جامعة بغداد

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قسم الرياضيات

المرحلة الثانيه

التفاضل المتقدم

اعداد

د. وفاء رحيم حسين

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Chapter Six

Sequences and Series

Sequence:

Sequence is a function that its domain is natural numbers (N) and the sequence was writing as following:

$$\{(n, f(n)) : n = 1,2,3, \dots\}$$

The sequence represented as follows:

$$< a_n > \; or \; \{a_n\} \; or \; \{f(n)\} \; or \; a(n) \; or \; a_n$$

Example 1: Find all terms of the following sequences:

1. $a_n = 3 \quad \forall n \in N$, **2.** $a_n = (-1)^n \quad \forall n \in N$,

3. $a_n = n - 1 \;\; \forall n \in N$ and **4.** $a_n = \frac{1}{n} \;\; \forall n \in N$

Solution: **1.** $a_1 = 3, a_2 = 3, \; a_3 = 3, \; , \dots \; a_n = 3, \; \dots$

2. $a_1 = (-1)^1 = -1, a_2 = (-1)^2 = 1, \; a_3 = (-1)^3 = -1,$
$\dots \; a_n = (-1)^n, \; \dots$

3. $a_1 = 1 - 1 = 0, a_2 = 2 - 1 = 1, \quad a_3 = 3 - 1 = 2, \quad , \dots \; a_n = n - 1,$
$\dots$

4. $a_1 = \frac{1}{1} = 1, a_2 = \frac{1}{2} \quad a_3 = \frac{1}{3}, \; , \dots \; a_n = \frac{1}{n}, \; \dots$

The Graph of Sequence:

There are two methods of represent the Sequence:

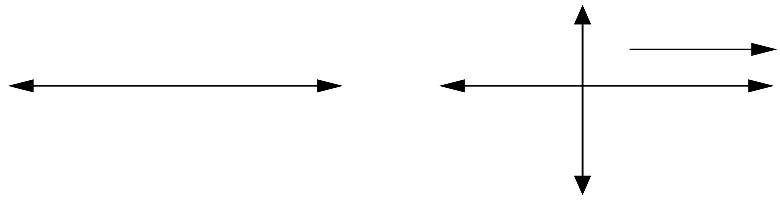
- The sequence a_n represented by using number line.
- The sequence a_n represented as a pair of point (n, a_n).

Example 2: Graph the following sequences by using two methods:

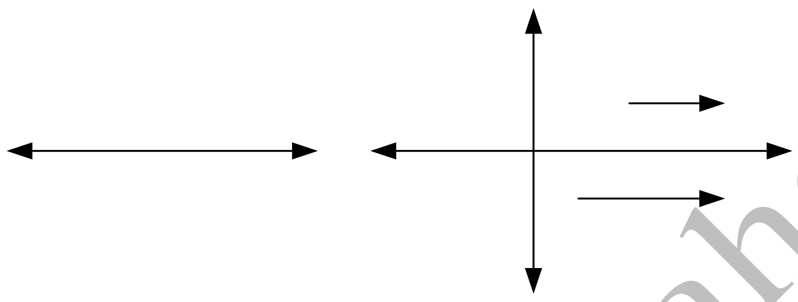
1. $a_n = 4 \quad \forall n \in N$, **2.** $a_n = (-1)^n \quad \forall n \in N$,

3. $a_n = n \;\; \forall n \in N$ and **4.** $a_n = (-1)^{n+1} \frac{1}{n} \;\; \forall n \in N$

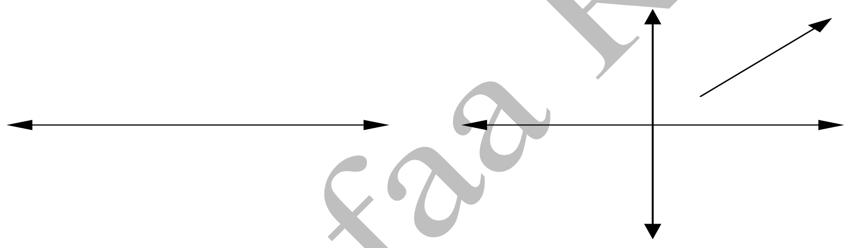
Solution: **1.** $a_n = \{4,4,4,4, \dots , 4 , \dots\}$



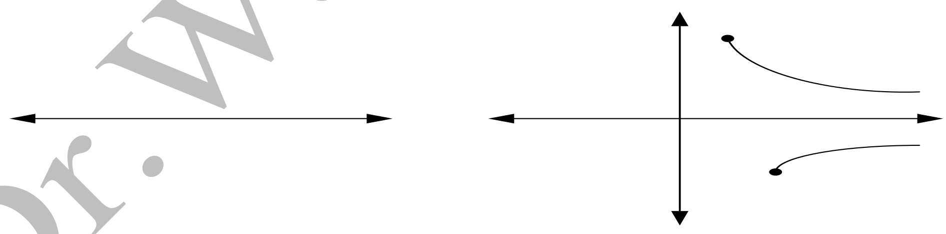
2. $a_n = \{-1,1,-1,1, \dots , (-1)^n , \dots\}$



3. $a_n = \{1,2,3,4, \dots , n , \dots\}$



4. $a_n = \{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \frac{-1}{6} \dots , (-1)^{n+1} \, \frac{1}{n} , \dots\}$



<u>Convergent and Divergent Sequences:</u>

Let a_n be a sequence and L is a real number, a_n is convergent to L if

$$\forall \epsilon > 0, \exists k \in N \ni |a_n - L| < \epsilon \quad \forall n > k$$

a_n is convergent to L is represented by $a_n \to L$ and this means L is the limit of a_n ($\lim_{n\to\infty} a_n = L$).

On the other hand the sequence a_n is divergent if

$$\exists \epsilon > 0, \forall k \in N \ \ni |a_n - L| \geq \epsilon \quad \forall n > k$$

If the limit of a_n is not exist then a_n is divergent.

