جامعة بغداد
كلية التربية للعلوم الصرفة- ابن الهيثم
قسم الرياضيات
المرحلة الثانيه

التفاضل المتقدم

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2016-2017

Chapter Three

Vectors and Parametric Equations

Parametric Equations:

In Cartesian plane can represent the curve equation as follows:

$$y = f(x), x = g(y), or\ F(x, y) = 0$$

For example the circle equation of centre (0,0) and radius r represent as follows:

$$y = \pm\sqrt{r^2 - x^2} \qquad [y = f(x)]$$

$$x = \pm\sqrt{r^2 - y^2} \qquad [x = f(y)]$$

$$x^2 + y^2 - r^2 = 0 \qquad [F(x, y) = 0]$$

But there is other way to represent curve equation by using parameter, by define x and y depend on this parameter and these equations are called parametric equations.

If *t* is the parameter then $x = f(t), and\ y = g(t)$, these equations are parametric equations of parameter *t*.

Example 1: Find parametric equations of the curve $y^2 = 4px$ using as parameter the slope $t = \frac{dy}{dx}$?

Solution: $y^2 = 4px \rightarrow 2y\frac{dy}{dx} = 4p \rightarrow \frac{dy}{dx} = \frac{2p}{y} \rightarrow t = \frac{2p}{y} \rightarrow$

$$y = \frac{2p}{t} \ldots\ldots\ldots.1$$

Put equation 1 in original equation

$$\frac{4p^2}{t^2} = 4px$$

$$x = \frac{p}{t^2} \ldots\ldots\ldots\ldots 2$$

$x = \frac{p}{t^2}$ and $y = \frac{2p}{t}$ are parametric equations

Example 2: Find parametric equations of the circle
$y^2 + x^2 = a^2, y > 0$ using as parameter the slope of the tangent at (x,y)?

Solution: $y^2 = a^2 - x^2 \rightarrow y = \pm\sqrt{a^2 - x^2}$

When $y > 0 \rightarrow y = \sqrt{a^2 - x^2} \rightarrow \frac{dy}{dx} = \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)$

$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2-x^2}} \rightarrow t = \frac{-x}{\sqrt{a^2-x^2}} \rightarrow t^2 = \frac{x^2}{a^2-x^2} \rightarrow t^2a^2 - t^2x^2 - x^2 = 0$

$x^2(t^2 + 1) = t^2a^2 \rightarrow x^2 = \dfrac{t^2a^2}{t^2 + 1} \rightarrow$

$$x = \frac{ta}{\sqrt{t^2 + 1}} \ldots\ldots\ldots..1$$

Put equation 1 in original equation

$y = \sqrt{a^2 - \frac{t^2a^2}{t^2+1}} \rightarrow y = \sqrt{\frac{a^2t^2+a^2-t^2a^2}{t^2+1}}$

$$y = \frac{a}{\sqrt{t^2+1}} \ldots\ldots\ldots 2$$

$x = \frac{ta}{\sqrt{t^2+1}}$ and $y = \frac{a}{\sqrt{t^2+1}}$ are parametric equations

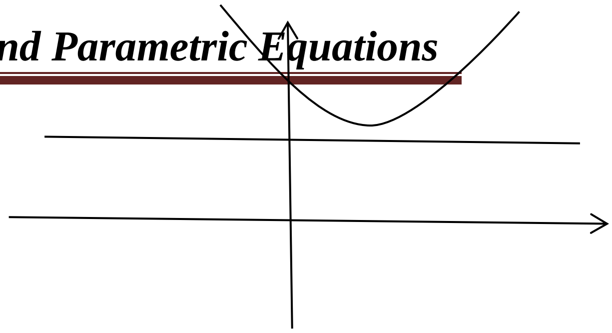
Example 3: Find Cartesian equation and sketch the following curve:

$x = t + 1, y = t^2 + 4$ when $0 \le t < \infty$?

Solution: $t = x - 1 \rightarrow y = (x - 1)^2 + 4 \rightarrow (x - 1)^2 = y - 4$

is Parabola equation

C(h.k)=(1,4),

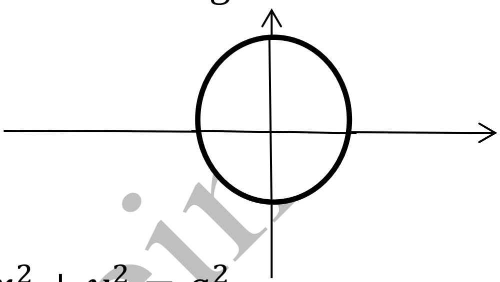


$F(h,k+p)=(1,\frac{17}{4})$ $[4p=1 \to p=\frac{1}{4}]$,

and $y=k-p \to y=\frac{15}{4}$

Example 4: Find Cartesian equation and sketch the following curve:

$x=acos\theta, y=asin\theta$ when $0 \le \theta \le 2\pi$?



Solution: $\cos\theta=\frac{x}{a}$, $sin\theta=\frac{y}{a}$

$$cos^2\theta+sin^2\theta=\frac{x^2}{a^2}+\frac{y^2}{a^2} \to 1=\frac{x^2+y^2}{a^2} \to x^2+y^2=a^2$$

This is the circle equation when C(0,0) and radius a.

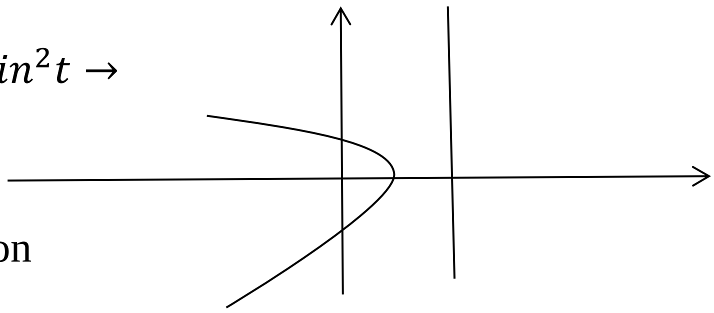
Example 5: Find Cartesian equation and sketch the following curve: $x=\cos t, y=sin\ t$ when $0 \le t \le 2\pi$? (**H.W.**).

Example 6: Find Cartesian equation and sketch the following curve: $x=\cos 2t, y=sin\ t$ when $0 \le t \le 2\pi$

Solution: $sin^2t=\frac{1}{2}(1-cos2t) \to 2sin^2t=1-cos2t \to$

$cos2t=1-2sin^2t \to x=1-2sin^2t \to$

$x=1-2y^2 \to 2y^2=1-x \to$



$y^2=-\frac{1}{2}(x-1)$ is Parabola equation

C(1,0), $-4p=-\frac{1}{2} \to p=\frac{1}{8}$, F(h-p,k)=(1-$\frac{1}{8}$,0)=($\frac{7}{8}$,0), x=h+p=1+$\frac{1}{8}=\frac{9}{8}$

Exercises:

1- Find Cartesian equation and sketch the following curves:

a) $x=t^2+t, y=t^2-t$ when $-\infty < t < \infty$

b) $x=2+\frac{1}{t}, y=2-t$ when $0 \le t < \infty$

c) $x=2t+3, y=4t^2-9$ when $-\infty < t < \infty$

d) $x=3-2\,\mathrm{sech}\,t, y=4-3\tanh t$ when $-\infty < t < \infty$

e) $x = 2 + 4\sin t, y = 3 - 2\cos t$ when $0 \le t \le 2\pi$

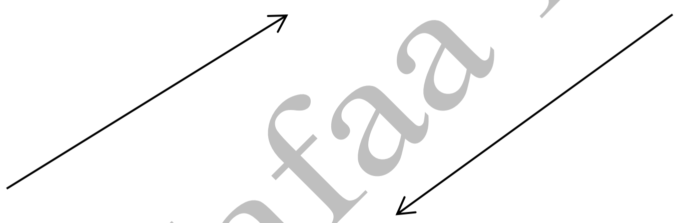
f) $x = \sec t, y = tant$ when $\frac{-\pi}{2} \le t \le \frac{\pi}{2}$

g) $x = \cosh t,\ y = \sinh t$ when $0 \le t < \infty$

2- Find parametric equation of the circle $y^2 + x^2 = a^2, y > 0$ using as parameter θ defined by the equation $x = a\tanh\theta$?

