جامعة بغداد

كلية التربية للعلوم الصرفة / ابن الهيثم

# التفاضل والتكامل

## قسم الرياضيات

### المرحلة الاولى

### الفصل الاول والثاني

اساتذة المادة

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# Chapter One "The Real Numbers" $\mathbb{R}$ "

## The Subsets of $\mathbb{R}$

{1} Natural Numbers, denoted by $\mathbb{N}$ s.t.

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

{2} Integer Numbers, denoted by $I$ or $\mathbb{Z}$ s.t.

$I$ or $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

{3} Rational Numbers: are all numbers of the form $\frac{p}{q}$, where $p, q$ are integers and $q \neq 0$. and denoted by $\mathbb{Q}$.

$\mathbb{Q} = \{x \in \mathbb{R} ; x = \frac{p}{q}, p, q \in \mathbb{Z} ; q \neq 0\}$

Ex. $\frac{1}{2}, \frac{5}{3}, 0, 2, \frac{56}{10}$, etc.

Note: The Rational Numbers can be written as decimal form ( $\frac{1}{3} = 0.333$ , $\frac{1}{4} = 0.25$ )

{4} Irrational Numbers: A number which is not rational is said to be irrational.

( Ex. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi = 3.14$ ) and denoted by $\mathbb{Q}'$

Note: $\phi \subseteq \mathbb{N} \subset I \subset \mathbb{Q} \subset \mathbb{R}$ , $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$.

## Properties of Real Numbers with Addition $(\mathbb{R}, +)$

let $a, b, c \in \mathbb{R}$ then :-

{1} $a+b \in \mathbb{R}$ (closure)

{2} $a+b = b+a$ (commutative)

{3} $a+(b+c) = (a+b)+c$ (associative)

{4} $a+0 = 0+a = a$ (identity element)

{5} $\exists\, (-a) \in \mathbb{R}$ such that $a+(-a) = (-a)+a = 0$ (additive inverse).

## Properties of Real Numbers with Multiplication $(\mathbb{R}, \cdot)$

Let each of $a, b, c$, are real numbers then :-

{1} $a \cdot b \in \mathbb{R}$ (closure property)
{2} $a \cdot b = b \cdot a$ (commutative)
{3} $a(b \cdot c) = (ab) \cdot c$ (associative)
{4} $1 \cdot a = a \cdot 1 = a$ (Multiplicative identity)

{5} $$\left.\begin{aligned} a \cdot (b+c) &= ab + ac \\ (b+c) \cdot a &= ba + ca \end{aligned}\right\} \text{(Distributive)}$$

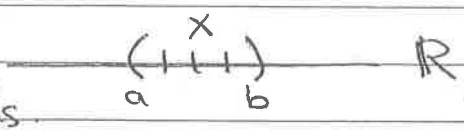
{6} $\exists\, a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = a \cdot \frac{1}{a} = 1$ (Multi. inverse)

## Intervals :-

1- Finite intervals :- Let each of $a, b \in \mathbb{R} \ni a < b$ then :-

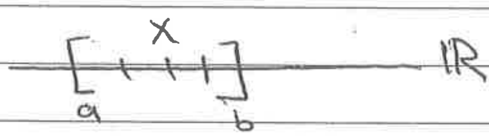
(a) open interval $= \{x \in \mathbb{R} : a < x < b\} = (a, b)$

Note : $a \notin (a,b)$ , $b \notin (a,b)$
$a, b$ is called end points intervals.



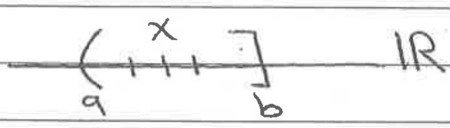

(b) closed interval $= \{x \in \mathbb{R} : a \leq x \leq b\} = [a, b]$

Note : $a \in [a,b]$ , $b \in [a,b]$



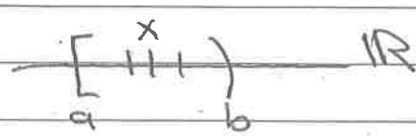

(c) The half open interval =

$\{x \in \mathbb{R} : a < x \leq b\} = (a, b]$




$b \in (a,b]$ , $a \notin (a,b]$

$\{x \in \mathbb{R} : a \leq x < b\} = [a, b)$

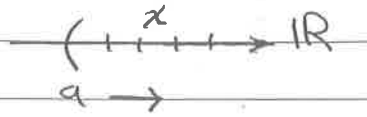



$b \notin [a,b)$ , $a \in [a,b)$

## (2) Infinite intervals :-

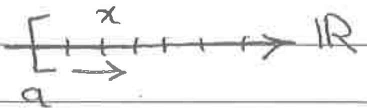
$a < x < \infty$

a) $\{x \in \mathbb{R} : x > a\} = (a, \infty)$



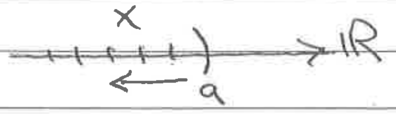

$a \leq x < \infty$

b) $\{x \in \mathbb{R} : x \geq a\} = [a, \infty)$



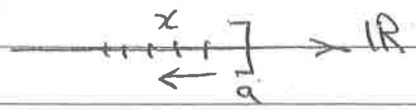

$-\infty < x < a$

c) $\{x \in \mathbb{R} : x < a\} = (-\infty, a)$




$-\infty < x \leq a$

d) $\{x \in \mathbb{R} : x \leq a\} = (-\infty, a]$




e) $\{x \in \mathbb{R} : -\infty < x < \infty\} = (-\infty, \infty) = \mathbb{R}$


**Inequalities** Let each of $a, b \in \mathbb{R}$; a number (b) is greater than a number (a) and denoted by

$$b > a \quad \text{if} \quad b - a > 0 \quad (\text{positive number}).$$

## Solving inequalities :-

Obtaining all values of $x$ for which the inequality is true.

Properties : Let $a, b, c \in \mathbb{R}$ then :

1. if $a < b$ Then $a + c < b + c$
2. if $a < b$ , $c > 0$ Then $a.c < b.c$
3. if $a < b$ , $c < 0$ Then $a.c > b.c$

Linear Inequalities

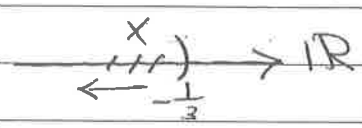
Ex ① : Solve the inequality $3(x+2) < 5$

Sol :- $3(x+2) < 5 \Rightarrow 3x + 6 < 5$

$\Rightarrow 3x < 5 - 6$

$\Rightarrow 3x < -1$

$\Rightarrow x < -\frac{1}{3}$




∴ Solution set : $\{x \in \mathbb{R} : x < -\frac{1}{3}\} = (-\infty, -\frac{1}{3})$

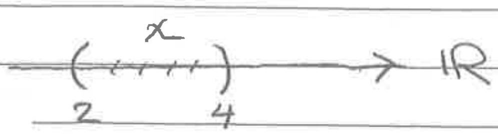
Ex ② Solve the inequality $7 < 2x + 3 < 11$

Sol :- $7 < 2x + 3 < 11 \Rightarrow -3 + 7 < 2x < -3 + 11$

$\Rightarrow 4 < 2x < 8$

$\Rightarrow 2 < x < 4$

∴ The Solution set : $\{x \in \mathbb{R} : 2 < x < 4\} = (2, 4)$



## Nonlinear Inequalities

**Ex①:** Solve the inequality $x^2 < 25$

**Sol** $x^2 < 25 \Rightarrow x^2 - 25 < 0$

$\Rightarrow (x-5)(x+5) < 0$

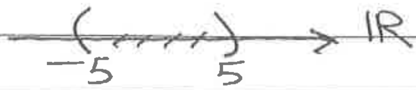
- Since the result is negative then there are two possibilites:

negative $(-) \longrightarrow$ either $(+\cdot-)$
or $(-\cdot+)$

**either**

$(x+5) > 0 \;\wedge\; (x-5) < 0$

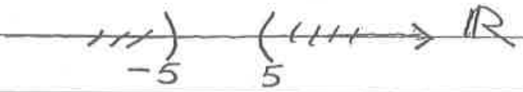
$x > -5 \;\wedge\; x < 5$




$\therefore$ The sol. set $(-5,5)$

**or** $(x+5) < 0 \;\wedge\; (x-5) > 0$

$x < -5 \;\wedge\; x > 5$




$\therefore$ The sol. set $\emptyset$

$\therefore$ The sol. set for inequality is $= (-5,5) \cup \emptyset = (-5,5)$

**Ex②:** Solve The inequality $x^2 - 5x - 6 > 0$

**Sol:** $x^2 - 5x - 6 > 0 \Rightarrow (x-6)(x+1) > 0$

either $(+\cdot+) \longleftarrow$ Positive $(+)$
$(-\cdot-)$ or

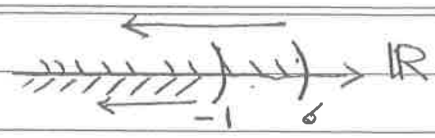
**either** $(x-6) > 0 \;\wedge\; (x+1) > 0$

$x > 6 \;\wedge\; x > -1$

$S_1 = \{x \in \mathbb{R} \,;\, x > 6\} = (6, \infty)$

or : $x-6<0 \wedge x+1<0$

$x<6 \wedge x<-1$



$S_2=\{x\in\mathbb{R} : x<-1\}=(-\infty,-1)$

$\therefore S=S_1\cup S_2$

$=\{x\in\mathbb{R} : x>6 \vee x<-1\}$

$=(6,\infty)\cup(-\infty,-1)$

$=\mathbb{R}\setminus[-1,6]$

## Absolute Value

The absolute value of a real number $x$ is denoted by $|x|$ and is defined by :

$$|x|=\sqrt{x^2}=\begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x & \text{if } x<0 \end{cases}$$

Ex : $|-8|=8$, $|9|=9$, $|0|=0$

### Properties of Absolute value

(1) $|-a|=|a|$

Proof: $|-a|=\sqrt{(-a)^2}=\sqrt{a^2}=|a|$

(2) $||a||=|a|$

Proof: $||a||=\sqrt{|a|^2}=\sqrt{a^2}=|a|$

(3) $|a \cdot b| = |a| \cdot |b|$

Proof

$$|a \cdot b| = \sqrt{(ab)^2} = \sqrt{a^2 \cdot b^2} = \sqrt{a^2} \cdot \sqrt{b^2} = |a| \cdot |b|$$

(4) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ ; $b \neq 0$

Proof :

$$\left|\frac{a}{b}\right| = \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}$$

(5) $|a+b| \leq |a| + |b|$.

Solving Absolute Value Inequalities :

From the previous definition of $|x|$ we reach :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Which means that ~~if~~ when $x = 0 \Rightarrow |x| = x$.

Thus, the absolute value of any real number is a real non-negative number.

Geometrically, the absolute value of number $x$ is the distance point which represents $x$ about the origin point (o).

In general, $|a-b|$ is the distance between $a$, $b$ on the real number line.

- To Solve the inequality $|x|<a$ s.t. a real number and $x \in \mathbb{R}$

if $x \geq 0 \Rightarrow |x| = x$
but $|x|<a \Rightarrow x<a$ , $(-\infty, a)$

if $x<0 \Rightarrow |x| = -x$
but $|x|<a \Rightarrow -x<a \Rightarrow x>-a$ , $(-a, \infty)$

$\therefore S = \{x \in \mathbb{R} : -a<x<a\} = (-a, a)$

$\therefore \{x \in \mathbb{R} : |x|<a\} = \{x \in \mathbb{R} : -a<x<a\} = (-a, a)$

$\{x \in \mathbb{R} : |x| \leq a\} = \{x \in \mathbb{R} : -a \leq x \leq a\} = [-a, a]$

- To solve the inequality $|x|>a$ s.t. $x \in \mathbb{R}$.

if $x \geq 0 \Rightarrow |x| = x$
but $|x|>a \Rightarrow x>a$ , $(a, \infty)$

if $x<0 \Rightarrow |x| = -x$
but $|x|>a \Rightarrow -x>a \Rightarrow x<-a$ , $(-\infty, -a)$

$\therefore S = \{x \in \mathbb{R} : x>a \text{ or } x<-a\}$
$= (a, \infty) \cup (-\infty, -a)$
$= \mathbb{R} \setminus [-a, a]$

But if we have:
$\{x \in \mathbb{R} : |x| \geq a\} = \{x \in \mathbb{R} : x \geq a \text{ or } x \leq -a\}$
$= [a, \infty) \cup (-\infty, -a]$
$= \mathbb{R} \setminus (-a, a)$.

Ex ⊛: Find the solution set for the inequality :

{1} $|x| > 3$

Sol

$\{x \in \mathbb{R} : x > 3 \text{ or } x < -3\} = (3, \infty) \cup (-\infty, -3)$
$= \mathbb{R} \setminus [-3, 3]$

{2} $|x| \leq 4$

Sol $\{x \in \mathbb{R} : |x| \leq 4\} = \{x \in \mathbb{R} : -4 \leq x \leq 4\}$
$= [-4, 4]$

{3} $|x - 4| < 5$

Sol

$\{x : |x-4| < 5\} = \{x : -5 < x-4 < 5\}$
$= \{x : -1 < x < 9\} = (-1, 9)$

{4} $|7 - 4x| \geq 1$

$\{x : |7-4x| \geq 1\} = \{x : 7-4x \geq 1 \text{ or } 7-4x \leq -1\}$
$= \{x : -4x \geq -6 \text{ or } -4x \leq -8\}$
$= \{x : x \leq \frac{3}{2} \text{ or } x \geq 2\}$

$= (-\infty, \frac{3}{2}] \cup [2, \infty)$

$= \mathbb{R} \setminus (\frac{3}{2}, 2)$

## Exercises of chapter One :-

Q1// write the following sets equivalent intervals, and test of these intervals open or closed or half open:-

1- $\{x : -20 \leq x \leq -12\}$ , 2- $\{x : -3 \leq x < 4\}$

3- $\{x : -1 < x < 10\}$ .

