جامعة بغداد
كلية التربية للعلوم الصرفة- ابن الهيثم
قسم الرياضيات
المرحلة الثانيه

التفاضل المتقدم

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2016-2017

Chapter one

Conic Sections

The distance between two points $p_1(x_1,y_1)$ and $p_2(x_2,y_2)$ is:

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

The distance between point p(x,y) and line $Ax + By + C = 0$ is:

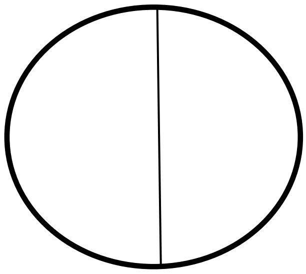
$$d = \frac{|\boldsymbol{Ax + By + C}|}{\sqrt{\boldsymbol{A^2 + B^2}}}$$

The general equation of second degree is:

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

Such that A,B,C,D,E and F are constants.

1- <u>The Circle</u>



The set of points in a plane whose distance from some fixed center point is constant radius value.

Let c(h,k) and r are the center and the radius of circle respectively. If p(x,y) be a point on this circle then the standard equation for circle is:

$$cp = r \text{ where } r > 0$$

and

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

or

$$\boldsymbol{(x-h)^2 + (y-k)^2 = r^2}$$

Note: if the center of circle is (0,0) then $\boldsymbol{x^2 + y^2 = r^2}$

Example1: Find the center and radius of the circle has the following equation: $x^2 + y^2 - 4x + 6y - 3 = 0$

Solution: $(x^2 - 4x + 4) + (y^2 + 6y + 9) = 4 + 9 + 3$

$$(x - 2)^2 + (y + 3)^2 = 16$$

The center is (2,-3) and the radius equal 4.

Example2: Find the equation of circle if the center (-1,1) and through tangent the line $x + 2y = 4$

Solution: $d = \frac{|Ax+By+C|}{\sqrt{A^2+B^2}} = \frac{|1(-1)+2(1)-4|}{\sqrt{1^2+2^2}} = \frac{3}{\sqrt{5}}$

$$(x + 1)^2 + (y - 1)^2 = \frac{9}{5}$$

Exercises:

1- Prove that the loci of the point p(x,y) is circle. If the distance between p and A(6,0) equal twice the distance between p and B(0,3)?

2- Find an equation for the circle through the point (1,0),(0,1) and (2,2)?

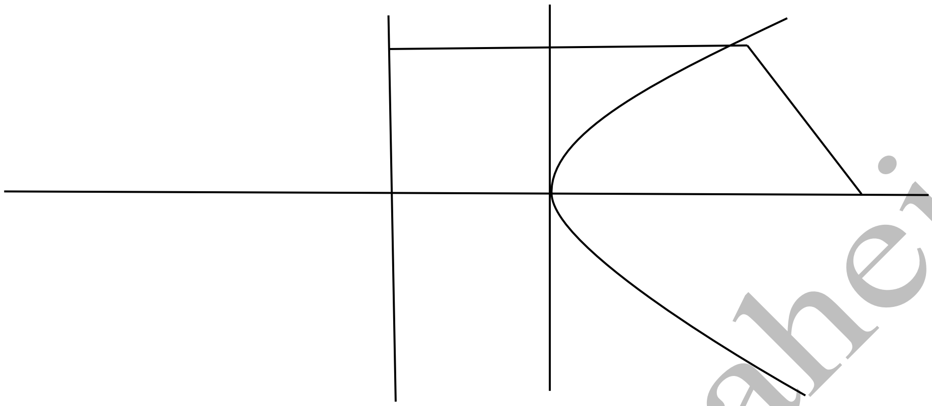
3- Find an equation for the circle through the point (2,3),(3,2) and (-4,3)?

4- Find an equation for the circle centered at (-2,1) that passes through the point (1,3). Is the point (1.1,2.8) inside, outside, or on the circle?

5- Find equations for the tangents to the circle: $(x - 2)^2 + (y - 1)^2 = 5$, at the points where the circle crosses the coordinate axes?

2- Parabola

Parabola is a set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane. The fixed point is called focus (F) and the fixed line is called directrix (L).

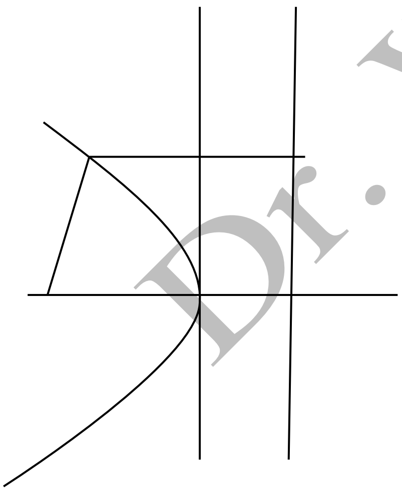


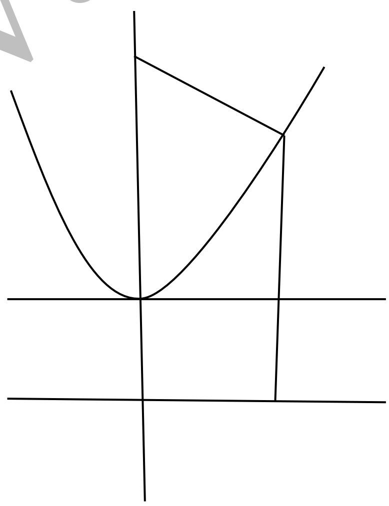
$PF = PQ$

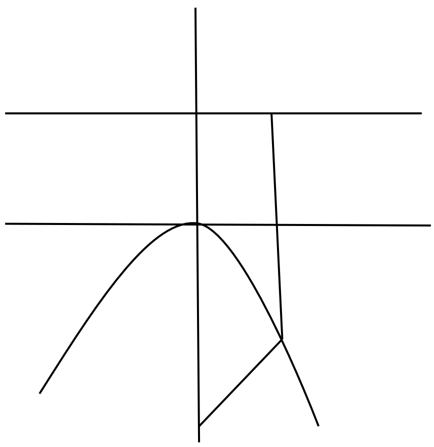
$\sqrt{(x-p)^2+(y-0)^2} = \sqrt{(x+p)^2+(y-y)^2}$

$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$

$y^2 = 4px \quad , x \geq 0$ is symmtric with $\mathrm{x-axis}$







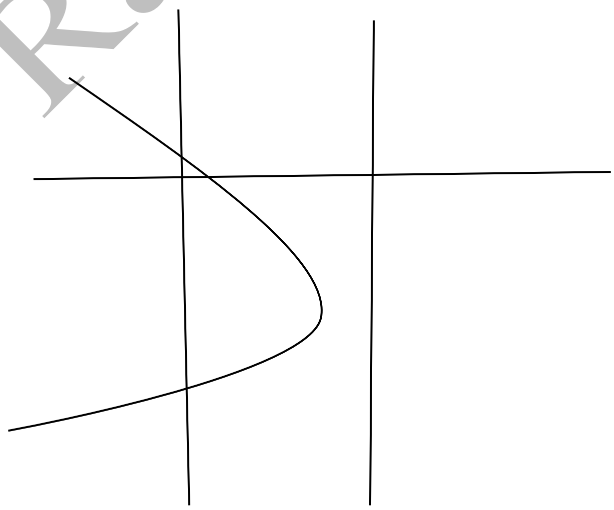
$y^2 = -4px$ $\qquad$ $x^2 = 4py$ $\qquad$ $x^2 = -4py$

