جامعة بغداد

كلية التربية للعلوم الصرفة- ابن الهيثم

قسم الرياضيات

المرحلة الثانيه

التفاضل المتقدم

اعداد

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Chapter Five

Multiple Integrals

Finding an Integral is the reverse of finding Derivative. So Integral and Derivative are opposites.

Ordinary Integral:

If $y = f(x)$ then

$$F(x) = \int_A f(x) \leftrightarrow \frac{d(F(x))}{dx} = f(x)$$

Double Integral:

- If $w = f(x, y)$ then

$\int_A \int f(x, y) dxdy$ or $\int_A \int f(x, y) dydx$

Triple Integral:

- If $w = f(x, y, z)$ then

$\int \int_A \int f(x, y, z) dxdydz$ or $\int \int_A \int f(x, y, z) dydxdz$ or $\int \int_A \int f(x, y, z) dzdydx$

$\int \int_A \int f(x, y, z) dxdzdy$ or $\int \int_A \int f(x, y, z) dydzdx$ or $\int \int_A \int f(x, y, z) dzdxdy$

- If $w = f(x, y, z, \dots)$ then

$$\int \int \int_A \dots \int f(x, y, z, \dots) dxdydz \dots$$

Integral Methods:

In Multiple Integral use the same rules and properties of Ordinary Integral but when Integral to one variable the other variables will be constants. We are explaining as follows:

Double Integral:

$\iint f(x,y)dxdy$, in this case Integral to *x* and *y* is constant, follow by Integral to *y* and *x* is constant.

or $\iint f(x,y)dydx,$ in this case Integral to *y* and *x* is constant, follow by Integral to *x* and *y* is constant.

Triple Integral:

$\iiint f(x,y,z)dxdydz$, in this case Integral to *x* but *y* and *z* are constants, follow by Integral to *y* but *x* and *z* are constants, follow by Integral to *z* but *x* and *y* are constants.

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Types of Multiple Integrals:

There are two types of Multiple Integral:

1- Indefinite Multiple Integral no specific values and the results of this Integral is function plus the 'Constant of Integration'. It is there because of all the functions whose same derivative.

The Indefinite Double Integral is defining as follows:

$$\iint f(x,y)dxdy = \int [g(x,y) + C_1]dy = h(x,y) + C_2$$

2- Definite Multiple Integral (with upper and lower limits) has start and end values, in other words there is an interval (a to b) and the result of this Integral is constant.

The Definite Double Integral is defining as follows:

$\int_A \int f(x,y)dA$ (Where A is a region of defined x and y)

Definite Multiple Integral:

There are two kinds of Definite Multiple Integral depends on value of A:

1- The region A is geomantic shapes like Square, Rectangle and so on. The x and y variables are defined as follows:

$$A:\ a \le x \le b, \qquad c \le y \le d$$

$$\int_A \int f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

2- The region A are functions for variables x and y but it is not geomantic shape. There are two kinds and it is defending as follows:

a-

$$A:\ a \le x \le b\ , \qquad g_1(x) \le y \le g_2(x)$$

$$\int_A \int f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

b-

$$A: \; h_1(y) \le x \le h_2(y)\,, \qquad c \le y \le d$$

$$\int_A\int f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Properties of Multiple Integrals:

1- If $f(x,y) \ge 0 \;\; in\; A \;\; \to \int_A\int f(x,y)dA \ge 0$

2- If $f(x,y) \ge g(x,y) \;\; \forall x,y \;\in A \;\to \int_A\int f(x,y)dA \ge \int_A\int g(x,y)dA$

3- If $A = A_1 + A_2 \;\to$

$\int_A\int f(x,y)dA = \int_{A_1}\int f(x,y)dA_1 + \int_{A_2}\int f(x,y)dA_2$

4- $\int_A\int kf(x,y)dA = k\int_A\int f(x,y)dA$

5- $\int_A\int [f(x,y) \pm g(x,y)]dA = \int_A\int f(x,y)dA \pm \int_A\int g(x,y)dA$

Example 1: Calculate the following Double Integrals:

1- $\int_A\int f(x,y)dA$ for $f(x,y) = 1 - 6x^2y \; and \;\; A: 0 \le x \le 2, -1 \le y \le 1,$

2- $\int_0^{\pi}\int_0^{x} xsiny\; dydx,$ and

3- $\int_0^{\pi}\int_0^{sinx} y\; dydx$ **(H.W.)**

Solution: **1-** $\int_{-1}^{1}\int_0^{2}(1 - 6x^2y)dxdy = \int_{-1}^{1}[x - 2x^3y]_{x=0}^{x=2}dy =$

$\int_{-1}^{1}[2 - 16y]\,dy = [2y - 8y^2]_{y=-1}^{y=1} = [2 - 8] - [-2 - 8] = -6 + 10 = 4$

Reversing the order or Integration givens the same answer:

