $f o f^{-1} = i_A$ (**H.W.**)

3. If $f: A \to B$ be a bijective mapping then $f^{-1}: B \to A$ is a bijective mapping.

4. If $f: A \to B,\ g: B \to C, h: C \to D$ then $(hog)of = ho(gof)$

5. If $f: A \to B$ is bijective and $g: B \to C$ is bij. then $(gof)^{-1} = f^{-1}og^{-1}$

<u>Proof</u> 1: Let $f: A \to B$ is bijective and $f^{-1}: B \to A$

T.P. $f^{-1}of = i_A$

$f^{-1}of: A \to A$ is defined

1) T.P. $D_{f^{-1}of} = \mathrm{D}_{\mathrm{i_A}}$

$f^{-1}of: A \to A \Longrightarrow D_{f^{-1}of} = A$

$i_A: A \to A \Longrightarrow \mathrm{D}_{\mathrm{i_A}} = A$

$\therefore \quad D_{f^{-1}of} = \mathrm{D}_{\mathrm{i_A}}$

2) T.P. $(f^{-1}of)(x) = i_A(x) \quad \forall x \in A$

$(f^{-1}of)(x) = f^{-1}\big(f(x)\big) \ldots\ldots (\mathbf{1})$

Since f is a map. $\Longrightarrow \ \forall x \in A, \exists y \in B\ s.t.\ f(x) = y$ **.....(2)**

Substitute (2) in (1) $\Longrightarrow (f^{-1}of)(x) = f^{-1}(y) = x$

$\therefore\ (f^{-1}of)(x) = x = i_A(x)$

From (1) and (2) $\Longrightarrow\ f^{-1}of = i_A$

Next we prove $fof^{-1} = i_B$

fof^{-1}:$B \to B$ is defined

1) T.P. $D_{f\ of^{-1}} = \mathrm{D}_{\mathrm{i_B}}$

$fof^{-1}: B \to B \Longrightarrow D_{f\ of^{-1}} = B$

$i_B: B \to B \Longrightarrow \mathrm{D}_{\mathrm{i_B}} = B$

$\therefore \quad D_{f \ \ of^{-1}} = \mathrm{D}_{\mathrm{i_B}}$

2) T.P. $(fof^{-1})(y) = i_B(y) \qquad \forall y \in B$

$(fof^{-1})(y) = f\big(f^{-1}(y)\big) \ldots \ldots .(\mathbf{1})$

Since f^{-1} is a map. $\Longrightarrow\ \forall y \in B, \exists x \in A\ s.t. f^{-1}(y) = x$ **.......(2)**

Substitute (2) in (1) $\Longrightarrow\ (fof^{-1})(y) = f(x) = y$

$\therefore\ (f^{-1}of)(y) = y = i_B\ (y)$

From (1) and (2) $\Longrightarrow\ f^{-1}of = i_B$

<u>**Proof**</u> **3:** . Let $f: A \to B$ be a bijective mapping (1-1 and onto)

T.P. $f^{-1}: B \to A$ is a bijective mapping.

f^{-1} is 1-1? Let $y_1, y_2 \in B$ s.t. $f^{-1}(y_1) = f^{-1}(y_2)$ T.P. $y_1 = y_2$

since $y_1, y_2 \in B = R_f \Longrightarrow \exists x_1, x_2 \in A$ s.t. $f(x_1) = y_1, f(x_2) = y_2$ [f is onto

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $f^{-1}(f(x_1)) = f^{-1}(y_1)$ and $f^{-1}(f(x_2)) = f^{-1}(y_2)$ [$f^{-1}$is well defined]

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $f^{-1}\big(f(x_1)\big) = f^{-1}(f(x_2) \qquad [f^{-1}(y_1) = f^{-1}(y_2)\ ]$

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $(f^{-1}of)(x_1) = (f^{-1}of)(x_2)$ [def. of $f^{-1}of$]

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $i_A(x_1) = i_A(x_2)$ [from part (1)]

$\Longrightarrow x_1 = x_2$

$\Longrightarrow f(x_1) = f(x_2)$ [f is well defined]

$\Longrightarrow y_1 = y_2$

$\therefore f^{-1}$is 1-1

$\boldsymbol{f^{-1}}$ **is onto?** T.P. $R_{f^{-1}} = A$

$R_{f^{-1}} = \{x \in A : x = f^{-1}(y)\} = \{x \in A : f(x) = y\} = A$

$\therefore f^{-1}$ is onto

f^{-1} is 1-1 and onto $\Longrightarrow$ f^{-1} is bijective

<u>Proof</u> 4: let $f: A \to B,\ g: B \to C, h: C \to D$ **T.P.** $(hog)of = ho(gof)$

1) $D_{(hog)of} = D_{ho(gof)} = A$

2) T.P. $\forall x \in A,\ [(hog)of](x) = [ho(gof)](x)$

$[(hog)of](x) = (hog)(f(x)$ (def. of o)

$= h(g(f(x))$ (def. of o)

$= h\big(gof(x)\big)$

$= [ho(gof)](x)$

$\therefore\ (hog)of = ho(gof)$

Proof 5: Let $f: A \to B$ is bijective and $g: B \to C$ is bijective

T.P. $(gof)^{-1} = f^{-1}og^{-1}$

Let $h = gof: A \to C$ T.P. $h^{-1} = f^{-1}og^{-1}$

T.P. $hoh^{-1} = i_C$

$hoh^{-1} = (gof)o(f^{-1}og^{-1})$

$= go(fof^{-1})og^{-1}$ [o is associative]

$= goi_Bog^{-1}$ $[fof^{-1} = i_B]$

$= gog^{-1}$ $[goi_B = g]$

$= i_C$ $[gog^{-1} = i_C]$

$\therefore\ (gof)^{-1} = f^{-1}og^{-1}$

Direct Image الصورة المباشرة

Let $f: A \to B$ be a mapping and $C \subseteq A$. Then the **direct image** of C under f is defined as

$$\boxed{f(C) = \{y \in B;\ \exists x \in C\ s.t.\ y = f(x)\}}$$

الصورة المباشرة $f(C)$ هي مجموعة جزئية من المجال المقابل B والتي كل عنصر فيها هو صورة لعنصر او اكثر من عناصر المجموعة C الجزئية من المجال. وتسمى $f(C)$ الصورة المباشرة ل Cبفعل التطبيق f.

$$y \in f(C) \Leftrightarrow \exists x \in C\ s.t.\ y = f(x)$$

$$y \notin f(C) \Leftrightarrow \forall x \in C\ s.t.\ y \neq f(x)$$

<u>Example 4.27</u>: Let $f: N\backslash\{1\} \to N\ s.t. f(n) = n^2 - 1$

Let $C = \{2,3,4\}$. Find $f(C)$

<u>Solution</u>:

$f(C) = \{f(2), f(3), f(4)\} = \{3,8,15\}$

<u>Theorem</u> 4.28: Let $f: A \to B$ be a mapping, let C, D are subsets of A. Then:

1. If $C \subseteq D$ then $f(C) \subseteq f(D)$

2. $f(C \cap D) \subseteq f(C) \cap f(D)$ (**H.W.**)

3. If f is injective (1-1) then $f(C \cap D) = f(C) \cap f(D)$

4. $f(C \cup D) = f(C) \cup f(D)$

5. $f(C)\backslash f(D) \subseteq f(C\backslash D)$

6. $f(C\backslash D) \subseteq f(C)$ (**H.W.**)

<u>Proof</u>1: Let $C \subseteq D$ T.P. $f(C) \subseteq f(D)$

Let $y \in f(C) \Longrightarrow\ y \in B;\ \exists x \in C\ s.t.\ y = f(x)$ (def. of $f(C)$)

$\Longrightarrow\ y \in B;\ \exists x \in D\ s.t.\ y = f(x)$ ($C \subseteq D$)

$\Longrightarrow$ y ∈ f(D) (def. of $f(D)$)

$\therefore\ f(C) \subseteq f(D)$

<u>Proof</u>3: Suppose f is 1-1 T.P. $\underbrace{f(C \cap D) \subseteq f(C) \cap f(D)}_{(1)} \wedge$

$\underbrace{f(C) \cap f(D) \subseteq f(C \cap D)}_{(2)}$

From (2), $f(C \cap D) \subseteq f(C) \cap f(D)$ ….(1) (**H.W.**)

T.P. $f(C) \cap f(D) \subseteq\ f(C \cap D)$

Let $y \in\ f(C) \cap f(D)$ T.P. $y \in\ f(C \cap D)$

$y \in\ f(C) \cap f(D) \Longrightarrow\ y \in\ f(C)\ \wedge y \in f(D)$ (def. of ∩)

$\Longrightarrow\ \exists\ x_1 \in C, y = f(x_1)\ \wedge\ \exists\ x_2 \in D, y = f(x_2)$ (def. of direct image)

$\Longrightarrow\ y = f(x_1) = f(x_2)$

$\Longrightarrow\ x_1 = x_2$ (f is 1-1)

$\Longrightarrow\ \exists\ x = x_1 = x_2 \in C \cap D$ s.t. $y = f(x) \in f(C \cap D)$ (def. of direct image)

$\therefore\ f(C) \cap f(D) \subseteq\ f(C \cap D)$ ….(2)

From (1) and (2), $f(C) \cap f(D) =\ f(C \cap D)$

<u>Proof</u>4: T.P. $\underbrace{f(C \cup D) \subseteq f(C) \cup f(D)}_{(1)} \wedge \underbrace{f(C) \cup f(D) \subseteq f(C \cup D)}_{(2)}$

Let $y \in f(C \cup D) \Leftrightarrow y \in B; \exists x \in C \cup D \; s.t. \; y = f(x)$ (def. of $f(C \cup D)$)

$\Leftrightarrow y \in B; \exists x \in C \vee x \in D \; s.t. \; y = f(x)$ (def. of $\cup$)

$\Leftrightarrow y \in B; [\exists x \in C \; s.t. \; y = f(x)] \vee [x \in D \; s.t. \; y = f(x)]$

$\Leftrightarrow y \in f(C) \vee y \in f(D)$ (def. of direct image)

$\Leftrightarrow y \in f(C) \cup f(D)$ (def. of $\cup$)

$\therefore f(C \cup D) = f(C) \cup f(D)$

<u>Proof</u>5: T.P. $f(C) \setminus f(D) \subseteq f(C \setminus D)$

Let $y \in f(C) \setminus f(D) \Rightarrow y \in f(C) \wedge y \notin f(D)$ (def. of \)

$\Rightarrow \exists x \in C \; s.t. \; y = f(x) \wedge \forall x \in D; \; y \neq f(x)$ (def. of direct image)

$\Rightarrow \exists x \in C \; s.t. \; y = f(x) \wedge x \notin D; \; y = f(x)$

$\Rightarrow \exists x \in C \wedge x \notin D; \; y = f(x)$

$\Rightarrow y \in f(C \setminus D)$

$\therefore f(C) \setminus f(D) \subseteq f(C \setminus D)$

Inverse Image الصورة العكسية

Let $f: A \to B$ be a mapping and $D \subseteq B$. Then the **inverse image** of D under f is defined as

$$\boxed{f^{-1}(D) = \{x \in A: f(x) \in D\}}$$

الصورة العكسية $f^{-1}(D)$ هي مجموعة جزئية من المجال A والتي تنتمي صورة كل عنصر فيها الى المجموعة D الجزئية من المجال. وتسمى $f^{-1}(D)$ الصورة العكسية ل D بفعل التطبيق f.

$$x \in f^{-1}(D) \Leftrightarrow x \in A \ s.t. \ f(x) \in D$$

$$x \notin f^{-1}(D) \Leftrightarrow x \in A \ s.t. \ f(x) \notin D$$

<u>Example 4.29</u>: Let $f: N\backslash\{1\} \to N \ s.t. f(n) = n^2 - 1$

Let $D = \{2,3,4,8\}$. Find $f^{-1}(D)$

Solution: $f^{-1}(D) = \{n \in N\backslash\{1\}: f(n) \in \{2,3,4,8\}\}$

$= \{n \in N\backslash\{1\}: \ n^2 - 1 = 2 \vee n^2 - 1 = 3 \vee n^2 - 1 = 4 \vee n^2 - 1 = 8\}$

$= \{n \in N\backslash\{1\}: \ n^2 = 3 \vee n^2 = 4 \vee n^2 = 5 \vee n^2 = 9\}$

$= \{n \in N\backslash\{1\}: \ n=\sqrt{3} \notin N \vee n=2 \in N \vee n = \sqrt{5} \notin N \vee n = 3 \in N\}$

$f^{-1}(D) = \{2,3\}$

<u>Example 4.30</u>: Let $f: R \to R \ s.t. f(x) = x^2 - 2$. Find $f^{-1}(\{2,7\})$

Solution: $f^{-1}(\{2,7\}) = \{x \in R: f(x) \in \{2,7\}\}$

$= \{x \in R: x^2 - 2 = 2 \ or \ x^2 - 2 = 7\} = \{2, -2, 3, -3\}$

Theorem 4.31: Let $f: A \to B$ be a mapping, let $E \subseteq A$ and C, D are subsets of B. Then:

1. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

2. If $C \subseteq D$ then $f^{-1}(C) \subseteq f^{-1}(D)$ (**H.W.**)

3. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ (**H.W.**)

4. $f^{-1}(C \backslash D) = f^{-1}(C) \backslash f^{-1}(D)$

5. $f^{-1}(C \backslash D) \subseteq f^{-1}(C)$ (**H.W.**)

6. $E \subseteq f^{-1}\big(f(E)\big)$ (**H.W.**)

7. If f is 1-1 if and only if $E = f^{-1}\big(f(E)\big)$

8. $f\ \big(f^{-1}(C)\big) \subseteq C$ (**H.W.**)

9. If f is onto if and only if $f\ \big(f^{-1}(C)\big) = C$

Proof1: Let $x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in C \cap D$ (def. of f^{-1})

$\Leftrightarrow f(x) \in C \wedge f(x) \in D$ (def. of $\cap$)

$\Leftrightarrow x \in f^{-1}(C) \wedge x \in f^{-1}(D)$ (def. of f^{-1})

