جامعة بغداد

كلية التربية للعلوم الصرفة / ابن الهيثم

التفاضل والتكامل

قسم الرياضيات

المرحلة الاولى

الفصل الرابع

اساتذة المادة

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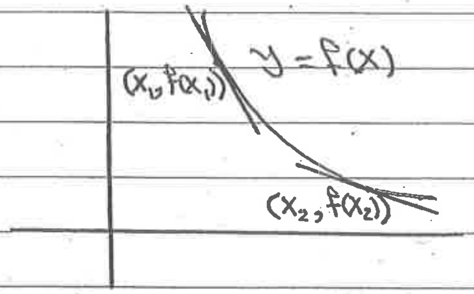
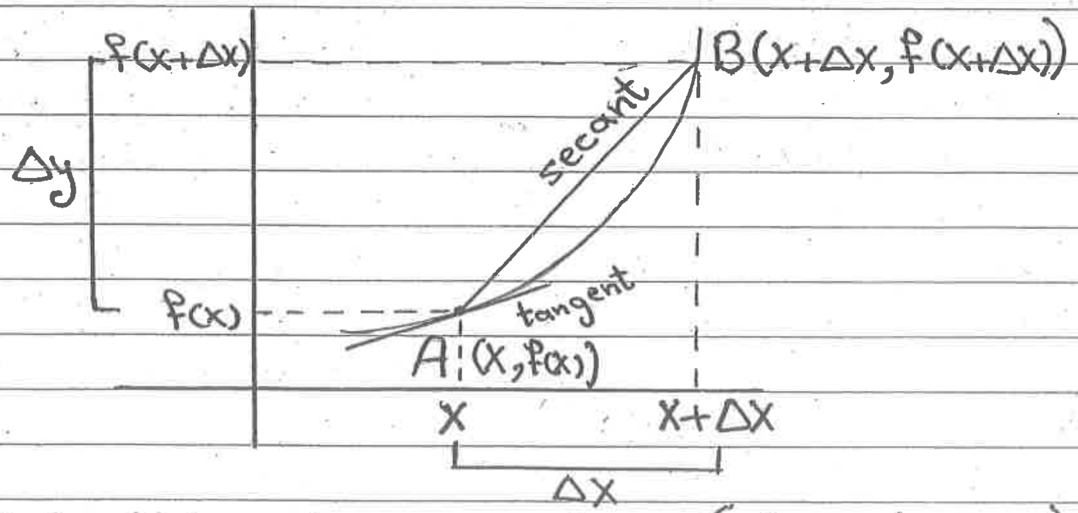
Chapter Four "Differentiation"

Each point on the curve $y=f(x)$ there is a single straight tangent at that point; The slop of straight tangent of the curve $y=f(x)$ at the point $(x,f(x))$ it represents aderivative at that point.



let $A(x,f(x))$ fixed point on the curve; and $B(x+\Delta x, f(x+\Delta x))$ is another point therefore $\Delta y = f(x+\Delta x) - f(x)$



Note that at Δx decreasing length (close to zero) the straight secant AB more and more applicability begins on the straight tangent at the point $(x,f(x))$, this means that slop straight secant AB be equal to slop straight tangent at the point $(x,f(x))$, that's When $(\Delta x \to 0)$, knowing that slop straight tangent at the point $(x,f(x))$ represents aderived function at that point.

$$m_{tan} = \lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

y' , $f'(x)$, $\frac{dy}{dx}$, $\frac{df(x)}{dx}$

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

Note: When the value of the limit exist then the function called differentiable function, and f' called the derivative of f at x.

Ex{1} let $f(x) = 4x - 2$, find $f'(x)$ by definition.

sol $y' = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$f(x) = 4x - 2$, $f(x+\Delta x) = 4(x+\Delta x) - 2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{4(x+\Delta x) - 2 - [4x - 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{4x} + 4\Delta x - 2 - \cancel{4x} + 2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} 4 = 4$$

Ex{2} let $f(x) = \sqrt{x}$, find the equation of the tangent line and normal line at the point $(4,2)$ by definition.

Sol $m_{tan}\Big]_{(4,2)} = f'(x)\Big]_{(4,2)}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x \left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta x \neq 0}} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$m_{tan}\Big]_{(4,2)} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$(y - y_1) = m_{tan}(x - x_1)$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1$$

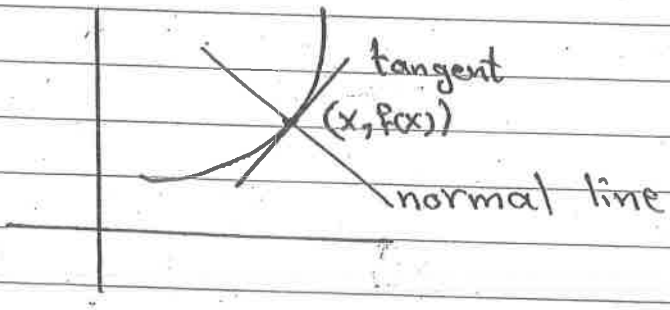
$$m_{\perp} = \frac{-1}{m_{tan}} = \frac{-1}{\frac{1}{4}} = -4$$

$$(y - y_1) = m_{\perp}(x - x_1)$$

$$y - 2 = -4(x - 4)$$

$$y = -4x + 18$$

Definition: the normal line to a curve is the line that is perpendicular to the tangent of the curve at a particular point.



Exc: Find $f'(x)$ by definition:

1- $f(x) = x^3$.

2- $f(x) = x^2 + \frac{1}{x}$.

3- let $f(x) = x^2$, find the equation of the tangent line and normal line at the point $(3,9)$ by definition.

4- using definition to prove that $f'(x) = m$ for $f(x) = y = mx + b$.

5- find the tangent line at $(6,3)$ for $y = \sqrt{x+3}$.

Theorem: Every function is differentiable at X_o, then f is continuous at X_o.

