الفصل الأول

التحليل العقدي

Definition (A Complex Number)

A Complex number Z is defined as an order pair (x,y) of real numbers x,y. It can be written in the form $Z = x + iy$, $i^2 = -1$.

The real number x is said to be the real Component (real part).

The real number y is said to be the imaginary Component (The Cofficient of the imaginary part), and i is Called imaginary unit, $Re Z = x$, $Im Z = y$.

Remarks

(1) The Complex numbers includs all the real numbers. Since if x is real number, then $x = x + 0.i$.

$Re(x) = x$, $Im(x) = 0$.

(2) The imaginary unit i is Complex number, $i = 0 + 1.i$

$Re(i) = 0$, $Im(i) = 1$

(3) If Z is Complex number s.t. $Re Z = 0$ and $Im(Z) \neq 0$

Then z is called Pure imaginary number.

(4) let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then $z_1 = z_2$ iff $x_1 = x_2$ and $y_1 = y_2$.

(5) let $z = x + iy$. Then $z = 0$ iff $x = y = 0$.

Now, let $\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(a,b) : a,b \in \mathbb{R}\}$

$$= \{a + ib : a,b \in \mathbb{R} , i^2 = -1\}.$$

Fundemental operations on $\mathbb{C}$: العمليات الأساسية على المجموعة $\mathbb{C}$.

let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

The sum of z_1 and z_2 is the complex number difined as follows: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. But $-z_2 = -x_2 - iy_2$, thus the deference $z_1 - z_2$ is defined as follows $z_1 - z_2 = z_1 + (-z_2) = (x_1 - x_2) + i(y_1 - y_2)$.

The product of z_1 and z_2 is the complex number defined as follows: $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

Now, $z_2^{-1} = \left(\frac{x_2}{x_2^2 + y_2^2} , \frac{-y_2}{x_2^2 + y_2^2}\right)$. Thus the division $\frac{z_1}{z_2}$ is defined as follows:

$$\frac{Z_1}{Z_2} = Z_1 \cdot Z_2^{-1} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right) \quad ; Z_2 \neq 0$$

From all these, we obtained $(\mathbb{C}, +, \cdot)$ be a field which is called the field of complex numbers. And $(1,0)$ is the identity with respect $\cdot$ on $\mathbb{C}$.

Some properties on operations بعض الخواص على العمليات

(1) Commutative property $Z_1 + Z_2 = Z_2 + Z_1$, $Z_1Z_2 = Z_2Z_1$

(2) associative property $(Z_1 + Z_2) + Z_3 = Z_1 + (Z_2 + Z_3)$, $(Z_1Z_2)Z_3 = Z_1(Z_2Z_3)$

(3) Distributive property $Z_1 \cdot (Z_2 + Z_3) = Z_1Z_2 + Z_1Z_3$

(4) $0 = (0, 0)$ is the unique aditi. identity and,

$1 = (1, 0)$ " " " a mutiple. identity.

(5) $\forall\ Z = (x, y)$, $\exists$ a unique aditive inverse which is $-Z = (-x, -y)$

(6) $\forall\ Z \neq 0$, $\exists\ Z^{-1}$ s.t. $ZZ^{-1} = 1$ and Z^{-1} is called the mutiplica. inverse,

(7) $\frac{Z_1}{Z_2} = Z_1Z_2^{-1}$, $Z_2 \neq 0$

(8) $\frac{1}{Z_1Z_2} = \left(\frac{1}{Z_1}\right)\left(\frac{1}{Z_2}\right)$; $Z_1 \neq 0$, $Z_2 \neq 0$, $Z_1Z_2 \neq 0$

(9) $\dfrac{z_1+z_2}{z_3} = \dfrac{z_1}{z_3} + \dfrac{z_2}{z_3}$; $z_3 \neq 0$ and

$$\frac{z_1 z_2}{z_3 z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) \; ; \; z_3 \neq 0 ,\; z_4 \neq 0 ,\; z_3 z_4 \neq 0$$

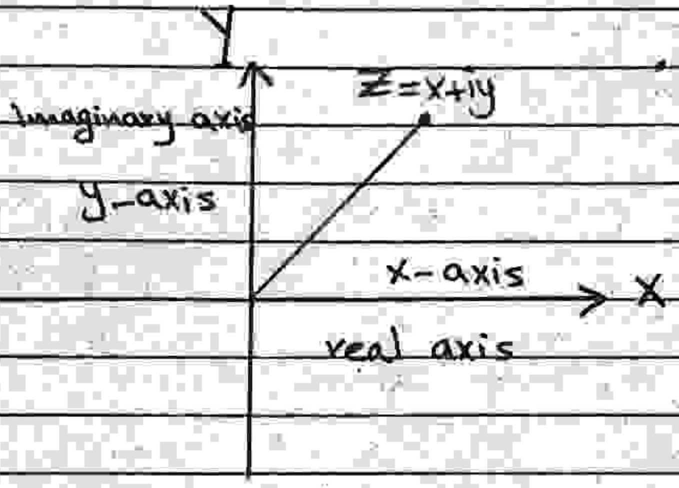
(10) If $z_1 z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$

Geometric Representation of $\mathbb{C}$ — التمثيل الهندسي للأعداد العقدية

Complex numbers can be represented as points in the plane, using the correspondence $x+iy \longleftrightarrow (x,y)$. The represention is known as the Argand diagram or complex plane. The real complex numbers lie on the x-axis, which is then called the real axis, while the imaginary numbers lie on the y-axis, which is known as the imaginary axis. The complex numbers with positive imaginary part lie in the upper half plane, while those with negative imaginary part lie in the lower half plane.

اي كل عدد عقدي $Z=x+iy$ يقابله نقطة واحدة في المستوي xy وهي النقطة (x,y).
وكذلك يمكن اعتبار العدد العقدي Z هو قطعة مستقيم

متجه من نقطة الاصل الى النقطة (x,y) ويسمى المستوي بالمستوي العقدي حيث يمثل فيه x الاحداثي الحقيقي و y المحور الخيالي .



Z plane or xy plane

Example: The Complex number $Z = 1 - 2i$ Correspondence the point $(1,-2)$ in the Z plane and the Complex number $Z = 0 + 0i = (0,0)$ Correspondence the origin point.

Now, we have the following examples:

Example(1): Find the value of the following problem:

$$\frac{(1+i)(-1+2i)+(2-i)}{2-3i} - 2i$$

Solution: $\frac{(1+i)(-1+2i)+(2-i)}{(2-3i)} - 2i = \frac{(-3+2)}{2-3i} - 2i = \frac{-1}{2-3i} - 2i$

$$= \frac{-1-2i(2-3i)}{2-3i} = \frac{-1-4i-6}{2-3i} = \frac{-(7+4i)}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{-(14-12)-(8i+21i)}{4+9}$$

$$= \frac{-2}{13} - \frac{29}{13} i$$

Example (1) : Find the value of $\left(\frac{1}{2-3i} \cdot \frac{1}{1+i}\right)$

Solution :

let $Z_1 = 2-3i$ and $Z_2 = 1+i$

Then we want to find $\frac{1}{Z_1} \cdot \frac{1}{Z_2} = Z_1^{-1} \cdot Z_2^{-1}$

Now, $Z_1^{-1} = \left(\frac{2}{13} + \frac{3}{13} i\right) \cdot \left(\frac{1}{2} - \frac{1}{2} i\right) = \frac{5}{26} + \frac{1}{26} i$

Example (2) : Show that if the product $Z_1 Z_2$ is Zero then so is at least one of the factors Z_1 and Z_2 .

Solution: let $Z_1 Z_2 = 0$ and $Z_1 \neq 0$ where

$Z_1 = (x_1, y_1)$, $Z_2 = (x_2, y_2)$

(Since $\because Z_1 \neq 0$, then either $x_1 \neq 0$ or $y_1 \neq 0$)

we take $x_1 \neq 0$

Now, $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2 , x_1 y_1 + x_2 y_1) = (0, 0)$

$\Leftarrow x_1 x_2 - y_1 y_2 = 0$ and $x_1 y_2 + x_2 y_1 = 0 \rightarrow$ ②

Then $x_1 x_2 - y_1 y_2 = 0 \Longrightarrow x_2 = \frac{y_1 y_2}{x_1} \Longrightarrow$ by subsit in ② ,

we get $x_1 y_2 + \frac{y_1 y_2}{x_1} y_1 = 0 \Longrightarrow x_1 y_2 + \frac{y_1^2 y_2}{x_1} = 0$

$\Longrightarrow \frac{y_2 (x_1^2 + y_1^2)}{x_1} = 0 \Longrightarrow y_2 = 0$ and from ① we obtian

$x_2 = 0$ and hence $z_2 = 0$

Problems : (H.W)

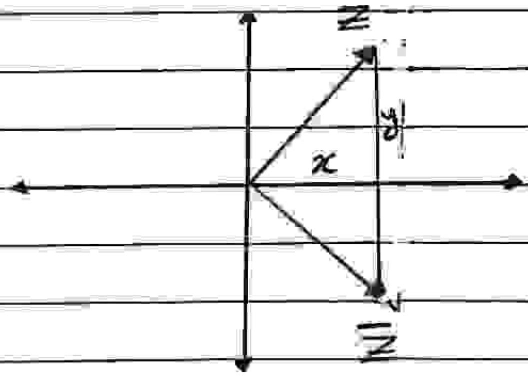
1. Solve the equation $z^2 + z + 1 = 0$

2. Show that $\dfrac{5}{(1-i)(2-i)(3-i)} = \frac{1}{2} i$

Complex Conjugate المرافق العقدي

Definition If $z = x + iy$, the complex conjugate of z is the complex number defined $\bar{z} = x - iy$.

Geometrically, the complex conjugate of z is obtained by reflecting z in the real axis, see the following figure.



The following properties of the complex conjugate are easy to verify :

1. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$;

(2) $\overline{-Z} = -\overline{Z}$ and $\overline{\overline{Z}} = Z$

(3) $\overline{Z_1 - Z_2} = \overline{Z}_1 - \overline{Z}_2$

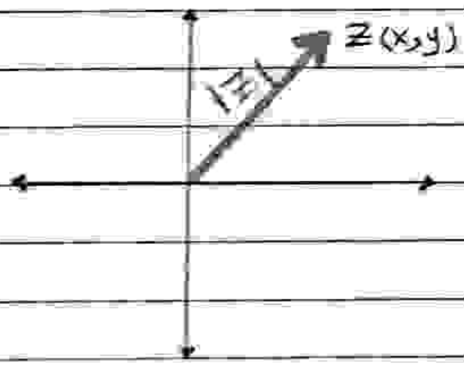
(4) $\overline{Z_1 Z_2} = \overline{Z}_1 \, \overline{Z}_2$

(5) $\overline{\left(\frac{1}{Z}\right)} = 1/\overline{Z}$

(6) $\overline{(Z_1 / Z_2)} = \overline{Z}_1 / \overline{Z}_2$

(7) $\forall \; Z \in \mathbb{C}$, then $Re(Z) = \frac{Z + \overline{Z}}{2}$, $Im(Z) = \frac{Z - \overline{Z}}{2i}$

Definition 3 (القيمة المطلقة للعدد Z) : The modulus or absolute value of Complex number $Z = x + iy$ is defined as $|Z| = \sqrt{x^2 + y^2}$



Geometrically, هندسياً

The number $|Z|$ is the distance between the point (x, y) and the origin.

More generally, $|Z_1 - Z_2|$ is the distance between Z_1 and Z_2 in the Complex plane. For

$$|Z_1 - Z_2| = |(x_1 + iy_1) - (x_2 + iy_2)| = |(x_1 - x_2) + i(y_1 - y_2)|$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} .$$

The following properties of the modulus بعض الخصائص الجبرية

1. $|z|^2 = |Re(z)|^2 + |Im(z)|^2$

2. $|z|^2 = z.\bar{z}$

3. $|z_1 z_2| = |z_1||z_2|$

4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$; $z_2 \neq 0$

5. $|z| = |\bar{z}| = |-z|$

ملاحظة/ سوف نستخدم في برهان الخاصية ③ و ④ و ⑤ العلاقة $|z|^2 = z.\bar{z}$

Proof (3) :

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) \quad \text{by } ②$$

$$= (z_1 . z_2)(\bar{z}_1 . \bar{z}_2)$$

$$= (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 . |z_2|^2$$

Then, we obtain بأخذ الجذر التربيعي للطرفين

$|z_1 z_2| = |z_1||z_2|$ because $|z|$ is non-negative real no.

proof (4)

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{z_1}{z_2}\right)\left(\frac{\bar{z}_1}{\bar{z}_2}\right) = \frac{z_1 . \bar{z}_1}{z_2 . \bar{z}_2} = \frac{|z_1|^2}{|z_2|^2}$$

بجذر الطرفين نحصل على

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Proof (5) :

$$|-Z| = \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} = |Z|$$

$$|\bar{Z}| = \sqrt{x^2 + y^2} = |Z|$$

The Inquality relations   علاقات التراجح

1. $|Z| \geq 0$

2. $Re(Z) \leq |Re(Z)| \leq |Z|$ and $Im(Z) \leq |Im(Z)| \leq |Z|$.

3. $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$ and it's generalization

$$\left|\sum_{k=1}^{n} Z_k\right| \leq \sum_{k=1}^{n} |Z_k|$$

4. $\big||Z_1| - |Z_2|\big| \leq |Z_1 + Z_2|$

Proof (3) : To prove triangle inquality (للتراجح المثلثية)

$$|Z_1 + Z_2|^2 = (Z_1 + Z_2)\overline{(Z_1 + Z_2)} = (Z_1 + Z_2)(\bar{Z}_1 + \bar{Z}_2)$$

$$= |Z_1|^2 + |Z_2|^2 + Z_2\bar{Z}_1 + Z_1\bar{Z}_2$$

$$= |Z_1|^2 + |Z_2|^2 + Z_1\bar{Z}_2 + \overline{Z_1\bar{Z}_2}$$

$\left(\text{since } Re\,Z = \frac{Z + \bar{Z}}{2}\right)$ then $= |Z_1|^2 + |Z_2|^2 + 2Re(Z_1\bar{Z}_2)$

$$\leq |Z_1|^2 + 2|Z_1\bar{Z}_2| + |Z_2|^2 = |Z_1|^2 + 2|Z_1||\bar{Z}_2| + |Z_2|^2$$

$$= |Z_1|^2 + 2|Z_1||Z_2| + |Z_2|^2$$

بأخذ الجذر التربيعي للطرفين نحصل على المطلوب $= (|Z_1| + |Z_2|)^2$

Problems

1. Show that :

a. $\overline{iz} = -i\bar{z}$

b. $\overline{\left(\frac{(2+i)^2}{3-4i}\right)} = 1$

c. $|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}\,|2z+5|$

2. prove that

a. $\overline{iz} = -i\bar{z}$

b. z is real iff $\bar{z} = z$.

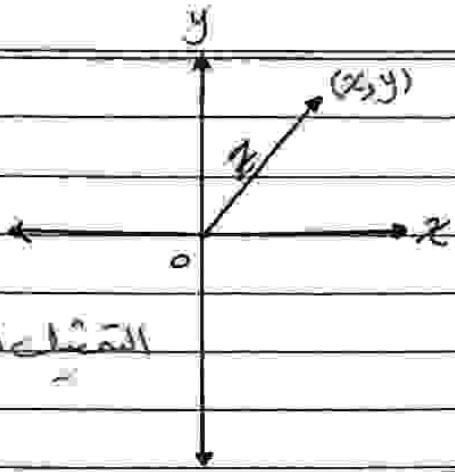
The Polar Form for the Complex Number

الصيغة القطبية للعدد المركب .

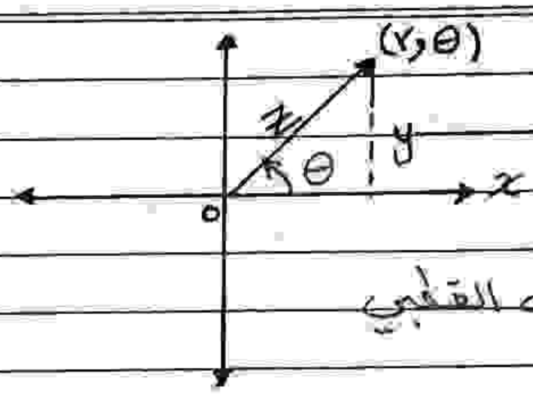
let r, θ be the polar coordinates of the point representing z , where $r > 0$:

$x = r\cos\theta$, $y = r\sin\theta$ and the complex number z can be written as follows

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) \quad \text{——} \quad (1)$$



التمثيل الهندسي



التمثيل القطبي

where $r = |Z| = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$

The angle θ is <u>the argument</u> of Z denoted by

$\arg Z$, $\arg Z = \theta \mp 2k\pi$, $k = 0,1,2,\ldots$

But $\arg Z$ is multiple-valued because in equation (1)

$\sin\theta$, $\cos\theta$ are periodic with period 2π.

If $Z \neq 0$, there is just one value of θ in any

given interval (فترة) $\theta_0 \leq \theta \leq \theta_0 + 2\pi$.

If $Z = 0$, then $r = 0$ and θ is arbitrary

اختيارية

الزاوية للعدد المركب Z (argument Z)
وهي الزاوية الاعتيادية رمزها arg Z وهي زاوية ليست
وحيدة القيم لأن هذه القيم تختلف عن بعضها بمقدار دورة
كاملة أو مضاعفاتها فهي زاوية متعددة القيم

θ : هي الزاوية التي يصنعها العدد المركب مع المحور X وهي
ازاحة السعة Z عن المحور الحقيقي X.

Remark: There is no polar representation for the Complex number $Z = 0$.

Example

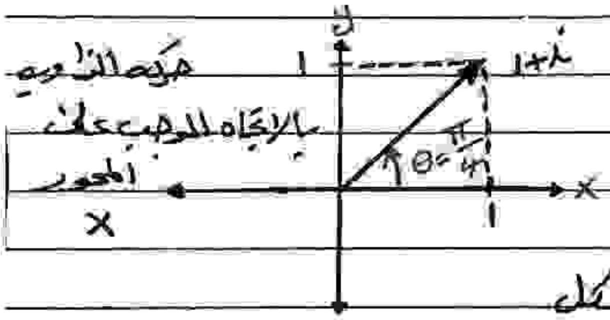
Find the polar form of the Complex number $Z = 1 + i$

Solution:

$$r = \sqrt{1+1} = \sqrt{2} \quad , \quad \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{1}{1} = \tan^{-1}(1)$$

Then $1 + i = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

يقع هذا العدد في الربع الأول



في حالة استخدام زوايا غير معلومة
أي قلنا إذا كان العدد العقدي بالشكل

$$Z = \sqrt{2} + 5i$$

وإن

$\theta = \tan^{-1}\frac{5}{\sqrt{2}}$ فلا نستخرج قيمة الزاوية وتبقى بهذا الشكل

Proposition

let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

(1) $Z_1 \cdot Z_2 = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$

(2) $\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$

Proof(2): $\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$

$$= \frac{r_1}{r_2} \frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\{(\cos\theta_2)^2 + (\sin\theta_2)^2\} = 1}$$

$$= \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$$

Definition Principle Argument of Complex Number

الزاوية الرئيسية للعدد العقدي

There is another angle for the Complex number which is Called the principle Argument and denoted by Arg Z. Since this angle is unique value or is one valued and is represented by the interval $(-\pi, \pi]$

That is, $-\pi < \operatorname{Arg} Z \leq \pi$

معنى الكلام اعلاه

ان العدد المركب يمتلك زاوية اخرى تدعى الزاوية الرئيسية. انها زاوية وحيدة القيمة والتي تتواجد في الفترة نصف المفتوحة $(-\pi, \pi]$.

Properties of argument

1. $\arg(1/Z) = -\arg Z$
2. $\arg(Z_1 \cdot Z_2) = \arg Z_1 + \arg Z_2$ (H.W)

3. $\arg(Z_1/Z_2) = \arg Z_1 - \arg Z_2$ (H.w)

4. $\arg(\bar{Z}) = -\arg Z$ هذه هي خواص الزاوية الاعتيادية وقد تنطبق بعض من هذه الخواص على الزاوية الرئيسية والبعض منها قد لا ينطبق.

Euler's Formula صيغة أويلر

Since $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

IF we take $x = i\theta$, then we have $e^{i\theta} = \cos\theta + i\sin\theta$

Thus $Z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ (Euler's Formula).

Proof (1) : $\arg(1/Z) = \arg\left(\frac{1}{re^{i\theta}}\right) = \arg\left(\frac{1}{r}e^{-i\theta}\right) = -\theta = -\arg Z$

Proof (4) : $\arg(\bar{Z}) = -\arg Z$.

let $Z = re^{i\theta} \Longrightarrow \bar{Z} = re^{-i\theta}$

$\therefore \arg(\bar{Z}) = \arg(re^{-i\theta}) = -\theta = -\arg Z$.

Example

Write the Complex number $Z = -1 - i$ by the polar Form and find Arg Z.