$\int_0^{2}\int_{-1}^{1}(1 - 6x^2y)dydx$=$\int_0^{2}[y - 3x^2y^2]_{y=-1}^{y=1}dx =$

$$\int_0^2 [(1-3x^2)-(-1-3x^2)]dx = \int_0^2 2dx = 2x\Big]_{x=0}^{x=2} = 4$$

2- $\int_0^{\pi}\int_0^{x} x\sin y\,dydx = \int_0^{\pi}[x(-\cos y)]_{y=0}^{y=x}dx = \int_0^{\pi}[x(-\cos x+\cos 0)]dx$

$$= \int_0^{\pi}[-x\cos x + x)]dx = -\int_0^{\pi} x\cos x\,dx + \int_0^{\pi} xdx$$

$$u = x, dv = \cos x dx \ \rightarrow du = dx, v = \sin x$$

$$= -[x\sin x]_0^{\pi} + \int_0^{\pi} \sin x dx + \frac{x^2}{2}\Big]_{x=0}^{x=\pi}$$

$$= -(\pi\sin\pi - 0) - \cos x\Big]_{x=0}^{x=\pi} + [\frac{\pi^2}{2} - 0]$$

$$=0 + [-\cos\pi + \cos 0] + \frac{\pi^2}{2} = [1+1] + \frac{\pi^2}{2} = 2 + \frac{\pi^2}{2}$$

<u>**Example 2**</u>**:** Calculate the following Triple Integrals:

$1-\ \int_0^1\int_x^1\int_0^{y-x} dz\,dy\,dx$, and

2- $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y}^{2-x^2-y^2} dz\,dx\,dy$ **(H.W.)**

Solution: 1-

$\int_0^1\int_x^1\int_0^{y-x} dz\,dy\,dx = \int_0^1\int_x^1[\int_0^{y-x} dz]dy\,dx = \int_0^1\int_x^1 z]_{z=0}^{z=y-x} dy\,dx =$
$\int_0^1[\int_x^1 (y-x)dy]\,dx = \int_0^1[\frac{y^2}{2} - xy]_{y=x}^{y=1} dx = \int_0^1[\frac{1}{2} - x - \frac{x^2}{2} + x^2]\,dx =$
$\int_0^1(\frac{1}{2} - x + \frac{x^2}{2})dx = \frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6}]_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$

The equivalent Integrals:

In Definite Multiple Integral, we can write equivalent integral to this integral by change upper and lower limits and the answer still the same. The equivalent integral is using in two cases: If ask in the question or if we cannot find the answer of the Multiple Integral.

Example 3: Calculate the Equivalent Double Integrals for the following Double Integrals:

1- $\int_0^2 \int_1^{e^x} dydx,$ **2-** $\int_0^2 \int_{x^2}^{2x} dydx,$ and **3-** $\int_0^1 \int_{\sqrt{y}}^1 dxdy$ **(H.W.)**

Solution: **1-** $\int_0^2 [\int_1^{e^x} dy]dx = \int_0^2 y]_{y=1}^{y=e^x} dx = \int_0^2 [e^x - 1]dx =$

$[e^x - x]_{x=0}^{x=2} = e^2 - 2 - e^0 + 0 = e^2 - 3$

To find the Equivalent Integral

$x = 2 \quad , y = e^x \rightarrow x = 2, y = e^2$

$x = 0 \quad , y = 1 \rightarrow \quad x = lny, \quad y = 1$

$$\int_1^{e^2} [\int_{lny}^2 dx]dy = \int_1^{e^2} x]_{x=lny}^{x=2} dy = \int_1^{e^2} [2 - lny]dy$$

$= 2\int_1^{e^2} dy - \int_1^{e^2} lnydy, \qquad u = lny, dv = dy \rightarrow du = \frac{dy}{y}, v = y$

$=2y]_{y=1}^{y=e^2} - \; ylny]_{y=1}^{y=e^2} + \int_1^{e^2} y\frac{dy}{y} = 2e^2 - 2 - e^2lne^2 + ln1 +$

$\int_1^{e^2} dy = 2e^2 - 2 - 2\,e^2 + ln1 + y]_{y=1}^{y=e^2} = -2 + e^2 - 1 = e^2 - 3$

2- $\int_0^2 [\int_{x^2}^{2x} dy] dx = \int_0^2 y]_{y=x^2}^{y=2x} dx = \int_0^2 [2x - x^2] dx =$

$$2\int_0^2 x dx - \int_0^2 x^2 dx = x^2 - \frac{x^3}{3}]_{x=0}^{x=2} = 4 - \frac{8}{3} = \frac{4}{3}$$

To find the Equivalent Integral

$x = 2 \quad , y = 2x \rightarrow \quad y = 4, x = \sqrt{y}$

$x = 0 \quad , y = x^2 \rightarrow \quad y = 0, x = \frac{y}{2}$

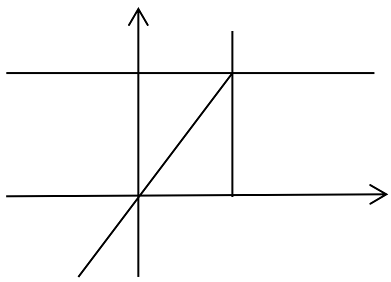
$$\int_0^4 [\int_{\frac{y}{2}}^{\sqrt{y}} dx] dy = \int_0^4 x]_{x=\frac{y}{2}}^{x=\sqrt{y}} dy = \int_0^4 \left[\sqrt{y} - \frac{y}{2}\right] dy =$$

$$\int_0^4 \sqrt{y} dy - \int_0^4 \frac{y}{2} dy = \frac{2}{3} y^{\frac{3}{2}} - \frac{y^2}{4}]_{y=0}^{y=4} = \frac{2}{3}(\sqrt{4})^3 - \frac{16}{4}$$

$$= \frac{16}{3} - 4 = \frac{4}{3}$$

<u>Example 4</u>: Calculate the equivalent Double Integral:

$\int_0^2 \int_y^2 e^{x^2} dxdy$



Solution:

$x = 2 \quad , y = 2 \rightarrow \quad x = 2, y = x$

$x = y \quad , y = 0 \rightarrow \quad x = 0, y = 0$

$$\int_0^2 [\int_0^x e^{x^2} dy] dx = \int_0^2 e^{x^2} y]_{y=0}^{y=x} dx = \int_0^2 e^{x^2} x dx = \frac{1}{2} e^{x^2}]_{x=0}^{x=2} =$$

$$\frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1)$$

Exercises:

1- Sketch the regain of integration and Evaluate the integral for each one :

1. $\int_0^2 \int_0^2 (4 - y^2)\, dydx$, 2. $\int_\pi^{2\pi} \int_0^\pi (sinx + cosy))\, dxdy$, and 3. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dxdy$

2- Integrate *f* over the given regain:

1. Square: $f(x, y) = {}^1/_{xy}$ $\;over\ the\ square\ 1 \le x \le 2,$ $1 \le y \le 2$
2. Triangle: $f(x, y) = ycosxy\;\ over\ the\ triangle$ $0 \le x \le \pi, 0 \le y \le 1$

3- Evaluate the integral for each one:

1. $\int_0^\pi \int_x^\pi \frac{siny}{y}\, dydx$, 2. $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\, dxdy$, and 3. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dydx}{y^4 + 1}$

4- Calculate the equivalent Double Integral for each one:

1. $\int_0^1 \int_1^{e^x} dydx$, 2. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} ydydx$, and 3. $\int_{-2}^1 \int_{x^2+4}^{3x+2} dydx$

5- Evaluate the following integrals:

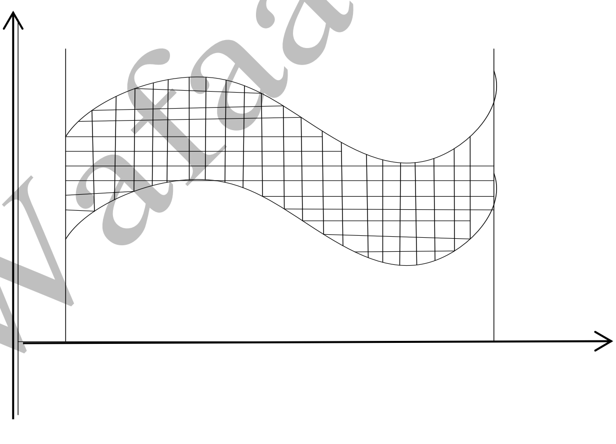
1. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dzdydx,$
2. $\int_0^1 \int_0^\pi \int_0^\pi ysinzdxdydz,$

3. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\cos(u+v+w)\,dudvdw\ (uvw\ space)$,
4. $\int_0^1\int_0^1\int_0^1(x^2+y^2+z^2)dzdydx$,
5. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}}dzdydx$,
6. $\int_1^e\int_1^e\int_1^e\frac{1}{xyz}dxdydz$, and

<u>Applications for Multiple Integrals:</u>

<u>The Area of The Regain in Cartesian Space:</u>

Find the area of the regain A bounded by two curves. If $y=f_1(x)$, $y=f_2(x)$ and limited by two lines $x=a, x=b$. Divided the regain A into n small rectangles with same size and find the area of each one as follows:



$$\Delta A_i=\Delta x_i\Delta y_i\,,\quad i=1,2,3,\dots,n$$

Then the overall area is approximate to:

$$A\cong\sum_{i=1}^{n}\Delta A_i=\sum_{i=1}^{n}\Delta x_i\Delta y_i\rightarrow A=\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x_i\Delta y_i$$

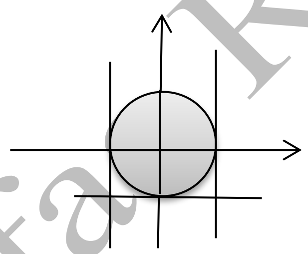
By using the fundamental theory of integration

$$\rightarrow A = \int_{x=a}^{x=b} \int_{y=f_1(x)}^{y=f_2(x)} dydx = \int_{y=c}^{y=d} \int_{x=g_1(y)}^{x=g_2(y)} dxdy$$

$$\boldsymbol{A = \int_{a}^{b} \int_{f_1(x)}^{f_2(x)} dydx = \int_{c}^{d} \int_{g_1(y)}^{g_2(y)} dxdy}$$

<u>Example 5</u>: Find the area of regain A bounded by the semicircle $y = \sqrt{a^2 - x^2}$ and the line $y = -a$, and the lines $x = \pm a$.