Proof: To prove $\lim_{X \to X_o} f(x) \stackrel{?}{=} f(X_o)$

i.e $\lim_{X \to X_o} [f(x) - f(X_o)] \stackrel{?}{=} 0$

suppose that $\Delta X = X - X_o \Rightarrow X = X_o + \Delta X$

$f(x) = f(X_o + \Delta X)$

When $X \to X_o$, then $\Delta X \to 0$

$$\lim_{X \to X_o} [f(x) - f(X_o)] = \lim_{\Delta X \to 0} [f(X_o + \Delta X) - f(X_o)]$$

$$= \lim_{\Delta X \to 0} \left[\frac{f(X_o + \Delta X - f(X_o)}{\Delta X} \cdot \Delta X \right]$$

$$= \lim_{\Delta X \to 0} \frac{f(X_o + \Delta X) - f(X_o)}{\Delta X} \cdot \lim_{\Delta X \to 0} \Delta X.$$

$$= f'(X_o) \cdot 0$$

$$= 0$$

Note: The inverse of the above theorem is not true, if this function f continuous at the point X_o, it is not necessary to be differentiable at that point as in the example:

let $f(x) = |x|$, $X_o = 0$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{|x + \Delta x| - |x|}{\Delta X} \qquad |\Delta x| = \begin{cases} \Delta x & \text{if } \Delta x \geq 0 \\ -\Delta x & \text{if } \Delta x < 0 \end{cases}$$

$$f'(0) = \lim_{\Delta X \to 0} \frac{|0 + \Delta x| - |0|}{\Delta X} = \lim_{\Delta X \to 0} \frac{|\Delta x|}{\Delta X} \quad L^+ = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1, \quad L^- = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta X} = -1$$

$L^+ \neq L^- \Rightarrow$ limit is not exist.

$\therefore f$ is not differentiable function at $X_o = 0$.

Derivation properties:

Theorem ① let $f(x)=c$, c is aconstant, then $f'(x)=0$.

Proof: By definition

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{c-c}{\Delta x}$$

$f(x)=c$
$f(x+\Delta x)=c$

$$=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=\lim_{\Delta x\to 0} 0 \quad =0$$

Ex: let $f(x)=-5$, then $f'(x)=0$.

Theorem ② let f is differentiable function at the point x, and let c is aconstant, then $(c.f)$ is adifferentiable function at x; $(c.f)'(x)=c.f'(x)$

Proof: By definition

$$(c.f)'(x)=\lim_{\Delta x\to 0}\frac{(c.f)(x+\Delta x)-(c.f)(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{c.f(x+\Delta x)-c.f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{c\left[f(x+\Delta x)-f(x)\right]}{\Delta x}$$

$$=c.\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=c.f'(x)$$ by suppose f is differentiable function at x. $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Ex: let $f(x)=3x$, then $f'(x)=3$.

Theorem (3) let $f(x)$, $g(x)$ are differentiable functions with respect to x, then $(f+g)$ is differentiable function with respect to x; $(f+g)'(x) = f'(x) + g'(x)$.

Proof: By definition

$$(f+g)'(x) = \lim_{\Delta x \to 0} \frac{(f+g)(x+\Delta x) - (f+g)(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - [f(x) + g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - f(x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\{f(x+\Delta x) - f(x)\} + \{g(x+\Delta x) - g(x)\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

by suppose f, g are differentiable functions at x.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

Ex: let $f(x) = 2x$, $g(x) = 1$

$$(f+g)'(x) = f'(x) + g'(x) = (2x)' + (1)' = 2(1) + 0 = 2$$

Remark: let $f_1, f_2, f_3, \ldots, f_n$ are differentiable functions at x, then

$$(f_1 \mp f_2 \mp f_3 \mp \cdots \mp f_n)'(x) = f_1'(x) \mp f_2'(x) \mp \cdots \mp f_n'(x)$$

Theorem ④ let $f(x)=X^n$, and let n be apositive integer number, then:

$$f'(x) = nX^{n-1}$$

Proof: By definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{d}{dx}(x^n) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{x+\Delta x - x}$$

نضيف ونطرح
X الى المقام

let $x=b$, $x+\Delta x = a$

$$\frac{(x+\Delta x)^n - x^n}{(x+\Delta x)-x} = \frac{a^n - b^n}{a-b}$$

$$= \frac{\cancel{(a-b)}\overbrace{(a^{n-1}+a^{n-2}b+\cdots\cdots+ab^{n-2}+b^{n-1})}^{n\text{- times}}}{\cancel{(a-b)}}$$

$$= (a^{n-1}+a^{n-2}b+\cdots\cdots+ab^{n-2}+b^{n-1})$$

$$= (x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + (x+\Delta x)^{n-3}x^2 + \cdots + (x+\Delta x)x^{n-2} + x^{n-1}$$

$$\therefore = \lim_{\Delta x \to 0}\left[(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + (x+\Delta x)x^{n-2} + x^{n-1}\right]$$

$$= \lim_{\Delta x \to 0}(x+\Delta x)^{n-1} + \lim_{\Delta x \to 0}(x+\Delta x)^{n-2} \cdot \lim_{\Delta x \to 0} x + \cdots + \lim_{\Delta x \to 0}(x+\Delta x)\lim_{\Delta x \to 0} x^{n-2} + \lim_{\Delta x \to 0} x^{n-1}$$

$$= x^{n-1} + x^{n-2}\cdot x + \cdots\cdots + x \cdot x^{n-2} + x^{n-1}$$

$$= \underbrace{x^{n-1} + x^{n-1} + x^{n-1} + \cdots\cdots + x^{n-1} + x^{n-1}}_{n\text{- times}} = nX^{n-1}$$

Ex: let $f(x)=X^6$, find $f'(x)$. $f'(x) = 6X^5$.

Theorem (5)

let $f(x)$, $g(x)$ be two differentiable functions at x, then $(f.g)$ is differentiable function at x,

$$(f.g)'(x) = f(x).g'(x) + g(x).f'(x).$$

Proof:

By definition

$$(f.g)'(x) = \lim_{\Delta x \to 0} \frac{(f.g)(x+\Delta x) - (f.g)(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x).g(x+\Delta x) - f(x).g(x)}{\Delta x}$$

بإضافة وطرح $f(x).g(x+\Delta x)$ إلى البسط

$$= \lim_{\Delta x \to 0} \frac{\overset{(1)}{f(x+\Delta x).g(x+\Delta x)} - \overset{(2)}{f(x).g(x)} + \overset{(2)}{f(x).g(x+\Delta x)} - \overset{(1)}{f(x).g(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g(x+\Delta x)\left[f(x+\Delta x) - f(x)\right] + f(x)\left[g(x+\Delta x) - g(x)\right]}{\Delta x}$$

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$$= \lim_{\Delta x \to 0} \left[g(x+\Delta x) \frac{f(x+\Delta x) - f(x)}{\Delta x} + f(x) \frac{g(x+\Delta x) - g(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} g(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} f(x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= g(x) \cdot f'(x) + f(x) \cdot g'(x)$$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

تمارين

Theorem ⑤ let $f(x)$, $g(x)$ be two differentiable functions at x, then $(f\cdot g)$ is differentiable function at x;

$$(f\cdot g)'(x) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Proof: By definition

$$(f\cdot g)'(x) = \lim_{\Delta x\to 0} \frac{(f\cdot g)(x+\Delta x) - (f\cdot g)(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)\cdot g(x+\Delta x) - f(x)\cdot g(x)}{\Delta x}$$

