جامعة بغداد

كلية التربية للعلوم الصرفة- ابن الهيثم

قسم الرياضيات

المرحلة الثانيه

التفاضل المتقدم

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2016-2017

Chapter Four

Partial Derivative

In Cartesian plane when $y = f(x)$.

f is continuous function if satisfied three conditions:

1. $f(x_0)$ is exists,
2. $\lim_{x\to x_0} f(x)$ is exists,
3. $\lim_{x\to x_0} f(x) = f(x_0)$.

f is differentiation if f is continuous function. The *differentiation* of function f is defined as $y' = \frac{dy}{dx}$.

But if $w = f(x, y)$. The partial derivative is defined for each variable:

- The partial derivative of x represented as follows:

$$\frac{\partial w}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

- The partial derivative of y represented as follows:

$$\frac{\partial w}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y$$

<u>Definition:</u>

If $f: E \to E'$ is continuous function, $w = f(x, y)$ and $p_0(x_0, y_0)$ then the partial derivative of x is defined is:

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} = f_x = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \quad \dots\dots\dots 1$$

And the partial derivative of y is defined is:

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \dots \dots \dots 2$$

But the partial derivative of x in the point $p_0(x_0, y_0)$ is:

$$(f_x)_{p_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \dots \dots \dots 3$$

And the partial derivative of y in the point $p_0(x_0, y_0)$ is:

$$(f_y)_{p_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \dots \dots \dots 4$$

Example 1: Find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ if a) $w = f(x, y) = 2xy$ and

b) $w = f(x, y) = x^2 - xy$ (**H.W.**)?

Solution: a)

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2(x + \Delta x)y - 2xy}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2xy + 2y\Delta x - 2xy}{\Delta x} = \lim_{\Delta x \to 0} \frac{2y\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2y = 2y$$

$$\frac{\partial w}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=\lim_{\Delta y\to 0}\frac{2x(y+\Delta y)-2xy}{\Delta y}$$

$$=\lim_{\Delta y\to 0}\frac{2xy+2x\Delta y-2xy}{\Delta y}=\lim_{\Delta y\to 0}\frac{2x\Delta y}{\Delta y}$$

$$=\lim_{\Delta y\to 0}2x=2x$$

Notes:

- The results of equations 3 and 4 are constant.
- All rules of ordinary differentiation are applied but when make partial derivative of one variable the other variables are held constants.
- The definitions of partial derivatives of functions of more than two variables are like the definitions for functions of two variables.
  - The partial derivative of x represented as follows:

    $\frac{\partial f}{\partial x}$ or f_x
  - The partial derivative of y represented as follows:

    $\frac{\partial f}{\partial y}$ or f_y
  - The partial derivative of z represented as follows:

    $\frac{\partial f}{\partial z}$ or f_z
  - ....

<u>Example 2</u>: Find $\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}$ if a) $w=f(x,y)=ysinxy$,

[b) $w=f(x,y)=cosx\ e^{2y}$, and c) w $=f(x,y)=\frac{2y}{y+cosx}$ (**H.W.**)]?

Solution: a) $\frac{\partial w}{\partial x} = y^2 cosxy, \quad \frac{\partial w}{\partial y} = yxcosxy + sinxy$

<u>Example 3</u>: Find $(\frac{\partial f}{\partial x})_{p_0}, (\frac{\partial f}{\partial y})_{p_0}$ if a) $f(x,y) = x^2 - 8xy, p_0 = (-1,1)$ and b) $f(x,y) = x^3 + 3xy + y - 1, p_0 = (4,-5)$(**H.W.**)?

Solution: a) $\frac{\partial f}{\partial x} = 2x - 8y \rightarrow \left(\frac{\partial f}{\partial x}\right)_{p_0} = 2(-1) - 8(1) = -2 - 8 = -10$

$$\frac{\partial f}{\partial y} = -8x \rightarrow \left(\frac{\partial f}{\partial y}\right)_{p_0} = -8(-1) = 8$$

<u>Example 4</u>: Find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ if a) $w = f(x,y,z) = 2xy^2 + xz^2 + zy + 2$ and b) $w = f(x,y,z) = xz^2 + y + 5$ (**H.W.**)?