$\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D)$ (def. of $\cap$)

$\therefore f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Proof4: Let $x \in f^{-1}(C \backslash D) \Leftrightarrow f(x) \in C \backslash D$ (def. of f^{-1})

$\Leftrightarrow f(x) \in C \wedge f(x) \notin D$ (def. of $\backslash$)

$\Leftrightarrow x \in f^{-1}(C) \wedge x \notin f^{-1}(D)$ (def. of f^{-1})

$\Leftrightarrow x \in f^{-1}(C) \backslash f^{-1}(D)$ (def. of $\backslash$)

$\therefore f^{-1}(C \backslash D) = f^{-1}(C) \backslash f^{-1}(D)$

<u>Proof</u>7: $\Longrightarrow$) Let f is 1-1 T.P. $E \subseteq f^{-1}\big(f(E)\big) \quad \wedge \; f^{-1}\big(f(E)\big) \subseteq E$

From part (6), $E \subseteq f^{-1}\big(f(E)\big)$ …….(1)

T.P. $f^{-1}\big(f(E)\big) \subseteq E$

Let $x \in f^{-1}\big(f(E)\big) \Longrightarrow f(x) \in f(E)$ (def. of inverse image)

$\Longrightarrow \exists$ e $\in$ E s.t. $f(x) = f(e)$

$\Longrightarrow x = e \in E$ (f is 1-1)

$\Longrightarrow x \in E$

$\therefore \quad f^{-1}\big(f(E)\big) \subseteq E$ ……(2)

From (1) and (2), $f^{-1}\big(f(E)\big) = E$

$\Longleftarrow$) Assume that $f^{-1}\big(f(E)\big) = E$ T.P. f is 1-1

Suppose f is not 1-1 برهان غير مباشر

$\exists x_1, x_2 \in A$ s.t $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

Let $E = \{x_1\} \Longrightarrow x_1 \in E \Longrightarrow f(x_1) \in f(E)$ (def. of direct image)

$\Longrightarrow f(x_2) \in f(E)$ ($f(x_1) = f(x_2)$)

$\Longrightarrow f(x_1) \in f(E)$ and $f(x_2) \in f(E)$

$\Longrightarrow x_1 \in f^{-1}\big(f(E)\big) = E$ and $x_2 \in f^{-1}\big(f(E)\big) = E$

$\Longrightarrow \{x_1, x_2\} \in E = \{x_1\}$ تناقض

f is 1-1

<u>Proof</u>9: $\Longrightarrow$) Let f is onto T.P. $f\ \big(f^{-1}(C)\big) \subseteq C \quad \wedge$ C $\subseteq f\ \big(f^{-1}(C)\big)$

From part (8), $f\ \big(f^{-1}(C)\big) \subseteq C$ …….(1)

T.P. $C \subseteq f\left(f^{-1}(C)\right)$

Let $y \in C \Longrightarrow \exists x \in A$ s.t. $y = f(x)$ (f is onto)

$\Longrightarrow x \in f^{-1}(y) \in f^{-1}(C)$ (def. of inverse image)

$\Longrightarrow f(x) = y \in f(f^{-1}(C))$

$\therefore$ $C \subseteq f\left(f^{-1}(C)\right)$ .....(2)

From (1) and (2), $C = f\left(f^{-1}(C)\right)$

$\Longleftarrow$) Assume that $f\left(f^{-1}(C)\right) = C$ T.P. f is onto

Assume f is not onto برهان بالتناقض

$\exists y \in B - f(A) \Longrightarrow y \neq f(x) \quad \forall x \in A$

$\Longrightarrow f^{-1}(y) \neq x \ \forall x \in A$

Let $C = \{y\} \Longrightarrow f^{-1}(C) = f^{-1}(\{y\}) = \emptyset$

$\Longrightarrow f\left(f^{-1}(C)\right) = f(\emptyset) = \emptyset$

$\Longrightarrow f\left(f^{-1}(C)\right) \neq C$ تناقض مع الفرض

$\therefore$ f is onto

Remark 4.32: : Let $f: A \rightarrow B$ be a mapping, let $E \subseteq A$ and $C \subseteq B$. Then in general

1. $A \neq f^{-1}\left(f(A)\right)$

2. $B \neq f\left(f^{-1}(B)\right)$

For example,

Let $A = \{1,2,3\}, B = \{4,5,6,7\}$

$f: A \rightarrow B$ s.t. $f(1) = f(2) = 4$

$$f(3) = 6$$

Let $E = \{1,3\} \subseteq A$ and $C = \{4,5\} \subseteq B$

$$f(E) = \{4,6\} \Longrightarrow f^{-1}\big(f(E)\big) = f^{-1}(\{4,6\}) = \{1,2,3\} \neq E$$

$$\Longrightarrow f^{-1}\big(f(E)\big) \neq E$$

Also, $f^{-1}(C) = \{1,2\} \Longrightarrow f\big(f^{-1}(C)\big) = f(\{1,2\}) = \{4\} \neq C$

$$\Longrightarrow f\big(f^{-1}(C)\big) \neq C$$

Chapter Five: Cardinality of Sets قدرة المجموعات

Chapter Five Contents:

1. Equivalent of Two Sets التكافؤ بين مجموعتين

2. Finite and infinite sets المجموعات المنتهية وغير المنتهية

3. Countably infinite sets المجموعات الغير النتهية القابلة للعد

Equivalent of Two Sets التكافؤ بين مجموعتين

Two sets A and B are called **equivalent** $(\boldsymbol{A} \approx \boldsymbol{B})$ if and only if there exist a bijective map between them.

المجموعتان A و B تسمى مجموعتين **متقابلتين** او **متكافئتين** اذا وجدت دالة متقابلة تربط بينهما

Mathematically,

$\boldsymbol{A} \approx \boldsymbol{B} \Leftrightarrow \exists f \ s.t. f: A \rightarrow B$ is **bijective**

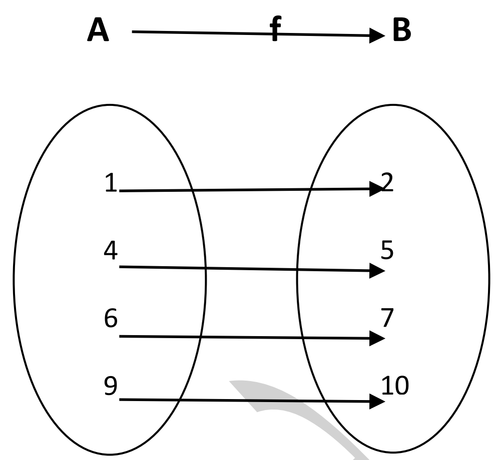
$\boldsymbol{A} \not\approx \boldsymbol{B} \Leftrightarrow \forall f \ s.t. f: A \rightarrow B$ is **not bijective**

Remark 5.1: If $\boldsymbol{A} \approx \boldsymbol{B}$, we say that there is a **"one to one correspondence"** between A and B.

Example 5.2: Let $A = \{1,4,6,9\}$, $B = \{2,5,7,10\}$. Show that $A \approx B$. Is $B \approx A$?

Solution: Define $f: A \rightarrow B \ s.t. f(1) = 2, \ f(4) = 5, f(6) = 7, f(9) = 10$

In general, $f(x) = x + 1 \ \ \forall x \in A$



1-1? $\forall x_1, x_2 \in A, x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2) \Longrightarrow\ f$ is 1-1

Onto? $R_f = \{2,5,7,10\} = cod_f$

$\therefore\ f$ is bijective $\Longrightarrow\ A \approx B$ (there is a one to one correspondence between A and B)

Give another bijective function that makes $\boldsymbol{A \approx B}$?

Define $h: A \to B\ s.t.\ h(1) =\ \ ,\ h(4) =\ \ , h(6) =\ \ , h(9) =$

Is $B \approx A$?

Define $g: B \to A\ s.t\ \ g(2) = 1, g(5) = 4, g(7) = 6, g(10) = 9$

Or $g(x) = x - 1\ \ \forall x \in B$

g is 1-1 and onto (check)

$\therefore\ g$ is bijective $\Longrightarrow\ B \approx A$

<u>طريقة ثانية لبرهان ان $\boldsymbol{B \approx A}$</u>

Since $A \approx B \Longrightarrow\ \exists f^{-1}: B \to A\ s.t.\ f^{-1}(y) = x$

لإيجاد قيمة x

Let $y = x + 1 \implies \mathrm{x} = y - 1$

Substitute the value of x in $f^{-1}(y)$

$\therefore\ f^{-1}: B \to A\ s.t.\ f^{-1}(y) = y - 1$

Show that f^{-1} is bijective (H.W.)

$\therefore\ B \approx A$

<u>Example</u> 5.3: Let $A = \{x, y, z\}$, $B = \{0, -1\}$

Is $A \approx B$? Is $B \approx A$?

<u>Solution</u>:

$A \approx B \Leftrightarrow \exists f\ s.t. f: A \to B$ is bijective

Let $f(x) = 0, f(y) = -1, f(z) = -1$

This relation is not 1-1 mapping because two element $y, z \in A$ have the same image.

In fact each $f: A \to B$ is not 1-1$\implies f$ is not bijective $\implies$ A is not equivalent to B.

$B \approx A \Leftrightarrow \exists g\ s.t. g: B \to A$ is bijective

Let $g(0) = x, g(-1) = y$

This relation is not onto mapping because $R_g \neq cod_g = A$

In fact each $g: B \to A$ is not onto$\implies g$ is not bijective $\implies$ B is not equivalent to A.

<u>Example</u> 5.4: Show that $N \approx B = \{5,10,15,20, \dots\}$

<u>Solution</u>: Define $f: N \to B \; s.t. \; f(x) = 5x \;\; \forall x \in N$

1-1? Let $x_1, x_2 \in N \; s.t. \; f(x_1) = f(x_2) \Longrightarrow 5x_1 = 5x_2$

$\Longrightarrow \mathrm{x}_1 = \mathrm{x}_2$

$\Longrightarrow$ f is injective

Onto? $R_f = \{y \in B: y = 5x, x \in N\} = \{5,10,15,20, \dots..\} = B$

$\therefore f$ is onto

$\therefore \; N \approx B$

<u>Example</u> 5.5: (**H.W.**) Show that $N \approx \{2,4,6, \dots..\}$

$$N \approx \{-2,-4,-6, \dots..\}$$

$$Z^- = \{-1,-2,-3, \dots.\} \approx N$$

<u>Theorem</u> 5.6: The equivalent relation ($\approx$) on sets is an equivalence relation

<u>Proof</u>:

1. $\approx$ reflexive? T.P. $A \approx A$ for any set A

$\exists i_A: A \to A \;\; s.t. \; i(x) = x \;\; \forall x \in A$

i_A is bijective $\Longrightarrow A \approx A \;\; \forall A$

$\therefore \approx$ reflexive

2. $\approx$ symmetric? Let A and B are two sets such that $A \approx B$ **T.P.** $B \approx A$

$A \approx B \Longrightarrow \exists f \; s.t. \; f: A \to B$ is bijective

Since f is bij. $\Longrightarrow \exists f^{-1}: B \to A$ such that f^{-1} is bij.

$$\Longrightarrow B \approx A$$

$\approx$ is symmetric

3. $\approx$ transitive? Let A, B and c are sets such that $A \approx B$ and $B \approx C$ **T.P.** $A \approx C$

$A \approx B \Longrightarrow \exists f\ s.t. f: A \rightarrow B$ is bijective

$B \approx C \Longrightarrow \exists g\ s.t. g: B \rightarrow C$ is bijective

$\therefore\ \exists gof: A \rightarrow C$ is bij. (by theorem 4.25 (chapter4))

$\therefore\ A \approx C$

$\therefore \approx$ is transitive

$\therefore \approx$ is an equivalence relation

Finite and Infinite Sets المجموعات المنتهية وغير المنتهية

A set A is said to be **finite** if A is empty or if A contains exactly m elements where
m is a positive integer; otherwise A is **infinite.**

المجموعة A تسمى **منتهية** اذا كانت مجموعة خالية او اذا كانت تحوي على عدد منتهي m من العناصر

وفيما عدا ذلك تكون المجموعة A **غير منتهية**.

Remark 5.7: The number of the elements in a finite set A is called the **size of $\boldsymbol{A}$** and is denoted by $n(A)$ or $\#(A)$ or $|A|$.

Example5:

Let $A = \{1,2\}$ finite set $\Longrightarrow \mathrm{n}(A) = 2$ عدد عناصر المجموعة

Let $A = \{\quad\}$ finite set $\Longrightarrow \#(A) = 0$

Let $A = \{\emptyset\}$ finite set $\Longrightarrow \#(A) = 1$

Let $A = \{1, \emptyset, \{1,3\}, [0,1], N\}$ finite set $\Longrightarrow |A| = 5$

Remark 5.8: If A is an infinite set $\Longrightarrow |A|$ or $\#(A)$ is not defined (does not exist)

Example 5.9:

Let $A = Z$ infinite set $\Longrightarrow |A|$ is not defined

Let $A = (9,40]$ infinite set $\Longrightarrow \#(\mathrm{A})$ does not exist

Cardinality of Finite Sets قدرة المجموعات المنتهية

Let $J_m = \{1,2,\dots,m\}$ be a set. A set A is called finite of size m $(n(A) = m)$ if and only if $A \approx J_m = \{1,2,\dots,m\}$. The positive number m is called **the cardinality of A**.

اذا كانت المجموعة منتهية فان عدد عناصرها يسمى **قدرة المجموعة**

Mathematically,

$$n(A) = m \Leftrightarrow A \approx \{1,2,\dots,m\} = J_m$$
$$n(A) = m \Leftrightarrow \exists f: A \to \{1,2,\dots,m\} \text{ s.t. } f \text{ is bijective}$$

Example 5.10: Let $A = \{\alpha, \beta, \sin(x)\}$. Find the cardinality of A.