Solution:



$$A = \int_{-a}^{a} [\int_{-a}^{\sqrt{a^2-x^2}} dy]dx = \int_{-a}^{a} y]_{y=-a}^{y=\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} \left[\sqrt{a^2 - x^2} + a\right] dx = \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx + \int_{-a}^{a} adx$$

$x = asin\theta, dx = acos\theta d\theta,$

$if\ x = a \rightarrow a = asin\theta \rightarrow sin\theta = 1 \rightarrow \theta = \frac{\pi}{2},$

$if\ x = -a \rightarrow -a = asin\theta \rightarrow sin\theta = -1 \rightarrow \theta = \frac{-\pi}{2}$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2 sin\theta}\ .acos\theta d\theta + ax\Big]_{x=-a}^{x=a}$$

$$= a^2 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - sin^2\theta}\ .cos\theta d\theta + a(a+a) = a^2 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos\theta . cos\,\theta d\theta + 2a^2$$

$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^2\theta d + 2a^2 = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 + cos2\theta)d\theta + 2a^2$$

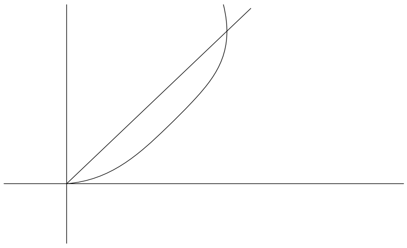
$$= \frac{a^2}{2}\Big[\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}{}_0 d\,\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos2\theta d\theta\Big] + 2a^2$$

$$= \frac{a^2}{2}\left[\theta + \frac{1}{2}sin2\theta\right]_{\theta=\frac{-\pi}{2}}^{\theta=\frac{\pi}{2}} + 2a^2 = \frac{a^2}{2}\Big[\frac{\pi}{2} - \frac{1}{2}sin\pi + \frac{\pi}{2} - \frac{1}{2}\sin(-\pi)\Big] + 2a^2$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} - 0 + \frac{\pi}{2} - 0\right] + 2a^2 = \frac{a^2\pi}{2} + 2a^2 = a^2\Big[\frac{\pi}{2} + 2\Big]$$

<u>Example 6</u>: Find the area of regain A bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution: The intersection point of two curves are (0, 0) and (1, 1).

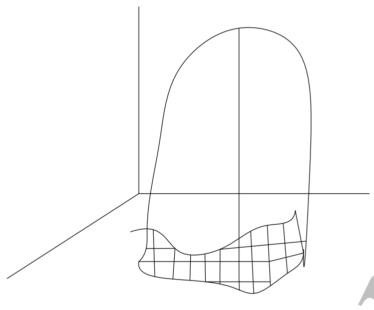


$$A = \int_0^1 \Big[\int_{x^2}^{x} dy\Big]dx = \int_0^1 y\Big]_{y=x^2}^{y=x} = \int_0^1 [x - x^2]dx = \frac{x^2}{2} - \frac{x^3}{3}\Big]_{x=0}^{x=1}$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

The Volume of the Solid in Space by Using Double Integrations:

Find the volume of the solid when its base in the *xy* plane and the top of it bounded by the surface $S: z = f(x, y)$. Divided the solid into *n* small partitions that stands directly above the base ΔA_i in the *xy* plane and its height is z_i. The total volume of the solid is approximate to:



$\Delta v_i = \Delta A_i z_i = \Delta x_i \Delta y_i z_i$

$$v \cong \sum_{i=1}^{n} \Delta v_i \cong \sum_{i=1}^{n} \Delta x_i \Delta y_i z_i \cong \sum_{i=1}^{n} z_i \Delta x_i \Delta y_i$$

$$v = \lim_{n\to\infty} \sum_{i=1}^{n} z_i \Delta x_i \Delta y_i = \lim_{n\to\infty} \sum_{i=1}^{n} z_i \Delta y_i \Delta x_i$$

By using the fundamental theory of integration

$\rightarrow v = \int_A \int z \, dxdy = \int_A \int z \, dydx$

If the base of the solid bounded by two curves. If $y = f_1(x), y = f_2(x)$ and limited by two lines $x = a, x = b$ then

$$\boldsymbol{v = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y) dydx = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) dxdy}$$

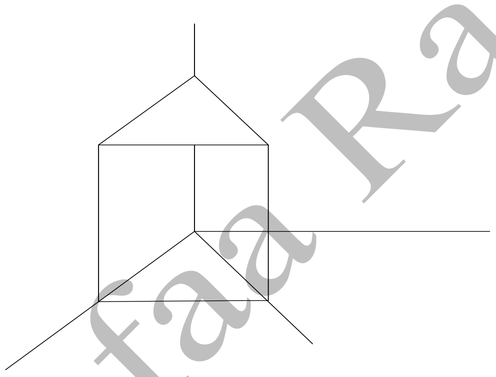
<u>Example 7</u>: Find the volume under the plane: $z = 4 - x - y$ over the rectangular regain $A: 0 \leq x \leq 2$ and $0 \leq y \leq 1$.

