جامعة بغداد
كلية التربية للعلوم الصرفة- ابن الهيثم
قسم الرياضيات
المرحلة الثانيه

التفاضل المتقدم

اعداد
د. وفاء رحيم حسين

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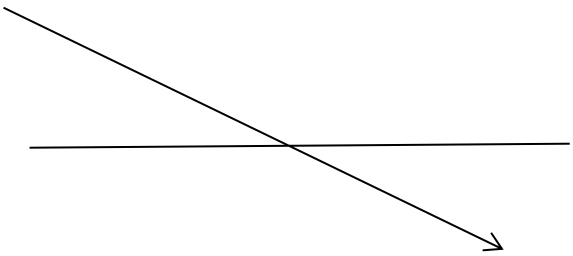
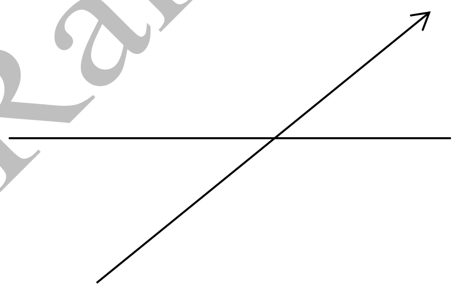
Chapter Two

Polar Coordinates

Cartesian Coordinates: any point in the plane can be represented as a pair of real numbers (x,y).

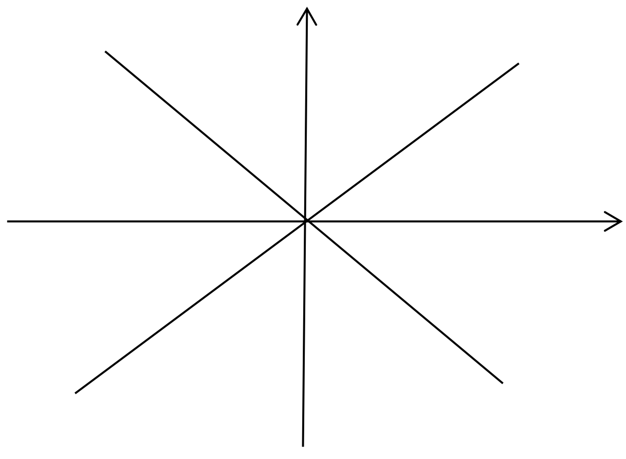
Polar Coordinates: in the plane fix an origin O (called the pole) and an initial ray from O. Then each point P can be located by assigning to it a polar coordinate (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.





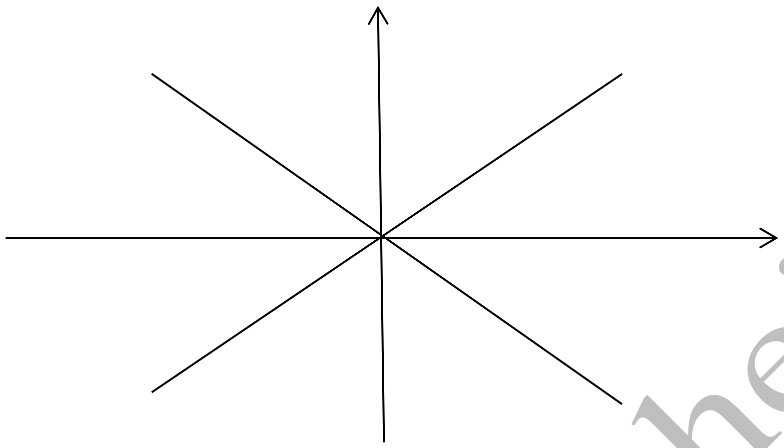
So r is positive when measured on ray and negative when going backward (opposite direction) but θ is positive when measured counterclockwise and negative when measured clockwise.

Polar Coordinates are not unique, for example, $(2, \frac{7\pi}{6}) = (-2, \frac{\pi}{6})$ and $(2, \frac{\pi}{6}) = (2, \frac{-11\pi}{6})$.

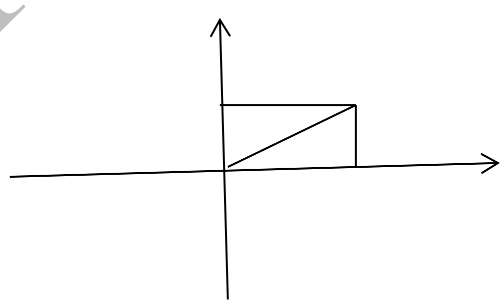


Example 1: Plot the following points?

$p_1(2,\frac{\pi}{4})$, $p_2(-2,\frac{\pi}{4})$, $p_3(2,\frac{-\pi}{4})$, $p_4(-2,\frac{-\pi}{4})$, $p_5(2,\frac{\pi}{2})$, $p_6(-2,\frac{\pi}{2})$, $p_7(2,0)$, $p_8(-2,0)$



The Relation between Cartesian Coordinate (x,y) and Polar Coordinate (r, θ):



$\frac{x}{r} = cos\theta \rightarrow \boldsymbol{x = rcos\theta}$

$\frac{y}{r} = sin\theta \rightarrow \boldsymbol{y = rsin\theta}$

$x^2 + y^2 = r^2 \rightarrow \boldsymbol{r = \sqrt{x^2 + y^2}}$

$\frac{rsin\theta}{rcos\theta} = \frac{y}{x} \rightarrow \frac{sin\theta}{cos\theta} = \frac{y}{x} \rightarrow tan\theta = \frac{y}{x} \rightarrow \boldsymbol{\theta = \tan^{-1}\frac{y}{x}}$

Example 2: converting the following Cartesian coordinate to polar coordinate: a)$x^2 + y^2 - 6x = 0$,

$b)x^2 + (y-3)^2 = 0$ $(\boldsymbol{H.W.})$?

Solution: $a)\ x = rcos\theta, y = rsin\theta$

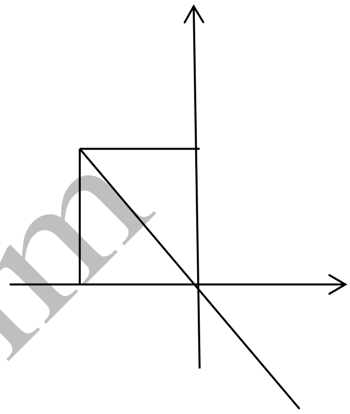
$(rcos\theta)^2 + (rsin\theta)^2 - 6rcos\theta = 0$

$r^2(cos^2\theta + sin^2\theta) - 6rcos\theta = 0$

$r^2 - 6r\cos\theta = 0$

$r = 6\cos\theta$

Example 3: Plot and transfer the points $p_1\left(-2, \frac{-\pi}{6}\right)$ and $p_2\left(2, \frac{\pi}{4}\right)$ $(\boldsymbol{H.W.})$ to Cartesian coordinate?

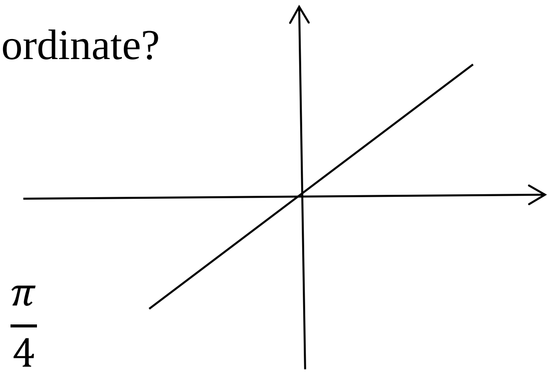


Solution: $x = r\cos\theta \rightarrow x = -2\cos\frac{-\pi}{6} = -2(\frac{\sqrt{3}}{2}) = -\sqrt{3}$

$y = r\sin\theta \rightarrow y = -2\sin\frac{-\pi}{6} = 2\sin\frac{\pi}{6} = 2(\frac{1}{2}) = 1$

$p\ (-2, \frac{-\pi}{6}) \rightarrow p(-\sqrt{3}, 1)$

Example 4: Plot and transfer the points $p_1(1,1), p_2(-1,-\sqrt{3})(\boldsymbol{H.W.})$ to polar coordinate?



Solution: $x^2 + y^2 = r^2 \rightarrow 1 + 1 = r^2 \rightarrow$

$2 = r^2 \rightarrow r = \pm\sqrt{2}$

$\theta = \tan^{-1}\frac{y}{x} \rightarrow \theta = \tan^{-1}1 \rightarrow 1 = \tan\theta \rightarrow \theta = \frac{\pi}{4}$

Example 5: Find Cartesian equation to polar equation

$a)\ r\cos\left(\theta - \frac{\pi}{3}\right) = 3, [b)\ r^2 = 2r\cos\theta, c)\ r^2 = 2\sin 2\theta].(\boldsymbol{H.W.})$

Solution: $a) r\cos(\theta - \frac{\pi}{3}) = 3 \rightarrow r(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}) = 3 \rightarrow$
$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 3 \rightarrow \frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3 \rightarrow x + \sqrt{3}y = 6$

Example 6: Find intersection points of the following two curves (a constant):) r=a(1+cos θ), r=a(1-sin θ)

Solution: a(1+cos θ)= a(1-sin θ)

1+cos θ = 1-sin $\theta \rightarrow \frac{sin\theta}{cos\theta} = -1 \rightarrow \tan\theta = -1$

$\theta = \pi - \frac{\pi}{4} \rightarrow \ \theta = 135°$

$\theta = 2\pi - \frac{\pi}{4} \rightarrow \ \theta = 315°$

r=a(1+cos 135°) $\rightarrow$ r $= \text{a}\left(1 - \frac{1}{\sqrt{2}}\right)$,

r=a(1+cos 315°) $\rightarrow$ r $= \text{a}\left(1 + \frac{1}{\sqrt{2}}\right)$,

(a $\left(1 - \frac{1}{\sqrt{2}}\right), 135°$),(a $\left(1 + \frac{1}{\sqrt{2}}\right), 315°$).