Solution: Let $J_3 = \{1,2,3\}$ T.P. $A \approx J_3$

Define $f: A \to J_3$ $s.t.$ $f(\alpha) = 1, f(\beta) = 2,$ $f(\sin(x)) = 3$

It is clear that f is 1-1 and onto $\Longrightarrow n(A) = 3$

Theorem 5.11: Any two finite sets have the same cardinal number if and only if there is a bijective map. between them

i.e., Let A and B be two finite sets. Then $n(A) = n(B) \Leftrightarrow A \approx B$

Proof: $\Longrightarrow$) let $n(A) = n(B) = m$ $\boldsymbol{T.P.}$ $A \approx B$

$n(A) = m \Longrightarrow A \approx \{1,2,\dots,m\}$....(1) (def. of $n(A)$)

$n(B) = m \Longrightarrow B \approx \{1,2,\dots,m\}$....(2) (def. of $n(B)$)

From (1) &(2)

$A \approx \{1,2,\dots,m\} \wedge B \approx \{1,2,\dots,m\}$

$\Longrightarrow A \approx \{1,2,\dots,m\} \wedge \{1,2,\dots,m\} \approx B$ [$\approx$ symmetric]

$\Longrightarrow A \approx B$ [$\approx$ transitive]

$\Longleftarrow$) Suppose $A \approx B$ **T.P.** $n(A) = n(B)$

Let $n(A) = m \Longrightarrow A \approx \{1,2,\dots,m\}$ (def. of $n(A)$)

$\Longrightarrow A \approx B \wedge A \approx \{1,2,\dots,m\}$

$\Longrightarrow B \approx A \wedge A \approx \{1,2,\dots,m\}$ [$\approx$ symmetric]

$\Longrightarrow B \approx \{1,2,\dots,m\}$ [$\approx$ transitive]

$\Longrightarrow n(B) = m$

$\Longrightarrow \mathrm{n}(A) = n(B)$

Theorem 5.12: Let A be a finite set and $A_1, A_2, \dots, A_n$ be a partition of A. Then

$n(A) = n(A_1) + \cdots + n(A_n)$ (**بدون برهان**)

Theorem 5.13: Let A and B be finite sets. Then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Proof: Take $A \backslash B, A \cap B, B \backslash A$ as partition for $A \cup B$

$\Longrightarrow \mathrm{n}(A \cup B) = n(A \backslash B) + n(A \cap B) + n(B \backslash A)$ ....(1) (by theorem 5.12)

Take $A\backslash B, A\cap B$, as partition for $A \implies n(A)=n(A\backslash B)+n(A\cap B)$ (by theorem 5.12)

$\implies n(A\backslash B)= n(A)-n(A\cap B)$…(2)

Take $B\backslash A$, $A\cap B$, as partition for $B \implies n(B)=n(B\backslash A)+n(A\cap B)$ (by theorem 5.12)

$\implies n(B\backslash A)= n(B)-n(A\cap B)$…(3)

Substitute (2)&(3) into (1)

$$\implies \mathrm{n}(A\cup B)$$
$$=n(A)-n(A\cap B)+n(A\cap B)+n(B)-n(A\cap B)$$

$$\implies \mathrm{n}(A\cup B)=n(A)+n(B)-n(A\cap B)$$

<u>**Theorem**</u> **5.14:** Let A and B be finite sets. Then $n(A\times B)=n(A).n(B)$

<u>**Proof**</u>**:**

A is finite set $\implies n(A)=m$ s.t. $A=\{a_1,a_2,\dots,a_m\}$

B is finite set $\implies n(B)=n$ s.t. $B=\{b_1,b_2,\dots,b_n\}$

$A\times B=\{(a,b): a\in A, b\in B\}$

$A\times\{b_1\}=\{(a_1,b_1),(a_2,b_1),\dots,(a_m,b_1)\}$ $A\times\{b_2\}=\{(a_1,b_2),(a_2,b_2),\dots,(a_m,b_2)\}$

$A\times\{b_n\}=\{(a_1,b_n),(a_2,b_n),\dots,(a_m,b_n)\}$

$\implies A\times B=A\times\{b_1\}\cup A\times\{b_2\}\cup\dots\cup A\times\{b_n\}$

Such that $A\times\{b_i\}\cap A\times\{b_j\}=\emptyset \quad \forall i\neq j$

$\therefore\ A\times\{b_1\}, A\times\{b_2\},\dots, A\times\{b_n\}$ is a partition for $A\times B$

$$n(A \times B) = n(A \times \{b_1\}) + n(A \times \{b_2\}) + \cdots + n(A \times \{b_n\})$$

$$= \underbrace{m + m + \cdots + m}_{n-times} = mn = n(A).n(B)$$

Cardinality of infinite Sets قدرة المجموعات غير المنتهية

Let A be an infinite set. Then the cardinality of A is not a finite positive number. The cardinality of A is denoted by $\aleph_0$

المجموعة غير المنتهية تكون لها قدرة ولكن القدرة هنا لاتكون عدد معرف موجب

ترمز القدرة المجموعة الغير منتهية الى حجم تلك المجموعة

<u>Example</u> 5.15: The cardinal number of N is denoted by $\aleph_0$

i.e., $n(N) = \#(N) = \aleph_0$

Countably Infinite Set المجموعة غير المنتهية القابلة للعد

An infinite set A is called **countable** if it is equivalent to the set of natural number. Thus, the cardinality of an infinite countable set is $\aleph_0$

المجموعة غير المنتهية تكون قابلة للعد اذا كافئت مجموعة الاعداد الطبيعية

Mathematically,

A is countable infinite set $\Leftrightarrow A \approx N \Leftrightarrow n(A) = n(N) = \aleph_0$
A is not countable infinite set $\Leftrightarrow A \approx N$

<u>Example</u> 5.16: Show that N is countably infinite

Solution: T. P. $N \approx N$

Define $f: N \to N\ s.t.\ f(x) = i_N(x) = x \quad \forall x \in N$

f is bijective (**prove**!)

$\therefore\ N \approx N$

<u>Example</u> 5.17: Show that the set of even positive numbers is countably infinite

Solution: T. P. $E^+ \approx N$

يمكن ايجاد دالة متقابلة بين المجموعتين اما بالتخمين او من خلال ايجاد معادلة مستقيم اذا علمت منه نقطتان

$\boldsymbol{E^+}$	0	2	4	6	8	10…
$\boldsymbol{N}$	1	2	3	4	5	6…

Let $f(0) = 1$ and $f(2) = 2$

$(x_1, y_1) = (0,1)$ and $(x_2, y_2) = (2,2)$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\frac{y-1}{x} = \frac{1}{2} \Longrightarrow \mathrm{y} - 1 = \frac{x}{2}$

$\Longrightarrow \mathrm{y} = \frac{\mathrm{x}}{2} + 1$

$\therefore\ f: E^+ \to N$ s.t. f(x)= $\frac{\mathrm{x}}{2} + 1$

f is bijective (**prove**!)

$\therefore\ E^+ \approx N$

<u>Example</u> 5.18: (H.W.) Show that the set of odd positive numbers is countably infinite

Solution: T. P. $N \approx O^+$

$\boldsymbol{N}$	1	2	3	4	5	6…….
$\boldsymbol{O^+}$	1	3	5	7	9	11 …..

Let $f(1) = 1$ and $f(2) = 3$

$(x_1, y_1) = (1{,}1)$ and $(x_2, y_2) = (2{,}3)$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\frac{y-1}{x-1} = \frac{2}{1} \Longrightarrow \mathrm{y} - 1 = 2x - 2$

$\Longrightarrow \mathrm{y} = 2\mathrm{x} - 1$

$\therefore\ f: N \to O^+$ s.t. f(x)= $2\mathrm{x} - 1$

f is bijective (**prove**!)

$\therefore\ N \approx O^+$

<u>Example</u> 5.19: (**H. W.)** Prove that $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots.\}$ is countably infinite set.

<u>Example</u> 5.20: (H. W.) Prove that $A = \{1, \sqrt{2}, \sqrt{3}, \dots\}$ is countably infinite set

<u>Example</u> 5.21: Prove that $A = Z$ is countable infinite set

Solution: T. P. $Z \approx N$

T. P. $\exists f: Z \to N \ s.t. f$ is bijective

Z	0	1	-1	2	-2	3	-3…
N	1	2	3	4	5	6…	

1) Let $x \in Z \ s.t. x \geq 0$

$f(0) = 1, \ f(1) = 3$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

$\frac{y-1}{x} = \frac{2}{1} \Longrightarrow \mathrm{y} - 1 = 2x$

$\Longrightarrow \mathrm{y} = 2\mathrm{x} + 1$…..(1)

2) Let $x \in Z \ s.t. x < 0$

$f(-1) = 2, \ f(-2) = 4$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\frac{y-2}{x+1} = \frac{2}{-1} \Longrightarrow \mathrm{y} - 2 = -2(x+1)$

$\Longrightarrow \mathrm{y} = -2\mathrm{x}$.....(2)

From (1) &(2), define $f: Z \to N\ s.t.\ f(x) = \begin{cases} 2x+1, & x \geq 0 \\ -2x, & x < 0 \end{cases}$

Show that f is bijective

$\therefore\ Z \approx N$

<u>Example</u> 5.22: Prove that $A_k = \{0, \mp k, \mp 2k, \mp 3k, \dots\}$ is countably infinite set

Solution:

سنبرهن بان $A_k \approx Z$ وبما ان $Z \approx N$ فحسب خاصية التعدي $A_k \approx N$

If $k = 1 \Longrightarrow \mathrm{A_k} = \mathrm{A_1} = \{0, \mp 1, \mp 2, \mp 3, \dots\} = \mathrm{Z}$

$A_1 \approx Z$ because $\exists f: A_1 \to Z\ s.t.\ f(x) = x$ is bijective

If $k = 2 \Longrightarrow \mathrm{A_k} = \mathrm{A_2} = \{0, \mp 2, \mp 4, \mp 6, \dots\}$

A_2	0	2	-2	4	-4	6 ...
Z	0	1	-1	2	-2	3...

$A_2 \approx Z$ because $\exists f: A_2 \to Z\ s.t.\ f(x) = \frac{x}{2}$ is bijective

If $k = 3 \Longrightarrow \mathrm{A_k} = \mathrm{A_3} = \{0, \mp 3, \mp 6, \mp 9, \dots.\}$

A_3	0	3	-3	6	-6	9 …
Z	0	1	-1	2	-2	3…

$A_3 \approx Z$ because $\exists f: A_3 \to Z\ s.t.\ f(x) = \frac{x}{3}$ is bijective

In general, $A_k \approx Z$ because $\exists f: A_k \to Z\ s.t.\ f(x) = \frac{x}{k}$, $k \neq 0$ is bijective

Since, $A_k \approx Z\ \wedge Z \approx N \Longrightarrow A_k \approx N$ ($\approx$ is transitive)

$\therefore A_k$ is countable infinite set

<u>Theorem</u> 5.23: Any infinite subset of an infinite countable set is countable

i.e., If A is countable infinite set and $B \subseteq A$ then B is countable set

اي مجموعة جزئية من مجموعة قابلة للعد تكون قابلة للعد

<u>Theorem</u> 5.24: If A is countably infinite set then $A \cup \{a\}$ is also countably infinite set.

Proof: Let A be a countably infinite set $\Longrightarrow A \approx N$

$\exists f: A \to N$ bijective $s.t.\ f(a_1) = 1,,,, f(a_2) = 2,\ \ f(a_3) = 3, \dots.$

A	a_1	a_2	a_3…..
N	1	2	3…….

T.P. $A \cup \{a\} \approx N$

Define $g: A \cup \{a\} \to N$ s.t. $g(a) = 1, \ \ g(a_1) = 2, \ \ g(a_2) = 3, \ \ g(a_3) = 4, \ldots.$

$A \cup \{a\}$	a	a_1	a_2	a_3…..
N	1	2	3	4…….

$g(a) = 1$

$g(a_1) = 2 = 1 + 1 = f(a_1) + 1$

$g(a_2) = 3 = 2 + 1 = f(a_2) + 1$

$g(a_3) = 4 = 3 + 1 = f(a_3) + 1$

In general,

$$g(x) = \begin{cases} 1, & x = a \\ f(x) + 1, & x \neq a \end{cases}$$

T.P. g is bijective

g is 1-1? Let $x_1, x_2 \in A \cup \{a\} \ s.t. \ g(x_1) = g(x_2)$ T.P. $x_1 = x_2$

$$g(x_1) = g(x_2) \ \Longrightarrow \begin{cases} 1 = 1, & x_1 = a = x_2 \\ f(x_1) + 1 = f(x_2) + 1, & x_1 \neq a \ and \ x_2 \neq a \end{cases}$$

$$\Longrightarrow \ f(x_1) + 1 = f(x_2) + 1$$

$$\Longrightarrow \ f(x_1) = f(x_2)$$

$\Longrightarrow \mathrm{x}_1 = x_2$ (f is 1-1)

$\therefore \ g$ is 1-1

g is onto? $R_g = \{y \in N : \exists x \in A \cup \{a\}, y = g(x)\}$

$=N$

g is onto

g is bijective

$\therefore\ A \cup \{a\} \approx N$

$\therefore\ A \cup \{a\}$ is countably infinite set

<u>Remark</u> 5.25:

1) Every finite set is countable

2) Q the set of rational numbers is countably infinite set

3) R the set of real numbers is uncountable infinite set

Chapter Six: Binary Operations العمليات الثنائية

Chapter Six Contents:

1. Binary operation العملية الثنائية

2. Properties of binary operations خواص العمليات الثنائية

3. Group, Ring and Field الزمرة والحلقة والحقل

Definition 6.1: Let A be a non empty set. Any mapping from $A \times A$ into A is called **a binary operation on A**. The binary operation is denoted by the symbols $*$, #, $, o, ….

Mathematically,

$*: A \times A \rightarrow A$ is a binary operation iff 1. $*((a,b)) = a * b \in A$ (**closure condition**) 2. if $a, b, c, d \in A$ s.t. $(a,b) = (c,d)$ then $a * b = c * d$ (**well defined condition**).

Example 6.2: Let $A = \{0,1,-1\}$, let * be an operation on A such that

$a * b = b^2 \qquad \forall a, b \in A$

Is * binary operation on A?