<u>Example 3</u>: Determine each of the following sequence if convergent or divergent and find their limits:

1. $a_n = n - 1 \quad \forall n \in N$, 2. $a_n = \frac{1}{n} \quad \forall n \in N$,
3. $a_n = \frac{1}{2^{n-1}} \quad \forall n \in N$, 4. $a_n = 1 + \frac{(-1)^n}{n} \quad \forall n \in N$and
5. $a_n = \frac{n^2-n}{3n^2+n} \quad \forall n \in N$

Solution: **1.** $\text{Lim}_{n\to\infty} a_n = \lim_{n\to\infty}(n-1) = \infty - 1 = \infty \to a_n$ is divergent

2. $\text{Lim}_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0 \to a_n$ is convergent ($a_n \to 0$)

3. $\text{Lim}_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{2^{n-1}}\right) = \lim\limits_{n\to\infty} \frac{1}{2^n\, 2^{-1}} = \frac{2}{\infty} = 0 \to a_n$ is convergent ($a_n \to 0$)

4. $\text{Lim}_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right) = \lim\limits_{n\to\infty} \begin{cases} 1 + \frac{1}{n} \ if\ n\ even \\ 1 - \frac{1}{n} \ if\ \ n\ odd \end{cases}$

$$= \begin{cases} \lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n} \ if\ n\ is\ even \\ \lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} \frac{1}{n} \ if\ n\ is\ odd \end{cases} = \begin{cases} 1 + 0\ if\ n\ is\ even \\ 1 - 0\ if\ n\ is\ odd \end{cases} \to$$

$\lim_{n\to\infty} a_n \ = 1 \to a_n$ is convergent($a_n \to 1$)

5. $\text{Lim}_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n^2-n}{3n^2+n}\right) = \lim_{n\to\infty}\left(\frac{\frac{n^2}{n^2} - \frac{n}{n^2}}{\frac{3n^2}{n^2} + \frac{n}{n^2}}\right) =$

$\lim_{n\to\infty}\left(\frac{1-\frac{1}{n}}{3+\frac{1}{n}}\right) = \frac{1-0}{3+0} = \frac{1}{3} \to a_n$ is convergent ($a_n \to \frac{1}{3}$)

<u>Property 1:</u>

1. If the sequence is convergent then the limit of this sequence is unique.
2. If a_n, $b_n$and c_n be three sequences such that $a_n \leq b_n \leq c_n$ and $\text{Lim}_{n\to\infty} a_n = \text{Lim}_{n\to\infty} c_n = L$ then $\text{Lim}_{n\to\infty} b_n = L$

3. If $f: D \to E$ be a function define and continuous on R and a_n be a sequence in D such that $a_n \to L, L \in D$(this means f is define on a_n and continuous at L) then $f(a_n) \to f(L)$.
4. If the two sequences a_n and b_n are convergent such that $\text{Lim}_{n\to\infty} a_n = A$ and $\text{Lim}_{n\to\infty} b_n = B$ then:
   - $\text{Lim}_{n\to\infty}(a_n \pm b_n) = \text{Lim}_{n\to\infty} a_n \pm \text{Lim}_{n\to\infty} b_n = A \pm B$
   - $\text{Lim}_{n\to\infty}(ka_n) = k\,\text{Lim}_{n\to\infty} a_n = kA$ (k is scalar)
   - $\text{Lim}_{n\to\infty}(a_n \,.\, b_n) = \text{Lim}_{n\to\infty} a_n \,.\, \text{Lim}_{n\to\infty} b_n = AB$
   - $\text{Lim}_{n\to\infty} \frac{a_n}{b_n} = \frac{\text{Lim}_{n\to\infty} a_n}{\text{Lim}_{n\to\infty} b_n} = \frac{A}{B}$

<u>Example 4</u>: Find the limit of the following sequences:

1. $a_n = \frac{\cos(n)}{n}$, **2.** $b_n = \sqrt{\frac{n+1}{n}}$ and **3.** $c_n = \sqrt[n]{2} = 2^{\frac{1}{n}}$ (**H.W.**)

Solution: **1.** $-1 \le \cos(\text{n}) \le 1 \to \frac{-1}{n} \le \frac{\cos(n)}{n} \le \frac{1}{n}$

$$\lim_{n\to\infty}\left(\frac{-1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

By using Property 1(2) , $\lim\limits_{n\to\infty}\left(\frac{\cos(\text{n})}{n}\right) = 0$

2. Let $a_n = \frac{n+1}{n}$ and $f(x) = \sqrt{x}$

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{\frac{n}{n} + \frac{1}{n}}{\frac{n}{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1 \to \lim_{n\to\infty} a_n = 1$$

By using Property 1(3), $(a_n) \to f(1)$ $\left(\sqrt{\frac{n+1}{n}} \to 1\right) \to \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = 1$

<u>Lopital Rule:</u> If $f(x) and\ g(x)$ are two derivative functions at x_0 then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

The rule is used when the result of limit is equal to: $\infty, 0^0, 1^\infty, \frac{\infty}{\infty}, \frac{0}{0}, \ldots.$

<u>Example 5</u>: Find

1. $\lim\limits_{n\to\infty} \frac{n}{3^n}$,

2. $\lim\limits_{n\to\infty}(\ln(n)-\ln(n+1))$,
3. $\lim_{n\to\frac{\pi}{4}}(1-\tan n)\sec(2n)$,and
4. $\lim\limits_{n\to\infty}\frac{ln\, n}{n}$ (**H. W.**),

Solution: 1. $\lim\limits_{n\to\infty}\frac{n}{3^n}=\frac{\infty}{\infty}$

$$\lim_{n\to\infty}\frac{n}{3^n}=\lim_{n\to\infty}\frac{1}{3^n.\,ln3}=\lim_{n\to\infty}\frac{1}{3^n}.\lim_{n\to\infty}\frac{1}{ln3}=\frac{1}{\infty}.\frac{1}{ln3}=0$$