	Equation	Focus(F)	Directrix	Axis
1	$\boldsymbol{y^2 = 4px}$ $\boldsymbol{(y-k)^2 = 4p(x-h)}$	*(p,0)* *(h+p,k)*	$x=-p$ $x=h-p$	*x-axis*
2	$\boldsymbol{y^2 = -4px}$ $\boldsymbol{(y-k)^2 = -4p(x-h)}$	*(-p,0)* *(h-p,k)*	$x=p$ $x=h+p$	*x-axis*
3	$\boldsymbol{x^2 = 4py}$ $\boldsymbol{(x-h)^2 = 4p(y-k)}$	*(0,p)* *(h,k+p)*	$y=-p$ $y=k-p$	*y-axis*
4	$\boldsymbol{x^2 = -4py}$ $\boldsymbol{(x-h)^2 = -4p(y-k)}$	*(0,-p)* *(h,k-p)*	$y=p$ $y=k+p$	*y-axis*

Example3: Discuss and sketch the following equation:

$$y^2+6y+2x+5=0$$

Solution:



$(y+3)^2+2x-4=0$

$(y+3)^2=-2x+4$

$(y+3)^2=-2(x-2)$

$2=4p$

$p=\frac{1}{2}$

V(h,k)=(2,-3) , F(h-p,k)= (2- $\frac{1}{2}$*,-3)= (* $\frac{3}{2}$*,-3)*

$x=h+p=2+\frac{1}{2}=\frac{5}{2}$

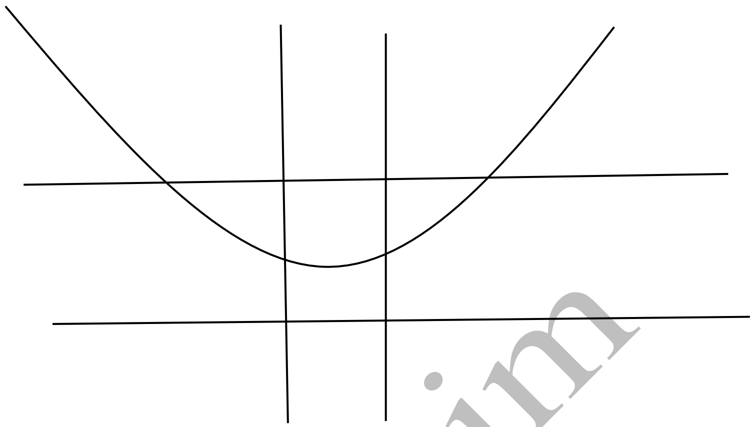
Example4: Find the focus, vertex and sketch of the following parabola:

$$x^2+2x+1-4y-4=0$$

Solution: $(x+1)^2=4(y+1)$

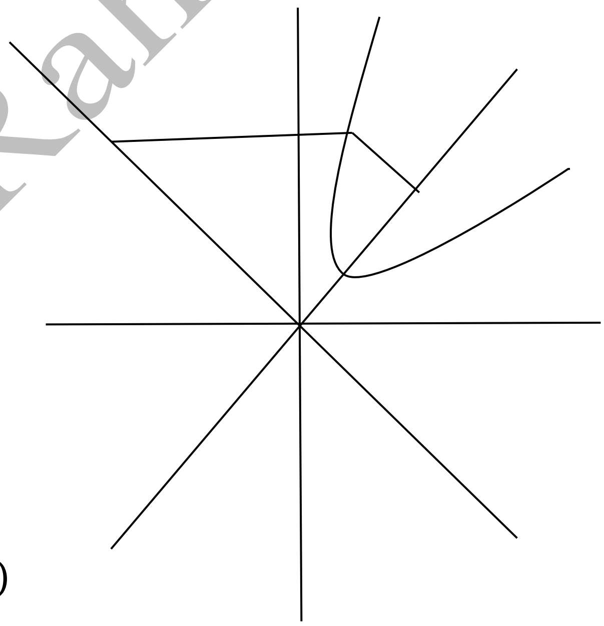
V(h,k)=(-1,-1) , $p = 1$, *F(h,k+p)= (-1,-1+1)=* $(-1,0)$

$y = k - p = -1 - 1 = -2$



Example5: Find an equation and sketch the parabola that the focus (2,2) and the directrix is the line $x + y = 0$.

Solution:



$d_1 = d_2$

$\frac{|Ax+By+c|}{\sqrt{A^2+B^2}} = \sqrt{(x-2)^2 + (y-2)^2}$

$\frac{|x+y|}{\sqrt{1^2+1^2}} = \sqrt{x^2 - 4x + 4 + y^2 - 4y + 4}$

$\frac{|x+y|}{\sqrt{2}} = \sqrt{x^2 - 4x + 4 + y^2 - 4y + 4}$

$(x+y)^2 = 2(x^2 - 4x + 4 + y^2 - 4y + 4)$

$x^2 + 2xy + y^2 = 2x^2 - 8x + 16 + 2y^2 - 8y$

$x^2 + y^2 - 2xy - 8x - 8y + 16 = 0$

Exercises:

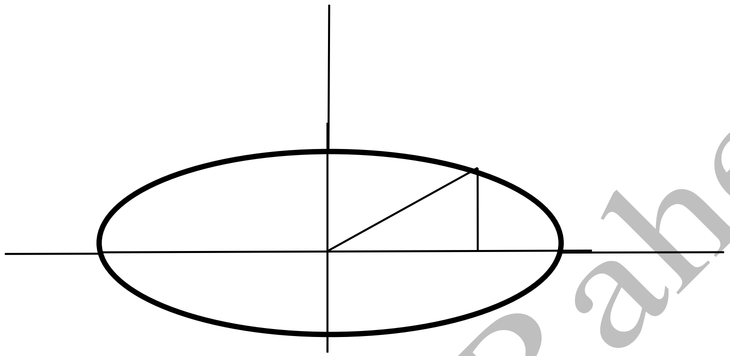
1- Find an equation, focus and directrix of each of the following Parabola, then sketch each one:
$y^2 = 12x$, $y^2 = -2x$, $x^2 = 6y$, $x^2 = -8y$, $y = 4x^2$,
$y = -8x^2$, $x = 2y^2$, $x = -3y^2$.

2- Find focus, vertex , directrix and sketch of each of the following Parabola:

$$a.\,(x+1)^2 = -4(y-3),\ b.\,(y+2)^2 = 8(x-1)$$

3- <u>Ellipse</u>

Ellipse is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are called foci(F_1,F_2) of the Ellipse.