Solution:

Closure? Let $a, b \in A \Longrightarrow a * b \in A$?

$\forall a, b \in A \Longrightarrow a * b = b^2 \in A \Longrightarrow *$ is closure

Well defined? Let $a, b, c, d \in A$ s.t. $(a, b) = (c, d) \Longrightarrow a * b = c * d$?

Since $(a, b) = (c, d) \Longrightarrow a = c \wedge b = d$

$a * b = b^2$ (def. of *)

$= d^2$ $(b = d)$

$= c * d$ $(\text{def. of } *)$

$\therefore *$ is well defined

$\therefore *$ is a binary operation on A

Example 6.3: Let $A = \{0,1,-1\}$, let * be an operation on A such that

$a * b = a - b \quad \forall a, b \in A$

Is * binary operation on A?

Solution:

Closure? Let $a, b \in A \Longrightarrow a * b \in A$?

If $a, b \in A \Longrightarrow a * b = a - b \notin A$

Take $a = 1, b = -1 \Longrightarrow a * b = a - b = 1 + 1 = 2 \notin A$

$\therefore *$ is not closure

$\therefore$ * is not a binary operation on A

Example 6.4: Let $A = N$, let # be an operation on N such that

$a \# b = a - b \quad \forall a, b \in N$

Is # binary operation on N?

Solution: Closure? Let $a, b \in N \Longrightarrow a \# b \in N$?

If $a, b \in N \Longrightarrow a \# b = a - b \notin N$

Take $a = 2, b = 5 \Longrightarrow a \# b = a - b = 2 - 5 = -3 \notin N$

$\therefore$ # is not closure

$\therefore$ # is not a binary operation on N

<u>Example</u> 6.5: (**H. W.**) Let $A = Z$, let * be an operation on Z such that

$a * b = a + b + 1 \quad \forall a, b \in Z$

Is * binary operation on Z?

<u>Example</u> 6.6: (**H. W.**) Let $A = E$, let * be an operation on E such that

$a * b = 2ab \quad \forall a, b \in E$

Is * binary operation on E?

<u>Example</u> 6.7: (**H. W.**) Let $A = O$, let * be an operation on O such that

$a * b = a + b \quad \forall a, b \in O$

Is * binary operation on O?

<u>Example</u> 6.8:

1. Let $A = N$, $a * b = a + b \quad \forall a, b \in N$

   "+" is a binary operation on N الجمع عملية ثنائية على مجموعة الاعداد الطبيعية

2. "+" is a binary operation on Z, R, Q, E
3. "-" is a binary operation on Z, R, Q, E
4. "×" is a binary operation on Z, R, Q, E, O, N
5. "÷" is a binary operation on $R\backslash\{0\}, Q\backslash\{0\}$
6. "÷" is not a binary operation on $N\backslash\{0\}, Z\backslash\{0\}$

Properties of Binary Operations خواص العمليات الثنائية

1. Commutative Binary Operation العملية الثنائية الابدالية

A binary operation * on a set A is called **commutative** iff $a * b = b * a \quad \forall a, b \in A$

Example 6.9:

''+'' is a commutative binary operation on N, Z, R, Q, E

''.'' is a commutative binary operation on N, Z, R, Q, E

''-'' is not commutative binary operation on N, Z, R, Q, E

Example 6.10: Let $a * b = a + b + ab \quad \forall a, b \in Z$. Is * commutative binary operation on Z?

Solution: * binary operation?

Closure? Let $a, b \in Z \Longrightarrow a + b \in Z \Longrightarrow \mathrm{a + b + ab \in Z}$

∴ * is closure

well-defined? Let $a, b, c, d \in Z$ s.t. $(a, b) = (c, d) \Longrightarrow a * b = c * d$?

Since $(a, b) = (c, d) \Longrightarrow a = c \wedge b = d$

$\Longrightarrow a * b = a + b + ab$ (def. of *)

$= c + d + cd \quad (a = c \wedge b = d)$

$= c * d$

∴ $*$ is well defined

∴ * is a binary operation

Commutative? $a * b = a + b + ab = b + a + ba = b * a$

$*$ is commutative

<u>Example</u> 6.11: Let $a\$b = a \;\; \forall a, b \in Q$. Is $ commutative binary operation on Q?

<u>Solution</u>: $ binary operation? (**H.W.**)

Comm.? $a\$b = a$ and $b\$a = b$

$\Longrightarrow a \neq b$

Take $a = \frac{1}{3}$ and $b = 5$

$a\$b = \frac{1}{3}$ and $b\$a = 5$

<u>Example</u> 6.12: Let $A * B = A \cup B \quad \forall A, B \in P(X)$. Is $\cup$ commutative binary operation on $P(X)$?

<u>Solution</u>: * binary operation?

Closure? Let $A, B \in P(X) \Longrightarrow A * B = A \cup B \subseteq X \Longrightarrow A * B \in \mathrm{P(X)}$

$\therefore$ $\cup$ is closure

well-defined? Let $A, B, C, D \in P(X)$ s.t. $(A, B) = (C, D) \Longrightarrow A * B = C * D$?

$(A, B) = (C, D) \Longrightarrow A = C \;\wedge\; B = D$

$\Longrightarrow A * B = A \cup B$ (def. of *)

$= C \cup D \quad (A = C \;\wedge\; B = D)$

$= C * D$

$\therefore$ * is well defined

$\therefore$ * is a binary operation

Commutative? (**H.W.**)

2. Associative Binary Operation العملية الثنائية التجميعية

A binary operation * on a set A is called **associative** if and only if

$$(a * b) * c = a * (b * c) \quad \forall a, b, c \in A$$

Example 6.13: Let $a.b = a + b - 2 \quad \forall a, b \in Z$. Is "." associative, commutative binary operation on Z?

Solution: "." binary operation? (**H.W.**)

Associative? Let $a, b, c \in Z, (a.b).c = a.(b.c)$?

$(a.b).c = (a + b - 2).c$ (def. of .)

$= (a + b - 2) + c - 2$ (def. of .)

$= a + b + c - 4 \ldots.(1)$

$a.(b.c) = a.(b + c - 2)$ (def. of .)

$= a + (b + c - 2) - 2$ (def. of .)

$= a + b + c - 4 \ldots.(2)$

From (1)&(2), $(a.b).c = a.(b.c)$

Commutative? (**H.W.**)

Example 6.14: (**H.W.**) Let $A * B = A \cup B \quad \forall A, B \in P(X)$. Is $\cup$ associative binary operation on $P(X)$?

Example 6.15: (**H.W.**) Let $A * B = A \cap B \quad \forall A, B \in P(X)$. Is $\cap$ associative, commutative binary operation on $P(X)$?

3. Distributive Property خاصية التوزيع

Let * and # are two binary operations on a set A. Then *** is distributive over # from the left** if and only if

$$a * (b \# c) = (a * b) \# (a * c) \quad \forall a, b, c \in A$$

Also, *** is distributive over # from the right** if and only if

$$(b \# c) * a = (b * a) \# (c * a) \quad \forall a, b, c \in A$$

Remark 6.16:

1. $a * (b \# c) \neq (b \# c) * a$ (in general)

2. If $a * (b \# c) = (b \# c) * a$ then we say that $*$ **is distributive over** #

Example 6.17: Let $*$ be a binary operation on Z such that

$$a * b = a \ \ \forall a, b \in Z$$

Let # be a binary operation on Z such that $a \# b = a + b - 2 \ \ \forall a, b \in Z$

Is * distributive over # from left and from right?

*** distributive over # from left ?**

We must show if $a * (b \# c) = (a * b) \# (a * c) \quad \forall a, b, c \in Z$

$a * (b \# c) = a$ (def. of *).....(1)

$(a * b) \# (a * c) = a \# a$ (def. of *)

$= a + a - 2$ (def. of #)

$= 2a - 2$ ....(2)

From (1) and (2), $a * (b\#c) \neq (a * b)\#(a * c)$

∴ * is not distributive over # from left

*** distributive over # from right ?**

We must show if $(b\#c) * a = (b * a)\#(c * a) \quad \forall a, b, c \in A$

$(b\#c) * a = b\#c$ (def. of *)

$= b + c - 2$ (def. of #) ….(1)

$(b * a)\#(c * a) = b\#c$ (def. of *)

$= b + c - 2$ (def. of #) ….(2)

From (1) and (2), $(b\#c) * a = (b * a)\#(c * a)$

∴ * is distributive over # from right

<u>**Example**</u> **6.18: (H.W.)** Let $*$ be a binary operation on N such that $a * b = ab \ \forall a, b \in N$

Let # be a binary operation on N such that $a\#b = a + b \ \forall a, b \in N$

Is * distributive over # from left and from right?

<u>**Example**</u> **6.19: (H.W.)** Let $*$ be a binary operation on $P(X)$ such that $A * B = A \cup B \ \forall A, B \in P(X)$

Let # be a binary operation on $P(X)$ such that $A\#B = A \cap B \ \forall A, B \in P(X)$

Is * distributive over # from left and from right?

<u>Definition</u>: The Identity Element العنصر المحايد

Let $*$ be a binary operation on a set A and $e \in A$, then e is called **the identity element of A** if and only if $a * e = e * a = a \quad \forall a \in A$

<u>Example</u> 6.20:

1. ''0'' is the identity element of the sets Z, Q, R with respect to (w.r.t.) (+)

الصفر هو العنصر المحايد للمجموعات Z, Q, R بالنسبة لعملية الجمع

$i.e., a + 0 = 0 + a \quad \forall a \in Z, Q, R$

2.''0'' is not the identity element of the sets Z, Q, R with respect to (w.r.t.) (-)

الصفر لايمثل العنصر المحايد للمجموعات Z, Q, R بالنسبة لعملية الطرح

$i.e., \ \exists a \in N, Z, Q, R \ \ s.t. \ \ a - 0 \neq 0 - a$

3. ''1'' is the identity element of the sets N, Z, Q, R w.r.t. (.)

الواحد هو العنصر المحايد للمجموعات N, Z, Q, R بالنسبة لعملية الضرب

$i.e., a.1 = 1.a \quad \forall a \in N, Z, Q, R$

4. ''1'' is not the identity element of the sets $Q - \{0\}, R - \{0\}$ with respect to (w.r.t.) (/)

الواحد لايمثل العنصر المحايد للمجموعات $Q - \{0\}, R - \{0\}$ بالنسبة لعملية القسمة

$i.e., \ \exists a \in Q - \{0\}, R - \{0\} \ \ s.t. \ \frac{a}{1} \neq \frac{1}{a}$

<u>Example</u> 6.21: Let # be a binary operation on $R\backslash\{-1\}$ such that

$a\#b = a + b + ab \quad \forall a, b \in R\backslash\{-1\}$. Find the identity element of $R\backslash\{-1\}$ with respect to #.

<u>Solution</u>: Let e be the identity element of E s.t. $a\#e = e\#a = a \quad \forall a \in R\backslash\{-1\}$

We must find e?

$a\#e = a \Rightarrow a + e + ae = a$ (def. of #)

$\Rightarrow \mathrm{e} + \mathrm{ae} = 0$

$\Rightarrow \mathrm{e}(1 + \mathrm{a}) = 0$

Either $e = 0$

or $1 + a = 0 \Rightarrow \mathrm{a} = -1 \notin R\backslash\{-1\}$ يهمل

$\therefore e = 0 \in R\backslash\{-1\}$ ....(1)

$e\#a = a \Rightarrow e + a + ea = a$ (def. of #)

$\Rightarrow \mathrm{e} + \mathrm{ea} = 0$

$\Rightarrow \mathrm{e}(1 + \mathrm{a}) = 0$

Either $e = 0$

or $1 + a = 0 \Rightarrow \mathrm{a} = -1 \notin R\backslash\{-1\}$ يهمل

$\therefore e = 0 \in R\backslash\{-1\}$....(2)

From (1) and (2), $e = 0$

Example 6.21: (H. W.) Let $*$ be a binary operation on N such that

$a * b = a + b + ab \quad \forall a, b \in N$. Find the identity element of N with respect to *.

Example 6.22: (**H. W.**) Let $*$ be a binary operation on N such that

$a * b = a + b - 1 \quad \forall a, b \in N$. Find the identity element of N with respect to *.

Example 6.23: Let $*$ be a binary operation on $P(X)$ such that

$A * B = A \cup B \quad \forall A, B \in P(X)$. Find the identity element of $P(X)$ with respect to *.

Solution: Let e be the identity element of $P(X)$ s.t. $A * e = e * A = A \quad \forall A \in P(X)$

$e = \emptyset$ because $A \cup \emptyset = \emptyset \cup A = A \quad \forall A \in P(X)$

Example 6.24: **(H. W.)** Let $*$ be a binary operation on $P(X)$ such that

$A * B = A \cap B \quad \forall A, B \in P(X)$. Find the identity element of $P(X)$ with respect to $*$.

Theorem 6.25: Let e is the identity element of a set A with respect to *, then e is unique.

Proof: Let e is the identity element of a set A with respect to *

Suppose e' is another identity of A w.r.t. *

Since e is the identity $\Longrightarrow e * e' = e' * e = e'$....(1)

Since e' is the identity $\Longrightarrow\Longrightarrow e' * e \quad = e \quad * e' = e$....(2)

From (1) and (2), $e = e'$

$\therefore$ e is unique

Definition: The Inverse Element العنصر النظير

Let $*$ be a binary operation on a set A and e is the identity element of A. Let $a \in A$, then $b \in A$ is called **the inverse element of $\boldsymbol{a}$** if and only if $a * b = b * a = e$.

The inverse element b is denoted by a^{-1}. So

$$a * a^{-1} = a^{-1} * a = e$$

Example 6.26: Find the inverse element of each element in Z, Q, R w.r.t "+"

Solution: The identity element $e = 0$

$$a * a^{-1} = 0 \Rightarrow a + a^{-1} = 0 \Rightarrow \mathrm{a}^{-1} = -\mathrm{a} \quad \forall \mathrm{a} \in \mathrm{Z}, \mathrm{Q}, \mathrm{R}$$

AND,

$$a^{-1} * a = 0 \Rightarrow a^{-1} + a = 0 \Rightarrow \mathrm{a}^{-1} = -\mathrm{a} \quad \forall \mathrm{a} \in \mathrm{Z}, \mathrm{Q}, \mathrm{R}$$

$\therefore a^{-1} = -a \quad \forall \mathrm{a} \in \mathrm{Z}, \mathrm{Q}, \mathrm{R}$

Example 6.27: Find the inverse element of each element in $Q\backslash\{0\}, R\backslash\{0\}$ w.r.t "."