Solution: a) $\frac{\partial w}{\partial x}$=$2y^2 + z^2$, $\frac{\partial w}{\partial y}$=$4xy + z$, and $\frac{\partial w}{\partial z}$=$2xz + y$

<u>Exercises:</u>

1. Find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for the following functions:

   1. $w = e^x siny$, 2. $w = sin^2(x - 3y)$, 3. $w = tan^{-1}\left({}^{y}/_{x}\right)$,

   4. $w = \ln(x + y)$5. $w = x^y$, 6. $w = \log_y x$, 7. $w = \sqrt{x^2 + y^2}$,

   8. $w = \frac{x+y}{xy-1}$, 9. $w = cos^2(3x - y^2)$ and 10. $w = e^{xy} lny$.

2. Find the partial derivatives of each function with respect to each variable:

   1. $f(x,y,z) = \sinh(xy - z^2)$, 2. $f(x,y,z) = yz\ln(xy)$,

   3. $f(x,y,z) = sec^{-1}(xyz)$,

4.$f(x,y,z,w) = x^2 e^{2y+4z}\cos(3w)$, 5. $f(x,y,u,v) = \frac{x^2-y^2}{u^2+v^2}$,

6. $f(x,y,r,s) = \sin(2x)\cosh(5r) + sinh(3y)\cos(2s)$

Tangent Plane and Normal Line:

If $S: z = f(x,y)$ be a surface and $p_0 = (x_0, y_0, z_0)$ be the point on this surface then:

- The plane intersect with surface at p_0 is called *Tangent Plane* of surface at $p_{0.}$
- The line orthogonal on tangent plane at p_0 is called *Normal Line* of surface at p_0.

- If P_1, P_2 be two tangent planes of surface at p_0 and v_1, v_2 be two vectors on P_1, P_2 respectively:

$N = v_1 \times v_2$ ($N \perp v_1, N \perp v_2$)

$$N = (\frac{\partial f}{\partial x})_{p_0} i + (\frac{\partial f}{\partial y})_{p_0} j - k \rightarrow N = (A,B,C), A = (\frac{\partial f}{\partial x})_{p_0}, B = (\frac{\partial f}{\partial y})_{p_0}, C = -1$$

The equation of Tangent Plane to the surface at the point $p_0$is:

$$\boldsymbol{P}: \boldsymbol{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

Where $p_0 = (x_0, y_0, z_0)$ be the point at the plane, N be a vector orthogonal on this plane and $A = (\frac{\partial f}{\partial x})_{p_0}, B = (\frac{\partial f}{\partial y})_{p_0}, C = -1$

- If L be a line orthogonal on surface at the point $p_0 = (x_0, y_0, z_0)$and $N \backslash\backslash L$ then the equation of Normal Line to the surface at the point p_0 is:

$$L: \frac{\boldsymbol{(x - x_0)}}{\boldsymbol{A}} = \frac{\boldsymbol{(y - y_0)}}{\boldsymbol{B}} = \frac{\boldsymbol{(z - z_0)}}{\boldsymbol{C}}$$

Where $A = (\frac{\partial f}{\partial x})_{p_0}, B = (\frac{\partial f}{\partial y})_{p_0}, C = -1.$

Notes:

- If $S: z = f(x, y)$be the surface equation and $N(A, B, C)$ then

$$A = (\frac{\partial f}{\partial x})_{p_0}, B = (\frac{\partial f}{\partial y})_{p_0}, C = -1.$$

- If $S: x = g(y, z)$be the surface equation and $N(A, B, C)$ then

$$A = -1, B = (\frac{\partial g}{\partial y})_{p_0}, C = (\frac{\partial g}{\partial z})_{p_0}.$$

- If $S: y = h(x, z)$be the surface equation and $N(A, B, C)$ then

$$A = (\frac{\partial h}{\partial x})_{p_0}, B = -1, C = (\frac{\partial h}{\partial z})_{p_0}.$$

<u>Example 5</u>: Find the plane that is tangent to the surface $y = x^2 - xz + z^2$ at the point $p_0(1, -1, 1)$ and find the normal line to the surface at p_0?