Vectors

Vectors are encountered in physics, as line segments whose direction is the direction of a force (or position, velocity) and whose length (or magnitude) gives the intensity of the force but the vector in geometry is used to represent the position of a point in relation to another point (i.e. a vector in space is specified by an initial point A and an end point B and is denoted by $\overrightarrow{AB}$).

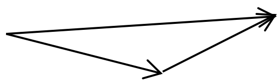


A vector of length 1 is called a **unit vector** and is denoted by *u* but a vector of length 0 is called a **zero vector** and is denoted by O.

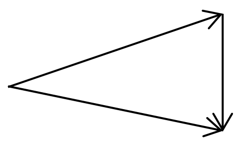
Properties of Vectors

1- Tow vectors are equal if they have the same length and direction.

2- a) Add two vectors is also a vector.



b) Subtract two vectors is also a vector.

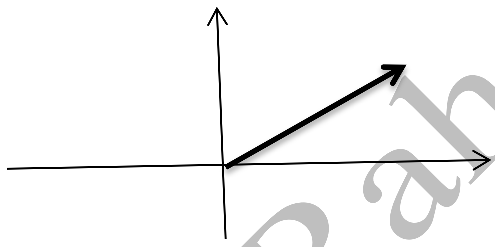


3- If a vector *v* and a scalar *c* (scalar is a real number , it can be positive ,negative, or zero) then *cv* is also a vector and the length of *cv* is *c* time of length *v*:
   a) cv is same direction of *v* if *c* positive number.
   b) cv is opposite direction of *v* if *c* negative number.

Definition of Vector in Cartesian plane:

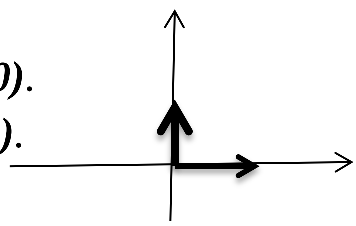
A vector *v* in two-dimensional (Cartesian plane) equal to the vector with initial point at the origin O(0,0) and end point p(x,y).

$v=\overrightarrow{Op}$=(x,y)



Notes:

1- The length of vector *v* is denoted by |*v*|.
2- A unit vector on the x-axis is denoted by *i* and ***i=(1,0)***.
3- A unit vector on the y-axis is denoted by *j* and ***j=(0,1)***.

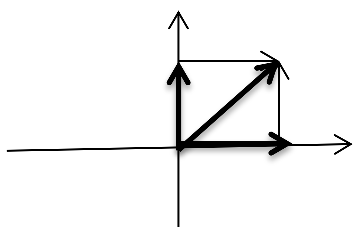


4- If *a* and *b* two scalars then:
   a) *ai* is a vector on the x-axis or parallel to x and the length of *ai* is *a* units.
   b) *bj* is a vector on the y-axis or parallel to y and the length of *bj* is *b* units.

5- It can represent any vector in Cartesian plane by using two vectors *i* and *j*.
   If *v*=(*a*,*b*), *ai* is a vector on x-axis, the length of *ai* is *a* units and *bj* is a vector on the y-axis, the length of *bj* is *b* units then a vector *v* is sum of two vectors *ai* and *bj*.

v(a,b)↔ ***v=ai+bj***

By using Pythagorean Theorem, |***v***|=$\sqrt{\boldsymbol{a^2+b^2}}$



6- A zero vector is denoted by *0* and ***0=(0,0)*** $\leftrightarrow \mathbf{0} = \mathbf{0}\boldsymbol{i} + \mathbf{0}\boldsymbol{j}$.

Vectors Operations:

If v_1, v_2 are two vectors in plane defined as follow:

$$v_1 = (a_1, b_1) \leftrightarrow v_1 = a_1 i + b_1 j,$$

$$v_2 = (a_2, b_2) \leftrightarrow v_2 = a_2 i + b_2 j$$

1- Addition

$$\boldsymbol{v_1 + v_2 = (a_1 + a_2, b_1 + b_2) \leftrightarrow v_1 + v_2 = (a_1 + a_2)i + (b_1 + b_2)j}$$

2- Subtraction

$$\boldsymbol{v_1 - v_2 = (a_1 - a_2, b_1 - b_2) \leftrightarrow v_1 - v_2 = (a_1 - a_2)i + (b_1 - b_2)j}$$

3- Scalar Multiplication

If *v=(a,b)* be a vector and *c* is scalar then *cv* is defined as follow:

$$\boldsymbol{cv=(ca,cb) \leftrightarrow cv = (ca)i + (cb)j}$$

and $|\boldsymbol{cv}|=\boldsymbol{c}|\boldsymbol{v}|$

Properties of Vector Operations:

Let v_1, v_2, and v_3 be vectors and *a*, *b* be scalars

- $v_1+v_2=v_2+v_1$
- $(v_1+v_2)+v_3=v_1+(v_2+v_3)$
- $v_1+0=v_1$
- $v_1+(-v_1)=0$
- $0\, v_1=0$
- $1\, v_1=v_1$
- $a(bv_1)=(ab)\, v_1$
- $a(v_1+v_2)=av_1+av_2$
- $(a+b)\, v_1=av_1+bv_1$

Notes:

1- *Equal two vectors*

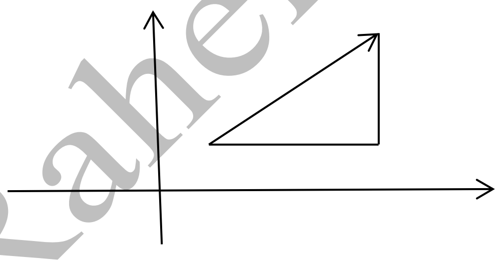
$$\boldsymbol{v_1 = v_2 \leftrightarrow (a_1, b_1) = (a_2, b_2) \leftrightarrow a_1 = a_2} \textbf{ and } \boldsymbol{b_1 = b_2}$$

2- *Parallel of vectors*

$$\boldsymbol{v_1 // v_2 \leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = c}, c \text{ is scalar}$$

Vector Between Two Points:

Let $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ be two points in the plane.