2. $\lim\limits_{n\to\infty}[\ln(n)-\ln(n+1)]=\infty-\infty$

$$\lim_{n\to\infty}ln\frac{n}{n+1}=\lim_{n\to\infty}(\ln(n)-\ln(n+1))$$

$$=\lim_{n\to\infty}\ln(n)-\lim_{n\to\infty}\ln(n+1)=\lim_{n\to\infty}\frac{1}{n}-\lim_{n\to\infty}\frac{1}{n+1}$$

$$=0-0=0$$

3. $\lim_{n\to\frac{\pi}{4}}(1-\tan n)\sec(2n)=\lim\limits_{n\to\frac{\pi}{4}}\frac{1-\tan n}{\cos 2n}=\frac{0}{0}$

$$\lim_{n\to\frac{\pi}{4}}(1-\tan n)\sec(2n)=\lim_{n\to\frac{\pi}{4}}\frac{1-\tan n}{\cos 2n}=\lim_{n\to\frac{\pi}{4}}\frac{-sec^2 n}{-2\sin 2n}$$

$$=\frac{sec^2\frac{\pi}{4}}{2\sin\frac{\pi}{2}}=\frac{2}{2}=1$$

Special Cases:

1. $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$,
2. $\lim\limits_{n\to\infty}(1\pm\frac{x}{n})^n=e^{\pm x}$,
3. $\lim\limits_{n\to\infty}x^n=0, |x|<1$,
4. $\lim\limits_{n\to\infty}\sqrt[n]{x}=1,\ x>0$,
5. $\lim\limits_{n\to\infty}\frac{x^n}{n!}=0$,
6. $\lim\limits_{n\to\infty}\frac{ln\, n}{n}=0$.

Example 6: Find

1. $\lim_{n\to\infty} \frac{\ln(n^2)}{n}$, 2. $\lim_{n\to\infty} \sqrt[n]{n^2}$, 3. $\lim_{n\to\infty} \sqrt[n]{3n}$, 4. $\lim_{n\to\infty} (-\frac{1}{2})^n$,

5. $\lim_{n\to\infty} (\frac{n-2}{n})^n$, 6. $\lim_{n\to\infty} \frac{100^n}{n!}$.

Solution: 1- $\lim_{n\to\infty} \frac{\ln(n^2)}{n} = \lim_{n\to\infty} \frac{2\ln(n)}{n} = 2(0) = 0$

2- $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} (n^{1/n})^2 = (1)^2 = 1$

3- $\lim_{n\to\infty} \sqrt[n]{3n} = \lim_{n\to\infty} \left(3^{1/n}\right)(\sqrt[n]{n}) = 1(1) = 1$

4- $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$

5- $\lim_{n\to\infty} (\frac{n-2}{n})^n = \lim_{n\to\infty} (1 - \frac{2}{n})^n = e^{-2}$

6- $\lim_{n\to\infty} \frac{100^n}{n!} = 0$

Exercises:

Determine each of the following sequence if convergent or divergent, Find all terms and graph each one:

1. $a_n = \frac{2n+1}{1-3n} \quad \forall n \in N,$
2. $a_n = sin\, n \quad \forall n \in N,$
3. $a_n = 1 + (-1)^n \quad \forall n \in N,$
4. $a_n = e^{-n} \quad \forall n \in N,$
5. $a_n = e^n \quad \forall n \in N,$
6. $a_n = \frac{1+(-1)^n}{n} \quad \forall n \in N,$
7. $a_n = \sin(\frac{\pi}{2} + \frac{1}{n}) \quad \forall n \in N$

Series:

➢ ***The Infinite Series:***

Let a_n *and* s_n be two sequences such that:

$$s_1 = a_1,$$

$$s_2 = a_1 + a_2,$$

$$s_3 = a_1 + a_2 + a_3,$$

.

.

.

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

The s_n is called **infinite series** and represented as follows:

$$\sum_{n=1}^{\infty} a_n$$

The $\sum_{n=1}^{k} a_n$ is a partial sums sequence of s_n.

Convergent and Divergent Infinite Series:

The infinite series $\sum_{n=1}^{\infty} a_n$ is convergent if s_n is convergent. If $\sum_{n=1}^{\infty} a_n$ is convergent this means it is having sum and it is sum as follows:

$$\sum_{n=1}^{\infty} a_n = L \leftrightarrow \lim_{n\to\infty} s_n = L$$

The infinite series $\sum_{n=1}^{\infty} a_n$ is divergent if s_n is divergent. If $\sum_{n=1}^{\infty} a_n$ is divergent this means it is not having sum.

Example 7:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3}{(10)^n} = \frac{3}{10} + \frac{3}{(10)^2} + \frac{3}{(10)^3} + \cdots + \frac{3}{(10)^n} + \cdots$$

$$s_1 = \frac{3}{10}\ , s_2 = \frac{3}{10} + \frac{3}{(10)^2}, s_3 = \frac{3}{10} + \frac{3}{(10)^2} + \frac{3}{(10)^3} \dots$$

$s_n = \frac{3}{10} + \frac{3}{(10)^2} + \frac{3}{(10)^3} + \cdots + \frac{3}{(10)^n}$.......(1)

$\frac{1}{10} s_n = \frac{3}{(10)^2} + \frac{3}{(10)^3} + \frac{3}{(10)^4} + \cdots + \frac{3}{(10)^{n+1}}$.......(2)

Subtraction (2) from (1)

$$s_n - \frac{1}{10} s_n = \frac{3}{10} - \frac{3}{(10)^{n+1}} \to (1 - \frac{1}{10}) s_n = \frac{3}{10}(1 - \frac{1}{(10)^n})$$

$$\to \frac{9}{10} s_n = \frac{3}{10}\left(1 - \frac{1}{(10)^n}\right) \to \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{(10)^n}\right)$$

$$= \frac{1}{3}(1 - 0) = \frac{1}{3}$$

$\sum_{n=1}^{\infty} a_n$ is convergent

- ***<u>The Geometric Series:</u>***

The series $a + ra + r^2 a + \cdots + r^n a + \cdots$ such that $(a \neq 0)$ where (r is the ratio between the value and previous one). This series is called **geometric** series and represented as follows:

$$\sum_{n=0}^{\infty} ar^n$$

<u>Example 8</u>:

1. 1+1+1+1+.... Is geometric series such that $a = 1, r = \frac{1}{1} = 1$.