Solution: The identity element $e = 1$

$$a * a^{-1} = 0 \Rightarrow a.a^{-1} = 1 \Rightarrow \mathrm{a}^{-1} = \frac{1}{\mathrm{a}} \quad \forall \mathrm{a} \in Q\backslash\{0\}, R\backslash\{0\}$$

AND,

$$a^{-1} * a = 0 \Rightarrow a^{-1} .a = 1 \Rightarrow \mathrm{a}^{-1} = \frac{1}{\mathrm{a}} \quad \forall \mathrm{a} \in Q\backslash\{0\}, R\backslash\{0\}$$

$\therefore a^{-1} = \frac{1}{a} \quad \forall \mathrm{a} \in Q\backslash\{0\}, R\backslash\{0\}$

Example 6.28: Let # be a binary operation on $Z\backslash\{-1\}$ such that

$a\#b = a + b + ab \quad \forall a, b \in Z\backslash\{-1\}$. Find the inverse element of each element in $Z\backslash\{-1\}$ (if exist).

Solution: From **Example 6.21,** $e = 0$

Let $a \in Z\backslash\{-1\}$ and $a^{-1}$is the inverse of a

$\Rightarrow a * a^{-1} = a^{-1} * a = e$

$a * a^{-1} = e \Rightarrow a + a^{-1} + aa^{-1} = 0$

$\Rightarrow \mathrm{a} + \mathrm{a}^{-1}(1 + \mathrm{a}) = 0$

$\Rightarrow \mathrm{a}^{-1} = -\frac{\mathrm{a}}{1+\mathrm{a}}$

$a^{-1} * a = e \Rightarrow a^{-1} + a + a^{-1}a = 0$

$$\Longrightarrow \mathrm{a} + \mathrm{a}^{-1}(1 + \mathrm{a}) = 0$$

$$\Longrightarrow \mathrm{a}^{-1} = -\frac{\mathrm{a}}{1+\mathrm{a}} \in Q$$

بصورة عامة نظير كل عدد في $Z\backslash\{-1\}$ هو عدد نسبي ماعدا $a = 0, -2$ فان نظيرهما عدد صحيح

If $a = 0 \Longrightarrow a^{-1} = 0 \in Z\backslash\{-1\} \Longrightarrow 0^{-1} = 0$

If $a = -2 \Longrightarrow \mathrm{a}^{-1} = \frac{2}{-1} = -2 \Longrightarrow \mathrm{a}^{-1} = -2 \in Z\backslash\{-1\}$

If a =3$\Longrightarrow \mathrm{a}^{-1} = -\frac{3}{4} \notin Z\backslash\{-1\}$

$\therefore$ $a = 3$ has no inverse

$\therefore \forall a \neq 0, -2$, $a^{-1} \notin Z\backslash\{-1\}$

Example 6.29: **(H. W.)** Let $*$ be a binary operation on $Q\backslash\{0\}$ such that

$a * b = 2ab \quad \forall a, b \in Q\backslash\{0\}$. Find the identity element of $Q\backslash\{0\}$ with respect to *.

Find the inverse of each element in $Q\backslash\{0\}$ (if exist).

Example 6.30: **(H. W.)** Let $*$ be a binary operation on Z such that

$a * b = a + b + 5 \quad \forall a, b \in Z$. Find the inverse of each element of Z with respect to *.

Group الزمرة

Let G be a non empty set and * be a binary operation on G. The pair $(G,*)$ is called **group** if and only if * is associative, there is an identity element and each element has an inverse.

Mathematically,

(G,*) is called **group** iff

1. $G \neq \emptyset$

2. * is a binary operation on G

3. * is associative on G

4. ∃ identity element $e \in G$ s.t. $a * e = e * a = a$

5. $\forall a \in G, \exists a^{-1} \in G$ s.t. $a * a^{-1} = a^{-1} * a = e$

<ins>**Remark**</ins> **6.30:** If $(G,*)$ is a group and * is a commutative then $(G,*)$ is called **commutative group**.

Mathematically,

A group $(G,*)$ is called **commutative** iff $a * b = b * a \quad \forall a, b \in G$

<ins>**Example**</ins> **6.31:** Show that $(Z, +)$ is a commutative group

1. $Z \neq \emptyset$

2. + is associative binary operation on Z

3. $\exists e = 0 \in Z \; s.t. \; a + 0 = 0 + a = a \quad \forall a \in Z$

4. $\exists a^{-1} = -a \in Z \quad \forall a \in Z \; s.t. \; a + a^{-1} = a^{-1} + a = 0$

$\therefore (Z, +)$ is a group

$\forall a, b \in Z \quad a + b = b + a$

$\therefore (Z, +)$ is a commutative group

<u>Example</u> 6.31:

$(Q, +)$ is a comm. group

$(R, +)$ is a comm. group

$(N, +)$ is not a group

$(Z, .)$ is not a group

$(O, +)$ is not a group

$(R\backslash\{0\}, .)$ is a group

$(R, .)$ is not a group

<u>Example</u> 6.32: Show that $(Z, *)$ is a commutative group such that $a * b = a + b - 5$

<u>Solution</u>:

1. **Closure:** let $a, b \in Z \Longrightarrow \mathrm{a} * \mathrm{b} = a + b - 5 \in Z \Longrightarrow$ closure is true

well-defined: let $a, b, c, d \in Z$ s.t. $(a, b) = (c, d) \Longrightarrow a * b = c * d$?

Since $(a, b) = (c, d) \Longrightarrow a = c \wedge b = d$

$\Longrightarrow a * b = a + b - 5$ (def. of *)

$= c + d - 5 \quad (a = c \wedge b = d)$

$= c * d$

$\therefore *$ is well defined

$\therefore$ * is a binary operation

2. **associative (H.W.)**

3. **Identity:** let $a \in Z$ we find $e \in Z$ such that $a * e = e * a = a$

$a * e = a$

$\Longrightarrow a + e - 5 = a$

$\Longrightarrow e = 5 \in Z$ ...(1)

Similarly, $e * a = a$

$\Longrightarrow e + a - 5 = a$

$\Longrightarrow e = 5 \in Z$ ...(2)

From (1) &(2) , $e = 5$

4. **Inverse:** $\forall a \in Z$, we find $a^{-1} \in Z$ such that $a * a^{-1} = a^{-1} * a = e$

$a * a^{-1} = e$

$\Longrightarrow a + a^{-1} - 5 = 5$

$\Longrightarrow a^{-1} = 10 - a \in Z$ ...(1)

Similarly, $a^{-1} * a = e$

$\Longrightarrow a^{-1} + a - 5 = 5$

$\Longrightarrow a^{-1} = 10 - a \in Z$ ...(2)

From (1) &(2) , $a^{-1} = 10 - a$

$\therefore$ $(Z,*)$ is a group

Commutative: (**H.W.**)

Example 6.33: Is $(P(X),\cup)$ group?

Solution:

1. $\cup$ is a binary operation (see Example 6.12)

2. $\cup$ is associative (see Example 6.14)

3. $\exists \emptyset \in P(X)$ s.t. $A \cup \emptyset = \emptyset \cup A = A$

$\therefore e = \emptyset$

4. **Inverse:** $\forall A \in P(X)$, we find $A^{-1} \in P(X)$ such that $A \cup A^{-1} = A^{-1} \cup A = \emptyset$

If $A = \emptyset$ then $A^{-1} = \emptyset$ s.t. $A \cup A^{-1} = \emptyset$

When $A \neq \emptyset$ then there is no inverse to A

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$\therefore$ $(P(X),\cup)$ is not a group

Example 6.34: (H.W.) Is $(P(X),\cap)$ group?

Is $(P(X),\setminus)$ group?

Example 6.35: Let $F(A) = \{f, f: A \to A$ is bijective map.$\}$

let * be an operation on $F(A)$ s.t. $f * g = fog$

Is $(F(A),*)$ commutative group?

Solution:

1. **Closure:** let $f, g \in F(A) \Longrightarrow f * g = fog \in F(A)$?

$f \in F(A) \Longrightarrow f: A \to A$ is bijective

$g \in F(A) \Longrightarrow g: A \to A$ is bijective

$\therefore fog: A \to A$ is bijective

Closure is true

well-defined: let $f_1, f_2, g_1, g_2 \in F(A)$ s.t. $(f_1, f_2) = (g_1, g_2) \Longrightarrow f_1 * f_2 = g_1 * g_2$?

Since $(f_1, f_2) = (g_1, g_2) \Longrightarrow f_1 = g_1 \wedge f_2 = g_2$

$\Longrightarrow f_1 * f_2 = f_1 o f_2$ (def. of *)

$= g_1 o g_2 \quad (f_1 = g_1 \wedge f_2 = g_2)$

$= g_1 * g_2$

$\therefore *$ is well defined

$\therefore$ * is a binary operation

2. **associative:** $\forall f, g, h \in F(A)$

$(fog)oh = fo(goh)$ (by theorem 4.26(4), chapter4)

3. **Identity:** $\exists i_A: A \to A$ is bijective such that $foi_A = i_Aof = f \ \ \forall f \in F(A)$(by thm 4.25, ch4)

$\therefore e = i_A$

4. **Inverse:** $\forall f \in F(A) \Longrightarrow f: A \to A$ is bijective

$\exists f^{-1}: A \to A$ is bijective $\Longrightarrow f^{-1} \in F(A)$

Such that $fof^{-1} = f^{-1}of = i_A$ (by thm 4.26(2), ch4)

$\therefore (F(A), o)$ is a group

Commutative:

Since $f: A \to A$ and $g: A \to A$

$fog = gof$

$\therefore (F(A), o)$ is a commutative group

Semi Group شبه الزمرة

Let A be a non empty set and * be a binary operation on A. The pair $(A,*)$ is called **semi group** if and only if * is associative.

Mathematically,

(A,*) is called **semi group** iff

1. $A \neq \emptyset$

2. * is a binary operation on A

3. * is associative on A

<u>Example</u> 6.36:

$(\mathrm{N}, +)$ is a semi group but not a group

$(\mathrm{Z}, .)$ is a semi group but not a group

Remark 6.37: Every group is a semi group

Ring الحلقة

Let R be a non empty set and * and # be two binary operations on R. The ordered triple $(R,*,\#)$ is called **ring** if and only if

1. $R \neq \emptyset$
2. $(R,*)$ is a commutative group
3. $(R,\#)$ is a semi group
4. # is distributed over * (from left and right)

Example 6.38: $(Z, +, .)$ is a ring

1. $(Z, +)$ is a commutative group
2. $(Z, .)$ is a semi-group
3. $a.(b+c) = a.b + a.c \quad \forall a, b, c \in R$ (distribution from left)

   $(b+c).a = b.a + c.a \quad \forall a, b, c \in R$ (distribution from right)

Example 6.39: $(Q, +, .)$ is a ring

$(R, +, .)$ is a ring

Commutative Ring الحلقة الابدالية

A ring $(R,*,\#)$ is called **commutative** iff $a\#b = b\#a \quad \forall a, b \in R$

الابدال يجب ان يتحقق على العملية الثانية

<u>Example</u> 6.40: $(Z, +, .)$ is a commutative ring because $a.b = b.a \quad \forall a, \in Z$

$(Q, +, .)$ is a commutative ring

$(R, +, .)$ is a commutative ring

Ring with Identity Element الحلقة ذات العنصر المحايد

A triple $(R, *, \#)$ has an identity element with respect to (#) if and only if

$$a\#e = e\#a = a \quad \forall a \in R$$

<u>Example</u> 6.41: $(Z, +, .), (Q, +, .), (R, +, .)$ are rings with $e = 1$ because $a.1 = 1.a$

Ordered Ring الحلقة المرتبة

A triple $(R, *, \#)$ is called **totally ordered ring** if and only if there is a totally ordered relation such that

1. $(R, *, \#)$ is a ring

2. The relation is totally ordered relation T.O.R

3. $\forall a, b \in R$ if $a \quad b$ then $a * c \quad b * c, \quad \forall c \in Z$

4. $\forall a, b \in R$ if $a \quad b$ then $a\#c \quad b\#c, \quad \forall c \geq 0$

The totally ordered relation is denoted by $(R, *, \#, \quad)$

<u>Example</u> 6.42: $(Z, +, ., \leq)$ is a totally ordered ring since

1. $(Z, +, .)$ is a ring (from Example 6.38)

2. $(Z, \leq)$ is T.O.R (see Example 3.85, Chapter 3)

3. $\forall a, b \in Z$ and $c \in Z$ if $a \leq b$ **T.P.** $a + c \leq b + c$

Let $a \leq b \Longrightarrow a = b - r,\ r \geq 0$

$\Longrightarrow \mathrm{a + c = (b + c) - r},\ \ c \in Z$ and $r \geq 0$

$\Longrightarrow\ a + c \leq b + c$

4. $\forall a, b \in Z$ and $c \geq 0,$ if $a \leq\ b$ then $a.c \leq\ b.c$

Let $a \leq b \Longrightarrow a = b - r,\ r \geq 0$

$\Longrightarrow \mathrm{a.c = b.c - cr} \quad \forall c \geq 0$

$\Longrightarrow \mathrm{a.c = b.c - cr} \quad cr \geq 0$

$\Longrightarrow\ a.c \leq b.c$

Example 6.43: (H.W.) Show that $(Z, +, ., \geq)$ is a totally ordered ring

Definition: Field الحقل

Let F be a non empty set and * and # be two binary operations on F. The ordered triple $(F, *, \#)$ is called **field** if and only if

1. $F \neq \emptyset$

2. $(F, *)$ is a commutative group

3. $(F \backslash \{e\}, \#)$ is a commutative group

Where e is the identity w.r.t. *

4. # is distributed over * (from left and right)