$\overrightarrow{p_1p_2}=\Delta xi + \Delta yj$ or $\overrightarrow{p_1p_2}=(\Delta x, \Delta y)$

$\boldsymbol{\overrightarrow{p_1p_2}=(x_2 - x)i + (y_2 - y_1)j}$

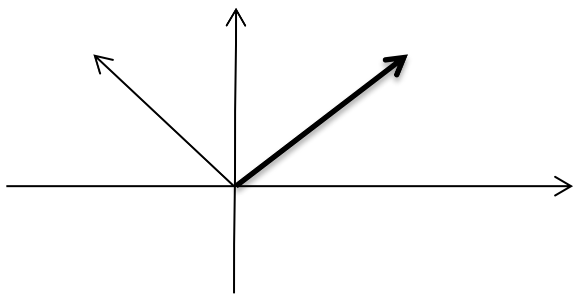
or $\boldsymbol{\overrightarrow{p_1p_2}=(x_2 - x_1, y_2 - y_1)}$

The length of $\overrightarrow{p_1p_2}$

$$|\boldsymbol{\overrightarrow{p_1p_2}}| = \sqrt{(\Delta \boldsymbol{x})^2 + (\Delta \boldsymbol{y})^2} = \sqrt{\boldsymbol{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

The Direction of Vector:

Let *v*=*ai*+*bj* and θ the angle between *v* and the x-axis and denoted by Dir *v*.



$$\boldsymbol{Dir\ v = \tan\theta = \frac{b}{a}}$$

$$\boldsymbol{Dir\ \overrightarrow{p_1p_2} = \frac{y_2 - y_1}{x_2 - x_1}}$$

Example 7: Let $p_1(2,3)$ and $p_2(4,5)$ be two points in the plane, Find $\overrightarrow{p_1p_2}$, $\overrightarrow{p_2p_1}$, $\overrightarrow{p_1p_2} + \overrightarrow{p_2p_1}$, $Dir\ \overrightarrow{p_1p_2}$, $Dir\ \overrightarrow{p_2p_1}$, $|\overrightarrow{p_1p_2}|$and $|\overrightarrow{p_2p_1}|$.

Solution: $\overrightarrow{p_1p_2}=(x_2 - x_1, y_2 - y_1)$=(4-2,5-3)=(2,2)=2i+2j

$\overrightarrow{p_2p_1}=(x_1 - x_2, y_1 - y_2)$=(2-4,3-5)=(-2,-2)=-2i-2j

$\overrightarrow{p_1p_2}+\overrightarrow{p_2p_1}$ =(2-2,2-2)=(0,0)=(2i+2j)+(-2i-2j)=(2-2)i+(2-2)j=0i+0j

$\overrightarrow{p_1p_2}+\overrightarrow{p_2p_1}$=0

$Dir\ \overrightarrow{p_1p_2}=\frac{\boldsymbol{y_2-y_1}}{\boldsymbol{x_2-x_1}}=\frac{5-3}{4-2}=\frac{2}{2}=1$

$Dir\ \overrightarrow{p_2p_1}=\frac{\boldsymbol{y_1-y_2}}{\boldsymbol{x_1-x_2}}=\frac{3-5}{2-4}=\frac{-2}{-2}=1$

$|\overrightarrow{p_1p_2}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-2)^2+(5-3)^2}$
$=\sqrt{2^2+2^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$

$|\overrightarrow{p_2p_1}|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{(2-4)^2+(3-5)^2}$
$=\sqrt{(-2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$

Example 8: Prove that the unit vector *u* can represent as follow:

$$u=cos\theta i+sin\theta j$$

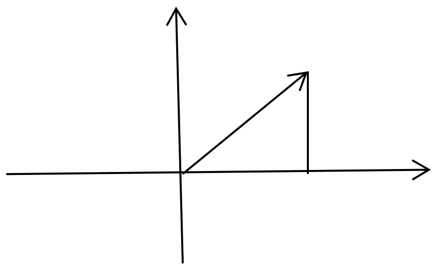
When θ the angle between *u* and the positive x-axis.

Proof: let $u=\overrightarrow{op}$ and *p(x,y)* any point in plane such that |*u*|=1

u=xi+yj

$u=cos\theta i+sin\theta j$ [$x=cos\theta, y=sin\theta$]

$|u|=\sqrt{cos^2\theta+sin^2\theta}=\sqrt{1}=1$



Example 9: Let *v=ai+bj* find the unit vector *u* has the same direction of *v*.

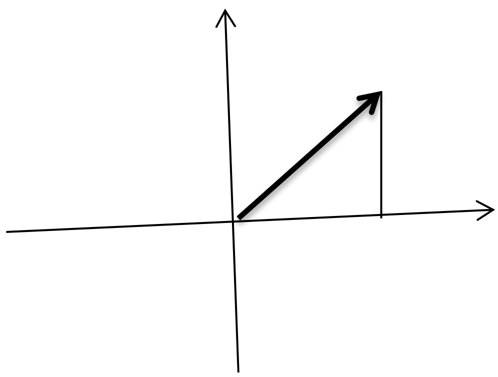
Solution: |*v*|= $\sqrt{a^2+b^2}$

$\frac{v}{|v|}=\frac{ai+bj}{\sqrt{a^2+b^2}}=\frac{a}{\sqrt{a^2+b^2}}i+\frac{b}{\sqrt{a^2+b^2}}j$

In Δomp

$cos\theta=\frac{a}{\sqrt{a^2+b^2}}, sin\theta=\frac{b}{\sqrt{a^2+b^2}}$

$\frac{v}{|v|}=cos\theta i+sin\theta j=u, \boldsymbol{u}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$



Example 10: Find the unit vector *u* has the same direction with *v =3i-4j*.

Solution:

$$u = \frac{v}{|v|} = \frac{3i-4j}{\sqrt{3^2+4^2}} = \frac{3i-4j}{\sqrt{9+16}} = \frac{3i-4j}{\sqrt{25}} = \frac{3i-4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$$|u| = \sqrt{(\frac{3}{5})^2 + (\frac{-4}{5})^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$$

$$Dir\ v = \frac{b}{a} = \frac{-4}{3}$$

$$Dir\ u = \frac{{}^{-4}/_{5}}{{}^{3}/_{5}} = \frac{-4}{3}$$

$$Dir\ v = Dir\ u$$

Example 11: Find the length and the angle that each one makes with the positive x-axis of the following vector:

1) i+j, 2) 2i-3j, 3) -2+3j, 4) 5i+12j, 5) -5i-12j, and 6) $\sqrt{3}$i+j

Solution: 1) v= i+j→v=(1,1)

$|v| = \sqrt{(1)^2 + (1)^2} = \sqrt{2},$

$Dir.v = \frac{b}{a} = \frac{1}{1} = 1 \rightarrow tan\theta = Dir\ v = 1 \rightarrow$

$\theta = \tan^{-1} 1 \rightarrow \theta = 45°$

2) v= 2i-3j→v=(2,-3)

$|v| = \sqrt{(2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13},$

$Dir.v = \frac{-3}{2} \rightarrow tan\theta = Dir.v = \frac{-3}{2} \rightarrow$

$\theta = \tan^{-1}\frac{-3}{2} \rightarrow \theta = -\tan^{-1}\frac{3}{2} \rightarrow \theta = -56.3°$

5) v= -5i-12j→v=(-5,-12)

$|v| = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25+144} = \sqrt{169}$ =13,

$Dir.v = \frac{-12}{-5} \rightarrow tan\theta = Dir.v = \frac{12}{5} \rightarrow$

$\theta = \tan^{-1}\frac{12}{5} \rightarrow \theta = 67.3°$

Exercises:

1- Find the unit vector u has the same direction with v=3i+4j.