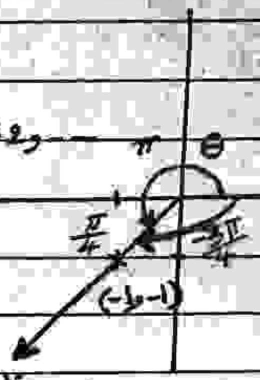
solution: $r = |Z| = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-1}{-1} = \tan^{-1} 1 = \frac{5\pi}{4}$

$\therefore -1-i=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

$\therefore \arg z=\frac{5\pi}{4}+2k\pi \qquad k=0,\mp1,\mp2,\dots$

$\therefore \operatorname{Arg}(-1-i)=\frac{5\pi}{4}-2\pi=\frac{-3\pi}{4}$



$\theta=\pi+\frac{\pi}{4}=\frac{4\pi+\pi}{4}=\frac{5\pi}{4}$

لاحظ

$\arg z=\operatorname{Arg} z+2k\pi$

$k=1$

$\operatorname{Arg} z=\arg z-2\pi=\frac{5\pi}{4}-2\pi=\frac{-3\pi}{4}$

Problems (H.W)

prove or disprove the following:

1. $\operatorname{Arg} z_1z_2 \neq \operatorname{Arg} z_1+\operatorname{Arg} z_2$

2. $\operatorname{Arg} \bar{z} \neq -\operatorname{Arg} z \quad ; \quad z\neq 0$

Powers and Roots of Complex Numbers

القوى والجذور للأعداد العقدية

هذا الموضوع يعتبر مهم جداً في ايجاد القوى والجذور العقدية خصوصاً عندما تكون القوى كبيرة القيمة ويصعب ايجادها بالحل الاعتيادي لذلك نلجأ الى حلها بالصيغة القطبية .

IF $z=re^{i\theta}$ is a non zero Complex number and n is any integer, then

$Z^n = (re^{i\theta})^n = r^n e^{in\theta}$ which is proved by mathematical iduction on n.

In Case $r=1$, then $Z^n = e^{in\theta}$ and by using Euler's formula, we get

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

This statement is Called De Moivre's theorem

هذه العلاقة تسمى مبرهنة دي موفر وهي تستخدم لإيجاد الجذور للأعداد العقدية بصورة عامة اي حساب جذور المعادلات التي من نوع $Z^n = 1$ والتي تكون لها n من الجذور

In general, the roots of the equation $Z^n = 1$ (where $Z \neq 0$ is a complex number and n is a positive integers) are Called the nth roots of unity)

ولإيجاد هذه الجذور نبدأ بالصيغة القطبية لـ Z^n وللعدد 1 اي ان

$$Z^n = 1 \Longrightarrow r^n e^{in\theta} = 1 \cdot e^{i0}$$

$$\Longrightarrow r^n = 1 \quad \text{and} \quad n\theta = 0 + 2k\pi$$

$$\therefore r = 1 \quad \text{and} \quad n\theta = 2k\pi \Longrightarrow \theta = \frac{2k\pi}{n} \ ; \ k = 0, 1, \ldots$$

let $w = e^{\frac{2\pi}{n} i} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

Then the roots are: $w^0, w^1, w^2, \ldots, w^n$

هندسياً

هذه الجذور هي رؤوس مضلع منتظم عدد اضلاعه
n مرسوم داخل دائرة الوحدة (اي دائرة مركزها نقطة الاصل
ونصف قطرها = 1).

Example: Find the roots of

$$Z = (i)^{1/4}.$$

<u>solution:</u>

$$Z = (i)^{1/4} \Longrightarrow Z^4 = i$$

put $Z = re^{i\theta} = 1 \cdot e^{i\frac{\pi}{2}}$

$\therefore r^4 = 1 \Longrightarrow r = 1$ and $4\theta = \frac{\pi}{2} + 2K\pi$, $K = 0, 1, 2, 3, ---$

$\therefore \theta = \frac{\pi}{8} + \frac{K\pi}{2}$; $K = 0, 1, 2, 3, ---$

$\therefore Z = 1 \cdot e^{i\left(\frac{\pi}{8} + \frac{K\pi}{2}\right)} = \cos\left(\frac{\pi}{8} + \frac{K\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{K\pi}{2}\right)$

if $K = 0 \Longrightarrow Z_1 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$

if $K = 1 \Longrightarrow Z_2 = \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}$

if $K = 2 \Longrightarrow Z_3 = \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}$

if $K = 3 \Longrightarrow Z_4 = \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}$.

problem: Find the solution of $Z^3 + 8 = 0$

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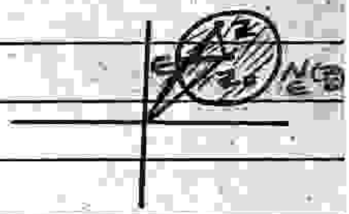
Regions in the Complex plane المناطق في المستوى العقدي

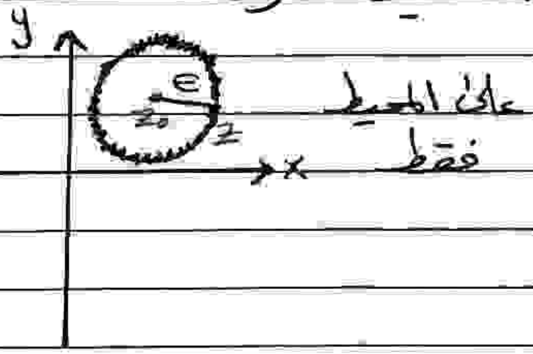
Definition let z_0 be a point in $\mathbb{C}$ and let $\epsilon > 0$ be a real number. Then

1. The set $N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$ is called a neighbourhood (جوار) of z_0, it is the set of all points inside a circle of radius ϵ and center at z_0 (but not on the circle) (اي ليس على محيط الدائرة)



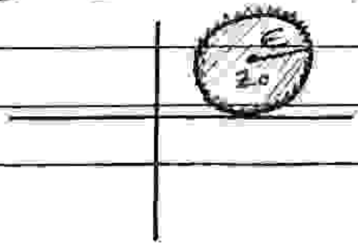
2. The set $C_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| = \epsilon\}$ is called a circle of radius ϵ and center at z_0.

تمثل مجموعة النقاط على محيط الدائرة



3. The set $D_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$ is called a disk of radius ϵ and center at z_0.

وهي تمثل مجموعة النقاط داخل الدائرة وعلى محيطها



Definition open set المجموعة المفتوحة

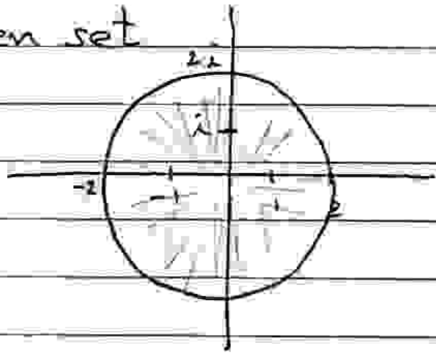
Let S be a subset of $\mathbb{C}$. Then S is called open set if for each point $z \in S$, there exists $N_\epsilon(z)$ s.t. $N_\epsilon(z) \subset S$.

Definition Closed Set المجموعة المغلقة

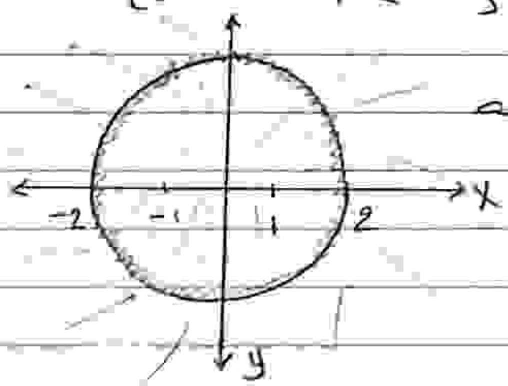
Let F be a subset of $\mathbb{C}$. F is called closed set if the Complement of F in $\mathbb{C}$ is an open set. That is, $F^c = \mathbb{C} - F$ is an open set in $\mathbb{C}$.

Examples

1. The set $\{z : |z| < 2\}$ is an open set.



2. The set $\{z : |z| \leq 2\}$ is closed set



المتممة
مجموعة
مفتوحة

3. The union of any collection of open sets in $\mathbb{C}$ is also an open set in $\mathbb{C}$. (H.W)

4. The union of a finite collection of closed sets in $\mathbb{C}$ is also a closed set in $\mathbb{C}$ (H.W)

5. The intersection of a finite (any) collection of open (closed) set in $\mathbb{C}$ is also open (closed) set in $\mathbb{C}$ (H.w)

6. The plane $\mathbb{C}$ and $\emptyset$ are always open and closed sets in the same time.

Definitions: let S be a subset of $\mathbb{C}$. Then

نقطة داخلية لـ S

1. let $z_0 \in \mathbb{C}$, z_0 is called an interior point for S, if there exists $N_\epsilon(z_0) \subset S$.

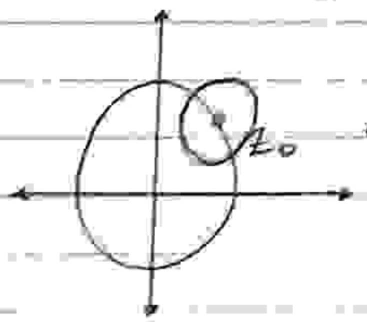
لاحظ كل نقطة داخلية تقع داخل S
ولكن ليس كل نقطة في S هي تعتبر داخلية بالنسبة لـ S

2. let $z_0 \in \mathbb{C}$, z_0 is called an exterior point for S

نقطة خارجية لـ S

if there exists $N_\epsilon(z_0)$ s.t $N_\epsilon(z_0) \cap S \neq \emptyset$

لاحظ ان كل نقطة خارجية تقع خارج S
ولكن ليست كل نقطة تقع خارج S هي خارجية "مثلاً"
بالنسبة لـ S



انظر z_0 ليست خارجية وهي
لا تنتمي الى S

Definition: let $z_0 \in \mathbb{C}$, z_0 is called a boundary point for S, if $\forall N_\epsilon(z_0)$, then $N_\epsilon(z_0) \not\subset S$ and

نقطة حدودية

$\underset{\epsilon}{N}(z_0) \cap S \neq \emptyset$. The set of all boundary points for S is called the boundary of S written by the symbol $B(S)$. That is,

$$B(S) = \{ z : \underset{\epsilon}{N}(z) \not\subset S \text{ and } \underset{\epsilon}{N}(z) \cap S \neq \emptyset \}$$

Remarks: let S be a subset of $\mathbb{C}$. Then

1. S is open set in $\mathbb{C}$ iff every point of S is an interior point.

2. S is closed set in $\mathbb{C}$ iff S contains all its boundary points. $\left\{ \begin{array}{l} \text{i.e. iff} \\ B(S) \subset S \end{array} \right\}$

3. let $z_0 \in \mathbb{C}$. Then z_0 is called an accumulation point (or limit point or cluster point) for S, if
   نقطة تجمع او تراكم

   $\forall \underset{\epsilon}{N}(z_0)$, $(\underset{\epsilon}{N}(z_0) \cap S) - \{z_0\} \neq \emptyset$

   اي يعني كل جوار حول النقطة z_0 يكون تقاطعه مع المجموعة S
   عدا النقطة z_0 يجب ان لا يساوي مجموعة خالية اي يكون
   الجوار يحوي نقطة واحدة فقط تختلف عن z_0 .

4. S is closed iff S contains all of its accumulation points
   (تكون S مغلقة اذا احتوت S على جميع نقاط تجمعها)

Remark There are sets which are not open and closed

هنا يعني ان هناك مجموعات تكون نقاط حدودية لها تقع داخل المجموعة فهي غير مفتوحة كما ان هناك نقاط حدودية تقع خارج المجموعة فبالتالي فهي غير مغلقة

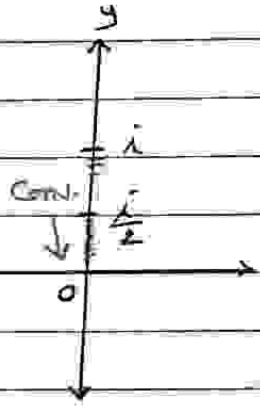
For example: $\{Z : 0 < |Z| \leq 1\}$

Example about accumulation point

let $S = \{Z_n \,;\, Z_n = i\frac{1}{n} \,;\, n = 1, 2, \ldots\}$

$= \{i, \frac{i}{2}, i\frac{1}{3}, \ldots\} \xrightarrow[\text{to Zero}]{\text{Convergent}} Z = 0$

$\therefore$ The only accu. pt. of S is $Z = 0$



Definition: A subset S of $\mathbb{C}$ is Called bounded, if $\exists$ a positive integer K such that s.t.
مقيده

$|Z| < K$ for each $Z \in S$.

اي ان S تكون مقيدة اذا استطعنا ان نجد عدد صحيح موجب K بحيث نحصر جميع نقاط المجموعة S داخل دائرة مركزها نقطة الاصل ونصف قطرها K.

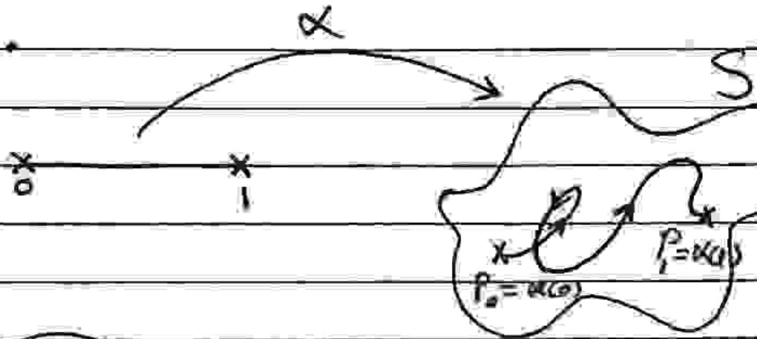
Definition let S be a subset of $\mathbb{C}$. S is said to be a Connected set (مجموعة متصلة) if for every two points, there exists a path (درب) joining them Contained entierly in $\mathbb{C}$.

يقال عن مجموعة انها متصلة اذا استطعنا ربط كل نقطتين من نقاط S بمسار او يسمى درب بحيث يقع ذلك المسار كلياً في S.

Definition: let $S \subseteq \mathbb{C}$, a path in S is a Continuous map $\alpha : [0,1] \to X$, $\alpha(0)$ is Called the initial point of the path, $\alpha(1)$ is Called the terminal point of the path.

نقطة بداية

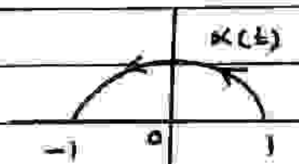
نقطة نهاية



عند التكلم عن path لا نستطيع كتابة صيغة للدالة ولكن يمكن رسمها دائماً بهذا الشكل

Example: let $\alpha(t) = (\cos \pi t, \sin \pi t)$
$= \cos \pi t + i \sin \pi t \quad 0 \le t \le 1$

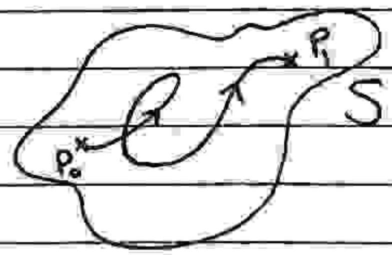
$\therefore \alpha : [0,1] \to \mathbb{C}$ s.t $\alpha(0) = 1$
$\alpha(1) = -1$



رسمت هكذا لان $\alpha(t)$ زاويته πt
اما اذا كانت $\frac{\pi}{2}$ فانها تصبح ربع دائرة.

متصلة دربياً

Definition: let $S \subseteq \mathbb{C}$, S is said to be path wise Connected if for each two points $P_0, P_1 \in S$, there exists a path α in S s.t. $\alpha(0) = P_0$, $\alpha(1) = P_1$



Example: $\mathbb{C}$ is path wise Connected

$\alpha(t) = (1-t)P_0 + tP_1 \quad 0 \le t \le 1$

يمثل خط مستقيم في $\mathbb{C}$

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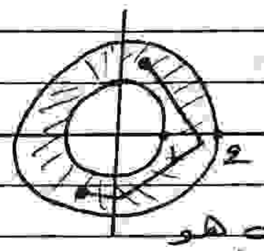
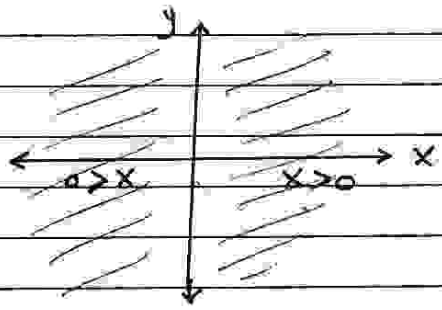
Examples: 1. The set $\{Z : 1<|Z|<2\}$ is a Connected set, since



ملاحظة / المسار أو الدرب هو
عبارة عن عدد من قطع المستقيمات المتصلة بالنهايات

2. The set $\{Z : \operatorname{Re} Z > 0 \cup \operatorname{Re} Z < 0\}$ is not Connected

محور السينات لم يدخل
اذن لا يمكن ان تكون
متصلة



Definitions: منطقة

1. A Connected open set is Called a domain
المجموعة المتصلة والمفتوحة تسمى منطقة — منطقة

2. A region in $\mathbb{C}$ is defined to be a domain with all or same of its boundary points or without of them

(المنطقة عبارة عن منطقة مع بعض نقاط الحدودية أو كلها أو بدونها)

Examples 1. $|Z|<1$ is a domain
2. $1<|Z|<2$ = =

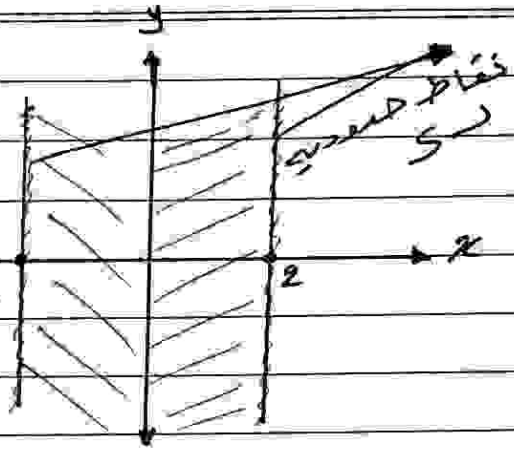
3. $\mathbb{R} \subset \mathbb{C}$ is not a domain, since $\mathbb{R}$ is Connected but not open.

4. $1 \le |Z| \le 2$ is not region, since it is closed but it is not Connected.

5. $S = \{ z \in \mathbb{C} : |\mathrm{Re}\, z| \le 2 \}$

S is closed

مغلقة لان تحوي جميع نقاطها الحدودية وغير مفتوحة



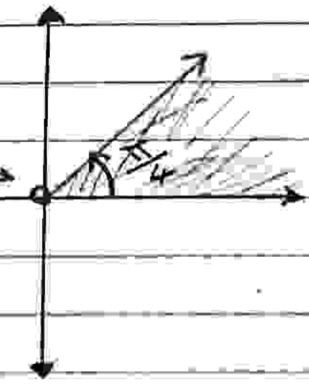
6. let $S = \{ z \in \mathbb{C} : |z| > 0 \ , \ 0 \le \arg z \le \frac{\pi}{4} \}$

since $z_0 = (0,0)$ and $\mathrm{Re}\, z_0 = 0$

$\because |z| > 0 \Rightarrow x^2 + y^2 > 0$

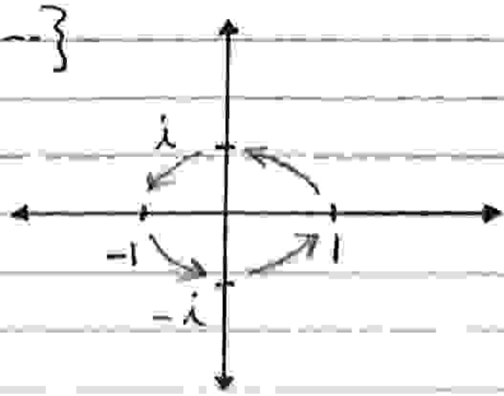
ليست منطقة (domain) لانها
ليست مفتوحة ولكنها متصلة

فجوة عند الصفر



7. let $S = \{ z : z_n = (i)^n \ , \ n = 1, 2, 3, \ldots \}$

<u>Solution</u>: if
$n=1 \longrightarrow z_1 = i$
$n=2 \longrightarrow z_2 = -1$
$n=3 \longrightarrow z_3 = -i$
$n=4 \longrightarrow z_4 = 1$
القيم تتكرر $\quad n=5 \longrightarrow z_5 = i$



نلاحظ مركزها منتظمة
لكنها غير تراكمية حول نقطة معينة

Definition let S be a subset of $\mathbb{C}$. The closure of S denoted by $\bar{S}$ and $\bar{S} = S \cup B(S)$.

Example let $S = \{z : |z| < 1\}$. Then $\bar{S} = S \cup B(S)$

$$\therefore \bar{S} = \{z : |z| < 1\} \cup \{z : |z| = 1\} = \{z : |z| \leq 1\}.$$

Problems: Draw each of the following sets, which of them, open, closed and domain.