Solution: $S: y = h(x, z) \rightarrow A = \left(\frac{\partial h}{\partial x}\right)_{p_0}, B = -1, C = \left(\frac{\partial h}{\partial z}\right)_{p_0}$

$A = \left(\frac{\partial h}{\partial x}\right)_{p_0} = (2x - z)_{(1,-1,1)} = 2 - 1 = 1, B = -1,$

$C = \left(\frac{\partial h}{\partial z}\right)_{p_0} = (-x + 2z)_{(1,-1,1)} = -1 + 2 = 1$

$$P: A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$P: 1(x - 1) - 1(y + 1) + 1(z - 1) = 0$

$x - 1 - y - 1 + z - 1 = 0 \rightarrow x - y + z - 3 = 0$

$$L: \frac{(x-x_0)}{A} = \frac{(y-y_0)}{B} = \frac{(z-z_0)}{C}$$

$$L: \frac{(x-1)}{1} = \frac{(y+1)}{-1} = \frac{(z-1)}{1}$$

Exercises:

Find the plane that is tangent to the given surface at the given point and find the normal line to the surface at the given point:

1. $z = \sqrt{9-x^2-y^2}$, $(1,-2,2)$
2. $y = x^2 + z^2$, $(3,4,25)$
3. $z = \tan^{-1}\frac{y}{x}$, $(1,1,\frac{\pi}{4})$
4. $x = e^{2y-z}$, $(1,1,2)$
5. $z = \frac{x}{\sqrt{x^2+y^2}}$, $(3,-4,{}^{3}/_{5})$

Linear Approximate :

Let $S: w = f(x,y)$ where $f: D \to E$, $(x_0,y_0) \in D$ such that

$w_0 = f(x_0,y_0)$, $p_0(x_0,y_0,w_0)$ be a point on the surface and let

$p(x,y,w)$ be a point on the tangent plane of surface in p_0 then the equation of plane is:

$P: A(x-x_0) + B(y-y_0) + C(w-w_0) = 0$

Such that $A = (f_x)_{p_0}, B = (f_y)_{p_0}, C = -1$

$\to (f_x)_{p_0}(x-x_0) + (f_y)_{p_0}(y-y_0) - (w-w_0) = 0$

$\to (w-w_0) = (f_x)_{p_0}(x-x_0) + (f_y)_{p_0}(y-y_0)$

$$\boldsymbol{w = w_0 + (f_x)_{p_0}(x - x_0) + (f_y)_{p_0}(y - y_0)}$$

The Linear Approximate to the function $w = f(x, y)$ at the point $p_0(x_0, y_0, w_0)$ such that $w_0 = f(x_0, y_0)$ is denoted by w_{tan}.

❖ When the function f is more than two variables:

If $w = f(x, y, z)$ then The Linear Approximate to the function $w = f(x, y, z)$ at the point $p_0(x_0, y_0, z_0, w_0)$ such that $w_0 = f(x_0, y_0, z_0)$ is denoted by w_{line}:

$$\boldsymbol{w = w_0 + (f_x)_{p_0}(x - x_0) + \left(f_y\right)_{p_0}(y - y_0) + (f_z)_{p_0}(z - z_0)}$$

<u>Example 6</u>: Find the linear approximate to the function $w = f(x, y, z) = x^2 y e^z$ at $p_0(1, -2, ln2\ )$?

Solution:

$$w_{line} = w_0 + (f_x)_{p_0}(x - x_0) + \left(f_y\right)_{p_0}(y - y_0) + (f_z)_{p_0}(z - z_0)$$

$$w_0 = f(x_0, y_0, z_0) = f(1, -2, ln2) = (1)^2(-2)e^{ln2} = (-2)(2) = -4$$

$$f_x = \frac{\partial f}{\partial x} = 2xye^z \rightarrow\ (f_x)_{p_0} = 2(1)(-2)e^{ln2} = 2(-2)2 = -8$$

$$f_y = \frac{\partial f}{\partial y} = x^2 e^z \rightarrow\ (f_y)_{p_0} = (1)^2 e^{ln2} = 1(2) = 2$$

$$f_z = \frac{\partial f}{\partial z} = x^2 y e^z \rightarrow\ (f_z)_{p_0} = (1)^2(-2)e^{ln2} = (-2)2 = -4$$

$$w_{line} = (-4) + (-8)(x - 1) + (2)(y + 2) + (-4)(z - ln2) \rightarrow$$

$$w_{line} = -4 - 8x + 8 + 2y + 4 - 4z + 4ln2 \rightarrow$$

$$w_{line} = -8x + 2y - 4z + 4ln2 + 8$$

<u>Example 7</u>: Find the linear approximate of the given function f at the given point p_0 : **(H.W.)**