$F_1(c,0)$, $F_2(-c,0)$

$$pF_1 + pF_2 = constant$$

$$\sqrt{(x-c)^2+(y-0)^2} + \sqrt{(x+c)^2+(y-0)^2} = 2a \quad \text{where} \quad a > 0$$

$$\sqrt{x^2-2cx+c^2+y^2} = 2a - \sqrt{x^2+2cx+c^2+y^2}$$

$$x^2-2cx+c^2+y^2 = 4a^2 - 4a\sqrt{x^2+2cx+c^2+y^2} + x^2+2cx+c^2+y^2$$

$$4cx+4a^2 = 4a\sqrt{x^2+2cx+c^2+y^2}$$

$$cx+a^2 = a\sqrt{x^2+2cx+c^2+y^2}$$

$$c^2x^2+2ca^2x+a^4 = a^2x^2+2ca^2x+a^2c^2+a^2y^2 \quad \text{where} \quad a > c$$

$$(a^2-c^2)x^2+a^2y^2 = (a^2-c^2)a^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1 \qquad \text{where} \quad a>c$$

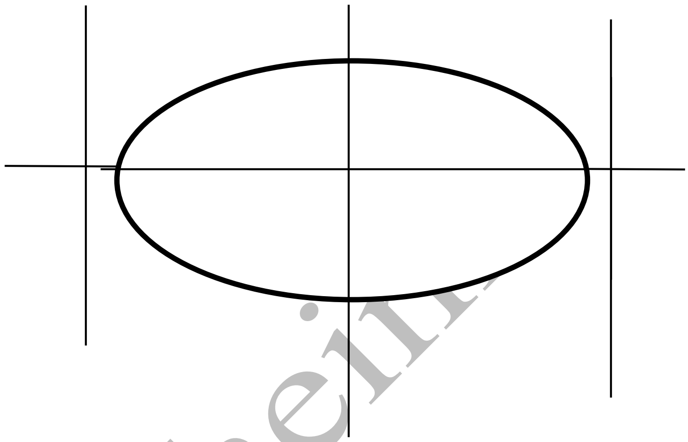
$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad \text{where} \quad a>b,\ b^2 = a^2-c^2$$

The Ellipse equation of x-axis with center (0,0) and foci ($\pm c, 0$).

$e = \frac{c}{a}$, $\boldsymbol{c < a} \rightarrow e < 1$ (Eccentricity)

The Directrix L_1, L_2 are:

$x = \pm\frac{a}{e}$ or $x = \pm\frac{a^2}{c}$



If $x = 0$

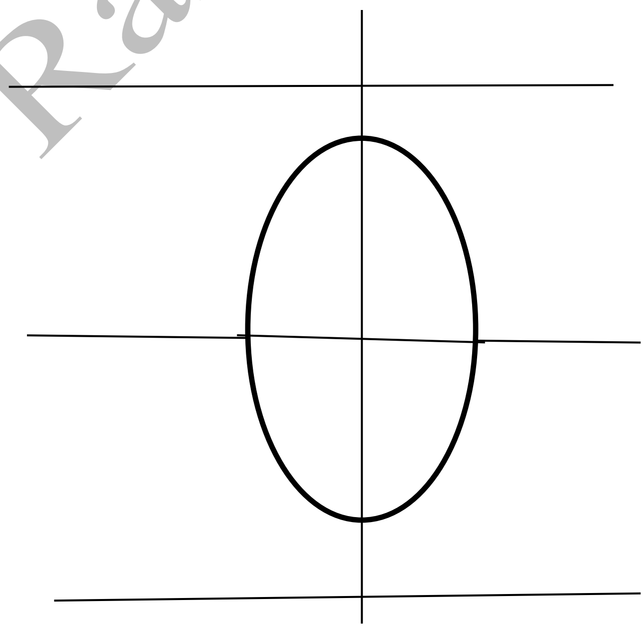
(0,-b)و (0,b)

If $y = 0$

(-a,0)و (a,0)

But The Ellipse equation of y-axis with center (0,0) and foci $(0, \pm c)$.

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$



If $x = 0$

(0,-a)و (0,a)

If $y = 0$

(-b,0)و (b,0)

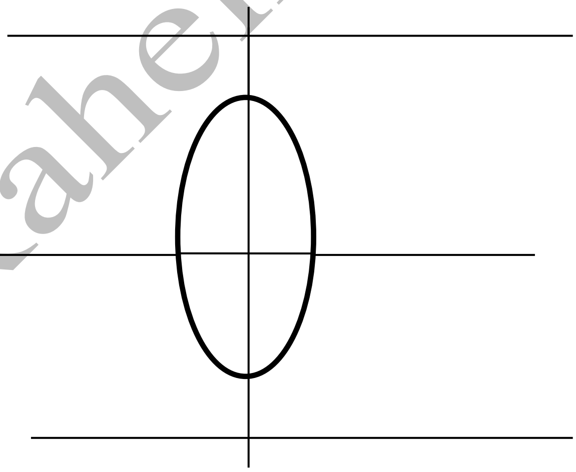
The Directrix L_1, L_2 are:

$y = \pm\frac{a}{e}$ or $y = \pm\frac{a^2}{c}$

x- axis	y- axis
$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ $F_1(c,0)$, $F_2(-c,0)$ Centre C(0,0) $A_1(a,0)$, $A_1(-a,0)$ $B_1(0, b)$, $B_2(0, -b)$ $x = \pm\frac{a}{e}$	$\boldsymbol{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$ $F_1(0,c)$, $F_2(0, -c)$ Centre C(0,0) $A_1(0, a)$, $A_2(0,-a)$ $B_1(b,0)$, $B_2(-b,0)$ $y = \pm\frac{a}{e}$

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $F_1(h+c,k)$, $F_2(h-c,k)$ Centre $C(h,k)$ $A_1(h+a,k)$, $A_1(h-a,k)$ $B_1(h,k+b)$, $B_2(h,k-b)$ $x=h\pm\frac{a}{e}$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ $F_1(h,k+c)$, $F_2(h,k-c)$ Centre $C(h,k)$ $A_1(h,k+a)$, $A_2(h,k-a)$ $B_1(h+b,k)$, $B_2(h-b,k)$ $y=k\pm\frac{a}{e}$

Example6: Find athe equation of Ellipse that the focus F(0,0) and a=3,C(0,2)?



Solution:

$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$

$b^2=a^2-c^2 \rightarrow b^2=9-4 \rightarrow b=\pm\sqrt{5}$

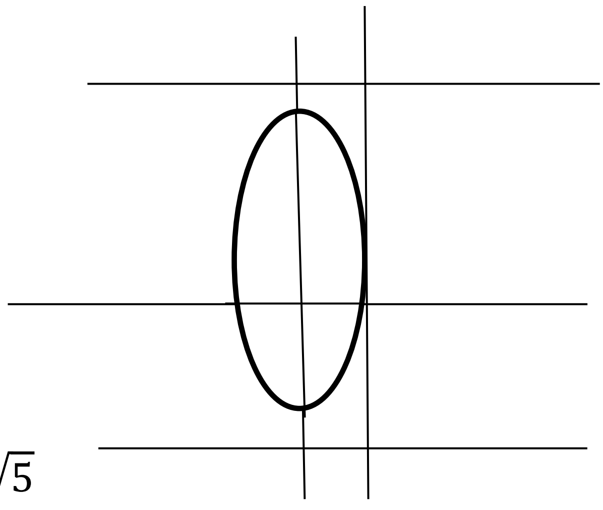
$\frac{(y-2)^2}{9}+\frac{(x-0)^2}{5}=1$

$e=\frac{c}{a}=\frac{2}{3}$

Example7: Discuss and sketch the following equation:

$$9x^2+4y^2+36x-8y+4=0$$

Solution:



$9(x^2+4x+4)+4(y^2-2y+1)=36$

$9(x+2)^2+4(y-1)^2=36$

$\frac{(x+2)^2}{4}+\frac{(y-1)^2}{9}=1$

$a^2=9 \rightarrow a=\pm3, b^2=4 \rightarrow b=\pm2$

$c^2=a^2-b^2 \rightarrow c^2=9-4=5 \rightarrow c=\pm\sqrt{5}$

$h = -2, k = 1 \rightarrow C(-2,1) centre\ of\ Ellipse$

$F_1(h, k + c), F_2(h, k - c) \rightarrow F_1(-2,1 + \sqrt{5}), F_2(-2,1 - \sqrt{5})$

$A_1(h, k + a), A_2(h, k - a) \rightarrow A_1(-2,1 + 3), A_2(-2,1 - 3) \rightarrow A_1(-2,4), A_2(-2, -2)$

$B_1(h + b, k), B(h - b, k) \rightarrow B_1(-2 + 2,1), B_2(-2 - 2,1) \rightarrow B_1(0,1), B_2(-4,1)$

$y = k \pm \frac{a^2}{c} \rightarrow y = 1 \pm \frac{9}{\sqrt{5}}$

<u>Exercises</u>:

1- Find an equation of Ellipse if the vertices (1,1),(1,7) , (3,4)and (-1,4)?