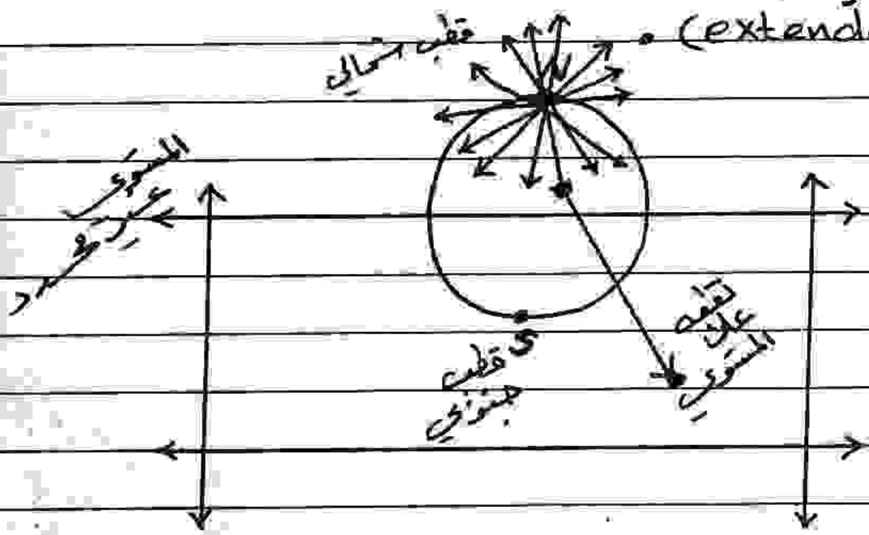
1. $\{z : |z-4| \geq |z|\}$
2. $\{z : 0 < |z - z_0| < \delta\}$
3. $\{z : \operatorname{Im} z > 1\}$
4. Find the closure of the set $\{z : |z| > 0, -\pi < \arg z < \pi\}$

The extended Complex plane
المستوي العقدي الموسع

ال (∞) تأتي في حقل الاعداد المركبة من كوننا نتحرك على المستوي وليس على خط الاعداد فان المستوي يمتد الى جميع الاتجاهات.

لنأخذ كرة لتمثل تطبيق من نقاط N (القطب الشمالي) الى نقاط S وهذا الاسقاط يسمى بالاسقاط المجسم (Stereographic projection)

ان الاسقاط من القطب الشمالي N بخط مستقيم ثم يخترق الكرة
بنقطة ويخرج من الكرة فيقابل نقطة على سطح المستوي.
بمعنى آخر: ان اي نقطة على سطح الكرة
يقابلها نقطة على سطح المستوي.
اما بالنسبة للنقطة (N) فان اي مستقيم يمر منها وفي اي اتجاه
يقابل نقطة بعيدة جداً تسمى (∞) اللانهاية.
بهذه الطريقة وسعنا حقل الاعداد
المركبة فاصبح المستوي يسمى مستوي
الاعداد المركبة الموسع $\mathbb{C} \cup \{\infty\}$
(extended complex plane).
كما تسمى هذه الكرة،
كرة ريمان
(Riemann sphere)



س/ هل يوجد جوار للـ (∞)
الجواب
نعم وهو

$$N_\epsilon(\infty) = \{z \in \mathbb{C} : |z| > \frac{1}{\epsilon}\}$$

(حيث $\epsilon > 0$)

اي ان جوار هذه النقطة هي النقاط التي تقع خارج القرص المغلق

$$\{z \in \mathbb{C} : |z| \leq \frac{1}{\epsilon}\}$$

س/ عند دخول (∞) الى المستوي هل يضيف صفة تبولوجية أو تحليلية الى الحقل

ج/ نعم والصفة هي ان تكون (غير مقيدة)

بصورة عامة: كل مجموعة غير مقيدة فان (∞) نقطة تراكمية لهذه المجموعة.

Functions of Complex Variable
دوال المتغير المركب

Def: let S be a set of Complex numbers. We say that f is a function on S, if f is a rule which assigns for each complex number z in S one and only one complex number w which is called the value of f at z written by $w = f(z)$. The set S is called the domain of f.

Example: let $f(z) = \frac{1}{z}$ defined on $S = \{z : \operatorname{Im} z > 1\}$

هنا قد ذكر المجموعة التي تكون الدالة معرفة عندها والتي تمثل (domain of f) اما اذا لم يذكر منطلق الدالة أو لا يعطي المجموعة S في السؤال معناه يجب ان نأخذ اكبر مجموعة ممكنة تتحرك فيه الدالة فمثلاً :-

① $f(z) = \frac{1}{z}$ $\qquad D_f = \mathbb{C} - \{0\}$

② $f(z) = z^2$ $\qquad D_f = \mathbb{C}$

③ $f(z) = \frac{1}{z-3}$ $\qquad D_f = \mathbb{C} - \{3\}$

Note :- If $w = f(z)$ (function of Complex variable) $z = x+iy$ and if $w = u+iv$, then each of u and v depends on the two real variables x and y.

That is, $w = f(z)$ Can be represent by the pair of real functions with real variables.

$$f(z) = u(x,y) + iv(x,y)$$

For examples: $f(z) = \frac{1}{z}$, $|z| > 0$

يعني $|z|$ هو طول المتجه وهو عدد حقيقي غير سالب وان z هي ليست نقطة الاصل.

$$\therefore f(z) = z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$\therefore u(x,y) = \frac{x}{x^2+y^2} \qquad v(x,y) = \frac{-y}{x^2+y^2}$$

② $w = f(z) = z^2 \longrightarrow w = (x+iy)^2 = (x^2 - y^2) + 2ixy$

$u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$

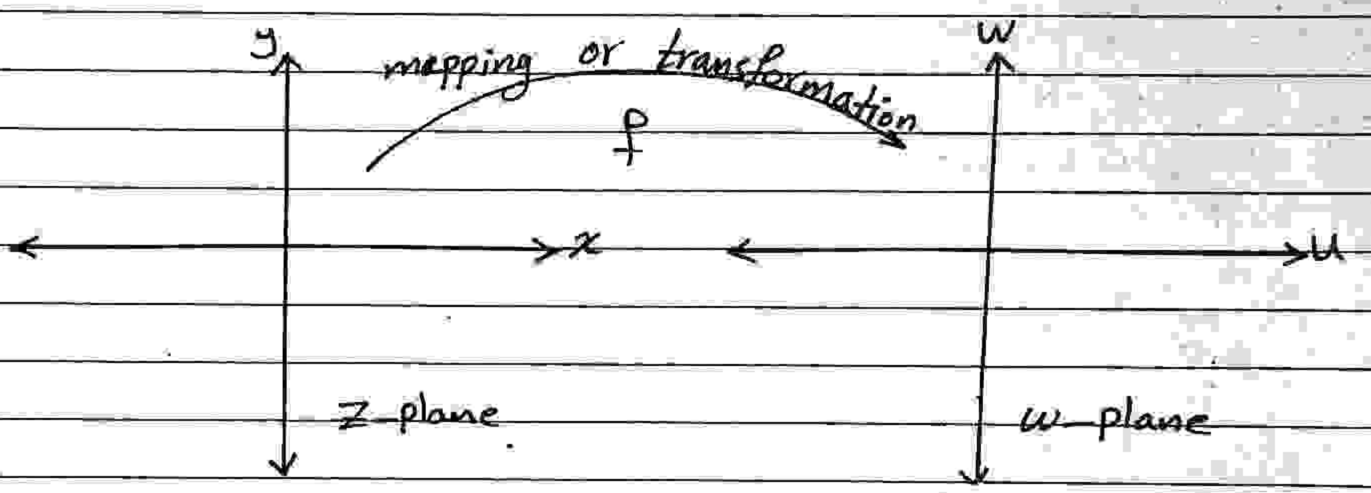
ملاحظة : في حقل الاعداد العقدية لا يمكن رسم الدوال لان الاعداد في مستويين

That is, $w = f(z)$
$z = (x,y) \in \mathbb{R} \times \mathbb{R}$

$w = (u, v) \in \mathbb{R} \times \mathbb{R}$

$(z, w) = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

وحلاً لهذه المشكلة نرسم المستويين ، كل مستوى وحده
ويحدد النقطتين w و z ولهذا تطلق على الدالة العقدية
$w = f(z)$ تطبيقاً أو تحويلاً mapping أو transformation ونقوم
برسم صور المنحنيات والمناطق بدلاً من رسم صورة النقاط .



Definitions :-

1. A function $w = f(z)$ is Called a single valued function in its domain S, if for each $z \in S$ there is only one value for $f(z)$. For example $f(z) = z^3 + 3$

2. A function $w = f(z)$ is Called multiple valued function in its domain S, if for each $z \in S$ there are many values of $f(z)$. For example $f(z) = z^{\frac{1}{3}}$.

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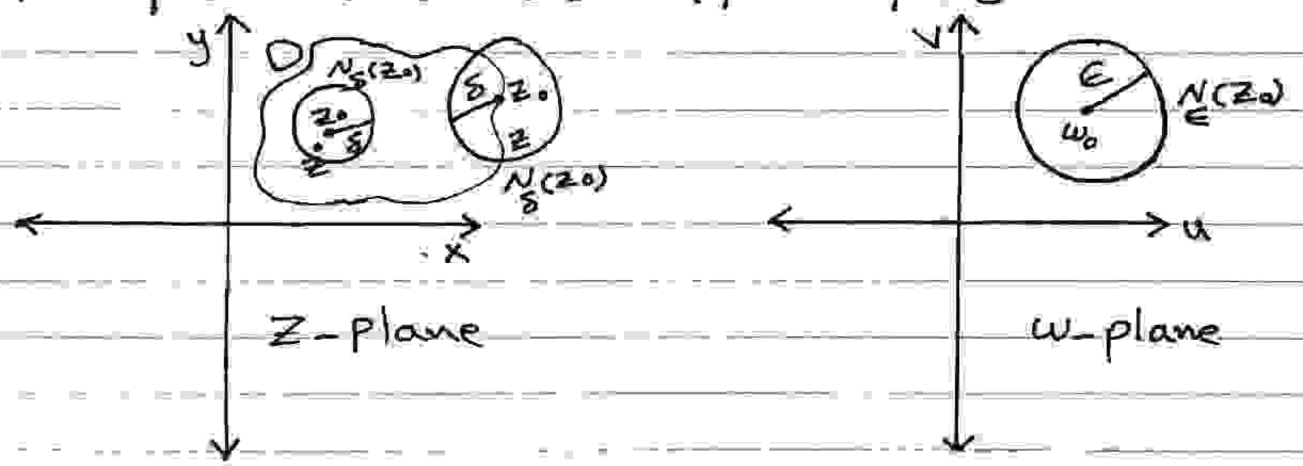
limit of a function in Complex variable

"الغايـة"

Def :- let f be a function defined on a region D and let z_0 be a point in D or on its boundary. we say that the limit of f at z_0 is the number w_0 (written by $\lim_{z \to z_0} f(z) = w_0$) if for each real number $\delta > 0$ such that

$|f(z) - w_0| < \epsilon$ whenevers $0 < |z - z_0| < \delta$.



ملاحظه / في التعريف اعلاه نلاحظ ان اقتراب z من z_0 ليس له اتجاه معين.

Example(1): Prove that by using the definition of limit $\lim_{z \to (1-i)} (x + i(2x+y)) = 1+i$

Solution: Given $\epsilon > 0$, to find $\delta > 0$ s.t

$$|x+i(2x+y)-(1+i)| < \epsilon \quad \text{when} \quad 0 < |Z-(1-i)| < \delta$$

$$|x+i(2x+y)-(1+i)| = |(x-1)+i(2x+y-1)|$$

$$\leqslant |x-1| + |2x+y-1| = |x-1| + |2x-2+y+1|$$

$$\leqslant |x-1| + 2|x-1| + |y+1| = 3|x-1| + |y+1| < \epsilon$$

if $3|x-1| < \frac{\epsilon}{2}$ and $|y+1| < \frac{\epsilon}{2}$

$\therefore |x-1| < \frac{\epsilon}{6}$ and $|y+1| < \frac{\epsilon}{2}$

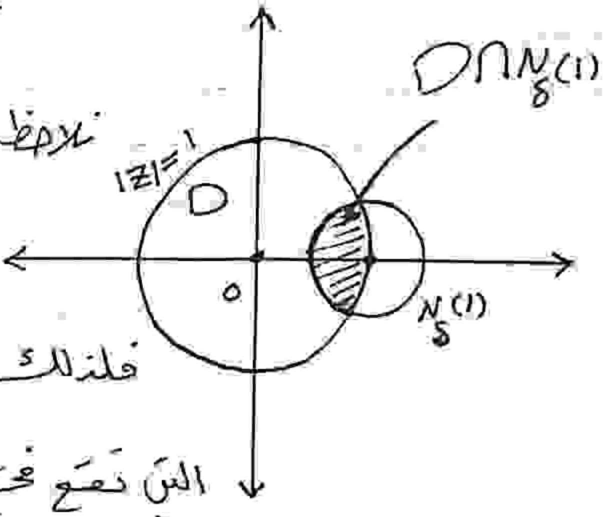
Now, $|Z-(1-i)| = |x+iy-1+i| = |(x-1)+i(y+1)|$

$$\leqslant |x-1| + |y+1| = \frac{\epsilon}{6} + \frac{\epsilon}{2} = \frac{4\epsilon}{6} = \frac{2}{3}\epsilon$$

we choose $\delta = \frac{2}{3}\epsilon$

<u>Example (2):</u> If $f(Z) = \frac{iZ}{2}$ is defined on $|Z| < 1$.

Prove that $\lim\limits_{Z \to 1} f(Z) = \frac{i}{2}$



نلاحظ هنا ان النقطه $Z_0 = 1$ هي نقطه

حدودية بالنسبة للمنطقة $|Z| < 1$

فلذلك سوف نأخذ في التعريف جميع النقاط

التي تقع في منطقة التقاطع .

Solution:

Choose $\epsilon > 0$, to find a real number $\delta > 0$

let $|f(z) - w_0| = \left|\frac{iz}{2} - \frac{i}{2}\right| = \left|\frac{i(z-1)}{2}\right| = \frac{|i||z-1|}{2} = \frac{|z-1|}{2}$

$< \frac{2\epsilon}{2} = \epsilon$

نأخذ $\delta = 2\epsilon$

$\therefore$ we take $\delta = 2\epsilon$ implies that $|f(z) - f(1)| < \epsilon$

for all z in $|z-1| < 2\epsilon$

$$\therefore \lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$$

Exercises:- (1) Prove that $\lim_{z \to i} z^2 = -1$.

(2) Prove that $\lim_{z \to 2i} (2x + y^2) = 4i$

In Ex ① and Ex ② use the definition of limit.

Example: Show that $\lim_{z \to 0} f(z)$ does not exists where $f(z) = \frac{\bar{z}}{z}$

Solution:- $\lim_{z \to z_0} \frac{\bar{z}}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - iy}{x + iy}$

اذا كان الاقتراب خلال المحور الحقيقي (أي $y = 0$)

$$\Rightarrow \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{x} = 1$$

اذا كان الاقتراب خلال المحور العمودي (أي $x = 0$)

$\therefore \lim_{\substack{x \to 0 \\ y \to 0}} \frac{-iy}{iy} = -1$ نلاحظ ان قيمة الغاية غير متساوية فالغاية غير موجودة

Limit in the extended Plane

Def: The set $N_\epsilon(\infty) = \{ z \in \mathbb{C} : |z| > \frac{1}{\epsilon} \}$ is called a neighborhood of the point at infinity. That is, $N_\epsilon(\infty)$ is the set of all points in $\mathbb{C}$ which are outside the disk $|z| \leq \frac{1}{\epsilon}$.

Def:

(1) $\lim_{z \to z_0} f(z) = \infty$ iff $\lim_{z \to z_0} \frac{1}{f(z)} = 0$. That means for each positive real number k (however k is large), there exists a real number $\delta > 0$ such that

$|f(z)| > k$ whenever $0 < |z - z_0| < \delta$.

Example: Prove that $\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty$

Solution:

by above definition, we have to show that

$\lim_{z \to 1} (z-1)^3 = 0$

let $\epsilon > 0$ be a real number. To find a real number $\delta > 0$ such that $|(z-1)^3 - 0| < \epsilon$ whenever $0 < |z-1| < \delta$

$|(z-1)^3| = |z-1|^3 < \epsilon$ if $0 < |z-1| < \sqrt[3]{\epsilon}$

Choose $\delta = \sqrt[3]{\epsilon} \Rightarrow \lim_{z \to 1} (z-1)^3 = 0$

and hence $\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty$.

2. $\lim_{z \to \infty} f(z) = w_0$ iff $\lim_{z \to 0} f(\frac{1}{z}) = w_0$. That means, $\forall \epsilon > 0$, $\exists \delta > 0$ s.t $|f(z) - w_0| < \epsilon$ whenever $|z| > \frac{1}{\delta}$. (where ϵ, δ are real numbers).

Example : Show that $\lim_{z \to \infty} \frac{1}{z^2} = 0$

Solution :

Given $\epsilon > 0$ be a real number. To find a real number $\delta > 0$ s.t. $|f(\frac{1}{z}) - 0| < \epsilon$ whenever $0 < |z - 0| < \delta$

$|f(\frac{1}{z})| = |\frac{1}{\frac{1}{z^2}}| = |z^2| = |z|^2 = |z||z| < \epsilon$ if $|z| < \sqrt{\epsilon}$.

we choose $\delta = \sqrt{\epsilon} \Rightarrow |\frac{1}{z^2} - 0| < \epsilon$ when $|z| > \frac{1}{\sqrt{\epsilon}} = \frac{1}{\delta}$

3. $\lim_{z \to \infty} f(z) = \infty$ iff $\lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$. That is, for each positive real number K (however it is large), there exists a positive real number δ s.t. $|f(z)| > K$ whenever $|z| > \frac{1}{\delta}$.

Example : Prove that $\lim_{z \to \infty} z^2 = \infty$

proof : by definition (3)

since $\frac{1}{f(\frac{1}{z})} = \frac{1}{\frac{1}{z^2}} = z^2$. To find $\lim_{z \to 0} z^2 = 0$

Given $K > 0$ a real number and positive . To find

a positive real no. δ

let $|f(z)| > K \Rightarrow |z^2| > K \Rightarrow |z|^2 > K \Rightarrow |z| > \sqrt{K}$

when $|z| > \frac{1}{\delta}$

Choose

$\delta = \sqrt{K} \Rightarrow |z^2| > K$ when $|z| > \frac{1}{\sqrt{K}} = \frac{1}{\delta}$.

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Some Properties of limits بعض خواص الغايات

Theorem (1): let $w = f(z)$ is defined on a region D and z_0 in D or on the ∂D (boundary of D) s.t. $\lim_{z \to z_0} f(z)$ exists. Then this limit is unique.

Proof: suppose that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = w_1$ and $w_0 \neq w_1$.

let $\epsilon = \frac{1}{2}|w_1 - w_0|$ ----→ ①

Since $\lim_{z \to z_0} f(z) = w_0 \Rightarrow \exists\, \delta > 0$ s.t. $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Since $\lim_{z \to z_0} f(z) = w_1 \Rightarrow |f(z) - w_1| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

But $|w_1 - w_0| = |w_1 - f(z) + f(z) - w_0|$

$$\leq |w_1 - f(z)| + |f(z) - w_0|$$

$$< \epsilon + \epsilon = \frac{1}{2}|w_1 - w_0| + \frac{1}{2}|w_1 - w_0|$$

$$= |w_1 - w_0|$$

$\therefore |w_1 - w_0| < |w_1 - w_0|$ C!

$\therefore w_1 = w_0 \Rightarrow$ The limit is unique (if it exists)

Theorem (2) :- let $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then $\lim\limits_{z \to z_0} f(z) = w_0$ iff $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

Proof :- $\Longrightarrow$) Assume that $\lim\limits_{z \to z_0} f(z) = w_0$

$\therefore \forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(z) - w_0| < \epsilon$ when $0 < |z - z_0| < \delta$.

$|f(z) - w_0| = |u(x,y) + iv(x,y) - (u_0 + iv_0)|$

$= |(u(x,y) - u_0) + i(v(x,y) - v_0)| < \epsilon$ when

$0 < |(x - x_0) + i(y - y_0)| < \delta$

$\because$ $|u(x,y) - u_0| \leq |(u(x,y) - u_0) + iv((x,y) - v_0)|$ and

$|v(x,y) - v_0| \leq |(u(x,y) - u_0) + i(v(x,y) - v_0)|$

$|Re z| \leq |z|$
$|Im z| \leq |z|$

$\therefore$ $|u(x,y) - u_0| < \epsilon$ and $|v(x,y) - v_0| < \epsilon$

when $(x - x_0)^2 + (y - y_0)^2 < \delta^2$ which implies

$\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Conversely : Suppose that $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and
$\Longleftarrow$)

$\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

$\therefore \forall \epsilon > 0$, $\exists \delta_1 > 0$ and $\delta_2 > 0$ such that :

$|u(x,y) - u_0| < \frac{\epsilon}{2}$ when $(x-x_0)^2 + (y-y_0)^2 < \delta_1$

and $|v(x,y) - v_0| < \frac{\epsilon}{2}$ when $(x-x_0)^2 + (y-y_0)^2 < \delta_2$

Choose $\delta = \min\{\delta_1, \delta_2\}$ implies that :

$$|(u(x,y) - u_0) + i(v(x,y) - v_0)| \leq |u(x,y) - u_0| + |v(x,y) - v_0|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ when}$$

$$0 < |(x-x_0) + i(y-y_0)| < \delta$$

$$\therefore \lim_{z \to z_0} (u(x,y) + iv(x,y)) = u_0 + iv_0 \Longrightarrow \lim_{w \to w_0} f(z) = w_0$$

As applications of above theorem, we have the following :

<u>Theorem (3) :-</u> let $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = t_0$.