2. $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$ Is geometric series such that $a = 1, r = \frac{1}{5}$.

<u>Convergent and Divergent Geometric Series:</u>

Let $\sum_{n=0}^{\infty} ar^n$ be a geometric series,

$\sum_{n=0}^{\infty} ar^n = a + ra + r^2 a + \cdots + r^n a + \cdots$

To find: $s_1 = a, s_2 = a + ar, s_3 = a + ar + ar^2,\ldots,$

$s_n = a + ar + ar^2,\ldots, ar^{n-1}$………(1)

$rs_n = ar + ar^2 + ar^3,\ldots, ar^n$……..(2)

Subtraction (2) from (1)

$(1-r)s_n = a - ar^n \to s_n = \dfrac{a - ar^n}{1-r}$

- If $r = 0$ then $s_n = a$
- If $r = 1$ then $s_n = na$
- If $|r| < 1$ then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a-ar^n}{1-r} = \frac{a}{1-r}$ and s_n is convergent to $\frac{a}{1-r}$
- If $|r| > 1$ then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a-ar^n}{1-r} = -\infty$ and s_n is divergent

The geometric series $\sum_{n=0}^{\infty} ar^n$ is

- ❖ Convergent if $|r| < 1$ and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
- ❖ Divergent if $|r| \geq 1$

<u>**Example 9**</u>**:** Determine which of the following series is convergent and which of them is divergent:

1. $3 + 6 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$
2. $1 - 2 + 4 - 8 + \cdots$

Solution: **1.** $3 + 6 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = 3 + 5 + (1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots)$

$a = 1, r = \frac{1}{3}$ (Geometric Series) $\to |r| < 1$(the series convergent)

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$3 + 5 + \frac{3}{2} = 8 + \frac{3}{2} = \frac{16+3}{2} = \frac{19}{2} = 9\,\frac{1}{2}$$

2- $\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}(1)(-2)^n$

$a = 1, r = -2$ (Geometric Series) $\rightarrow$

$|r| = |-2| = 2 > 1$ (series divergent)

Property 2:

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series such that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ then:
   - $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$
   - $\sum_{n=1}^{\infty}(ka_n) = k\sum_{n=1}^{\infty} a_n = kA$ (k is scalar)
2. If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} ca_n$ is also divergent $\forall c \neq 0$
3. If $\sum_{n=1}^{\infty} a_n$ is infinite series and the limit of a_n is not exists then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 10: Determine each of the following series if convergent or divergent:

1. $\sum_{n=1}^{\infty} n^2$

2. $\sum_{n=1}^{\infty}(-1)^{n+1}$

3. $\sum_{n=1}^{\infty}[(\frac{1}{2})^n - (\frac{1}{3})^n]$

Solution: **1.** $\sum_{n=1}^{\infty} n^2$ is divergent since $\lim_{n\to\infty} n^2 = \infty$ (Property 2(3)).

2. $\sum_{n=1}^{\infty}(-1)^{n+1}$ is divergent since $\lim_{n\to\infty}(-1)^{n+1}$ is not exists (Property 2(3)).

$a_n = (-1)^{n+1} = \{1, -1, 1, -1, \ldots\}$

$$\lim_{n\to\infty}(-1)^{n+1} = \begin{cases} \lim\limits_{n\to\infty} 1 & if\ n\ odd \\ \lim\limits_{n\to\infty}(-1) & if\ n\ even \end{cases} = \begin{cases} 1 & if\ n\ odd \\ -1 & if\ n\ even \end{cases}$$

3. $\sum_{n=1}^{\infty}(\frac{1}{2})^n = \frac{1}{2} + \frac{1}{2}(\frac{1}{2}) + \frac{1}{2}(\frac{1}{2})^2 + \frac{1}{2}(\frac{1}{2})^3 + \cdots$ is geometric series such that $a = r = \frac{1}{2}$, $|r| = \frac{1}{2} < 1$ then $\sum_{n=1}^{\infty}(\frac{1}{2})^n$ is convergent and $\sum_{n=1}^{\infty}(\frac{1}{2})^n = \frac{a}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$.

$\sum_{n=1}^{\infty}(\frac{1}{3})^n = \frac{1}{3} + \frac{1}{3}(\frac{1}{3}) + \frac{1}{3}(\frac{1}{3})^2 + \frac{1}{3}(\frac{1}{3})^3 + \cdots$ is geometric series such that $a = r = \frac{1}{3}$, $|r| = \frac{1}{3} < 1$ then $\sum_{n=1}^{\infty}(\frac{1}{3})^n$ is convergent and $\sum_{n=1}^{\infty}(\frac{1}{3})^n = \frac{a}{1-r} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2}$.

$$\sum_{n=1}^{\infty}[(\frac{1}{2})^n - (\frac{1}{3})^n] = 1 - \frac{1}{2} = \frac{1}{2}$$

Exercises:

Determine each of the following series if convergent or divergent:

1. $\sum_{n=1}^{\infty} a_n\,,\ a_n = 1$
2. $1 + 2(7) + 2(7)^2 + 2(7)^3 + \cdots$.
3. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$
4. $\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{2^n}{5^n}$

➢ ***The Positive Series:***

Let $\sum_{n=1}^{\infty} a_n$ be infinite series then $\sum_{n=1}^{\infty} a_n$ is called **positive** if $a_n \geq 0 \ \ \forall n$ and s_n is called increase sequence

$s_1 = a_1, s_2 = a_1 + a_2, s_3 = a_1 + a_2 + a_3, ...$

$s_n = a_1 + a_2 + a_3, ..., a_n$

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq \cdots$$

Tests of Positive Series:

1. Comparison Test:

Let $\sum_{n=1}^{\infty} a_n$ be positive infinite series then $\sum_{n=1}^{\infty} a_n$ is convergent if there exist positive infinite series $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$. Otherwise $\sum_{n=1}^{\infty} a_n$ is divergent if there exist positive infinite series $\sum_{n=1}^{\infty} c_n$ such that $\sum_{n=1}^{\infty} c_n$ is divergent and $a_n \geq c_n$.