2- Le w=(3,-2), and v=(-2,5), represent each vector in the form (ai+bj) and find the length of each one:

a) 3w, b) -2v, c) w+v, d) w-v, e) 2w-3v, f) -2w+5v, g) $\frac{3}{5}$w+$\frac{4}{5}$v, and h) $\frac{-5}{13}$w+$\frac{12}{13}$v

3- Represent each vector in the form (ai+bj):

a) The vector $\overrightarrow{p_1p_2}$, where p_1=(1,3) and p_2=(2,-1).

b) The vector from the point A=(2,3) to the origin.

c) The vector $\overrightarrow{op}$, where o is the origin and p is the midpoint of the vector $\overrightarrow{p_1p_2}$ where p_1=(2,-1) and p_2=(4,3).

d) The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ where A=(1,-1), B=(2,0), C=(-1,3), and D=(-2,2).

e) The unit vector that makes an angle $\theta = \frac{2\pi}{3}$ with the positive x-axis.

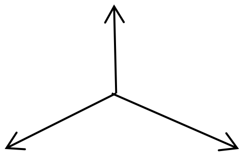
f) The unit vector that makes an angle $\theta = \frac{-3\pi}{4}$ with the positive x-axis.

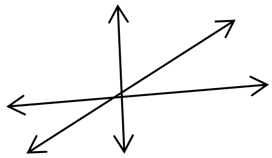
g) The unit vector obtained by rotating the vector (0,1), $120°$ counterclockwise about the origin.

h) The unit vector obtained by rotating the vector (1,0), $135°$ counterclockwise about the origin.

Coordinate Space:

There are three planes in space (xy, xz and yz) and these planes called *Coordinate planes* such that: xy $\perp$ xz $\perp$ yz and these planes intersect in coordinate axes (xx', yy' and zz') but the coordinate axes intersect in origin point. The coordinate planes are divided space into eight parts.





p(x,y,z) is point in the space, the origin point is O(0,0,0).

There are eight options of point in space: (x,y,z),(-x,y,z),(x,-y,z), (x,y,-z),(-x,-y,z), (-x,y,-z),(x,-y,-z),)(-x,-y,-z).

The distance between two points $p_1(x_1,y_1,z_1)$ and $p_2(x_2,y_2,z_2)$in space is:

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

<u>Equations and Shapes in Space:</u>

In space can represent the equation as follows:

$$z = f(x, y), y = g(x, z), or\ F(x, y, z) = 0$$

<u>Surface:</u>

All points p(x,y,z) are satisfied the equation $F(x, y, z) = 0$ is called *surface.* For example, Ax+By+Cz+D=0 (when A,B,C and D are real numbers) is equation surface of space.

<u>Sphere:</u>

Sphere is geometric shape of the set of points p(x,y,z) whose distances from fixed point in the space is constant . The fixed point and constant distance are called center and radius of the Sphere respectively.

To find equation of sphere in space, let *C(h,k,l)* and *r* are center and radius of sphere respectively. P(x,y,z) is any point in sphere

PC=r

$$\sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2} = r$$

$$(\boldsymbol{x}-\boldsymbol{h})^{2}+(\boldsymbol{y}-\boldsymbol{k})^{2}+(\boldsymbol{z}-\boldsymbol{l})^{2}=\boldsymbol{r}^{2}$$

Example 12: Find the centre and radius of the sphere:

$$x^2+y^2+z^2+4x-4z=0$$

Solution: $(x^2+4x+4)+y^2+(z^2-4z+4)=4+4$

$$(x+2)^2+y^2+(z-2)^2=8$$

C(-2,0,2), $r=\sqrt{8}$

Exercises:

1- Find the centre and radius of the following spheres:
   $a)$ $x^2+y^2+z^2-2x+2y=0$
   b) $x^2+y^2+z^2-2z=0$
   c) $3x^2+3y^2+3z^2+2y-4z=9$
   d) $2x^2+2y^2+2z^2+x+y+z=0$
2- Find the locus of P if d_1 is the distance between P(x,y,z) and origin, d_2 is the distance between P and Q(0,0,3), and $d_1=2d_2$.
3- Find the distance between P(x,y,z) and
   a) x-axis , b) y-axis, c) z-axis, d) xy- plane, e) xz-plane, and f) zy-plane.

Vectors in Coordinate Space:

A vector *v* in space equal to the line between origin point O(0,0,0) and end point P(a,b,c).

$$\boldsymbol{v}=\overrightarrow{\boldsymbol{OP}}=(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$$

- A unit vector on the x-axis is denoted by *i* and ***i=(1,0,0).***
- A unit vector on the y-axis is denoted by *i* and ***j=(0,1,,0).***
- A unit vector on the z-axis is denoted by *i* and ***k=(0,0,1).***
  If a,b,c are real numbers

- ai is a vector on the x-axis or parallel to x and the length of ai is a units.
- bj is a vector on the y-axis or parallel to y and the length of bj is b units.
- ck is a vector on the z-axis or parallel to z and the length of ck is c units.

Let v=(a,b,c) is a vector

$$\boldsymbol{v(a,b,c)} \leftrightarrow \boldsymbol{v=ai+bj+ck}$$

The *length* of vector *v* is denoted by |*v*| and defined as follows:

$$|\boldsymbol{v}|=\sqrt{\boldsymbol{a^2+b^2+c^2}}$$

The vector between two points $p_1(x_1, y_1, z_1)$ and $p_2(x_2, y_2, z_2)$ in space is:

$\overrightarrow{p_1p_2}=\Delta xi+\Delta yj+\Delta zk$ or $\overrightarrow{p_1p_2}=(\Delta x, \Delta y, \Delta z)$

$$\overrightarrow{\boldsymbol{p_1p_2}}=(\boldsymbol{x_2-x_1})\boldsymbol{i}+(\boldsymbol{y_2-y_1})\boldsymbol{j}+(\boldsymbol{z_2-z_1})\boldsymbol{k} = (\boldsymbol{x_2-x_1}, \boldsymbol{y_2-y_1}, \boldsymbol{z_2-z_1})$$

The *length* of vector $\overrightarrow{p_1p_2}$ in space is denoted by $|\overrightarrow{p_1p_2}|$ and defined as follows:

$$|\overrightarrow{\boldsymbol{p_1p_2}}|=\sqrt{(\boldsymbol{x_2-x_1})^2+(\boldsymbol{y_2-y_1})^2+(\boldsymbol{z_2-z_1})^2}$$

The middle of $\overrightarrow{p_1p_2}$ is $p_3\left(\frac{\boldsymbol{x_1+x_2}}{\boldsymbol{2}}, \frac{\boldsymbol{y_1+y_2}}{\boldsymbol{2}}, \frac{\boldsymbol{z_1+z_2}}{\boldsymbol{2}}\right)$

A unit vector *u* in space has the same direction of *v* is defined as follows:

$$\boldsymbol{u}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

<u>Example 13:</u> Find the length of the following vectors:

a) 3i+6j+2k , b) i+4j-8k, c) 2i+j-2k, and 9i-2j+6k

Solution: a) $|v|=\sqrt{a^2+b^2+c^2}$

$$|v|=\sqrt{3^2+6^2+2^2}=\sqrt{9+36+4}=\sqrt{49}=7$$

Example 14: Find the unit vector has the same direction of the vector 4i+3j+12k?