1. $w = f(x, y) = (2 + xy)\cos y$, $p_0\left(2, \frac{\pi}{2}\right)$
2. $w = f(x, y, u, v) = \frac{(x^2+y^2)}{(u^2-v^2)}$, $p_0(1, -1,0,2\ )$

<u>Approximated Value (Δw):</u>

$$w_{tan} = w_0 + (f_x)_{p_0}(x - x_0) + \left(f_y\right)_{p_0}(y - y_0)$$

$w_{tan} - w_0 = (f_x)_{p_0}\Delta x + \left(f_y\right)_{p_0}\Delta y$, if $w_{tan} - w_0 = \Delta w$

$$\boldsymbol{\Delta w_{tan} = (f_x)_{p_0}\Delta x + \left(f_y\right)_{p_0}\Delta y}$$

The Δw_{tan} is called the Approximate Value of the tangent plane of surface at the point p_0.

- ❖ If the function f is more than two variables then:

$$\boldsymbol{\Delta w_{line} = (f_x)_{p_0}\Delta x + \left(f_y\right)_{p_0}\Delta y + (f_z)_{p_0}\Delta z}$$

The $\Delta w_{line}$is called the Approximate Value of the tangent plane of surface at the point p_0.

<u>Notes:</u>

To find the approximate value for the function (Δw) be using the following two rules:

1. $\boldsymbol{\Delta w = \Delta w_{tan} + \epsilon_1\Delta x + \epsilon_2\Delta y}$, when $\epsilon_1, \epsilon_2, \Delta x, \Delta y \to 0$,
   $\Delta w \to \Delta w_{tan}, p \to p_0$
2. $\boldsymbol{\Delta w = \Delta w_{line} + \epsilon_1\Delta x + \epsilon_2\Delta y + \epsilon_3\Delta z,}$ when $\epsilon_1, \epsilon_2, \epsilon_3, \Delta x, \Delta y, \Delta z \to 0$,
   $\Delta w \to \Delta w_{line}, p \to p_0$

<u>Example 8</u>: Find the approximate value of the function $w = f(x,y) = x^2 - xy + y^2$ at the point $p_0(1,-2)$ when $\Delta x = 0.01$, $\Delta y = -0.02, \epsilon_1 = 0.09$ and $\epsilon_2 = 0.002$?

Solution: $\Delta w = \Delta w_{tan} + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \Delta w_{tan} = (f_x)_{p_0}\Delta x + \left(f_y\right)_{p_0}\Delta y$

$f_x = 2x - y \rightarrow (f_x)_{p_0} = 2(1) - (-2) = 4$

$f_y = -x + 2y \rightarrow \left(f_y\right)_{p_0} = -1 + 2(-2) = -5$

$\Delta w_{tan} = 4(0.01) - 5(-0.02) = 0.04 + 0.1 = 0.14$

$\Delta w = 0.14 + 0.09(0.01) + 0.002(-0.02) = 0.1408$

<u>Example 9</u>: Find the approximate value of the function $w = f(x,y) = x^2 - xy + y^2$ at the point $p_0(1,-2\ )$ when $\Delta x = 0.01$ and $\Delta y = -0.02$?

Solution: $w + \Delta w = f(x + \Delta x, y + \Delta y)$

$$w + \Delta w = (x + \Delta x)^2 - (x + \Delta x)(y + \Delta y) + (y + \Delta y)^2$$
$$= x^2 + 2x\Delta x + (\Delta x)^2 - xy - y\Delta x - x\Delta y - \Delta x\Delta y + y^2 + 2y\Delta y + (\Delta y)^2$$

$w = x^2 - xy + y^2$, $w + \Delta w - w = \Delta w$

$$\Delta w = 2x\Delta x + (\Delta x)^2 - y\Delta x - x\Delta y - \Delta x\Delta y + 2y\Delta y + (\Delta y)^2$$
$$= (2x - y)\Delta x + (-x + 2y)\Delta y + (\Delta x)^2 + (\Delta y)^2 - \Delta x\Delta y$$
$$= \Delta w_{tan} + (\Delta x)^2 + (\Delta y)^2 - \Delta x\Delta y$$
$$= 0.14 + (0.01)^2 + (-0.02)^2 - (0.01)(-0.02) = 0.1407$$

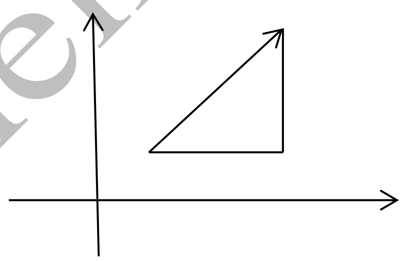
The Directional Derivative:

➢ ***The Directional Derivative in Cartesian Plane:***

Let $w = f(x, y)$ where $f: D \to E$, $(x, y) \in D$ and D is xy plane.

