جامعة بغداد

كلية التربية للعلوم الصرفة / ابن الهيثم

التفاضل والتكامل

قسم الرياضيات

المرحلة الاولى

الفصل الثالث

اساتذة المادة

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Chapter Three

Limits and Continuty

Definition of Limit: If the values of function f of x approach the value L as x approaches c. we say f has limit L as x approaches c and we can write as:

$\lim_{x \to c} f(x) = L$ (Read

Example: Let $f(x) = x^2 + 3$, find the limit of f when x approaches to 2.

Sol.

from right $x > 2$	x	3	2.5	2.3	2.1	2.01	2.001	2.0001	
	$f(x)$	12	9.25	8.25	7.44	7.040	7.004	7.0004	$\cong 7$
from left $x < 2$	x	1	1.2	1.4	1.5	1.9	1.99	1.999	
	$f(x)$	4	4.44	4.96	5.95	5.98	6.98	6.999	$\cong 7$

we can note the x approach to 2 from left and right $\Rightarrow$ Then $f(x)$ approach to 7,

Therefore, we can say that $\lim_{x \to 2} f(x) = 7$

(2)

ex 2

If $f(x)$ is the identity function $f(x) = x$, Then for any value of c

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x) = c$$

ex 3 If f is the constant function $f(x) = k$

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k$$

Properties of limits

If $\lim_{x \to c} f_1(x) = L_1$, $\lim_{x \to c} f_2(x) = L_2$ Then

1. $\lim_{x \to c} [f_1(x) + f_2(x)] = L_1 + L_2$
2. $\lim_{x \to c} [f_1(x) - f_2(x)] = L_1 - L_2$
3. $\lim_{x \to c} [f_1(x) \cdot f_2(x)] = L_1 \cdot L_2$
4. $\lim_{x \to c} k \cdot f_2(x) = k \cdot L_2$ (k is any number)
5. $\lim_{x \to c} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$ if $L_2 \neq 0$

The limits are all taken as $x \to c$, and L_1 and L_2 are real number

③

ex 4 $\lim_{x \to 2} \frac{x^2+2x+4}{x+2}$

sol

$\frac{(2)^2+2(2)+4}{2+2} = \frac{12}{4} = 3$

ex 5

$\lim_{x \to 5} \frac{x^2-25}{3(x-5)}$

$= \lim_{x \to 5} \frac{(x+5)\cancel{(x-5)}}{3\cancel{(x-5)}}$

$= \lim_{x \to 5} \frac{x-5}{3} = \frac{5+5}{3} = \frac{10}{3}$

ex 6 find the Limit of $f(x) = \frac{(2+h)^2-4}{h}$

when x approaches to 0 (zero) ?

sol

$\lim_{h \to 0} \frac{(2+h)^2-4}{h} = \lim_{h \to 0} \frac{4+4h+h^2-4}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$

$= \lim_{h \to 0} (4+h) = 4+0 = 4$

ex 7 $\lim_{n \to 0} \frac{\sqrt{4+n}-2}{n}$

$= \lim_{n \to 0} \left(\frac{\sqrt{4+n}-2}{n} \cdot \frac{\sqrt{4+n}+2}{\sqrt{4+n}+2} \right)$

$= \lim_{n \to 0} \frac{4+n-4}{n(\sqrt{4+n}+2)} = \lim_{n \to 0} \frac{\cancel{n}}{\cancel{n}(\sqrt{4+n}+2}$

$$= \frac{1}{\sqrt{4+0}+2} = \frac{1}{4}$$

Ex find the limit of $f(x) = \frac{\sqrt{4+n}-2}{n}$

when x approaches to 5

$$L = \lim_{x \to 5} \frac{\sqrt{4+n}-2}{n} = \frac{\sqrt{4+5}-2}{5}$$

$$= \frac{1}{5}$$

Right-hand limits and left-hand limits.

Sometimes the value of a function $f(x)$ tend to diferent limits as x approaches a number c from different sides.
when this ~~ap~~ happens. we call the limit of f as x approaches c from the right the "right-hand limit" of f at c. and the limit as x approaches c from the left the "Left-hand limit" of f at c.

$\lim_{x \to c^+} f(x)$ (the limit of f as x approaches c from the right)

$\lim_{x \to c^-} f(x)$ (" " " " " from Left ")

(5)

ex1 show that $\lim\limits_{x \to 0} |x| = 0$

solution :-

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

$\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^+} x = 0$

$\lim\limits_{x \to 0^-} |x| = \lim\limits_{x \to 0^-} (-x)$

$= -\lim\limits_{x \to 0^-} x$

$= -0 = 0.$

$\therefore \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x)$

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} |x| = |0| = 0$

Not :- A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left hand limits at c exist and are equal :-

Mathematically :-

$\lim f(x) = L \iff \lim\limits_{x \to c^+} f(x) = L$ and $\lim\limits_{x \to c^-} f(x) = L$.