Solution:

$$u = \frac{v}{|v|}$$

$$\mathrm{u} = \frac{4\mathrm{i} + 3\mathrm{j} + 12\mathrm{k}}{\sqrt{4^2 + 3^2 + 12^2}} = \frac{4\mathrm{i} + 3\mathrm{j} + 12\mathrm{k}}{\sqrt{169}} = \frac{4\mathrm{i} + 3\mathrm{j} + 12\mathrm{k}}{13}$$
$$= \frac{4}{13}i + \frac{3}{13}j + \frac{12}{13}k$$

$$|\,u| = \sqrt{(\tfrac{4}{13})^2 + (\tfrac{3}{13})^2 + (\tfrac{12}{13})^2} = \sqrt{\frac{16}{169} + \frac{9}{169} + \frac{144}{169}} = \sqrt{\frac{169}{169}} = 1$$

Properties of Vectors in space

Let $v_1 = (a_1, b_1, c_1)$ and $v_2 = (a_2, b_2, c_2)$ be vectors in space then

- Zero vector is O=(0,0,0)
- *Equal*

$$v_1 = v_2 \leftrightarrow (a_1, b_1, c_1) = (a_2, b_2, c_2)$$
$$\leftrightarrow a_1 = a_2 \wedge\ b_1 = b_2\ \wedge c_1 = c_2$$

- *Parallel*

$$v_1 \backslash\backslash v_2 \leftrightarrow v_1 = sv_2\ (s\ scalar)$$
$$\leftrightarrow (a_1, b_1, c_1) = s(a_2\ , b_2, c_2)$$
$$\leftrightarrow (a_1, b_1, c_1) = (sa_2\ , sb_2, sc_2)$$
$$\leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = s$$

- *Addition*

$$v_1 + v_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$
$$= (a_1 + a_2)i + (b_1 + b_2)j + (c_1 + c_2)k$$

- *Subtraction*

$$v_1 - v_2 = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$$
$$=(a_1 - a_2)i + (b_1 - b_2)j + (c_1 - c_2)k$$

- *Scalar Multiplication*

If c is scalar then cv_1 is defined as follow:

$$cv_1 = (ca_1, cb_1, cc_1)$$

Properties of Vector Operations:

Let A, B, and C be vectors and a, b be scalars

- $A+B=B+A$,
- $(A+B)+C=A+(B+C)$,
- $A+0=A$,
- $A+(-A)=0$,
- $0\,A=0$,
- $1\,A=A$,
- $a(bA)=(ab)\,A$,
- $a(A+B)=aA+aB$,
- $(a+b)\,A=aA+bA$

- *Product*
  - ***Dot(Scalar) Product***

$$\boldsymbol{v_1.v_2 = |v_1||v_2|cos\theta}$$

$$cos\theta = \frac{v_1.v_2}{|v_1||v_2|}$$

Properties of dot product:

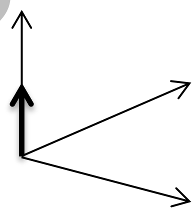
Let $A, B, C and\ \ D$ be vectors in space:

- $A.B = B.A$
- $A.(B+C) = (A.B) + (A.C)$
- $(A+B).(C+D) = (A.C) + (A.D) + (B.C) + (B.D)$

- $A.A = |A|^2$ (**Prove**)
    - $i.i = j.j = k.k = 1$
- $A \perp B$ $(Orthogonal$ or $Perpendicular) \leftrightarrow A.B = 0$ (**Prove**)
    - $i \perp j \perp k \rightarrow i.j = i.k = j.k = 0$
- If $v = a_1 i + b_1 j + c_1 k, and\ w = a_2 i + b_2 j + c_2 k$ then $v.w = a_1 a_2 + b_1 b_2 + c_1 c_2$

➢ ***Cross (Vector) Product***

$\boldsymbol{v_1 \times v_2 = N|v_1||v_2|sin\theta}$ (Where N is a unit vector perpendicular of $v_1\ and\ v_2$**)**



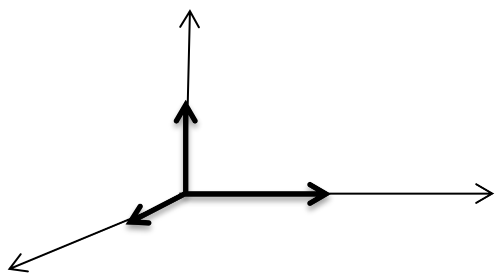
If n is a unit vector perpendicular of plane then

$$|\boldsymbol{v_1 \times v_2}| = |\boldsymbol{v_1}||\boldsymbol{v_2}|\boldsymbol{sin\theta} \quad (|N|=1)$$

<u>Properties of cross product:</u>

Let $A, B, C and\ \ D$ be vectors in space:

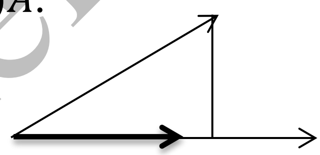
- $A \times B \neq B \times A$ (**Prove**)
- $A \times (B + C) = (A \times B) + (A \times C)$
- $(A + B) \times (C + D) = (A \times C) + (A \times D) + (B \times C) + (B \times D)$
- $A \times A = O$ (**Prove**)
    - $i \times i = j \times j = k \times k = O$
- $A \perp B$ $(Orthogonal$ or $Perpendicular) \leftrightarrow A \times B = N|A||B|$ (**Prove**)
    - $i \perp j \perp k \rightarrow i \times j = k = -j \times i \rightarrow j \times i = -k$
      $i \times k = -j = -k \times i \rightarrow k \times i = j$
      $j \times k = i = -k \times j \rightarrow k \times j = -i$



- Nonzero vectors v and w are parallel $\leftrightarrow$ $v\times w=0$
- *Projection*

  Let A and B be two vectors the $\vec{C}$ is vector projection of A and B and denoted by $Proj_A B=\vec{C}$ and $Proj_A B=(\frac{B.A}{|A|^2})A$.

  Scalar projection of A and B =$|Proj_A B|$



<u>Example 15:</u> Find $v_1.v_2$ *and* the angle between two vectors a) v_1=2i+j and v_2=i+2j-k and b) v_1=i-2j-2k and v_2=6i+3j-2k (**H.W.**)?

Solution: $v_1.v_2=a_1a_2+b_1b_2+c_1c_2\rightarrow v_1.v_2=2(1)+1(2)+0(-1)=2+2=4$

$$cos\theta=\frac{v_1.v_2}{|v_1||v_2|}\rightarrow cos\theta=\frac{4}{\sqrt{2^2+1^2+0^2}\sqrt{1^2+2^2+(-1)^2}}$$

$$=\frac{4}{\sqrt{5}\sqrt{6}}\rightarrow\theta=cos^{-1}\frac{4}{\sqrt{5}\sqrt{6}}$$

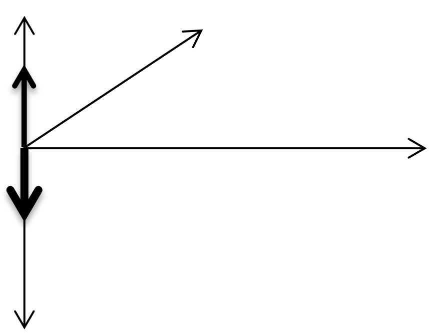
<u>Example 16</u>: Prove that $A\times B=-B\times A$?