There are some series:

- ***The Harmonic Series:***

This kind of series is always divergent and it is define as follows:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

All the following series are divergent:

$\sum_{n=1}^{\infty} \frac{2}{n}, \sum_{n=1}^{\infty} \frac{1}{3n}, \sum_{n=1}^{\infty} \frac{5}{4n}$, ... (by using Property 2(2)).

- ***The P- series:***

This kind of series is defined as follows:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^P}$$

This kind of series is convergent or divergent depends on the value of P:

- If $P > 1$ then the series is convergent.
- If $P < 1$ then the series is divergent.
- If $P = 1$ then the series is Harmonic, so this series is divergent.

Property 3: If $\sum_{n=1}^{\infty} a_n$ be infinite series and $\sum_{n=1}^{\infty} a_n$ is convergent then a_n is convergent too. The converse of this property is not true (if a_n is convergent then $\sum_{n=1}^{\infty} a_n$ is not always convergent) for example, $a_n = \frac{1}{n}$ is convergent to 0 but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

<u>Example 11</u>: Determine each of the following series if convergent or divergent:

1. $\sum_{n=1}^{\infty}\frac{7}{n}$, 2. $\sum_{n=1}^{\infty}\frac{1}{11n}$, 3. $\sum_{n=1}^{\infty}\frac{1}{n!}$, 4. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$,

5. $\sum_{n=1}^{\infty}a_n = 5+\frac{1}{3}+2+\frac{1}{7}+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots$, [6. $\sum_{n=1}^{\infty}\frac{1}{4n}$, and

7. $\sum_{n=1}^{\infty}\frac{sin^2 n}{2^n}$ **(H.W.)**]

Solution:

1. $\sum_{n=1}^{\infty}\frac{7}{n} = 7\sum_{n=1}^{\infty}\frac{1}{n}$ is Harmonic series and its divergent

$\sum_{n=1}^{\infty}\frac{7}{n}$ is divergent (by Property 2(2) in page 11).

2. $\sum_{n=1}^{\infty}\frac{1}{11n} = \frac{1}{11}\sum_{n=1}^{\infty}\frac{1}{n}$ is Harmonic series and its divergent

$\sum_{n=1}^{\infty}\frac{1}{11n}$ is divergent (by Property 2(2) in page 11).

3. $$\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\frac{1}{n!} = 1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots = 1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots$$

$$\sum_{n=1}^{\infty}b_n = 1+\sum_{n=1}^{\infty}\frac{1}{2^n} = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

$a_n \le b_n, \forall n$, and $\sum_{n=1}^{\infty}b_n = \sum_{n=0}^{\infty}(1)(\frac{1}{2})^n$ is Geometric series $(a=1, r=\frac{1}{2}) \to |r| = \frac{1}{2} < 1 \to \sum_{n=1}^{\infty}b_n$is convergent$\to \sum_{n=1}^{\infty}\frac{1}{n!}$ Is convergent too (depends on Comparison Test).

4. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ is P- series because $P = \frac{1}{2} < 1 \to \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent

5. $\sum_{n=1}^{\infty}a_n = 5+\frac{1}{3}+1+\frac{1}{7}+(1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots) = 5+\frac{1}{3}+1+\frac{1}{7}+\sum_{n=1}^{\infty}\frac{1}{n!}$

By 3. In same example $\rightarrow \sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent $\rightarrow \sum_{n=1}^{\infty} a_n$is convergent

<u>The limit of Comparison Test:</u>

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$are two infinite series such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k \text{ where } k \text{ is real number and } k \neq 0$$

- $\sum_{n=1}^{\infty} a_n$ is convergent $\leftrightarrow \sum_{n=1}^{\infty} b_n$is convergent.
- $\sum_{n=1}^{\infty} a_n$ is divergent $\leftrightarrow \sum_{n=1}^{\infty} b_n$is divergent.

<u>Example 12</u>: Determine each of the following series if convergent or divergent:

1. $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$, 2.$\sum_{n=1}^{\infty} sin\frac{1}{n}$, 3.$\sum_{n=1}^{\infty} \frac{2n}{3n^2-4n+1}$, 4. $\sum_{n=1}^{\infty} \frac{2n^3+100n^2+1000}{\frac{1}{6}n^6-n+2}$,

[5. $\sum_{n=1}^{\infty} n^{\frac{-n-1}{n}}$, 6. $\sum_{n=1}^{\infty} \frac{100+n}{n^3+2}$, and 7. $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ **(H.W.)**]

Solution:

1. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n-1} = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots$, let $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ is Geometric series $(a = 1, r = \frac{1}{2}) \rightarrow |r| = \frac{1}{2} < 1 \rightarrow \sum_{n=1}^{\infty} b_n$is convergent.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1 \neq 0$$

$\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ is convergent

2. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} sin\frac{1}{n}$, $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{sin\frac{1}{n}}{\frac{1}{n}} = \lim_{\frac{1}{n}\to 0} \frac{sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0$(Since $\lim_{x\to 0} \frac{sinx}{x} = 1$)

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series and divergent $\rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