<u>Example</u> 6.44: $(R, +, .)$ is field since

1. $(R, +)$ is comm. Group

2. $(R\backslash\{0\}, .)$ is a commutative group

3. (.) dist. Over (+)

<u>Example</u> 6.45: (**H.W.**) Show that $(Q, +, .)$ is field

Ordered field الحقل المرتب

A triple $(F, *, \#)$ is called **totally ordered field** if and only if there is a totally ordered relation such that

1. $(F, *, \#)$ is a field

2. The relation is totally ordered relation T.O.R

3. $\forall a, b \in R$ if a b then $a * c$ $b * c$, $\forall c \in Z$

4. $\forall a, b \in R$ if a b then $a\#c$ $b\#c$, $\forall c \geq 0$

The totally ordered relation is denoted by $(F, *, \#, \quad)$

<u>Example</u> 6.46: $(R, +, ., \leq)$ is a totally ordered field since

1. $(R, +)$ is a comm. Group

2. $(R\backslash\{0\}, .)$ is a comm. Group

3. (.) is distributive over (+)

4. $(R, \leq)$ is T.O.R (see Example 3.86, Chapter 3)

5. $\forall a, b \in R$ and $c \in Z$ if $a \leq b$ then $a + c \leq b + c$ (see example 6.42)

6. $\forall a, b \in R$ and $c \geq 0$, if $a \leq b$ then $a . c \leq b . c$ (see example 6.42)

<u>Example</u> 6.47: show that $(R, +, ., \geq)$ is a totally ordered field

show that $(Q, +, ., \geq)$ is a totally ordered field

show that $(Q, +, ., \leq)$ is a totally ordered field

Chapter Seven: Number Systems الانظمة العددية

Chapter Seven Contents:

1. Natural Number System نظام الاعداد الطبيعية

2. Integer Number System نظام الاعداد الصحيحة

يتناول الفصل الحالي الانظمة العددية المتعارف عليها التي نستخدمها في حياتنا اليومية. ان اكبر واشمل نظام عددي هو نظام الاعداد الحقيقية والذي يتضمن الانظمة التالية:

1) نظام الاعداد الطبيعية N

2) نظام الاعداد الصحيحة Z

3) نظام الاعداد النسبية Q

هناك نظام عددي اخر وهو نظام الاعداد المركبة اوالمعقدة C

تاريخيا اول مجموعة تم تكوينها هي مجموعة الاعداد الطبيعية N ثم مجموعة الاعداد الصحيحة Z ومن الاعداد الصحيحة تم الحصول على الاعداد النسبية Q ومن الاعداد النسبية يمكن الحصول على الاعداد الحقيقية R واخيرا نكون الاعداد المعقدة C من R.

ولتكوين اي مجموعة من هذه المجموعات يجب تعريف عمليتي الجمع والضرب على تلك المجموعة ومن ثم اعطاء خواص الجمع والضرب كمبرهنات ثم نعين ترتيبا على المجموعة ونعين خواصه.

سنبدأ اولا باستعراض كيفية تكوين نظام الاعداد الطبيعية N

The Natural Number: نظام الاعداد الطبيعية

تم تكوين نظام الاعداد الطبيعية بالاعتماد على مجموعة من البديهيات (عبارات رياضية لاتحتاج الى برهان) والتي تسمى **بديهيات بيانو** (Peano's Axioms) والتي تعود لمكتشفها العالم الايطالي **(Giuseppe Peano)**

Peano's Axioms: بديهيات بيانو

1. $1 \in N$

2. $\exists \alpha : N \to N \backslash \{1\}$ bijective map s.t. $\alpha(n) = n^* = n + 1 \quad \forall n \in N$

n^* is called the **successor التابع** of n

3. Induction Axiom بديهية الاستقراء

$\forall\, M \subseteq N$; If $[(1 \in M \wedge \mathrm{k} \in M)$ then $k^* \in M\,]$ then $M = N$

Sum on N الجمع على مجموعة الاعداد الطبيعية

ان عملية الجمع على الاعداد الطبيعية تم تعريفها كالتالي

1. $n + 0 = n \quad \forall\, n \in N$

2. $\forall n \in N$ define $n + 1 = n^*$

3. $\forall n, m \in N$ define $m + n^* = (m + n)^*$

Example 7.1: Use the definition above to find $2 + 1$ and $2 + 3$

$2 + 1 = 2^* \quad [n + 1 = n^*]$

$= 3$

$2 + 3 = 2 + 2^* \quad [n + 1 = n^*]$

$= (2 + 2)^* \quad [m + n^* = (m + n)^*\ ]$

$= (2 + 1^*)^* = (2 + 1)^{**} \quad = 2^{***} \quad = 3^{**} \ = 4^* = 5$

Example 7.2: Find $5 + 4$ (**H.W.**)

Theorem 7.3:

1. $n+1=1+n=n^* \quad \forall n \in N$

2. $(m+n)+p=m+(n+p) \quad \forall m,n,p \in N$

3. $m+n=n+m \qquad \forall n,m \in N$ (**H. W.**)

4. $m+n=p+n$ iff $m=p$

ملاحظة: سنعتمد على تحقيق الشرط الثالث من بديهيات بيانو لبرهان المبرهنة اعلاه

Proof 1: Let $M \subseteq N$ s.t. $M=\{m \in N: m+1=1+m=m^*\}$ T.P. $M=N$ (Axiom3)

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $m=1 \in M$

i.e., **T. P.** $1+1=1+1=2$

$m+1=1+1=1^*=2$ ....(1)

$1+m=1+1=1^*=2$ ....(2)

From (1) and (2), $1+1=1+1=2 \Longrightarrow 1 \in M$

2. Suppose $k \in M$ ($k+1=1+k=k^*$)....(3)

3. T. P. $k^* \in M$ ($\text{T.P.}\, \text{k}^*+1=1+\text{k}^*=\text{k}^{**}$)

$k^*+1=k^{**}$ ....(4)

$1+k^*=(1+k)^*=(k+1)^*$ (from 3, $1+k=k+1$)

$= k^{**}$...(5)

From (4) and (5) $\Longrightarrow \mathrm{k}^{**} \in \mathrm{M}$

Hence, $M = N$ (Axiom3)

$\Longrightarrow m + 1 = 1 + m = m^* \quad \forall m \in M$

Proof 2: Let $M \subseteq N$ s.t. $M = \{p \in N: (n + m) + p = n + (m + p) \quad \forall m, n \in N\}$

T.P. $M = N$

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $1 \in M$ (T.P. $(n + m) + 1 = n + (m + 1)$)

$(n + m) + 1 = (n + m)^* = n + m^* = n + (m + 1)$

2. Suppose $k \in M$ ($(n + m) + k = n + (m + k)$)

3. T. P. $k^* \in M$ $\left(\mathrm{T.P.}\,(n + m) + k^* = n + (m + k^*)\right)$

$$(n + m) + k^* = \left((n + m) + k\right)^* = \left(n + (m + k)\right)^* = n + (m + k)^*$$
$$= n + (m + k^*)$$

$\therefore\ k^* \in M$

$\therefore M = N$

Proof 4: $\Longrightarrow$) Suppose $m + n = p + n$ **T. P.** $m = p$

Let $M \subseteq N$ s.t. $M = \{n \in N: m + n = p + n \Longrightarrow \mathrm{m} = \mathrm{p} \quad \forall m, p \in N\}$ T.P. $M = N$

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $n = 1 \in M$

i.e., $m + 1 = p + 1$ $\mathbf{T.P.}\ \mathrm{m = p}$

$m + 1 = p + 1$

$m^* = p^*$

$\alpha(m) = \alpha(p)$ (by axiom2)

$m = p$ (α is 1-1)$\Longrightarrow 1 \in M$

2. Suppose $k \in M$ ($m + k = p + k \Longrightarrow \mathrm{m = p}$)

3. T. P. $k^* \in M$ ($\mathrm{if\ m + k^* = p + k^*}$ $\mathbf{T.P.}\ \mathrm{m = p}$)

$m + k^* = p + k^*$

$(m + k)^* = (p + k)^*$

$\Longrightarrow \alpha(\mathrm{m + k}) = \alpha(\mathrm{p + k})$

$\Longrightarrow m + k = p + k$ (α is 1-1)

$\Longrightarrow m = p$ (from2)

$\Longrightarrow \mathrm{k^* \in M}$

Hence, $M = N$ (Axiom3)

$\Longleftarrow$) Let $m = p$ **T.P.** $m + n = p + n$

Let $M \subseteq N$ s.t. $M = \{n \in N: \mathrm{m = p} \Longrightarrow m + n = p + n\ \forall m, p \in N\}$ T.P. $M = N$

1. T. P. $n = 1 \in M$

Let $m = p$ T.P. $m + 1 = p + 1$

$m = p \Longrightarrow \ \alpha(m) = \alpha(p)$ (α is well defined)

$\Longrightarrow m^* = p^*$ (by axiom 2)

$\Longrightarrow \mathrm{m} + 1 = p + 1$

$\therefore 1 \in M$

2. Suppose $k \in M$ ($if\ \mathrm{m} = \mathrm{p} \Longrightarrow m + k = p + k$)

3. T. P. $k^* \in M$ ($\mathrm{if\ m = p}$ $\mathbf{T.P.}\,\mathrm{m + k^* = p + k^*}$)

$m = p \ \Longrightarrow m + k = p + k$ (from2)

$\Longrightarrow \alpha(\mathrm{m + k}) = \alpha(\mathrm{p + k})$ (α is well defined)

$\Longrightarrow (m + k)^* = (p + k)^*$ (by axiom2)

$\Longrightarrow \mathrm{m} + k^* = p + k^*$

$\Longrightarrow k^* \in M$

$\therefore M = N$

Multiplication on N عملية الضرب على مجموعة الاعداد الطبيعية

ان عملية الضرب على الاعداد الطبيعية تم تعريفها كالتالي

1. $n.0 = n \quad \forall\, n \in N$

1. $n.1 = n \quad \forall\, n \in N$

2. $m.n^* = m.(n + 1) = m.n + m \quad \forall\, n, m \in N$

__Example__ **7.4:** Use the definition above to find 2.3 and 5.4

2.3=$2.2^* = 2.2 + 2$ $\qquad (m.n^* = m.n + m)$

$= 2.1^* + 2$

$= (2.1 + 2) + 2 \quad (m.n^* = m.n + m)$

$= (2 + 2) + 2$

$= (2 + 1^*) + 2 \quad (n^* = n + 1)$

$= (2 + 1)^* + 2$

$= 2^{**} + 2 \quad (n^* = n + 1)$

$= 3^* + 2$

$= 4 + 2 = 4 + 1^* = (4 + 1)^* = 4^{**} = 5^* = 6$

Find 5.4 (**H. W.**)

<u>Theorem 7.5</u>:

1. $n.1 = 1.n = n \quad \forall n \in N$
2. $m^*.n = m.n + n \quad \forall n, m \in N$
3. $p.(m + n) = p.m + p.n \quad \forall n, m, p \in N$
4. $(m.n).p = m.(n.p) \quad \forall n, m, p \in N$ (**H. W.**)
5. $m.n = n.m \quad \forall n, m \in N$
6. $(m + n).p = m.p + n.p \quad \forall n, m, p \in N$ (**H. W.**)
7. If $m.p = n.p$ then $m = n$ (**H. W.**)

<u>Proof1</u>: Let $M \subseteq N$ s.t. $M = \{n \in N: n.1 = 1.n = n\}$ T.P. $M = N$ (Axiom3)

شرط بيانو الثالث يتضمن التالي

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $n = 1 \in M$

i.e., **T. P.** $1.1 = 1.1 = 1$

$1.1 = 1 \Longrightarrow 1 \in M$

2. Suppose $k \in M$ $(k.1 = 1.k = k)$….(1)

3. T. P. $k^* \in M$ $(\mathrm{T.P.\,k^*}.1 = 1.\mathrm{k}^* = \mathrm{k}^*)$

$k^*.1 = k^*$ (n..1=n) ….(2)

$1.k^* = 1.k + 1$

$= k.1 + 1$ $(1.k = k.1)$

$= k + 1 = k^*$…(3)

From (2) and (3) $\Longrightarrow \mathrm{k}^* \in \mathrm{M}$

Hence, $M = N$ (Axiom3)

$\Longrightarrow n.1 = 1.n = n \quad \forall n \in M$

<u>Proof2</u>: Let $M \subseteq N$ s.t. $M = \{n \in N\colon\ m^*.n = m.n + n \quad \forall m \in N\}$ T.P. $M = N$ (Axiom3)

شرط بيانو الثالث يتضمن التالي

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $n = 1 \in M$

i.e., **T. P.** $m^*.1 = m.1 + 1$

$m^*.1 = m^*$ $[n.1 = n]$ …(1)

$m.1 + 1 = m + 1$ $[m.1 = m]$

$= m^*$…..(2)

From (1) and (2) $\Longrightarrow 1 \in M$

2. Suppose $k \in M$ $(m^*.k = m.k + k)$….(3)

3. T. P. $k^* \in M$ $(\mathrm{T.P.}\ \mathrm{m^*.k^*} = \mathrm{m.k^*} + \mathrm{k^*})$

$\mathrm{m^*.k^*} = m^*.k + m^*$

$= (m.k + k) + m^*$ (from 3)

$= (m.k + k) + (m + 1)$

$= m.k + (k + m) + 1$ (+ is associative)

$= m.k + (m + k) + 1$ (+ is commutative)

$= (m.k + m) + (k + 1)$ (+ is associative)

$= \mathrm{m.k^*} + \mathrm{k^*}$

$\Longrightarrow \mathrm{k^*} \in \mathrm{M}$

Hence, $M = N$ (Axiom3)

$m^*.n = m.n + n$ $\forall n, m \in N$

<u>**Proof**</u>**3:** Let $M \subseteq N$ s.t. $M = \{n \in N: p.(m + n) = p.m + p.n\ \ \forall m, p \in N\}$

T.P. $M = N$ (Axiom3)