Proof: $A\times B=N|A||B|sin\theta$

$B\times A=N|A||B|\sin(-\theta)$

$B\times A=-N|A||B|\sin\theta$

$A\times B=-B\times A$



<u>Example 17:</u> Find $v\times w$ If $v=a_1i+a_2j+a_3k, and$ $w=b_1i+\ b_2j+b_3k$?

Solution:

$v \times w = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k)$

$=(a_1 i + a_2 j + a_3 k) \times b_1 i + (a_1 i + a_2 j + a_3 k) \times b_2 j +$
$(a_1 i + a_2 j + a_3 k) \times b_3 k = a_1 b_1 (i \times i) + a_2 b_1 (j \times i) +$
$a_3 b_1 (k \times i) + a_1 b_2 (i \times j) + a_2 b_2 (j \times j) + a_3 b_2 (k \times j) +$
$a_1 b_3 (i \times k) + a_2 b_3 (j \times k) + a_3 b_3 (k \times k)$
$= (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k$

Or

$$v \times w = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$= (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k$$

<u>Example 18:</u> Find the unit vector perpendicular to both vectors A=i+2j-k and B=i-j+k?

Solution:

$$A \times B = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} k$$

$$= (2-1) i - (1+1) j + (-1-2) k = i - 2j - 3k$$

$$u = \frac{A \times B}{|A \times B|} = \frac{i - 2j - 3k}{\sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{i - 2j - 3k}{\sqrt{14}}$$

$$= \frac{1}{\sqrt{14}} i - \frac{2}{\sqrt{14}} j - \frac{3}{\sqrt{14}} k$$

$$|u| = \sqrt{(\tfrac{1}{\sqrt{14}})^2 + (\tfrac{2}{\sqrt{14}})^2 + (\tfrac{3}{\sqrt{14}})^2} = \sqrt{\tfrac{1}{14} + \tfrac{4}{14} + \tfrac{9}{14}} = \sqrt{\tfrac{14}{14}} = 1$$

<u>Example 19:</u> Find the vector projection of vectors A=i+j and B=2i-j+3k and find $| \; proj_A^B \; |$?

Solution: $Proj_A B = \left(\frac{B.A}{|A|^2}\right) A = \frac{(2)(1)+(-1)(1)+(3)(0)}{1^2+1^2+0^2}(i + j) =$
$\frac{2-1}{2}(i + j) = \frac{1}{2}(i + j) = \frac{1}{2}i + \frac{1}{2}j$

$|Proj_A B| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$

Example 20: Find the $A \times B$ and $B \times A$ if a) A=2i+j+k and B=-4i+3j+k , [b) A=2i-2j-k and B=i+j+k , c) A=2i and B=-3j, and d) A=$\frac{3}{2}$i-$\frac{1}{2}$j+k and B=i+j+2k](**H.W.**) ?

Solution: a)

$$A \times B = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$
$$= -2i - 6j + 10k$$

$$B \times A = \begin{vmatrix} i & j & k \\ -4 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} i - \begin{vmatrix} -4 & 1 \\ 2 & 1 \end{vmatrix} j + \begin{vmatrix} -4 & 3 \\ 2 & 1 \end{vmatrix} k$$
$$= 2i + 6j - 10k$$

Example 21: Find a vector that is perpendicular to both vectors A=i+j+k and B=i+j (**H.W.**)?

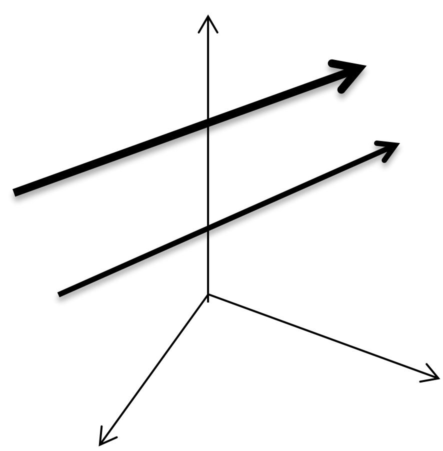
Equation of Line:

Let L is a line in space through the point $p_1(x_1, y_1, z_1)$ and parallel to vector $v = Ai + Bj + Ck$ .To find the equation for the line L, if *p(x,y,z)* is any point on L then

$\overrightarrow{p_1p} = (x - x_1, y - y_1, z - z_1),$

$\overrightarrow{p_1p} \backslash\backslash v \rightarrow \overrightarrow{p_1p} = tv\ (\ t\ is\ a\ scalar) \rightarrow$

$(x - x_1, y - y_1, z - z_1) = t(A, B, C) \rightarrow$



$(x - x_1, y - y_1, z - z_1) = (tA, t, B, tC) \rightarrow$

$(x - x_1) = tA, (y - y_1) = tB, (z - z_1) = tC \rightarrow$

$\dfrac{(x - x_1)}{A} = t\,, \dfrac{(y - y_1)}{B} = t\,, \dfrac{(z - z_1)}{C} = t \;\rightarrow$

$$\textbf{\textit{Line}}: \frac{(\boldsymbol{x}-\boldsymbol{x_1})}{\boldsymbol{A}} = \frac{(\boldsymbol{y}-\boldsymbol{y_1})}{\boldsymbol{B}} = \frac{(\boldsymbol{z}-\boldsymbol{z_1})}{\boldsymbol{C}} = \boldsymbol{t}$$

(Line Equation through the point p_1 and parall to vector v)

- ❖ If $A, B, C \neq 0$ then $\frac{(x-x_1)}{A} = \frac{(y-y_1)}{B} = \frac{(z-z_1)}{C}$
- ❖ If one of A, B, C equal to zero then
  - If A=0 then $\frac{(y-y_1)}{B} = \frac{(z-z_1)}{C} \wedge (x - x_1) = 0.$
  - If B=0 then $\frac{(x-x)}{A} = \frac{(z-z_1)}{C} \wedge (y - y_1) = 0.$
  - If C=0 then $\frac{(x-x_1)}{A} = \frac{(y-y_1)}{B} \wedge (z - z_1) = 0.$

- ❖ To find parametric equations for the line, if *t* is a parameter then

$$\frac{(x - x_1)}{A} = \frac{(y - y_1)}{B} = \frac{(z - z_1)}{C} = t \rightarrow$$

$$\frac{(x - x_1)}{A} = t \rightarrow (x - x_1) = tA \rightarrow x = At + x_1$$

$$\frac{(y - y_1)}{B} = t \rightarrow (y - y_1) = tB \rightarrow \; y = Bt + y_1$$

$$\frac{(z - z_1)}{C} = t \rightarrow (z - z_1) = tC \rightarrow \; z = Ct + z_1$$

$\boldsymbol{x = At + x_1} \wedge \; \boldsymbol{y = Bt + y_1} \wedge \boldsymbol{z = Ct + z_1}$ (Parametric equations of parameter t)

<u>Example 22:</u> Find the line through the points P(1,2,-1) and Q(-1,0,1), and find parametric equations ?