Note:

1- If r=0 for any value of θ then (0, θ) is the origin point.
2- If r=a (a is a constant) for all value of θthen these points are represented a circle of centre (0,0) and radius a.
3- θ=A (A is an angle) for all value of r the these points are represented a line through origin point.

Exercises: 1- Plot and converting the following points to Cartesian coordinates:

a.(3, $\frac{\pi}{4}$), b.(-3, $\frac{\pi}{4}$), c.(3, $\frac{-\pi}{4}$), d.(-3, $\frac{-\pi}{4}$)

2-Graph the locus of point p(r, θ) Whose polar coordinate satisfy the given equation inequality or inequalities.

a) r=2, b) r>2, c) r<2, d)1<r<2, e) $0 \leq \theta \leq 30°, r \geq 0$,

f) $\theta = 120°, r \leq -2$, g) $\theta = 60°, 1 \leq r \leq 3$,

h) $\theta = 495°, r \geq -1$.

3- Graph the loci:

a) rcos θ =2, b) rsin θ =-1, c) rcos(θ - 60°)=3, e) rcos(30° - θ)=0.

4-Show that (-2, $\frac{3\pi}{4}$) is on the curve $r = 2sin2\theta$ and ($\frac{1}{2}, \frac{3\pi}{2}$) is on the curve r= -sin($\frac{\theta}{3}$).

5-Show that these equations represent the same curve:

r =cos θ +1, r = cos θ -1.

6- Find some intersections of the following curves (a constant):

a) r=a(1+cos θ), r=a(1-cos θ) b)r=a(1+sin θ), r=2acosθ

Draw Curve in Polar Coordinate:

$$F(\theta) = r \quad or \;\; F(r,\theta) = 0$$

- **Symmetry about the x-axis (polar axis)**

$$F(r,\theta) = F(r,-\theta) \; or \; F(r,\theta) = F(-r,\pi-\theta)$$

For example, r =3cos$\theta = 3\cos(-\theta)$

- **Symmetry about the origin**

$$F(r,\theta) = F(-r,\theta) \text{ or } F(r,\theta) = F(r,\pi+\theta)$$

For example, r^2 =3cosθ , → $(-r)^2 = 3\cos\theta$

- **Symmetry about the y-axis**

$$F(r,\theta) = F(r,\pi-\theta) \text{ or } F(r,\theta) = F(-r,-\theta)$$

For example, $r = 3sin\theta \;\; = 3\sin(\pi-\theta)$

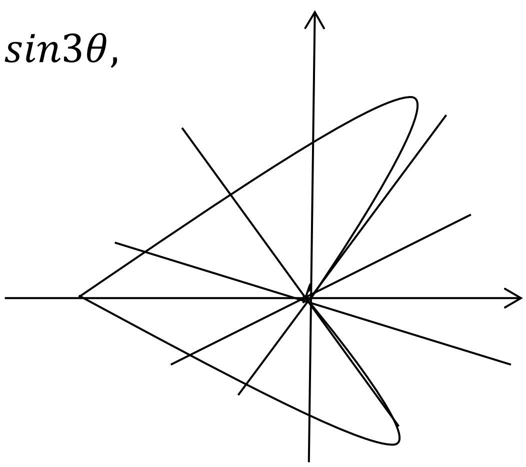
Example 7: Discuss and sketch the following curves:

1)$r = a(1-cos\theta)$, 2)$r = a(1-sin\theta)$,

3) $r^2 = 2a^2cos2\theta$, 4) $r = asin2\theta$, 5) $r = sin3\theta$,

6) $r = \theta, and$ 7) $r = -\theta$.

Solution: 1) ***Cardioid curve***



- $0 \le \theta \le 2\pi$
- $F(r,\theta) = F(r,-\theta)$

(Symmetry about the polar axis)

- $-1 \leq cos\theta \leq 1 \rightarrow 0 \leq 1 - cos\theta \leq 2 \rightarrow$

$0 \leq a(1 - cos\theta) \leq 2a \rightarrow 0 \leq r \leq 2a$

- Intersection points

If $\theta = 0 \rightarrow r = 0$ $(0,0)$

If $\theta = \frac{\pi}{2} \rightarrow r = a$ $(a, \frac{\pi}{2})$

If $\theta = \frac{3\pi}{2} \rightarrow r = a$ $(a, \frac{3\pi}{2})$

If $\theta = \pi \rightarrow r = 2a$ $(2a, \pi)$

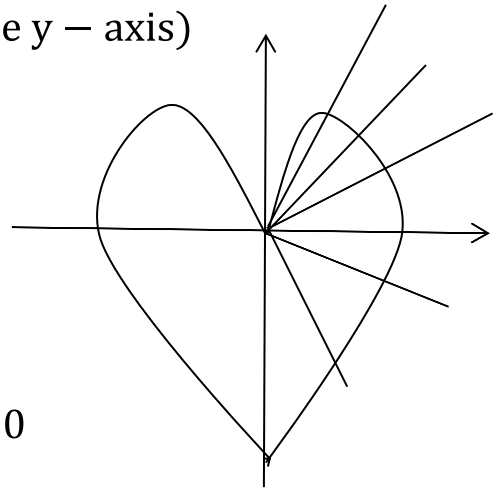
- $\frac{dr}{d\theta} = asin\theta \rightarrow \frac{dr}{d\theta} > 0$ $when\ sin\theta > 0$

$(sin\theta > 0\ in\ the\ interval\ [0, \pi]) \rightarrow increases\ when\ \theta \in [0, \pi]$

	P_1	P_2	P_3	P_4	P_5
θ	30°	60°	90°	120°	150°
r	0.15a	0.5a	a	1.5a	1.85a

2) ***Cardioid curve***

- $0 \leq \theta \leq 2\pi$
- $-1 \leq sin\theta \leq 1 \rightarrow 0 \leq 1 - sin\theta \leq 2 \rightarrow 0 \leq a(1 - sin\theta) \leq 2a \rightarrow 0 \leq r \leq 2a$
- $a(1 - \sin(\pi - \theta) = a(1 - sin\theta)$
  $F(r, \theta) = F(r, \pi - \theta)$(Symmetry about the y − axis)



- Intersection points

If $\theta = 0 \rightarrow r = a$ $(a, 0)$

If $\theta = \frac{\pi}{2} \rightarrow r = 0$ $(0, \frac{\pi}{2})$

If $\theta = \frac{3\pi}{2} \rightarrow r = 2a$ $(2a, \frac{3\pi}{2})$

If $\theta = \pi \rightarrow r = a$ (a, π)

- $\frac{dr}{d\theta} = -acos\theta \rightarrow \frac{dr}{d\theta} \leq 0$ $when\ cos\theta \geq 0$
-

$\left(cos\theta \geq 0\ in\ the\ interval\ \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right) \rightarrow decreases\ when\ \theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

	P_1	P_2	P_3	P_4	P_5
θ	30°	45°	60°	$-30°$	$-45°$
r	0.5a	0.3a	0.1a	1.5a	1.7a

3) ***Lemniscate Curve***

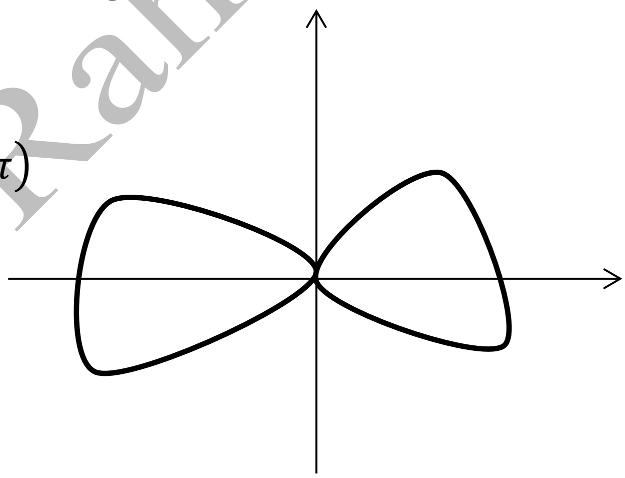
- $\frac{-\pi}{2} \leq 2\theta \leq \frac{\pi}{2} \rightarrow \frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$
- $-2a^2 \leq r^2 \leq 2a^2$
- $F(r,\theta) = F(r,-\theta), F(r,\theta) = F(-r,\theta), F(r,\theta) = F(-r,-\theta)$

(Symmetry about the polar axis, y-axis and origin)

- Intersection points

If $\theta = 0 \rightarrow r = \sqrt{2}a \quad (\sqrt{2}a, 0)$

If $\theta = \pi \rightarrow r = -\sqrt{2}a \quad (-\sqrt{2}a, \pi)$



- $2r\frac{dr}{d\theta} = -4a^2 sin2\theta \rightarrow$

$\frac{dr}{d\theta} = \frac{-2a^2}{r} sin2\theta$

$\frac{dr}{d\theta} < 0 \; when \; sin2\theta > 0$

$0 \leq 2\theta \leq \pi \rightarrow 0 \leq \theta \leq \frac{\pi}{2} \rightarrow decreases \; when \; \theta \in [0, \frac{\pi}{4}]$