3. $\sum_{n=1}^{\infty} \frac{2n}{3n^2-4n+1} = \sum_{n=1}^{\infty} \frac{\frac{2n}{n^2}}{\frac{3n^2}{n^2}-\frac{4n}{n^2}+\frac{1}{n^2}} = \sum_{n=1}^{\infty} \frac{\frac{2}{n}}{3-\frac{4}{n}+\frac{1}{n^2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\frac{2}{n}}{3-\frac{4}{n}+\frac{1}{n^2}}}{\frac{2}{n}} = \lim_{n\to\infty} \frac{1}{3-\frac{4}{n}+\frac{1}{n^2}} = \frac{1}{3-0+0} = \frac{1}{3} \neq 0$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n}$ is harmonic series and divergent $\rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

4. $\sum_{n=1}^{\infty} \frac{3n^3+100n^2+10}{\frac{1}{6}n^6-n+2} = \sum_{n=1}^{\infty} \frac{\frac{3n^3}{n^6}+\frac{100n^2}{n^6}+\frac{10}{n^6}}{\frac{\frac{1}{6}n^6}{n^6}-\frac{n}{n^6}+\frac{2}{n^6}} = \sum_{n=1}^{\infty} \frac{\frac{3}{n^3}+\frac{100}{n^4}+\frac{10}{n^6}}{\frac{1}{6}-\frac{1}{n^5}+\frac{2}{n^6}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\frac{3}{n^3}+\frac{100}{n^4}+\frac{10}{n^6}}{\frac{1}{6}-\frac{1}{n^5}+\frac{2}{n^6}}}{\frac{3}{n^3}} = \lim_{n\to\infty} \frac{\frac{3}{n^3}+\frac{100}{n^4}+\frac{10}{n^6}}{\frac{1}{6}-\frac{1}{n^5}+\frac{2}{n^6}} . \frac{n^3}{3} = \frac{1}{\frac{1}{6}} = 6 \neq 0$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^3}$ is P- series and $P = 3 > 1 \rightarrow \sum_{n=1}^{\infty} b_n$ is convergent$\rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

2. <u>The Ratio Test:</u>

Let $\sum_{n=1}^{\infty} a_n$ be positive series such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = k \quad \text{where } k \text{ is scaler}$$

- If $k < 1$ then the series is convergent.
- If $k > 1$ then the series is divergent.
- If $k = 1$ then the series maybe convergent or divergent.

<u>Example 13</u>: Determine each of the following series if convergent or divergent by using Ratio test:

1. $\sum_{n=1}^{\infty}\frac{1}{n}$, 2. $\sum_{n=1}^{\infty}\frac{n!\,n!}{(2n)!}$, 3. $\sum_{n=1}^{\infty}\frac{1}{n^2}$, and 4.$\sum_{n=1}^{\infty}\frac{3^n}{4^n.\sqrt{n}}$

Solution:

1. $a_n=\frac{1}{n},\ a_{n+1}=\frac{1}{n+1}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{n+1}.\frac{n}{1}=$

$\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=\frac{1}{1+0}=1\to\sum_{n=1}^{\infty}a_n$ either convergent or divergent $\to\sum_{n=1}^{\infty}\frac{1}{n}$ is harmonic series $\to\sum_{n=1}^{\infty}\frac{1}{n}$is divergent

2. $a_n=\frac{n!\,n!}{(2n)!},\ a_{n+1}=\frac{(n+1)!(n+1)!}{(2n+2)!}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{(n+1)!(n+)!}{(2n+2)!}}{\frac{n!\,n!}{(2n)!}}=\lim_{n\to\infty}\frac{(n+1)!(n+1)!}{(2n+2)!}.\frac{(2n)!}{n!\,n!}=$

$\lim_{n\to\infty}\frac{(n+1)\,n!(n+1)n!}{(2n+2)(2n+1)(2n)!}.\frac{(2n)!}{n!\,n!}=\lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)}=$

$\lim_{n\to\infty}\frac{(n+1)(n+1)}{2(n+1)(2n+1)}=\frac{1}{2}\lim_{n\to\infty}\frac{n+1}{2n+1}=\frac{1}{2}\lim_{n\to\infty}\frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}+\frac{1}{n}}=\frac{1}{2}\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2+\frac{1}{n}}=$

$\frac{1}{2}.\left(\frac{1+0}{2+0}\right)=\frac{1}{2}.\frac{1}{2}=\frac{1}{4}<1\to\ \sum_{n=1}^{\infty}\frac{n!\,n!}{(2n)!}$ is convergent

3. $a_n=\frac{1}{n^2},\ a_{n+1}=\frac{1}{(n+1)^2}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{1}{(n+1)^2}.\frac{n^2}{1}=$

$\lim_{n\to\infty}\frac{n^2}{(n+1)^2}=\lim_{n\to\infty}\frac{n^2}{n^2+2n+1}=\lim_{n\to\infty}\frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}=\lim_{n\to\infty}\frac{1}{1+\frac{2}{n}+\frac{1}{n}}=$

$\frac{1}{1+0+0}=1\to\sum_{n=1}^{\infty}a_n$ either convergent or divergent $\to\sum_{n=1}^{\infty}\frac{1}{n^2}$ is P-series and $P=2>1\to\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent

4. $a_n=\frac{3^n}{4^n.\sqrt{n}},\ a_{n+1}=\frac{3^{(n+1)}}{4^{(n+1)}.\sqrt{n+1}}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{3^{(n+1)}}{4^{(n+1)}.\sqrt{n+1}}}{\frac{3^n}{4^n.\sqrt{n}}}=\lim_{n\to\infty}\frac{3^{(n+1)}}{4^{(n+1)}.\sqrt{n+1}}.\frac{4^n.\sqrt{n}}{3^n}=$$

$$\lim_{n\to\infty}\frac{3^n.3}{4^n.4.\sqrt{n+1}}.\frac{4^n.\sqrt{n}}{3^n}=\frac{3}{4}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}=\frac{3}{4}\lim_{n\to\infty}\sqrt{\frac{n}{n+1}}=\frac{3}{4}\lim_{n\to\infty}\sqrt{\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}}=$$

$$\frac{3}{4}\lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}}=\frac{3}{4}.\sqrt{\frac{1}{1+0}}=\frac{3}{4}(1)=\frac{3}{4}<1\to\sum_{n=1}^{\infty}\frac{3^n}{4^n\sqrt{n}}$$ is convergent

3. ***The n^th^ Root Test:***

Let $\sum_{n=1}^{\infty}a_n$ be infinite positive series such that

$$\lim_{n\to\infty}\sqrt[n]{a_n}=k \text{ where } k \text{ is scaler}$$

- If $k<1$ then the series is convergent.
- If $k>1$ then the series is divergent.
- If $k=1$ then the series maybe convergent or divergent.