Solution:

$$\overrightarrow{PQ} = (-1-1, 0-2, 1+1) = (-2,-2,2)$$

$$\frac{(x-x_1)}{A}=\frac{(y-y_1)}{B}=\frac{(z-z_1)}{C}\rightarrow$$

$$L\colon \frac{(x-1)}{-2}=\frac{(y-2)}{-2}=\frac{(z+1)}{2}$$

The parametric equations are:

$$\frac{(x-1)}{-2}=\frac{(y-2)}{-2}=\frac{(z+1)}{2}=t\rightarrow$$

$x-1=-2t\ \rightarrow x=-2t+1$

$y-2=-2t\ \rightarrow y=-2t+2$

$z+1=2t\ \rightarrow z=2t-1$

Example 23: Find the line through the point p(3,-4,-1) and parallel to the vector $v=i+j+k$, and find parametric equations?

Solution:

$$\frac{(x-x_1)}{A}=\frac{(y-y_1)}{B}=\frac{(z-z_1)}{C}\rightarrow$$

$$L\colon \frac{(x-3)}{1}=\frac{(y+4)}{1}=\frac{(z+1)}{1}$$

The parametric equations are:

$$\frac{(x-3)}{1}=\frac{(y+4)}{1}=\frac{(z+1)}{1}=t\rightarrow$$

$x-3=t\ \rightarrow x=t+3$

$y+4=t\ \rightarrow y=t-4$

$z+1=t\ \rightarrow z=t-1$

Example 24: Find the distance from the point P(0,0,12) to the line (x=4t, y=-2t, z=2t)?

Solution: $v=Ai+Bj+Ck=4i-2j+2k$, the line through Q(0,0,0)

$\overrightarrow{PQ}$ = (0-0,0-0,0-12) = (0,0,-12)=-12k

$$\overrightarrow{PQ} \times v = \begin{vmatrix} i & j & k \\ 0 & 0 & -12 \\ 4 & -2 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -12 \\ -2 & 2 \end{vmatrix} i - \begin{vmatrix} 0 & -12 \\ 4 & 2 \end{vmatrix} j + \begin{vmatrix} 0 & 0 \\ 4 & -2 \end{vmatrix} k$$

$$= -24i - 48j$$

$$d = \frac{|\overrightarrow{PQ} \times v|}{|v|} = \frac{\sqrt{576 + 2304}}{\sqrt{16 + 4 + 4}} = \frac{\sqrt{2880}}{\sqrt{24}}.$$

Exercises:

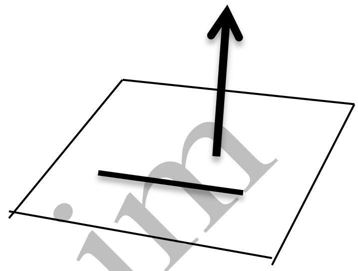
1. Find the line through the points and find parametric equations for these lines:
   a. P(-2,0,3) and Q(3,5,-2),
   b. P(1,-1,2) and Q(-1,-4,-2),and
   c. P(1,2,0) and Q(1,1,-1)?
2. Find the line through
   a. The point p(2,1,-2) and parallel to the vector $v = i + j + 2k,$ and find parametric equations for this line.
   b. The point p(2,3,1) and parallel to the vector $v = 3i - 2j + 6k,$ and find parametric equations for this line.
3. Find the line through the origin and parallel to the vector $v = 2j + k,$ and find parametric equations for this line.
4. Find the distance from the point to the line
   a. p(0,0,0) ; x=5+3t, y=5+4t, z=-3-5t
   b. p(2,1,3) ; x=2+2t, y=1+6t, z=3
   c. p(-1,4,3) ; x=10+4t, y=-3, z=4t
5. Find the distance from the origin to the line
   $$\frac{(x-2)}{3} = \frac{(y-1)}{4} = \frac{(z-2)}{5}$$

Equation of Plane:

To find the equation for the plane through the point $p_1(x_1, y_1, z_1)$ and perpendicular on the vector N *(A,B,C)* , if *p(x,y,z)* is any point on the plane , the vector $\overrightarrow{p_1p} = (x - x_1, y - y_1, z - z_1)$ inside the plane

$N \perp \overrightarrow{p_1p} \ \rightarrow \ N \,.\, \overrightarrow{p_1p} = 0 \ \rightarrow$

$(A, B, C).(x - x_1, y - y_1, z - z_1) = 0 \rightarrow$



Plane: $\boldsymbol{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}$

(Plane Equation through the point p_1 and perpendicular on vector N)

Notes:

1. $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \rightarrow$

   $Ax - Ax_1 + By - By_1 + Cz - Cz_1 = 0 \rightarrow$

   $Ax + By + Cz = (Ax_1 + By_1 + Cz_1) \rightarrow$

   $\boldsymbol{Ax + By + Cz = D}$ (D is scalar) [General Plane Equation perpendicular on vector N]

2. Let P_1, P_2 be two planes in space:

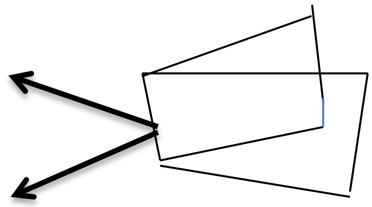
$$P_1 = A_1x + B_1y + C_1z = 0$$
$$P_2 = A_2x + B_2y + C_2z = 0$$

And $N_1 = (A_1, B_1, C_1), N_2 = (A_2, B_2, C_2)$,
Such that $P_1 \perp N_1$ and $P_2 \perp N_2$
There are two options:

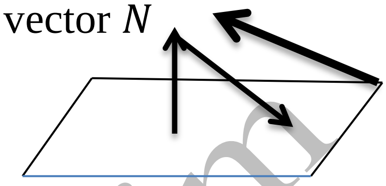
- $P_1 \parallel P_2 \ \rightarrow \ N_1 \parallel N_2 \rightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$
- $P_1 \perp P_2 \rightarrow N_1 \perp N_2 \rightarrow N_1 . N_2 = 0 \rightarrow A_1A_2 + B_1B_2 + C_1C_2 = 0$

3.
   - The angle between two planes is an angle between two vectors N_1, N_2 perpendicular on planes.

$$\cos\theta = \frac{N_1.N_2}{|N_1|.|N_2|} \rightarrow \boldsymbol{\theta = cos^{-1}\theta \frac{N_1.N_2}{|N_1|.|N_2|}}$$



- The angle between the line and plane is an angle between the vector v is parallel to line and the vector N perpendicular on plane.



$$\boldsymbol{\cos\theta = \frac{v.N}{|v|.|N|} \rightarrow \theta = cos^{-1}\theta \frac{v.N_2}{|v|.|N_2|}}$$

<u>Example 25:</u> Find the plane through the point p(0,2,-1) and perpendicular to the vector $N = i + 2j + 2k$?

Solution: *Plane:* $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

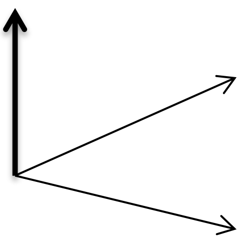
$$1(x - 0) + 2(y - 2) + 2(z + 1) = 0$$

$$x + 2y + 2z - 2 = 0$$

$$x + 2y + 2z = 2$$

<u>Example 26:</u> Find the plane through the points p_1(1,1,-1)), p_2 (2,0,2) and p_3(0,-2,1)?