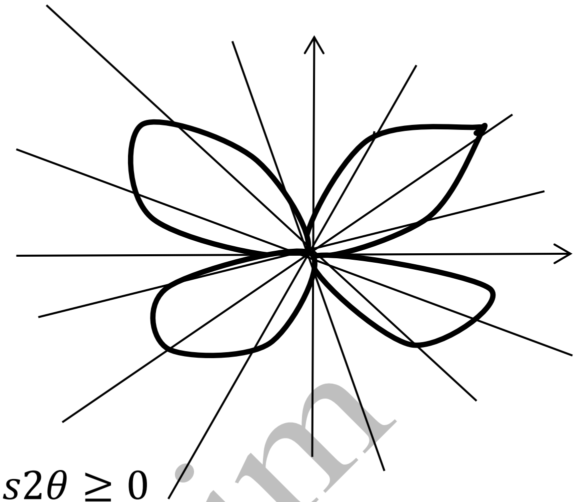
	P_1	P_2
θ	30°	45°
r	$\pm$a	0

4) ***Rose curve***

- $0 \leq \theta \leq 2\pi$
- $-a \leq r \leq a$
- $F(r,\theta) = F(r, \pi - \theta)$(Symmetry about the y- axis)

- Intersection points

If $\theta = 0 \rightarrow r = 0$ (0,0)



If $\theta = \frac{\pi}{2} \rightarrow r = 0$ $(0, \frac{\pi}{2})$

If $\theta = \frac{3\pi}{2} \rightarrow r = 0$ $(0, \frac{3\pi}{2})$

If $\theta = \pi \rightarrow r = 0$ $(0, \pi)$

- $\frac{dr}{d\theta} = 2a\cos 2\theta \rightarrow \frac{dr}{d\theta} \geq 0$ *when* $\cos 2\theta \geq 0$

$\left(\cos 2\theta \geq 0 \text{ in the interval } \frac{-\pi}{2} \leq 2\theta \leq \frac{\pi}{2}\right) \rightarrow$ *increases when* $\theta \in [\frac{-\pi}{4}, \frac{\pi}{4}]$

	P_1	P_2	P_3	P_4
θ	30°	45°	-30°	-45°
r	0.86a	a	-0.86a	-a

5) ***Rose curve***

- $0 \leq \theta \leq 2\pi$
- $-1 \leq r \leq 1$
- $F(r, \theta) = F(r, \pi - \theta)$

(Symmetry about the y- axis)

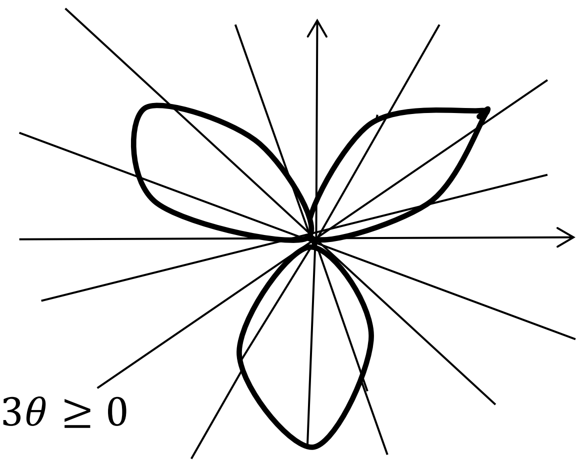
- Intersection points

If $\theta = 0 \rightarrow r = 0$ (0,0)

If $\theta = \frac{\pi}{2} \rightarrow r = -1$ $(-1, \frac{\pi}{2})$

If $\theta = \frac{3\pi}{2} \rightarrow r = 0$ $(1, \frac{3\pi}{2})$

If $\theta = \pi \rightarrow r = 0$ $(0, \pi)$



- $\frac{dr}{d\theta} = 3\cos 3\theta \rightarrow \frac{dr}{d\theta} \geq 0$ *when* $\cos 3\theta \geq 0$

$\left(\cos 3\theta \geq 0 \text{ in the interval } \frac{-\pi}{2} \leq 3\theta \leq \frac{\pi}{2}\right) \rightarrow$ *increases when* $\theta \in [\frac{-\pi}{6}, \frac{\pi}{6}]$

	P_1	P_2
θ	30°	-30°
r	1	-1

6) *Spiral curve*

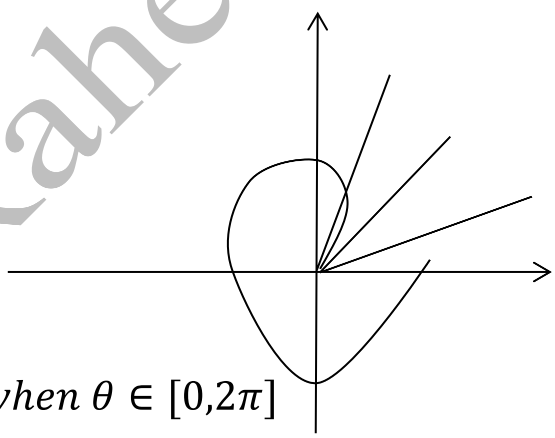
- $0 \le \theta \le 2\pi$
- $0 \le r \le 2\pi \to 0 \le r \le 6 \ (\pi = 3.14)$
- No Symmetry
- Intersection points

If $\theta = 0 \to r = 0$ (0,0)

If $\theta = \frac{\pi}{2} \to r = 1.5$ $(1.5, \frac{\pi}{2})$

If $\theta = \frac{3\pi}{2} \to r = 4.5$ $(4.5, \frac{3\pi}{2})$

If $\theta = \pi \to r = 3.14$ $(3.14, \pi)$



- $\frac{dr}{d\theta} = 1 \to \frac{dr}{d\theta} \ge 0 \ \to increases\ when\ \theta \in [0,2\pi]$

	P_1	P_2	P_3
θ	30°	45°	60°
r	0.52	0.78	1.04

7) *Spiral curve*

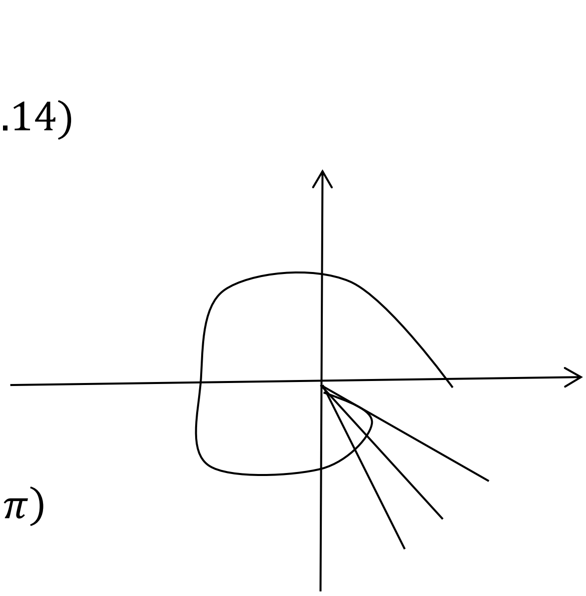
- $-2\pi \le \theta \le 0$
- $0 \le r \le 2\pi \to 0 \le r \le 6 \ (\pi = 3.14)$
- no Symmetry
- Intersection points

If $\theta = 0 \to r = 0$ (0,0)

If $\theta = \frac{-\pi}{2} \to r = 1.5$ $(1.5, \frac{-\pi}{2})$

If $\theta = \frac{-3\pi}{2} \to r = 4.5$ $(4.5, \frac{-3\pi}{2})$

If $\theta = -\pi \to r = 3.14$ $(3.14, -\pi)$



- $\frac{dr}{d\theta} = -1 \rightarrow \frac{dr}{d\theta} \leq 0 \ \rightarrow decreases\ when\ \theta \in [-2\pi, 0]$

	P_1	P_2	P_3
θ	-30°	$-45°$	-60°
r	0.52	0.78	1.04

Note:

The curves in polar coordinates as follows:

- ***Cardioid Curves***

r=$a \pm bcos\theta \ \ or$ r=$a \pm bsin\theta$

- ***Rose Curves***

$r = asin(n\theta) \ \ or \ \ r = acos(n\theta)$

- If n=1 then the curve is a circle.
- If n is an odd number then the number of leaves equal n.
- If n is an even number then the number of leaves equal 2n.