* How to find the limit to the roots functions

* If c is start or end point to the domain of f, we must find the right-hand limits and left-hand limit. If not, we solve directly

ex1 find the limit to $f(x) = \sqrt{2x-6}$ when c approaches to 3.

sol $2x-6 \geq 0 \Rightarrow 2x \geq 6 \Rightarrow x \geq 3$

$D_f = [3, \infty)$ since $c = 3$

$L^+ = \lim\limits_{x \to c^-} f(x)$,

$= \lim\limits_{x \to 3^+} (\sqrt{2 \cdot 4 - 6}) = \sqrt{2}$

$L^- = \lim\limits_{x \to 3^-} f(x) \notin D_f$

$\therefore \lim\limits_{x \to 3} f(x)$ not exist

ex2 find the limit of $f(x) = \sqrt[3]{x^3 - 8}$

when x approaches to 2.

sol $D_f = \mathbb{R}$

$L = \lim\limits_{x \to 2} \sqrt{x^3 - 8} = \sqrt[3]{8-8} = 0$

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ex(3) find the limit of $f(x) = \frac{\sqrt{x-2}}{x-2}$ when x approaches to 2.

Sol

$x-2 \geq 0 \Longrightarrow x \geq 2$

and

$x-2 \neq 0 \Longrightarrow x \neq 2$

$$D_f = (2, \infty)$$

$$L = \lim_{x\to 2} \frac{\sqrt{x-2}}{x-2} = \lim_{x\to 2} \frac{1}{\sqrt{x-2}}$$

$$L^+ = \lim_{x\to 2^+} \frac{1}{\sqrt{x-2}} = +\infty$$

$$L^- = \lim_{x\to 2^-} \frac{1}{\sqrt{x-2}} \quad \text{not exist}$$

$$\therefore L = \lim_{x\to 2} \frac{\sqrt{x-2}}{x-2} \quad \text{not exist}$$

~~If f(x) is a~~

ex1 $L = \lim_{x\to 0} |x| = \lim_{x\to 0} \begin{cases} x & \text{if } x \geq 0 \\ -x & x < 0 \end{cases}$

$$L^+ = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

$$L^- = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

$$\therefore L^+ = L^- \Longrightarrow L = \lim_{x\to 0} |x| = 0$$

ex find the limit of $f(x)=|x|$, when x approaches to -1

sol $L = \lim_{x \to -1} |x| = |-1| = 1$

ex find the limit $f(x) = \begin{cases} y+2 & y \geq 0 \\ 2 & y < 0 \end{cases}$ when the y approaches to 1

sol $L = \lim_{y \to 1} (y+2) = 1 + 2 = 3$

* Limits involving infinity

Although there is no real number infinity. The word infinity is useful for describing how some function behave when their domains or range exceed all bounds.

limits as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$

the function $f(x) = \frac{1}{x}$

is defined for all real number except $x = 0$

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We summarize these facts by saying

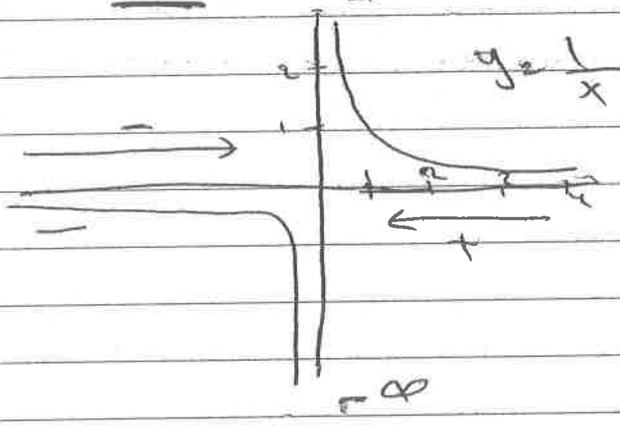
1- As x approache ∞, $\frac{1}{x}$ approaches to 0

2- As x approaches 0 from the right, $\frac{1}{x}$ approaches to ∞

3- As x approaches 0 from the left, $\frac{1}{x}$ approaches $-\infty$

4- As x approaches to $-\infty$, $\frac{1}{x_\infty}$ approaches 0

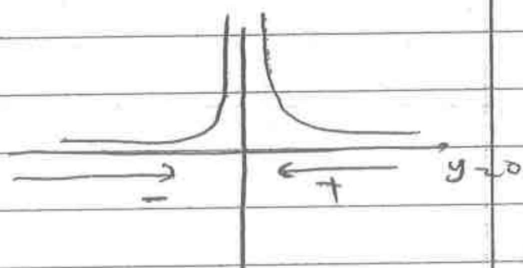
ex ① $y = \frac{1}{x}$



ex 2

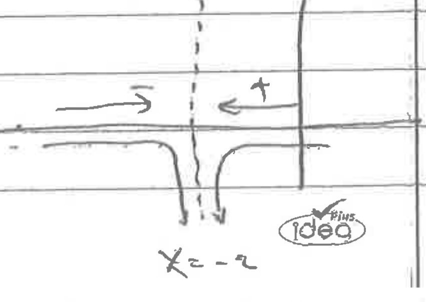
$$\lim_{x \to \infty} \frac{1}{x} = 0 , \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