2- Prove that the tangent of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point p(x_1,y_1)is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$?

3- Find the equation of an Ellipse of eccentricity $\frac{2}{3}$ if the line $x = 9$ is one directrix and the focus is at (4,0)?

4- Find the equation of Ellipse having the center C, focus F and sketch graph:

1. C(0,0), F(0,2), a=4
2. C(0,0), F(-3,0), a=5
3. C(0,2), F(0,0), a=4
4. C(-3,0), F(-3,-2), a=4

5- Find and sketch the center, vertices and foci of the Ellipse:

1)$25x^2 + 9y^2 - 100x + 54y - 44 = 0$

2) $9x^2 + 4y^2 = 36$

3. $4x^2 + 9y^2 = 144$

4- Hyperbola

Hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are called foci(F_1,F_2) of the Hyperbola.

$F_1(-c,0)$, $F_2(c,0)$

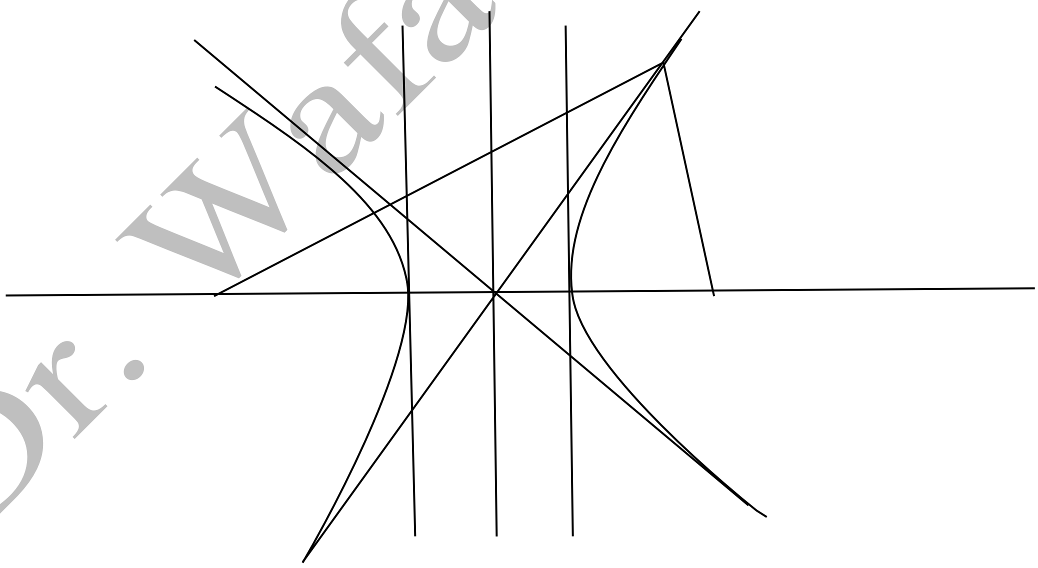
$$pF_1 - pF_2 = 2a$$

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a \quad \text{where} \quad a > 0$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1 \qquad \text{where} \quad c>a,\; b^2 = c^2 - a^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The Hyperbola equation the foci on x-axis with center (0,0), intersection with x-axis at ($\pm$a,0)and its asymptotes $y = \pm\frac{bx}{a}$



<u>Eccentricity</u>: $e = \frac{c}{a}$, $\boldsymbol{c > a}$

<u>Directrix:</u> $x = \pm\frac{a}{e}$

x- axis	y- axis
$\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$ $F_1(c,0)$, $F_2(-c,0)$ Centre C(0,0) $A_1(a,0)$, $A_1(-a,0)$ $B_1(0,b)$, $B_2(0,-b)$ $y=\pm\frac{b}{a}x$ $x=\pm\frac{a}{e}==\pm\frac{a^2}{c}$	$\boldsymbol{\frac{y^2}{a^2}-\frac{x^2}{b^2}=1}$ $F_1(0,c)$, $F_2(0,-c)$ Centre C(0,0) $A_1(0,a)$, $A_2(0,-a)$ $B_1(b,0)$, $B_2(-b,0)$ $y=\pm\frac{a}{b}x$ $y=\pm\frac{a}{e}=\pm\frac{a^2}{c}$
$\boldsymbol{\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1}$ $F_1(h+c,k)$, $F_2(h-c,k)$ Centre C(h,k) $A_1(h+a,k)$, $A_1(h-a,k)$ $B_1(h,k+b)$, $B_2(h,k-b)$ $(y-k)=\pm\frac{b}{a}(x-h)$ $x=h\pm\frac{a}{e}$ $x=h\pm\frac{a^2}{c}$	$\boldsymbol{\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1}$ $F_1(h,k+c)$, $F_2(h,k-c)$ Centre C(h,k) $A_1(h,k+a)$, $A_2(h,k-a)$ $B_1(h+b,k)$, $B_2(h-b,k)$ $(y-k)=\pm\frac{a}{b}(x-h)$ $y=k\pm\frac{a}{e}$ $y=k\pm\frac{a^2}{c}$

<u>Example8:</u> Sketch the following hyperbolas:

$$1.\ \frac{y^2}{9}-\frac{x^2}{16}=1 \qquad 2.\ [\frac{x^2}{16}-\frac{y^2}{9}=1\ (\mathbf{H.W.})]$$

Solution: 1**.** C(0,0)

$a^2=9\rightarrow a=\pm3\ , b^2=16\rightarrow b=\pm4$

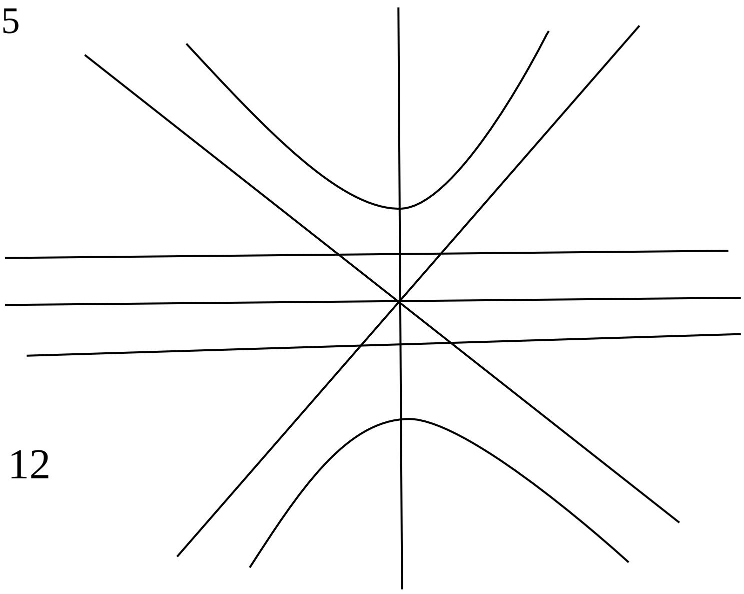
$c^2=a^2+b^2\rightarrow c=\sqrt{a^2+b^2}$=±5