- ***Lemniscate Curves***

$r^2 = cos2\theta \ \ or\ r^2 = sin2\theta$

- ***Spiral Curves***

$r = a\,\theta$

Exercises: Discuss and sketch the following curves:

1) $r = a(1 + sin\theta)$, 2) $r = a(1 + cos\theta)$, 3) $r = acos2\theta$
4) $r = a(1 + 2sin\theta)$, 5) $r = a(1 + 2cos\theta)$, 6) $r = a(2 + sin\theta)$
7) $r = a(2 + cos\theta)$, 8) $r^2 = 2a^2sin2\theta$, 9) $r = sin2\theta$,
10) $r^2 = sin2\theta$, 11) $r^2 = cos2\theta$ and 12) $r = cos3\theta$

Example 8: Find the points on the following curve:

$$r = a(1 + \cos\theta)$$

Solution:

a- If the tangent is parallel to x-axis

$$\frac{dy}{dx} = 0$$

y=rsin $\theta \rightarrow y = a(1 + cos\ \theta)sin\theta$

$\frac{dy}{d\theta}$=acos $\theta + acos^2\theta - asin^2\theta \rightarrow \frac{dy}{d\theta}$=acos $\theta + acos2\theta$

x=rcos $\theta \rightarrow x = a(1 + cos\ \theta)cos\theta$

$\frac{dx}{d\theta}$=-asin $\theta + 2acos\theta(-sin\theta) \rightarrow \frac{dx}{d\theta}$=-asin $\theta - asin2\theta$

$$\frac{{dy}/{d\theta}}{{dx}/{d\theta}} = \frac{dy}{dx}$$

$$\frac{acos\theta + acos2\theta}{-asin\theta - asin2\theta} = \frac{dy}{dx}$$

$$\frac{acos\theta + acos2\theta}{-asin\theta - asin2\theta} = 0$$

$$acos\theta + acos2\theta = 0]\ \frac{1}{a}$$

$$cos\theta + cos2\theta = 0$$

$$cos\theta + 2cos^2\theta - 1 = 0$$

$$2cos^2\theta + cos\theta - 1 = 0$$

$$(2\ cos\theta - 1)(\ cos\theta + 1) = 0$$

$$2cos\theta - 1 = 0 \quad or\ \ cos\theta + 1 = 0$$

$$cos\theta = \frac{1}{2} \text{ or } cos\theta = -1$$

$$\theta = 60°, -60° \quad \text{or } \theta = \pi$$

r=a(1+cos 60°) or r=a(1+cos π)

r=a(1+$\frac{1}{2}$) or r=a(1-1)

r=$\frac{3}{2}a$ or r=0

$(\frac{3}{2}a, \mp 60°), (0, \pi)$

b- If the tangent is parallel to y-axis (**H.W.**)

<u>Example 9:</u> Sketch and find intersection points for the following pair of curves: 1. $r = a(1 + cos\theta), r = 3acos\theta$, *and*
2. $r = (1 + cos\theta), r^2 = 4cos\theta$ (**H.W**.)

Solution: intersection points for two curves

$$a(1 + cos\theta) = 3acos\theta \,] \div a$$

$$1 + cos\theta = 3cos\theta$$

$$2cos\theta = 1$$

$$cos\theta = \frac{1}{2}$$

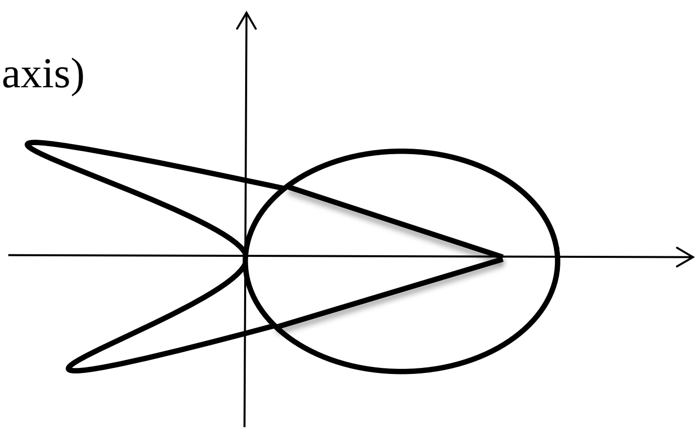
$$\theta = \mp\frac{\pi}{3}$$

If $\theta = \frac{\pi}{3} \rightarrow r = a\left(1 + cos\frac{\pi}{3}\right) \rightarrow r = \frac{3a}{2}$

If $\theta = \frac{-\pi}{3} \rightarrow r = a\left(1 + cos\frac{-\pi}{3}\right) \rightarrow r = \frac{3a}{2}$

$(\frac{3a}{2}, \frac{\pi}{3}), (\frac{3a}{2}, -\frac{\pi}{3})$

$r = 3acos\theta$ (Symmetry about the x-r axis)



- $0 \leq \theta \leq 2\pi$

- $-3a \le r \le 3a$
- Intersection points

If $\theta = 0 \rightarrow r = 3a$ $(3a, 0)$

If $\theta = \pi \rightarrow r = -3a$ $(-3a, \pi)$

If $\theta = \frac{\pi}{2} \rightarrow r = 0$ $(0, \frac{\pi}{2})$

If $\theta = \frac{3\pi}{2} \rightarrow r = 0$ $\left(0, \frac{3\pi}{2}\right)$

- $\frac{dr}{d\theta} = -3a\sin\theta \rightarrow \frac{dr}{d\theta} < 0$ $when\ \sin\theta > 0 \rightarrow decreases\ when\ \theta \in [0, \pi]$

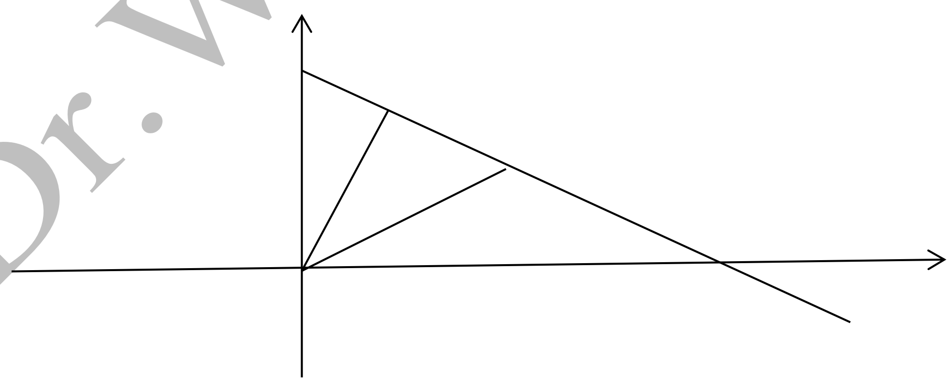
	P_1	P_2	P_3	P_4	P_5	P_6
θ	30°	45°	60°	120°	150°	170°
r	2.5a	2.1a	1.5a	-1.5a	-2.5a	-2.9a

The Polar Equation for Line:

Let *a* be the length of perpendicular line from pole (origin) to line L and α be an angle between perpendicular line and x-axis. To find the equation of L when P(r, θ) & Q(a, α) be two points on L.

$\Delta OQP, < POQ = \theta - \alpha$

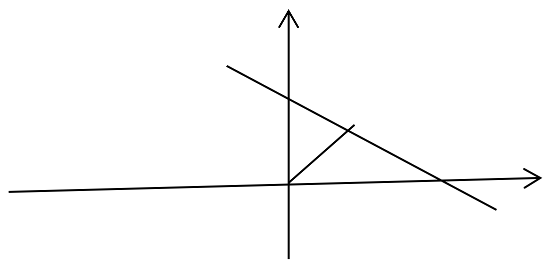
$\cos(\theta - \alpha) = \cos(< POQ) \rightarrow \boldsymbol{r\cos(\theta - \alpha) = a}$



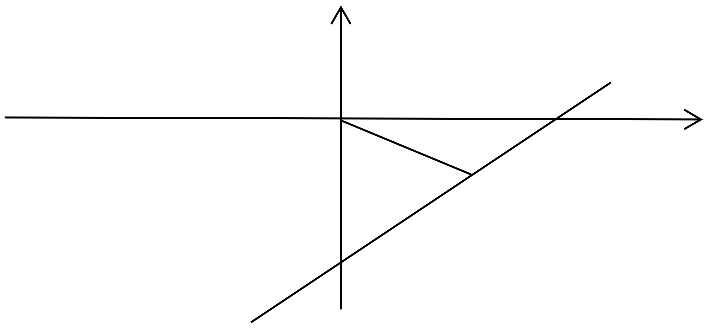
Example 10: Sketch the following straight line

$a)\ r\cos\left(\theta - \frac{\pi}{4}\right) = 3, b)\ r\cos\left(\theta + \frac{\pi}{6}\right) = 2$

Solution: a) Q(3, $\frac{\pi}{4}$) $\rightarrow \alpha = \frac{\pi}{4}, a = 3$



b) $Q(2, -\frac{\pi}{6}) \rightarrow \alpha = -\frac{\pi}{6}, a = 2$



Special cases of Polar Equation for Line

1) If the line L is parallel to y-axis

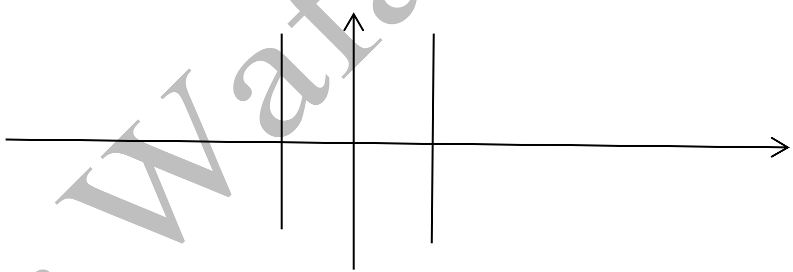
$\alpha = \pi \quad or \quad \alpha = 0$

If $\alpha = 0 \rightarrow rcos(\theta - 0) = a \rightarrow rcos\theta = a$

If $\alpha = \pi \rightarrow rcos(\theta - \pi) = a \rightarrow rcos(\pi - \theta) = a \rightarrow$
$-rcos\theta = a \rightarrow rcos\theta = -a$

The Polar Equation for Line is parallel to y-axis as follows:

$$\boldsymbol{r}\mathbf{cos}\boldsymbol{\theta} = \mp\mathbf{a}$$



2) If the line L is parallel to x-axis

$\alpha = \frac{\pi}{2}$ or $\alpha = \frac{3\pi}{2}$

If $\alpha = \frac{\pi}{2}$

$r\cos(\theta - \frac{\pi}{2}) = a \rightarrow r\cos(-\left(\frac{\pi}{2} - \theta\right) = a \rightarrow rsin\theta = a, \ \left(a, \frac{\pi}{2}\right)$