Q2// Give a description of the following intervals as sets:-

$(3,5)$ , $(0,6)$ , $[2,7]$ , $[-1,0)$

Q3// Find the Solu. set of the following inequalities :-

1. $x(x-3) > 4$
2. $-2 < \frac{1}{x}$ ; $x \neq 0$
3. $x^2 \geq 25$
4. $x^2 - 2x - 24 < 0$
5. $-7 \leq -3x + 5 \leq 14$
6. $\frac{x}{x-3} < 4$
7. $\frac{x^2+2x-35}{x+2} > 0$
8. $6x - 4 > 7x + 2$
9. $x^2 \leq 16$
10. $3x^2 > 2x + 5$
11. $x^2 > 5x + 6$
12. $\frac{x-3}{x+2} < 5$
13. $\frac{1}{x-2} > \frac{2}{x+3}$

Q4// Find the Solu. set of the following inequalities :-

1) $|x| \geq 5$
2) $|x| < 2$
3) $|3x+3| \geq 2$
4) $1 \leq \left|\frac{x-3}{1-2x}\right| \leq 2$
5) $|x-3|^2 - 4|x-3| = 12$
6) $\left|\frac{2-x}{x-3}\right| \geq 4$
7) $|x+1| < |3x+4|$
8) $\frac{|2-x|}{3x} < 1$
9) $\left|\frac{3+2x}{3x}\right| \leq 1$
10) $|x-1| \geq 6$
11) $|2-2x| \leq 7$
12) $\left|\frac{4}{2x+1}\right| \leq 3$ .

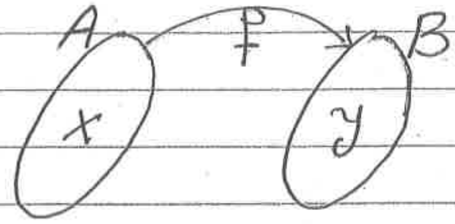
//

# Chapter Two : The functions :

**The functions:** let $A$ and $B$ be two non-empty sets, the relation that assigns to every element $x \in A$, with a unique value $y \in B$ is called a function.

i.e $\boxed{f: A \longrightarrow B \; ; \forall x \in A, \exists!\, y \in B \ni f(x) = y}$

**Notes:**



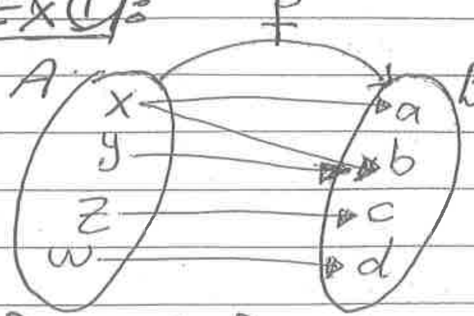

{1} $A = \text{Domain} = D_f$

$B = \text{Co-domain} = \text{Co-}D_f$

{2} the set of all images $f(x) = y$, $\forall x \in D_f$ is called the Rangs of $f$

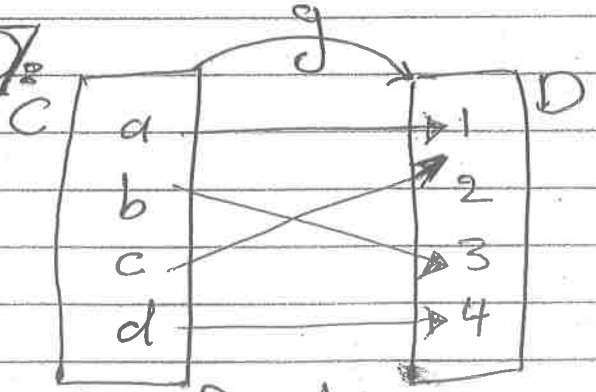
i.e $R_f = \{ f(x) = y \; ; x \in D_f \}$

**Ex (1):**




$f$ is not function since $f(x) = a$ and $f(x) = b$

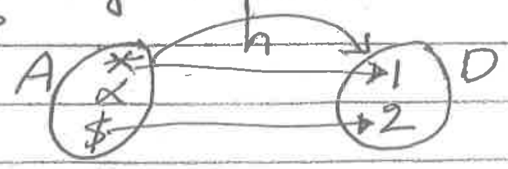
**Ex (2):**




$g$ is a function

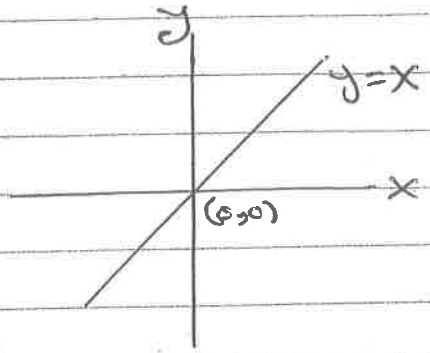
$R_g = \{1, 3, 4\}$

**Ex (3):**




$h$ is not function since $\alpha \in A$ and $\alpha$ hasnot image.

$\boxed{Ex\,4}$: linear function $y = x$



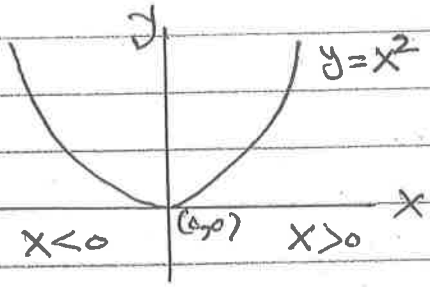

$y = f(x) = x \quad , \quad f: \mathbb{R} \longrightarrow \mathbb{R}$.

$D_f = \mathbb{R} = \{x \in \mathbb{R} ; -\infty < x < \infty\}$

$R_f = \mathbb{R} = \{y \in \mathbb{R} ; -\infty < x < \infty\}$

is function.

$\boxed{Ex\,5}$: $y = x^2$



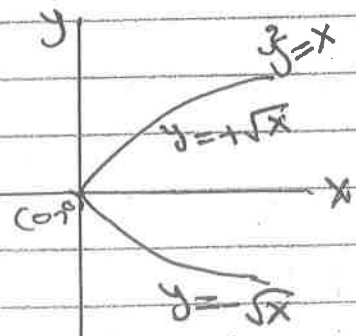

$y = f(x) = x^2$

$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+ = \{y \in \mathbb{R} ; y \geq 0\} = [0, \infty)$

is function.

$\boxed{Ex\,6}$: $y^2 = x$

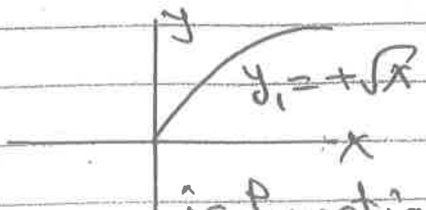



$\sqrt{y^2} = \sqrt{x} \Rightarrow |y| = \sqrt{x} \Longrightarrow y = \mp\sqrt{x}$

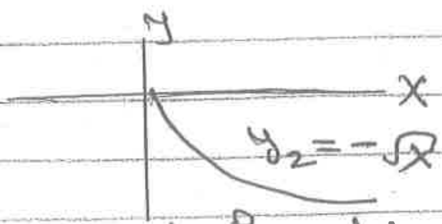
$\forall x \in D_f \; , \exists \mp\sqrt{x}$ two image for every $x$.

not function.

But if




is function




is function

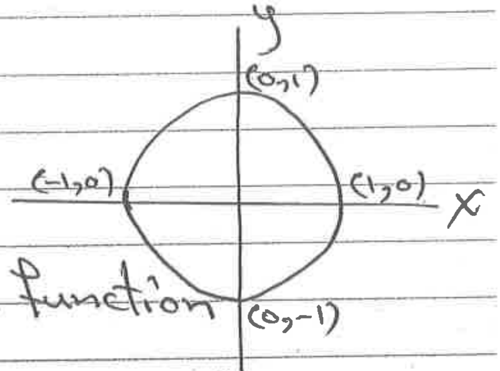
$\boxed{Ex\ 7}$: $2y+3x=5$

$2y=5-3x \Rightarrow y=\dfrac{5-3x}{2}$ function.

$\boxed{Ex\ 8}$: $\dfrac{x}{y}=2$

$y=\dfrac{1}{2}x$ function.

$\boxed{Ex\ 9}$: is $x^2+y^2=1$ function?




$y^2=1-x^2 \Rightarrow y=\mp\sqrt{1-x^2}$ is not function

since $\forall x\in D_f$, $\exists \mp\sqrt{1-x^2}$ two image for every $x$.

Now if

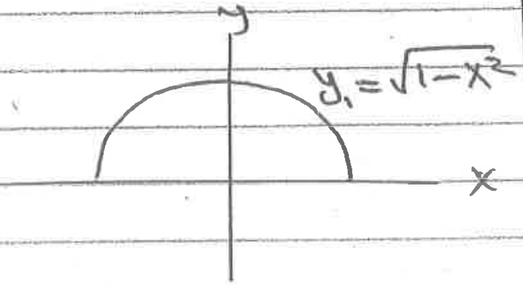
$y_1=f_1(x)=\sqrt{1-x^2}$

$1-x^2\geq 0 \Rightarrow x^2\leq 1 \Rightarrow -1\leq x\leq 1$

$D_{f_1}=[-1,1]$

$R_{f_1}=[0,1]$

is function




$x^2=1-y^2$

$\sqrt{x^2}=\sqrt{1-y^2}$

$x=\mp\sqrt{1-y^2}$

$1-y^2\geq 0$

$y^2\leq 1$

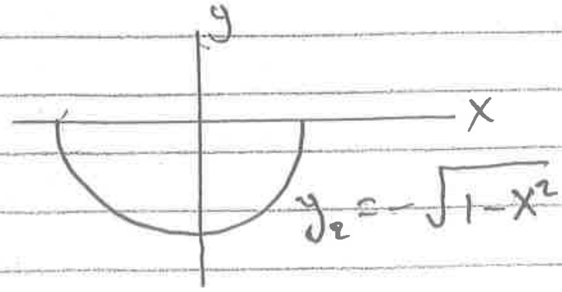
$\underbrace{-1\leq y}_{y_2}\ \overset{0}{}\ \underbrace{\leq 1}_{y_1}$

$y_2=f_2(x)=-\sqrt{1-x^2}$

$D_{f_2}=[-1,1]$

$R_{f_2}=[-1,0]$

is function



How to find the Domain and the Rang of function:

① The domain of all polynomail or odd root is all real numbers.

$\boxed{Ex ①}$ $f(x) = x^3 + 2x^2 + 3x - 5$.

$D_f = \mathbb{R}$ ; $R_f = \mathbb{R}$.

$\boxed{Ex ②}$ $g(x) = \sqrt[3]{x^7 - 1}$.

$D_g = \mathbb{R}$ ; $R_g = \mathbb{R}$.

② the domain of even root such as square roots is all real numbers that the expression under the radical to greater than or equal to zero.

$\boxed{Ex ①}$: $f(x) = \sqrt{x^2 - 4}$

$x^2 - 4 \geqslant 0 \Rightarrow (x-2)(x+2) \geqslant 0$

$x - 2 \geqslant 0 \wedge x + 2 \geqslant 0 \Rightarrow x \geqslant 2 \wedge x \geqslant -2$

$\Rightarrow [2, \infty)$

$x - 2 \leqslant 0 \wedge x + 2 \leqslant 0 \Rightarrow x \leqslant 2 \wedge x \leqslant -2$

$\Rightarrow (-\infty, -2]$

$D_f = (-\infty, -2] \cup [2, \infty) = \mathbb{R} \setminus (-2, 2)$

$y^2 \geqslant 0 \Rightarrow y \in \mathbb{R} \Rightarrow R_f = \mathbb{R}^+$

or $x^2 - 4 \geqslant 0 \Rightarrow x^2 \geqslant 4 \Rightarrow |x| \geqslant 2 \Rightarrow x \geqslant 2$ or $x \leqslant -2$

$D_f = \mathbb{R} \setminus (-2, 2)$

$\boxed{Ex②}$: $g(x) = -\sqrt{2x-1}$

$2x-1 \geq 0 \Rightarrow 2x \geq 1 \Rightarrow x \geq \frac{1}{2} \Rightarrow D_g = [\frac{1}{2}, \infty)$

$y = -\sqrt{2x-1}$

$y^2 = 2x-1$

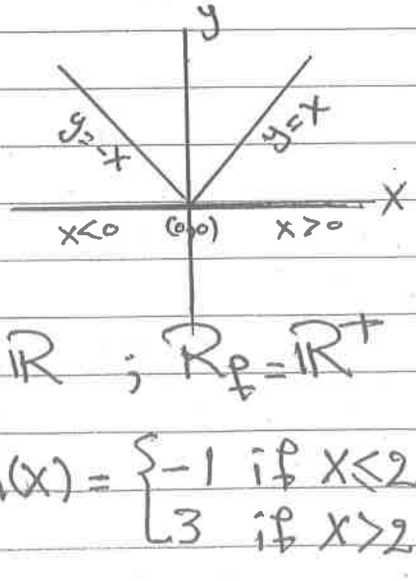
$2x = y^2+1$

$x = \frac{y^2+1}{2}$

$y \in \mathbb{R} \Rightarrow R_g = \mathbb{R}^- = (-\infty, 0]$

③ Piecewis functions that is defined by more than one formule; such functions are written using the brace ($\{$) ; such as signum function, absolute value function ...., etc.
The domain of these function are the restrictions of the functions.