$F_1(0,5)$, $F_2(0,-5)$

$A_1(0,3)$, $A_1(0,-3)$

$B_1(4,0)$, $B_2(-4,0)$

$y=\pm\frac{a}{b}x\rightarrow y=\pm\frac{3}{4}x$



$$y = \pm\frac{a^2}{c} \rightarrow\ y = \pm\frac{9}{5}$$

Example9: Sketch the following hyperbolas:

1. $9(x-2)^2 - 4(y+3)^2 = 36$ [2. $4(x-2)^2 - 9(y+3)^2 = 36$

3. $4x^2 - 5y^2 - 16x + 10y + 31 = 0$]**H.W.**

Solution: 1. $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

C(2,-3)

$a^2 = 4 \rightarrow a = 2\ , b^2 = 9 \rightarrow b = \pm 3$

$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2} = \pm\sqrt{13}$

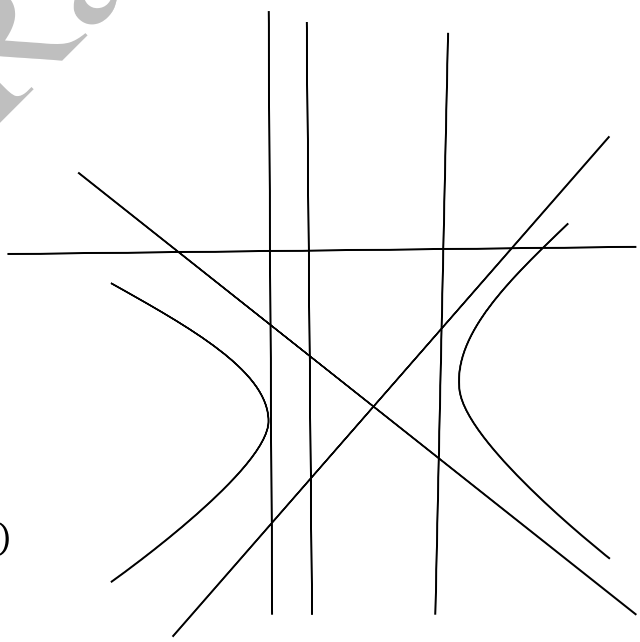
$F_1(2+\sqrt{13},-3)$, $F_2(2-\sqrt{13},-3)$

$A_1(2+2,-3)$, $A_1(2-2,-3) \rightarrow A_1(4,-3)$, $A_1(0,-3)$

$B_1(2,-3+3)$, $B_2(2,-3-3) \rightarrow B_1(2,0)$, $B_1(2,-6)$

$$x = h + \pm\frac{a^2}{c} \rightarrow\ x = 2 \pm \frac{4}{\sqrt{13}}$$

$$(y-k) = \pm\frac{b}{a}(x-h) \rightarrow (y+3) = \pm\frac{3}{2}(x-2)$$



Exercises:

1- Prove that tangent of Hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ at the points $p_1(x_1,y_1)$ is $b^2xx_1 - a^2yy_1 = a^2b^2$?

2- Find an equation of Hyperbola from the given in formation, $F_1(0,0)$, $F_2(0, 4)$ and pass through the point (12,9)?

3- Find an equation of Hyperbola that the eccentricity $\sqrt{2}$

and the vertices$(0, \pm 2)$?

4- Find an equation, foci, directrix, and asymptotes of each of the following Hyperbola, then sketch each one:

$x^2 - y^2 = 1$, $9x^2 - 16y^2 = 144$, $8x^2 - 2y^2 = 16$, $64x^2 - 36y^2 = 2304$, $8y^2 - 2x^2 = 16$, $y^2 - x^2 = 8$.

5- The Hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, is shifted two units to the right to generate the Hyperbola $\frac{(x-2)^2}{16} - \frac{y^2}{9} = 1$. Find the center, foci, vertices and asymptotes of new Hyperbola and sketch it.

6- Find the center, foci, vertices and asymptotes of Hyperbola $\frac{(y+2)^2}{4} - \frac{x^2}{5} = 1$ and sketch it.

7- Find and sketch the center, foci, vertices and radius as appropriate of the conic sections:

$x^2 + 4x + y^2 = 12$, $2x^2 + 2y^2 - 28x + 12y + 114 = 0$

$x^2 + 2x + y - 3 = 0$, $y^2 - 4y - 8x - 12 = 0$

$x^2 + 5y^2 + 4x = 1$, $9x^2 + 6y^2 + 6y = 0$ *and*, $x^2 + 2y^2 - 2x - 4y = -1$.

<u>Note:</u>

The general equation of conic section:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Using the Discriminant $B^2 - 4AC$

If $B^2 - 4AC = 0 \rightarrow Parabola$

If $B^2 - 4AC < 0 \rightarrow Ellipse$

If $B^2 - 4AC > 0 \rightarrow Hyperbola$

Example10: Use the Discriminant $B^2 - 4AC$ to decide whether the following equations represent parabola, ellipse, or hyperbola:

1- $x^2 + 4y^2 + 20x + y = 4$
2- $xy + y^2 - 3x = 5$
3- $x^2 - 2xy + y^2 - 6x = 2$

Solution: 1. $A = 1, B = 0, C = 4 \to B^2 - 4AC = 0 - 4(1)(4) = -16 < 0\ (Ellipse)$
2. $A = 0, B = 1, C = 1 \to B^2 - 4AC = 1 - 4(0)(1) = 1 > 0\ (Hyperbola)$
3. $A = 1, B = -2, C = 1 \to B^2 - 4AC = 4 - 4(1)(1) = 0\ (Parabola)$

Quadratic Equation and Rotations

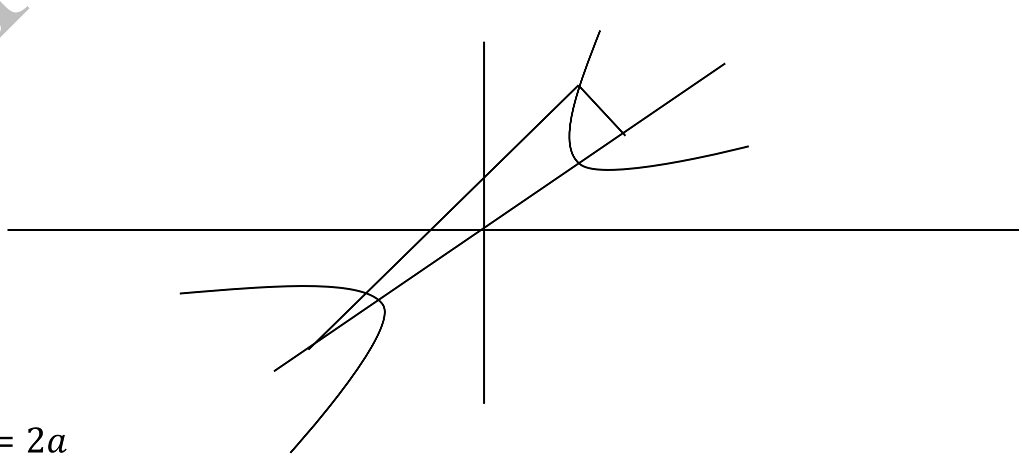
The general equation of second degree is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

This equation could be Circle or Parabola or Ellipse or Hyperbola (except few cases). In all these cases Bxy is unavailable (i.e. $B = 0$).

If $B \neq 0$

Example11: Let F_2(a,a), F_1(-a,-a) be two foci of hyperbola and rotated angle $\alpha = 45°$.