Solution: $v=\int_0^2[\int_0^1(4-x-y)dy]dx=\int_0^2(4y-xy-\frac{y^2}{2}]_{y=0}^{y=1}dx$

$$=4\int_0^2 dx-\int_0^2 xdx-\int_0^2\frac{1}{2}dx=4x-\frac{x^2}{2}-\frac{x}{2}\Big]_{x=0}^{x=2}$$

$$=8-\frac{4}{2}-\frac{2}{2}=8-2-1=5$$

<u>Example 8</u>: Find the volume of the solid whose base is the triangle in the *xy* plane bounded by the x-axis, the line $y=x$ and the line $x=1$ and whose top lies in the plane $z=x+y+1$.



Solution: $v=\int_0^1[\int_0^x(x+y+1)dy]dx=\int_0^1(xy+\frac{y^2}{2}+y]_{y=0}^{y=x}dx$

$$=\int_0^1 x^2dx+\int_0^1\frac{x^2}{2}dx+\int_0^1 xdx=\frac{x^3}{3}+\frac{x^3}{6}+\frac{x^2}{2}\Big]_{x=0}^{x=1}$$

$$=\frac{1}{3}+\frac{1}{6}+\frac{1}{2}=\frac{2+1+3}{6}=\frac{6}{6}=1$$

<u>The Volume of the Solid in Space by Using Triple Integrations:</u>

Find the volume of the solid when the whole of solid in the *space* and it bounded by two surfaces: $z_1=f_1(x,y), z_2=f_2(x,y)$ or the solid is result of intersection these two surfaces. Divided the solid into *n* small

cuboid, each one with width, length, and height $\Delta x_i, \Delta y_i, and\ \Delta z_i$. The volume of the solid is approximate to:

$\Delta v_i = \Delta A_i \Delta z_i = \Delta x_i \Delta y_i \Delta z_i$

$$v \cong \sum_{i=1}^{n} \Delta v_i \cong \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i$$

$$v = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i$$

By using the fundamental theory of integration

$\rightarrow v == \int_A \int \int dx\, dydz == \int_A \int \int dydxdz = \cdots$

If the base of the solid bounded by curves: $z = f_1(x,y), z = f_2(x,y), y = h_1(x), y = h_2(x)$ and limited by two lines $x = a, x = b$ then

$\boldsymbol{v = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dydx = \int_c^d \int_{g_1(y)}^{g_2(y)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dxdy}$**=...**

<u>Example 9</u>: Find the volume enclosed between two surfaces:

$z = 0, z = x + 2$ and bounded by the cylinder $x^2 + 4y^2 = 4$.

Solution: from cylinder equation: $x^2 + 4y^2 = 4$

$x^2 + 4y^2 = 4 \rightarrow x^2 = 4 - 4y^2 = 4(1 - y^2) \rightarrow x = \pm 2\sqrt{1 - y^2}$

If $x = 0 \rightarrow 4y^2 = 4 \rightarrow y^2 = 1 \rightarrow y = \pm 1$

$$v = \int_{-1}^{1} \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} \int_{0}^{x+2} dz\, dxdy = \int_{-1}^{1} \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} z\Big]_{z=0}^{z=x+2} dxdy =$$

$$= \int_{-1}^{1} \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} [x+2] dxdy = \int_{-1}^{1} \left(\frac{x^2}{2} + 2x\right]_{x=-2\sqrt{1-y^2}}^{x=2\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{1} [\frac{4(1-y^2)}{2} + 4\sqrt{1-y^2} - \frac{4(1-y^2)}{2} + 4\sqrt{1-y^2}]dy$$

$$= 8\int_{-1}^{1} \sqrt{1-y^2}\ dy$$

$y = sin\theta, dy = cos\theta d\theta,$

$if\ y = 1 \rightarrow 1 = sin\theta \rightarrow \theta = \frac{\pi}{2},$

$if\ y = -1 \rightarrow -1 = sin\theta \rightarrow sin\theta = -1 \rightarrow \theta = \frac{-\pi}{2}$

$$v = 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - sin\theta^2}\ cos\theta d\theta = 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos\theta\,.\,cos\theta d\theta = 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^2\theta\ d\theta$$

$$= 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 + cos2\theta)d\theta = 4[\theta + \frac{1}{2}sin2\theta]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}$$

$$= 4\left[\frac{\pi}{2} + \frac{1}{2}sin\pi + \frac{\pi}{2} - \frac{1}{2}\sin(-\pi)\right] = 4\pi$$

Exercises:

1- Find the area of the region bounded by the given curves and lines, by double integrals:

- The coordinate axes and the line $x + y = a$
- The x-axis , the curve $y = e^x$ and the lines $x = 0, x = 1$
- The y-axis , the line $y = 2x$ and the line $y = 4$
- The curve $y^2 + x = 0$ and the line $y = x + 2$
- The curves $x = y^2, x = 2y - y^2$
- The Parabola $x = y - y^2$ and the line $x + y = 0$

2- Find the volume of the solid whose base is the region in the xy-plane bounded by the Parabola $y = 4 - x^2$ and the line $y = 3x$, and whose top is bounded by the plane $z = x + 4$.

3- Find the volume if the base of a solid is the region in the y-plane bounded by the circle $x^2 + y^2 = a^2$,while the top is bounded by the Parabola $az = x^2 + y^2$.