$\boxed{Ex①}$: find the domain of $|x|$.




$$f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

$D_f = \mathbb{R}$ ; $R_f = \mathbb{R}^+$

$\boxed{Ex②}$: find the $D_h$ and $R_h$ $\ni y = h(x) = \begin{cases} -1 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$

$D_h = \mathbb{R}$ ; $R_h = \{-1, 3\}$

$\boxed{Ex\,③}$: $f(x) = y = |x+3|$ find $D_f$ and $R_f$.

$$|x+3| = \begin{cases} x+3 & \text{if } x+3>0 \Rightarrow x>-3 \\ 0 & \text{if } x+3=0 \Rightarrow x=-3 \\ -(x+3) & \text{if } x+3<0 \Rightarrow x<-3 \end{cases}$$

$D_f = \mathbb{R}$ ; $R_f = \mathbb{R}^+$.

$\boxed{Ex\,④}$ $f(x) = \begin{cases} x & \text{if } x<-2 \\ x+1 & \text{if } -2 \leq x \leq 1 \\ x^2 & \text{if } x>1 \end{cases}$ , find $D_f$ and $R_f$.

$x<-2 \vee -2 \leq x \leq 1 \vee x>1$

$(-\infty,-2) \cup [-2,1] \cup (1,\infty) = \mathbb{R} = D_f$

$y<-2 \vee -1 \leq y \leq 2 \vee y>1$

$(-\infty,-2) \cup [-1,2] \cup (1,\infty) = \mathbb{R} \setminus [-2,-1) = R_f$

$\boxed{Ex\,⑤}$: $f(t) = |t-2|$ , find $D_f$ and $R_f$.

$$|t-2| = \begin{cases} t-2 & \text{if } t>2 \\ 0 & \text{if } t=2 \\ 2-t & \text{if } t<2 \end{cases}$$

$D_f = \mathbb{R}$ ; $R_f = \mathbb{R}^+$.

④ The domain of Rational function is all real number except the value of $x$ which make the denominator zero.

i.e $\dfrac{\text{Numerator}}{\text{denominator}} = \dfrac{①\ ,\ ②\ ,\ ③}{①\ ,\ ②\ ,\ ③}$

$x \neq 0$ , $x \neq 0$ , $x \neq 0$

$D_{rational} = D_{Numerator} \cap D_{denominator}$.

$\boxed{Ex①}$: $f(x) = \dfrac{x}{x^2-1}$

$x^2-1 \neq 0 \Rightarrow x^2 \neq 1 \Rightarrow \sqrt{x^2} \neq 1 \Rightarrow |x| \neq 1 \Rightarrow x \neq \mp 1$

$D_f = \mathbb{R} \setminus \{-1, 1\}$

$y = \dfrac{x}{x^2-1} \Rightarrow x = yx^2 - y \Rightarrow \underset{a}{\underline{y}}x^2 \underset{b}{\underline{-}}x \underset{c}{\underline{-y}} = 0$

$x = \dfrac{1 \mp \sqrt{1+4y^2}}{2y}$ ; $x = \dfrac{-b \mp \sqrt{b^2-4ac}}{2a}$

$2y \neq 0 \Rightarrow y \neq 0$

$1+4y^2 \geq 0 \Rightarrow y^2 \geq -\frac{1}{4} \Rightarrow y^2 \geq 0$

$y \in \mathbb{R}$

$R_f = \mathbb{R} \setminus \{0\}$

$\boxed{Ex②}$: $g(x) = \dfrac{2-x}{\sqrt{1-x}}$

$1-x > 0 \Rightarrow 1 > x$ , $D_f = (-\infty, 1)$.

$y = \dfrac{2-x}{\sqrt{1-x}} \Rightarrow y\sqrt{1-x} = 2-x$

$y^2(1-x) = (2-x)^2$

$y^2 - y^2x = 4 - 4x + x^2$

$x^2 - 4x + y^2x + 4 - y^2 = 0$

$\underbrace{x^2}_{ax^2} + \underbrace{(-4+y^2)x}_{bx} + \underbrace{(4-y^2)}_{c} = 0$

$x = \dfrac{-(-4+y^2) \mp \sqrt{(-4+y^2)^2 - 4(1)(4-y^2)}}{2(1)}$

$= \dfrac{4-y^2 \mp \sqrt{16-8y^2+y^4-16+4y^2}}{2}$

$$= \frac{4-y^2 \mp \sqrt{y^4-4y^2}}{2}$$

$$y^4 - 4y^2 \geqslant 0 \Rightarrow y^2(y^2-4) \geqslant 0$$

$+ - + - +$

$$y^2 \geqslant 0 \wedge y^2 - 4 \geqslant 0 \Rightarrow y^2 \geqslant 0 \wedge y^2 \geqslant 4$$
$$\Rightarrow y \in \mathbb{R} \wedge |y| \geqslant 2$$

$$R_f = [2,\infty) \cup (-\infty,-2] = \mathbb{R} \setminus (-2,2).$$

$\boxed{Ex③}$: $h(x) = \sqrt[3]{\frac{x+1}{x-2}}$

$$\sqrt[3]{x-2} \neq 0 \Rightarrow x-2 \neq 0 \Rightarrow x \neq 2 \quad ; \quad D_h = \mathbb{R} \setminus \{2\}.$$

$$y^3 = \frac{x+1}{x-2} \Rightarrow (x-2)y^3 = x+1$$
$$xy^3 - 2y^3 - x - 1 = 0$$
$$(y^3-1)x = 2y^3+1$$
$$x = \frac{2y^3+1}{y^3-1}$$

$$y^3 - 1 \neq 0 \Rightarrow y^3 \neq 1 \Rightarrow y \neq 1$$

$$R_h = \mathbb{R} \setminus \{1\}.$$

$\boxed{Ex④}$: Find the Domains and Ranges for the following functions:

$$y = f(x) = \frac{1}{x^2+1} + 3$$

$$x^2+1 \neq 0 \Rightarrow x^2 \neq -1 \Rightarrow D_f = \mathbb{R}.$$

$$y = \frac{1}{x^2+1} + 3 \Rightarrow y-3 = \frac{1}{x^2+1}$$

$$(y-3)(x^2+1) = 1$$

$$yx^2 + y - 3x^2 - 3 = 1$$

$$(y-3)x^2 + y - 3 = 1$$

$$(y-3)x^2 = 4-y$$

$$x^2 = \frac{4-y}{y-3}$$

$$x = \mp\sqrt{\frac{4-y}{y-3}}$$

$$\frac{+\ \geq 0}{+\ > 0} \to +$$

$$\frac{-\ \leq 0}{-\ < 0}$$

$$4-y \geq 0 \wedge y-3 > 0 \quad \vee \quad 4-y \leq 0 \wedge y-3 < 0$$

$$y \leq 4 \wedge y > 3 \quad \vee \quad y \geq 4 \wedge y < 3$$

3 4 3 4

$$(3,4] \quad \cup \quad \emptyset \quad = (3,4] = R_f$$

**Ex 5** Find the domain and range of the following function :-

$$f(x) = \sqrt{x^2+25}$$

**Sol** : $x^2+25 \geqslant 0 \Rightarrow x^2 \geqslant -25 \Rightarrow x^2 \geqslant 0$

$\therefore D_f = \mathbb{R}$

$R_f = ?$

$$y = \sqrt{x^2+25}$$

$$y^2 = x^2+25$$

$$x^2 = y^2-25$$

$$|x| = \sqrt{y^2-25}$$

$$x = \mp\sqrt{y^2-25}$$

$$y^2-25 \geqslant 0 \Rightarrow y^2 \geqslant 25 \Rightarrow |y| \geqslant 5$$

$$y \geqslant 5 \quad \vee \quad \underbrace{y \leqslant -5}_{\text{رفض}}$$

$\therefore D_f = [5, \infty)$ .

Homework: find the domains and ranges for the following function.

① $f(x) = \sqrt{\frac{1}{x} - 2}$ ② $h(x) = \frac{\sqrt{x+1}}{x-1}$

③ $g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$ ④ $l(x) = \frac{x+1}{|x-5|}$

## Algebraic of function

let $f$ and $g$ be two functions, then:-

{1} Equality of functions:

$f$ and $g$ are equality iff ① $D_f = D_g$
and ② $f(x) = g(x)$

{2} The sum of functions:

The sum of $f$ and $g$ is anew function with the domain $D_f \cap D_g = D_{f+g}$ and

$$(f+g)(x) = f(x) + g(x).$$

{3} the difference of functions:-

the difference between $f$ and $g$ is anew function with the domain $D_{f-g} = D_f \cap D_g$ and

$$(f-g)(x) = f(x) - g(x).$$

## {4} The Product of functions:

Is anew function with the domain $D_{f.g} = D_f \cap D_g$ and

$$(f.g)(x) = f(x) . g(x).$$

## {5} The division of functions:

Is anew function with domain

$$D_{f/g} = D_f \cap D_g \setminus \{x \in R ; g(x) = 0\}$$

and $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$D_{g/f} = D_f \cap D_g \setminus \{x \in \mathbb{R} ; f(x) = 0\}$$

and $$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$$

$\boxed{Ex}$: Which of the following functions are equal to the function $f(x) = \frac{x - 2x^2}{x}$

① $g(x) = 1 - 2x$

$D_g = \mathbb{R}$ ; $D_f = \mathbb{R} \setminus \{0\}$

$D_f \neq D_g \implies f \neq g$.

② $h(x) = \frac{x^2 - 2x^3}{x^2}$

$D_h = \mathbb{R} \setminus \{0\}$.

$$h(x) = \frac{x^2 - 2x^3}{x^2} = \frac{x(x-2x^2)}{x \cdot x} = \frac{x-2x^2}{x} = f(x)$$

$\therefore\ h = f$.

③ $l(x) = \sqrt{1-4x+4x^2}$

$$\sqrt{1-4x+4x^2} = \sqrt{(1-2x)(1-2x)} = \sqrt{(1-2x)^2} = |1-2x|$$

$D_l = \mathbb{R}$. $\Rightarrow f \neq l$.

④ $w(x) = \dfrac{(x^3+x)(1-2x)}{x(1+x^2)}$

$x(1+x^2) \neq 0 \Rightarrow x \neq 0 \vee 1+x^2 \neq 0 \Rightarrow D_w = \mathbb{R} \setminus \{0\}$
$x^2 \neq -1$

$$w(x) = \frac{(x^3+x)(1-2x)}{x(1+x^2)} = \frac{x(x^2+1)(1-2x)}{x(1+x^2)} = \frac{x-2x^2}{x} = f(x)$$

$\therefore\ f = w$.

Homework: Are the following functions equals:

① $f(x) = \dfrac{2x^2+4x}{6x^2}$ , $g(x) = \dfrac{6x^3+12x^2}{6x^3}$

② $V(x) = \dfrac{\sqrt{x+1}}{x^3}$ , $w(x) = \dfrac{\sqrt[3]{x^2-1}}{\sqrt{x^2}}$

③ $h(x) = \dfrac{2x^2+3x^{-2}}{8x}$ , $l(x) = \dfrac{2x^3+3x^{-1}}{8x^2}$

$\boxed{Ex}$: If $f(x)=\sqrt{x+1}$, $g(x)=\sqrt{4-x^2}$ find $f+g$, $f-g$, $f.g$, $f/g$, $g/f$ with domain for all.

Sol: $f(x)=\sqrt{x+1}$

$x+1 \geq 0 \Rightarrow x \geq -1 \Rightarrow D_f=[-1,\infty)$

$g(x)=\sqrt{4-x^2}$

$4-x^2 \geq 0 \Rightarrow x^2 \leq 4 \Rightarrow |x| \leq 2$

$\Rightarrow -2 \leq x \leq 2 \Rightarrow D_g=[-2,2]$

$(f+g)(x) = f(x)+g(x) = \sqrt{x+1}+\sqrt{4-x^2}$

$(f-g)(x) = f(x)-g(x) = \sqrt{x+1}-\sqrt{4-x^2}$

$(f.g)(x) = f(x).g(x) = \sqrt{x+1}.\sqrt{4-x^2} = \sqrt{(x+1)(4-x^2)}$

$D_{f+g} = D_{f-g} = D_{f.g} = D_f \cap D_g = [-1,\infty) \cap [-2,2]$
$= [-1,2]$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{\sqrt{4-x^2}} = \sqrt{\frac{x+1}{4-x^2}}$

$D_{f/g} = D_f \cap D_g \setminus \{x\in\mathbb{R} ; g(x)=0\}$

$= [-1,2] \setminus \{x\in\mathbb{R} ; \sqrt{4-x^2}=0\}$

$= [-1,2] \setminus \{-2,2\} = [-1,2)$

$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4-x^2}}{\sqrt{x+1}} = \sqrt{\frac{4-x^2}{x+1}}$

$D_{g/f} = D_f \cap D_g \setminus \{x\in\mathbb{R} ; f(x)=0\}$

$= [-1,2] \setminus \{x\in\mathbb{R} ; \sqrt{x+1}=0\}$

$= [-1,2] \setminus \{-1\} = (-1,2]$

## Composition Function :

Let f, g be two functions, be such that the range of(g) subset of domain (f) then exist function (fog) define the following formula:

$$(f\circ g)(x) = f(g(x))$$

$$Dom(f\circ g) = \{x : g(x) \in Dom\, f \wedge x \in Dom\, g\}$$

To discuss $(g\circ f)$ : let f, g be two function be such that $R_f \subseteq D_g$ then :-

$$(g\circ f)(x) = g(f(x))$$

$$Dom(g\circ f) = \{x : f(x) \in Dom\, g \wedge x \in Dom\, f\}$$

**Note** : $f\circ g \neq g\circ f$

**Ex ①**: Let $f(x) = \sqrt{x}$ , $g(x) = x^2+1$ , Find

$f\circ g$ , $g\circ f$ ?