بإضافة وطرح $f(x)\cdot g(x+\Delta x)$ الى البسط

$$= \lim_{\Delta x\to 0} \frac{\overset{①}{f(x+\Delta x)\cdot g(x+\Delta x)} - f(x)\cdot \overset{②}{g(x)} + f(x)\cdot \overset{②}{g(x+\Delta x)} - \overset{①}{f(x)\cdot g(x+\Delta x)}}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{g(x+\Delta x)\left[f(x+\Delta x) - f(x)\right] + f(x)\left[g(x+\Delta x) - g(x)\right]}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \left[g(x+\Delta x)\frac{f(x+\Delta x) - f(x)}{\Delta x} + f(x)\frac{g(x+\Delta x) - g(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0} g(x+\Delta x)\cdot \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x\to 0} f(x)\cdot \lim_{\Delta x\to 0}\frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= g(x)\cdot f'(x) + f(x)\cdot g'(x)$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Ex: let $f(x) = (x^3+2)(1-x^2)$, find $f'(x)$

$$f'(x) = (x^3+2)\frac{d}{dx}(1-x^2) + (1-x^2)\cdot\frac{d}{dx}(x^3+2)$$

$$= (x^3+2)\cdot(-2x) + (1-x^2)\cdot(3x^2).$$

Remark: let f, g, h are differentiable functions at x, then:

$$(f\cdot g\cdot h)'(x) = f(x)\cdot g(x)\cdot h'(x) + f(x)\cdot h(x)\cdot g'(x) + g(x)\cdot h(x)\cdot f'(x)$$

Ex: Let $f(x)=(x^3+2)(1-x^2)$, find $f'(x)$

$$f'(x)=(x^3+2)\frac{d}{dx}(1-x^2)+(1-x^2)\frac{d}{dx}(x^3+2)$$
$$=(x^3+2)\cdot(-2x)+(1-x^2)(3x^2)$$
$$=-2x(x^3+2)+3x^2(1-x^2)$$

Remark: Let f, g, h are differentiable functions at x, then

$$(f\cdot g\cdot h)'(x)=f(x)(g\cdot h)'(x)+(g\cdot h)(x)\cdot f'(x)$$
$$=f(x)\{g(x)\cdot h'(x)+h(x)\cdot g'(x)\}+g(x)\cdot h(x)\cdot f'(x)$$
$$=f(x)\,g(x)\,h'(x)+f(x)\,h(x)\,g'(x)+g(x)\,h(x)\,f'(x)$$
$$=f'(x)\cdot g(x)\cdot h(x)+g'(x)\cdot f(x)\cdot h(x)+h'(x)\cdot f(x)\cdot g$$

Theorem (6)

* : Let $f(x)$, $g(x)$ be two differentiable functions at x, if $g(x)\neq 0$, then $\left(\frac{f}{g}\right)$ is differentiable function at x:

$$\left(\frac{f}{g}\right)'(x)=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{(g(x))^2}$$

Proof

: By definition

$$\left(\frac{f}{g}\right)'(x)=\lim_{\Delta x\to 0}\frac{\left(\frac{f}{g}\right)(x+\Delta x)-\left(\frac{f}{g}\right)(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left(\left\{\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}\right\}\cdot\frac{1}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{g(x)\cdot f(x+\Delta x) - f(x)\cdot g(x+\Delta x)}{g(x)\cdot g(x+\Delta x)} \cdot \frac{1}{\Delta x} \right)$$

بإضافة وطرح $f(x+\Delta x)\cdot g(x+\Delta x)$ الى البسط

$$= \lim_{\Delta x \to 0} \frac{g(x)\cdot f(x+\Delta x) - f(x)\cdot g(x+\Delta x) + f(x+\Delta x) g(x+\Delta x) - f(x+\Delta x) g(x+\Delta x)}{\Delta x \cdot g(x)\cdot g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{-f(x+\Delta x)\{g(x+\Delta x) - g(x)\} + g(x+\Delta x)\{f(x+\Delta x) - f(x)\}}{\Delta x \cdot g(x)\cdot g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \left\{ \frac{-f(x+\Delta x)}{g(x)\, g(x+\Delta x)} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \frac{g(x+\Delta x)}{g(x)\cdot g(x+\Delta x)} \frac{f(x+\Delta x) - f(x)}{\Delta x} \right.$$

$$= \lim_{\Delta x \to 0} \frac{-f(x+\Delta x)}{g(x)\cdot g(x+\Delta x)} \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{1}{g(x)} \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{-f(x)}{g(x)\cdot g(x)} g'(x) + \frac{1}{g(x)} f'(x)$$

by suppose f, g are differentible functions at x.

$$= \frac{-f(x)\cdot g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)}$$

$$= \frac{-f(x)\cdot g'(x) + g(x)\cdot f'(x)}{(g(x))^2}$$

$$= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

Ex₂ Let $f(x) = \frac{x+1}{x}$, find $f'(x)$

$$f'(x) = \frac{x\cdot 1 - (x+1)\cdot 1}{x^2} = \frac{\cancel{x} - \cancel{x} - 1}{x^2} = \frac{-1}{x^2}$$

Corollary

Let $y=f(x)=x^{-n}$, for $n\in Z_{+}$, $x\neq 0$, then
$$y'=f'(x)=\frac{dx^{-n}}{dx}=-n\,x^{-n-1}$$

Proof: $\frac{dx^{-n}}{dx}=\frac{d}{dx}\left(\frac{1}{x^n}\right)$

$$=\frac{x^n\frac{d(1)}{dx}-(1)\cdot\frac{dx^n}{dx}}{(x^n)^2} \quad \text{by Thm (6)}$$

$$=\frac{x^n\cdot 0-n\cdot x^{n-1}}{x^{2n}} \quad \text{by Thm (1) \& (4)}$$

$$=\frac{-n\,x^{n-1}}{x^{2n}}$$

$$=-n\,x^{n-1}\cdot x^{-2n}$$

$$=-n\,x^{-n-1}$$

Ex. Let $f(x)=-5x^{-3}$, find $f'(x)$

$$f'(x)=-5(-3x^{-4})=15x^{-4}$$

$$\frac{d}{dx}(x^r)=r\,x^{r-1}\quad\forall r\in\mathbb{R}$$

generalization

Theorem (4) Derivation of the composition function (chain Rule)

Let g is adifferentiable function at x, f is adifferentiable function at $g(x)$, and let $h = f\circ g$, then h is adifferentiable function at x,

$$h'(x) = f'(g(x)) \cdot g'(x)$$

proof: By definition

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(f\circ g)(x+\Delta x) - (f\circ g)(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(u+\Delta u) - f(u)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{f(u+\Delta u) - f(u)}{\Delta x} \cdot \frac{\Delta u}{\Delta u} \right)$$

نفرض أن $u = g(x)$

$u + \Delta u = g(x+\Delta x)$

$\Delta u = g(x+\Delta x) - u$

$\Delta u = g(x+\Delta x) - g(x)$

عندما $\Delta x \to 0$

فإن $\Delta u \to 0$

$$= \lim_{\Delta x \to 0} \frac{f(u+\Delta u) - f(u)}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

، أي $\frac{du}{dx} = \frac{dg(x)}{dx} = g'(x)$

$$= \lim_{\Delta u \to 0} \frac{f(u+\Delta u) - f(u)}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$= f'(u) \cdot g'(x)$ by suppose g is differentiable function at x.