$|pF_1 - pF_2| = 2a$

$pF_1 - pF_2 = 2a$

$\sqrt{(x+a)^2 + (y+a)^2} - \sqrt{(x-a)^2 + (y-a)^2} = 2a$

$x^2 + 2ax + a^2 + y^2 + 2ay + a^2 = 4a^2 + 4a\sqrt{(x-a)^2 + (y-a)^2} + x^2 - 2ax + a^2 + y^2 - 2ay + a^2$

$4ax + 4ay - 4a^2 = 4a\sqrt{(x-a)^2 + (y-a)^2}] \div 4a$

$x + y - a = \sqrt{(x-a)^2 + (y-a)^2}$

$[(x+y) - a]^2 = (x-a)^2 + (y-a)^2$

$x^2 + 2xy + y^2 - 2ax - 2ay + a^2 = x^2 - 2ax + a^2 + y^2 - 2ay + a^2$

$2xy = a^2$

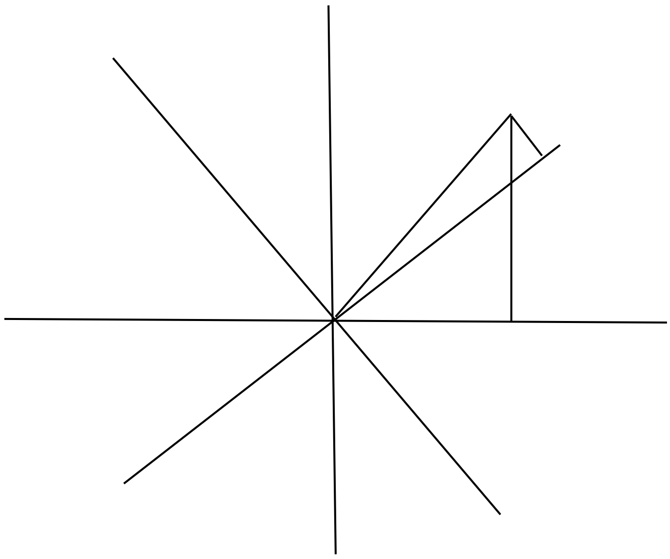
The foci of Hyperbola equation on xy with $45°$. In this example, rotated original axis (x and y) by $45°$(counterclockwise).

To find the new axis x' and $y'$from rotated original axis with rotated angle α (counterclockwise).

In Δopm

$x = om = op\cos\ (\alpha + \theta)$
$y = pm = op\sin\ (\alpha + \theta)$

.opm' في المثلث



$x' = om' = op\cos\theta$

$y' = pm' = op\sin\theta$

$x = op\,(\cos\alpha\ cos\theta - \sin\alpha\sin\theta)$

$y = op\,(\cos\theta\ sin\alpha + \sin\theta\cos\alpha)$

$x = op\cos\alpha\ cos\theta - op\sin\alpha\sin\theta$

$y = op\cos\theta\ sin\alpha + op\sin\theta\cos\alpha$

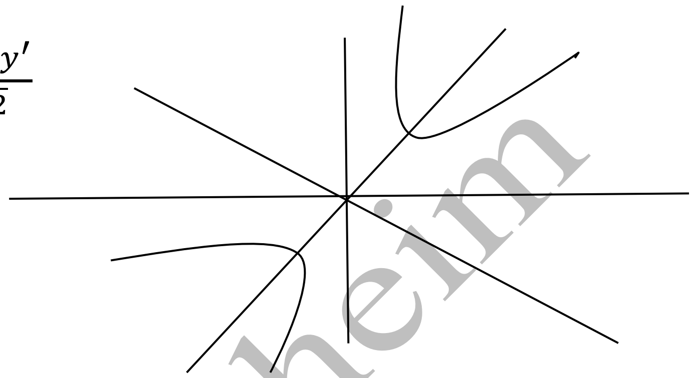
$$\boldsymbol{x = x'\cos\alpha\ - y'\sin\alpha}$$

$$\boldsymbol{y = x' \, sin\alpha + y' \cos\alpha}$$

In the last example, $2xy = a^2$ and rotated angle $\alpha = 45°$.

$x = x' \cos\alpha - y' \sin\alpha \rightarrow x = \frac{x'-y'}{\sqrt{2}}$

$y = x' \sin\alpha + y' \cos\alpha \rightarrow y = \frac{x'+y'}{\sqrt{2}}$



$2xy = a^2$

$2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) = a^2$

$(x')^2 - (y')^2 = a^2$

$$\frac{(x')^2}{a^2} - \frac{(y')^2}{a^2} = 1$$

Hyperbola

$c = \sqrt{a^2 + b^2} = \mp\sqrt{2}a$

$F_1{}'\left(\sqrt{2}a, 0\right), F_2{}'(-\sqrt{2}a, 0)$

$x' = \mp\sqrt{2}a, \; y' = 0$

$$\left.\begin{aligned} x &= \frac{\sqrt{2}a-0}{\sqrt{2}} = a \\ y &= \frac{\sqrt{2}a-0}{\sqrt{2}} = a \end{aligned}\right\} F_1(a,a), \qquad \left.\begin{aligned} x &= \frac{-\sqrt{2}a-0}{\sqrt{2}} = -a \\ y &= \frac{-\sqrt{2}a-0}{\sqrt{2}} = -a \end{aligned}\right\} F_2(-a,-a)$$

$A_1{}'(a, 0), A_2{}'(-a, 0)$

$x' = \mp a, \; y' = 0$

$$\left.\begin{aligned} x &= \frac{a-0}{\sqrt{2}} = \frac{a}{\sqrt{2}} \\ y &= \frac{a+0}{\sqrt{2}} = \frac{a}{\sqrt{2}} \end{aligned}\right\} A_1\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right), \qquad \left.\begin{aligned} x &= \frac{-a-0}{\sqrt{2}} = \frac{-a}{\sqrt{2}} \\ y &= \frac{-a+0}{\sqrt{2}} = -\frac{a}{\sqrt{2}} \end{aligned}\right\} A_2\left(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$$

$B_1{}'(0,a), B_2{}'(0,-a)$

$x' = 0,\ \ y' = \mp a$

$$\left.\begin{aligned} x &= \frac{0-a}{\sqrt{2}} = \frac{-a}{\sqrt{2}} \\ y &= \frac{0+a}{\sqrt{2}} = \frac{a}{\sqrt{2}} \end{aligned}\right\} B_1\left(\frac{-a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right), \qquad \left.\begin{aligned} x &= \frac{0+a}{\sqrt{2}} = \frac{a}{\sqrt{2}} \\ y &= \frac{0-a}{\sqrt{2}} = \frac{-a}{\sqrt{2}} \end{aligned}\right\} B_2\left(\frac{a}{\sqrt{2}},-\frac{a}{\sqrt{2}}\right)$$

Note:

If applied two equations of rotated axis on equation of degree two, then will find new equation of degree two as follows:

$A'x'^2+B'x'y'+C'y'^2+D'x'+E'y'+F'=0$

The relation between old and new equations will be as follows:

$$\boldsymbol{A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha}$$

$$\boldsymbol{B' = B(\cos^2\alpha - \sin^2\alpha) + 2(C-A)\sin\alpha\cos\alpha}$$

$$\boldsymbol{C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha}$$

$$\boldsymbol{D' = D\cos\alpha + E\sin\alpha}$$

$$\boldsymbol{E' = -D\sin\alpha + E\cos\alpha}$$

$$\boldsymbol{F' = F}$$

To find the *rotated angle* α for the equation:

$B' = B(cos^2\alpha - sin^2\alpha) + 2(C-A)sin\alpha\cos\alpha$

$B' = B\cos 2\alpha + (C-A)sin2\alpha$ if $B' = 0$

$0 = B\cos 2\alpha + (C-A)sin2\alpha$

$\frac{cos2\alpha}{sin2\alpha} = \frac{A-C}{B}$

$$\boldsymbol{cot2\alpha = \frac{A-C}{B}}$$

Example12: Decided whether the conic section with equation

$x^2 + xy + y^2 = 3$ represents a Parabola, an Ellipse, or Hyperbola.