If $\alpha = \frac{3\pi}{2}$

$r\cos(\theta - \frac{3\pi}{2}) = a \rightarrow r\cos(-\left(\frac{3\pi}{2} - \theta\right)) = a \rightarrow rsin\theta = -a,\ \left(a, \frac{3\pi}{2}\right)$

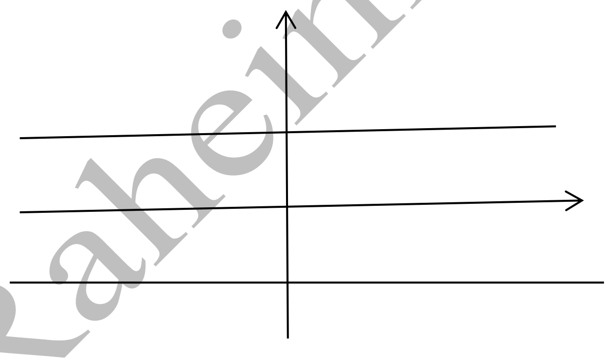
The Polar Equation for Line is parallel to x-axis as follows:

$$\boldsymbol{r}\mathbf{sin}\boldsymbol{\theta} = \mp\mathbf{a}$$

Example 11: Sketch the following straight line:

$a)r\sin\theta = 4, r\sin\theta = -4\ , c)r\cos\theta = 4\ and\ d)r\cos\theta = -4$

Solution:



If $\alpha = \frac{\pi}{2} \rightarrow a = 4\ \ ,\ \ (4, \frac{\pi}{2})$

If $\alpha = \frac{3\pi}{2} \rightarrow a = -4\ ,\ (4, \frac{3\pi}{2})$

3) If the line L through the origin

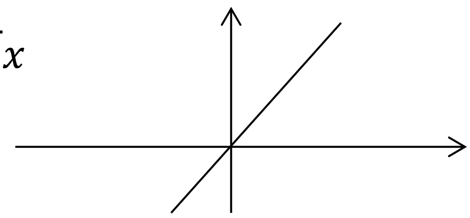
The Polar Equation for Line L through the origin as follows:

$$\boldsymbol{\theta}\mathbf{=A}$$

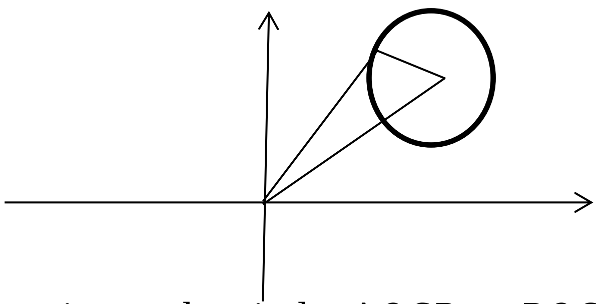
Example 12: Sketch the following straight line: $\theta = \frac{\pi}{3}$?

$tan^{-1}\frac{y}{x} = \frac{\pi}{3} \rightarrow \frac{y}{x} = tan\frac{\pi}{3} \rightarrow \frac{y}{x} = \sqrt{3} \rightarrow y = \sqrt{3}x$

$\theta = \frac{\pi}{3} or\ y = \sqrt{3}x$



Polar Equation for the Circle of radius a Centered at C (b,β°):



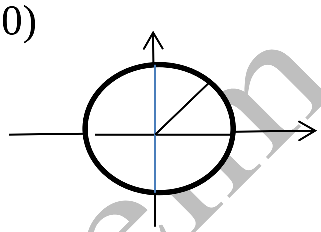
P(r, θ) be a point on the circle, $\Delta OCP, < POC = \theta - \beta$

The polar equation for the circle of radius a centered at C (b,β)

$$\boldsymbol{a^2 = b^2 + r^2 - 2brcos(\theta - \beta)}$$

Notes:

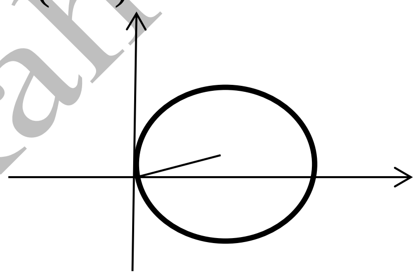
- If the circle's centre is origin point(b=0)



$a^2 = b^2 + r^2 - 2brcos(\theta - \beta)$

$$\boldsymbol{a^2 = r^2}$$

- If the circle passes through the origin (a=b)



$a^2 = b^2 + r^2 - 2brcos(\theta - \beta)$

$a^2 = a^2 + r^2 - 2arcos(\theta - \beta)$] a=b

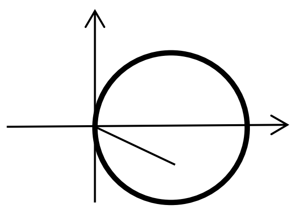
$r^2 = 2arcos(\theta - \beta)$]÷r

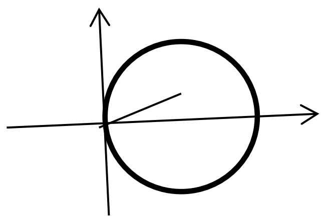
$$\boldsymbol{r = 2acos(\theta - \beta)}$$

Example 13: Sketch $a)\ r = 6\cos\left(\theta - \frac{\pi}{4}\right), b) r = 4\cos(\theta + \frac{\pi}{6})$

Solution: a) 2a=6→ $a = 3, C\ (3, \frac{\pi}{4})$

b) 2a=4→ $a = 2, C\ \left(2, \frac{-\pi}{6}\right)$

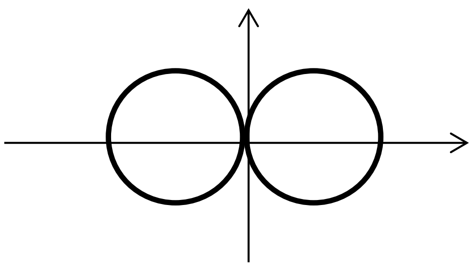




- If the circle passes through the origin(a=b) centered on x-axis

$\beta = 0$ or $\beta = \pi$

If b=a → $r = 2acos(\theta - \beta)$



If $\beta = 0 \rightarrow r = 2acos(\theta - 0) \rightarrow r = 2acos\theta$ (Equation for circle through the origin centered C(a,0)on the positive x-axis)

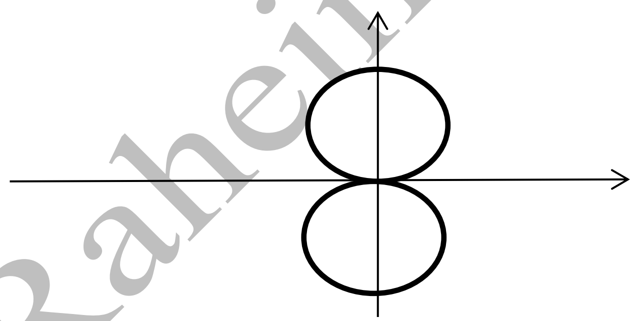
If $\beta = \pi \to r = 2acos(\theta - \pi) \to r = -2acos\theta$ (Equation for circle through the origin centered C(a,π)on the negative x-axis)

$\boldsymbol{r = \mp 2acos\theta}$ **(**Equations of circles pass through the origin centered on x-axis)

Example 14: Sketch $r^2 = 4a^2cos^2 \ \theta$

Solution: $r^2 = 4a^2cos^2 \ \theta \to r = \mp 2acos\theta$

- If the circle passes through the origin(a=b) centered on y-axis



$\beta = \frac{\pi}{2}$ or $\beta = \frac{3\pi}{2}$

If b=a $\to r = 2acos(\theta - \beta)$

If $\beta = \frac{\pi}{2} \to r = 2acos\left(\theta - \frac{\pi}{2}\right) \to r = 2asin\theta$

(Equation for circle through the origin centered C(a,$\frac{\pi}{2}$)on the positive y-axis)

If $\beta = \frac{3\pi}{2} \to r = 2acos\left(\theta - \frac{-\pi}{2}\right) \to r = -2asin\theta$

(Equation for circle through the origin centered C(a,$\frac{3\pi}{2}$)on the negative y-axis)

$$\boldsymbol{r = \mp 2asin\theta}$$

(Equations of circles pass through the origin centered on y-axis)

Example 15: Sketch $a)\, r^2 = 9cos^2 \ \theta, b)\, r^2 = 4a^2sin^2 \ \theta$

Solution: $a)\, r^2 = 9cos^2 \ \theta \to r = \mp 3cos\theta$

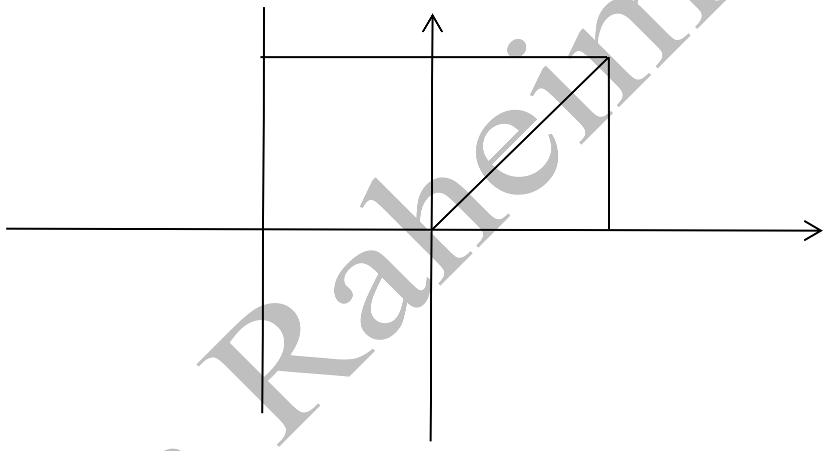
$b)\, r^2 = 4a^2sin^2 \ \theta \to r = \mp 2asin\theta$