$= f'(g(x)) \cdot g'(x)$

Ex: Let $h(x) = (2x^3 + x^2 - 5x + 1)^{15}$, find $h'(x)$.

Sol.: Let $g(x) = 2x^3 + x^2 - 5x + 1$, $f(x) = x^{15}$.

$$h'(x) = f'(g(x)) \cdot \underbrace{g'(x)}_{\text{مشتقة داخل القوس}}$$

$$f'(x) = 15x^{14}$$

$$f'(g(x)) = 15(g(x))^{14} = 15(2x^3 + x^2 - 5x + 1)^{14}$$

$$g'(x) = 6x^2 + 2x - 5$$

$$h'(x) = 15(2x^3 + x^2 - 5x + 1)^{14} \cdot (6x^2 + 2x - 5).$$

Corollary: Let f is a differentiable function at x, and let $y = (f(x))^n$, for $n \in \mathbb{Z}_1$, then

$$\frac{dy}{dx} = n\,(f(x))^{n-1} \cdot f'(x)$$

EXC, Find $\frac{dy}{dx}$?

1. $y=\left(\frac{x^2+3}{x+1}\right)^4$

2. $y=(2\sqrt{x}-1)^3$

3. $y=\sqrt{3-x^2}$

4. $y=\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}$

5. $y=(x^3+2)^2(1-x^2)^3$

6. $y=\frac{(1+2x^3)(1+x^4)}{x^2}$

7. $y=\sqrt{x+\sqrt{1+\sqrt{x}}}$

8. $y=x^3-\frac{1}{x^2+1}$

9. $y=\frac{\sqrt[4]{x^2+1}}{(x+2)^4}$

10. $y=x^2(x^2+1)^{-\frac{1}{3}}$

11. Let $f(x)=x$, $g(x)=x^2$, whats the value of x that makes the tangent line of two curves are parallel.

12. Let $f(x)=\frac{1}{\sqrt{x}}$, what the value of x that make the tangent of the curve when its parallel to line $x+8y=10$

Chain Rule

- Let $y=f(x)$, $x=g(t)$, find $\frac{dy}{dt}$

$$\boxed{\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}}$$

- Let $y=f(t)$, $t=g(x)$, find $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}}$$

- Let $y=f(t)$, $x=g(t)$, find $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Ex: ① Let $y=3x-1$, $x=2t$, find $\frac{dy}{dt}$

Sol: $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$

$$=(3)\cdot(2)$$
$$=6$$

or $y=3x-1$

$$=3(2t)-1$$
$$y=6t-1$$
$$\frac{dy}{dt}=6$$

EX (2)

: Let $y = t^2 - 1$, $x = 2t + 3$, find $\frac{dy}{dx}$ at $t=1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\cancel{2}t}{\cancel{2}}$$

$$= t$$

$$\left.\frac{dy}{dx}\right]_{t=1} = 1$$

EXC

1. Let $y = x^3 - 3x^2 + 5x - 4$, $x = t^2 + t$, find $\frac{dy}{dt}$
Find $\frac{dy}{dx}$:

2. Let $y = u^3 + 1$, $u = x^2 + 3$

3. $y = 3t^2 - 1$, $x = 6t - 1$

4. $y = \frac{t^2}{1+t}$, $x = \frac{t}{2+t}$

5. $y = t^2$, $x = \frac{t}{1-t}$

6. $y = z^{2/3}$, $z = x^2 + 1$

7. $y = w^2 - w^{-1}$, $w = 3x$

8. $y = 2v^3 + \frac{2}{v^3}$, $v = (2x+2)^{2/3}$

9. $y = \frac{u^2}{u^2+1}$, $u = \sqrt{2x+1}$

١٨

Implicit differentiation

EX: $x^2 + xy + y^5 = 0$

we derive implicity for x considering y implicit function of x

$$2x\frac{dx}{dx} + \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 5y^4\frac{dy}{dx} = 0$$

$$2x + xy' + y + 5y^4y' = 0$$

$$xy' + 5y^4y' = -2x - y$$

$$(x + 5y^4)y' = -2x - y$$

$$y' = \frac{-2x - y}{x + 5y^4}$$

we derive implicity for y (considering x implicit function of y.)

$$2x\frac{dx}{dy} + \left(x\frac{dy}{dy} + y\frac{dx}{dy}\right) + 5y^4\frac{dy}{dy} = 0$$

$$2x\,x' + x + yx' + 5y^4 = 0$$

$$x + 5y^4 = -2xx' - yx'$$

$$x + 5y^4 = (-2x - y)x'$$

$$x' = \frac{x + 5y^4}{-2x - y}.$$

Note that: $x' = \dfrac{1}{y'}$

Ex

: Find the equation of the tangent line and normal line of the curve $x^2+y^2=2$ at the point $(1,1)$.

Sol: $2x\frac{dx}{dx}+2y\frac{dy}{dx}=0$

$2x+2yy'=0$

$2yy'=-2x$

$y'=\frac{-\cancel{2}x}{\cancel{2}y}=\frac{-x}{y}$

$y'=m\Big]_{(1,1)}=\frac{-1}{1}=-1$

$(y-y_1)=m(x-x_1)$

$y-1=-1(x-1)$

$y-1=-x+1$

$y=-x+2$ the eqn of the tangent line.

$m_\perp\Big]_{(1,1)}=\frac{-1}{m\Big]_{(1,1)}}$

$m_\perp\Big]_{(1,1)}=\frac{-1}{-1}=1$

$(y-y_1)=m(x-x_1)$

$y-1=1\ (x-1)$

$y=x$ the eqn of the normal line

EXC

1- Find the slope of the tangent line of the curve $x^2+xy+y^2=7$ at the point $(1,2)$.