4- Find the volume of:

- The Tetrahedron bounded by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate plane (a,b,c positive)
- The solid bounded by the Elliptic ,Paraboloids $z = x^2 + 9y^2$, $z = 18 - x^2 - 9y^2$
- The common to the two Cylinder $x^2 + y^2 = a^2,\ x^2 + z^2 = a^2$

Integrals in Polar Coordinates:

If $f(r, \theta)$ is a function defined on regain *R* and bounded by two arrows $\theta = \alpha, \theta = \beta$ and two curves $r = r_1(\theta), r = r_2(\theta)$. Then Double Integral in Polar Coordinates as follows:

$$\int_A\int \boldsymbol{f(r,\theta)dA} = \int_{\alpha}^{\beta}\int_{r_1(\theta)}^{r_2(\theta)} \boldsymbol{f(r,\theta)rdrd\theta} \quad \text{where } A \text{ inside } R$$

The Area in Polar Coordinates:

By using Jacobian transformation to transform Double Integral from Cartesian plane to Polar Coordinates as follows:

If $x = f(u, v), y = g(u, v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Then the Double Integral is:

$$\int_A \int \emptyset(x,y)dxdy = \int_A \int \emptyset[f(u,v), g(u,v)] \frac{\partial(x,y)}{\partial(u,v)} dudv$$

The Area in Polar Coordinates by using Jacobian transformation as follows:

$x = rcos\theta, y = rsin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = rcos^2\theta + rsin^2\theta = r(cos^2\theta + sin^2\theta) = r$$

$$\int\int f(x,y)dxdy = \int\int f(rcos\theta, rsin\theta) \frac{\partial(x,y)}{\partial(r,\theta)} drd\theta$$

$$\int\int f(x,y)dxdy = \int\int f(rcos\theta, rsin\theta) rdrd\theta$$

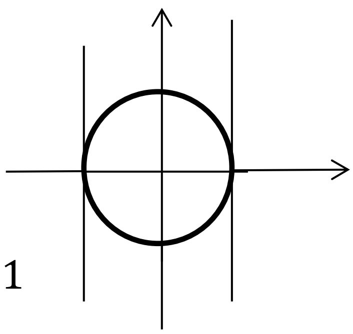
$dxdy = rdrd\theta$

$A = \int_A \int \boldsymbol{dxdy}$ [the area in Cartesian plane**)**

$A = \int_A \int \boldsymbol{rdrd\theta}$ [the area in Polar Coordinates **)**

<u>**Example 10**</u>**:** Find the value of Double Integral in the Polar Coordinates $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2) dydx$

Solution:



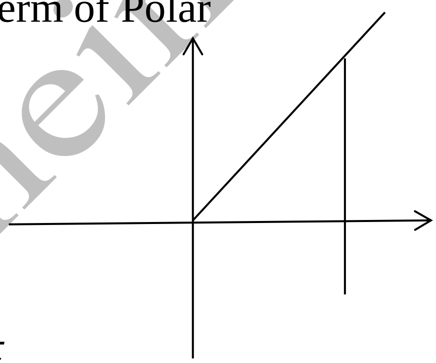
$y = \pm\sqrt{1-x^2} \rightarrow y^2 = 1 - x^2 \rightarrow x^2 + y^2 = 1 \rightarrow r^2 = 1$

$\rightarrow r = \pm 1 \rightarrow r = 0 \;\; and \; r = 1, \theta = 0 \; and \; \theta = 2\pi$

$$\int_0^{2\pi}\int_0^{1} r^2 r dr d\theta = \int_0^{2\pi}\int_0^{1} r^3 dr d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big]_{r=0}^{r=1} d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} d\theta = \frac{\theta}{4}\Big]_{\theta=0}^{\theta=2\pi} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Example11: Find an equivalent Double Integral in term of Polar Coordinates $\int_0^2 \int_0^x y dy dx$



Solution:

$x = r cos\theta, y = r sin\theta$

$y = x \rightarrow \; r sin\theta = \; r cos\theta \rightarrow cos\theta = sin\theta \rightarrow \theta = \frac{\pi}{4}$

$x = 2 \rightarrow \; r cos\theta = \; 2 \rightarrow r = \frac{2}{cos\theta} \rightarrow r = 2sec\theta$

$r = 0 \; and \; r = 2sec\theta, \theta = 0 \; and \; \theta = \frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}}\int_0^{2sec\theta} r sin\theta . r dr d\theta = \int_0^{\frac{\pi}{4}}\int_0^{2sec\theta} r^2 sin\theta . dr d\theta = \int_0^{\frac{\pi}{4}} \frac{r^3}{3}\Big]_{r=0}^{r=2sec\theta} sin\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{8}{3} sec^3\theta sin\theta d\theta = \frac{8}{3}\int_0^{\frac{\pi}{4}} \frac{sin\theta}{cos\theta} sec^2\theta d\theta = \frac{8}{3}\int_0^{\frac{\pi}{4}} tan\theta sec^2\theta d\theta =$$

$$\frac{8}{3}\frac{tan^2\theta}{2}\Big]_{\theta=0}^{\theta=\frac{\pi}{4}} = \frac{4}{3}\left[tan^2\frac{\pi}{4} - tan^2 0\right] = \frac{4}{3}[1-0] = \frac{4}{3}$$