<u>Example 14</u>: Determine each of the following series if convergent or divergent by using n^{th} Root Test:

1. $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$, *and* 2. $\sum_{n=1}^{\infty}\frac{1}{n^n}$

Solution:

1. $a_n=\frac{2^n}{n^2}$,

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{2^n}{n^2}}=\lim_{n\to\infty}\frac{\sqrt[n]{2^n}}{\sqrt[n]{n^2}}=$$

$$\lim_{n\to\infty}\frac{2}{(n)^{\frac{2}{n}}}=\lim_{n\to\infty}\frac{2}{n^{\frac{1}{n}}.n^{\frac{1}{n}}}=\lim_{n\to\infty}\frac{2}{\sqrt[n]{n}\sqrt[n]{n}}==\lim_{n\to\infty}\frac{2}{\sqrt{1}\sqrt{1}}=$$

$$\frac{2}{(1)(1)}(since\ \lim_{n\to\infty}\sqrt[n]{n}=1)=2>1\to\sum_{n=1}^{\infty}\frac{2^n}{n^2}$$ is divergent

2. $a_n=\frac{1}{n^n}$,

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{n^n}} = \lim\limits_{n\to\infty} \frac{1}{n} = 0 < 1 \to$

$\sum_{n=1}^{\infty} \frac{1}{n^n}$ is convergent

Exercises:

Determine each of the following series if convergent or divergent:

1. $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$, 2. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, 3. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$, 4. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$,
   5. $\sum_{n=1}^{\infty} \frac{(n+3)!}{3^n\ n!\ 3!}$, and 6. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

The Alternating series:

The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 \ldots$ is called alternating series.

Property 4:

Let $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is alternating series then this series is convergent if satisfied the following conditions:

- $a_n > 0\ \ \forall n,$
- $a_{n+1} \le a_n \quad \forall n$
- $\lim\limits_{n\to\infty} a_n = 0$

The Conditionally Convergent:

If the series is convergent by satisfied the conditions of property 4 then this series is called have **conditionally convergent**.

The Absolutely Convergent:

Property 5:

1. If $\sum_{n=1}^{\infty} a_n$ be infinite series (maybe positive or negative or alternating terms) and $\sum_{n=1}^{\infty} |a_n|$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent too. The converse of this property is not true (if $\sum_{n=1}^{\infty} a_n$ is convergent then $\sum_{n=1}^{\infty} |a_n|$ is not always convergent) for example,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent (by Property 4). but $\sum_{n=1}^{\infty} |\frac{(-1)^n}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) is divergent.

2. If $\sum_{n=1}^{\infty} a_n$ is infinite series and $\lim_{n\to\infty} a_n = L \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

- The series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.
- Is not necessary the sum of absolute series equal to sum of origin series. For example, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ is geometric series where $a = 1, r = -\frac{1}{2} \to |r| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1 \to$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ is convergent and $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{2+1}{2}} = \frac{1}{\frac{3}{2}} =$ $\frac{2}{3}$ $(\sum_{n=0}^{\infty} a_n = \frac{a}{1-r})$, but $\sum_{n=0}^{\infty} |\frac{(-1)^n}{2^n}| = \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is geometric series where $a = 1, r = \frac{1}{2} \to |r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1 \to \sum_{n=0}^{\infty} |\frac{(-1)^n}{2^n}|$ is convergent and $\sum_{n=0}^{\infty} |\frac{(-1)^n}{2^n}| = \frac{1}{1-\frac{1}{2}} =$ $\frac{1}{\frac{2-1}{2}} = \frac{1}{\frac{1}{2}} = 2$
 $(\sum_{n=0}^{\infty} |a_n| = \frac{a}{1-r})$.
- To find convergent or divergent for absolute series (that have positive and negative terms) by using tests for positive series.

<u>Example 15</u>: Determine each of the following series if convergent or divergent:

1. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5n+1}$ $, and$ 2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Solution:

1. $\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{5n+1} \right| = \sum_{n=1}^{\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{n}{5n+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{5n}{n}+\frac{1}{n}} =$ $\lim_{n\to\infty} \frac{1}{5+\frac{1}{n}} = \frac{1}{5+0} = \frac{1}{5} \neq 0$ (By Properties 5(2)) $\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{5n+1} \right|$ is divergent.

2. $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=0}^{\infty} \frac{1}{n^2}$ is P-series with $P = 2 > 1 \rightarrow$ $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right|$ is convergent and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolute convergent.

<u>Example 16</u>: Determine each of the following series if absolutely or conditionally convergent:

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, [2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$, and 3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ **(H.W.)**]

Solution:

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series and it is divergent$\rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not absolutely convergent but it have conditionally convergent because:

- $a_n = \frac{1}{n} > 0 \ \forall n,$
- $a_{n+1} = \frac{1}{n+1} \leq a_n = \frac{1}{n} \quad \forall n$
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

➢ ***<u>Power series:</u>***

The series $\sum_{i=0}^{\infty} a_i \, x^i = \quad a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$ (where $a_0, a_1, a_2, a_3, \ldots$ are constants) is called **power series**.

We can represent any function by using power series of n terms as follows:

$f(x) = \sum_{i=0}^{n} a_i \, x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \quad \ldots \ldots (1)$

For find the constants $a_0, a_1, a_2, , \dots , a_n$

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$,

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots + n a_n x^{n-1}$,

$f''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \dots + n(n-1) a_n x^{n-2}$,

.