ex 3 $\lim_{x \to 0} \frac{1}{x^2} = \infty$



ex 4 $\lim_{x \to -2} \frac{-1}{(x+2)^2} = -\infty$

$$= \frac{-1}{0^+}$$



Not $\lim_{x \to 0} f(k) = k$, $\lim_{x \to \mp\infty} f(k) = k$

ex

$\lim_{x \to 0} f(4) = 4$, $\lim_{x \to +\infty} f(3) = 3$

Limits of Rational function as $x \to \mp\infty$

to find the limit of rational function as $x \to \mp\infty$ (when the limit exists), we divide the numerator and denominator by the highest power of x in the denominator

ex1 $\lim_{x \to \infty} \dfrac{x-2}{2x^2-7x+5}$

$= \lim_{x \to \infty} \dfrac{\frac{x}{x^2} - \frac{2}{x^2}}{\frac{2x^2}{x^2} - \frac{7x}{x^2} + \frac{5}{x^2}}$

$= \lim_{x \to \infty} \dfrac{\frac{1}{x} - \frac{2}{x^2}}{2 - \frac{7}{x} + \frac{5}{x^2}}$

$= \dfrac{0}{2-0+0} = \dfrac{0}{2} = 0$

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ex 2

$$\lim_{x \to \infty} \frac{x^2+2x+1}{5x^2+2} =$$

$$= \lim_{x \to \infty} \frac{x^2/x^2 + \frac{2x}{x^2} + \frac{1}{x^2}}{5x^2/x^2 + \frac{2}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1 + 2/x + 1/x^2}{5 + 2/x^2}$$

$$= \frac{1+0}{5+0} = \frac{1}{5}$$

ex 3 $\lim_{x \to \infty} \frac{x^3+x^2+2}{x^2+1}$

$$= \lim_{x \to \infty} \frac{\frac{x^3}{x^2} + \frac{x^2}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{x + 1 + 2/x^2}{1 + \frac{1}{x^2}}$$

$\lim_{x \to \infty} x = \infty$

$$= \frac{\infty + 1 + 0}{1+0} = \infty$$

Note

even the degree of denominator less than the degree of numerator, we divide by the highest power of x in the denominator.

ex 4 $\lim_{x \to \infty} \frac{-4x^3 + 7x}{2x^2 - 3x - 10}$

$= \lim_{x \to \infty} \frac{-4x^3/x^2 + 7x/x^2}{2x^2/x^2 - 3x/x^2 - 10/x^2}$

$= \lim_{x \to \infty} \frac{-4x + 7/x}{2 - 3/x - 10/x^2}$

$= \frac{-\infty + (0)}{2 - (0) - (0)} = -\infty$

Homework — find the limit for the following functions

1. $\lim_{x \to \infty} \frac{2x + 3}{5x + 7}$

2. $\lim_{x \to 0^+} \frac{1}{3x}$

3. $\lim_{x \to -\infty} \frac{10x^5 + x^4 + 31}{x^6}$

4. $\lim_{x \to 2} \frac{1}{x^2 - 4}$

5. let $f(x) = \begin{cases} \frac{x-2}{x-1} & x \le 0 \\ \frac{1}{x^2} & x > 0 \end{cases}$

find the limit of $f(x)$ at $x_0 = -\infty$, $x_0 = \infty$, $x_0 = {}^+0$ and $x_0 = {}^-0$

6. $\lim_{x \to \infty} \left(\frac{-x}{7x + 4} + \frac{5x + 2}{2x^3 - 1} \right)$

7. let $f(x) = \begin{cases} 3 - x & x < 2 \\ \frac{x}{2} & x > 2 \end{cases}$

find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$

Does $\lim_{x \to 2} f(x)$ exist?

Continuous functions

we say f is continuous function in the point c
if there is no any interrupt in this point
and f is continuous function in a certain interval
if there is no any interrupt in this interaval

Def f is continuous function at x_0
if and only if all three of the following
conditions :-

1- $f(x_0)$ exist

2- $\lim_{x \to x_0} f(x)$ exist

3- $\lim_{x \to x_0} f(x) = f(x_0)$

Note The Limit in the continuous test is to be two sided if x_0 is an interior point of the domain of f; it is to be the appropriate one-sided limit if x_0 is an end point of the domain.

ex 1

IS $f(x) = x^2$ Continuous at $x = 2$

sol

$D_f = \mathbb{R}$

1- $f(2) = 2^2 = 4$

2- $\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4$

$\therefore$ $f(x)$ is continuous at $x = 2$

How to test Continuity Function.