1) $1 \in M$? T.P. $p.(m + 1) = p.m + p.1$ $\forall m, p \in N$

$p.(m + 1) = pm + p$ $[m.(n + 1) = m.n + m]$

$= p.m + p.1 \quad (n.1 = n)$

2) Suppose $k \in M$ $[p.(m+k) = p.m + p.k \ \ \forall m, p \in N]$

3) T.P. $k^* \in M \quad (T.P. \quad p.(m+k^*) = p.m + p.k^*)$

$p.(m+k^*) = p.(m+k)^* \qquad [m + n^* = (m+n)^*]$

$= p.(m+k) + p \qquad [m.n^* = m.n + n]$

$= (p.m + p.k) + p \qquad$ [from 2]

$= p.m + (p.k + p) \qquad$ [+ is associative]

$= p.m + p.k^* \qquad [m.n^* = m.n + n]$

$\Longrightarrow \ k^* \in M$

Hence, $M = N$

$\Longrightarrow \ p.(m+n) = p.m + p.n \ \ \forall m, n, p \in N$

<u>**Proof**</u>**5:** Let $M \subseteq N$ s.t. $M = \{n \in N\colon \ m.n = n.m \ \ \forall m \in N\}$

T.P. $M = N$ (Axiom3)

1) $1 \in M$? T.P. $m.1 = 1.m \ \ \forall m \in N$

$m.1 = 1.m = m$ [from 1]

2) Suppose $k \in M$ $[k.m = m.k \ \ \forall m \in N]$

3) T.P. $k^* \in M \quad (T.P. \quad m.k^* = k^*.m)$

$m.k^* = m.k + m$

$= k.m + m \qquad [k.m = m.k \ \ \forall m \in N]$

$= k^*.m \qquad$ [from 2]

$\Longrightarrow \ k^* \in M$

Hence, $M = N$

$\Longrightarrow \quad m.n = n.m \quad \forall m, n \in N$

Definition 7.6: Let $x \in N$, then

1. $x^0 = 1$

2. $x^1 = x$

3. $x^{n+1} = x^n . x^1 \quad \forall n \in N$

Theorem 7.7: Let $x, y, n, m \in N$. Then:

1. $x^m . x^n = x^{m+n}$

2. $(x^m)^n = x^{nm}$ **(H. W.)**

3. $(x.y)^n = x^n . y^n$ **(H. W.)**

Proof1: : Let $M \subseteq N$ s.t. $M = \{n \in N: \ x^m . x^n = x^{m+n} \ \forall m \in N\}$

T.P. $M = N$ (Axiom3)

1) $1 \in M$? T.P. $x^m . x^1 = x^{m+1} \ \forall m \in N$

$x^m . x^1 = x^m . x \quad [x^1 = x]$

$= x^{m+1} \quad [x^{n+1} = x^n . x^1]$

2) Suppose $k \in M$ $[x^m . x^k = x^{m+k} \ \forall m \in N]$

3) T.P. $k^* \in M$ $\left(T.P. \quad x^m . x^{k^*} = x^{m+k^*} \ \forall m \in N\right)$

$x^m . x^{k^*} = x^m . x^{k+1}$

$= x^m . x^k . x^1$

$= x^{m+k} . x^1$ (from2)

$= x^{m+k+1}$

$= x^{m+k^*}$

$\Longrightarrow \quad k^* \in M$

Hence, $M = N$

$\Longrightarrow \quad x^m . x^n = x^{m+n} \quad \forall m, n \in N$

Ordered on N الترتيب على الاعداد الطبيعية

The relation $<$ is defined as follows

$m < n \ \leftrightarrow \exists! r \in N \ s.t. m + r = n$

$m \leq n \leftrightarrow m = n \vee m < n \quad \forall m, n \in N$

__**Theorem 7.8**__:

1. $\forall m, n \in N$, either $m = n$ or $m < n$ or $n < m$ (بدون برهان) قانون الانقسام الثلاثي

2. If $m \leq n$ and $n \leq m$ then $n = m$ **(see Example 3.75 (anti symm.), ch3)**

3. If $m < n$ and $n < p$ then $m < p$ **(see Example 3.75 (transitive), ch3)**

4. $m < n \ \leftrightarrow m + p < n + p \quad \forall p \in N$ **(see Example 6.42, ch6)**

5. $m < n \ \leftrightarrow m.p < n.p \quad \forall p \in N$ **(see Example 6.42, ch6)**

6. If $p \leq m \leq p^*$ then $m = p$ or $m = p^*$

__**Proof**__**6:** Assume $m \neq p$ and $m \neq p^*$ برهان غير مباشر

$\Longrightarrow p < m$ and $m < p^*$

$p < m \Longrightarrow \quad \exists k \in N \ s.t. p + k = m$....(1)

$m < p^* \Longrightarrow \quad \exists r \in N \ s.t. m + r = p^*$....(2)

Substitute (1) in (2) $\Longrightarrow \ p^* = p + k + r$

$\Longrightarrow\ p+1=p+k+r$

$\Longrightarrow 1=k+r$ تناقض لايوجد عددين طبيعيين حاصل جمعهما واحد

$\Longrightarrow m=p$ or $m=p^*$

Least Element العنصر الاصغر

Let $\emptyset \neq M \subseteq N$ and $a \in N$. Then a is called **least element** in M if $a \leq x \quad \forall x \in M$.

Example 7.9: Let $M=\{3,4,5\} \subseteq N$

Then 3 is the least element of M because $3 \leq x \quad \forall x \in M$

Well-Ordered Set المجموعة المرتبة جيدا

Let $\emptyset \neq M \subseteq N$ and $a \in N$. Then M is called **well-ordered** if each non empty subset of M contains least element

المجموعة M تكون مرتبة ترتيبا جيدا اذا كانت كل مجموعة جزئية وغير خالية منها تمتلك عنصر اصغر

Integer Numbers System نظام الاعداد الصحيحة

ظهرت الحاجة الى تكوين نظام عددي اوسع من نظام الاعداد الطبيعية عندما وجد بان نظام الاعداد الطبيعية غير قادر على ايجاد حلول لمعادلات من النوع التالي

$5+x=2$ or $10+y=1 \quad \forall x,y \in N$

لذلك تم ايجاد نظام عددي أوسع يحوي الحل ل هكذا نوع من المعادلات ألا وهو نظام الاعداد الصحيحة والتي يرمز لها بالرمز Z أو I

Construction of Integer Numbers انشاء مجموعة الاعداد الصحيحة

Definition 7.10: Let N be the set of natural numbers then

$$N \times N = \{(n, m): n, m \in N\}$$

Define a relation $\sim$ on $N \times N$ as

$(n_1, m_1) \sim (n_2, m_2)$ iff $n_1 + m_2 = m_1 + n_2$ $\forall\ (n_1, m_1), (n_2, m_2) \in N \times N$

Example 7.11: $(6,3) \sim (10,7)$ since $6 + 7 = 3 + 10$

$(5,2) \sim (8,4)$ since $5 + 4 \neq 2 + 8$

Theorem 7.12: The relation $\sim$ on $N \times N$ is an equivalence relation

Proof:

Reflexive? T.P. $(n, m) \sim (n, m)$ $\forall n, m \in N$

T.P. $n + m = m + n$ (def. of $\sim$)

Since $n + m = m + n$ (+ is commutative on N)

$\therefore\ (n, m) \sim (n, m)$

Symmetric? Let $(n, m) \sim (r, s)$ T. P. $(r, s) \sim (n, m)$

Since $(n, m) \sim (r, s)$ $\Rightarrow \mathrm{n} + s = m + r$ (def of $\sim$)

$\Rightarrow \mathrm{s} + \mathrm{n} = \mathrm{r} + \mathrm{m}$ (+ is commutative on N)

$\Rightarrow r + m = s + n$

$\Rightarrow\ (r, s) \sim (n, m)$

Transitive? Let $(n, m) \sim (r, s)$ and $(r, s) \sim (p, q)$ T. P. Let $(n, m) \sim (p, q)$

$(n,m)\sim(r,s) \Rightarrow \mathrm{n}+s=m+r$ (def. of ~) ....(1)

$(r,s)\sim(p,q) \Rightarrow r+q=s+p$ (def. of ~) ....(2)

By summing up (1) and (2)

$\Rightarrow n+s+r+q=m+r+s+p$

$\Rightarrow n+(s+r)+q=m+(s+r)+p$ (+ comm. and assoc. on N)

$\Rightarrow n+q+(s+r)=m+p+(s+r)$ (+ comm. on N)

$\Rightarrow n+q=m+p$ (by theorem 7.5(4), قاتون الحذف)

$(n,m)\sim(p,q)$

∴ ~ is an equivalence relation

Definition 7.13: Let $\sim$ be a relation on $N\times N$. Since $\sim$ is an equivalence relation, then we can define an equivalence classes on $N\times N$

$H=\{[(n,m)]:\ n,m\in N\}=\{[n,m]:\ n,m\in N\}$

Where $[n,m]=\{(r,s):(n,m)\sim(r,s)\}$

Definition 7.14: The set of all different equivalence classes on $N\times N$ form the set of integer numbers

Mathematically,

$Z^{+}=\{[n,m]:\ n>m\ and\ n,m\in N\}$

$0=\{[n,m]: n=m\ and\ n,m\in N\}$

$Z^{-}=\{[n,m]: n<m\ and\ n,m\in N\}=\{-[m,n]: m>n\}$

Example 7.15: $[\mathbf{3},\mathbf{1}]=\mathbf{2}$; $[\mathbf{2},\mathbf{3}]=-[\mathbf{3},\mathbf{2}]=-\mathbf{1}$; $[\mathbf{3},\mathbf{3}]=\mathbf{0}$

Example 7.16: Find $[(2,3)], [(10,7)], [0,0]$

$[(\mathbf{2},\mathbf{3})]=[2,3]=\{(r,s)\in N\times N:(r,s)\sim(2,3)\}$

$=\{(r,s):r+3=s+2\}$

$=\{(r,s):\ r+1+2=s+2\}$

$=\{(r,s):r+1=s,\ \ r,s\in N\}$

$=\{(1,2),(2,3),(3,4),\ldots.\}=-1$ (from def.7)

من خواص صفوف التكافؤ ان العناصر الموجودة في صف تكافؤ معين لها صفوف تكافؤ متساوية.
اي ان

$[2,3]=[(1,2)]=[3,4]=[(4,5)]=\cdots$

$[\mathbf{10},\mathbf{7}]=\{(r,s)\in N\times N:(r,s)\sim(10,7)\}$

$=\{(r,s):r+7=s+10\}$

$=\{(r,s):\ r+7=s+3+7\}$

$=\{(r,s):r=s+3,\ \ r,s\in N\}$

$=\{(4,1),(5,2),(6,3),\ldots.\}=3$ (from def.7)

$\therefore [10,7]=[4,1]=[5,2]=[6,3]=\cdots$ $[\mathbf{0},\mathbf{0}]=\{(r,s)\in N\times N:(r,s)\sim(0,0)\}$

$=\{(r,s):r+0=s+0\}$

$=\{(r,s):\ r=s\}$

$=\{(1,1),(2,2),(3,3),\ldots.\}=0$ (from def.7)

$\therefore [0,0]=[1,1]=[2,2]=[3,3]=\cdots$

كل عدد صحيح يمكن كتابته على شكل صف تكافؤ

Addition and Multiplication on Z

Let $x, y \in Z$ such that $x = [n, m]$ and $y = [r, s]$. Define

$x +_Z y = [n, m] +_Z [r, s] = [n + r, m + s]$

$x ._Z \; y = [n, m] ._Z [r, s] = [nr + ms, ns + mr]$

<u>Example 7.17</u>: Find $(3) +_Z (-5)$, $(-1) +_Z (-3)$, $(-2) ._Z (4)$

$(3) +_Z (-5) = [5,2] +_Z [1,6] = [5 + 1, 2 + 6]$ (def. of $+_Z$)

$= [6,8] = -2$

$(-1) +_Z (-3) = [5,6] +_z [1,4]$

$= [5 + 1, 6 + 4]$ (def. of $+_Z$)

$= [6,10] = -4$

$(-2) ._Z (4) = [2,4] ._Z [8,4]$

$= [(2.8) + (4.4), (2.4) + (4.8)]$ (def. of $._Z$)

$= [16 + 16, 8 + 32]$

$= [32, 40] = -8$

<u>Example 7.18</u>: (H. W.) Find $(-3) +_Z (4)$, $(-5) ._Z (-1)$

<u>Theorem 7.19</u>: Properties of the sum on Z

1. $x +_Z y = y +_Z x \quad \forall x, y \in Z$

2. $(x +_Z y) +_Z p = x +_Z (y +_Z p) \quad \forall x, y, p \in Z$

3. $\exists 0 \in Z$ such that $x +_Z 0 = 0 +_Z x = x \quad \forall x \in Z$

4. $x +_Z p = y +_Z p \Leftrightarrow x = y \quad \forall x, y, p \in Z$

5. $\forall x \in Z, \exists y \in Z \ s.t. \ x +_Z y = 0$

Proof1: T.P. $x +_Z y = y +_Z x \quad \forall x, y \in Z$

Let $x = [n, m]$ and $y = [r, s]$ such that $n, m, r, s \in N$

$x +_Z y = [n, m] +_Z [r, s]$

$= [n + r, m + s]$ (def. of $+_Z$)

$= [r + n, s + m]$ (+ is comm. on N)

$= [r, s] +_Z [n, m]$ (def. of $+_Z$)

$= y +_Z x$

Proof3: T.P. $\exists 0 \in Z$ such that $x +_Z 0 = 0 +_Z x = x \quad \forall x \in Z$

Let $x = [n, m]$ and $0 = [r, s]$ such that $n, m, r, s \in N$

$x +_Z 0 = x \Longrightarrow [n, m] +_Z [r, s] = [n, m]$

$\Longrightarrow [n + r, m + s] = [n, m]$ (def. of $+_Z$)

$\Longrightarrow (n + r, m + s) \sim (n, m)$ ($[\mathrm{a}] = [\mathrm{b}] \Longrightarrow a \sim b$)

$\Longrightarrow n + r + m = m + s + n$ (def. of $\sim$)