**Sol.** : $f(x) = \sqrt{x}$ , $D_f = \mathbb{R}^+ = [0, \infty)$

$y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow R_f = \mathbb{R}^+ = [0, \infty)$

$g(x) = x^2+1$ , $D_g = \mathbb{R}$

$y = g(x) = x^2+1 \Rightarrow y = x^2+1 \Rightarrow x^2 = y-1$

$\Rightarrow x = \mp\sqrt{y-1}$

$\Rightarrow y-1 \geq 0 \Rightarrow y \geq 1$ , $R_g = [1, \infty)$

To find $f \circ g$ must be $R_g \subseteq D_f$ ?

$[1, \infty) \subseteq [0, \infty) \Rightarrow \therefore f \circ g$ exist.

$\therefore (f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^2+1}$

$(f \circ g)(x) = \sqrt{x^2+1}$

$D_{f \circ g} = \{x : x \in D_g \wedge g(x) \in D_f\}$

$= \{x : x \in \mathbb{R} \wedge x^2+1 \in \mathbb{R}^+\}$

$= \{x : x \in \mathbb{R} \wedge x \in \mathbb{R}\} = \mathbb{R}$

But since $x^2+1 \geq 0 \Rightarrow x^2 \geq -1 \Rightarrow x^2 \geq 0 \Rightarrow x \in \mathbb{R}$

And to find $g \circ f$ must be $R_f \subseteq D_g$ ?

$\mathbb{R}^+ \subseteq \mathbb{R} \Longrightarrow \therefore g \circ f$ exist.

$\therefore (g \circ f)(x) = g(f(x)) = (\sqrt{x})^2 + 1 = x + 1$

$(g \circ f)(x) = x + 1$

$D_{g \circ f} = \{x : x \in D_f \wedge f(x) \in D_g\}$

$= \{x : x \in \mathbb{R}^+ \wedge \sqrt{x} \in \mathbb{R}\}$

$= \{x : x \in \mathbb{R}^+ \wedge x \in \mathbb{R}^+\} = \mathbb{R}^+$

$x \geq 0 \Rightarrow x \in \mathbb{R}^+$

Ex ② : Let $f(x) = \sqrt{x-4}$ , $g(x) = \dfrac{x+1}{3-x}$ , find $f \circ g$ , $g \circ f$ ?

Sol $f(x) = \sqrt{x-4}$

$x - 4 \geq 0 \Rightarrow x \geq 4$ , $D_f = [4, \infty)$

$y = \sqrt{x-4} \Rightarrow y^2 = x-4 \Rightarrow x = y^2 + 4$ , $R_f = \mathbb{R}^+$

$g(x) = \dfrac{x+1}{3-x}$

$3 - x \neq 0 \Rightarrow x \neq 3$ , $D_g = \mathbb{R} \setminus \{3\}$

$y = \dfrac{x+1}{3-x} \Rightarrow x+1 = y(3-x) \Rightarrow x+1 = 3y - xy$

$\Rightarrow x + xy = 3y - 1$

$\Rightarrow x = \dfrac{3y-1}{1+y}$

$1 + y \neq 0 \Rightarrow y = -1$

$\therefore R_g = \mathbb{R} \setminus \{-1\}$

Now To find $f \circ g$

is $R_g \subseteq D_f$ ?

$\mathbb{R} \setminus \{-1\} \not\subseteq [4, \infty) \Rightarrow \therefore f \circ g$ is not exist

To find $g \circ f$ is $R_f \subseteq D_g$ ?

$\mathbb{R}^+ \not\subseteq \mathbb{R} \setminus \{3\} \Rightarrow \therefore g \circ f$ is not exist.

Ex ③: Let $f(x) = |x|$, $g(x) = -x$, find $f \circ g$, $g \circ f$.

Sol: $f(x) = |x|$

$D_f = \mathbb{R}$, $R_f = \mathbb{R}^+$

$g(x) = -x$

$D_g = \mathbb{R}$, $R_g = \mathbb{R}$

Now to find $f \circ g$ is $R_g \subseteq D_f$ ?

$\mathbb{R} \subseteq \mathbb{R} \Rightarrow f \circ g$ is exist.

$(f \circ g)(x) = f(g(x))$

$= f(-x)$

$= |-x|$

$D_{f \circ g} = \{x \in \mathbb{R} ; x \in D_g \wedge g(x) \in D_f\}$

$= \{x \in \mathbb{R} ; x \in \mathbb{R} \wedge -x \in \mathbb{R}\}$

$= \mathbb{R}$

Now to find $g \circ f$ is $R_f \subseteq D_g$ ?

$\mathbb{R}^+ \subseteq \mathbb{R} \Rightarrow g \circ f$ is exist.

$(g \circ f)(x) = g(f(x)) = g(|x|) = -|x|$

$D_{g \circ f} = \{x \in \mathbb{R}, x \in D_f \wedge f(x) \in D_g\}$

$= \{x \in \mathbb{R}, x \in \mathbb{R} \wedge |x| \in \mathbb{R}\} = \mathbb{R}$.

Ex ④: Let $f(x) = \frac{x}{x+2}$ , $g(x) = \frac{x-1}{x}$ , find $f\circ g$ , $g\circ f$.

Sol: $f(x) = \frac{x}{x+2}$ , $x+2 \neq 0 \Rightarrow x \neq -2$

$D_f = \mathbb{R} \setminus \{-2\}$

$y = \frac{x}{x+2} \Rightarrow x = y(x+2)$

$\Rightarrow x = yx + 2y$

$\Rightarrow x - yx = 2y \Rightarrow x = \frac{2y}{1-y}$

$1-y \neq 0 \Rightarrow y \neq 1$

$R_f = \mathbb{R} \setminus \{1\}$

$g(x) = \frac{x-1}{x}$ , $x \neq 0$ , $D_g = \mathbb{R} \setminus \{0\}$

$y = \frac{x-1}{x} \Rightarrow x-1 = yx$

$x - yx = 1$

$x = \frac{1}{1-y}$ , $1-y \neq 0 \Rightarrow y \neq 1$

$\therefore R_g = \mathbb{R} \setminus \{1\}$

Now to find $f\circ g$ is $R_g \subseteq D_f$ ?

$\mathbb{R} \setminus \{1\} \not\subseteq \mathbb{R} \setminus \{-2\} \Rightarrow f\circ g$ is not exist.

Now to find $g\circ f$ is $R_f \subseteq D_g$ ?

$\mathbb{R} \setminus \{1\} \not\subseteq \mathbb{R} \setminus \{0\} \Rightarrow g\circ f$ is not exist.

Ex ⑤: Let $f(t) = t^2$, find two functions $g, h$ such that:

$(f \circ g)(x) = (f \circ h)(x) = x^2 - 10x + 25$.

Sol: $(f \circ g)(x) = f(g(x)) = (g(x))^2$ ; since $f(t) = t^2$

but $(f \circ g)(x) = x^2 - 10x + 25 = (x-5)^2$

$\therefore (g(x))^2 = (x-5)^2$

$\Rightarrow (g(x))^2 - (x-5)^2 = 0$

$\Rightarrow [g(x) - (x-5)][g(x) + (x-5)] = 0$

either $g(x) - (x-5) = 0 \Rightarrow g(x) = x-5$

or $g(x) + (x-5) = 0 \Rightarrow g(x) = -(x-5)$

Since $(f \circ g)(x) = (f \circ h)(x)$

$\Rightarrow g(x) = x-5$ , $h(x) = -(x-5)$

$(f \circ g)(x) = f(g(x)) = f(x-5) = (x-5)^2 = x^2 - 10x + 25$

$(f \circ h)(x) = f(h(x)) = f(-(x-5)) = [-(x-5)]^2 = x^2 - 10x + 25$

$\therefore g(x) = h(x) = \mp(x-5)$.

Exercises: Q1/ Find $f \circ g$, $g \circ f$ for the following functions:

1. $f(x) = \sqrt{x-1}$ , $g(x) = \sqrt{1-x}$
2. $f(x) = x+1$ , $g(x) = 2x$
3. $f(x) = -\sqrt{x}$ , $g(x) = x^2+1$
4. $f(x) = 2x+4$ , $g(x) = \frac{1}{2}x - 2$
5. $f(x) = x^2$ , $g(x) = 2x+3$
6. $f(x) = x^3$ , $g(x) = \sqrt{1-x}$

Q2// If the function $f(x) = 2x+1$ , find $g$ such that

$$(f \circ g)(x) = x^3 .$$

Home work: find each of $f+g$, $f-g$, $f\cdot g$, $f/g$, $g/f$ then find the domain of each them.

① $f(x) = x^2$ , $g(x) = x+1$

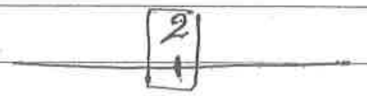
② $f(x) = x^3 + x$ , $g(x) = \dfrac{1}{\sqrt{x+1}}$

③ $f(x) = \dfrac{x}{x+1}$ , $g(x) = \dfrac{x-1}{x}$

## The greatest integer function:

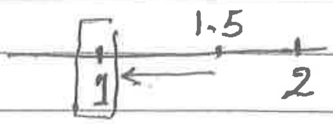
The function whose value at any number $x$ is the greatest integer less than or equal to $x$ is called the greatest integer function, denoted by $[\ ]$ s.t $[x] \leq x$

$\boxed{Ex}$: $[2] = 2$

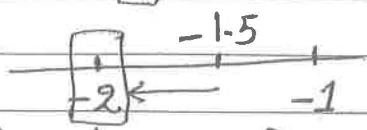



$[\text{integer}] = \text{integer}$

$[1.5] = 1$

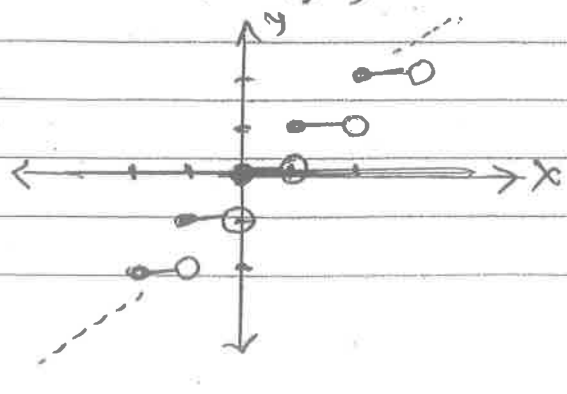



$[-1.5] = -2$




The graph to this function $f(x) = [x] = n$ , $\forall x \in [n, n+1)$ , $\forall n \in I$

| $x$ | $[x]$ | closed | open |
|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ |
| $-2 \leq x < -1$ | $-2$ | $(-2,-2)$ | $(-1,-2)$ |
| $-1 \leq x < 0$ | $-1$ | $(-1,-1)$ | $(0,-1)$ |
| $0 \leq x < 1$ | $0$ | $(0,0)$ | $(1,0)$ |
| $1 \leq x < 2$ | $1$ | $(1,1)$ | $(2,1)$ |
| ⋮ | ⋮ | ⋮ | ⋮ |



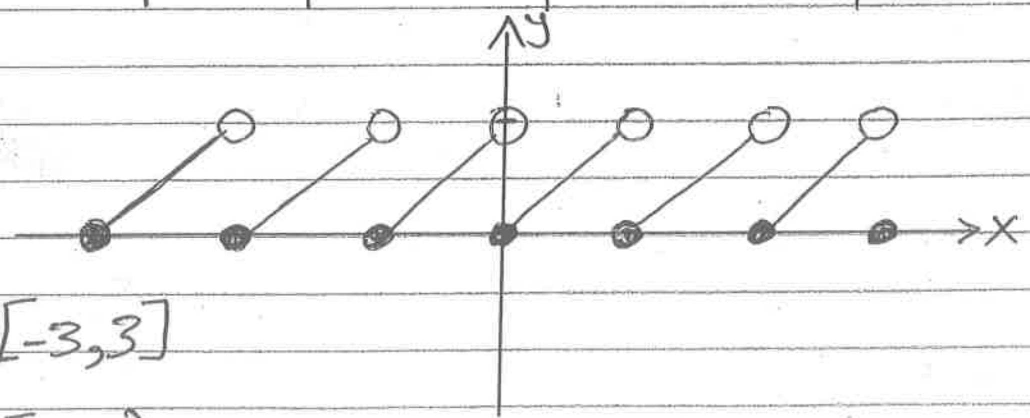
It is called also step function.

$D_f = \mathbb{R}$ , $R_f = I$

Ex: sketch Graph the following function.