1- polynomials are continuou at every point.

ex

IS $f(x) = x^2 + 3x + 5$ continuous at $x = 1$

sol $1 \in D_f = \mathbb{R}$

$f(1) = 1^2 + 3 + 5 = 9$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 3x + 5) = 1 + 3 + 5 = 9$

$\therefore \lim_{x \to 1} f(x) = f(1)$

$\therefore$ f is continuous at $x = 1$

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2- Rational functions are continuous if x_0 belong to domain of function (f)

ex IS $f(x) = \frac{x^3 - 2x^2 + 5x - 1}{x^2 - 3x + 2}$ continuous at $x = 1$ and $x = 0$

Sol

$x^2 - 3x + 2 = (x-2)(x-1) \neq 0$
either $x \neq 2$ or $x \neq 1$

$\Rightarrow \therefore D_f = \mathbb{R} \setminus \{2, 1\}$

$\therefore 1 \notin \mathbb{R} \setminus \{1, 2\}$

Therefore $f(1)$ not

$\therefore f(x)$ not continuous

when $x = 0$

$0 \in \mathbb{R} \setminus \{2, 1\}$

$f(0) = \frac{0 - 0 + 0 - 1}{0 - 0 + 2} = \frac{-1}{2}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^3 - 2x^2 + 5x - 1}{x^2 - 3x + 1} = \frac{-1}{2}$

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore f$ is continuous function.

If $f(x)$ is root function, $f(x)$ is continuous if :-

x_0 belong to domain f

ex Is $f(x)=\sqrt{x-x^2}$ at $x_0=\frac{1}{2}$

sol $x-x^2 \geq 0 \Rightarrow$

either $x \geq 0 \wedge 1-x \geq 0 \Rightarrow x \geq 0 \wedge x \leq 1$

$\therefore 0 \leq x \leq 1$

or $x \leq 0 \wedge 1-x \leq 0 \Rightarrow x \leq 0 \wedge x \geq 1$

$\Rightarrow \phi$

$D_f=[0,1]$

Therefore $x_0=\frac{1}{2} \in D_f=[0,1]$, then we find the continuous of f.

$f(\frac{1}{2})=\sqrt{\frac{1}{2}-\frac{1}{4}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$

$\lim_{x \to \frac{1}{2}} f(x)=\lim_{x \to \frac{1}{2}} \sqrt{x-x^2}=\sqrt{\frac{1}{2}-\frac{1}{4}}=\sqrt{\frac{1}{4}}$

$=\sqrt{\frac{1}{2}}$

$\lim_{x \to \frac{1}{2}} f(x)=f(\frac{1}{2})$

$\therefore$ f continuous at $x=\frac{1}{2}$

(17)

(b) If x_0 is bounded point of domain of f, ~~It~~ must satisfy the Last condition.

ex Is $f(x) = \sqrt{x - x^2}$ at $x = 0$ and $x = 1$

sol $D_f = [0, 1]$

(1) $L^+ = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x - x^2} = 0$

$f(0) = \sqrt{0 - 0} = 0$

$\therefore$ f is continuous at 0

(2) $L^- = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{x - x^2} = \sqrt{1 - 1} = 0$

$f(1) = \sqrt{1 - 1} = 0$

$\therefore$ f is continuous at 1

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ex IS $f(x) = \begin{cases} x+1 & x \geq 0 \\ 1 & x < 0 \end{cases}$ when $x = 0$

Sol, $0 \in D_f = \mathbb{R}$

when $f(0) = 0+1 = 1$

$L^+ = \lim_{x \to 0^+} (x+1) = 0+1 = 1$

$L^- = \lim_{x \to 0^-} (1) = 1$

$L^+ = L^-$

$\therefore$ Limit of $f(x)$ is exist

$\lim_{x \to 0} f(x) = 1 = f(0)$

$\therefore$ f is continuous at $x = 0$

ex2 IS $f(x) \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$ continuous at $x = 2$

Sol, $D_f = \mathbb{R}$
$2 \in D_f$

$f(2) = 3$, $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2}$

$= \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$= 4$

$\lim f(x) \neq f(2)$

$\therefore$ f is not continuous at $x = 2$

idea plus

Note

Can we redefine the f in pag 18 ex2 to make it continuous at $x_0 = 2$

sol Yes, we can that as a following definition.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$$

$f(2) = 4$, $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \frac{x^2-4}{x-2} = 4$

$\therefore \lim\limits_{x \to 2} f(x) = 4 = f(2)$

$\therefore f(x)$ is continuous at $x_0 = 2$

ex3 $f(x) = \begin{cases} \frac{x^3-1}{x-1} & x \neq 1 \\ k & x = 1 \end{cases}$ is continuous at $x_0 = 1$, find the value of k

sol $D_f = \mathbb{R}$

$\because f$ is continuous at $x_0 = 1$

$\therefore f(1) = k$

$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \frac{x^3-1}{x-1} = 3$

$\lim\limits_{x \to 1} f(x) = f(1) = 3 \implies k = 3$

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Note If the function (f) is not continuous at certain point (x_0) in the domain, f called discontinuous at this point (X_0)

ex Sgn x discontinuous at $x=0$

since $Sgn(0)=0$

$$\lim_{x\to 0^+} Sgn(x)=1$$

$$\lim_{x\to 0^-} Sgn(x)=-1$$

$\therefore L^+ \neq L^-$

$\lim_{x\to 0}$ sgn (x) not exist

ex IS $f(x)=[x]$ continuous at $x=\frac{1}{2}$, $x=-3$

sol $f(-3)=[-3]=-3$

$$L^+ = \lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} [x] = -3$$

$$L^- = \lim_{x\to -3^-} [x] = -4$$

$$L^+ \neq L^-$$

$\therefore$ the limit of f is not exist.