Suppose that $p_0(x_0, y_0)$ and $p_1(x_1, y_1)$ be two points in D such that

$w_0 = f(x_0, y_0)$ and $w_1 = f(x_1, y_1)$. If $\overrightarrow{p_0p_1}$ (is a moving vector or path) and Δs= change of length $\overrightarrow{p_0p_1}$ then the directional derivative of function $f(x, y)$ is:



$$\frac{dw}{ds} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s} = \lim_{\Delta s \to 0} \frac{w_1 - w_0}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$\left(\frac{\boldsymbol{dw}}{\boldsymbol{ds}}\right)_{\mathbf{0}} = \lim_{p_1 \to p_0} \frac{\boldsymbol{f(x_1, y_1) - f(x_0, y_0)}}{\sqrt{\boldsymbol{(x_1 - x_0)^2 + (y_1 - y_0)^2}}}$$

➢ ***The Directional Derivative in Space:***

Let $w = f(x, y, z)$ where $f: D \to E$, $(x, y, z) \in D$ and D is the space.

Suppose that $p_0(x_0, y_0, z_0)$ and $p_1(x_1, y_1, z_1)$ be two points in D such that $w_0 = f(x_0, y_0, z_0)$ and $w_1 = f(x_1, y_1, z_1)$. If $\overrightarrow{p_0p_1}$ (is a moving vector or path) and Δs= change of length $\overrightarrow{p_0p_1}$ then the directional derivative of function $f(x, y, z)$ is:

$$\frac{dw}{ds} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s} = \lim_{\Delta s \to 0} \frac{w_1 - w_0}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}$$

$$\left(\frac{\boldsymbol{dw}}{\boldsymbol{ds}}\right)_{\mathbf{0}} = \lim_{p_1 \to p_0} \frac{\boldsymbol{f(x_1, y_1, z_1) - f(x_0, y_0, z_0)}}{\sqrt{\boldsymbol{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}}$$

Notes:

1. $(\frac{dw}{ds})_0 = (f_x)_0 cos\,\theta + (f_y)_0 sin\theta,$

Where θ is the angle between $\overrightarrow{p_0p_1}$ and x=axis.

2. $(\frac{dw}{ds})_0 = (f_x)_0 cos\,\alpha + (f_y)_0 cos\beta + (f_z)_0 cos\gamma,$

Where $\theta, \beta, and\ \gamma$ are the direction angles of $\overrightarrow{p_0p_1}$.

3. Let $\theta, \beta, and\ \gamma$ are the direction angles of $\overrightarrow{p_0p_1}$, u is a unit vector has the same direction of $\overrightarrow{p_0p_1}$ and $u = cos\alpha i + cos\beta j + cos\gamma k.$
4. $\frac{dw}{ds} = u.v$, where $u = cos\alpha i + cos\beta j + cos\gamma k,$

   $v = (f_x)_0 i + (f_y)_0 j + (f_z)_0 k.$

Example 10: Find the directional derivative of function $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at the point $p_0(1,1,1\ )$ and in the direction of vector $A = i + j + k$?