Polar Equation for Conic Section if place one focus F at the Origin point and the Directrix L parallel to y-axis:

L: X=-k

If P(r, θ) ay point on conic section, by definition of conic section

$\frac{PF}{PQ} = e$



$$PF = ePQ = eP'Q'$$

$$\boldsymbol{PF = e(Q'F + FP')}$$

$$\boldsymbol{PF = e(k + rcos\theta)}$$

$$\boldsymbol{r = ek + ercos\theta}$$

$$\boldsymbol{r(1 - ecos\theta) = ek}$$

$$r = \frac{ek}{1-ecos\theta}$$ when *x=-k* *or* $rcos\,\theta = -k$

Special Cases:

1) If *e=1* then the Conic Section is Parabola :

$$\boldsymbol{r = \frac{k}{1 - cos\theta}}$$

In some cases:

$$sin^2\frac{\theta}{2} = \frac{1}{2}(1 - cos\theta) \rightarrow 2sin^2\frac{\theta}{2} = 1 - cos\theta \rightarrow$$

$$\frac{1}{2sin^2\frac{\theta}{2}} = \frac{1}{1 - cos\theta} \rightarrow \frac{k}{2sin^2\frac{\theta}{2}} = \frac{k}{1 - cos\theta} \rightarrow$$

$$r = \frac{k}{2sin^2\frac{\theta}{2}} \rightarrow \boldsymbol{r = \frac{k}{2} csc^2 \frac{\theta}{2}}$$

Parabola	
The Parabola section with focus (0,0) and directrix is parallel to y-axis, x=-k	$r = \frac{k}{1-cos\theta}$
The Parabola section with focus (0,0) and directrix is parallel to y-axis, x=k	$r = \frac{k}{1+cos\theta}$
The Parabola section with focus (0,0) and directrix is parallel to x-axis, y=-k	$r = \frac{k}{1-sin\theta}$
The Parabola section with focus (0,0) and directrix is parallel to x-axis, y=k	$r = \frac{k}{1+sin\theta}$

2) If 0<*e<1* then the Conic Section is Ellipse :

If $e=\frac{1}{2}$

$$\boldsymbol{r} = \frac{\frac{1}{2}k}{1-\frac{1}{2}cos\theta} = \boldsymbol{\frac{k}{2-cos\theta}}$$

The denominator is always positive because score of *cos* not more than 1.

3) If *e>1* then the Conic Section is Hyperbola :

If *e*=2

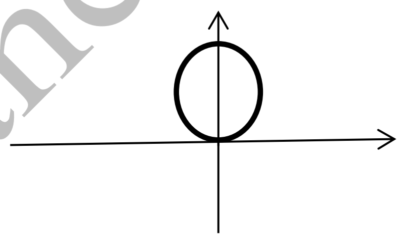
$$\boldsymbol{r = \frac{2k}{1-2cos\theta}}$$

	Ellipse	***Hyperbola***
The conic section is intersect with negative x-axis, first focus is (0,0) and directrix is parallel to y-axis	$r = \frac{k}{2-cos\theta}$	$r = \frac{2k}{1-2cos\theta}$
The conic section is intersect with positive x-axis, first focus is (0,0)	$r = \frac{k}{2+cos\theta}$	$r = \frac{2k}{1+2cos\theta}$

and directrix is parallel to y-axis		
The conic section is intersect with negative y-axis, first focus is (0,0) and directrix is parallel to x-axis	$r = \frac{k}{2 - sin\theta}$	$r = \frac{2k}{1 - 2sin\theta}$
The conic section is intersect with positive y-axis, first focus is (0,0) and directrix is parallel to x-axis	$r = \frac{k}{2 + sin\theta}$	$r = \frac{2k}{1 + 2sin\theta}$

Example 16: Determine the polar equation and sketch the given curves: 1)$x^2 + y^2 - 2y = 0$, and 2) $y^2 = 4ax + 4a^2$

Solution: 1) $r^2 = 2rsin\theta \to r = 2\ sin\theta$



2) $r^2 sin^2\theta = 4arcos\theta + 4a^2$

$r^2(1 - cos^2\theta) = 4arcos\theta + 4a^2$

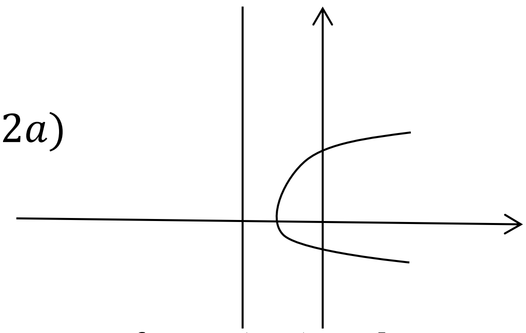
$r^2 - r^2cos^2\theta = 4arcos\theta + 4a^2$

$r^2 = r^2cos^2\theta + 4arcos\theta + 4a^2$

$r^2 = (rcos\theta + 2a)^2 \to r = \mp(rcos\theta + 2a)$

If $r = rcos\theta + 2a \to r(1 - cos\theta) = 2a$

$r = \frac{2a}{1 - cos\theta}$



The Parabola section is intersect with negative x-axis, focus (0,0) and directrix is parallel to y-axis.

x=-k=-2a

$\theta \neq 0, if\ \theta = \frac{\pi}{2} \to r = 2a\ or\ \theta = \frac{3\pi}{2} \to r = 2a$

$if\ \theta = \pi \to r = a$

Or $r = -rcos\theta - 2a \to r(-1 - cos\theta) = 2a$

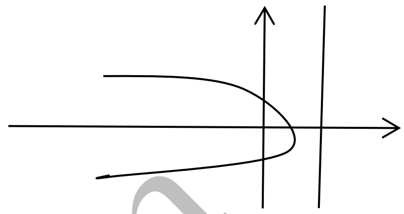
$$r = \frac{-2a}{1 + cos\theta}$$

The Parabola section is intersect with positive x-axis, focus (0,0) and directrix is parallel to y-axis.

x=k=2a

$\theta \neq \pi, if\ \theta = \frac{\pi}{2} \rightarrow r = -2a\ or\ \theta = \frac{3\pi}{2} \rightarrow r = -2a$

$if\ \theta = 0 \rightarrow r = a$



Example 17: Determine the Cartesian equation and sketch the following curves: 1) $r = sin2\theta$, 2) $r = \frac{8}{1-2cos\theta}$

Solution: 1) $\sqrt{x^2+y^2} = 2sin\theta cos\theta \rightarrow \sqrt{x^2+y^2} = 2\frac{rsin\theta}{r} \cdot \frac{rcos\theta}{r} \rightarrow$

$\sqrt{x^2+y^2} = 2\frac{y}{r} \cdot \frac{x}{r} \rightarrow \sqrt{x^2+y^2} = 2\frac{y}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} \rightarrow$

$\sqrt{x^2+y^2} = \frac{2xy}{x^2+y^2} \rightarrow (x^2+y^2)^{\frac{3}{2}} = 2xy \rightarrow (x^2+y^2)^3 = 4x^2y^2$

2) $\sqrt{x^2+y^2} = \frac{8}{1-\frac{2x}{r}} \rightarrow \sqrt{x^2+y^2} = \frac{8}{1-\frac{2x}{\sqrt{x^2+y^2}}} \rightarrow$

$\sqrt{x^2+y^2} = \frac{8}{\frac{\sqrt{x^2+y^2-2x}}{\sqrt{x^2+y^2}}} \rightarrow \sqrt{x^2+y^2} = \frac{8\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}-2x}] \div \sqrt{x^2+y^2} \rightarrow$

$\sqrt{x^2+y^2} - 2x = 8 \rightarrow \sqrt{x^2+y^2} - 2x - 8 = 0$

Exercises:

1) Determine the polar equation and sketch the given curves:
   a) $(x^2+y^2)^2 + 2ax(x^2+y^2) - a^2y^2 = 0$,
   b) $xcosa + ysina = b(a\ and\ b\ are\ constants)$
2) Determine the Cartesian equation and sketch the following curves:
   a) $r = 4cos\theta$, b) $r = 6sin\theta$, c) $r = a(1+sin\theta), and$

d) $r^2 = 2a^2 cos2\theta$

3) Sketch the following loci:

$a)\ r = a + acos\,\theta\,,\ \ b)\ r = 2\,cos(\theta + 45°),$

$c)\ r = 4\,csc\,\theta\,, \qquad d)\ r = 5\,sec\,\theta\,, and$

$e)\ r = 3si\,n\,\theta$

4) Find polar equation for each conic sections:

1) e=1, x=2, 2) e=1, y=2, 3) e=5, y=-6, 4) e=2 x=4, 5) e=$\frac{1}{2}$, x=1,

6) e=$\frac{1}{4}$, x=-2, 7) e=$\frac{1}{5}$, x=-10, and 8)) e=$\frac{1}{3}$, y=6.