$\therefore f(x)$ is discontinuous at $x_0=-3$

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Another way to solve the example when $x_0 = \frac{1}{2}$

① $f(\frac{1}{2}) = [\frac{1}{2}] = 0$

$L^+ = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} [x] = 0$

$L^- = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} [x] = 0$

$\therefore \lim_{x \to \frac{1}{2}} f(x) = 0 = f(\frac{1}{2})$

$\therefore f$ is continuous at $x = \frac{1}{2}$

Generally, greatest integer function is discontinuous for all integer number (I).

Algebraic Properties of Continuous Function

A. If the functions f and g are continuous at $x = x_0$, then the following combinations are continuous at $x = x_0$

1. Sums $f+g$
2. Differences $f-g$
3. Products $f \cdot g$
4. Quotients f/g (provided $g(c) \neq 0$)

5- Constant multiples: $k \cdot g$ (any number k)

6- If f is continuous at x_0 and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at x_0.

B- Any polynomilal is continuous on $\mathbb{R}$

C- Any constant function is continuous on $\mathbb{R}$

ex 1 find the intervales in which the following functions are continuous.

1- $f(x) = x^4 - 3x^3 + 22x^2 - 6x + 17$

f is continuous on $\mathbb{R}$ since f is polynomilal

2- $f(x) = \dfrac{2}{x^4 \; 4x+3}$, $x^2 \; 4x+3 = (x$

$x^2 - 4x + 3 = (x-1)(x-3) \neq 0$

$\Rightarrow x \neq 1 \; , \; x \neq 3$

$\therefore$ f is continuous on $\mathbb{R} \setminus \{1, 3\}$

3- $f(x) = \sqrt{x}$

$x \geq 0$

$\therefore f$ is continuous on $[0, \infty)$

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ex 2

Graph $f(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < 2 \\ 4 & x \geq 2 \end{cases}$

and prove that f is continuous on $\mathbb{R}$

Sol $D_f = \mathbb{R}$

1- when $x < 0 \Rightarrow f(x) = 0$

$\therefore$ f is continuous since f is constant function from $(-\infty, 0)$

2- when $0 \leq x \leq 2$, $f(x) = x^2$

$\therefore$ f is continuous on $[0, 2)$ since f is polynomial

3- when $x \geq 2$, $f(x) = 4$

$\therefore$ f is continuous since f is constant function on $[2, \infty)$

To find the continuity of function at $x_0 = 0$ and $x_0 = 2$

1) when $x_0 = 0 \Rightarrow f(0) = 0$

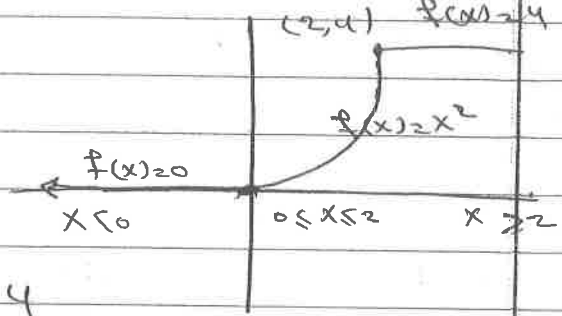
$L^+ = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$

$L^- = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (0) = 0$

$\therefore L^+ = L^-$ and $\lim_{x \to 0} f(x) = 0 = f(0)$

$\therefore$ f is continuous at $x_0 = 0$

2- When $x_0 = 2$



$f(2) = 4$

$L^+ = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 4 = 4$

$L^- = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$

$L^+ = L^- = 4$

$\therefore f(2) = \lim_{x \to 2} f(x) = 4$

$\therefore f$ is continuous at $x_0 = 2$

$\therefore f$ is continuous on $\mathbb{R}$

Therom:- If f is continuous at $x_0 = d$ and $\lim_{x \to x_0} g(x) = d$ Then

$$\lim_{x \to x_0} (f \circ g)(x) = f\left(\lim_{x \to x_0} g(x)\right) = f(d)$$

ex Find $\lim_{x \to 3} \sqrt{2(2-x) + x^3}$

sol let $f(x) = \sqrt{x}$ and $g(x) = 2(2-x) + x^3$

$(f \circ g)(x) = f(g(x)) = f(2(2-x) + x^3)$

$= \sqrt{2(2-x) + x^3}$

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$$\lim_{x \to 3} (f \circ g)_{(x)} = f\left(\lim_{x \to 3} (2(2-x) + x^3)\right)$$

$$= f(25) = 5$$

ex2 find $\lim_{x \to \frac{1}{3}} \left(\frac{9x^2-1}{3x-1}\right)^4$

sol Let $f(x) = x^4$, $g(x) = \frac{9x^2-1}{3x-1}$

Then

$$(f \circ g)_{(x)} = f(g(x)) = f\left(\frac{9x^2-1}{3x-1}\right) = \left(\frac{9x^2-1}{3x-1}\right)^4$$

$$\lim_{x \to \frac{1}{3}} (f \circ g)(x) = f\left(\lim_{x \to \frac{1}{3}} \frac{9x^2-1}{3x-1}\right)$$

$$= f\left(\lim_{x \to \frac{1}{3}} (3x+1)\right)$$

$$= f\left(3 \cdot \frac{1}{3} + 1\right)$$

$$= f(2)$$

$$= 2^4 = 16$$

Exercises

find the Limit for the following functions.