Then :-

1. $\lim_{z \to z_0} (f(z) \pm g(z)) = w_0 \pm t_0$
2. $\lim_{z \to z_0} (f(z) g(z)) = w_0 t_0$
3. $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{t_0}$

<u>Proof</u> : Exercise (H.W.)

Countinuity الاستمرارية

Def :- let $f(z)$ be a function defined in some neighbourhood of the point z_0. Then f is said to be Continuous function at z_0 iff $\lim_{z \to z_0} f(z) = f(z_0)$

That is, $f(z)$ is Cont. at z_0 iff $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| < \delta$.

Example : $f(z) = z^2$ is a Cont. function at the point $z_0 = 3$. Since $\lim_{z \to 3} f(z) = \lim_{z \to 3} z^2 = 9 = 3^2 = f(3)$

Also, from above example, we get the following :

Remark:

$f(z)$ is Continuous on a rigion D iff $f(z)$ is Cont. at each point of D.

Example : $f(z) = z + 1$ is Cont. function at $z_0 = 1$

Solution :- Choose $\epsilon > 0$, To find a real number $\delta > 0$ we have $|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| < \delta$

Now, $|z + 1 - 2| = |z - 1| < \delta = \epsilon$ Since $|z - 1| < \delta$

Then we choose $\boxed{\delta = \epsilon}$.

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Theorem: A function $f(z) = u + iv$ is continuous at the point $z_0 = x_0 + iy_0$ iff the functions u and v are continuous at the point (x_0, y_0).

proof: $\Rightarrow$) Suppose that $f(z)$ is continuous at z_0.

Then $\lim\limits_{z \to z_0} f(z) = f(z_0)$ (by def. of Cont.)

$\therefore \lim\limits_{z \to z_0} (u + iv) = f(z_0) = u_0 + iv_0$

$\Rightarrow \lim\limits_{(x,y) \to (x_0,y_0)} u = u_0$ and $\lim\limits_{(x,y) \to (x_0,y_0)} v = v_0$ (خواص الغايات)

$\therefore$ each of u and v are Conts. at $(x_0, y_0) = z_0$

$\Leftarrow$) suppose that each of u and v are Cont. at the point (x_0, y_0).

$\therefore \lim\limits_{(x,y) \to (x_0,y_0)} u = u_0$ and $\lim\limits_{(x,y) \to (x_0,y_0)} v = v_0$ (by def. of Cont.)

Now, $\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to z_0} (u + iv)$

$= \lim\limits_{(x,y) \to (x_0,y_0)} u + i \lim\limits_{(x,y) \to (x_0,y_0)} v$ (لخواص الغايات)

$= u_0 + iv_0$

$\therefore \lim\limits_{z \to z_0} f(z) = f(z_0)$

$\therefore f(z)$ is Cont. at z_0

Theorem : If $f(z)$ is a continuous function at a point z_0, then $|f(z)|$ is also a continuous function at z_0.

Proof : Since $f(z)$ is Cont. at z_0, then $\lim\limits_{z \to z_0} f(z) = f(z_0)$
(تعريف الاستمرارية)

and since $\lim\limits_{z \to z_0} |f(z)| = \left| \lim\limits_{z \to z_0} f(z) \right| = |f(z_0)|$

$\therefore$ $|f(z)|$ is Cont. at z_0.

Remark : If $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$ be a polynomail in z, then $f(z)$ is Cont. function on $\mathbb{C}$.

Theorem :- If $f(z)$ is a Cont. function on closed and bounded region D, then $f(z)$ is a bounded function on D.

Proof :- Since $f(z)$ is Continuous (by hypothesis) on D.

Then $|f(z)|$ is Continuous on D. (by above Theorem)

let $f(z) = u + iv$. Then $|f(z)| = \sqrt{u^2 + v^2}$ is Continuous on D. That is the real function $\sqrt{u^2+v^2}$ is Cont. on a closed and bounded region D.

$\therefore |f(z)|$ has a maximum value in D.

That is there exists a positive real number M

s.t. $|f(z)| \le M \quad \forall z \in D$.

$\therefore f(z)$ is a bounded function on D.

Theorem :- let $f(z)$ and $g(z)$ be Countinuous functions at z_0. Then :

(1) $f(z) \pm g(z)$ is Continuous at z_0

(2) $f(z) \cdot g(z)$ " " at z_0

(3) $\dfrac{f(z)}{g(z)}$ is Cont. at z_0 and $g(z_0) \neq 0$.

Proof :- (Exercise) برهانها يشبه برهان خواص النهايات سابقاً

Theorem : let f be a function defined in a neighbour. B of the point z_0 and let g be a function defined on a region D s.t. $f(B) \subset D$. If f Continuous at z_0 and g is Continuous at $f(z_0)$, then the Composite function $g \circ f$ is Continuous at z_0

(بدون برهان) والذي يريد ان يبحث عنه يجده في الكتب

Example: Show that the function $f(z) = xy^2 + i2xy$ is Cont. at every where

Solution: $u(x,y) = xy^2$ $\qquad$ $v(x,y) = 2xy$

Since the two real functions u and v are polynomails with respect to x and y and it is known that the polynomail is Cont. every where and hence $f(z)$ is Cont. function every where

That is, u is Cont. function
v is Cont. function

Then by previous (السابقة) theorem, we get $f(z) = u + iv$ is Cont.

المشتقات
Derivatives

Definition: Let $f(z)$ be a complex-valued function defined in a neighbourhood of z_0. Then the derivative of $f(z)$ at z_0 is given by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

where $\Delta z = z - z_0 \Rightarrow \Delta z = \Delta x + i\Delta y$

and $\Delta x = (x - x_0)$, $\Delta y = (y - y_0)$.

Remark: The function f is said to be differentiable at z_0 if $f'(z_0)$ exists.

Example: prove that $f(z) = 3z^2$ at the point $(0,0)$

Solution:-

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3(z+\Delta z)^2 - 3z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3(z^2 + 2z\Delta z + (\Delta z)^2) - 3z^2}{\Delta z} = \lim_{\Delta z \to 0} (6z + 3\Delta z) = 6z$$

والدالة هي قابلة للاشتقاق في كل $\mathbb{C}$ وليس فقط عند 0

$f'(0) = 0$

Example: Prove that the following function

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ \frac{(\bar{z})^2}{z} & \text{if } z \neq 0 \end{cases}$$ is not differentiable at $z = 0$

Solution :-

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$f'(0) = \lim_{z \to 0} \frac{(\bar{z}^2/z) - 0}{z} = \lim_{z \to 0} \left(\frac{\bar{z}}{z}\right)^2$$

if $z \to 0$ on the x-axis, then $y = 0$

$$f'(0) = \lim_{x \to 0} \left(\frac{x}{x}\right)^2 = 1$$

if $z \to 0$ through the line $x = y$.

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x - iy}{x + iy}\right)^2 = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{y - iy}{y + iy}\right)^2$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(1-i)^2}{(1+i)^2} = \frac{(1-i)^2}{(1+i)^2} = \frac{\cancel{1} - 2i - \cancel{1}}{\cancel{1} + 2i - \cancel{1}} = -1$$

Therefore $f'(0)$ is not exists.

Example :-

let $f(z) = |z|^2$ be a function. Then $f(z)$ is not differentiable at each point of $\mathbb{C}$ accept the point $z = 0$

Solution: $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{|z+\Delta z|^2 - |z|^2}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z} \right).$$

سوف نثبت ان المشتقة موجودة عند نقطة $z=0$

If $z = 0 \Rightarrow f'(z) = \lim\limits_{\Delta z \to 0} (\overline{\Delta z}) = 0$

ولنثبت ان $f(z)$ غير قابلة للاشتقاق عند اي نقطة اخرى غير $z=0$

If $\Delta z \to 0$ through the real axis, then $\Delta z = \Delta \bar{z}$ (i.e $y=0$)

and $f'(z) = \lim\limits_{\Delta z \to 0} \left(z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z} \right) = z + \bar{z}$

If $\Delta z \to 0$ through the imajunary axis, then $\Delta z = -\overline{\Delta z}$ (i.e $x=0$)

and $f'(z) = \lim\limits_{\Delta z \to 0} \left(z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z} \right) = \bar{z} - z$

$\therefore$ The limit is not equal and $f'(z)$ does not exists.

Theorem:- If f is a differentiable function at a point z_0, then f is Continuous at z_0, but the Converse is not true in general.

Proof: Since f is a differentiable function at z_0,

therefore $f'(z_0)$ exists and $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$.

we have to show that $\lim\limits_{z \to z_0} f(z) = f(z_0)$

$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)} \cdot (z - z_0) =$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0).$$

$$= f'(z_0) \cdot 0 = 0 \Rightarrow \lim_{z \to z_0} (f(z) - f(z_0)) = 0$$

$\therefore \lim\limits_{z \to z_0} f(z) = f(z_0)$ and hence f is a continuous function at z_0.

The converse is not true ingeneral. Consider the following example:

$f(z) = |z|^2$ is continuous function at each point of $\mathbb{C}$, but it is differentiable only at $z = 0$

To prove $f(z) = |z|^2$ is continuous function.

$f(z) = |z|^2 = x^2 + y^2$

$\therefore u(x,y) = x^2 + y^2 \qquad v(x,y) = 0$

each of u and v a cont. function on $\mathbb{C}$

$\therefore f(z)$ is cont. function on $\mathbb{C}$.

Theorem :-

1. If k is a constant, then $\frac{d}{dz}(k) = 0$

2. if $f(z) = z^n$, then $f'(z) = nz^{n-1}$

3. $\frac{d}{dz}(kf(z)) = kf'(z)$ where f is differentiable

4. If f and g are differentiable functions,

then :

(i) $\frac{d}{dz}(f(z)+g(z)) = f'(z)+g'(z)$

(ii) $\frac{d}{dz}(f(z)\,g(z)) = f(z)g'(z)+f'(z)\,g(z)$.

(iii) $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)f'(z)-f(z)g'(z)}{(g(z))^2}$; $g(z)\neq 0$

5. if $F(z) = g(f(z))$ Composit function

then :

$$F'(z) = g'(f(z))\,f'(z)$$

under the condition g is differentiable at $f(z)$.

ملاحظة : جميع قواعد الاشتقاق بالنسبة للدوال الحقيقية تنطبق
من حيث الشكل على الدوال العقدية والتطبيق .

Analytic Function الدالة التحليلية

Definition: A function f is said to be analytic at a point if f is differentiable at z_0 and there exists a neighbour. of z_0 s.t. f is differentiable at each point of this neigh.

- A function f is said to be analytic in a region D if f is analytic at each point of D.

Definition: A point z_0 is called a singular point (منفردة) for a function f if f is not analytic at z_0 but f is analytic at some points of each neigh. of z_0.

Definition: A singular point for a function f is called an isolated (منعزلة) singular point if there exists a neigh. of z_0 s.t. f is analytic at each of its points except at the point z_0 itself.

Example :

(1) let $f(z) = \frac{z+1}{z(z^2+4)}$. Then the points $z = 0, \pm 2i$ are isolated singular points for f.

That is f is analytic on $\mathbb{C} - \{0, \pm 2i\}$.

(2) $f(z) = \frac{1}{(1+z)^2}$. Then f is analytic at each point in $\mathbb{C}$ except $z = -1$ and $z = -1$ is singular point.

(3) $f(z) = z^2$ is analytic function and has no singular points.

Definition :- A function f is called an *entire function* [illegible] if f is analytic on the complex plane $\mathbb{C}$.

Example :- Every polynomail is an entire function.

For example : $f(z) = z^2$, $f(z) = z^5 + 3z + 1$ ... etc.

"Cauchy Riemann Equations"

[illegible]

Theorem :-

let $f(z) = u(x,y) + i v(x,y)$ be a function defined at a point $z_0 = x_0 + iy_0$ and in some neigh. of z_0. If f

is differentiable at z_0, then the partial derivatives of the functions u and v with respect to x and y are exist at (x_0, y_0) and satisfy the two equations

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad v_x(x_0, y_0) = -u_y(x_0, y_0)$$

and $f'(z_0) = u_x + i v_x \Big|_{(x_0,y_0)} = (v_y - i u_y)\Big|_{(x_0,y_0)}$.

That is, $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ and $f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$

وهي طريقة لكتابة المشتقة باستخدام معادلتا كوشي ريمان

Proof :- f is differentiable at $z_0 = (x_0, y_0)$, that is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Since the limit for the right side exists, then we can choose two ways :-

① when $\Delta y = 0$, that is $\Delta z = \Delta x$

$$\therefore f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$= (u_x + i v_x)\Big|_{(x_0, y_0)}$$

② when $\Delta x = 0$, that is $\Delta Z = i\Delta y$, then we have

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0,y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}$$

$$\therefore f'(z_0) = v_y(x_0,y_0) - i\, u_y(x_0,y_0)$$

$$\therefore \boxed{u_x = v_y \quad \text{and} \quad v_x = -u_y}$$ وهذه تسمى معادلات كوشي ريمان

Remark: The Converse of above theorem is not true in general, For example:

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ \dfrac{(\bar{z})^2}{z} & \text{if } z \neq 0 \end{cases}$$

Since Cauchy Riemann equation hold at $z=0$ however f is not diff. at $z=0$.

Remark: The Converse of theorem ⊛ holds if the partial derivatives of u and v with respect to x and y are Continuous at (x_0,y_0).

Theorem: If $f(z) = u(x,y) + i\,v(x,y)$ is defined at $z_0 = x_0 + iy_0$ and in some neigh. of z_0 s.t. the first order partial derivatives of u and v with respect to

x and y are defined in that neigh. and Continuous at (x_0, y_0).
and satisfy Cauchy-Riemann equations at (x_0, y_0), then f
is differentiable at z_0.

Proof :- $\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$

$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$

we have the partial derivatives for u and v
are Continuous niegh bour hood $\in$ for the point z_0

Then by using the mean value theorem, we get
نظرية القيمة المتوسطة

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$\Delta v = v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$$

such that $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ Converge to Zero when
Δx and Δy Converge to Zero.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta z} = \lim_{\Delta z \to 0} \frac{u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i v_x(x_0, y_0)\Delta x + i v_y(x_0, y_0)\Delta y + i\epsilon_3 \Delta x + i\epsilon_4 \Delta y}{\Delta z}$$

بتطبيق
by apply CauchRiemann equation, we get

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{U_x(x_0,y_0)\Delta x + iU_x(x_0,y_0)\Delta y - V_x(x_0,y_0)\Delta y + iV_x\Delta x + \epsilon_1\Delta x + \epsilon_2\Delta y + i\epsilon_3\Delta x + i\epsilon_4\Delta y}{\Delta z}$$

$$= \lim_{\Delta z \to 0} U_x(x_0,y_0) + iV_x(x_0,y_0) + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}$$

$\because$ $|\Delta x| \leq |\Delta z|$, $|\Delta y| \leq |\Delta z|$ and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ Converge to zero when $\Delta z \longrightarrow 0$.

That is , $f'(z_0) = U_x(x_0,y_0) + iV_x(x_0,y_0)$

Examples :

① $f(z) = \bar{z}$

$f(z) = x - iy$ $\qquad$ $U(x,y) = x \longrightarrow U_x = 1$, $U_y = 0$

$\qquad\qquad\qquad\qquad$ $V(x,y) = -y \longrightarrow V_x = 0$, $V_y = -1$

since $U_x \neq V_y$

$\therefore$ $f(z) = \bar{z}$ is not differ. on the plane $\mathbb{C}$.

② $f(z) = e^x \cos y + i e^x \sin y$

$U(x,y) = e^x \cos y \longrightarrow U_x = e^x \cos y$, $U_y = -e^x \sin y$

$V(x,y) = e^x \sin y \longrightarrow V_x = e^x \sin y$, $V_y = e^x \cos y$

look, $U_x = V_y$ and $V_x = -U_y$

and u_x, u_y, v_x, v_y are Continuous on the whole plane $\mathbb{C}$, then $f'(z)$ exists and which is equal to

$$f'(z) = e^x \cos y + i e^x \sin y$$

Cauchy_Riemann Equations in the Polar Representation

معادلتي كوشي - ريمان بالصيغة القطبية

let $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore f(z) = u(x,y) + iv(x,y) = u(r,\theta) + iv(r,\theta)$$

$$\therefore \boxed{u_r = \frac{v_\theta}{r} = \frac{1}{r} v_\theta} \quad , \quad \boxed{v_r = \frac{-1}{r} u_\theta}$$

Solution : IS an Exercise

Hint : $x = r\cos\theta$ $\quad y = r\sin\theta$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{-y}{\sqrt{x^2+y^2}}\right) = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\partial u}{\partial x}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta$$

Then Complete solution

ثم اكمل الحل
بنفس الطريقة لمشتقات V

Example: $f(z) = \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$$u(r,\theta) = \frac{\cos\theta}{r} \rightarrow u_r = \frac{-\cos\theta}{r^2}, \quad u_\theta = \frac{-\sin\theta}{r}$$

$$v(r,\theta) = \frac{-\sin\theta}{r} \rightarrow v_r = \frac{\sin\theta}{r^2}, \quad v_\theta = \frac{-\cos\theta}{r}$$

$\therefore$ The Cauchy Riemann equation hold and all partial derivatives $u_r, u_\theta, v_r, v_\theta$ are continuous

$$\therefore f'(z) = e^{-i\theta}\left(\frac{-\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = \frac{e^{-i\theta}}{r^2}(-e^{-i\theta})$$

$$= \frac{-1}{r^2 e^{2i\theta}} = \frac{-1}{z^2}.$$

Exercises: (H.w)

1. Examined any of the following functions are differentiable

(a) $f(z) = z\bar{z}$ (b) $f(z) = 2x + ixy^2$

2. where are $f'(z)$ of the following functions exists and find $f'(z)$.

(a) $f(z) = z\,\mathrm{Im}\,z$ (b) $f(z) = x^2 + iy^2$

Example: Let $f(z) = \frac{1}{z}$. Then find $f'(z)$ by using Cauchy-Riemann equations in the polar form.

Solution:-

$$f(z) = \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u(r,\theta) = \frac{\cos\theta}{r} \Longrightarrow u_r = \frac{-\cos\theta}{r^2}, \quad u_\theta = \frac{-\sin\theta}{r}$$

$$v(r,\theta) = \frac{-\sin\theta}{r} \Longrightarrow v_r = \frac{\sin\theta}{r^2}, \quad v_\theta = \frac{-\cos\theta}{r}$$

∴ تحقق معادلتي كوشي ـ ريمان بالصيغة القطبية والدوال u_r , u_θ , v_r , v_θ مستمرة .

$$f'(z) = e^{-i\theta}\left(\frac{-\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right)$$

$$= \frac{e^{-i\theta}}{r^2}\left(-e^{-i\theta}\right) = \frac{-1}{r^2 e^{2i\theta}} = \frac{-1}{z^2}.$$

((Harmonic Functions)) الدوال التوافقية

Definition:- let $f(x,y)$ be a real function in two variables x, y. Then f is said to be harmonic function in a region D, if the partial derivative of the first and second order of f w.r.t. x and y are

Continuous in D and satisfy Laplace's equation
معادلة لابلاس
$(f_{xx} + f_{yy} = 0)$.

Example :- let $f(x,y) = 2xy$. To prove that f is harmonic function.

$f_x = 2y$, $f_y = 2x$
$f_{xx} = 0$ $\quad$ $f_{yy} = 0$ since f_x, f_y, f_{xx}, f_{yy} are Continuous.

Also, $f_{xx} + f_{yy} = 0$ (Laplace's equation)

$\therefore f(x,y)$ is harmonic function.

Remark :- let $f(z) = u + iv$ be analytic function. Then the partial derivative of u and v of all orders are exists and Continuous, also from real analysis we obtain that $U_{xy} = U_{yx} \wedge V_{xy} = V_{yx}$ because the Continuouity of the higher derivative.
(بسبب استمرارية المشتقات العليا)

Theorem :- If $f(z) = u + iv$ is an analytic function on a region D, then each of $u(x,y)$ and $v(x,y)$ is a harmonic function.

Proof :- $f(z)$ is an analytic function on D, then $f(z)$ is a differ. on D and hence u and v satisfy Cauchy – Riemann equation in D.

i.e $u_x = v_y$ and $v_x = -u_y$. Also, $u_{xx} = v_{yx}$ and $v_{xx} = -u_{yx}$ and $u_{xy} = v_{yy}$, $v_{xy} = -u_{yy}$,

we obtain $u_{xx} + u_{yy} = v_{yx} - v_{xy} \Longrightarrow u_{xx} + u_{yy} = 0$

(since $v_{yx} = v_{xy}$ from abovious remark)

and by the same way, we get $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$

Therefore u and v are harmonic functions in D.

Definition : let each of $u(x,y)$ and $v(x,y)$ be a harmonic function in a region D. If u and v satisfy Cauchy – Riemann equations, then we say that v is a harmonic conjugate for u. (يسمى V المرافق التوافقي لـ U)

The following theorem gives a necessary and sufficient condition for a function $f(z)$ to be analytic

تعطي الشرط الضروري والكافي للدالة $f(z)$ لكي تكون دالة تحليلية

Theorem : A function $f(z) = u(x,y) + iv(x,y)$ is an analytic function in a region D iff v is a harmonic Conjugate of u in D.