$\Longrightarrow (n + m) + r = (n + m) + s$ (+ comm. and assoc)

$\Longrightarrow \mathrm{r} = s$ (By theorem 7.5(4))

$\therefore 0 = [r, s] = [r, r] = \{[1,1], [2,2], [3,3], \dots\}$

Similarly, $0 +_Z x = x$

$\therefore \exists 0 = [r, r] \ s.t. \ x +_Z 0 = 0 +_Z x = x \quad \forall x \in Z$

Proof4:

Let $x = [n, m]$, $y = [r, s]$ and $p = [h, k]$ such that $n, m, r, s, h, k \in N$

$x +_Z p = y +_Z p$

$\Leftrightarrow [n, m] +_Z [h, k] = [r, s] +_Z [h, k]$

$\Leftrightarrow [n + h, m + k] = [r + h, s + k]$ (def. of $+_Z$)

$\Leftrightarrow (n + h, m + k) \sim (r + h, s + k)$ ($[\mathrm{a}] = [\mathrm{b}] \Longrightarrow a \sim b$)

$\Leftrightarrow n + h + s + k = m + k + r + h$ (def. of ~)

$\Leftrightarrow n + s + (h + k) = m + r + (h + k)$ (+ comm. and assoc)

$\Leftrightarrow n + s = m + r$ (By theorem 7.5(4))

$\Leftrightarrow [n, m] \sim [r, s]$ (def. of ~)

$\Leftrightarrow [n, m] = [r, s]$ ($a \sim b \Longrightarrow [a] = [b]$)

$\Leftrightarrow x = y$

Proof5: T. P. $\forall x \in Z, \exists y \in Z \ s.t. x +_Z y = 0$

Let $x = [n, m]$ and $y = [r, s]$

Suppose $x +_Z y = 0$

$[n, m] +_Z [r, s] = [h, h]$

$[n + r, m + s] = [h, h]$ $(\text{def. of } +_Z)$

$(n + r, m + s) \sim (h, h)$ $([\mathrm{a}] = [\mathrm{b}] \Longrightarrow a \sim b)$

$n + r + h = m + s + h$ (def. of $\sim$)

$n + r = m + s$

$[n, m] \sim [s, r]$ (def. of $\sim$)

$[n, m] = [s, r]$ $(a \sim b \Longrightarrow [a] = [b])$

$x = -y = [s, r]$

$$\Longrightarrow \forall x \in Z, \exists y \in Z \ s.t. \ x +_Z y = [s, r] +_Z [r, s] = [s + r, r + s] = 0$$

Theorem 7.20: Properties of the multiplication on Z

1. $x ._Z \ y = y ._Z \ x \quad \forall x, y \in Z$ (**H. W.**)

2. $(x ._Z \ y) ._Z \ w = x ._Z (y ._Z \ w) \ \ \forall x, y, w \in Z$ (**H. W.**)

3. $\exists 1 \in Z$ such that $x ._Z \ 1 = 1 ._Z \ x = x \quad \forall x \in Z$

4. $x ._Z \ w = y ._Z \ w \Longleftrightarrow x = y \ \ \forall x, y, w \in Z$ (**H. W.**)

5. $x ._Z \ 0 = 0 ._Z \ x = 0 \quad \forall x \in Z$ (**H. W.**)

6. $x ._Z (y +_Z w) = x ._Z \ y +_Z x ._Z \ w \quad \forall x, y, w \in Z$

Proof3:

Let $x = [n, m]$ and $1 = [2,1] = [3,2] = \cdots$

T. P. $x ._Z \ 1 = 1 ._Z \ x = x \quad \forall x \in Z$

$x ._Z \ 1 = [n, m] ._Z [2,1]$

$= [2n + m, n + 2m]$ (def. of $._Z$)

$= [n + n + m, n + m + m]$

$= [n + m + n, n + m + m]$

$= [n + m, n + m] +_Z [n, m]$

$0 +_Z [n, m] = x$

Similarly, $1 ._Z \ x = x$ (**H. W.)**

Proof6: **T. P.** $x \cdot_Z (y +_Z w) = x \cdot_Z y +_Z x \cdot_Z w \quad \forall x, y, w \in Z$

Let $x = [n, m], y = [r, s]$ and $w = [p, q]$

$x \cdot_Z (y +_Z w) = [n, m] \cdot_Z ([r, s] +_Z [p, q])$

$= [n, m] \cdot_Z [r + p, s + q]$ (def. of $+_Z$)

$= [n.(r + p) + m.(s + q), n.(s + q) + m.(r + p)]$ (def. of $\cdot_Z$)

$= [nr + np + ms + mq, ns + nq + mr + mp]$

$= [(np + mq) + (nr + ms), (nq + mp) + (ns + mr)]$ (+ is comm. &asso.)

$= [np + mq, nq + mp] +_Z [nr + ms, ns + mr]$ (def. of $+_Z$)

$= ([n, m] \cdot_Z [p, q]) +_Z ([n, m] \cdot_Z [r, s])$ (def. of $\cdot_Z$)

$= (x \cdot_Z w) +_Z (x \cdot_Z y)$

Ordering on Z الترتيب على الاعداد الصحيحة

Let $x = [n, m]$ be an integer number then

1. x is called **positive integer number** if $m < n$.
2. x is called **negative integer number** if $-x$ is positive or if $n < m$.

Example 7.21: $x = [10, 3]$ is positive because $3 < 10$

$y = [1,5]$ is negative because $-y$ is positive (or because $1 < 5$)

Theorem 7.22: Let x and y are positive integer numbers then $x +_Z y$ is positive integer number

Proof: Let x be a positive integer number $\Longrightarrow \mathrm{x} = [n, m]\ s.t. m < n$ ....(1)

Let y be a positive integer number $\Longrightarrow \mathrm{y} = [r, s]\ s.t. s < r$ ....(2)

From (1)&(2) $\Longrightarrow m + s < n + r$ (by thm 7.8(4)) ...(3)

$x +_Z y = [n, m] +_Z [r, s] = [n + r, m + s]$ (def. of $+_Z$)

Since $m + s < n + r \Longrightarrow x +_Z y$ is positive integer number

Definition 7.23: Let x and y are two integer numbers such that $x = [n, m]$ and $y = [r, s]$. $y - x$ is positive integer number iff $x < y$

In other words,

$y - x = [r, s] - [n, m]$ is positive

$= [r, s] +_Z [m, n]$ $(-x = [m, n])$

$= [r + m, s + n]$ (def. of $+_Z$)

Since, $y - x$ is positive $\Longrightarrow \mathrm{s} + n < r + m$

Example 7.24: let $x = [2,3] = [n, m]$ and $y = [5,1] = [r, s]$. Is $\mathrm{x} < y$?

$s + n = 3 < 8 = r + m \Longrightarrow y - x$ is positive

$\Longrightarrow \mathrm{x} < y$

Example 7.25: let $x = [10, 5] = [n, m]$ and $y = [6,1] = [r, s]$

$s + n = 11 >$ 1=6-5 $\Longrightarrow$ x – y is positive

$\Longrightarrow$ y< x

Definition 7.26: Let x be an integer number. Then

1. x is a positive integer number $\Leftrightarrow x > 0$

2. x is a negative integer number $\Leftrightarrow x < 0$

3. $x \leq y$ means $x < y$ or $x = y$

4. $x \geq y$ means $x > y$ or $x = y$

Theorem 7.27: For each $x, y, p \in Z$

1. If $x > 0$ and $y > 0$ then $x \cdot_Z y > 0$ $(\mathbf{H. W.})$

2. If $x < 0$ and $y < 0$ then $x \cdot_Z y > 0$ **(H. W.)**

3. If $x < 0$ and $y > 0$ then $x \cdot_Z y < 0$

4. $x < y \Leftrightarrow x +_Z p < y +_Z p$

5. $x < y \Leftrightarrow x \cdot_Z p < y \cdot_Z p \quad \forall p \in Z^+$ **(H. W.)**

Proof3: Let $x = [n, m] < 0$, $y = [r, s] > 0$ T. P. $x \cdot_Z y < 0$

Since $x < 0 \Longrightarrow n < m \Longrightarrow m = n + k_1;\ k_1 \in N \ldots.(1)$

Since $y > 0 \Longrightarrow r > s \Longrightarrow r = s + k_2;\ k_2 \in N$ .....(2)

$x \cdot_Z$ y=$[n, m] \cdot_Z [r, s] = [nr + ms, ns + mr]$ (def. of $\cdot_Z$)

$= [n(s + k_2) + ms, ns + m(s + k_2)]$ (using (2))

$= [ns + nk_2 + ms, ns + ms + mk_2]$

$= [ns + nk_2 + (n + k_1)s, ns + (n + k_1)s + (n + k_1)k_2]$ (using (1))

$$= [ns + (n + k_1)s + nk_2,\ ns + (n + k_1)s + (n + k_1)k_2]$$

$$= ns + (n + k_1)s + nk_2 < ns + (n + k_1)s + nk_2 + k_1k_2$$

$\Rightarrow\ x \cdot_Z y < 0$

<u>Proof4</u>: $\Rightarrow$) Let $x = [n, m] < y = [r, s]$ T. P. $x +_Z p < y +_Z p$

Since $x < y \Rightarrow \mathrm{y} - \mathrm{x}$ is positive

$\Rightarrow [r, s] +_Z [m, n]$ is positive

$\Rightarrow [r + m, s + n]$ is positive

$\Rightarrow s + n < r + m$

$\Rightarrow \mathrm{r} + \mathrm{m} = \mathrm{s} + \mathrm{n} + \mathrm{k};\ \ \mathrm{k} \in \mathrm{N} \ldots\ldots (\mathbf{1})$

Let $p = [u, v]$; $u, v \in N$ T. P. $x +_Z p < y +_Z p$

T. P. $(y +_Z p) - (x +_Z p)$ is positive

T. P. $[r + u, s + v] - [n + u, m + v]$ is positive

T. P. $[r + u, s + v] +_Z [m + v, n + u]$ is positive

T. P. $[r + u + m + v, s + v + n + u]$ is positive

T. P. $\boldsymbol{r + u + m + v > s + v + n + u}$

From (1), $\mathrm{r} + \mathrm{m} = \mathrm{s} + \mathrm{n} + \mathrm{k};\ \ \mathrm{k} \in \mathrm{N}$

$\Rightarrow\ \mathrm{r} + \mathrm{m} + (\mathrm{u} + \mathrm{v}) = \mathrm{s} + \mathrm{n} + \mathrm{k} + (\mathrm{u} + \mathrm{v})$

$\Rightarrow (\mathrm{r} + \mathrm{u} + \mathrm{m} + \mathrm{v}) = (s + n + u + v) + k;\ \ k \in N$

$\Rightarrow\ \boldsymbol{r + u + m + v > s + v + n + u}$

$\Rightarrow\ x +_Z p < y +_Z p$

$\Leftarrow$) Suppose $x +_Z p < y +_Z p$ T. P. $x < y$

Since $x +_Z p < y +_Z p \Rightarrow\ r + u + m + v > s + v + n + u$

$$\Longrightarrow (\mathrm{r}+\mathrm{u}+\mathrm{m}+\mathrm{v}) = (s+n+u+v)+k;\quad k \in N$$

$$\Longrightarrow\ \mathrm{r}+\mathrm{m}+(\mathrm{u}+\mathrm{v}) = \mathrm{s}+\mathrm{n}+\mathrm{k}+(\mathrm{u}+\mathrm{v})$$

$$\Longrightarrow\quad \mathrm{r}+\mathrm{m} = \mathrm{s}+\mathrm{n}+\mathrm{k}$$

$$\Longrightarrow \mathrm{r}+\mathrm{m} > s+n$$

$\Longrightarrow [r+m, s+n]$ is positive

$\Longrightarrow [r,s] +_Z [m,n]$ is positive

$\Longrightarrow$ $\mathrm{y}-\mathrm{x}$ is positive $\Longrightarrow \mathrm{x} < y$

Embedding: الغمر

Let $(M,*,\#,R_1)$ and $(L,*',\#',R_2)$ are two number systems with the T.O.R relation R_1, R_2. We say that

$(M,*,\#,R_1)$ is **embedded مغمورة** in $(L,*',\#',R_2)$ if and only if there exist $\alpha: M \to L$ such that α is 1-1 and the following conditions hold: For all $n, m \in M$

1. $\alpha(n*m) = \alpha(n) *' \alpha(m)$

2. $\alpha(n\#m) = \alpha(n)\#'\alpha(m)$

3. $n\ R_1\ m \iff \alpha(n) R_2\, \alpha(m)$

Theorem 7.28: $(N \cup \{0\}, +, ., \leq)$ is embedded in $(Z, +, ., \leq)$

Proof: Define $\alpha: N \cup \{0\} \to Z$ s.t. $\alpha(n) = [n, 0] \quad \forall n \in N$

1. T. P. α is 1-1

Let $\alpha(n) = \alpha(m)$ T. P. $n = m$

$[n,0] = [m,0] \Longrightarrow (\mathrm{n},0) \sim (\mathrm{m},0)$

$\Longrightarrow n+0=m+0 \Longrightarrow \mathrm{n}=\mathrm{m} \Longrightarrow$ α is 1-1

2. T. P. $\alpha(n+m)=\alpha(n)+\alpha(m)$

$\alpha(n+m)=[n+m,0]=[n,0]+_Z[m,0]$ (def. of $+_Z$)

$= \alpha(n)+\alpha(m)$ (def. of α)

3. T. P. $\alpha(n.m)=\alpha(n).\alpha(m)$

$\alpha(n.m)=[n.m,0]=[n,0]._Z\ [m,0]$ (def. of $._Z$)

$= \alpha(n).\alpha(m)$ (def. of α)

4. T. P. $n<m \Leftrightarrow \alpha(n)<\alpha(m)$

$n<m \Leftrightarrow n+0<m+0$

$\Leftrightarrow [n,0]<[m,0]$

$\Leftrightarrow \alpha(n)<\alpha(m)$

$\therefore$ $(N\cup\{0\},+,.,\leq)$ is embedded in $(Z,+,.,\leq)$