1 - $\lim_{x \to \infty} \frac{1}{\sqrt{x}}$

$\lim_{x \to \infty} \sqrt{\frac{1}{x}} = \sqrt{\lim_{x \to \infty} \frac{1}{x}} = \sqrt{0} = 0$

2 - $\lim_{x \to \infty} \frac{x^2 + 2x + 3}{x^4 + 4x + 4} = \lim_{x \to \infty} \frac{x^2/x^4 + 2x/x^4 + 3/x^4}{x^4/x^4 + 4x/x^4 + 4/x^4}$

$= \lim_{x \to \infty} \frac{\frac{1}{x^2} + \frac{2}{x^3} - \frac{3}{x^4}}{1 + \frac{4}{x^3} + \frac{4}{x^4}}$

$= \frac{0 + 2(0) + 3(0)}{1 + 4(0) + 4(0)} = \frac{0}{1} = 0$

3 - $\lim_{x \to -\infty} \frac{4x^5 - 2x^2 + 5x + 1}{2x^5 + x^3 + 5} =$

$\lim_{x \to -\infty} \frac{4\, x^5/x^5 - 2x^2/x^5 + 5\, x/x^3 + 1/x^5}{2\, x^5/x^5 + x^3/x^5 + 5/x^3}$

$\lim_{x \to -\infty} \frac{4 - 2/x^3 + 5/x^4 + 1/x^5}{2 + 1/x^2 + 5/x^5}$

$= \frac{4 - 0 + 0 + 0}{2 + 0 + 0} = \frac{4}{2} = 2$

4 - $\lim_{x \to \infty} \frac{x^4 + 2x^3 + x + 1}{2x^5 + 5x - 3x^2} = \frac{0}{2} = 0$

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5- $$\lim_{x\to\infty} \frac{x^3-2x-5}{3x^2+5x+1}$$

$$= \lim_{x\to\infty} \frac{x^3/x^2 - 2x/x^2 - 5/x^2}{3x^2/x^2 + 5x/x^2 + 1/x^2}$$

$$= \lim_{x\to\infty} \frac{x - 2/x - 5/x^2}{3 + 5/x + 1/x^2}$$

$$= \frac{\infty - 0 - 0}{3 + 0 + 0} = \infty$$

6- $$\lim_{x\to 0} \frac{1}{x^3}$$

$$L^+ = \lim_{\substack{x\to 0^+ \\ x>0}} \frac{1}{x^3} = \frac{1}{0^+} = \infty$$

$$L^- = \lim_{\substack{x\to 0^- \\ x<0}} \frac{1}{x^3} = \frac{1}{0^-} = -\infty$$

$$L^+ \neq L^-$$

$\therefore \lim_{x\to 0} \frac{1}{x^3}$ is not exist

7- $$\lim_{x\to 0} \frac{1}{x^6}$$

$$L^+ = \lim_{x\to 0^+} \frac{1}{x^6} = \frac{1}{0^+} = +\infty$$

$$L^- = \lim_{\substack{x\to 0^- \\ x<0}} \frac{1}{x^6} = \frac{1}{0^+} = +\infty$$

$\therefore \lim_{x\to 0} \frac{1}{x^6} = +\infty$

$$L^+ = L^-$$

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8) $\lim_{x\to 3} \frac{4}{(x-3)^4}$

$L^+ = \lim_{x\to 3^+} \frac{4}{(x-3)^4} = \frac{4}{0^+} = +\infty$

$L^- = \lim_{x\to 3^-} \frac{4}{(x-3)^4} = \frac{4}{0^+} = +\infty$

$L^+ = L^-$

$\therefore \lim_{x\to 3} \frac{4}{(x-3)^4} = +\infty$

9) $\lim_{x\to 5} \frac{1}{(x-5)^3}$

$L^+ = \lim_{x\to 5^+} \frac{1}{(x-5)^3} = \frac{1}{0^+} = +\infty$

$L^- = \lim_{x\to 5^-} \frac{1}{(x-5)^3} = \frac{1}{0^-} = -\infty$

$L^+ \neq L^-$

$\therefore$ limit of f not exist.

10) $\lim_{x\to -1} \frac{2}{\sqrt{x+1}}$

$x+1 > 0 \Longrightarrow x > -1 \qquad D_f = (-1, \infty)$

$L^+ = \lim_{x\to -1^+} \frac{2}{\sqrt{x+1}} = \frac{2}{\sqrt{-1^+ + 1}} = \frac{2}{\sqrt{0^+}} = \frac{2}{0} = \infty$

$L^- = \lim_{x\to -1^-} \frac{2}{\sqrt{x+1}}$ the limit from left not exist

$\therefore \lim_{x\to -1} \frac{2}{\sqrt{x+1}}$ not exist.