Solution: A,B,C=1

$$cot2\alpha = \frac{A-C}{B} = \frac{1-1}{1} = 0$$

$$2\alpha = 90° \rightarrow \alpha = 45°$$

$$x = x'\cos\alpha - y'\sin\alpha \rightarrow x = \frac{x'-y'}{\sqrt{2}}$$

$$y = x'\sin\alpha + y'\cos\alpha \rightarrow y = \frac{x'+y'}{\sqrt{2}}$$

$$(\frac{(x'-y')}{\sqrt{2}})^2 + \left(\frac{(x'-y')}{\sqrt{2}}\right)\left(\frac{(x'+y')}{\sqrt{2}}\right) + (\frac{(x'+y')}{\sqrt{2}})^2 = 3\,] \times 2$$

$$x'^2 - 2x'y' + y'^2 + x'^2 - y'^2 + x'^2 + 2x'y' + y'^2 = 6$$

$$3x'^2 + y'^2 = 6$$

$$\frac{x'^2}{2} + \frac{y'^2}{6} = 1 \quad , a = \mp\sqrt{6} \;, b = \mp\sqrt{2}, \qquad c = \sqrt{a^2 - b^2} = \mp 2$$

$$F_1{}'(0,2), F_2{}'(0,-2)$$

$$e = \frac{c}{a} = \frac{2}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}} < 1$$

$$y' = \mp\frac{a}{e} = \mp\frac{\sqrt{6}}{\frac{\sqrt{2}}{\sqrt{3}}} = \mp 3 \qquad (Directrix)$$

If $y' = 3 \rightarrow x = \frac{x'-3}{\sqrt{2}}, y = \frac{x'+3}{\sqrt{2}}$

$\sqrt{2}\,x = x' - 3, \sqrt{2}\,y = x' + 3$

$\sqrt{2}\,x - \sqrt{2}\,y = -6$ (the equation of first Directrix)

If $y' = -3 \rightarrow x = \frac{x'+3}{\sqrt{2}}, y = \frac{x'-3}{\sqrt{2}}$

$\sqrt{2}\,x = x' + 3, \sqrt{2}\,y = x' - 3$

$\sqrt{2}\,x - \sqrt{2}\,y = 6$ (the equation of second Directrix)

$C(0,0)$ و $A'_1(0,\sqrt{6}), A'_2(0,-\sqrt{6}), B'_1(\sqrt{2},,0), B'_2(-\sqrt{2},0)$.

$F_1(-\sqrt{2},\sqrt{2}), F_2(\sqrt{2},-\sqrt{2})$, $A_1(-\sqrt{3},\sqrt{3}), A_2(\sqrt{3},-\sqrt{3})$,

$B_1(1,1), B_2(-1,-1)$.

Example13: Prove that the equation $x^4 + 6x^2y^2 + y^4 = 32$ become ${x'}^4 + {y'}^4 = 16$ after the coordinate axes angle rotation 45°.

Solution:

$x = x'\cos\alpha - y'\sin\alpha \rightarrow x = \frac{x'-y'}{\sqrt{2}}$

$y = x'\sin\alpha + y'\cos\alpha \rightarrow y = \frac{x'+y'}{\sqrt{2}}$

$$\frac{(x'-y')^4}{4} + 6\frac{(x'-y')^2}{2}\,\frac{(x'+y')^2}{2} + \frac{(x'+y')^4}{4} = 32\,] \times 4$$

$$(x'-y')^4 + 6(x'-y')^2(x'+y')^2 + (x'+y')^4 = 128$$

$$\begin{aligned} &{x'}^4 - 4{x'}^3y' + 6{x'}^2{y'}^2 - 4x'{y'}^3 + {y'}^4 + 6{x'}^4 - 12{x'}^2{y'}^2 + 6{y'}^4 \\ &\qquad + {x'}^4 + 4{x'}^3y' + 6{x'}^2{y'}^2 + 4x'{y'}^3 + {y'}^4 = 128 \end{aligned}$$

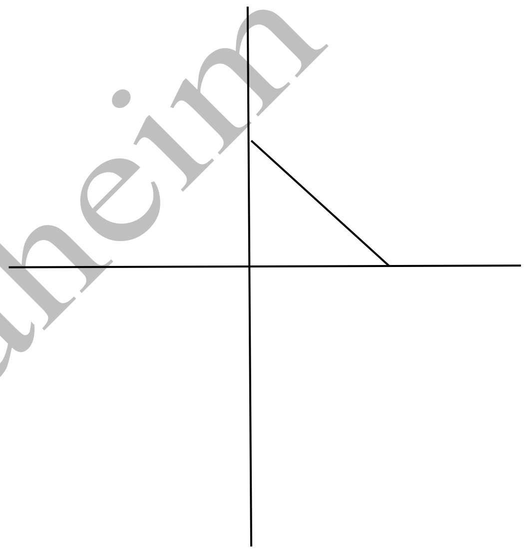
$8{x'}^4 + 8{y'}^4 = 128$

${x'}^4 + {y'}^4 = 16$

<u>Example14:</u> use the definition of ellipse conic to find ellipse equation that foci $F_1(-1,0)$, $F_2(0,\sqrt{3})$ and through the point $(1,0)$ and find the angle of rotation.

Solution:

$$pF_1 + pF_2 = 2a$$



$$\sqrt{(x+1)^2 + y^2} + \sqrt{x^2 + (y-\sqrt{3})^2} = 2a$$

$$\sqrt{(1+1)^2 + 0^2} + \sqrt{1^2 + \left(0-\sqrt{3}\right)^2} = 2a$$

$$2 + 2 = 2a \rightarrow a = 2$$

$$\sqrt{(x+1)^2 + y^2} + \sqrt{x^2 + (y-\sqrt{3})^2} = 4$$

$$\sqrt{(x+1)^2 + y^2} = 4 - \sqrt{x^2 + (y-\sqrt{3})^2}$$

$$x^2 + 2x + 1 + y^2 = 16 - 8\sqrt{x^2 + y^2 - 2\sqrt{3}y + 3} + x^2 + y^2 - 2\sqrt{3}y + 3$$

$$2x + 1 + 2\sqrt{3}y - 3 - 16 = -8\sqrt{x^2 + y^2 - 2\sqrt{3}y + 3}$$

$$2x + 2\sqrt{3}y - 18 = -8\sqrt{x^2 + y^2 - 2\sqrt{3}y + 3}$$

$$4x^2 + 8\sqrt{3}xy + 12y^2 - 72x - 72\sqrt{3}y + 324 = 64x^2 + 64y^2 - 128\sqrt{3}y + 192$$

$$-60x^2 + 8\sqrt{3}xy - 52y^2 - 72x + 56\sqrt{3}y - 132 = 0$$

$$15x^2 - 2\sqrt{3}xy + 13y^2 + 18x - 24\sqrt{3}y - 33 = 0$$

$$\cot 2\alpha = \frac{A-C}{B} = \frac{15-13}{-2\sqrt{3}} = \frac{2}{-2\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

α in quarter two and four

$2\alpha = 120° \rightarrow \alpha = 60°$

$2\alpha = -60° \rightarrow \alpha = -30°$

Example15: Prove that $A' + C' = A + C$ whatever the angle of rotation.