Proof :- $\Rightarrow$) Suppose that $f(z)$ is analytic function. Then by abovious theorem and Definition of a harmonic Conjucate, we get v is a harmonic Conjucate for u

$\Leftarrow$) If v is a harmonic Conjugate of u, then by Definition of harmonic Conjugate each of u and v is a harmonic function and the partial derivatives of u and v w.r.to x and y are Continuous and satisfy Cauchy-Riemann equations on D and hence $f(z)$ is differe. on D and hence f is analytic function on D.

Remarks :- let $f(z) = u(x,y) + iv(x,y)$ be a Complex fun. Then :

1. If $f(z)$ is analytic function in D, then v is a harmonic Conjugate for u.
2. If v is a harmonic Conjugate for u, then $f(z)$ is analytic function in D.

③ If V is a harmonic Conjugate for u in a region D.,
then it is not necessary (or it is not always) that u is a harmonic Conjugate for V, for example:
$f(z) = z^2$, since $f(z)$ is analytic function in $\mathbb{C}$,
then V is a harmonic Conjugate for u in $\mathbb{C}$ (by theorem)
But u is not harmonic Conjucate for V in $\mathbb{C}$
look $u = x^2 - y^2$, $V = 2xy$

$V_x = U_y \Longrightarrow 2y = -2y \Longrightarrow y = 0$

and

$V_y = -U_x \Longrightarrow 2x = -2x \Longrightarrow x = 0$

المساواة متحققة فقط عندما $x = y = 0$ أي في النقطة $(0,0)$

$\therefore$ U is not a harmonic Conjugate for V.

④ If V is a harmonic Conjugate for U in a region D,
then $-U$ is a harmonic Conjugate for V in D.

proof :- let V be a harmonic Conjugate for U in a region D. Then

$U_x = V_y \Longrightarrow V_y = -(-U_x)$ and

$U_y = -V_x \Longrightarrow V_x = -U_y$

$\therefore$ $-U$ is a harmonic Conjugate for V (since $-U$ and V satisfies Cauchy-Riemann equations and $-U$, V are harmonic functions.

(5) If V is a harmonic Conjugate for U and U is a harmonic Conjugate for V, then $f(z)$ is a Constant function.

proof: let V is a harmonic Conjugate for U. Then U and V are harmonic functions and $U_x = V_y$, $U_y = -V_x$

Also, U is a harmonic Conjugate for $V \Rightarrow V_x = U_y$, $V_y = -U_x$

$\therefore U_x = V_y = -U_x \Longrightarrow U_x = 0$ and $V_y = 0$, also

$U_y = -V_x = V_x \Longrightarrow V_x = 0$ and $U_y = 0$

$\therefore$ U and V are Constant function and hence $f(z)$ is a Constant function.

In the following, we explain the method which is using to find the harmonic Conjugate.

سوف نشرح الطريقه اللازمه لايجاد المرافق التوافقي

Example:

Find the harmonic Conjugate for the function $U(x,y) = y^3 - 3x^2 y$

Solution: $u_x = -6xy$

$\because u_x = v_y \;,\; u_y = -v_x$

$\therefore v_y(x,y) = -6xy$

$\therefore v(x,y) = \int -6xy \, dy + \phi(x)$

$= -3xy^2 + \phi(x)$

$v_x(x,y) = -3y^2 + \phi'(x) = -3y^2 + 3x^2$ (since $v_x = -u_y$
($= -3y^2 + 3x^2$)

$\therefore \phi'(x) = 3x^2 \Longrightarrow \phi(x) = x^3 + c$

$\therefore v(x,y) = -3xy^2 + x^3 + c$

نعتبر $C=0$ فأن

$v(x,y) = -3xy^2 + x^3$

هو مرافق توافقي الى u وان

$f(z) = y^3 - 3x^2y + i(x^3 - 3xy^2) = iz^3$

Exercise :- Find the harmonic Conjugate for the following function $u(x,y) = 2x(1-y)$.

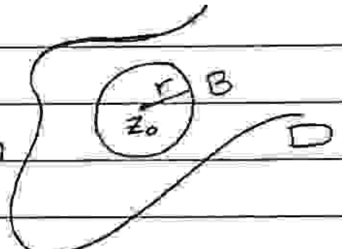
Taylor's Theorem

let D be a domain in $\mathbb{C}$, and let f be an analytic function on D. let z_0 be any point in D, then there exists a ball $B_r(z_0)$ in D and a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{s.t.} \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall z \in B(z_0)$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad \forall n \geq 0$$

Morever the series is unique.



proof: let $z_0 \in D$, since D is a domain $\exists\, B = B_r(z_0) \subseteq D$. Assume that $\partial B \subseteq D$.

Now, $\forall z \in B \Longrightarrow f(z) = \frac{1}{2\pi i} \oint_{\partial B} \frac{f(t)}{t-z}\, dt$ By C.I.F.

$$\frac{1}{t-z} = \frac{1}{(t-z_0)-(z-z_0)} = \frac{1}{(t-z_0)\left[1-\frac{z-z_0}{t-z_0}\right]} \qquad t \neq z_0$$

But $\left|\frac{z-z_0}{t-z_0}\right| < 1$, hence $\sum_{n=0}^{\infty} \left(\frac{z-z_0}{t-z_0}\right)^n$ is Converges

Then $$\frac{1}{1-\left(\frac{z-z_0}{t-z_0}\right)} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{t-z_0}\right)^n$$

$$\frac{1}{t-z} = \frac{1}{(t-z_0)\left[1-\frac{z-z_0}{t-z_0}\right]} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}}$$

and the series converges uniformaly, where

$\left| \frac{z-z_0}{t-z_0} \right| < 1$

$\therefore \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}}$ is converges uniformaly.

$$\frac{f(t)\,dt}{t-z} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}} f(t)\,dt.$$

$$\therefore f(z) = \frac{1}{2\pi i} \oint_{\gamma B} \frac{f(t)\,dt}{(t-z)} = \frac{1}{2\pi i} \oint \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}} f(t)\,dt$$

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i} \oint_{\gamma B} \frac{f(t)\,dt}{(t-z_0)^{n+1}}$$

$$\therefore a_n = \frac{1}{2\pi i} \oint_{\gamma B} \frac{f(t)\,dt}{(t-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!} \text{ by G.C.I.F}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

The proof of uniqeness

we know that

$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ (*)

Assume that $f(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots$ (**)

If $z = z_0 \Rightarrow f(z_0) = a_0$ from (*) and $f(z_0) = b_0$ from (**)

we have $a_0 = b_0$

$f'(z) = a_1 + 2a_2(z-z_0) + \cdots$

$f'(z) = b_1 + 2b_2(z-z_0) + \cdots$

$\therefore f'(z_0) = a_1$ and $f'(z_0) = b_1 \Rightarrow a_1 = b_1$

$\vdots$

a+c

Proposition :. Let D be a domain and f analytic in D, $f \not\equiv 0$. Let $Z(f) = \{Z \in D \,/\, f(Z) = 0\}$ = the set of zeros f in D. Then

(1) $Z(f)$ is a closed set
(2) $Z(f)$ does not have a Limit point in D (isolated points)

Corollary :. Let D be a bounded domain in $\mathbb{C}$ with $\partial D = B$. Let f be analytic in D if $f(Z) \neq 0$ $\forall Z \in B$, then f has only a finit number of zeros in D (i-e) is a finit set.

o~~~~~~o

Laurent Series : Let f be an analytic in the domain $r < |Z - Z_0| < R$, then f be represented in the form

$$f(Z) = \sum a_n (Z - Z_0)^n = a_0 + a_1 (Z - Z_0) + \cdots\cdots$$

when $a_n = \dfrac{1}{2\pi i} \oint_\gamma \dfrac{f(Z)}{(Z - Z_0)^{n+1}} dZ$ where γ is a simply connected contour that gives center $|Z - Z_0| = r$ and Lies in the domain.

Since $f(Z) = \cdots + \dfrac{a_{-2}}{(Z - Z_0)^2} + \dfrac{a_{-1}}{Z - Z_0} + a_0 + a_1 (Z - Z_0) + a_2 (Z - Z_0)^2 + \cdots\cdots$

is called Laurent Series

when $n = -1$ we have

$$a_{-1} = \frac{1}{2\pi i} \oint_{\gamma} f(z)\, dz$$

a_{-1} is called the residue of f at z_0

- If there exists the positive integer K s.t $a_{-n} = 0$ $\forall\, n > K$.

when $n > 0$, then z_0 is called a pole for f

- if $a_{-n} = 0$ $\forall\, n > K$ and $a_{-K} \neq 0$, then z_0 is a pole of f of order K.

- If z_0 is not a pole then z_0 is called essential singularity for f

- A pole of order 1 is called a simple pole and a pole of order 2 is called a double pole.

Theorem: If f has a pole of order K at z_0 then $\frac{1}{f}$ is analytic at z_0 and has a zero of order K at z_0. Conversely if f is analytic at z_0 and has a zero of order K at z_0, then $\frac{1}{f}$ has a pole of order K at z_0

The proof Laurent's Theorem

Proof: It is enough to prove that

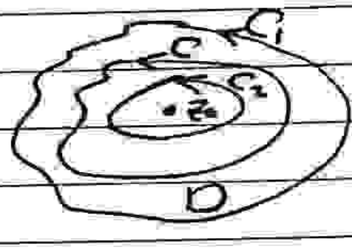
$$f(z) = \sum_{n=0}^{\infty} A_n (z-z_0)^n + \sum_{n=0}^{\infty} \frac{B_n}{(z-z_0)^n} \quad \text{where}$$

$$A_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} \quad n = 0, 1, 2, \dots$$

$$B_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}} \quad n=1,2,\ldots$$

By cauchy integral formula:-

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i}\oint_{C_2}\frac{f(t)}{t-z}\,dt \quad \forall z\in D.$$



$$\frac{1}{t-z} = \frac{1}{(t-z_0)-(z-z_0)} = \frac{1}{(t-z_0)\left[1-\frac{z-z_0}{t-z_0}\right]}$$

$$= \frac{1}{t-z_0}\cdot\frac{-1}{1-\frac{z-z_0}{t-z_0}} = \frac{1}{t-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{t-z_0}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(t-z_0)^{n+1}}$$ is absolutely and uniformly convergent in D.

Also $\frac{-1}{t-z} \leftarrow \frac{1}{z-t}$

$$= \frac{1}{(z-z_0)-(t-z_0)} = \sum_{n=0}^{\infty}\frac{(t-z_0)^n}{(z-z_0)^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{(t-z_0)^{-n}(z-z_0)^{n+1}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{(t-z_0)^{-n+1}(z-z_0)^n}$$

$$\therefore f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{t-z} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(t)\,dt}{t-z}$$

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} f(t)\,dt\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(t-z_0)^{n+1}} - \frac{1}{2\pi i}\oint_{C_2} f(t)\,dt\sum_{n=1}^{\infty}\frac{1}{(t-z_0)^{-n+1}(z-z_0)^n}$$

$$= \sum_{n=0}^{\infty}(z-z_0)^n \underbrace{\frac{1}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{(t-z_0)^{n+1}}}_{A_n} - \sum_{n=1}^{\infty}\frac{1}{(z-z_0)^n}\cdot\underbrace{\frac{1}{2\pi i}\oint_{C_2}\frac{f(t)\,dt}{(t-z_0)^{-n+1}}}_{B_n}$$

Example : Comput the Taylor series for the function $f(z) = \frac{3}{z+i}$ is the region $|z-i| < 2$ a round $z_0 = i$

Solution : $\frac{3}{z+i} = \frac{3}{z+i+i-i} = \frac{3}{2i+(z-i)} = \frac{3}{2i\left(1+\frac{z-i}{2i}\right)}$

$$= \frac{3}{2i} \cdot \frac{1}{1+\frac{z-i}{2i}}$$

$$\left|\frac{z-i}{2i}\right| = \frac{|z-i|}{|2i|} = \frac{|z-i|}{2} < 1$$

$$\therefore \frac{3}{2i} \cdot \frac{1}{1+\frac{(z-i)}{2i}} = 3\sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(2i)^{n+1}}$$

Example : Find laurent exp. of $f(z) = \frac{z}{z^2+1}$ in $0 < |z-i| < 2$.

$$f(z) = \frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)}$$

f is analytic except at $z_0 = \pm i$

$$\frac{z}{(z+i)(z-i)} = \frac{A}{(z+i)} + \frac{B}{(z-i)} = \frac{A(z-i)+B(z+i)}{(z+i)(z-i)}$$

$$= \frac{(A+B)z+(B-A)i}{(z+i)(z-i)}$$

$A+B=1$, $A-B=0$

$\Longrightarrow A=B=\frac{1}{2}$

$$\therefore f(z) = \frac{1/2}{z+i} + \frac{1/2}{z-i}$$

$$\frac{1}{z+i} = \frac{1}{z+i+i-i} = \frac{1}{2i+z-i} = \frac{1}{2i\left[1+\frac{z-i}{2i}\right]}$$

$$= \frac{1}{2i}\left[\frac{1}{1+\frac{z-i}{2i}}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^{n+1}}$$

$a_{-1} = Re(f) = \frac{1}{2}$ أخطأ العدد

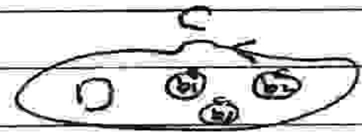
i is simple pole of f.

Residues Theorem

Let D be a simply connected domain bounded $\partial D = C$, f is analytic on D except at finit number of poles $\{b_1, b_2, \dots, b_n\}$. Assume f is cont. on $\partial D = C$.

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \text{The sum of residues of } f \text{ in } D.$$

$$= \sum_{j=1}^{n} Res_{b_j}(f)$$



proof :- $\forall\, b_j \in D$, $\exists$ a ball $B(b_j) \subseteq D$ s.t $1 \le j \le n$.

$$f(z) = (z-z_0)^{-n_j}\, g_j(z).$$

$g_j(z)$ is analytic in $B(b_j)$ at $g_j(z) \neq 0$ $\forall\, z \in \overline{B}(b_j)$

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_{i=1}^{n}\frac{1}{2\pi i}\oint_{\partial B_i} f(z)\,dz = \sum_{i=1}^{n} Res_{b_j}(f).$$

Some elementary analytic functions

بعض الدوال التحليلية البسيطة

① Exponential function :- الدالة الاسية

The exponential function which is denoted by $w = e^{z}$

or $w = \exp z$ and define as :

$w = e^{x} \cdot e^{iy} = e^{x}(\cos y + i \sin y)$.

Remarks :-

(1) If $y = 0$, then $w = e^{x}$ is real function

(2) If $x = 0$, then $w = e^{iy} = \cos y + i \sin y$ (Euler's formula)

(3) $\because$ e^{x} and e^{y} are real functions and defined on $\mathbb{R}$ (real numbers), then $w = e^{z}$ is defined on every $\mathbb{C}$.

(4) The range of $w = e^{z}$ is $\mathbb{C} - \{0\}$. That is $e^{z} \neq 0$ for each $z \in \mathbb{C}$, since

$|w| = |e^{z}| = |e^{x}(\cos y + i \sin y)| = |e^{x}| \, |\cos y + i \sin y|$

$= e^{x} \sqrt{\cos^2 y + \sin^2 y} = e^{x} \neq 0 \Rightarrow e^{z} \neq 0 \quad \forall z \in \mathbb{C}$

$\underbrace{}_{1}$

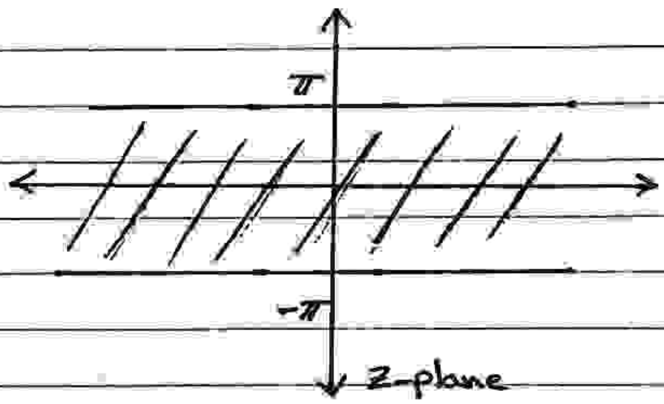
(5) $e^{z} = 1$ iff $z = 2k\pi i$; k is an integer.

(6) $e^{z_1} = e^{z_2}$ iff $z_1 = z_2 + 2k\pi i$; k is an integer.

(That is, e^{z} is a periodic function of Periode $2\pi i$)

(7) $w = e^{z}$ is analytic in $\mathbb{C}$. And $\frac{\partial}{\partial z} e^{z} = e^{z}$ $\forall z \in \mathbb{C}$.

(8) $w = e^{z}$ is not (1-1) function in $\mathbb{C}$. But e^{z} becomes (1-1) function in the region



(9) $e^{0} = 1$

(10) $(e^{z})^{n} = e^{nz}$ $\quad \forall n = 0, \mp 1, \mp 2, \mp 3, \ldots$

(11) $e^{z_1} \cdot e^{z_1} = e^{z_1 + z_2}$

(12) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

To find the Solution of $w = e^{z}$.

كيف نجد الحل للمعادلة $w = e^{z}$ لقيم z .

Since $e^{z} \neq 0$, then $w \neq 0$, by using the polar form for w, we obtain $w = re^{i\theta}$, $r = |w|$, $\theta = \arg w$

$re^{i\theta} = e^{x} \cdot e^{iy}$

$\therefore r = e^{x} \Longrightarrow x = \ln r = \ln |w|$

and $y = \theta + 2K\pi = \arg w + 2K\pi$; $K = 0, \mp 1, \mp 2, ---$

$$\boxed{\therefore z = \ln r + i(\theta + 2K\pi) = \ln |w| + i(\arg w + 2K\pi)}$$

Example :- Solve the equation $e^{z} = -1 - i\sqrt{3}$

Solution :- $w = -1 - i\sqrt{3}$

في الربع الثالث

$r = |w| = \sqrt{1^2 + (\sqrt{3})^2} = 2$ $\qquad \theta = \frac{\pi}{3} = 60^{\circ} \Rightarrow \pi + \frac{\pi}{3} = \frac{3\pi + \pi}{3} = \frac{4\pi}{3}$

$\theta = \arg w = \tan^{-1} \frac{-\sqrt{3}}{-1} = \tan^{-1} \sqrt{3} = \frac{4\pi}{3}$

$\therefore z = \ln |w| + i(\arg w + 2K\pi)$

$\therefore z = \ln 2 + i\left(\frac{4\pi}{3} + 2K\pi\right)$; $K = 0, \mp 1, \mp 2, ---$

(2) Logarithim function (الدالـه اللوغاريتمـيـة)

The logarithim function is the inverse of the exponential function. تعرف الداله اللوغاريتمية على انها معكوس الدالة الاسية

let $z = re^{i\theta}$, $r>0$. Then $\log z = \ln r + i\theta$

$\Rightarrow \log z = \ln|z| + i\arg z$

Remarks :-

1. The domain of the function $\log z$ is $\mathbb{C} - \{0\}$

2. $\log z$ is a multi-valued function دالة متعددة القيم

3. If $\theta = \phi + 2k\pi$ such that $(-\pi < \phi < \pi)$, then ϕ is Called the principle value of θ, that is

$$\text{Log } z = \ln r + i(\phi + 2k\pi) \quad , k = 0, \mp 1, \mp 2, \ldots$$

سوف نحصل على القيمة الرئيسية لـ θ

when $n=0$, then $\text{Log } z = \text{Log } r + i\phi$, $r>0$, $-\pi<\phi<\pi$

$\Rightarrow \text{Log } z = \ln r + i\text{Arg } z$.

4. The range of $\log z$ is the region $-\pi < Im\, w < \pi$

5. $\log z^{1/n} = \frac{1}{n}\log z$

6. $\log z_1 z_2 = \log z_1 + \log z_2$ $\quad z_1 \neq 0$, $z_2 \neq 0$ (H.W)

7. $\log \frac{z_1}{z_2} = \log z_1 - \log z_2$ $\quad , z_1 \neq 0 \wedge z_2 \neq 0$ (H.W)

8. $\frac{d}{dz}(\log z) = \frac{1}{z}$

because each of them is a linear combination of the exponential function e^{z} which is an entire function.

2. Each of $\sin z$ and $\cos z$ is periodic function with period 2π

3. Each of $\tan z$ and $\cot z$ = = = = = π

4. $\sec z$, $\csc z$ are periodic functions with period 2π.

Some properties of Trigonometric function

بعض خواص الدوال المثلثية

1. $\sin z = \sin x \cosh y + i \cos x \sinh y$

2. $\cos z = \cos x \cosh y - i \sin x \sinh y$

3. $|\sin z|^2 = \sin^2 x + \sinh^2 y$

4. $|\cos z|^2 = \cos^2 x + \sinh^2 y$

5. $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

6. $\sin z = 0 \iff z = k\pi$ $\quad k = 0, \mp 1, \mp 2, \ldots$

7. $\cos z = 0 \iff z = \frac{\pi}{2} + 2k\pi$ $\quad k = 0, \mp 1, \mp 2, \ldots$

8. $\frac{d}{dz} \sin z = \cos z$, $\frac{d}{dz} \cos z = -\sin z$

$\frac{d}{dz}(\tan z) = \sec^2 z$, $\frac{d}{dz}(\cot z) = -\csc^2 z$.

$$\frac{d}{dz}(\sec z) = \sec z \tan z$$

$$\frac{d}{dz}(\csc z) = -\csc z \cot z .$$

Hyporbolic Functions «الدوال الزائديـة»

we define :

$$\sinh z = \frac{e^{z} - e^{-z}}{2} , \quad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z} , \quad \coth z = \frac{\cosh z}{\sinh z}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} , \quad \operatorname{csch} z = \frac{1}{\sinh z}$$

Remarks :

(1) $\sinh z$, $\cosh z$ are analytic functions in $\mathbb{C}$.