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11 $\lim_{x \to \infty} \frac{1}{(x-2)^2} = \lim_{x \to \infty} \frac{1}{x^2-4x+4}$

$$= \lim_{x \to \infty} \frac{1/x^2}{x^2/x^2 - 4x/x^2 + 4/x^2}$$

$$= \frac{0}{1} = 0$$

12 $\lim_{x \to 0} \left(\frac{x(x+1)}{x} + |x| \right)$

$$= \lim_{x \to 0} \frac{\cancel{x}(x+1)}{\cancel{x}} + \lim_{x \to 0} |x|$$

$$= \lim_{x \to 0} (x+1) + \lim_{x \to 0} |x|$$

$$= (0+1) + 0 = 1$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$L^+ = \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

$$L^- = \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

$\therefore \lim_{x \to 0} |x| = 0$ Since $L^+ = L^- = 0$

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13) $\lim_{x \to 2} (x^2 - 2x + 3)$ H-W

(14) $\lim_{x \to 3} \dfrac{[x]^2 - 9}{x^2 - 9}$

x	$[x]$
$2 \leq x < 3$	2
$3 \leq x < 4$	3

$L^+ = \lim_{\substack{x \to 3^+ \\ x > 3}} \dfrac{[x]^2 - 9}{x^2 - 9}$

$= \dfrac{3^2 - 9}{0^+} = \dfrac{9 - 9}{0^+} = \dfrac{0}{\;}$ كمية صغيرة جداً موجبة very small value +

لأن $3\emptyset$

$L^- = \lim_{\substack{x \to 3^- \\ x < 3}} \dfrac{[x]^2 - 9}{x^2 - 9}$

$= \dfrac{(2)^2 - 9}{0^-} = \dfrac{-5}{0^-} = +\infty$

$L^+ \neq L^-$

$\therefore \lim_{x \to 3} \dfrac{[x]^2 - 9}{x^2 - 9}$ not exist.

(15) $\lim_{x \to 4^+} [[x]]$

$= [4] = 4$

$4 \leq x < 5$

$[x] = 4$

(16) $\lim_{x \to 0} \dfrac{|x+1|}{[x]}$

$$|x+1| = \begin{cases} x+1 & \text{if } x+1 > 0 \Rightarrow x > -1 \\ 0 & x+1 = 0 \Rightarrow x = -1 \\ -x-1 & x+1 < 0 \Rightarrow x < -1 \end{cases}$$

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$\lim_{x \to 0} [x] =$

x	$[x]$
$-1 \le x < 0$	-1
$0 \le x < 1$	0

$L^+ = \lim_{x \to 0^+} [x] = 0$

$L^- = \lim_{x \to 0^-} [x] = -1$

$\therefore \lim_{x \to 0} [x]$ not exist

$\therefore \lim_{x \to 0} \frac{|x+1|}{[x]}$ is not exist.

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$\lim_{x \to 4} ([x] - x)$

x	$[x]$
$4 \le x < 5$	4

$= \lim_{x \to 4^+} [x] - \lim_{x \to 4^+} x$

$= \lim_{x \to 4^+} 4 - 4 = 4 - 4 = 0$

18) $\lim_{x \to \infty} \frac{2x+3}{5x+7}$ H.W

19) $\lim_{x \to \infty} \frac{3x^2 - 6x}{5x+8}$ H.W

20) $\lim_{x \to \infty} \left(\frac{2x^2+3}{x^3+x-5} \cdot \operatorname{sgn}(x) \right)$

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$= \lim_{x \to \infty} \frac{2x^2+3}{x^3+x-5} \cdot \lim_{x \to \infty} \operatorname{sgn}(x)$

$= \lim_{x \to \infty} \frac{2x^2/x^3 + 3/x^3}{x^3/x^3 + x/x^3 - 5/x^3} \cdot \lim_{x \to \infty} (1)$

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$$= \lim_{x\to\infty} \frac{2/x + 3/x^3}{1 + 1/x^2 - 5/x^3} \cdot 1$$

$$= \frac{2(0) + 3(0)^3}{1 + (0)^2 + 5(0)^3} = \frac{0}{1} = 0$$

Some exercises on Continuity.