{1} $f(x) = x - [x]$ for $-3 \leq x \leq 3$

| $x$ | $[x]$ | $y = x - [x]$ | closed Point | open Point |
|---|---|---|---|---|
| $-3 \leq x < -2$ | $-3$ | $x+3$ | $(-3,0)$ | $(-2,1)$ |
| $-2 \leq x < -1$ | $-2$ | $x+2$ | $(-2,0)$ | $(-1,1)$ |
| $-1 \leq x < 0$ | $-1$ | $x+1$ | $(-1,0)$ | $(0,1)$ |
| $0 \leq x < 1$ | $0$ | $x$ | $(0,0)$ | $(1,1)$ |
| $1 \leq x < 2$ | $1$ | $x-1$ | $(1,0)$ | $(2,1)$ |
| $2 \leq x < 3$ | $2$ | $x-2$ | $(2,0)$ | $(3,1)$ |
| $3 = x$ | $3$ | $x-3$ | $(3,0)$ | |

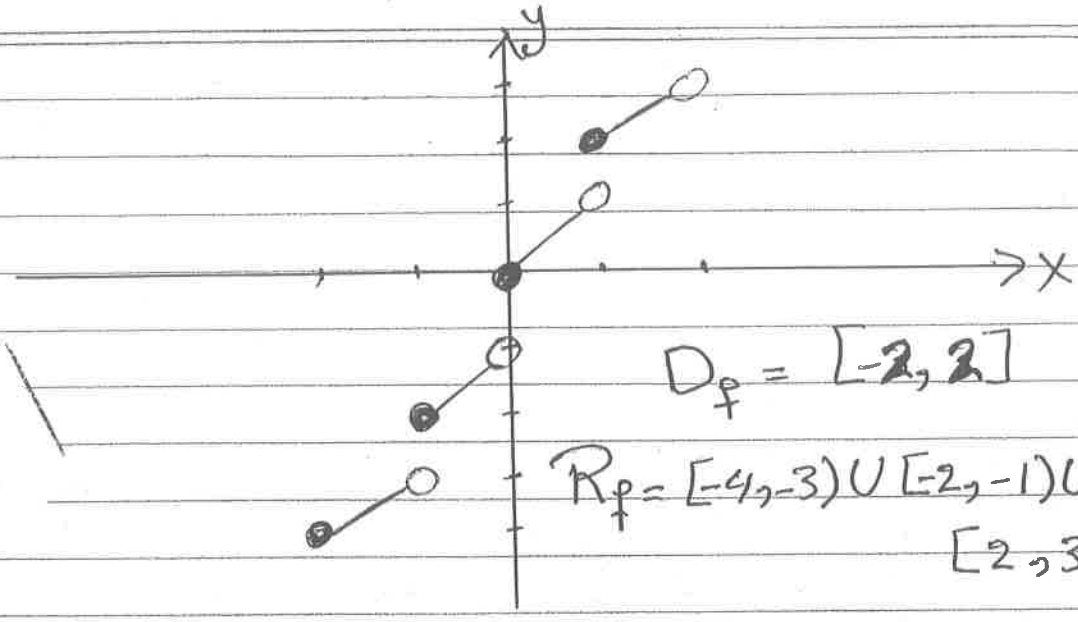



$D_f = [-3,3]$

$R_f = [0,1)$

{2} $g(x) = x + [x]$ for $-2 \leq x < 2$

| $x$ | $[x]$ | $y = x + [x]$ | closed Point | open point. |
|---|---|---|---|---|
| $-2 \leq x < -1$ | $-2$ | $x-2$ | $(-2,-4)$ | $(-1,-3)$ |
| $-1 \leq x < 0$ | $-1$ | $x-1$ | $(-1,-2)$ | $(0,-1)$ |
| $0 \leq x < 1$ | $0$ | $x$ | $(0,0)$ | $(1,1)$ |
| $1 \leq x < 2$ | $1$ | $x+1$ | $(1,2)$ | $(2,3)$ |

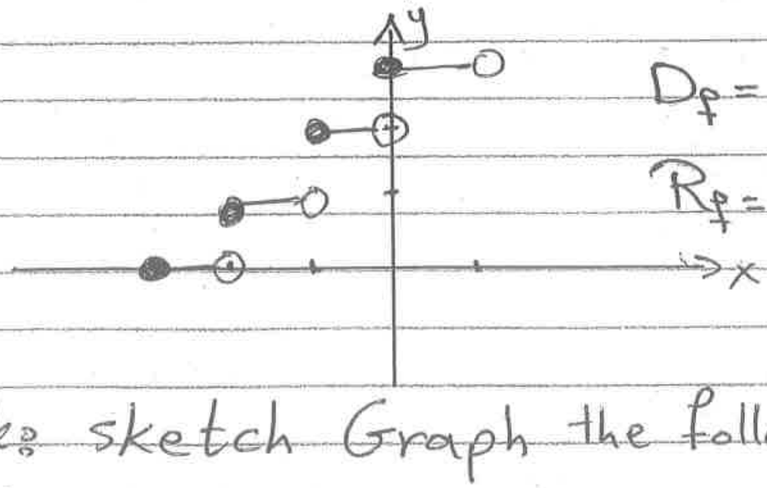


$D_f = [-2, 2]$

$R_f = [-4, -3) \cup [-2, -1) \cup [0, 1) \cup [2, 3)$.

③ $h(x) = [3 + x]$ for $-3 \leq x < 1$

| $x$ | $3+x$ | $y = [3+x]$ | closed Point | open Point. |
|---|---|---|---|---|
| $-3 \leq x < -2$ | $0 \leq 3+x < 1$ | 0 | $(-3, 0)$ | $(-2, 0)$ |
| $-2 \leq x < -1$ | $1 \leq 3+x < 2$ | 1 | $(-2, 1)$ | $(-1, 1)$ |
| $-1 \leq x < 0$ | $2 \leq 3+x < 3$ | 2 | $(-1, 2)$ | $(0, 2)$ |
| $0 \leq x < 1$ | $3 \leq 3+x < 4$ | 3 | $(0, 3)$ | $(1, 3)$ |



$D_f = [-3, 3)$

$R_f = \{0, 1, 2, 3\}$

Home work: sketch Graph the following function:

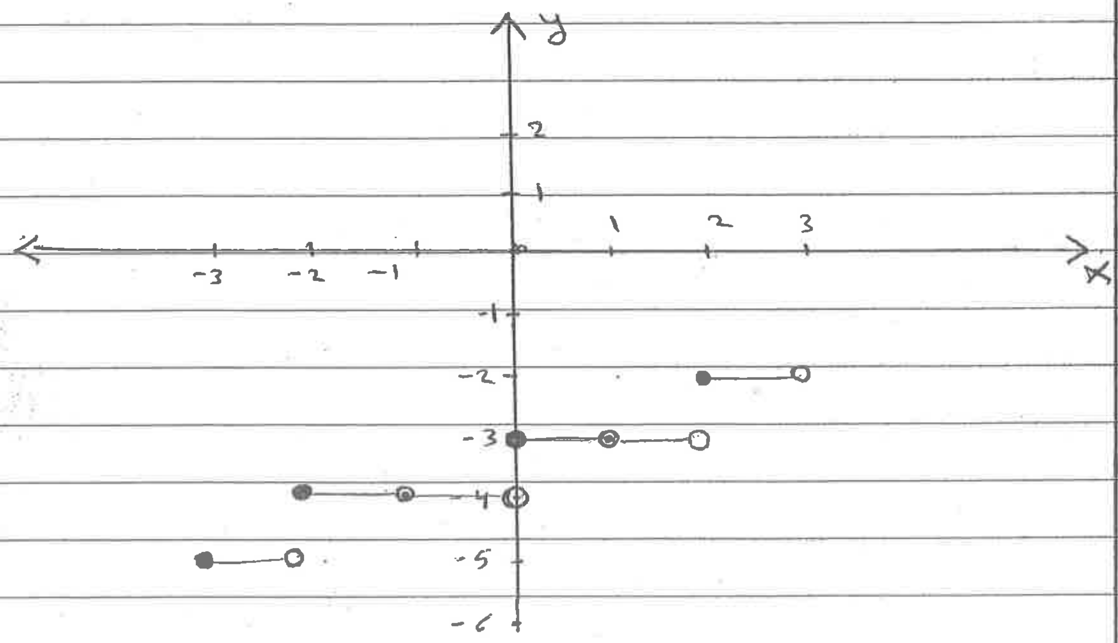
① $[x - 4]$ for $-2 \leq x \leq 2$

② $\left[\frac{x}{2}\right] - 1$ for $-2 \leq x < 2$

Ex :- Sketch Graph the following functions :-

(1) $f(x) = \left[\frac{x}{2}\right] - 3$ for $-3 \leq x \leq 3$

| $x$ | $\frac{x}{2}$ | $\left[\frac{x}{2}\right]$ | $y = \left[\frac{x}{2}\right] - 3$ | closed Point | open Point |
|---|---|---|---|---|---|
| $-3 \leq x \leq -2$ | $-3/2 \leq x/2 \leq -1$ | $-2$ | $-5$ | $(-3,-5)$ | $(-2,-5)$ |
| $-2 \leq x \leq -1$ | $-1 \leq x/2 \leq -1/2$ | $-1$ | $-4$ | $(-2,-4)$ | $(-1,-4)$ |
| $-1 \leq x \leq 0$ | $-1/2 \leq x/2 \leq 0$ | $-1$ | $-4$ | $(-1,-4)$ | $(0,-4)$ |
| $0 \leq x \leq 1$ | $0 \leq x/2 \leq 1/2$ | $0$ | $-3$ | $(0,-3)$ | $(1,-3)$ |
| $1 \leq x \leq 2$ | $1/2 \leq x/2 \leq 1$ | $0$ | $-3$ | $(1,-3)$ | $(2,-3)$ |
| $2 \leq x \leq 3$ | $1 \leq x/2 \leq 3/2$ | $1$ | $-2$ | $(2,-2)$ | $(3,-2)$ |




$D_f = [-3, 3)$

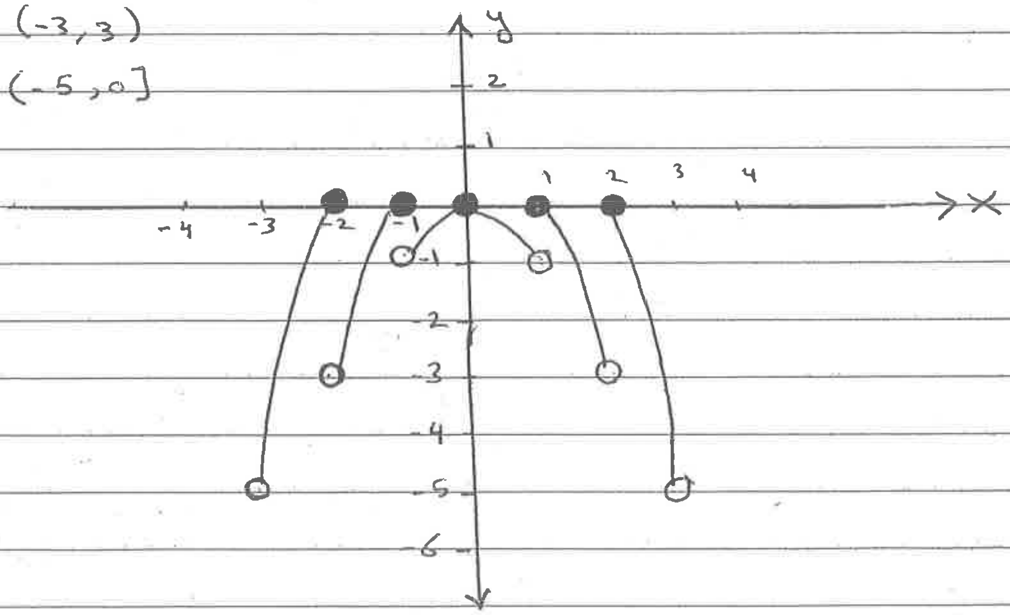
$R_f = \{-5, -4, -3, -2\}$.

(23) $f(x) = [x^2] - x^2$ for $-3 < x < 3$

| $x$ | $x^2$ | $[x^2]$ | $y = [x^2] - x^2$ | open point | |
|---|---|---|---|---|---|
| $-3 < x < -2$ | $4 < x^2 < 9$ | 4 | $4 - x^2$ | $(-3, -5)$ | $(-2, 0)$ |
| $-2 < x < -1$ | $1 < x^2 < 4$ | 1 | $1 - x^2$ | $(-2, -3)$ | $(-1, 0)$ |
| $-1 < x < 0$ | $0 < x^2 < 1$ | 0 | $-x^2$ | $(-1, -1)$ | $(0, 0)$ |
| $0 < x < 1$ | $0 < x^2 < 1$ | 0 | $-x^2$ | $(0, 0)$ | $(1, -1)$ |
| $1 < x < 2$ | $1 < x^2 < 4$ | 1 | $1 - x^2$ | $(1, 0)$ | $(2, -3)$ |
| $2 < x < 3$ | $4 < x^2 < 9$ | 4 | $4 - x^2$ | $(2, 0)$ | $(3, -5)$ |

$D_f = (-3, 3)$

$R_f = (-5, 0]$




closed point :