(since $\sinh z$, $\cosh z$ are linear Combinations of exponential functions) .

Some properties of hyporbolic function بعض خواص الدوال الزائدية

(1) $\frac{d}{dz}(\sinh z) = \cosh z$

(2) $\frac{d}{dz}(\cosh z) = \sinh z$

(3) $\frac{d}{dz}(\tanh z) = \operatorname{sech}^2 z$

(4) $\frac{d}{dz}(\coth z) = -\operatorname{csch}^2 z$

(5) $\frac{d}{dz}(\operatorname{sech} z) = -\operatorname{sech} z \tanh z$

(6) $\frac{d}{dz}(\operatorname{csch} z) = -\operatorname{csch} z \coth z$

(7) $\cosh^2 z - \sinh^2 z = 1$

(8) $\sinh z = \sinh x \cos y + i \cosh x \sin y$

(9) $\cosh z = \cosh x \cos y + i \sinh x \sin y$

(10) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

(11) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

(12) $\sinh(iz) = i \sin z$

(13) $\cosh(iz) = \cos z$

The Complex foundations الأسس المعقدية

let A and B be two Constant Complex numbers.

Then the Complex number A^B is defined as

$$A^B = e^{B \log A}$$

Example :-

let $A = 1+i$, $B = i$. Then evaluate A^B and B^A .

$$A^B = e^{B\log A} = e^{i(\ln r + i(\theta + 2k\pi))}$$

$$= e^{i(\ln\sqrt{2} + i(\frac{\pi}{4} + 2k\pi))}$$

$$= e^{i\ln(2)^{1/2}} \cdot e^{-(\frac{\pi}{4} + 2k\pi)}$$

To Compute B^A (H.W)

وبنفس الطريقه يمكن استخدام نفس الطريقه بالحل الحالات التاليه

فقط نستخدم الصيغه المناسبه الخاصه بكل حالة .

(1) If B is a Constant Complex number and A is a Complex variable not equal to Zero , we get $f(z) = z^B$. Then $f(z) = z^B = e^{B\log z}$.

(2) If A is a Constant Complex number and B is a Complex variable not equal to Zero , then we get $f(z) = B^z \Longrightarrow f(z) = e^{z\log A}$.

Transformation by Complex functions

التحويلات بدوال عقدية

(1) linear transformation التحويل الخطي

$$w = f(z) = Az + B$$

حيث A و B اعداد عقدية ثابتة

(P) اذا كانت $A = 1$ و $B \neq 0$ فان

$$w = z + B$$

$$\Rightarrow u + iv = (x + iy) + (b_1 + ib_2)$$

$$= (x + b_1) + i(y + b_2)$$

$$\Rightarrow u = x + b_1 \quad \text{and} \quad v = y + b_2$$

اي ان التحويل $z+B$ عبارة عن انسحاب بمقدار b_1 افقياً و b_2 عمودياً.

وعليه اذا كانت (x, y) نقطة في المنطقة S (في المستوي Z) فان صورتها تكون النقطة $(x+b_1, y+b_2)$ في المنطقة $f(S)$ [في المستوي w] وعليه فان صورة المنطقة S بفعل التحويل $z+B$ ستكون مشابهة للمنطقة الاصلية اي ان $z+B$ عبارة عن علاقة تطابق.

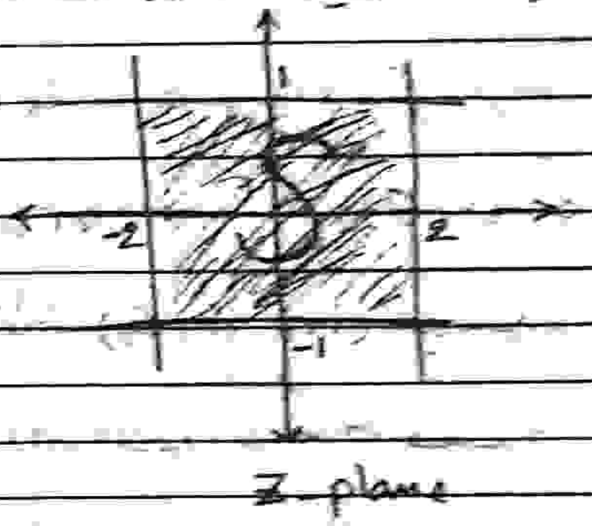
Example: Draw the set and find it image with drawing

$S=\{z: |x|\le 2 ,\ |y|\le 1\}$

تحت التحويل $f(z)=z+1-2i$

$|x|\le 2 \Rightarrow -2\le x\le 2$

$|y|\le 1 \Rightarrow -1\le y\le 1$



$f(z)=u+iv \ ,\ z=x+iy$

$u+iv = x+iy+1-2i$

$\quad = (x+1)+i(y-2)$

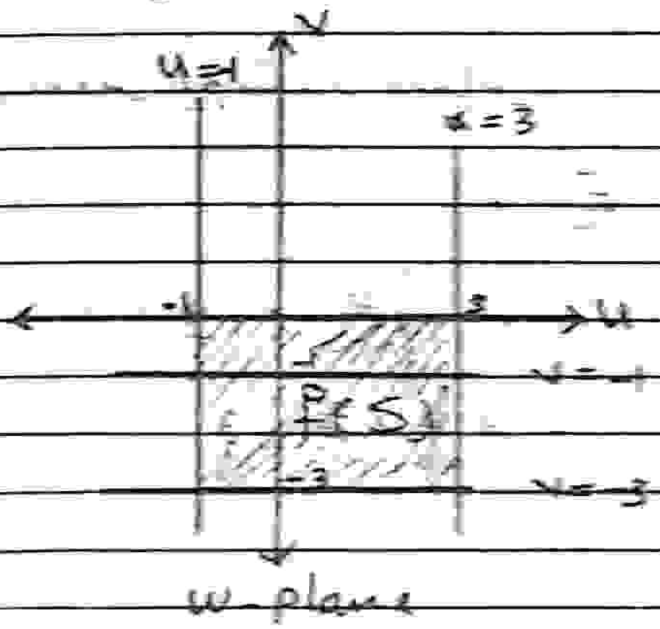
$u=x+1 \quad , \quad v=y-2$

$x=u-1 \quad , \quad y=v+2$

$-2\le u-1\le 2$ and $-1\le v+2\le 1$

$-1\le u\le 3 \qquad -3\le v\le -1$

$\therefore f(S)=\{w: -1\le u\le 3 \ ,\ -3\le v\le -1\}$



(ب) اذا كان $B=0 ,\ A\ne 0 ,\ A\ne 1$

$w=f(z)=Az$

$A = r_0 e^{i\theta_0}$, $z = re^{i\theta}$ and $w = \rho e^{i\varphi}$

$$\rho e^{i\varphi} = r_0 e^{i\theta_0} \cdot r e^{i\theta} = r_0 r \, e^{i(\theta+\theta_0)}$$

$$\Rightarrow \rho = rr_0 \quad \text{and} \quad \varphi = \theta + \theta_0$$

$$\therefore (r, \theta) \longmapsto (r_0 r, \theta + \theta_0)$$

أي أن صورة النقطة (r, θ) في المستوى Z هي النقطة $(rr_0, \theta+\theta_0)$ في المستوى W .

∴ التحويل AZ عبارة عن تدوير (Rotation) يصحبه تمدد (exposition) أو تقلص (Contraction) والنقطة $f(S)$ تكون مشابهة النقطة S .

ملاحظة :- التحويل $w = AZ + B$ يتم على مرحلتين

$$Z \rightarrow AZ \rightarrow AZ + B$$

اي ان التحويل $AZ + B$ عبارة عن تدوير مع تمدد او تقلص يعقبه انسحاب

Ex :

$$S = \{ z : x + y < 1 , y \geq 0 \}$$

$$f(z) = i\bar{z} + 3$$

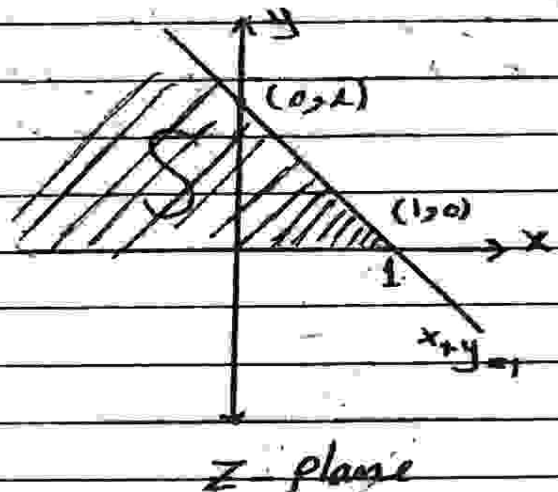
ارسم المنطقة S ، ثم صورتها بفعل f ثم ارسم $f(S)$

$$u + iv = i(x - iy) + 3$$
$$= y + 3 + ix$$

$$\Rightarrow u = y + 3 , \quad v = x$$

الدائرة وبالعكس اي ان $|z| < 1 \Rightarrow \frac{1}{|z|} = |w| > 1$

$\Rightarrow y = u - 3 \quad , \quad v = x$



$x + y \leq 1$

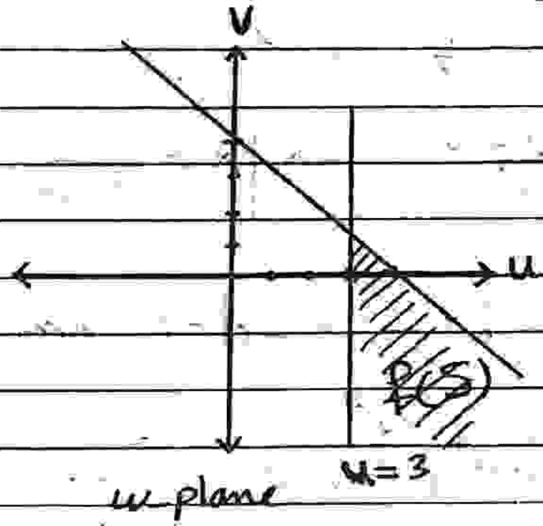
$\therefore u - 3 + v \leq 1$

$u + v \leq 4$

$\because y \geq 0 \Rightarrow u - 3 \geq 0 \Rightarrow u \geq 3$

$\therefore f(S) = \{w : u + v \leq 4 \ , \ u \geq 3\}$

تم التدوير بزاوية قائمة عكس عقارب الساعة لأن $i = e^{i\frac{\pi}{2}}$
مع سحب الى اليمين ٣ وحدات



تحويل بفعل $(\frac{1}{z})$

② The transformation by $\frac{1}{z}$

In polar : $w = \frac{1}{z} \Rightarrow \rho e^{i\alpha} = \frac{1}{r} e^{-i\theta}$

$\Rightarrow \rho = \frac{1}{r}$ and $\alpha = -\theta$

اي ان صورة النقطة (r, θ) بفعل التحويل $\frac{1}{z}$ هي النقطة $(\frac{1}{r}, -\theta)$

هذا يعني ان التحويل $\frac{1}{z}$ عبارة عن انعكاسين احدهما بالنسبة للمحور

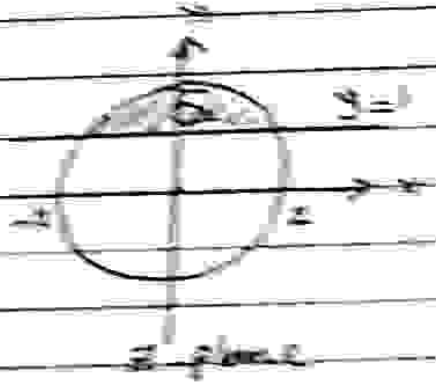
الحقيقي (وهو $\theta \to -\theta$) والآخر بالنسبة الى دائرة الوحدة (وهو
$r \to \frac{1}{r}$) فالنقاط الواقعة داخل الدائرة صورها نقاط خارج
الدائرة وبالعكس اي ان $|z| < 1 \Rightarrow |w| = \frac{1}{|z|} > 1$

[illegible]

Ex: Find and draw the image of the following set

$$S = \{z : |z| \le 2 \,,\, \operatorname{Im} z \ge 1\} \text{ by } f(z) = \frac{1}{z}$$

$w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow |z| = \frac{1}{|w|}$



Since $|z| \le 2 \Rightarrow \frac{1}{|w|} \le 2$

$\therefore |w| \ge \frac{1}{2}$

Since $\operatorname{Im} z = y \ge 1$ [illegible]

$\therefore \frac{-v}{u^2+v^2} \ge 1 \quad \left(\text{since } y = \frac{-v}{u^2+v^2}\right)$

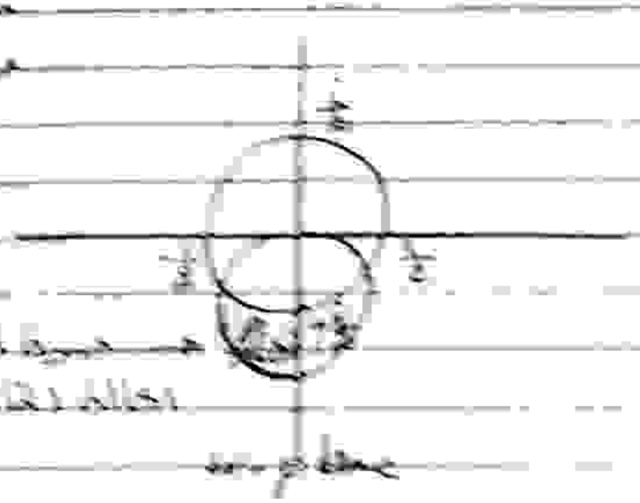
$\therefore -v \ge u^2+v^2 \Rightarrow u^2+v^2+v \le 0$

$u^2 + (v+\frac{1}{2})^2 \le \frac{1}{4} \Rightarrow |w + \frac{i}{2}| \le \frac{1}{2}$

دائرة مركزها $(0,-\frac{1}{2})$ ونصف قطرها $\frac{1}{2}$

$\therefore f(S) = \{w : |w| \ge \frac{1}{2} \text{ and } |w + \frac{i}{2}| \le \frac{1}{2}\}$

[illegible]



[illegible]

Transformation by the Function z^2 التحويل بواسطة الدالة z^2

$f(z) = z^2$

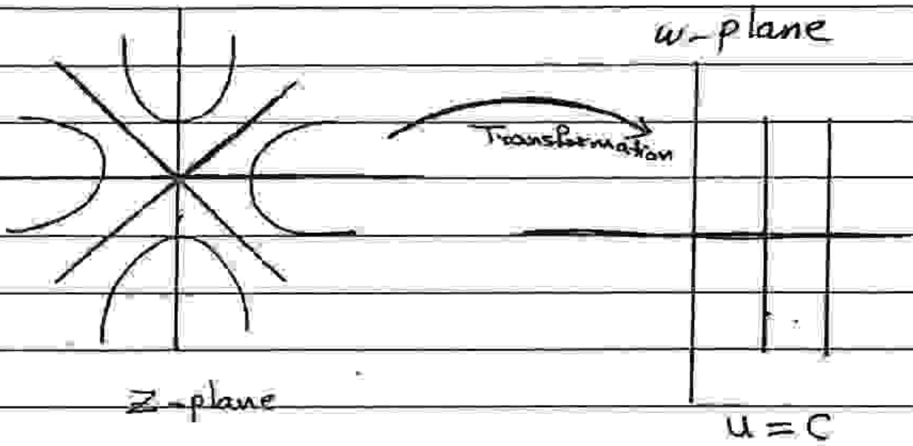
الآن لو مثلنا z^2 بالاعتماد على الاحداثيات المتعامدة

$f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + 2xyi$

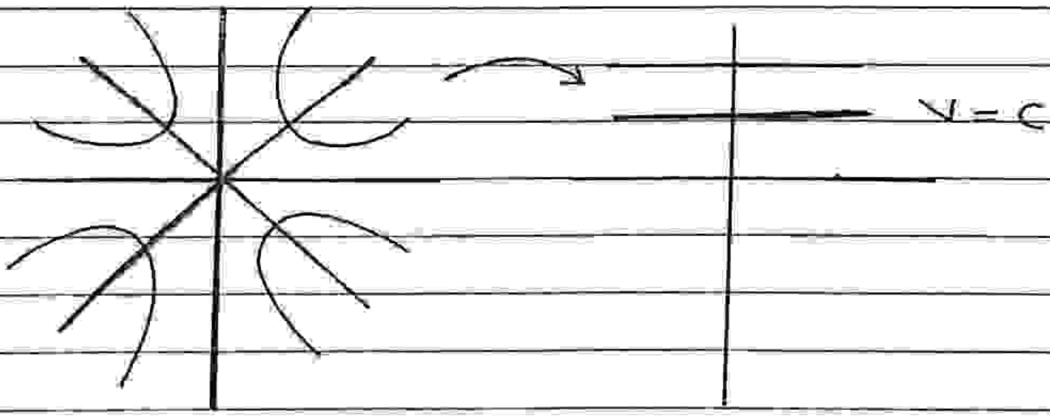
$\therefore u(x,y) = x^2 - y^2 \qquad v = 2xy$

اذا كانت u ثابت أي ان $x^2 - y^2 = c$

Then



Now, if $v = c$, that is $2xy = c$



اذا كان لدينا دوائر واشعه فان الدالة هي $f(z) = z^n$ (هنا الداله بصورة عامة)

كل جزء يذهب الى دائرة وكلما كبر الجزء كبرت الدائرة

Integrals of Complex Functions [illegible]

Definition: Let $f(t) = u(t) + iv(t)$ be a complex function in one real variable t defined on the interval $[a,b]$ with each of $u(t)$ and $v(t)$ is a sectionally continuous real function on $[a,b]$. Then the definite integral of f on $[a,b]$ denoted by $\int_a^b f(t)\,dt$ is defined by

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

[illegible]

[illegible]

[illegible]

Remark: The continuous function $z(t) = x(t) + iy(t)$ which is defined on $[a, b]$, (that is $a \le t \le b$) is called path and denoted it by C. $z(a)$ is called the initial point and $z(b)$ is called terminal point.

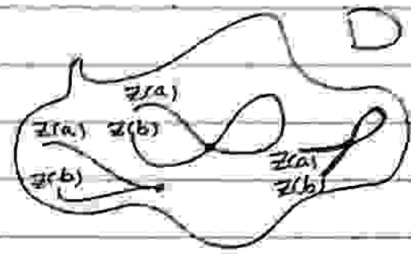
Now, we have the following:

1. If $z(a) = z(b)$, then C is called closed path. ([illegible])
2. If C is not intersection, that is $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$, then C is called simple path ([illegible])

3. If $Z(t_1) \neq Z(t_2)$ when $t_1 \neq t_2$ (اي لاتقاطع) , also $Z(a)=Z(b)$

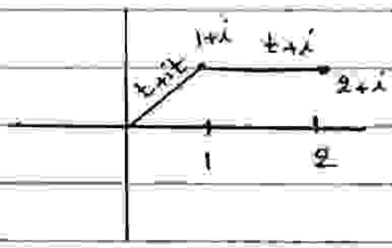
then the path (arc) C is Called simple closed path
منحني مغلق بسيط أو منحني جوردان

4. If the path is intersection with itself by one point , then the path is Called not simple path



Examples :-

1. $Z(t) = \begin{cases} t+it & 0 \le t \le 1 \\ t+i & 1 \le t \le 2 \end{cases}$

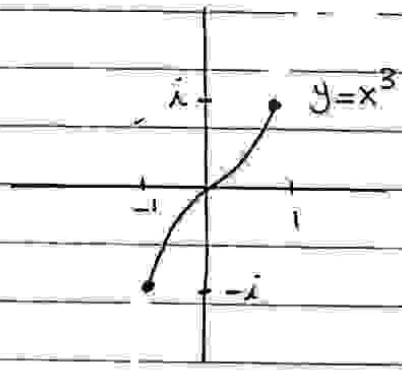


تمثل Z قطعة المستقيم الواصلة بين $z=0$ و $z=1+i$ متبوعة بقطعة مستقيم بين
$z=1+i$ و $z=2+i$ فلذلك Z تمثل درب بسيط

2. $Z(t) = \cos t + i \sin t \qquad 0 \le t \le 2$ Unit Circle
يمثل دائرة الوحدة مركزها $z=0$ وهي منحني بسيط مغلق

3. Draw the path (or the arc)
$C: Z(t) = t + it^3 \qquad -1 \le t \le 1$

Solution :- $x = t$ and $y = t^3 \Longrightarrow y = x^3$



The Parametric equation of a Circle

المعادلة الوسيطية للدائرة

The equation $|Z - Z_0| = r$ represent a Circle of radius r and Center at Z_0.