2- Find the slope of the tangent line for the circle equation $8x^2+8y^2=232$ at the point $(-5,2)$.

3- Find the equation of the normal line and the tangent line for the curve $xy^2-yx^2=0$ at the point $(1,1)$.

Find $\frac{dy}{dx}=y'$, $\frac{dx}{dy}=x'$.

4- $x^3y^2+2xy-x+3y=6$

5- $x^2+x^3=y+y^4$

6- $\frac{1}{x}+\frac{1}{y}=x+y$

7- $x^2-\sqrt{xy}+y^2=6$

8- $x^3+y^3-9xy=0$

9- $xy^2+yx^2=3y^3$

10- $2-y^3+x^2y=5$

11- $(1+x^2y)^3+x\sqrt{y}=9$

Higher - order derivatives

Let $y = f(x)$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = y' = y^{(1)}$$ المشتقة الأولى للدالة

$$f''(x) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x} = \frac{d^2y}{dx^2} = y'' = y^{(2)}$$ = الثانية =

$$f'''(x) = \lim_{\Delta x \to 0} \frac{f''(x+\Delta x) - f''(x)}{\Delta x} = \frac{d^3y}{dx^3} = y''' = y^{(3)}$$ = الثالثة =

$$\vdots$$

$$f^{(n)}(x) = \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x+\Delta x) - f^{(n-1)}(x)}{\Delta x} = \frac{d^ny}{dx^n} = y^{(n)} \; ; n \in \mathbb{N}$$ المشتقة النونية للدالة

Note

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$ Second derivative ,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$ third derivative

EX, Let $y = 2x^3 + x^2 - 1$

$$y' = 6x^2 + 2x$$

$$y'' = 12x + 2$$

$$y''' = 12$$

$$y^{(4)} = 0 , \ldots$$

EXC: Find y', y''

1. $x^5+3x^2y-6x=7$

2. $y=t^2+4$, $x=2t^2+3$

Find y', y'', y'''

1. $y=x^3-9x-5$

2. $y=x^4-4x^3+4x^2+1$

Rolle's Theorem

Let $f(x)$ be a continuous function at every point of the interval $[a,b]$, and f is differentiable on (a,b), if $f(a)=f(b)$, then $f'(c)=0$; $c\in(a,b)$.

EX: Let $f(x)=x^2-3x+2$ $[1,2]$

Sol: f is continuous function at $[1,2]$
f is differentiable function at $(1,2)$
$a=1$, $b=2$
$f(a)=f(1)=(1)^2-3(1)+2=-2+2=0$
$f(b)=f(2)=(2)^2-3(2)+2=4-6+2=-2+2=0$
$f(1)=f(2)$
$f'(x)=2x-3$

$f'(c) = 2c - 3$

$\therefore \exists\, c \in (1,2)$

S.t. $f'(c) = 0$

$2c - 3 = 0 \Rightarrow 2c = 3 \Rightarrow c = \frac{3}{2} \in (1,2).$

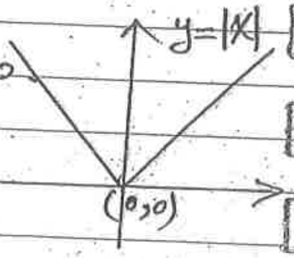
Ex

: Show that: $f(x) = 1 - |x|$ is not satisfy Rolle's theorem at $[-1,1]$.

Sol.: f is Cont. fun. at $[-1,1]$

f is not diff. fun. at $(-1,1)$

since f is not diff. fun. at $x = 0$



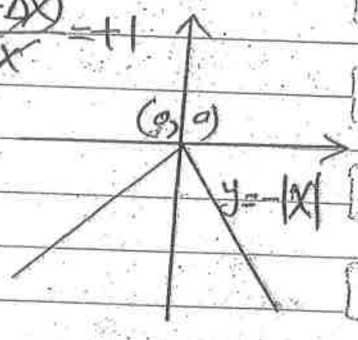
$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{1} - |x+\Delta x| - \cancel{1} + |x|}{\Delta x}$$

$$f'(0) = \lim_{\Delta x \to 0} \frac{-|x+\Delta x| + |x|}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-|\Delta x|}{\Delta x}$$

$$L^+ = \lim_{\Delta x \to 0^+} \frac{-\cancel{\Delta x}}{\cancel{\Delta x}} = -1$$

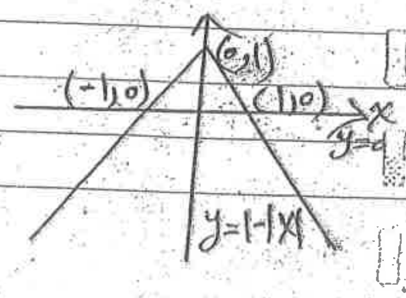
$$L^- = \lim_{\Delta x \to 0^-} \frac{-(-\cancel{\Delta x})}{\cancel{\Delta x}} = +1$$



$L^+ \neq L^-$

Limit is not exist.

$f'(0)$ not exist



EXC

1- Let $f(x)=(2-x)^2$, $x\in[0,4]$

2- $f(x)=9x+3x^2-x^3$, $x\in[-1,1]$

3- $f(x)=\begin{cases} x^2+1 & , x\in[-1,2] \\ -1 & , x\in(-4,-1) \end{cases}$

The Mean value theorem

Let $f(x)$ is a continuous on closed interval $[a,b]$, and differentiable on the interval (a,b). then there exist at least one point c in (a,b) at which

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

EX

Let $f(x)=x^2$ $[0,2]$

f is cont fun at $[0,2]$

f is diff fun at $(0,2)$

$a=0$, $b=2$

$f(a)=f(0)=0^2=0$

$f(b)=f(2)=2^2=4$

$f'(x)=2x$

$f'(c)=2c$

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{4-0}{2-0}=\frac{4}{2}=2$$

\\.

$2c = 2 \Rightarrow c = 1 \in (0, 2)$.

<u>Note</u>: the Rolle's theorem is special case of the Mean Value theorem.

<u>EX</u>: Let $f(x) = \sqrt{25 - x^2}$, $x \in [-4, 0]$.

f is Cont fun at $[-4, 0]$.
f is diff fun at $(-4, 0)$.