$x = -2 \Rightarrow y = [(-2)^2] - (-2)^2 = 4 - 4 = 0 \Rightarrow (-2, 0)$

$x = -1 \Rightarrow y = [(-1)^2] - (-1)^2 = 1 - 1 = 0 \Rightarrow (-1, 0)$

$x = 0 \Rightarrow y = [0] - 0 = 0 - 0 = 0 \Rightarrow (0, 0)$

$x = 1 \Rightarrow y = [(1)^2] - (1)^2 = 1 - 1 = 0 \Rightarrow (1, 0)$

$x = 2 \Rightarrow y = [(2)^2] - (2)^2 = 4 - 4 = 0 \Rightarrow (2, 0)$

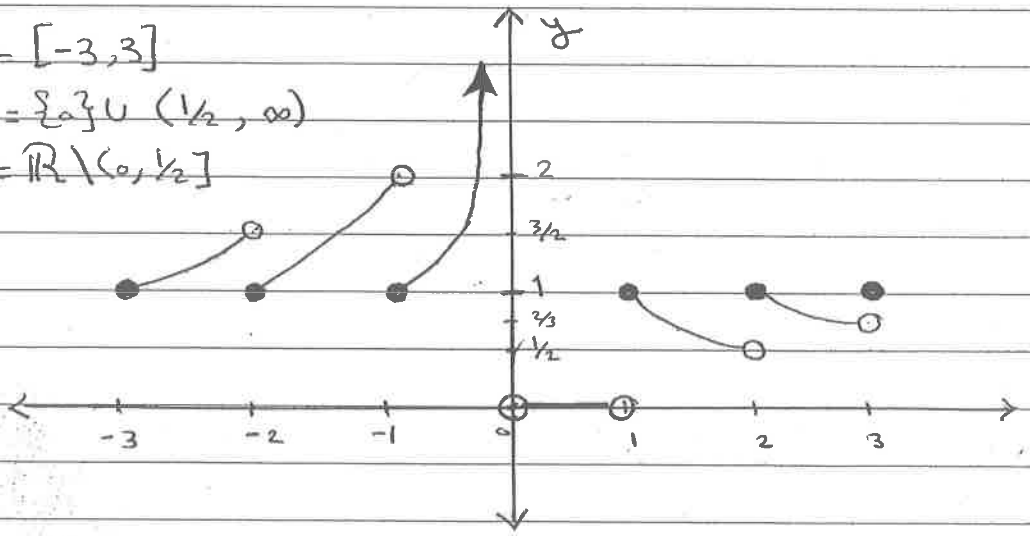
③ $y = \frac{[x]}{x}$ for $-3 \leq x \leq 3$, $x \neq 0$

| $x$ | $[x]$ | $\frac{[x]}{x}$ | closed point | open point |
|---|---|---|---|---|
| $-3 \leq x < -2$ | $-3$ | $-3/x$ | $(-3, 1)$ | $(-2, 3/2)$ |
| $-2 \leq x < -1$ | $-2$ | $-2/x$ | $(-2, 1)$ | $(-1, 2)$ |
| $-1 \leq x < 0$ | $-1$ | $-1/x$ | $(-1, 1)$ | $0 \to +\infty$ |
| $0 \leq x < 1$ | $0$ | $0$ | | $(0,0), (1,0)$ |
| $1 \leq x < 2$ | $1$ | $1/x$ | $(1, 1)$ | $(2, 1/2)$ |
| $2 \leq x < 3$ | $2$ | $2/x$ | $(2, 1)$ | $(3, 2/3)$ |
| $x = 3$ | $3$ | $3/x$ | $(3, 1)$ | |

$D_f = [-3, 3]$

$R_f = \{0\} \cup (1/2, \infty)$

$= \mathbb{R} \setminus (0, 1/2]$




Exercises: Sketch Graph the following functions:

① $f(x) = x + [-x]$ $\ni$ $-3 \leq x \leq 3$

② $f(x) = [2 + x]$ $\ni$ $-2 \leq x < 3$

③ $f(x) = [x^3]$ $\ni$ $-3 \leq x < 3$

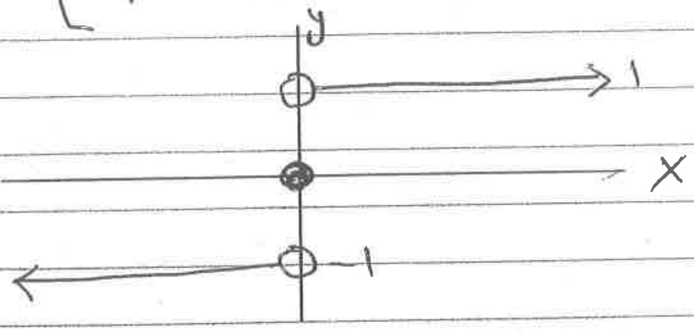
④ $f(x) = 4[x]$ $\ni$ $-3 \leq x < 3$

## Defination: Signum function

$$Sgn(x) = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

$D_f = \mathbb{R}$

$R_f = \{-1, 0, 1\}$




$\boxed{Ex1}$: Using Definition Sgn function to find value of $t$ of the following functions:

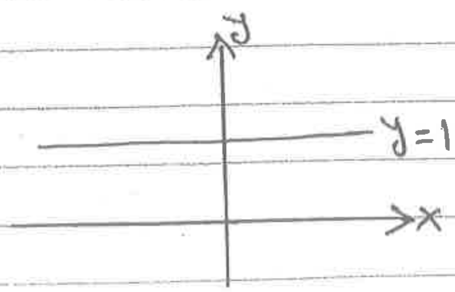
$Sgn(t^2+1)$ , $Sgn(t^2-1)$ , $t\,Sgn(t)$ .

$$y = Sgn(t) = \begin{cases} 1 & \text{if } t>0 \\ 0 & \text{if } t=0 \\ -1 & \text{if } t<0 \end{cases}$$

$\{1\}$ $$Sgn(t^2+1) = \begin{cases} 1 & \text{if } t^2+1>0 \Rightarrow t^2>-1 \Rightarrow t^2 \geq 0 \Rightarrow t\in\mathbb{R} \\ 0 & \text{if } t^2+1=0 \Rightarrow t^2=-1 \quad \text{d.n.e} \\ -1 & \text{if } t^2+1<0 \Rightarrow t^2<-1 \quad \text{d.n.e} \end{cases}$$

$\therefore Sgn(t^2+1) = 1 \quad \forall t\in\mathbb{R}$

$D = \mathbb{R}$ ; $R = \{1\}$

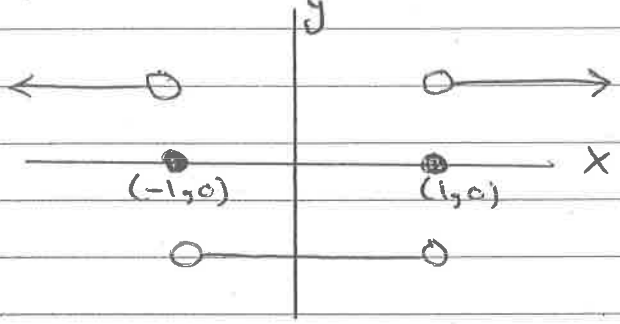


{2} $\operatorname{Sgn}(t^2-1) = \begin{cases} 1 & \text{if } t^2-1>0 \Rightarrow t^2>1 \Rightarrow |t|>1 \\ 0 & \text{if } t^2-1=0 \Rightarrow t^2=1 \Rightarrow |t|=1 \\ -1 & \text{if } t^2-1<0 \Rightarrow t^2<1 \Rightarrow |t|<1 \end{cases}$

$= \begin{cases} 1 & \text{if } t>1 \text{ or } t<-1 \\ 0 & \text{if } t=1 \text{ or } t=-1 \\ -1 & \text{if } -1<t<1 \end{cases}$

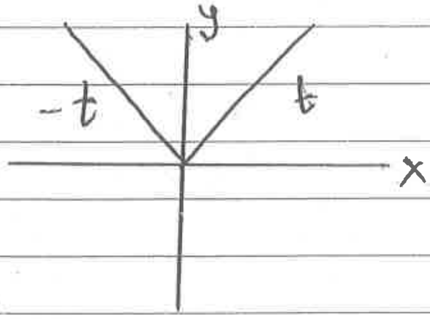
$D_f = \mathbb{R}$

$R_f = \{1, 0, -1\}$



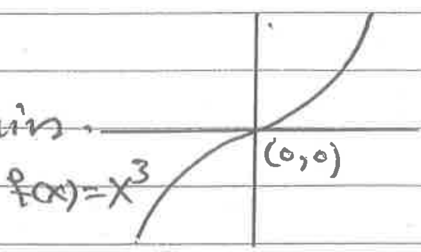

{3} $t \operatorname{Sgn}(t) = \begin{cases} t & \text{if } t>0 \\ 0 & \text{if } t=0 \\ -t & \text{if } t<0 \end{cases}$

$= |t|$




$D_f = \mathbb{R}$ , $R_f = \mathbb{R}^+$

Odd function: A function $f(x)$ is called odd function if
$$f(-x) = -f(x)$$
for every $x$ in the function's domain.




Ex: let $f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x) \Rightarrow f$ is odd function.

Even function: A function $f(x)$ is called even function if
$$f(-x) = f(x)$$
for every $x$ in the function's domain.

- إزاحة بيانات الدوال :-

ليس من السهل دائماً رسم بيانات الدوال فهناك قيماً منها يمكن رسمه بعد معرفة عدداً قليلاً من النقاط التي تقع عليه وقسماً آخر يحتاج رسمه لمعرفة عدداً أكبر من النقاط التي تقع عليه ، سنتعلم رسم بيانات دوال معينة هي في الحقيقة مشابهه لدوال يمكن رسم بياناتها بسهوله .

ملاحظة: اذا كانت $x \in R$ ؛ $y = f(x)$ فأن :-

(1) $g(x) = f(x) + c$ إزاحة نحو الاعلى ↑ c من الوحدات

(2) $k(x) = f(x) - c$ إزاحه نحو الاسفل ↓ c من الوحدات

(3) $h(x) = f(x + c)$ إزاحة نحو اليسار ← c من الوحدات

(4) $t(x) = f(x - c)$ إزاحة نحو اليمين → c من الوحدات

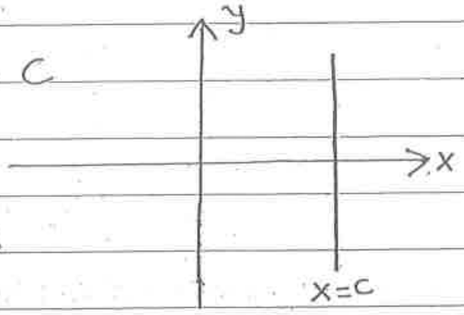
(5) $L(x) = -f(x)$ إنعكاس حول محور السينات x

(6) $m(x) = f(-x)$ إنعكاس حول محور الصادات y

اشكال دوال بصورة عامة :