Solution: $u = \frac{A}{|A|} = \frac{i+j+k}{\sqrt{(1)^2+(1)^2+(1)^2}} = \frac{i+j+k}{\sqrt{3}} = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$

$(f_x)_0 = (2x)_0 = 2, (f_y)_0 = (4y)_0 = 4, and\ (f_z)_0 = (6z)_0 = 6$

$v = (f_x)_0 i + (f_y)_0 j + (f_z)_0 k = 2i + 4j + 6k$

$\frac{dw}{ds} = u.v = (\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k).(\ 2i + 4j + 6k)$

$$= 2\left(\frac{1}{\sqrt{3}}\right) + 4\left(\frac{1}{\sqrt{3}}\right) + 6\left(\frac{1}{\sqrt{3}}\right) = \frac{12}{\sqrt{3}}$$

Exercises:

1- Find the directional derivatives of given function at the given point and in the same direction of the given vector A:

1. $f(x,y) = x\tan^{-1}\frac{y}{x}$, $p_0(1,1), A = 2i - j$
2. $f(x,y,z) = \ln\sqrt{x^2+y^2+z^2}$, $p_0\ (3,4,12), A = 3i + 6j - 2k$
3. $f(x,y,z) = xy + yz + zx$, $p_0\ (1,-1,2), A = 10i + 11j - 2k$

2- The directional derivative of a function $w = f(x,y)$ at $p_0(1,2)$ in the direction toward $p_1(2,3)$ is $2\sqrt{2}$ and in the direction toward $p_2(1,0)$ is (-3), What is the $\frac{dw}{ds}$ at p_0 in the direction toward the origin?

3- In which direction is the directional derivative of $f(x,y) = {(x^2 - y^2)}\big/{(x^2 + y^2)}$ at $(1,1)$ equal to zero?

The Chain Rule:

The function of one or more independent variables and each variable is a function of one or more other variables. The Chain Rule for functions of a single variable when

$$w = f(x)\ and\ x = g(t) then\ \frac{dw}{dt} = \frac{dw}{dx}.\frac{dx}{dt}$$

Or

$$w = f(x)\ and\ x = g(r,s) then\ \frac{\partial w}{\partial r} = \frac{dw}{dx}.\frac{\partial x}{\partial r}, \qquad \frac{\partial w}{\partial s} = \frac{dw}{dx}.\frac{\partial x}{\partial s}$$

For functions of two or more independent variables the Chain Rule has several forms. The form depends on how many variables are involved.

Case 1: Chain Rule for functions of two independent variables as follows:

If $w = f(x, y)$ and $x = x(t), y = y(t)$ then $w = w(t)$ and

$$\frac{\boldsymbol{dw}}{\boldsymbol{dt}} = \boldsymbol{f_x\big(x(t), y(t)\big).x'(t) + f_y\big(x(t), y(t)\big).y'(t)}$$

Or

$$\frac{\boldsymbol{dw}}{\boldsymbol{dt}} = \frac{\boldsymbol{\partial w}}{\boldsymbol{\partial x}}.\frac{\boldsymbol{dx}}{\boldsymbol{dt}} + \frac{\boldsymbol{\partial w}}{\boldsymbol{\partial y}}.\frac{\boldsymbol{dy}}{\boldsymbol{dt}}$$

The generalization of case 1:

If $w = f(x, y, z, \dots)$ and $x = x(t), y = y(t), z = z(t), \dots \dots$ then $w = w(t)$ and

$$\frac{\boldsymbol{dw}}{\boldsymbol{dt}} = \boldsymbol{f_x(x(t), y(t), z(t), \dots).x'(t) + f_y(x(t), y(t), z(t), \dots).y'(t) + f_z(x(t), y(t), z(t), \dots).z'(t)} + \cdots$$

Or

$$\frac{\boldsymbol{dw}}{\boldsymbol{dt}} = \frac{\boldsymbol{\partial w}}{\boldsymbol{\partial x}}.\frac{\boldsymbol{dx}}{\boldsymbol{dt}} + \frac{\boldsymbol{\partial w}}{\boldsymbol{\partial y}}.\frac{\boldsymbol{dy}}{\boldsymbol{dt}} + \frac{\boldsymbol{\partial w}}{\boldsymbol{\partial z}}.\frac{\boldsymbol{dz}}{\boldsymbol{dt}} + \dots$$

Example 10: Find $\frac{dw}{dt}$ if $w = xy \; and \; x = cost, y = sint$, what the derivative's value at $t = {}^{\pi}\!/_{2}$.

Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} = y(-sint) + x(cost) =$

$sint(-sint) + cost(cost) = -sin^2t + cos^2t = cos2t$

$$\left(\frac{dw}{dt}\right)_{t={}^{\pi}/_{2}} = \cos\left(2.{}^{\pi}\!/_{2}\right) = cos\pi = -1$$

Example11: Find $\frac{dw}{dt}$ if $w = xy + z \; and \; x = cost, y = sint \; , z = t$, what the derivative's value at $t = 0$.

Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dz}{dt} = y(-sint) + x(cost) + 1(1)$

$= sint(-sint) + cost(cost) + 1 = -sin^2t + cos^2t + 1 = cos2t + 1$

$$\left(\frac{dw}{dt}\right)_{t=0} = \cos 2(0) + 1 = cos0 + 1 = 1 + 1 = 2$$

<u>Case 2</u>: Chain Rule for functions of two independent variables as follows:

If $w = f(x, y)$ and $x = x(r, s), y = y(r, s)$ then $w = w(r, s)$ and

$$\frac{\partial w}{\partial r} = f_x\big(x(r, s), y(r, s)\big).x_r + f_y\big(x(r, s), y(r, s)\big).y_r,$$

$$\frac{\partial w}{\partial s} = f_x\big(x(r, s), y(r, s)\big).x_s + f_y\big(x(r, s), y(r, s)\big).y_s$$

Or

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial r}, \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s}$$

The generalization of case 2:

- If $w = f(x, y, z, \dots)$ and $x = x(r, s), y = y(r, s), z = z(r, s), \dots \dots$.. then $w = w(r, s)$ and

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial r} + \cdots, \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial s} + \cdots$$

- If $w = f(x, y, z, \dots)$ and $x = x(r, s, q, \dots), y = y(r, s, q, \dots)$, $z = z(r, s, q, \dots)$, ... then $w = w(r, s, q, \dots)$ and

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial r} + \cdots,$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial s} + \cdots,$$

$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial q} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial q} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial q} + \cdots,$$

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Example12: Find $\frac{\partial w}{\partial r}$ *and* $\frac{\partial w}{\partial s}$ in term of r *and* s if $w = x^2 + y^2$ *and* $x = r - s, y = r + s$.

Solution:1- $w = (r-s)^2 + (r+s)^2$

$$\frac{\partial w}{\partial r} = 2(r-s) + 2(r+s) = 4r, \frac{\partial w}{\partial s} = -2(r-s) + 2(r+s) = 4s.$$

2- $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial r}$, $\quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s}$

$= 2x(1) + 2y(1)$, $\quad = 2x(-1) + 2y(1)$

$= 2(r-s) + 2(r+s)$, $\quad = -2(r-s) + 2(r+s)$

$= 4r$, $\quad = 4s$

Example13: Find $\frac{\partial w}{\partial r}$ *and* $\frac{\partial w}{\partial s}$ in term of r *and* s if $w = xy + z$ *and* $x = r - s, y = r + s, z = r + 2s$.

Solution: $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial r}$, $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial s}$

$= y(1) + x(1) + 1(1)$, $\quad = y(-1) + x(1) + 1(2)$

$= (r+s) + (r-s) + 1$, $\quad = -(r+s) + (r-s) + 2$

$= 2r + 1$, $\quad = -2s + 2$

Exercises:

1- Find $\frac{dw}{dt}$ by using Chain Rule for the following functions:

a- $w = e^{2x+3y} cos4z$ *and* $x = \ln t, y = \ln(t^2+1), z = t$

b- $w = \frac{xy}{x^2+y^2}$ *and* $x = cosht, y = sinht$

c- $w = x^2 + y^2 + z^2$ *and* $x = e^t cost, y = e^t sint, z = e^t$

2- Find $\frac{dw}{dt}$ by using Chain Rule, check your answer by using different method and evaluate $\frac{dw}{dt}$ at given value of t for the following functions:

a- $w = x^2 + y^2, x = cost, y = sint; t = \pi$

b- $w = x^2 + y^2, x = cost + sint, y = cost - sint, ; t = 0$

c- $w = \frac{x}{z} + \frac{y}{z}, x = cos^2 t, y = sin^2 t, z = {}^1/_t;\ t = 3$

d- $w = ln(x^2 + y^2 + z^2), x = cost, y = sint, z = 4\sqrt{t};\ t = 3$

e- $w = 2ye^x - ln\, z, x = ln(t^2 + 1), y = tan^{-1} t, z = e^t;\ t = 1$

f- $w = z - sinxy, x = t, y = lnt, z = e^{t-1};\ t = 1$

3- Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in term of r and s by using Chain Rule and check your answer by using different method for the following functions:

a- $w = \sqrt{x^2 + y^2 + z^2}$ and $x = e^r cos\, s, y = e^r sin\, s, z = e^r$

b- $w = ln(x^2 + y^2 + 2z) and\ x = r + s, y = r - s, z = 2rs$

c- $w = x + 2y\ and\ x = \frac{r}{s}, y = r^2 + lns$

4- Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ in term of u and v by using Chain Rule and check your answer by using different method and evaluate $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ at given value of u, v for the following functions:

a- $w = 4e^x lny, x = \ln(u\ cosv), y = u\ sinv;\ (u, v) = (2, {}^\pi/_4)$

b- $w = tan^{-1}\left({}^x/_y\right), x = u\ cosv, y = u\ sinv;\ (u, v) = (1.3, {}^\pi/_6)$

c- $w = xy + yz + xz, x = u + v, y = u - v, z = uv;\ (u, v) = (^1/_2, 1)$

d- $w = \ln(x^2 + y^2 + z^2), x = ue^v \sin u, y = ue^v \cos u,$

$z = ue^v; (u, v) = (-2,0)$

5- Express $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ as a functions of x, y and z both by using Chain Rule and by expressing u directly in term of x, y and z before differentiating and evaluate $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at given point (x, y, z) for the following functions:

a- $u = \frac{p-q}{q-r}, p = x + y + z, q = x - y + z, r = x + y - z; (x, y, z) = (\sqrt{3}, 2,1)$

b- $u = e^{qr} sin^{-1} p, p = sinx, q = z^2 lny, r = {}^1/_z; (x, y, z) = (^\pi/_4, {}^1/_2, {}^{-1}/_2)$

Higher – Order Derivative:

1- If the function $y = f(x)$ can derivation *n* times for *x* then derivatives of f as follows:

$$\boldsymbol{y = f(x),}$$

$$\boldsymbol{y' = f'(x) = \frac{dy}{dx},}$$

$$\boldsymbol{y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},}$$

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$$\boldsymbol{y^{(n)} = f^{(n)}(x) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}}$$

<u>Example 14</u>: Find $y'and\ y''$ if $y = x^3 + 2x$.
Solution: $y' = 3x^2 + 2, y'' = 6x,\ y''' = 6,\ y'''' = 0.$

2- If the function $w = f(x, y, z, ...)$ can partial derivation *n* times for *x*, *y*, z,…then

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial w}{\partial x},$$
$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$
$$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right),$$

.
.
.

$$f_{xxx\dots x} = \frac{\partial^n f}{\partial x^n} = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right)$$

Same way for variables y, z,…

<u>Example 15</u>: Find $f_{xxxxx} and\ f_{yyyyy}$ if $w = f(x, y) = x^4 - 3y^2x + y^4 + 10.$
Solution: $f_x = 4x^3 - 3y^2, f_{xx} = 12x^2, f_{xxx} = 24x,$
$f_{xxxx} = 24, f_{xxxxx} = 0,$
$f_y = -6yx + 4y^3, f_{yy} = -6x + 12y^2, f_{yyy} = 24y,$
$f_{yyyy} = 24, f_{yyyyy} = 0$

3- Higher – Order partial derivatives for different variables as follows:

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$

$$\boldsymbol{f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),}$$

$$\boldsymbol{f_{xyz} = \frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = [\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\right)](\frac{\partial f}{\partial x})}$$

Note: Let $w = f(x, y)$ be a continuous function and have partial derivatives f_x, f_y then $f_{xy} = f_{yx}$.

Example16: Prove that $f_{xy} = f_{yx}$ such that

$$w = f(x, y) = x^4 - 3y^2 x + y^4 + 10$$

Proof: $f_x = 4x^3 - 3y^2$, $f_{xy} = -6y,$

$f_y = -6yx + 4y^3$, $f_{yx} = -6y$

$$f_{xy} = f_{yx}$$

Example17: Find $f_{xyz}\ and\ f_{zyx}$ if

$$w = f(x, y, z) = 3y^2 x + y^3 zx + x^2 z$$

Solution: $f_x = 3y^2 + y^3 z + 2xz, f_{xy} = 6y + 3y^2 z, f_{xyz} = 3y^2$

$$f_z = y^3 x + x^2, f_{zy} = 3y^2 x, f_{zyx} = 3y^2$$