$25 - x^2 \geq 0 \Rightarrow x^2 \leq 25$
$\Rightarrow |x| \leq 5$
$\Rightarrow -5 \leq x \leq 5$
$D_f = [-5, 5]$

$a = -4$, $b = 0$

$f(a) = f(-4) = \sqrt{25 - (-4)^2} = \sqrt{25 - 16} = \sqrt{9} = 3$

$f(b) = f(0) = \sqrt{25 - (0)^2} = \sqrt{25} = 5$

$f'(x) = \frac{1}{\cancel{2}} (25 - x^2)^{-\frac{1}{2}} \cdot -\cancel{2}x$

$f'(x) = \frac{-x}{\sqrt{25 - x^2}}$

$f'(c) = \frac{-c}{\sqrt{25 - c^2}}$

$\therefore \exists c$ s.t. $c \in (-4, 0)$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\frac{-c}{\sqrt{25 - c^2}} = \frac{5 - 3}{0 - (-4)} = \frac{2}{4} = \frac{1}{2}$

$\frac{-c}{\sqrt{25 - c^2}} \times \frac{1}{2}$

$-2C = \sqrt{25 - C^2}$ by squaring

$4C^2 = 25 - C^2$

$5C^2 = 25$

$C^2 = \frac{\cancel{25}^{5}}{\cancel{5}_{1}}$

$C^2 = 5$

$|C| = \sqrt{5}$

$C = \mp\sqrt{5} = \mp 2.2$

$C = +\sqrt{5} \notin (-4, 0)$

$C = -\sqrt{5} \in (-4, 0)$

EXC

1- Find the value of c which satisfy the Mean value theorem,

$f(x) = x^2 - 6x + 4, \quad x \in [-1, 7]$.

2- Let $f(x) = x^3 - 4x^2$, $f: [0, b] \to \mathbb{R}$
f satisfy the mean value theorem at $C = \frac{2}{3}$, find the value of b.

L' Hopital's Rule $\frac{0}{0}$:

Let f, g are differentiable functions at x_0 and if :-

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

i.e. $f(x_0) = g(x_0) = 0$ $\qquad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$

and $\lim_{x \to x_0} g'(x) = g'(x_0) \neq 0$

, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = K \text{ (constant)}$$

فأن :

and if $\lim_{x \to x_0} f'(x) = \lim_{x \to x_0} g'(x) = 0$

i.e. $f'(x_0) = g'(x_0) = 0$ $\qquad \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \frac{0}{0}$

We derive again :-

Then: $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \lim_{x \to x_0} \frac{f''(x)}{g''(x)} = C \text{ (Constant)}$

ملاحظة

: أن قاعدة لوبيتال يمكن تطبيقها عندما $x_0 = +\infty$ وكذلك

هنالك قاعدة مشابهة تدعى بقاعدة لوبيتال للصيغة $\frac{\infty}{\infty}$

وأن هاتين القاعدتين صحيحتين في حالة إيجاد الغاية من الجهة

اليمنى أو اليسرى.

EX ①

Find $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 1}$

$$\lim_{x\to 1}(x^2-3x+2)=0,\quad \lim_{x\to 1}(x^2-1)=0$$

$$\therefore \lim_{x\to 1}\frac{x^2-3x+2}{x^2-1}=\lim_{x\to 1}\frac{2x-3}{2x}=-\frac{1}{2}\ \text{exist}$$

EX (2) : Find $\displaystyle\lim_{x\to 0}\frac{2-\sqrt{x+4}}{x}=\frac{0}{0}$

$$=\lim_{x\to 0}\frac{-\frac{1}{2\sqrt{x+4}}}{1}=-\frac{1}{2}\lim_{x\to 0}\frac{1}{\sqrt{x+4}}$$

$$=-\frac{1}{2}\cdot\frac{1}{2}=-\frac{1}{4}$$

طريقة ثانية : الضرب بالمرافق

$$\lim_{x\to 0}\frac{2-\sqrt{x+4}}{x}=\lim_{x\to 0}\frac{(2-\sqrt{x+4})(2+\sqrt{x+4})}{x(2+\sqrt{x+4})}$$

$$=\lim_{x\to 0}\frac{4-(x+4)}{x(2+\sqrt{x+4})}=\lim_{x\to 0}\frac{\cancel{4}-x-\cancel{4}}{x(2+\sqrt{x+4})}$$

$$=\lim_{x\to 0}\frac{-\cancel{x}}{\cancel{x}(2+\sqrt{x+4})}=\lim_{x\to 0}\frac{-1}{2+\sqrt{x+4}}$$

$$=\frac{-1}{2+\sqrt{0+4}}=\frac{-1}{2+2}=-\frac{1}{4}$$

EXC

: Find the Limit if it exist :

1. $\lim_{x \to 2} \frac{x^2+2x-8}{x^2-9x+14}$

2. $\lim_{x \to 0} \frac{\sqrt{x+9}-3}{x}$

3. $\lim_{x \to 1} \frac{x^2+5x+4}{x^2-4x-5}$

4. $\lim_{x \to 0} \frac{4x^3+3x^2-8x+1}{x^3+2x^2+3x-6}$

5. $\lim_{x \to 1} \frac{x^2-3x+2}{x^2-1}$

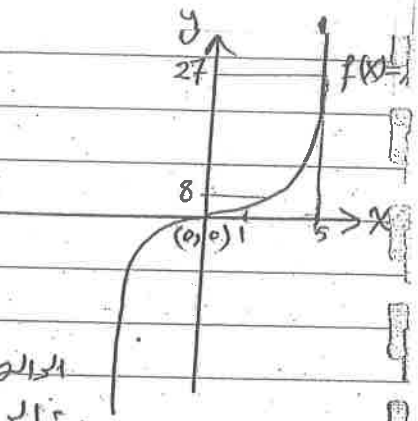
6. $\lim_{x \to 0} \frac{x^3+4x^2-5x}{x^3-2x}$

Increasing and decreasing functions

Definition :- A function f defined on interval $[a,b]$ is said to be increasing on $[a,b]$ if, $f(x_1) < f(x_2)$ $\forall$ x_1, x_2 s.t.:

$$a \leq x_1 < x_2 \leq b .$$

EX: Let $f(x) = x^3$, $[1,5]$



$$2 < 3$$
$$f(2) = 8$$
$$f(3) = 27$$
$$8 < 27$$
$$f(2) < f(3)$$

الدالة متزايدة على $[1,5]$

الدالة f متزايدة على $\mathbb{R}$.

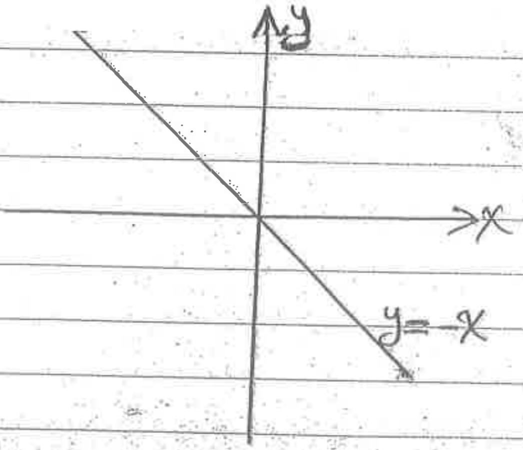
H.W.