Example 12: Find the area inside the Cardioid $r = a(1 - cos\theta)$

Solution: $0 \le \theta \le 2\pi$

$\theta = 0 \rightarrow \; r = a(1 - cos0) \rightarrow r = 0$

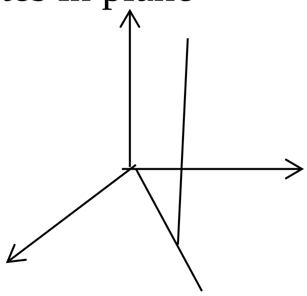
$r = 0$ and $r = a(1 - cos\theta), \theta = 0$ and $\theta = 2\pi$

$A = \int_0^{2\pi} \int_0^{a(1-cos0)} r\,drd\theta = \int_0^{2\pi} \frac{r^2}{2}\Big]_{r=0}^{r=a(1-cos0)} d\theta =$
$\frac{a^2}{2}\int_0^{2\pi}(1 - cos\theta)^2 d\theta = \frac{a^2}{2}\int_0^{2\pi}(1 - 2cos\theta + cos^2\theta)d\theta =$
$\frac{a^2}{2}\int_0^{2\pi}\left(1 - 2cos\theta + \frac{1}{2}(1 + cos2\theta)\right)d\theta =$

$$\frac{a^2}{2}\left[\theta - 2sin\theta + \frac{\theta}{2} + \frac{1}{4}sin2\theta\right]_{\theta=0}^{\theta=2\pi}$$
$$= \frac{a^2}{2}\left[2\pi - 2sin2\pi + \frac{2\pi}{2} + \frac{1}{4}sin4\pi - 0 + 2sin0 - 0 - \frac{1}{4}sin0\right]$$
$$= \frac{a^2}{2}[2\pi + \pi] = \frac{3a^2\pi}{2}$$

Volume in Cylindrical Coordinates:

$p(r, \theta, z)$ is point in space where(r, θ) is Polar Coordinates in plane $z = 0$



The Volume in Cylindrical Coordinates is:

$$V = \int_A\int\int \boldsymbol{r\,drd\theta dz}$$

Example 13: Find the volume by Triple Integral that is cut from the sphere $x^2 + y^2 + z^2 = 4a^2$ by the Cylinder $x^2 + y^2 = a^2$ by using Cylindrical Coordinates

Solution: $x^2 + y^2 + z^2 = 4a^2 \to z^2 = 4a^2 - x^2 - y^2 \to$

$z^2 = 4a^2 - (x^2 + y^2) \to z^2 = 4a^2 - r^2 \to z = \pm\sqrt{4a^2 - r^2}$

$x^2 + y^2 = a^2 \to r^2 = a^2 \to r = 0, a$

$\theta = 0, 2\pi$

$$V=\int_0^{2\pi}\int_0^{a}\int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} dz.rdr.d\theta=\int_0^{2\pi}\int_0^{a} z\Big]_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} rdr.d\theta=$$

$$\int_0^{2\pi}\int_0^{a}(\sqrt{4a^2-r^2}+\sqrt{4a^2-r^2})rdr.d\theta$$

$$=2\int_0^{2\pi}\int_0^{a} r(4a^2-r^2)^{\frac{1}{2}}\,dr.d\theta$$

$$=\frac{-2}{2}\int_0^{2\pi}\frac{2}{3}(4a^2-r^2)^{\frac{3}{2}}\Big]_{r=0}^{r=a} d\theta$$

$$=\frac{-2}{3}\int_0^{2\pi}[\,(4a^2-a^2)^{\frac{3}{2}}-(4a^2-0)^{\frac{3}{2}}]\,d\theta$$

$$=\frac{-2}{3}\int_0^{2\pi}[\,(a^3\left(\sqrt{3}\right)^3-(2a)^3]\,d\theta$$

$$=\frac{-2}{3}a^3\left(3\sqrt{3}-8\right)\int_0^{2\pi} d\theta=a^3\left(-2\sqrt{3}+\frac{16}{3}\right)\theta\Big]_{\theta=0}^{\theta=2\pi}$$

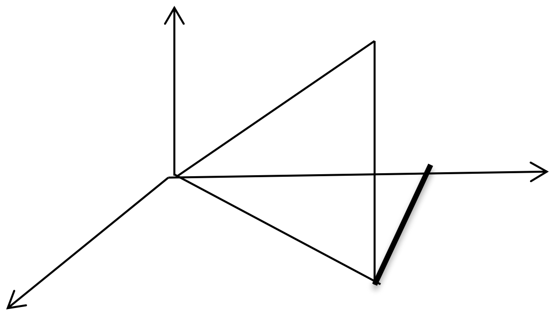
$$=(-4\sqrt{3}+\frac{32}{3})a^3\pi$$

Volume in Spherical Coordinates:

$p(\rho,\emptyset,\theta)$ the point in apace such that $\rho=|op|$ distance from o to p and $\rho\geq 0$.

$\emptyset$ is an angle between $\overrightarrow{op}$ and the positive z-axes, $0\leq\emptyset\leq\pi$.

θ is an angle in polar Coordinates, $0\leq\theta\leq 2\pi$.