Q1:- Test the functions continuity at x_0:

A- $f(x) = x^2$, $x_0 = 0$

$D_f = \mathbb{R}$

$0 \in D_f$ $\quad f(0) = (0)^2 = 0$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = (0)^2 = 0$

$\lim_{x\to 0} f(x) = f(0) = 0$

$\therefore$ f is continuous at $x = 0$

B: $f(x) = \frac{1}{2}$, $x_0 = 0$

$D_f = \mathbb{R}$

1. $0 \in D_f$ $\quad f(0) = \frac{1}{2}$

2. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$

3. $\lim_{x\to 0} f(x) = f(\frac{1}{2}) = \frac{1}{2}$

$\therefore$ f is continuous at $x_0 = 0$

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C. $f(x) = \frac{x^2 - 25}{x + 5}$ $\qquad x_0 = -5$

$D_f = \mathbb{R} \setminus \{-5\}$

1. $-5 \notin D_f$, $f(-5)$ not exist

$\therefore$ f is discontinuous at $x = -5$

D. $f(x) = [x]$, $x_0 = 2$

$D_f = \mathbb{R}$

1. $2 \in D_f$

$f(2) = [2] = 2$

2. $\lim\limits_{x \to 2} f(x)$

$L^+ = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} [x] = \lim\limits_{x \to 2^+} 2 = 2$

$L^- = \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} [x] = \lim\limits_{x \to 2^-} 1 = 1$

x	$[x]$
$1 \leq x < 2$	1
$2 \leq x < 3$	2

$L^+ \neq L^-$

$\therefore$ $\lim\limits_{x \to 2} f(x)$ is not exist

$\therefore$ f is discontinuous at $x_0 = 2$

E) $f(x) = \frac{|x^2 - 8|}{x - 9}$, $x_0 = -4$, H-W

Q2 IS the following functions are continuous at $x_0 = 0$. If these functions discontinuous, redefine the function to be continuous at $x_0 = 0$

A- $f(x) = \begin{cases} \frac{x^3 + 3x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$D_f = \mathbb{R}$

1- $0 \in D_f$, $f(0) = 0$

2- $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^3 + 3x}{x} = \lim_{x \to 0} \frac{\cancel{x}(x^2 + 3)}{\cancel{x}}$

$= \lim_{x \to 0} (x^2 + 3) = (0)^2 + 3 = 3$

3- $\lim_{x \to 0} f(x) \neq f(0)$, $3 \neq 0$

$\therefore$ f is discontinuous at $x_0 = 0$

we can redefine the function to make it continuous

$f(x) = \begin{cases} \frac{x^3 + 2x}{x} & x \neq 0 \\ 3 & x = 0 \end{cases}$

f is continuous at $x_0 = 0$

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B- $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$, $x_0 = 0$

$D_f = \mathbb{R}$

1- $0 \in \mathbb{R}$, $f(0) = 0$

2- $\lim_{x \to 0} f(x)$

$L^+ = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{\cancel{x}}{\cancel{x}} = \lim_{x \to 0^+} 1 = 1$

$L^- = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$

$\because L^+ \neq L^-$

$\therefore \lim_{x \to 0}$ is not exist

$\therefore$ f is discontinuous at $x_0 = 0$

and we can not redefine the function since the limit is not exist.

C- $f(x) = x^3$, $x_0 = 0$

$D_f = \mathbb{R}$

1- $0 \in D_f$, $f(0) = (0)^3 = 0$

2- $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^3 = (0)^3 = 0$

3- $\lim_{x \to 0} f(x) = f(0) = 0$

f is continuous at $x_0 = 0$

D) $f(x) = \begin{cases} \frac{1}{2} + x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$D_f = \mathbb{R}$

1- $0 \in D_f$, $f(0) = 1$

2- $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{1}{2} + x\right) = \frac{1}{2} + 0 = \frac{1}{2}$

3- $\lim_{x \to 0} f(x) \neq f(0)$

$\therefore$ f is discontinuous at $x_0 = 0$

We can redefine the function as

$f(x) = \begin{cases} \frac{1}{2} + x & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$

$\therefore$ f is continuous at $x_0 = 0$

E)- $f(x) = \begin{cases} \sqrt[3]{x} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$

$D_f = \mathbb{R}$

1) $0 \in D_f$ $f(0) = 0$

2) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sqrt[3]{x} = \sqrt[3]{\lim_{x \to 0} x} = \sqrt[3]{0} = 0$

3) $\lim_{x \to 0} f(x) = f(0) = 0$

$\therefore$ f is continuous at $x_0 = 0$

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Homework for Limits and continuity

Q1 find the limits for the following functions if exist.

1. $\lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$

2. $\lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right]$

3. $\lim_{x \to -2} \frac{x^3+8}{x+2}$

4. $\lim_{x \to -2} \frac{x^{10}+1024}{x+2}$

5. $\lim_{x \to \infty} \left(\sqrt{n^2+n} - n\right)$

6. $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x-4}$

7. $\lim_{x \to 0} \frac{2-\sqrt{x+4}}{x}$

8. $\lim_{x \to 4} \begin{cases} x^2 & x < 4 \\ \sqrt{x} & x \geq 4 \end{cases}$

Q2 let $f(x) = \frac{x-1}{\sqrt{x}-1}$ and $x \neq 1$,

find $f(1)$ which makes f is continuous at 1?

Q3 find all the points which the following functions are discontinuous.

1- $f(x) = \frac{1}{\sqrt{x-3}}$

2- $f(x) = \sqrt{x-3}$

3- $f(x) = \frac{1}{x-1}$

4- $f(x) = \frac{x+2}{x^2-5x+6}$

5- $f(x) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ 2-x^2 & |x| > 1 \end{cases}$