Let $f(x) = x$, prove that f is increasing function on $\mathbb{R}$.

Definition :- A function f defined on interval $[a,b]$ is said to be decreasing on $[a,b]$ if, $f(x_1) > f(x_2)$ $\forall$ x_1, x_2 s.t.:

$$a \leq x_1 < x_2 \leq b .$$

EX: Let $f(x) = -x$



$$2 < 3$$
$$f(2) = -2$$
$$f(3) = -3$$
$$-2 > -3$$
$$f(2) > f(3)$$

الدالة f متناقصة على $\mathbb{R}$.

H-W.

: Find the intervals that the function is increasing and decreasing:

1. $f(x) = |x|$

2. $f(x) = \frac{1}{x}$

Definition :- let f is defined and continuous function on $[a,b]$ and let $x_0 \in [a,b]$, then the point $(x_0, f(x_0))$ is said to be (critical point) of f iff :-

$f'(x_0) = 0$ or $f'(x_0)$ is not defined.

EX

: لتكن الدالة $f(x)=x^2$ دالة معرفة ومستمرة على الفترة المغلقة $[-1,1]$ 4

Sol

: $0 \in [-1,1]$

$f'(x) = 2x$

$f'(0) = 0$

$x = 0 \Rightarrow y = 0$

$\therefore (0,0)$ is critical point.

EX

: لتكن الدالة $f(x)=|x|$ معرفة ومستمرة على الفترة $[-1,1]$

$0 \in [-1,1]$

$f'(0)$ not exist.

$x = 0 \Rightarrow y = 0$

$\therefore (0,0)$ is critical point.

هناك علاقة بين كون الدالة متزايدة او متناقصة واشارة مشتقتها
والمبرهنة التالية توضح ذلك :-

Theorem :- Let f is adifferentiable function $\forall x$ s.t. $a<x<b$, and conts. $\forall x$ s.t. $a\leq x\leq b$ then :-

(a) f is increasing on $[a,b]$ if $f'(x)>0 \quad \forall \; x\in(a,b)$.

(b) f is decreasing on $[a,b]$ if $f'(x)<0 \quad \forall \; x\in(a,b)$.

ملاحظة

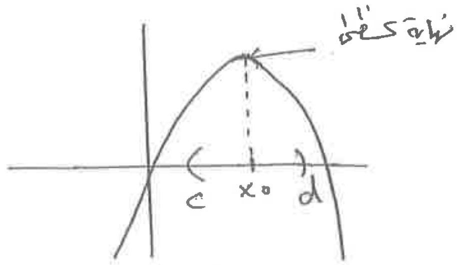
:- ان معرفة كون الدالة متزايدة أو متناقصة على فترة معينة يسهل لنا عملية رسم مخطط الدالة.

Maximum and Minimum points

The function f is said to be :-

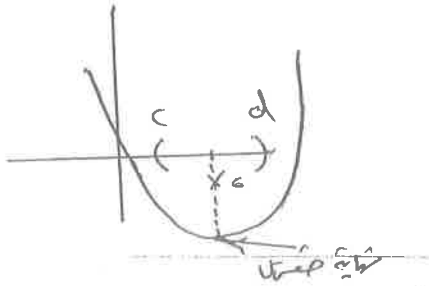
(a) Locally maximum point at x_0 if There is open interval (c,d) contain x_0 s.t :-

$f(x_0) \geq f(x) \quad \forall \quad x\in(c,d)$



(b) Locally minimum Point at x_0 if There is open interval (c,d) contain x_0 s.t :

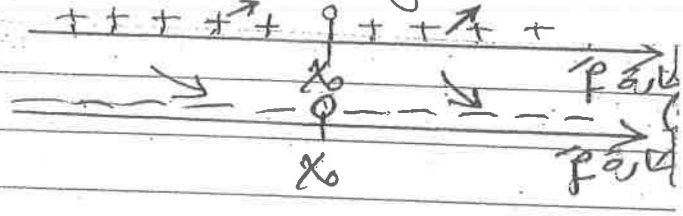
$f(x_0) \leq f(x) \quad \forall \quad x\in(c,d)$



(c) It is said to the largest value taken by the function f on $[c,d]$ in the (Absolute maximum point). and for the Lowest value at the (Absolute minimum point).

Theorem: Let f function has maximum Point or minimum point at x_0, then x_0 is a Critical point.

* the inverse theorem is not true. by the following example:



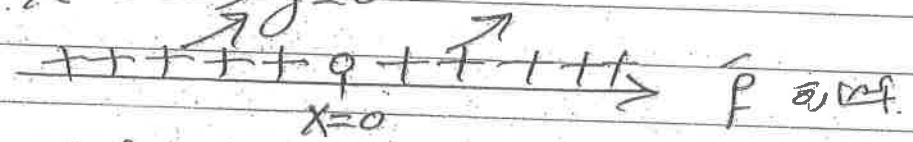
EX:

$f(x) = x^3$

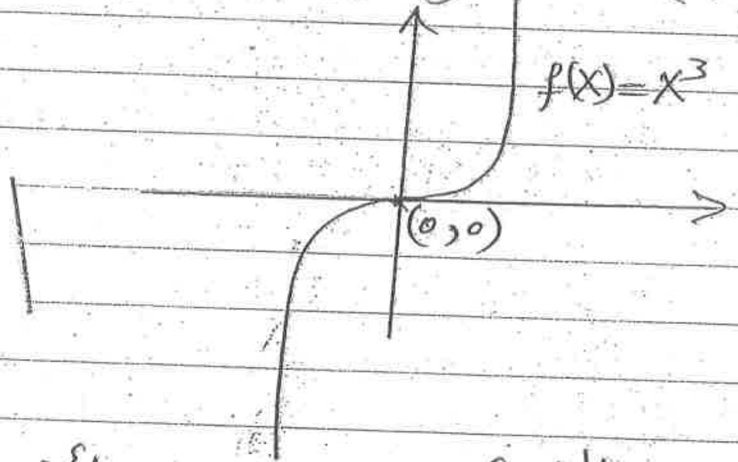
$f'(x) = 3x^2$ موجبة

$f'(x) = 0 \Longrightarrow 3x^2 = 0 \Longrightarrow x = 0$

$x = 0 \Longrightarrow y = 0$



نقطة حرجة وليست نهاية $(0,0)$

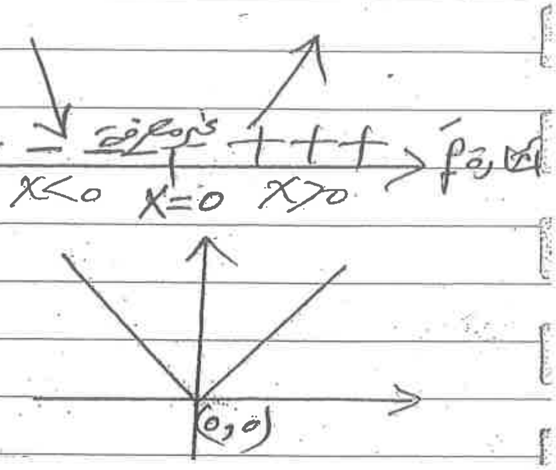


الدالة f ليس لها نهاية عظمى أو صغرى عند $x=0$ لأنه لا توجد فترة مفتوحة حولها كانت فيها تحوي الصفر تحقق خاصية النهاية العظمى أو الصغرى.