[illegible]

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[illegible] | معادلة دائرة مركزها نقطة الأصل ونصف قطرها ٢

[illegible]

[illegible] (أي معادلة محيط دائرة الوحدة)

The integration of Complex functions on path (or on a Contour)

التكامل المنحني أو التكامل على مسار

Definition :- The path $Z(t) = x(t) + iy(t)$ is Called a smooth path if

1. The functions $x(t)$ and $y(t)$ are differentiable on $(0,1)$

2. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \neq 0 \quad \forall \ 0<t<1$

~~Definition :- Z(t) is said to be a Contour if the Conditions~~

(1), (2) are satisfied on $[0,1]$ except at finite number of points.

Definition :- Let f be a Continuous function on D and let C be a Contour in D.

$f(z) = u(x,y) + iv(x,y)$ each of u and v is Conts. on D.

$z = x + iy \Rightarrow dz = dx + idy$.

$$f(z)\,dz \overset{\text{Def.}}{\equiv} (u(x,y) + iv(x,y))(dx + idy) =$$

$$[u(x,y)dx - v(x,y)dy] + i[u(x,y)dy + v(x,y)dx]$$

$$\therefore \int_C f(z)\,dz = \int_C \underbrace{u(x,y)dx - v(x,y)dy}_{\text{real part}} + i\int_C \underbrace{u(x,y)dy + v(x,y)dx}_{\text{Img part}}$$

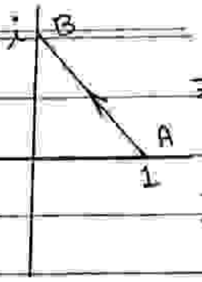
الآن اكتب القانون بدلالة t عندما نحول التكامل من $\int_C$ الى $\int_0^1$

فيصبح التكامل

$$\int_\alpha f(z)\,dz \overset{\text{def.}}{\equiv} \int_0^1 \left[u(x(t),y(t))\frac{dx}{dt} - v(x(t),y(t))\frac{dy}{dt}\right]dt + i\int_0^1 \left[u(x(t),y(t))\frac{dy}{dt} + v(x(t),y(t))\frac{dx}{dt}\right]dt$$

Example :- Find the value of integral $I = \int_C z\,dz$ where C is the line segment AB from $z=1$ to $z=i$.

Solution :- $f(z) = x + iy$
$\qquad\qquad = (1-t) + it$



$z(t) = x(t) + iy(t)$
$\quad = (1-t) + it$

$z(0) = 1$
$z(1) = i$

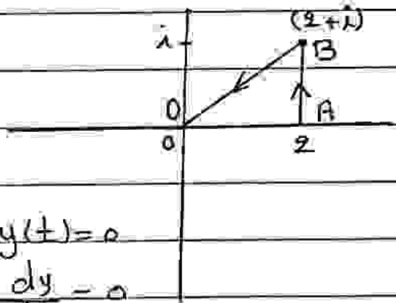
$u(x(t), y(t)) = 1 - t$
$v(x(t), y(t)) = t$

$$\int_C f(z)\,dz = \int_0^1 [(1-t)(-1) - t.1]\,dt + i\int_0^1 [(1-t).1 + t(-1)]\,dt$$

$$= \int_0^1 -dt + i\int_0^1 (1-2t)\,dt = -t\Big]_0^1 + i(t - t^2)\Big]_0^1 = -1$$

Example :- Find the value of the integral $I = \int_C z^2\,dz$

where C is the Contour OAB.



(1) on the path OA :

$z(t) = 2t + 0.i \qquad x(t) = 2t \qquad y(t) = 0$

Since $z(0) = 0 \qquad \dfrac{dx}{dt} = 2 \qquad \dfrac{dy}{dt} = 0$

$\quad z(1) = 2$

$f(z) = z^2 = (2t)^2 = 4t^2$

$$\int_{OA} f(z)\,dz = \int_0^1 (4t^2)(2)\,dt + i\int_0^1 (4t^2)(0) + 0 = \frac{8t^3}{3}\Big]^1 = \frac{8}{3}$$

(2) on the

$z(t) = 2 + it$

$x(t) = 2 \Longrightarrow \dfrac{dx}{dt} = 0$

$y(t) = t \Longrightarrow \dfrac{dy}{dt} = 1$

$f(z) = (x+iy)^2 = x^2 - y^2 + 2ixy = \underbrace{4 - t^2}_{u} + i\underbrace{4t}_{v}$

$$\therefore \int_{AB} f(z)\,dz = \int_0^1 (4-t^2)(0) - (4t).1\,dt + i\int_0^1 (4-t^2)\,1\,dt$$

$$= -2 + i\left(4-\frac{1}{3}\right) = -2 + i\,\frac{11}{3}$$

(3) on the path Bo

$z(t) = 2t + it$

$x(t) = 2t \Longrightarrow \frac{dx}{dt} = 2$

$y(t) = t \Longrightarrow \frac{dy}{dt} = 1$

$f(z) = x^2 - y^2 + i2xy$

$$= (2t)^2 - t^2 + 2i(2t)(t) = 3t^2 + i4t^2$$

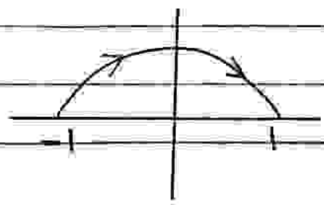
$$\int_{OB} f(z)\,dz = \int_0^1 [(3t^2)(2) - 4t^2]\,dt + i\int_0^1 [(3t^2) + (4t^2)(2)]\,dt$$

$$= \frac{2}{3} + i\,\frac{11}{3}.$$

$$\therefore \int_C f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BO} f(z)\,dz$$

$$\int_C f(z)\,dz = \frac{8}{3} - 2 + i\,\frac{11}{3} - \frac{2}{3} - i\,\frac{11}{3} = 0$$

Exercise: Find the value of integral $I = \int_C \bar{z}\,dz$ where C is the upper half of the Circle $|z| = 1$ from $z = -1$ to $z = 1$



Properties of line Integral

(خواص التكامل الخطي)

(1) $\int_C (\alpha f(z) \pm \beta g(z))\, dz = \alpha \int_C f(z)\,dz \pm \beta \int_C g(z)\,dz$

where α and β are Constant Complex numbers

(2) $\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$

لاحظ ان المسار $-C$ هو نفس المسار C ولكن بالاتجاه العاكس

(3) If the path C Consists of a finite number of arcs $C_1, C_2, \ldots, C_n$ joined in the end points, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

(4) If $f(z)$ is bounded function on a path C whose equation $C: z(t) = x(t) + iy(t)$, $a \le t \le b$,

then $\left| \int_C f(z)\,dz \right| \le ML$.

where M is an upper bound for the function $f(z)$ and L is the length of the path C.

That is, $|f(z)| \le M \quad \forall z \in C$

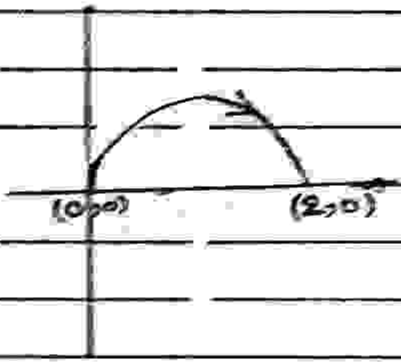
and The length of C is $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b |z'(t)|\, dt$

حيث ML تمثل حد (an upper bound) لقيمة $\left|\int_C f(z)\,dz\right|$

Example: Find $\left|\int_C e^{z}\,dz\right|$ when C is the upper half of the circle $|z-1|=1$ from $z=0$ to $z=2$

Solution: $|f(z)| = |e^{z}| = |e^{x+iy}|$

$= |e^{x}|\,|e^{iy}| = |e^{x}| \le e^{2}$ (لأن x تتغير من $0 \to 2$)



$\therefore C: z(t) = 1 + e^{it} \quad 0 \le t \le \pi$

$z'(t) = ie^{it}$

$\therefore L = \int_a^b |z'(t)|\,dt = \int_0^{\pi} |ie^{it}|\,dt = \int_0^{\pi} |i|\,|e^{it}|\,dt = \int_0^{\pi} 1.1\,dt$

$= t\Big]_0^{\pi} = \pi.$

$\therefore \left|\int_C e^{z}\,dz\right| \le \pi e^{2}.$

Proof (4):

$\left|\int_a^b f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right| \le \int_a^b |f(z(t))\,z'(t)|\,dt$

$= \int_a^b |f(z(t))|\,|z'(t)|\,dt$

$\le \int_a^b M|z'(t)|\,dt = M\underbrace{\int_a^b |z'(t)|\,dt}_{L} = ML.$

(5) $\displaystyle\int_C z^n\,dz = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n = -1 \end{cases}$

where C is the circle center at the origin and of radius r taken the positive direction.

نصف قطر الدائرة لا يؤثر على قيمة التكامل هنا.

(6) If C is the circle of center z_0 and radius r taken the positive direction, then $\displaystyle\int_C (z-z_0)^n\,dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$

Exerciese: Find the integrals of the following:

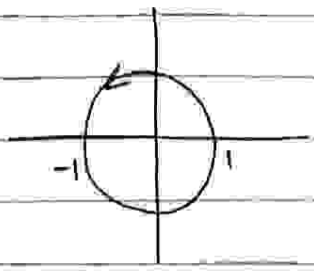
① $\displaystyle\int_C |z|^2\,dz$ ② $\displaystyle\int_C |z|\,dz$ ③ $\displaystyle\int_C y\,dz$

where C is the segment from $z=0$ to $z=1+i$

② Find the following integral:

$\displaystyle\int_C (x^2+iy^3)\,dz$, where C is the line from $z=1$ to $z=i$.

③ Find $\displaystyle\int_C x\,dz$, where C is unite circle



مبرهنة كرين

Green's Theorem :

وهي مبرهنة موجودة في التحليل الحقيقي سوف نوظفها في معرفة خواص اخرى للتكامل العقدي.

let each of $P(x,y)$ and $Q(x,y)$ be a real function defined on a region D such that their first order derivatives are continuous on $C \cup D$, where C is a semi-smooth simple closed path around the region D in the positive direction.

Then
$$\int_C (P\,dx + Q\,dy) = \iint_{C \cup D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Cauchy-Goursat Theorem

" مبرهنة كوشي - كورسا "

If $f(z)$ is an analytic function on and inside a semi-smooth simple closed path C in the positive direction, then $\int_C f(z)\,dz = 0$.

اذا كانت الدالة تحليلية داخل وعلى منحني مغلق شبه املس (كنتور) فان التكامل = صفر

Proof :

let $f(z) = u + iv \Rightarrow dz = dx + i\,dy$ where $z = x + iy$

Then
$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

$$= \int_C (\underbrace{u}_{P}\,dx \underbrace{- v}_{Q}\,dy) + i\int_C (\underbrace{v}_{P}\,dx + \underbrace{u}_{Q}\,dy) =$$

$$= \iint_{C\cup D} (-V_x - U_y)\,dx\,dy + i \iint_{C\cup D} (U_x - V_y)\,dx\,dy$$

But $f(z)$ is an analytic function on $C\cup D$. Therefore

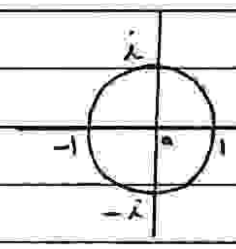
$U_x = V_y$ and $U_y = -V_x$. Then

$$\int_C f(z)\,dz = \iint_{C\cup D} (U_y - U_y)\,dx\,dy + i \iint_{C\cup D} (V_y - V_y)\,dx\,dy = 0 + i.0 = 0$$

$$\therefore \int_C f(z)\,dz = 0$$

Examples :- Compute the following :

1. $\oint_C (3z^2+1)\,dz$, where $C : |z| = 1$



بما أن الدالة $f(z)=(3z^2+1)$ تحليلية في داخل المنطقة
وعلى المحيط فإن التكامل على منحنى مغلق بسيط كنتور
= صفر

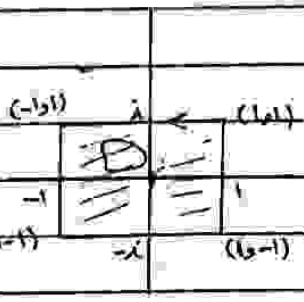
$$\therefore \oint_C (3z^2+1)\,dz = 0$$

2. $\oint_C \frac{2z}{z+3}\,dz$, where $C : |z| = 1$

Since $z+3=0 \Rightarrow z=-3$ إذن الدالة تحليلية في كل مكان
فقط عند $z=-3$ تكون الدالة غير تحليلية
ولكن $z=-3$ لا تنتمي للمنحنى C ولا توجد داخله $|z|=1$ $\therefore -3 \notin C : |z|=1$
فلذلك فإن التكامل لتلك الدالة = صفر

$$\therefore \oint_C \frac{2z}{z+3}\,dz = 0$$

3. $\oint_C \dfrac{(z^2+3z)e^z}{z^2+4}\,dz$, where C is given on the following figure



$z^2+4=0 \Rightarrow z^2=-4$

$\Rightarrow z=\mp 2i$. But $\mp 2i \notin C \cup D$

$\therefore \oint_C \dfrac{(z^2+3z)e^z}{z^2+4}\,dz = 0$

4. $\oint_C e^z dz$, where $C: |z| = 4$

since e^z is analytic function everywhere and C is Contour, then $\oint_C e^z dz = 0$

The Converse of Cauchy Goursat Theorem is not true in general, for example: $\int_C \dfrac{1}{z^2}\,dz = 0$, where $C: |z| = 2$

since $\dfrac{1}{z^2}$ is not analytic in C.

منطقة بسيطة الاتصال

Definition: A region D is Called simply Connected if the interior points of each semi smooth simple closed path in D lies in D. If D is not simply Connected it is Called multiply Connected (متعددة أو معقدة الاتصال)

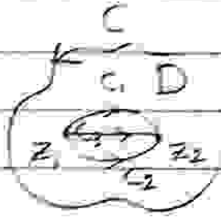
Scanned by CamScanner

Simply Connected

multiply Connected

$$D = C \cup C_1 \cup C_2 \cup C_3$$

Since



اذا خمنا z_1, z_2 نقطتين في منطقة بسيطة الاتصال لذلك فإن

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz - \int_{C_3} f(z)\,dz$$

اي التكامل على اي مسار له نفس القيمة

Cauchy Integral Formula (صيغة كوشي التكاملية)

let f be an analytic function function inside and on a simple semi-smooth closed path C in the positive direction and z_0 is a point inside C, then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$

That is, $\boxed{\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)}$

Examples: Compute the following:

1. $\int_C \frac{dz}{z(z+\pi i)}$ where $C: z(t) = -3i + e^{it}$ $0 \le t \le 2\pi$.

المنطقة هي دائرة مركزها $-3i$ ونصف قطرها $r=1$

Where $\frac{1}{z}$ is analytic function on C

and $z + \pi i = 0 \Rightarrow z = -\pi i$

at $z = -\pi i$ the function is not analytic

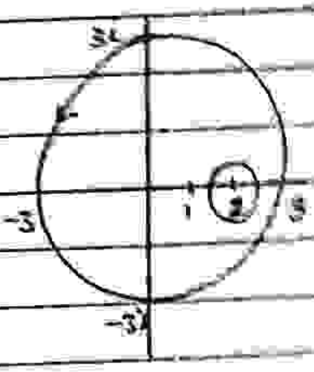
Since $\pi = 3.14$

[illegible]

$$\oint_C \frac{dz}{z(z+\pi i)} = \oint_C \frac{\frac{1}{z}}{z+\pi i}\, dz = 2\pi i\, f(-\pi i)$$

$$= 2\pi i \cdot \frac{1}{-\pi i} = \boxed{-2}$$

2. $\oint_C \frac{z^2\, dz}{z-2}$, where $C: |z| = 3$



$z-2=0 \Rightarrow z = 2$, $f(z) = z^2$

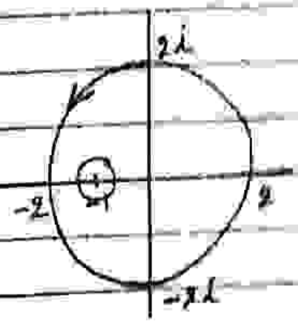
$$\Rightarrow \oint_C \frac{z^2\, dz}{z-2} = 2\pi i\, f(2)$$

$$= 2\pi i \cdot (2)^2 = 2\pi i \cdot 4$$

$$= 8\pi i.$$

3. $\oint_C \frac{e^z\, dz}{z+1}$ where $C: |z| = 2$

$f(z) = e^z$ and $z+1=0 \Rightarrow z = \boxed{-1}$



$$\oint_C \frac{e^z\, dz}{z+1} = 2\pi i \cdot e^{-1} = 2\pi i\, e^{-1}$$

The general Cauchy integral formula

صيغة كوشي العامة للتكامل

If f is an analytic function on and inside a simple semismooth closed path C ; the positive direction and if Z_0 is a point inside C, then

$$f^{(n)}(Z_0) = \frac{n!}{2\pi i} \int_C \frac{f(Z)}{(Z-Z_0)^{n+1}}\, dZ \qquad \forall n = 0, 1, 2, \ldots$$

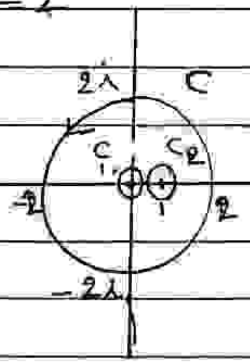
where $f^{(n)}(Z_0)$ denoted the nth-derivative of f at the point Z_0. تعني المشتقة من الرتبة n

That is,

$$\boxed{\int_C \frac{f(Z)}{(Z-Z_0)^{n+1}}\, dZ = \frac{2\pi i}{n!} f^{(n)}(Z_0)} \qquad \forall n = 0, 1, 2, \ldots$$

Example:

Evaluate $\displaystyle\int_C \frac{dZ}{(Z-1)^2 Z^2}$ where $C : |Z| = 2$



① $f(Z) = \dfrac{1}{(Z-1)^2}$ and f is not analytic in $Z = 0$

$$\therefore \int_{C_1} \frac{1/(Z-1)^2\, dZ}{Z^2} = 2\pi i\, f'(Z_0) = 2\pi i \cdot (-2) = \boxed{-4\pi i}$$

Since, $f(Z) = \dfrac{1}{(Z-1)^2} \Longrightarrow f'(Z) = \dfrac{(Z-1)^2 \cdot 0 - 1 \cdot 2}{(Z-1)^4} = \dfrac{-2}{(Z-1)^4}$

$$\therefore f'(0) = \frac{-2}{(0-1)^4} = \frac{-2}{1} = -2$$

② $f(Z) = \dfrac{1}{Z^2}$ and f is not analytic in $Z = 1$

$$f'(Z) = \frac{Z^2 \cdot 0 - 1 \cdot 2Z}{Z^4} = \frac{-2Z}{Z^4} \Longrightarrow f'(1) = \frac{-2}{1} = -2$$

$$\therefore \int_{C_2} \frac{1/z^2}{(z-1)^2}\, dz = 2\pi i \cdot f'(z_0)$$

$$= 2\pi i \cdot -2 = -4\pi i$$

$$\therefore \int_{C} \frac{dz}{(z-1)^2 z^2} = \int_{C_1} \frac{1/(z-1)^2}{z^2}\, dz + \int_{C_2} \frac{1/z^2}{(z-1)^2}\, dz$$

$$= -4\pi i + (-4\pi i) = -8\pi i$$

Exercises :- Evaluate the following:

1. $\int_{C} \frac{z^3 + e^z}{(z+\pi i)^3}\, dz$ where $C: z(t) = 4e^{it}$ $0 \leq t \leq 2\pi$

2. $\int_{C} \frac{dz}{(z-2)\, z^4}$ where $C: |z-3| = 2$

Serieses المتسلسلات

Complex Sequences متتابعات عقدية

Definition :- A complex sequence is a function from the set of natural numbers to the set of complex numbers. (that is, $f: N \longrightarrow \mathbb{C}$).

The symbol $\{Z_n\}_{n=1}^{\infty} = \{Z_1, Z_2, Z_3, \ldots\}$ to denote a sequence of complex numbers, also, Z_n is denoted to the nth-term (الحد النوني الذي يمثل الحد العام) of the sequence.

Examples :

(1) $\left\{\frac{i}{n}\right\}_{n=1}^{\infty} = \left\{i, \frac{i}{2}, \frac{i}{3}, \ldots \frac{i}{n}, \ldots\right\}$

(2) $\{(2i+1)^n\}_{n=1}^{\infty} = \{(2i+1), (2i+1)^2, (2i+1)^3, \ldots\}$.

Definition :- A sequence $\{Z_n\}_{n=1}^{\infty}$ is said to be Convergent sequence, if there exists a complex number Z such that $\lim_{n\to\infty} Z_n = Z$ (هذا يعني That is, $\forall \epsilon > 0$, $\exists$ a positive integer K s.t $|Z_n - Z| < \epsilon \quad \forall n > K$). Otherwise it is Called divergent.

Theorem :- If $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} z_n = w$, then

$z = w$ (أي بمعنى نقطة التقارب وحيدة)

Theorem :- If $z_n = x_n + iy_n$; $n = 1, 2, \ldots$ and $z = x + iy$,

then $\lim_{n\to\infty} z_n = \lim_{n\to\infty} (x_n + iy_n) = z = (x + iy)$ iff

$\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

قدمنا تلك المفاهيم البسيطة والمختصرة عن المتتابعات لكي نستفاد منها في دراسة وفهم المتسلسلات.