5) Sketch the following:

1) $r = \frac{1}{1+cos\theta}$, 2) $r = \frac{6}{2+cos\theta}$, 3) $r = \frac{25}{10-5cos\theta}$, 4) $r = \frac{4}{2-2cos\theta}$,

5) $r = \frac{12}{12+3sin\theta}$, 6) $r = \frac{8}{2-2sin\theta}$, 7) $r = \frac{400}{16+8sin\theta}$,

8)$r = 3\sec(\theta - \frac{\pi}{3})$, 9) $r = 4\sec(\theta + \frac{\pi}{6})$, 10) $r = 4si\,n\,\theta$,

11) $r = -2\,cos\,\theta$, 12) $r = 8(4 + \cos\theta)$, 13) $r = 8(4 + \sin\theta)$,

14) ${}^{1}/_{(1 - sin\theta)}$, 15) ${}^{1}/_{(1 + cos\theta)}$, 16) ${}^{2}/_{(1 + 2sin\theta)}$, and

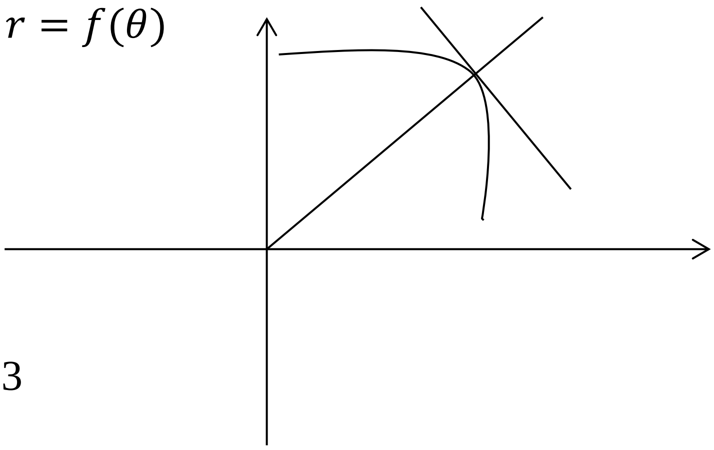
17) ${}^{2}/_{(1 + 2cos\theta)}$

The Angle ψ between Radius an Tangent of Curve

$\Delta OPQ, \varphi = \theta + \psi \rightarrow \psi = \varphi - \theta, r = f(\theta)$

$\rightarrow \tan\psi = \tan(\varphi - \theta)$

$$tan\psi = \frac{tan\varphi - tan\theta}{1 + tan\varphi tan\theta}$$



$$tan\psi = \frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{y}{x}.\frac{dy}{dx}} = \frac{\frac{dy/_{d\theta}}{dx/_{d\theta}} - \frac{y}{x}}{1 + \frac{y}{x}.\frac{dy/_{d\theta}}{dx/_{d\theta}}} = \frac{x\frac{dy}{d\theta} - y\frac{dx}{d\theta}}{x\frac{dx}{d\theta} + y\frac{dy}{d\theta}}$$

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta}$$

$$= \ rcos\theta\left(rcos\theta + sin\theta\frac{dr}{d\theta}\right) - rsin\theta(-rsin\theta$$

$$+ cos\theta\frac{dr}{d\theta})$$

$$= r^2cos^2\theta + rcos\theta sin\theta\frac{dr}{d\theta} + r^2sin^2\theta - rsin\theta cos\theta\frac{dr}{d\theta}$$

$$= r^2$$

Use same way to find $x\frac{dx}{d\theta} + y\frac{dy}{d\theta} = r\frac{dr}{d\theta}$

$$tan\psi = \frac{r^2}{r\frac{dr}{d\theta}} = \frac{r}{\frac{dr}{d\theta}} = \frac{rd\theta}{dr}$$

$$\boldsymbol{tan\psi = \frac{rd\theta}{dr}}$$

<u>Example 18</u>: Find ψ for the following curve: $r = a(1 - cos\theta)$

Solution: $\frac{dr}{d\theta} = asin\theta \ \rightarrow \frac{d\theta}{dr} = \frac{1}{asin\theta}$

$$tan\psi = \frac{rd\theta}{dr}$$

$$=a(1-\cos\theta).\frac{1}{a\sin\theta}=\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2\theta/_2}{2\sin^{\theta}/_2 .\cos^{\theta}/_2}=\frac{\sin^{\theta}/_2}{\cos^{\theta}/_2}$$

$$=\tan\frac{\theta}{2}\rightarrow\psi=\frac{\theta}{2}$$

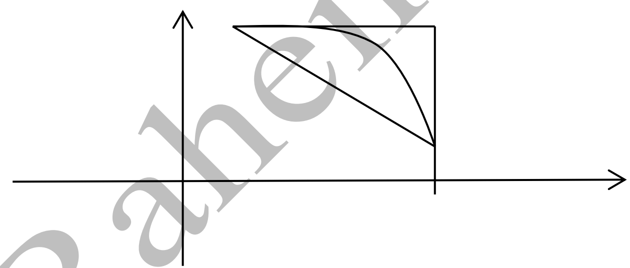
Length of Arc or Curves

s= length of arc between two points on curve in polar plane

ds (change of arc length)

$x=r\cos\theta, y=r\sin\theta$

$ds^2=dx^2+dy^2$



$ds^2=(-r\sin\theta d\theta+\cos\theta dr)^2+(r\cos\theta d\theta+\sin\theta dr)^2$

$$ds^2=r^2\sin^2\theta d\theta-2r\sin\theta\cos\theta d\theta dr+\cos^2\theta dr^2+r^2\cos^2\theta d\theta^2 \\ +2r\cos\theta\sin\theta d\theta dr+\sin^2\theta dr^2$$

$=(\sin^2\theta+\cos^2\theta)r^2d\theta^2+(\cos^2\theta+\sin^2\theta)dr^2$

$=r^2d\theta^2+dr^2$

$=r^2d\theta^2+\left(\frac{dr}{d\theta}.d\theta\right)^2$

$$ds=\sqrt{r^2+(\frac{dr}{d\theta})^2}\ d\theta$$

$$\int_{\alpha}^{\beta}ds=\int_{\theta=\alpha}^{\theta=\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$\boldsymbol{\int_{\alpha}^{\beta}ds=\int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\ d\theta}$$

<u>Example 19</u>: Find length of the cardioid curve:

$$r = a(1 - cos\theta) \text{ when } 0 \leq \theta \leq 2\pi$$

Solution:

$$\int_{\alpha}^{\beta} ds = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$\int_{0}^{2\pi} ds = \int_{0}^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$=\int_{0}^{2\pi} \sqrt{a^2(1 - cos\theta)^2 + a^2 sin^2\theta}\ d\theta$$

$$=\int_{0}^{2\pi} \sqrt{a^2 - 2a^2 cos\theta + a^2 cos^2\theta + a^2 sin^2\theta}\ d\theta$$

$$=a\int_{0}^{2\pi} \sqrt{2 - 2cos\theta}\ d\theta$$

$$=\sqrt{2}a\int_{0}^{2\pi} \sqrt{2}\sqrt{sin^2\frac{\theta}{2}}\ d\theta$$

$$=2a\int_{0}^{2\pi} sin\frac{\theta}{2}\, d\theta = 4a\int_{0}^{2\pi} sin\frac{\theta}{2} . \frac{1}{2} d\theta$$

$$=4a\left[-cos\frac{\theta}{2}\right]_{0}^{2\pi} = 4a\left[-cos\frac{2\pi}{2} + cos\frac{0}{2}\right] = 4a[1+1] = 8a$$

<u>Example 20</u>: Find length of the arc for the following curve:

$$r = asin^2\frac{\theta}{2} \quad \text{ when } 0 \leq \theta \leq \pi$$

Solution:

$$\int_{\alpha}^{\beta} ds = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$\int_0^{\pi} ds = \int_0^{\pi} \sqrt{(a sin^2 \frac{\theta}{2})^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$=\int_0^{\pi} \sqrt{a^2 sin^4 \frac{\theta}{2} + a^2 sin^2 \frac{\theta}{2} cos^2 \frac{\theta}{2}}\ d\theta$$

$$=\int_0^{\pi} \sqrt{a^2 sin^2 \frac{\theta}{2} (sin^2 \frac{\theta}{2} + cos^2 \frac{\theta}{2})}\ d\theta$$

$$=a \int_0^{\pi} sin \frac{\theta}{2}\ d\theta = a \int_0^{\pi} sin \frac{\theta}{2}\ d\theta = -2a cos \frac{\theta}{2}\Big]_0^{\pi} = -2a[cos \frac{\pi}{2} - \ cos \frac{0}{2}]$$

$$= -2a[0-1] = 2a$$

<u>**Example 21**</u>**:** Find length of the arc for the following curve:

$$r = a\theta^2 \quad \text{when}\ \ 0 \le \theta \le \pi$$

Solution:

$$\int_{\alpha}^{\beta} ds = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$\int_0^{\pi} ds = \int_0^{\pi} \sqrt{(a\theta^2)^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$=\int_0^{\pi} \sqrt{a^2\theta^4 + 4a^2\theta^2}\ d\theta$$

$$=\int_0^{\pi} \sqrt{a^2\theta^2(\theta^2+4)}\ d\theta$$

$$=a \int_0^{\pi} \theta\sqrt{\theta^2+4}\ d\theta = \frac{a}{2} . \frac{(\theta^2+4)^{3/2}}{\frac{3}{2}}\Big]_0^{\pi} = \frac{a}{3}\left[(\pi^2+4)^{3/2} - 4^{3/2}\right]$$

$$= \frac{a}{3}[(\pi^2+4)^{3/2} - 8]$$

<u>**Example 22**</u>**:** Find intersection points between following two curves and find the angle between their two tangents:

$$r = \frac{1}{1 - cos\theta} \quad , r = \frac{3}{1 + cos\theta}$$

Solution:

$$\frac{1}{1 - cos\theta} = \frac{3}{1 + cos\theta} \to 3 - 3cos\theta = 1 + \cos\theta \to 2 = 4cos\theta$$

$$\to cos\theta = \frac{1}{2} \to \theta = \pm\frac{\pi}{3}$$

When $r = \frac{1}{1-cos\theta}, \theta = \frac{\pi}{3}$

$$r = \frac{1}{1-cos\frac{\pi}{3}} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{2-1}{2}} = 2, \quad (2, \frac{\pi}{3})$$