Solution:

$A' + C' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha + A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha$

$= (A + C)\cos^2\alpha + (C + A)\sin^2\alpha$

$= (A + C)(\cos^2\alpha + \sin^2\alpha)$

$= (A + C)$

Example16: Prove that ${x'}^2 + {y'}^2 = R^2$ *is* $x^2 + y^2 = R^2$ whatever the angle of rotation (**H.W**.).

Exercises:

1- Prove that ${B'}^2 - 4A'C' = B^2 - 4AC$ whatever the angle of rotation.

2- Prove that ${D'}^2 + {E'}^2 = D^2 + E^2$ whatever the angle of rotation.

3- Prove that the coordinate axes angle rotation 45° and $A = C$ for the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ to produce deleting xy.

4- Find the equation of the curve $x^2 + 2xy + y^2 = 1$ after the coordinate axes rotation to produce $A = 1,\ A' = 0.$

5- Decided whether the equations represents a Parabola, an Ellipse, or a Hyperbola and find angle α:

a) $x^2 - xy + 3y^2 + x - y - 3 = 0,$

b) $2x^2 + xy - 3y^2 + 3x - 7 = 0,$

c) $x^2 - 4xy + 4y^2 - 5 = 0,$

d) $2x^2 - 12xy + 18y^2 - 49 = 0$

e) $3x^2 + 5xy + 2y^2 - 8y - 1 = 0$

f) $2x^2 + 7xy + 9y^2 + 20x - 86 = 0, and$

g) $9x^2 + 6xy + y^2 - 12x - 4y + 4 = 0$

5- Use the discriminant $B^2 - 4AC$ to decide whether the following equations:

a) $x^2 - 3xy + y^2 - x = 0,$

b) $3x^2 - 18xy + 27y^2 - 5x + 7y + 4 = 0,$

c) $2x^2 + 4xy - y^2 - 2x + 3y = 6,$

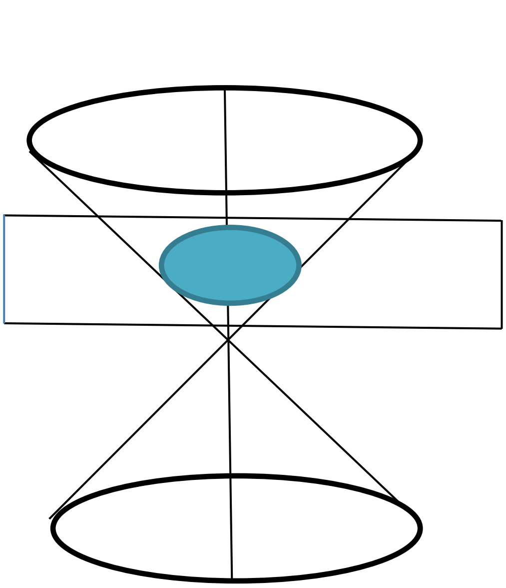
d) $x^2 + 4xy + 4y^2 - 3x = 6,$

e) $2x^2 - 4.9xy + 3y^2 - 4x = 7,$

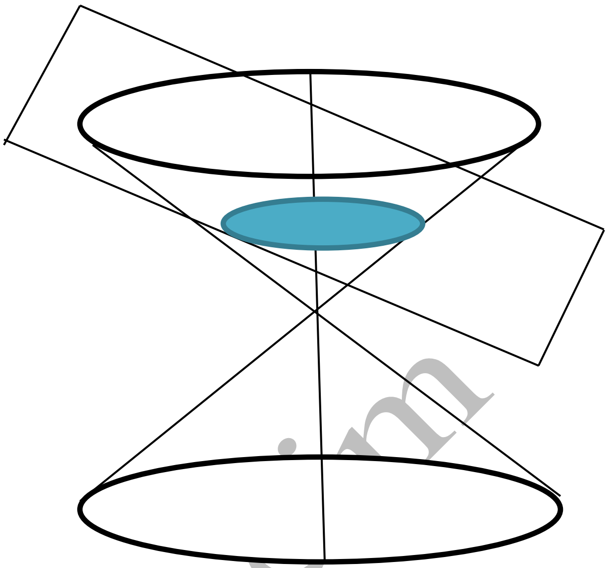
f) $3x^2 + 12xy + 12y^2 + 435x - 9y + 72 = 0,$

g) $3x^2 - \sqrt{15}\,xy + 2y^2 + x + y = 0, and$

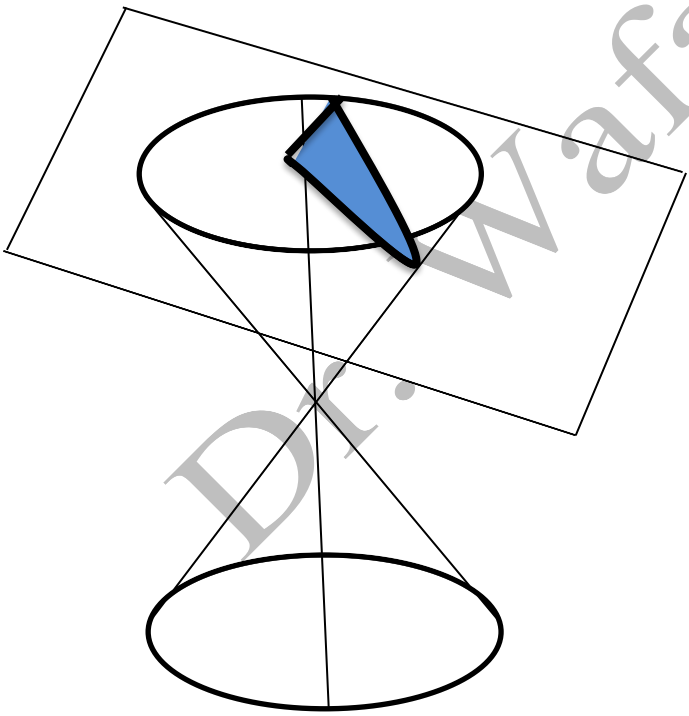
h) $3x^2 - 7xy + \sqrt{17}\,y^2 = 1$



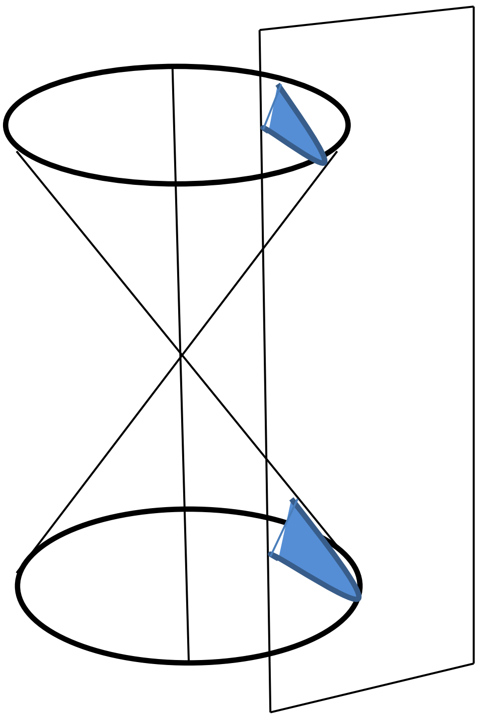
Circle



Ellipse



Parabola



Hyperbola