Serieses المتسلسلات

Definition :- let $\{z_n\}_{n=1}^{\infty}$ be a sequence. The sum

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$ is Called

an infinite series of Complex numbers. The term z_n is

Called the nth - term of the sequence of $\{z_n\}_{n=1}^{\infty}$

Note that : $\{z_n\}_{n=1}^{\infty}$ is a sequence. we put

$S_1 = z_1$
$S_2 = z_1 + z_2$
$S_3 = z_1 + z_2 + z_3$
$\vdots$
$S_n = z_1 + z_2 + \cdots + z_n$ (متتابعة المجاميع الجزئية) (S_n is Called the partial sum) of the series

Definition: A series $\sum_{n=1}^{\infty} z_n$ is said to be Convergent to S (written $\sum_{n=1}^{\infty} z_n = S$ if the sequence of the partial sum $\{S_n\}_{n=1}^{\infty}$ is Convergent to S. That is, $\lim_{n\to\infty} S_n = S$. (S is Called the sum of the series)

تكون المتسلسلة متقاربه اذا كانت متتابعه المجاميع الجزئيه متقاربه

But, if the series is not Convergent, it is Called divergent

Theorem: let $z_n = x_n + iy_n$ $\forall n = 1, 2, \ldots$ and let $S = X + iY$, $S_n = X_n + iY_n$. Then $\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} X_n = X$ and $\sum_{n=1}^{\infty} Y_n = Y$

Proof: $\Rightarrow$) suppose that $\sum_{n=1}^{\infty} z_n = S$. Therefore

$$\lim_{n\to\infty} S_n = S. \text{ Hence } \lim_{n\to\infty} (X_n + iY_n) = X + iY$$

$$\therefore \lim_{n\to\infty} X_n = X \quad \text{and} \quad \lim_{n\to\infty} Y_n = Y$$

$\Leftarrow$) let $\lim_{n\to\infty} X_n = X$ and $\lim_{n\to\infty} Y_n = Y$

$$\text{Then } \lim_{n\to\infty} X_n + i \lim_{n\to\infty} Y_n = X + iY$$

$$\Rightarrow \lim_{n\to\infty} (X_n + iY_n) = X + iY$$

هناك الكثير من الاختبارات التي تستعمل لاختبار تقارب متسلسلات الاعداد الحقيقية والتي سوف نستخدمها في موضوعنا هذا .

Tests of Convergence اختبارات التقارب

(1) Ratio Test اختبار النسبة

(حدودها غير سالبة)

let $\sum_{n=1}^{\infty} x_n$ be a series of real numbers with non-negative terms. let $r = \lim_{n \to \infty} \frac{x_{n+1}}{x_n}$. Then:

(a.) if $r < 1$, then the series $\sum_{n=1}^{\infty} x_n$ is Convergent

(b.) if $r > 1$, = = = = = divergent

(c.) if $r = 1$, يفشل الاختبار

(2) Root Test اختبار الجذر

let $\sum_{n=1}^{\infty} x_n$ be a series of non-negative real numbers and let $r = \lim_{n \to \infty} \sqrt[n]{x_n}$. Then:

(1) if $r < 1$, then the series is Convergent.

(2) if $r > 1$, = = = = divergent.

(3) Comparative Test اختبار المقارنة

(a) let $\sum_{n=1}^{\infty} X_n$ be a series of real numbers with non-negative terms and if $\sum_{n=1}^{\infty} Y_n$ be a Convergent series such that $X_n \leq Y_n \quad \forall n > M$ (M is a positive integer). Then $\sum_{n=1}^{\infty} X_n$ is Convergent.

(b) let $\sum_{n=1}^{\infty} X_n$ be a series of non-negative real numbers and if $\sum_{n=1}^{\infty} Y_n$ is divergent series with positive terms such that $X_n \geq Y_n \quad \forall n > M$. Then $\sum_{n=1}^{\infty} X_n$ is divergent.

(4) Test of Altarnating series اختبار المتسلسلات المتناوبة

If $\sum_{n=1}^{\infty} (-1)^n X_n$ is a series such that $X_n > 0 \quad \forall n$ and

(1) $\lim_{n \to \infty} X_n = 0$ and (2) $X_{n+1} \leq X_n \quad \forall n > M$, then

$\sum_{n=1}^{\infty} (-1)^n X_n$ is Convergent.

(5) Test of Geometric series اختبار المتسلسلات الهندسية

A series of the form $\sum_{n=1}^{\infty} ar^n$ is Called a geometric series

It is convergent if $|r|<1$, and divergent if $|r|>1$.

(5) Test of P_ series (اختبار المتسلسلات من النمط P)

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called P_ series, it is convergent if $P>1$, and divergent if $0<P\le 1$.

Examples :-

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+--- \text{ is divergent.}$$

since it is a p_ series and $P=1$.

Definition:- A series $\sum_{n=1}^{\infty} Z_n$ is called absoluitly convergent if the series $\sum_{n=1}^{\infty} |Z_n|$ is convergent.

Since, every absoluitly convergent series is convergent and the converse is not true ingeneral.

Power Series

متسلسلة القوى

Definition :- A series of the form $\sum_{n=0}^{\infty} a_n (Z-C)^n$ is Called a power series where $C, a_0, a_1, a_2, a_3, \ldots$ are Constant Complex numbers, C is Called the center of Series.

$$\sum_{n=0}^{\infty} a_n(Z-C)^n = a_0 + a_1(Z-C) + a_2(Z-C)^2 + \cdots .$$

A power Series is said to be Convergent at Z_0 if the series $\sum_{n=0}^{\infty} a_n (Z_0 - C)^n$ is Convergent.

Remark :- For any Power series $\sum_{n=0}^{\infty} a_n (Z-Z_0)^n$, $\exists$ real number $R \geqslant 0$ such that if :

$|Z-Z_0| < R$, then the series is Convergent for all Z and The series is divergent for all Z is satisfing $|Z-Z_0| > R$.

نصف قطر التقارب

The real number R is Called the radius of Convergence and the Circle which have a Center Z_0 and radius R is Called Circle of Convergence. دائرة أو منطقة التقارب

If $R=0$, then the series is Convergent only at the point z_0.

If $R=\infty$, then the series is Convergent for all Complex number z.

We Can find the radius of Convergence by the following methods:

1. By ratio test اختبار النسبة

1- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$, then $R=\frac{1}{L}$.

2. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, then $R=\infty$.

3. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $R=0$.

2. By the root test اختبار الجذر

$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}$, if $L\neq 0 \Rightarrow R=\frac{1}{L}$

if $L=0 \Rightarrow R=\infty$

if $L=\infty \Rightarrow R=0$

Example :- Find the radius and the circle of Convergence of the series.

$$\sum_{n=0}^{\infty}\left(\frac{6n+1}{2n+5}\right)^n (z-2i)^n$$

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{6n+1}{2n+5}=\lim_{n\to\infty}\frac{6+\frac{1}{n}}{2+\frac{5}{n}}=3$$

$R=\frac{1}{L}=\frac{1}{3}$ and $|z-2i|=\frac{1}{3}$ is the circle of Convergence.

Then, the series is Convergent for all z such that $|z-2i|<\frac{1}{3}$.

Example :- Find the radius of Convergence for the series $\sum_{n=0}^{\infty}\frac{2^n(z-i)^n}{n!}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{2^{n+1}\cdot n!}{2^n(n+1)!}=\lim_{n\to\infty}\frac{2}{n+1}=0$$

$\therefore R=\frac{1}{L}=\infty$ and the series is Convergent for all z such that $|z-i|<\infty$.

Taylor's Theorem :- مبرهنة تيلر

If $f(z)$ is analytic function inside the circle $C: |z-z_0|=r$ then for each z inside C

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0) \quad , \text{ where}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}$$

when $z_0=0$, then the series is Called Maclaurin Series متسلسلة ماكلورين .

Example :- Write Taylor series for the function $f(z)=\frac{1}{1-z}$ about $z=0$.

$f(z)$ is analytic function on $\mathbb{C}/\{1\}$, we Can choose $C: |z|=1$, then $\frac{1}{1-z}$ is analytic inside C. That is in the region $|z|<1$.

$$f(z)=\frac{1}{1-z}=\sum_{n=0}^{\infty} a_n (z-0)^n = \sum_{n=0}^{\infty} a_n z^n \quad \text{where}$$

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \forall\, n=0,1,2,\ldots$$

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$f(0) = 1$, $f'(z) = \frac{1}{(1-z)^2} \Rightarrow f'(0) = 1$, $f''(z) = \frac{2}{(1-z)^3} = 2 = 2!$

$f'''(z) = \frac{6}{(1-z)^4} = 6 = 3!$, ------

$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \Rightarrow f^{(n)}(0) = n!$

$\therefore \frac{1}{1-z} = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \text{----} = \sum_{n=0}^{\infty} z^n$

$= 1 + z + z^2 + z^3 + \text{——} + z^n + \text{——}$

Exercises : ① Write the function $\frac{1}{1-z^2}$ as a Taylor series about $z = 0$.

② write the function $\frac{1}{1+z}$ as a Taylor series about $z = 0$

③ write the function $\frac{1}{1-z}$ as a Taylor series about $z = -i$.

Lourent's Theorem : مبرهنة لورنت

If $f(z)$ is analytic function in the region $r_2 \leq |z-a| \leq r_1$ where $r_2 < r_1$, then for all z in region $r_2 < |z-a| < r_1$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \qquad n = 0, 1, 2, \ldots$

C_1 هو محيط الدائرة التي نصف قطرها X_1

$a_{-n} = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw \qquad n = 1, 2, \ldots$

حيث C_2 هو محيط الدائرة التي نصف قطرها X_2 .

Example :- Find Laurent's seres for the function

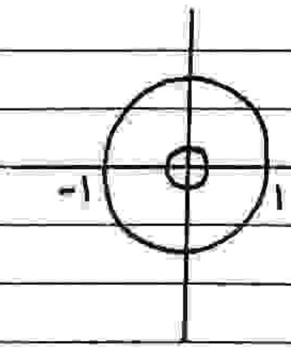
$f(z) = \frac{1}{z(1-z)}$ in the region $0 < |z| < 1$

Solution :- $\frac{1}{z(1-z)} = \frac{A}{z} + \frac{B}{1-z} = \frac{A - Az + Bz}{z(1-z)}$

$\Rightarrow B - A = 0 \Rightarrow A = B$

$\therefore A = 1 \Rightarrow B = 1$

$\therefore f(z) = \frac{1}{z} + \frac{1}{1-z} = \frac{1}{z} + \sum_{n=0}^{\infty} z^n$



Exersice :- write Laurent series for the function

$f(z) = z^2 e^{1/z}$ in the region $0 < |z| < 1$.

If $\{S_n(z)\}$ converges uniformaly, then we say that $\sum_{n=0}^{\infty} a_n z^n$ converges uniformaly. $\sum_{n=0}^{\infty} a_n z^n = \lim_{n\to\infty} S_n(z)$

Taylor's Theorem

Let D be a domain in $\mathbb{C}$, and let f be an analytic function on D. Let z_0 be any point in D, then there exists a ball $B_r(z_0)$ in D and a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ s.t $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \; \forall z \in B(z_0)$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} \, dz \quad \forall n \geq 0$$

Morever the series is unique.

proof :- Let $z_0 \in D$, since D is a domain.

$\exists \; B = B_r(z_0) \subseteq D$

Assume that $\partial B \subseteq D$

Now, $\forall z \in B \Rightarrow f(z) = \frac{1}{2\pi i} \oint_{\partial B} \frac{f(t)}{t-z} \, dt$ By C.I.F

$$\frac{1}{t-z} = \frac{1}{(t-z_0)-(z-z_0)} = \frac{1}{(t-z_0)\left[1 - \frac{z-z_0}{t-z_0}\right]} \qquad t \neq z_0$$

But $\left|\frac{z-z_0}{t-z_0}\right| < 1$, hence $\sum_{n=0}^{\infty} \left(\frac{z-z_0}{t-z_0}\right)^n$ is converges

Then $$\frac{1}{1-\left(\frac{z-z_0}{t-z_0}\right)} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{t-z_0}\right)^n$$

$$\frac{1}{t-z} = \frac{1}{(t-z_0)\left[1-\frac{z-z_0}{t-z_0}\right]} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}}$$

and the series converges uniformaly, where

$\left| \frac{z-z_0}{t-z_0} \right| < 1$

$\therefore \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}}$ is converges uniformaly.

$$\frac{f(t)}{t-z} dt = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}} f(t)\, dt.$$

$$\therefore f(z) = \frac{1}{2\pi i} \oint_{\partial B} \frac{f(t)}{(t-z)} dt = \frac{1}{2\pi i} \oint \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}} f(t) dt$$

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i} \oint_{\partial B} \frac{f(t)}{(t-z_0)^{n+1}} dt$$

$$\therefore a_n = \frac{1}{2\pi i} \oint_{\partial B} \frac{f(t)\, dt}{(t-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!} \text{ by G.C.I.F}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

The proof of uniqeness

we know that

$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots$ (*)

Assume that $f(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots\cdots$ (**)

If $z = z_0 \Rightarrow f(z_0) = a_0$ from (*) and $f(z_0) = b_0$ from (**)
we have $a_0 = b_0$

$f'(z) = a_1 + 2a_2(z-z_0) + \cdots\cdots$

$f'(z) = b_1 + 2b_2(z-z_0) + \cdots$

$\therefore f'(z_0) = a_1$ and $f'(z_0) = b_1 \Rightarrow a_1 = b_1$

⋮

etc

~~$a_{-1} = \frac{1}{2\pi} \oint_{\gamma} f(z)\,dz$~~

a_{-1} is called the residue of f at z_0

- If there exists the positive integer K s.t $a_{-n} = 0$ $\forall\, n > K$.

when $n > 0$, then z_0 is called a pole for f

- I. $a_{-n} = 0$ $\forall\, n > K$ and $a_{-K} \neq 0$, then z_0 is a pole of f of order K.

- If z_0 is not a pole then z_0 is called essential singularity for f

- A pole of order 1 is called a simple pole and a pole of order 2 is called a double pole.

Theorem:- If f has a pole of order K at z_0 then $\frac{1}{f}$ is analytic at z_0 and has a zero of order K at z_0. Conversely if f is analytic at z_0 and has a zero of order K at z_0, then $\frac{1}{f}$ has a pole of order K at z_0

~~The proof Laurent's Theorem~~

Proof:- It is enough to prove that

$$f(z) = \sum_{n=0}^{\infty} A_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{B_n}{(z-z_0)^n} \quad \text{where}$$

$$A_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} \quad n = 0, 1, 2, \ldots$$

$$B_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} \qquad n=1,2,\dots$$

By cauchy integral formula:-

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)}{t-z}\,dt \quad \forall z \in D.$$

$$\frac{1}{t-z} = \frac{1}{(t-z_0)-(z-z_0)} = \frac{1}{(t-z_0)\left[1-\dfrac{z-z_0}{t-z_0}\right]}$$

$$= \frac{1}{t-z_0} \cdot \frac{-1}{1-\dfrac{z-z_0}{t-z_0}} = \frac{1}{t-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{t-z_0}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}}$$ is absolutly and uniformaly convergent in D.

Also $\dfrac{-1}{t-z}$ $\frac{1}{z-t} \rightleftarrows$

$$\frac{-1}{t-z} = \frac{1}{(z-z_0)-(t-z_0)} = \sum_{n=0}^{\infty} \frac{(t-z_0)^n}{(z-z_0)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{(t-z_0)^{-n}(z-z_0)^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(t-z_0)^{-n+1}(z-z_0)^n}$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)\,dt}{t-z} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)\,dt}{t-z}$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} f(t)\,dt \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^{n+1}} - \frac{1}{2\pi i} \oint_{C_2} f(t)\,dt \sum_{n=1}^{\infty} \frac{1}{(t-z_0)^{-n+1}(z-z_0)^n}$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\frac{1}{2\pi i} \oint_{C_1} \frac{f(t)\,dt}{(t-z_0)^{n+1}}}_{A_n} - \sum_{n=1}^{\infty} \frac{1}{(z-z_0)^n} \cdot \underbrace{\frac{1}{2\pi i} \oint_{C_2} \frac{f(t)\,dt}{(t-z_0)^{-n+1}}}_{B_n}$$

<u>Example</u>:- $f(z) = \frac{z^2-2z+3}{z-2}$

$$f(z) = \frac{z(z-2)+3}{z-2} = z + \frac{3}{z-2} = (z-2)+2+\frac{3}{z-2}$$

$\therefore$ 2 is a simple pole

$$a_{-1} = 3 = \frac{1}{2\pi i}\oint f(z)\,dz$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{k+1}}{(z-z_0)^{k+1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$a_{-k} \neq 0$

f has a pole of order k at z_0.

$$f(z) = \frac{1}{(z-z_0)^k}\,[\underbrace{a_{-k} + a_{-k+1}(z-z_0) + \cdots}_{g(z)}]$$

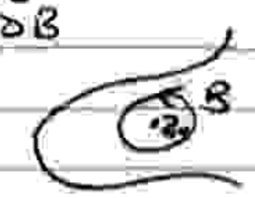
$$= (z-z_0)^{-k}\, g(z).$$

$\frac{1}{f}$ has a zero of order k at z_0.

$$f'(z_0) = (z-z_0)^{-k}\, g'(z) - k(z-z_0)^{-n-1}\, g(z).$$

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{k}{z-z_0}$$

$$\frac{1}{2\pi i}\oint_{\partial B}\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{\partial B}\frac{g'(z)}{g(z)}\,dz - \frac{k}{2\pi i}\overbrace{\oint_{\partial B}\frac{1}{z-z_0}\,dz}^{2\pi i}$$



$\oint \frac{g'(z)}{g(z)}\,dz = 0$ because $\frac{g'}{g}$ is analytic in B. and hence

$$\frac{1}{2\pi i}\oint_{\partial B}\frac{f'(z)}{f(z)} = -k$$

$$\frac{1}{z+i} = \frac{1}{z+i+i-i} = \frac{1}{2i+z-i} = \frac{1}{2i[1+\frac{z-i}{2i}]}$$

$$= \frac{1}{2i}\left[\frac{1}{1+\frac{z-i}{2i}}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^{n+1}}$$

$a_{-1} = Re(f) = \frac{1}{2}$

i is simple pole of f.

Residues Theorem

Let D be a simply connected domain bounded $\partial D = C$. f is analytic on D except at finit number of poles $\{b_1, b_2, \dots, b_n\}$. Assume f is cont. on $\partial D = C$.

$\frac{1}{2\pi i}\oint_C f(z)\,dz$ = The sum of residues of f in D.

$$= \sum_{j=1}^{n} Res_{b_j}(f)$$

Proof: $\forall\, b_j \in D$, $\exists$ a ball $B(b_j) \subset D$ st $1 \le j \le n$.

$$f(z) = (z-z_0)^{-m_j}\, g_j(z).$$

$g_j(z)$ is analytic in $B(b_j)$ st $g_j(z) \neq 0$ $\forall\, z \in \overline{B}(b_j)$

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_{j=1}^{n}\frac{1}{2\pi i}\oint_{\partial B_j} f(z)\,dz = \sum_{j=1}^{n} Res_{b_j}(f)$$

Remark: (1) Assum that f has a pole of order 1 at Z_0

$$f(Z) = \frac{a_{-1}}{Z-Z_0} + a_0 + a_1(Z-Z_0) + \cdots$$

$$(Z-Z_0) f(Z) = a_{-1} + a_0 \underbrace{(Z-Z_0)}_{\to 0} + a_1 \underbrace{(Z-Z_0)^2}_{\to 0} + \cdots$$

$$\therefore \lim_{Z \to Z_0} (Z-Z_0) f(Z) = a_{-1}$$

$$\boxed{\therefore \lim_{Z \to Z_0} (Z-Z_0) f(Z) = a_{-1}}$$

(2) Assum that f has a pole of order 2 at Z_0

$$f(Z) = \frac{a_{-2}}{(Z-Z_0)^2} + \frac{a_{-1}}{(Z-Z_0)} + a_0 + a_1(Z-Z_0) + \cdots$$

$$(Z-Z_0)^2 f(Z) = a_{-2} + a_{-1}(Z-Z_0) + a_0(Z-Z_0)^2 + a_1(Z-Z_0)^3 + \cdots$$

$$\frac{d}{dZ}(Z-Z_0)^2 f(Z) = a_{-1} + 2a_0(Z-Z_0) + \cdots$$

Thus

$$\boxed{\lim_{Z \to Z_0} \frac{d}{dZ}(Z-Z_0)^2 f(Z) = a_{-1}}$$

In general if Z_0 is a pole of order n for f.

$$\boxed{\therefore a_{-1} = \operatorname{Res}_{Z_0}(f) = \frac{1}{(n-1)!} \lim_{Z \to Z_0} \frac{d^{n-1}}{dZ^{n-1}} (Z-Z_0)^n f(Z)}$$

Let C be a simple closed path in $\mathbb{C}$.
f is cont. complex valued function defined on C.
$f(Z) \neq 0 \quad \forall Z \in \mathbb{C}$.

- we let Z goes around C once in the positive direction. Then $f(Z)$ will go around 0