.

$f^{(n)}(x) = n!\, a_n x^{n-n} = n!\, a_n$

When $x = 0$

$f(0) = a_0$, $f'(0) = a_1$, $f''(0) = 2a_2 \to a_2 = \frac{f''(0)}{2!}$,

$f^{(3)}(0) = 6a_3 \to a_3 = \frac{f^{(3)}(0)}{3!}$, $f^{(4)}(0) = 24 a_4 \to a_4 = \frac{f^{(4)}(0)}{4!}$, ...

$f^{(n)}(x) = n!\, a_n \to a_n = \frac{f^{(n)}(0)}{n!}$, ...

Put the values of $(a_0, a_1, a_2, \dots , a_n)$ in equation (1)

$$\boldsymbol{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n}$$

This equation is called the **Taylor polynomial of degree *n*** for function $f(x)$ at $x = 0$.

But if $n \to \infty$ the equation become as follows:

$$\boldsymbol{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \dots}$$

This equation is called the **Makloren series** for function $f(x)$ at $x = 0$.

<u>Example 17</u>: Find Taylor polynomial and Makloren series generated by the function $f(x) = e^x$ at $x = 0$.

Solution:

$$f(x) = e^x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$e^x = e^0 + e^0 x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \cdots + \frac{e^0}{n!}x^n$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ Is the Taylor polynomial

But $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$. Is the Makloren series

Exercises:

Find Taylor polynomials and Makloren series for the following functions at $x = 0$:

1. $f(x) = 3x^3 + 2$, 2. $f(x) = e^{-x}$, 3. $f(x) = \cosh x$,

4. $f(x) = \sinh x$, 5. $f(x) = \cos x$ and 6. $f(x) = \sin x$.

Where $\cosh x = \frac{1}{2}(e^x + e^{-x}), and\ \sinh x = \frac{1}{2}(e^x - e^{-x})$

➢ *Taylor Series:*

The power series is representing when $x = a$ as follows:

$\sum_{i=0}^{\infty} a_i\,(x-a)^i = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 \ldots + a_n(x-a)^n + \cdots$

We can represent any function of this kind of power series by using the same value of terms $(a_0, a_1, a_2, , \ldots, a_n, \ldots$ as follows:

$$\boldsymbol{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

This equation is called **extending Taylor polynomial** for $f(x)$ at $x = a$.

But if $n \to \infty$ the equation became as follows:

$$\boldsymbol{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots}$$

This equation is called the **Taylor series** for function $f(x)$ at $x = a$.

Example 18: Find Taylor series generated by the function

$f(x) = cos\, x$ at $x = 2\pi$.

Solution:

$$f(x) = \cos x = f(2\pi) + f'(2\pi)(x - 2\pi) + \frac{f''(2\pi)}{2!}(x - 2\pi)^2 + \frac{f^{(3)}(2\pi)}{3!}(x - 2\pi)^3 + \cdots$$

$[f(x) = \cos x \to f(2\pi) = \cos(2\pi) = 1, f'(x) = -\sin x \to f'(2\pi) = -\sin(2\pi) = 0, f''(x) = -\cos x \to f''(2\pi) = -\cos(2\pi) = -1,$
$f^{(3)}(x) = \sin x \to f^{(3)}(2\pi) = \sin(2\pi) = 0, f^{(4)}(x) = \cos x \to f^{(4)}(2\pi) = \cos(2\pi) = 1, \dots,$

$f^{(n)}(x) = \begin{cases} (-1)^n & if\ n\ even \\ 0 & if\ n\ is\ odd \end{cases} \to f^{(2n)}(2\pi) = (-1)^n]$

$$\cos x = 1 + 0 - \frac{1}{2!}(x - 2\pi)^2 + 0 + \frac{1}{4!}(x - 2\pi)^4 + 0 - \frac{1}{6!}(x - 2\pi)^6 \dots + \frac{1}{n!}(x - 2\pi)^n$$

$$\cos x = 1 - \frac{1}{2!}(x - 2\pi)^2 + \frac{1}{4!}(x - 2\pi)^4 - \frac{1}{6!}(x - 2\pi)^6 \dots + \frac{1}{(2n)!}(x - 2\pi)^{2n}$$

This is the extending Taylor polynomial for $\cos x$ at $x = 2\pi$

But $\cos x = 1 - \frac{1}{2!}(x - 2\pi)^2 + \frac{1}{4!}(x - 2\pi)^4 - \frac{1}{6!}(x - 2\pi)^6 \dots$

Is the Taylor series.

Convergent of power series:

Property 6:

Let $\sum_{n=0}^{\infty} a_n\, x^n$ be a power series:

- If the series $\sum_{n=0}^{\infty} a_n\, x^n$ is convergent at $x = b$ then it is convergent to every values of x less than b. This means the series is convergent for all values of x such that $|x| < b (b \neq 0)$.

- ❖ If the series $\sum_{n=0}^{\infty} a_n\, x^n$ is divergent at $x = c$ then it is divergent to every values of c such that $|x| > |c| (c \neq 0)$.

<u>Example 19</u>: Find the interval of convergent for the following series:

1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$, and 2. $\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2}$ **(H.W.)**

Solution:

$a_n = \frac{x^n}{n}$, $a_{n+1} = \frac{x^{(n+1)}}{n+1}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{(n+1)}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty} \left|\frac{x^n x}{n+1} \cdot \frac{n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{n\,x}{n+1}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}}\right| = |x| \lim_{n\to\infty} \left|\frac{1}{1 + \frac{1}{n}}\right| = |x|$$

By using the ratio test If $|x| < 1$ then the series is convergent→ $-1 < x < 1$

When $x = 1 \to \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series → the series is divergent. But $x = -1 \to \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ → the series is conditionally convergent (Property 4). the series is convergent→ $-1 \leq x < 1$.