Solution:



$\overrightarrow{p_1p_2}$ =(2-1,0-1,2+1)=(1,-1,3) = i-j+3k

$\overrightarrow{p_1p_3}$ =(0-1,-2-1,1+1)=(-1,-3,2) = -i-3j+2k

$$N=\overrightarrow{p_1p_2} \times \overrightarrow{p_1p_3} = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -3 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} k$$

$$= 7i - 5j - 4k$$

$N \perp \overrightarrow{p_1p_2}$ and $N \perp \overrightarrow{p_1p_3}$ then $N \perp$ plane of $(\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3})$

Plane Equation through the point p_1 and perpendicular on vector N

Plane: $A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \rightarrow$

$$7(x-1)-5(y-1)-4(z+1)=0 \rightarrow$$

$$7x-5y-4z-6=0$$

$$7x-5y-4z=6$$

Example 27: Find the angle between two planes:
$P_1: x+y=1, P_2: 2x+y-2z=2$?
Solution: $N_1=(1,1,0),\ N_2=(2,1,-2)$

$$\cos\theta=\frac{N_1.N_2}{|N_1|.|N_2|}=\frac{(1)(2)+(1)(1)+(0)(-2)}{\sqrt{1+1+0}\ \sqrt{4+1+4}}=\frac{2+1}{\sqrt{2}\ \sqrt{9}}=\frac{1}{\sqrt{2}}$$

$$\rightarrow\ \theta=cos^{-1}\frac{1}{\sqrt{2}}\rightarrow\ \theta=\frac{\pi}{4}$$

Example 28: Find the angle between the line $\frac{(x+1)}{2}=\frac{y}{3}=\frac{(z-3)}{6}$ and the plane $10x+2y-11z=3$?
Solution: $v=(2,3,6), N=(10,2,-11)$

$$\cos\theta=\frac{v.N}{|v|.|N|}=\frac{(2)(10)+(3)(2)+(6)(-11)}{\sqrt{4+9+36}\,.\sqrt{100+4+121}}$$

$$=\frac{20+6-66}{\sqrt{49}.\sqrt{225}}=\frac{-40}{(7)(15)}=\frac{-40}{105}=-0.38$$

$$\rightarrow\theta=cos^{-1}(-0.38)$$

Example 29: Find the distance from the point p(2,-3,4) to the plane x+2y+2z=13?
Solution: $N=(1,2,2)$,
Let L is a line through p and parallel to N
The line equation is:
$\frac{(x-2)}{1}=\frac{(y+3)}{2}=\frac{(z-4)}{2}$=t
x=t+2, y=2t -3 and z=2t+4
(t+2)+2(2t-3)+2(2t+4)=13 → t+2+4t-6+4t+8=13 → $9t+4=13$ →
9t=9→ $t=1$

x=1+2=3, y=2(1)-3=-1, z=2(1)+4=6

Q(3,-1,6), the distance between P and Q is:

d=$\sqrt{(3-2)^2+(-1+3)^2+(6-4)^2} = \sqrt{1+4+4}$=$\sqrt{9}$=3 unit

Example 30: Find an equation of plane P that pass through the two points p_1 (1,0,-1), p_2(-1,2,1) and parallel to the line of intersection of the planes, P_1: x+y-2z=0 and P_2: 2x-y+3z=0?

Solution:

N_1(1,1,-2), N_2(2,-1,3) , $N_1 \perp P_1$, $N_2 \perp P_2$

Let $V = N_1 \times N_2$= $\begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} i - \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} k$

=(3-2)i-(3+4)j+(-1-2)k=i-7j-3k

The vector V the line of intersection of the planes $P_1$and P_2, $V\backslash\backslash P$

$\overrightarrow{p_1p_2} = (-1-1,2-0,1+1) = -2i+2j+2k$

$N = \overrightarrow{p_1p_2} \times V$= $\begin{vmatrix} i & j & k \\ -2 & 2 & 2 \\ 1 & -7 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ -7 & -3 \end{vmatrix} i - \begin{vmatrix} -2 & 2 \\ 1 & -3 \end{vmatrix} j + \begin{vmatrix} -2 & 2 \\ 1 & -7 \end{vmatrix} k$

=(-6+14)i-(6-2)j+(14-2)k=8i-4j+12k

$P\!: A(x-x_1) + B(y-y_1) + C(z-z_1) = 0 \rightarrow$

$8(x-1) - 4(y-0) + 12(z+1) = 0 \rightarrow$

$8x - 8 - 4y + 12z + 12 = 0 \rightarrow 8x - 4y + 12z = -4$

$\rightarrow 2x - y + 3z = -1$

Exercises:

1. Find the plane through the points
   a) p_1(2,4,5) , p_2 (1,5,7) and p_3(-1,6,8).
   b) p_1(2,-1,-1) , p_2 (1,1,-1) and p_3(3,2,1).
2. Find the plane through the point p(1,2,-3) and perpendicular to the vector $v = i - 4j + k$?
3. Find the angle between two planes:

   $a) \; P_1 = 5x + y - z = 10, \; P_2 = x - 2y + 3z = -1$
   $b) \; P_1 = 3x - 2y + z = 3, P_2 = 2x + y - 2z = 6$

4. Find the distance from the point to the plane
   a. (0,1,1); $4y + 3z = -12{,}2$
   b. (0,0,0); $3x + 2y + 6z = 6,$
   c. (1,0,-1); $-4x + y + z = 4,$
   d. (0,-1,0); $2x + y + 2z = 4,$
   e. (2,2,3); $2x + y + 2z = 4.$
5. Find intersection point of the line to the plane and find the angle between the line and the plane:
   a. $\frac{(x-1)}{2} = \frac{(y+1)}{-1} = \frac{z}{3}$; $3x + 2y - z = 5.$
   b. $x = 1 - t, y = 3t, z = 1 + t$; $2x - y + 3z = 6.$
   c. $x = -1 + 3t, y = -2, z = 5t$; $2x - 3z = 7.$
6. Find a plane through p_1 (2,1,-1) and perpendicular the line of intersection of the planes: P_1: $2x + y - z = 3$ and P_2: $x + 2y + z = 2$?
7. Find a plane through the point p_1 (1,2,3) , p_1 (3,2,1) and perpendicular to the plane $4x - y + 2z = 7$?
8. Find a plane through the point p_1 (1,-1,3) and parallel to the plane $3x + y + z = 7$?
9. Find a plane through the point p_1 (2,4,5) and perpendicular to the line $x = 5 + t, y = 1 + 3t, z = 4t$?
10. Find a plane through the point p_1 (1,-2,1) perpendicular to the vector from the origin to p_1?
11. Find the point of intersection lines and find the plane determined by these lines:
    a. $x = 2t + 1, y = 3t + 2, z = 4t + 3,$ and $x = s + 2, y = 2s + 4,$ $z = -4s - 1.$
    b. $x = t, y = -t + 2, z = t + 1,$ and $x = 2s + 2, y = s + 3,$ $z = 5s + 6.$
    c. $\frac{(x-1)}{1} = \frac{(y-2)}{3} = \frac{z-3}{1}$, $\frac{(x-2)}{1} = \frac{(y-3)}{4} = \frac{z-4}{2}$.

12. Prove that intersection line of two planes: $P_1: x + 2y - 2z = 0$ and $P_2: 5x - 2y - z = 0$ is parallel to the line $\frac{(x+3)}{2} = \frac{y}{3} = \frac{z-1}{4}$ and find the plane determined by these two lines?