1) $x = c$

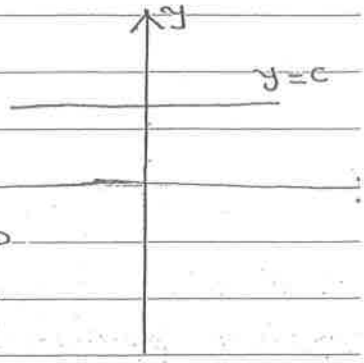
خط شاقولي




2) $y = c$

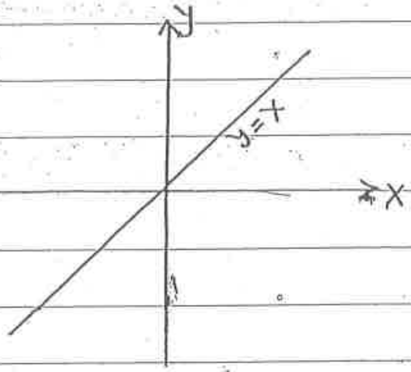
تسمى دالة ثابتة

خط افقي



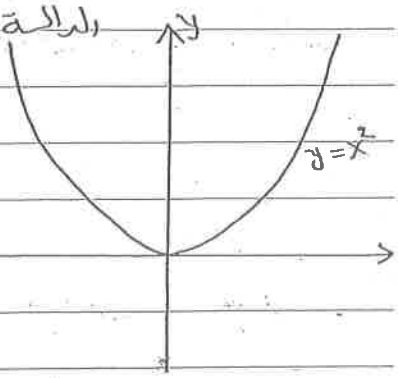

3) $f(x) = y = x$

الدالة خطية



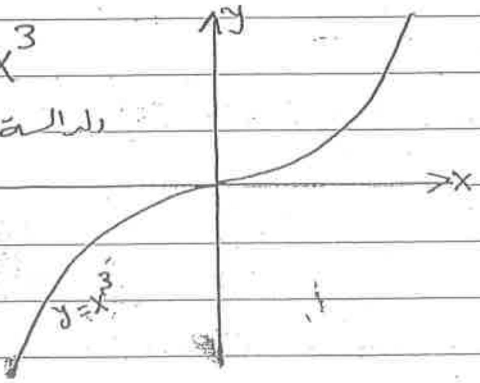

4) $y = x^2$

الدالة التربيعية



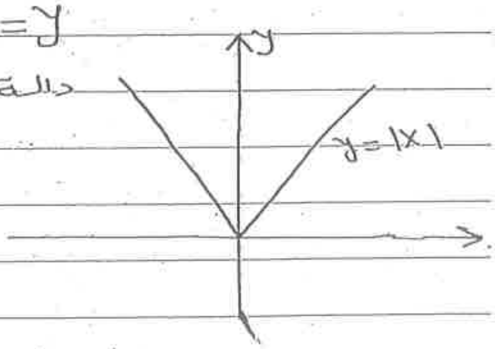

5) $y = x^3$

الدالة التكعيبية



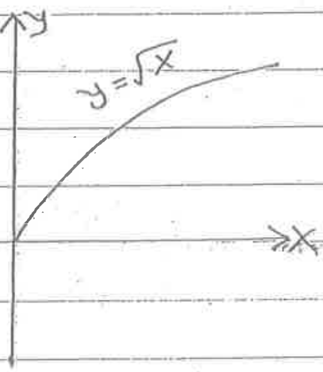

6) $|x| = y$

دالة المطلق



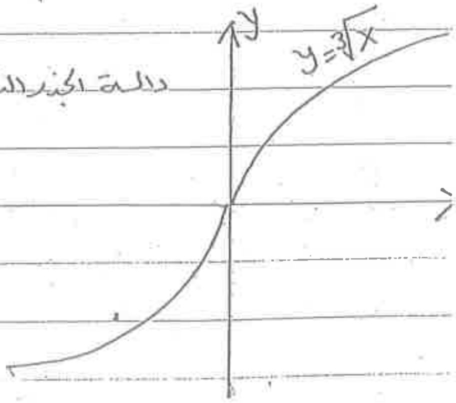

7) $y = \sqrt{x}$

دالة الجذر التربيعي



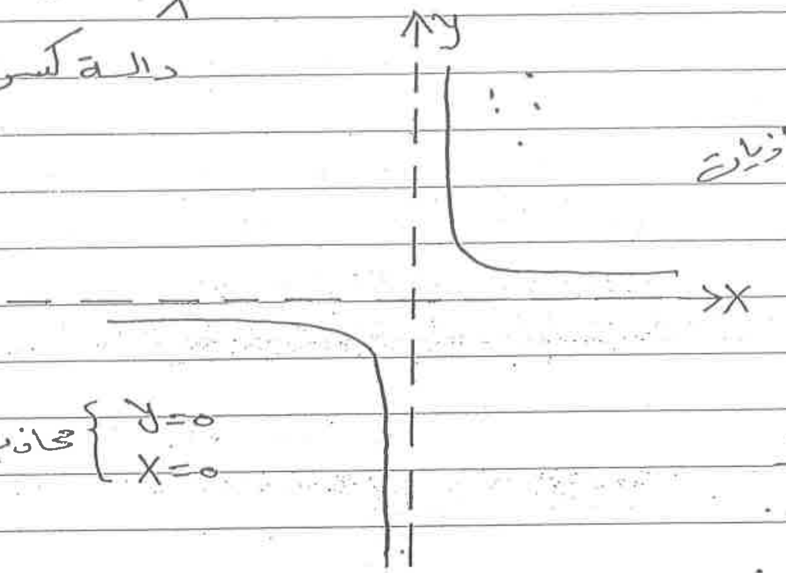

8) $y = \sqrt[3]{x}$

دالة الجذر التكعيبي



{9} $y = \frac{1}{X}$

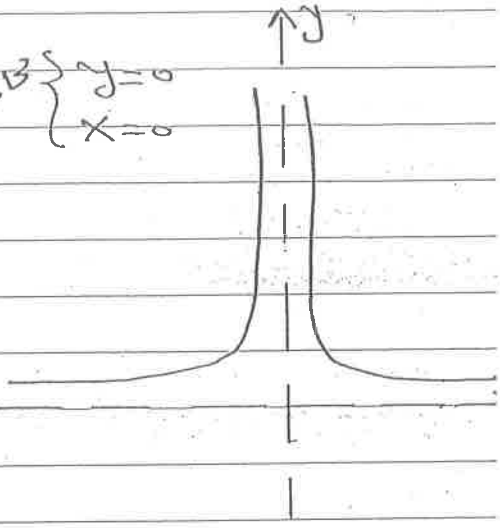
دالة كسرية



محاذيات $\begin{cases} y=0 \\ X=0 \end{cases}$

{10} $y = \frac{1}{X^2}$

محاذيات $\begin{cases} y=0 \\ X=0 \end{cases}$



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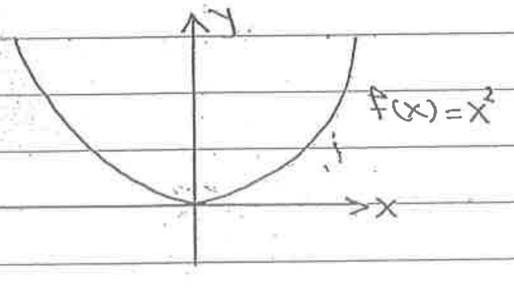
مثال (1): ارسم مخطط الدالة $f(x) = y = X^2$ ثم جد:

$g(x) = f(x)+1$ , $k(x) = f(x)-1$ , $h(x) = f(X+1)$ , $t(x) = f(X-1)$ , $L(x) = -f(x)$ , $m(X) = f(-X)$

sol

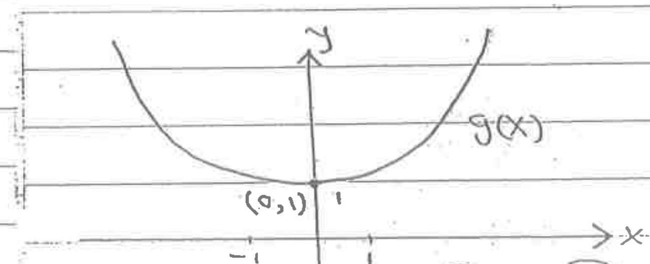




$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+ = [0, \infty)$

{1} $g(x) = f(x)+1 = X^2+1$

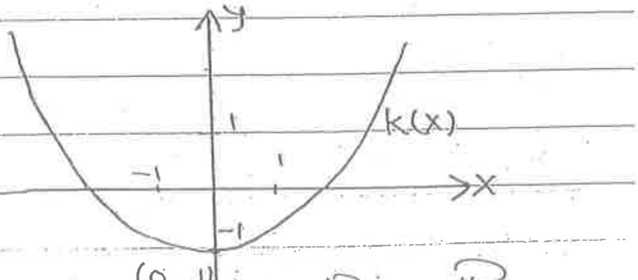




$D_g = \mathbb{R}$

$R_g = [1, \infty)$

{2} $K(X) = f(x)-1 = X^2-1$





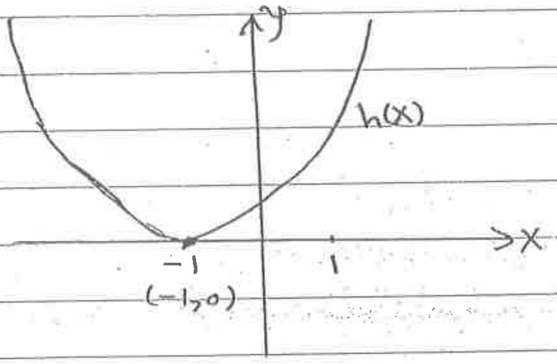
$D_k = \mathbb{R}$

$R_k = [-1, \infty)$

٢٥

$x+1=0 \Rightarrow x=-1$

{3} $h(x) = f(x+1) = (x+1)^2$

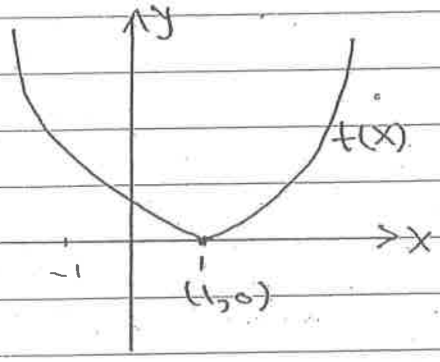



$D_h = \mathbb{R}$

$R_h = \mathbb{R}^+ = [0,\infty)$

$x-1=0 \Rightarrow x=1$

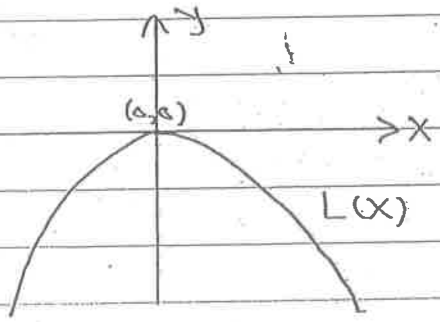
{4} $t(x) = f(x-1) = (x-1)^2$




$D_t = \mathbb{R}$

$R_t = \mathbb{R}^+ = [0,\infty)$

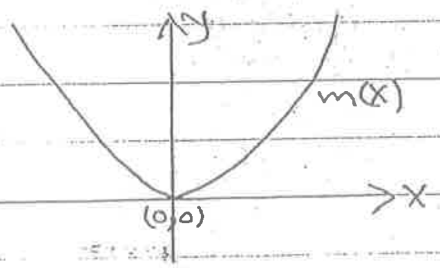
{5} $L(x) = -f(x) = -x^2$




$D_L = \mathbb{R}$

$R_L = \mathbb{R}^- = (-\infty,0]$

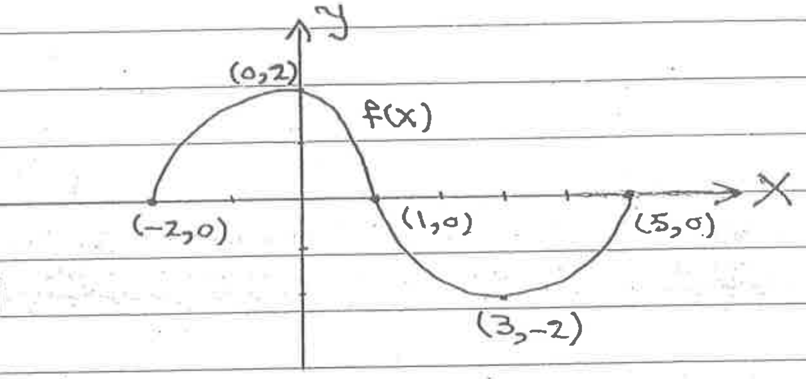
{6} $m(x) = f(-x) = (-x)^2$




$D_m = \mathbb{R}$

$R_m = \mathbb{R}^+ = [0,\infty)$

مثال (٤): أرسم مخطط الدالة $f(3-x)$ اذا كانت $f(x)$ معطاه في الشكل المجاور :