When $r = \frac{1}{1-cos\theta}, \theta = \frac{-\pi}{3}$

$$r = \frac{1}{1-cos\frac{-\pi}{3}} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{2-1}{2}} = 2, \quad (2, \frac{-\pi}{3})$$

$$tan\psi = r \Big/ \frac{dr}{d\theta}$$

$$tan\,\psi_1 = \frac{1}{1-cos\theta} \Big/ \frac{-sin\theta}{(1-cos\theta)^2}$$

$$= \frac{1}{1-cos\theta} \cdot \frac{(1-cos\theta)^2}{-sin\theta} = \frac{1-cos\theta}{-sin\theta}$$

When $\theta = \frac{\pi}{3} \to tan\,\psi_1 = \frac{1-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}}$

$$tan\,\psi_2 = \frac{3}{1+cos\theta} \Big/ \frac{3sin\theta}{(1+cos\theta)^2}$$

$$= \frac{3}{1+cos\theta} \cdot \frac{(1+cos\theta)^2}{3sin\theta} = \frac{1+cos\theta}{sin\theta}$$

When $\theta = \frac{\pi}{3} \rightarrow \ tan\,\psi_2 = \frac{1+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \sqrt{3}$

$$tan\,\psi = \frac{tan\,\psi_2 - tan\,\psi_1}{1+tan\,\psi_1 \ \ tan\,\psi_2} = \frac{\sqrt{3}+{}^{1}/_{\sqrt{3}}}{1-\sqrt{3}\ {}^{1}/_{\sqrt{3}}} = \infty \rightarrow \ \psi = \frac{\pi}{2}$$

When $\theta = -\frac{\pi}{3}$ (**H.W.**)

<u>Example 23</u>: Find intersection points of curve $r = a(1 + cos\theta)$ with horizontal tangent.

Solution: $\frac{dy}{dx} = 0$

$$\frac{{}^{dy}/_{d\theta}}{{}^{dx}/_{d\theta}} = 0 \rightarrow \frac{dy}{d\theta} = 0,$$

$$y = rsin\theta \rightarrow y = a(1 + cos\theta)sin\theta \rightarrow y = asin\theta + acos\theta sin\theta$$

$$\frac{dy}{d\theta} = acos\theta + acos\theta cos\theta - asin\theta sin\theta = 0$$

$$= acos\theta + acos^2\theta - asin^2\theta = 0$$

$$= acos\theta - a[sin^2\theta - cos^2\theta] = 0$$

$$= acos\theta - a[1 - cos^2\theta - cos^2\theta] = 0$$

$$= a[cos\theta + 2cos^2\theta - 1] = 0$$

$$= a[2cos^2\theta + cos\theta - 1] = 0$$

$$= a(cos\theta + 1)(2cos\theta - 1) = 0$$

If $\quad cos\theta + 1 = 0 \rightarrow cos\theta = -1 \rightarrow \theta = \pi$

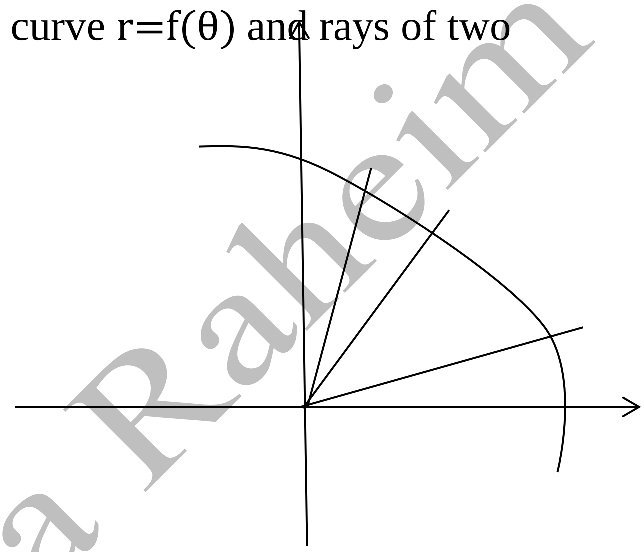
Or $2cos\theta - 1 = 0 \rightarrow cos\theta = \frac{1}{2} \rightarrow \theta = \pm\frac{\pi}{3}$

When $\theta = \pm\frac{\pi}{3} \rightarrow r = a\left(1 + cos \pm\frac{\pi}{3}\right) = a\left(1 + \frac{1}{2}\right) \rightarrow r = \frac{3a}{2}, (\frac{3a}{2}, \pm\frac{\pi}{3})$

$\theta = \pi \rightarrow r = a(1 + cos\pi) = a(1 - 1) \rightarrow r = 0, (0, \pi)$

The Area of Region in Polar Plane

The area A bounded between the curve r=f(θ) and rays of two angles $\alpha \le \theta \le \beta$



$r_i = f(\theta_i)$

$A_i = the\ area\ of\ OPQ$

$A_i = \frac{1}{2}{r_i}^2\ \Delta\theta_i = \frac{1}{2}\ (f(\theta_i))^2\ \Delta\theta_i$

$$\sum_{i=1}^{n} A = \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2\ \Delta\theta_i$$

$A = \lim_{\Delta\theta\to 0} \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2\ \Delta\theta = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta_i))^2 d\theta$

$$\boldsymbol{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta}$$

<u>Example 24</u>: Find the area under the curve $r^2 = 2a^2cos2\theta$ when $\frac{-\pi}{4} \le \theta \le \frac{\pi}{4}$

Solution:

$$A = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta = 2.\frac{1}{2}\int_{0}^{\frac{\pi}{4}} cos2\theta.2a^2 d\theta$$

$=a^2 \int_0^{\frac{\pi}{4}} cos2\theta.2d\theta = a^2 sin2\theta\Big]_0^{\pi/4} = a^2(sin\frac{2\pi}{4} - sin0)=a^2 sin\frac{\pi}{2} = a^2$

<u>Example 25</u>: Find the area under the curve $r = a(1 + cos\theta)$ when $0 \leq \theta \leq 2\pi$

Solution: $A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = 2.\frac{1}{2}\int_0^{\pi} a^2.(1 + cos\theta)^2 d\theta$

$= a^2 \int_0^{\pi}(1 + 2cos\theta + cos^2\theta)\, d\theta$

$$= a^2 \int_0^{\pi} [1 + 2cos\theta + \frac{1}{2}(1 + cos2\theta)]\, d\theta$$

$$= a^2[\int_0^{\pi} 1.d\theta + 2\int_0^{\pi} cos\theta.d\theta + \frac{1}{2}\int_0^{\pi} 1.d\theta + \frac{1}{4}\int_0^{\pi} cos2\theta\,.2d\theta]$$

$$= a^2 \left[\theta + 2sin\theta + \frac{1}{2}\theta + \frac{1}{4}sin2\theta\right]_0^{\pi}$$

$$= a^2 \left[\pi - 0 + 2(0 - 0) + \frac{1}{2}(\pi - 0) + \frac{1}{4}(0 - 0)\right]$$

$$= a^2 \left[\pi + \frac{1}{2}\pi\right] = \frac{3\pi}{2}a^2$$

Note: To find the area between two curves:
$r_1 = f_1(\theta), r_2 = f_2(\theta)$ *and* α, β are angles of intersection points.

$A = A_2 - A_1 = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta$, where α, β

$$\boldsymbol{A = \int_{\alpha}^{\beta} \frac{1}{2}[r_2^2 - r_1^2]d\theta}$$

<u>Example 26</u>: Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$

Solution:

$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}[r_2^2 - r_1^2]d\theta = 2\int_0^{\pi/2} \frac{1}{2}[r_2^2 - r_1^2]d\theta$

$$A = \int_0^{\pi/2} [1 - (1 - 2\cos\theta + cos^2\theta)]d\theta = \int_0^{\pi/2} (2\cos\theta - cos^2\theta)d\theta$$

$$= \int_0^{\pi/2} (2\cos\theta - \frac{1+cos2\theta}{2}) d\theta = \left[2sin\theta - \frac{\theta}{2} - \frac{sin2\theta}{4}\right]_0^{\pi/2}$$

$$= 2 - \frac{\pi}{4} = 2 - \frac{3.14}{4}$$

<u>Exercises:</u>

1) Find length of the following curves:
   - $r = asin^2\frac{\theta}{2}$ when $-\pi \le \theta \le 0$
   - $r = asin^3\frac{\theta}{3}$ when $0 \le \theta \le \pi$
   - $r = a(1 + cos\theta)$ when $0 \le \theta \le 2\pi$
   - $r = \theta^2$ when $0 \le \theta \le \sqrt{5}$
   - $r = 6(1 + \cos\theta)$ when $0 \le \theta \le \frac{\pi}{2}$
   - $r = 2(1 - \cos\theta)$ when $\frac{\pi}{2} \le \theta \le \pi$
   - $r = acos^3\frac{\theta}{3}$ when $0 \le \theta \le \frac{\pi}{4}$
2) Find the area inside the curve: $r = 4 + 2cos\theta$.
3) Find the area inside the curve: $r = 3acos\theta$ and outside the curve: $r = a(1 + cos\theta)$.
4) Find the area shared inside by the circles: $r = a$ and $r = 2asin\theta$.
5) Find the area of the curve $r^2 = 2a^2sin2$.