$x = r\cos\theta, y = r\sin\theta, \sin\emptyset = \frac{r}{\rho}, r = \rho\sin\emptyset$

$z = \rho\cos\emptyset, y = \rho\sin\emptyset\sin\theta, x = \rho\sin\emptyset\cos\theta$

$x^2 + y^2 + z^2 = \rho^2$

$$\boldsymbol{V = \int_A\int\int \rho^2 \sin\emptyset \, d\rho d\emptyset d\theta}$$

<u>Example 14</u>: Find the volume cut from the sphere $\rho = a$ by the cone $\emptyset = \alpha$ using the spherical coordinate.

Solution: $\theta: 0 \to 2\pi, \emptyset: 0 \to \alpha, \rho: 0 \to a$

$$V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho^2 \sin\emptyset d\rho d\emptyset d\theta$$

$$= \int_0^{2\pi}\int_0^{\alpha} \frac{\rho^3}{3}\sin\emptyset\Big]_{\rho=0}^{\rho=a} d\emptyset d\theta = \int_0^{2\pi} \frac{a^3}{3}\left(\int_0^{\alpha} \sin\emptyset d\emptyset\right) d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} -\cos\emptyset\Big]_{\emptyset=0}^{\emptyset=\alpha} d\theta = \frac{a^3}{3}\int_0^{2\pi}(-\cos\alpha + \cos 0)d\theta =$$

$$\frac{a^3}{3}\int_0^{2\pi}(1-\cos\alpha)d\theta = \frac{a^3}{3}(\theta - \theta\cos\alpha)\Big]_{\theta=0}^{\theta=2\pi} = \frac{a^3}{3}(2\pi - 2\pi\cos\alpha - 0)$$

$$= \frac{2a^3}{3}\pi(1-\cos\alpha)$$

<u>Note:</u>

In xyz space -Cylindrical:

$$x = rcos\theta, y = rsin\theta, z = z$$

In xyz space - Spherical:

$$x = \rho sin\emptyset cos\theta, y = \rho sin\emptyset sin\theta, z = \rho cos\emptyset$$

In Cylindrical - Spherical:

$$r = \rho sin\emptyset, \theta = \theta, z = \rho cos\emptyset$$

Area of Surface:

There are three kinds of Area of Surface:

1- If $S: z = f(x, y)$ such that the domain of f is $xy\ plane$ and f have partial derivatives f_x, f_y continuous in this domain, then the area of surface is:

$$\boldsymbol{S = \int_A\int \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dxdy}$$

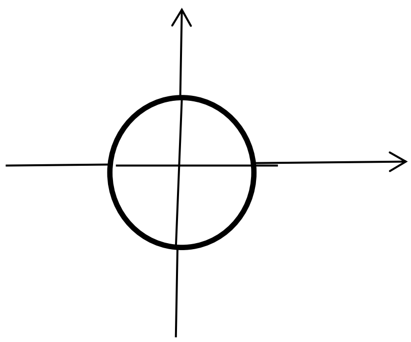
2- If $S: y = f(x, z)$ such that the domain of f is $xz\ plane$ and f have partial derivatives f_x, f_z continuous in this domain, then the area of surface is:

$$\boldsymbol{S = \int_A\int \sqrt{f_x{}^2 + f_z{}^2 + 1}\, dxdz}$$

3- If $S: x = f(y, z)$ such that the domain of f is $yz\ plane$ and f have partial derivatives f_y, f_z continuous in this domain, then the area of surface is:

$$\boldsymbol{S = \int_A\int \sqrt{f_y{}^2 + f_z{}^2 + 1}\, dydz}$$

<u>Example 15</u>: Find the Area of the Paraboloid $z = x^2 + y^2$ below the plane $z = 1$.



Solution: $S = \int_A\int \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dxdy$

$z = 1 \to x^2 + y^2 = 1, f_x = 2x, f_y = 2y$

$$S=\int_{x^2+y^2=1}\int \sqrt{(2x)^2+(2y)^2+1}\,dxdy$$

$r^2=1,\qquad 0\le r\le 1,\quad 0\le\theta\le 2\pi$

$$S=\int_0^{2\pi}\int_0^1\sqrt{4r^2+1}\,rdrd\theta=\frac{1}{8}\int_0^{2\pi}(4r^2+1)^{\frac{3}{2}}.\frac{2}{3}\Big]_0^1 d\theta$$

$$=\frac{1}{12}\int_0^{2\pi}[(4+1)^{\frac{3}{2}}-(0+1)^{\frac{3}{2}}]d\theta=\frac{1}{12}\int_0^{2\pi}[(\sqrt{5})^3-1]d\theta$$

$$=\frac{5\sqrt{5}-1}{12}\theta\Big]_0^{2\pi}=\frac{5\sqrt{5}-1}{6}\pi$$

Exercises:

Find the Area of the following:

1- Triangle cut from the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ by the coordinate planes.

2- Portion of the sphere $x^2+y^2+z^2=a^2$ that lies in the first octant.

3- Surface of that portion of the sphere $x^2+y^2+z^2=a^2$ that lies inside cylinder $x^2+y^2=ax$.

4- Portion of the cylinder $x^2+z^2=a^2$ that lies between the planes $y=\frac{a}{2}$ and $x=\frac{a}{2}$.

5- The area cut from the plane $z=cx$ to cylinder $x^2+y^2=a^2$.