$D_f = [-2,5]$

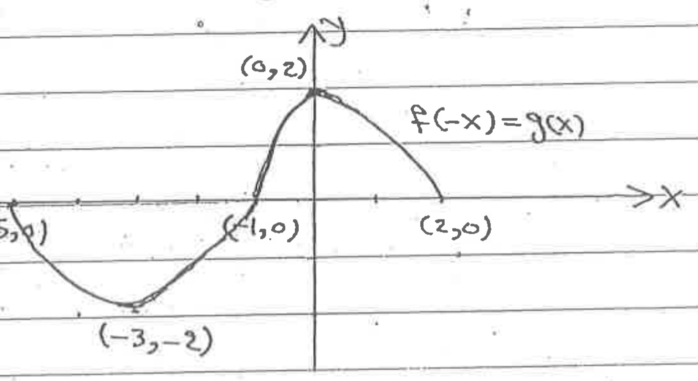
$R_f = [-2,2]$

sol

① نعكس المخطط حول محور y للحصول على مخطط $f(-x)$

$D_g = [-5,2]$

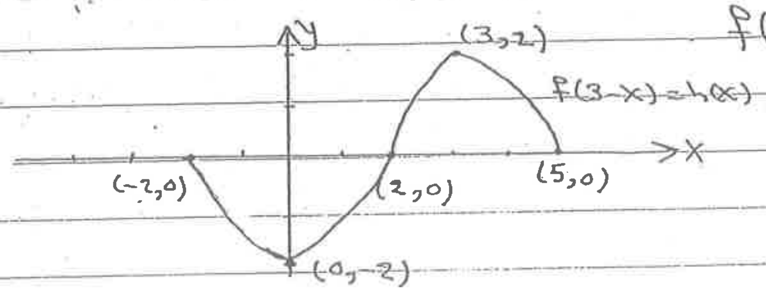
$R_g = [-2,2]$




② ازاحة المخطط الى اليمين ثلاث وحدات للحصول على مخطط الدالة $f(3-x)$

$D_h = [-2,5]$

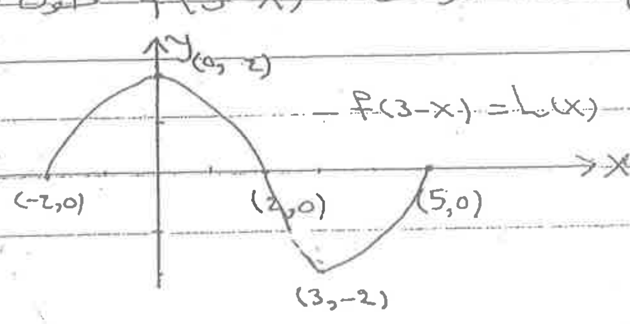
$R_h = [-2,2]$




③ نعكس مخطط الدالة $f(3-x)$ حول محور السينات

$D_h = [-2,5]$

$R_h = [-2,2]$

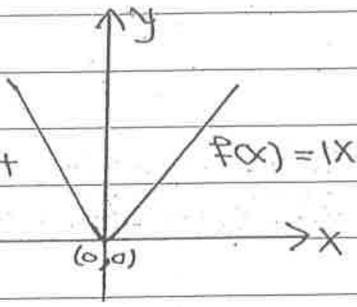


مثال (٣): أرسم مخطط الدالة $y = -|2-x| + 4$

{1} $f(x) = |x|$

$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+$

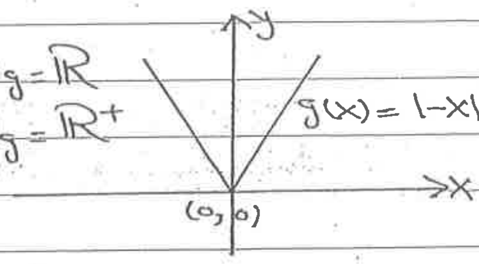



{2} $g(x) = |-x|$

نعكس مخطط الدالة حول محور y

$D_g = \mathbb{R}$

$R_g = \mathbb{R}^+$




{3} $h(x) = |2-x|$

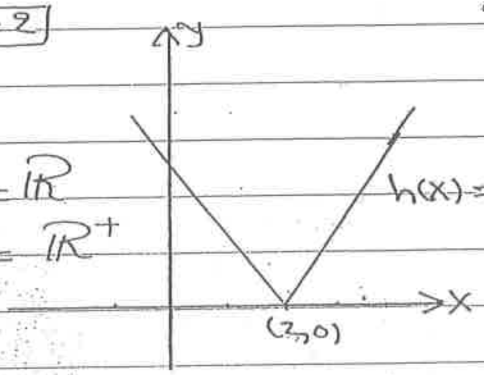
$2 - x = 0$

$\boxed{x = 2}$

ازاحة نحو اليمين

$D_h = \mathbb{R}$

$R_h = \mathbb{R}^+$

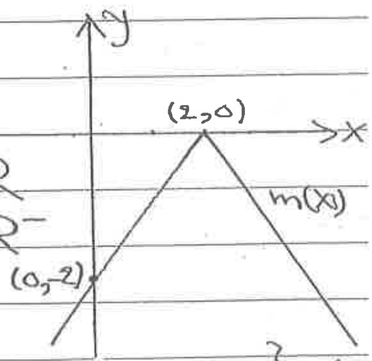



{4} $m(x) = -|2-x|$

نعكس مخطط الدالة حول محور x

$D_m = \mathbb{R}$

$R_m = \mathbb{R}^-$



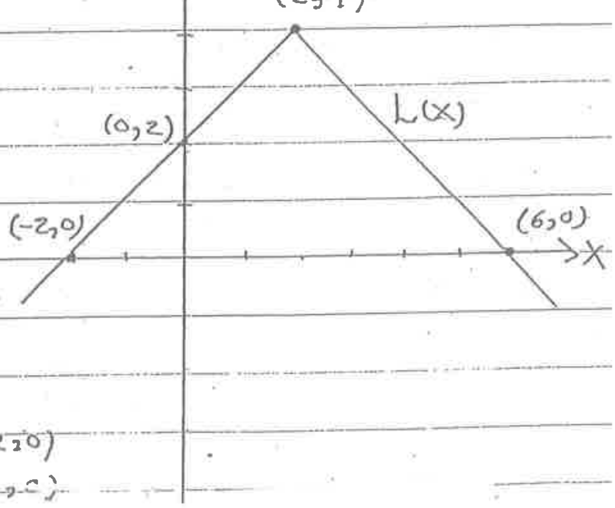

$x = 0$

$\Rightarrow y = -|2-0| = -|2| = -2 \Big\} \Rightarrow (0,-2)$

{5} $L(x) = -|2-x| + 4$

$D_L = \mathbb{R}$

$R_L = (-\infty, 4]$

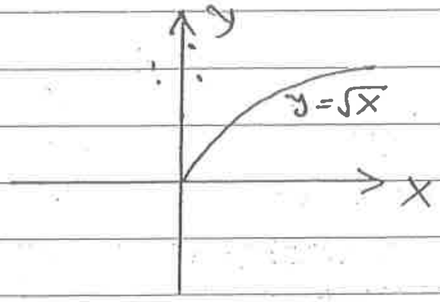



$\left.\begin{array}{l} x = 0 \\ y = -|2-0| + 4 = 2 \end{array}\right\} \Rightarrow (0,2)$

$y = 0$

$|2-x| = 4 \quad \left.\begin{array}{l} 2 - x = 4 \Rightarrow x = -2 \\ 2 - x = -4 \Rightarrow x = 6 \end{array}\right\} \Rightarrow \begin{array}{l} (-2,0) \\ (6,0) \end{array}$

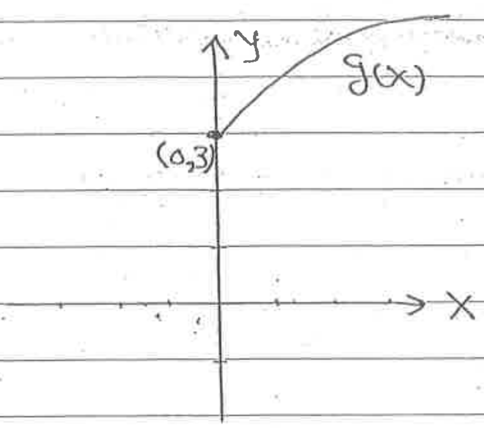
مثال (٤): الشكل الآتي يمثل مخطط الدالة $y=\sqrt{x}$ ، أرسم مخططا لكل من الدوال الآتية:




$D_f = \mathbb{R}^+$

$R_f = \mathbb{R}^+$

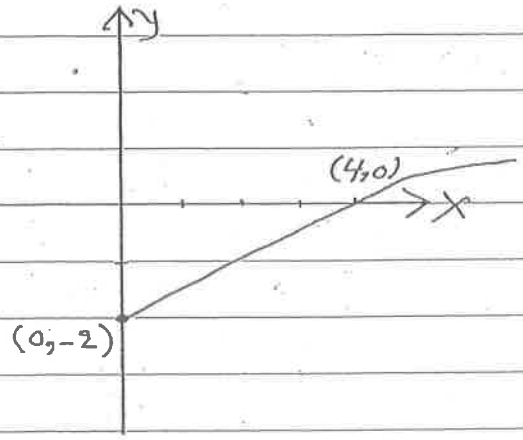
{1} $g(x) = \sqrt{x} + 3$




$D_g = \mathbb{R}^+$

$R_g = [3, \infty)$

{2} $h(x) = \sqrt{x} - 2$




$y = 0 \Rightarrow 0 = \sqrt{x} - 2$

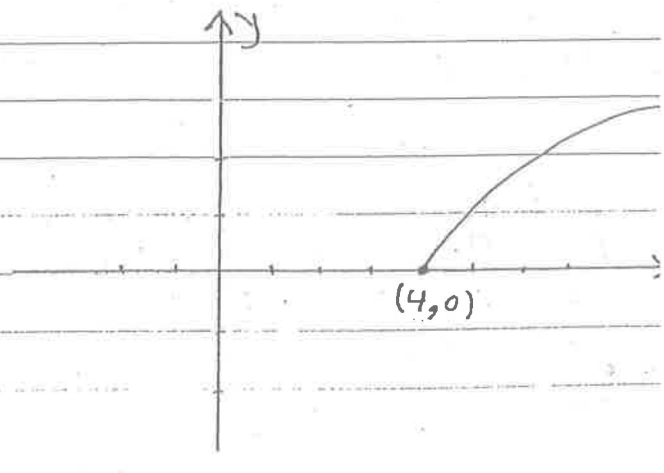
$\Rightarrow \sqrt{x} = 2$

$\Rightarrow x = 4 \Rightarrow (4,0)$

$D_h = \mathbb{R}^+$

$R_h = [-2, \infty)$

{3} $m(x) = \sqrt{x-4}$




$x - 4 = 0 \Rightarrow x = 4$

$D_m = [4, \infty)$

$R_m = \mathbb{R}^+$

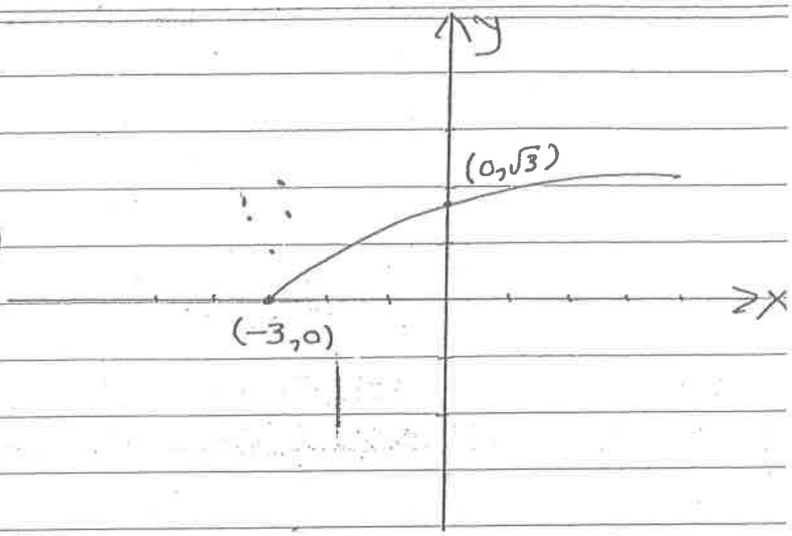
{4} $\sqrt{x+3} = t(x)$

$x+3=0 \Rightarrow x=-3$

$x=0 \Rightarrow y=\sqrt{3} \Rightarrow (0,\sqrt{3})$

$D_t = [-3,\infty)$

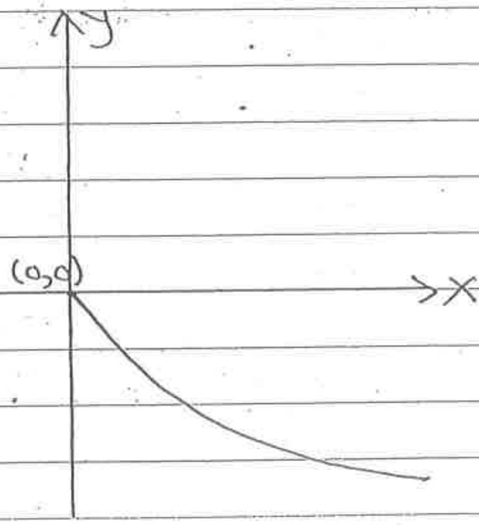
$R_t = \mathbb{R}^+$




{5} $h(x) = -\sqrt{x}$

$D_h = \mathbb{R}^+$

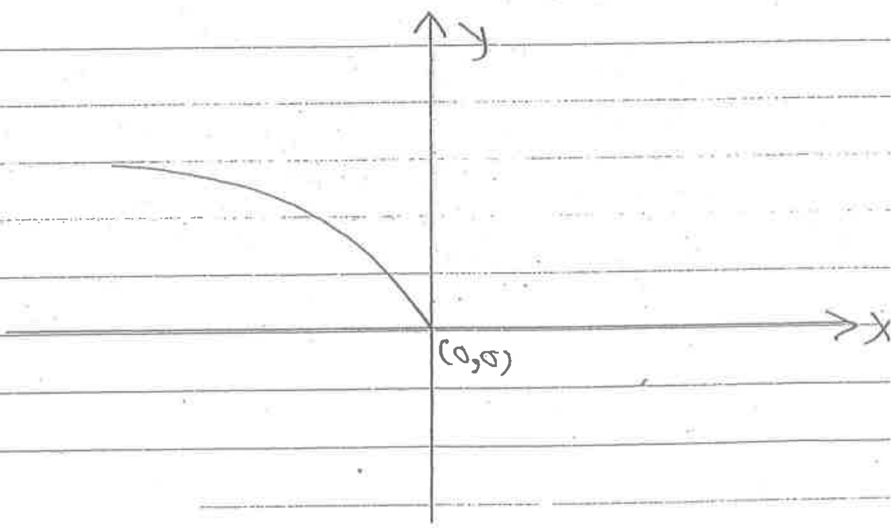
$R_h = \mathbb{R}^-$
$= (-\infty, 0]$




{6} $k(x) = \sqrt{-x}$

$D_k = \mathbb{R}^-$
$= (-\infty, 0]$

$R_k = \mathbb{R}^+$



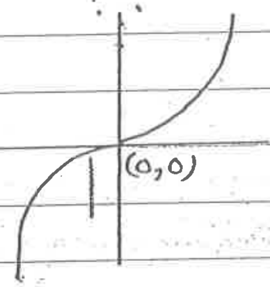
ع:

## تمارين ٤

س١/ الشكل التالي يمثل مخططاً للدالة $f(x)=x^3$ ، أرسم مخططاً للدوال الآتية:

$D_f=\mathbb{R}$

$R_f=\mathbb{R}$




1) $g(x)=(x-2)^3$

2) $h(x)=-x^3$

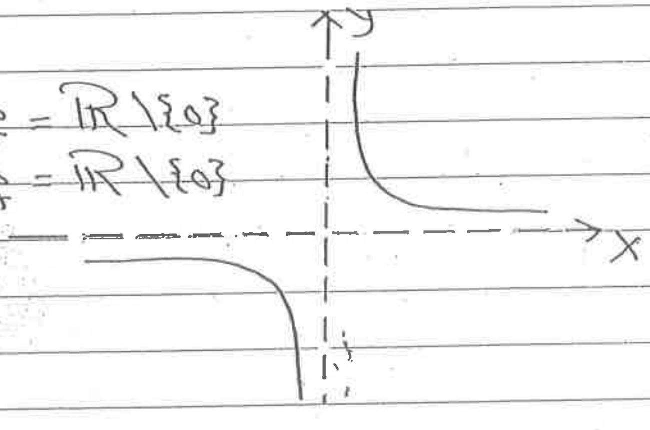
3) $k(x)=(4-x)^3+5$

س٢/ أرسم مخطط الدالة الآتية: $y=|-x+2|-3$ .

س٣/ مخطط الدالة $f(x)=\frac{1}{x}$ معطى في الشكل أدناه ، أرسم مخطط الدالة الآتية:

$D_f=\mathbb{R}\setminus\{0\}$

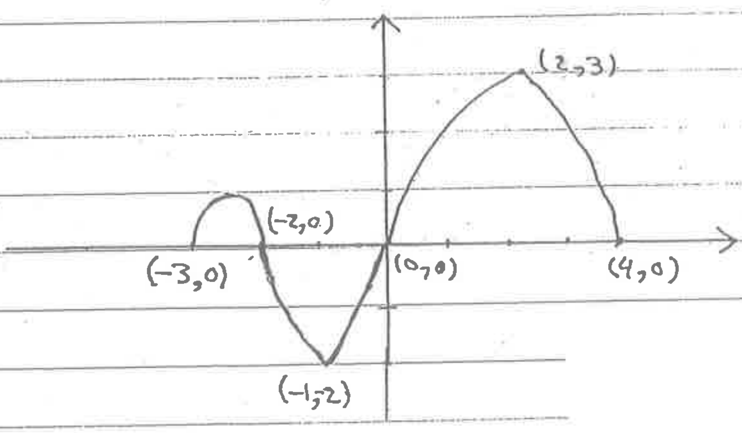
$R_f=\mathbb{R}\setminus\{0\}$




أ) $g(x)=\frac{1}{x+3}$

ب) $h(x)=2-\frac{1}{x}$

س٤/ الشكل التالي يمثل الدالة $f$ ، أرسم مخطط الدوال الآتية:




1) $f(x-2)$

2) $f(x+3)$

3) $-f(x)$