Ex : Let $f(x) = |x|$

$$= \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \text{غير معرف} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$



$f'(0)$ not exist.

: $(0,0)$ نقطة حرجة وليست زاوية.

Concavity

Def : Let f is a differentiable function at (a,b) then :-

(a) The function f is concave to the top (concavity) on (a,b) if f' increasing in this interval.

$f'' > 0 \Rightarrow f'$ increasing $\Rightarrow f$ concave to the top.

(b) The function f is concave to the down (convexity) on (a,b) if f' decreasing in this interval.

$f'' < 0 \Rightarrow f'$ decreasing $\Rightarrow f$ concave to the down.

ملاحظة

: لكي تكون الدالة f مقعرة الى الأعلى او الى الأسفل يجب أن تكون مشتقة الدالة موجودة.

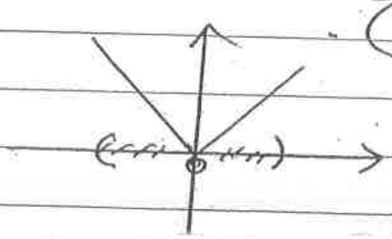
EX:

$f(x) = |x|$

$f'(0)$ not exist

∴ لايوجد مشتقة للدالة عند النقطة $(0,0)$

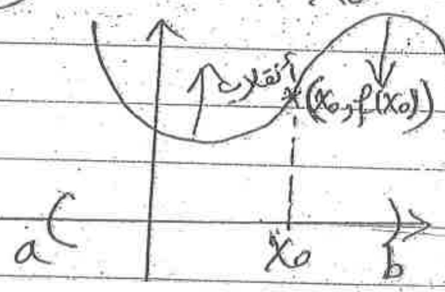
∴ لا يوجد تقعر او تحدب على أي فترة مفتوحة حاوية على النقطة $(0,0)$.



Inflection point (

Def: It is said to point x_0 inflection point for f if: $f'(x_0)$ exist and f change of direction of the curve at this point, which means that if there is (a,b) contain x_0 s.t. the curve concave to the top on (a,x_0) and concave to the down on (x_0,b) and conversely.

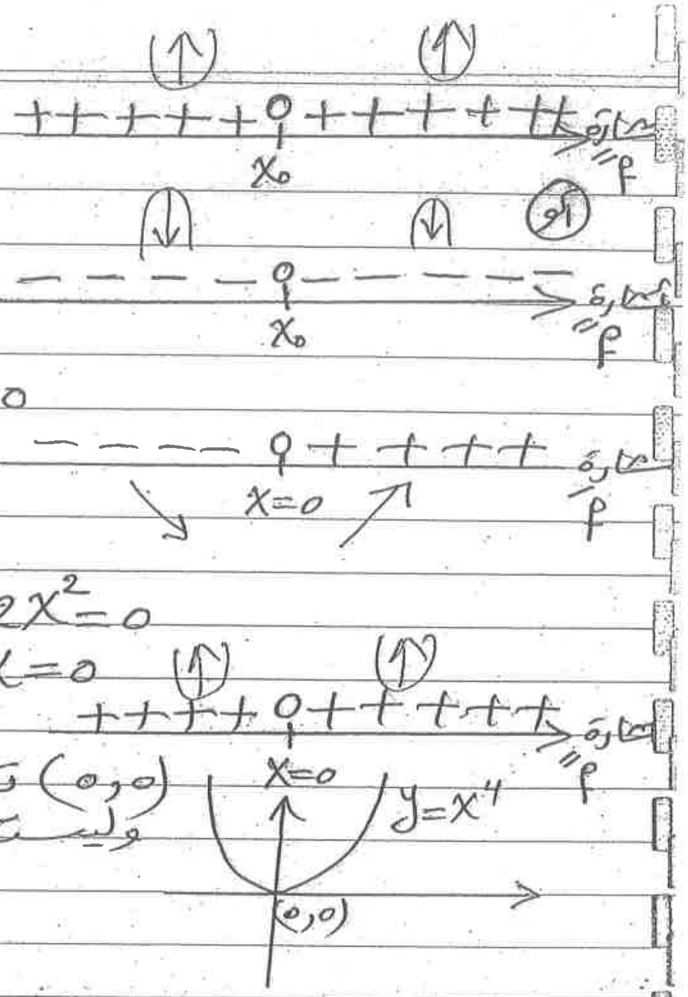
$f''(x_0)=0 \Longleftarrow$ نقطة أنقلاب x_0



Theorem: let f be a function and let x_0 inflection point then if $f''(x_0)$ exist then $f''(x_0)=0$,

the converse theorem is not true, by the following example

Ex, Let $f(x) = x^4$



Sol, $f'(x) = 4x^3$

$f'(x) = 0 \Longrightarrow x = 0$

$f''(x) = 12x^2$

$f''(x) = 0 \Longrightarrow 12x^2 = 0$

$\Longrightarrow x = 0$

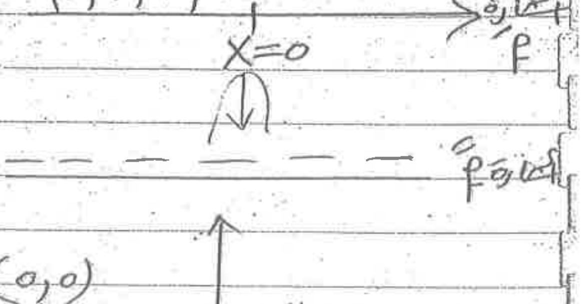
$x = 0 \Longrightarrow y = 0$

(0,0) نقطة نهاية صغرى (محلية)
وليست نقطة انقلاب.

Ex, Let $f(x) = -x^2$

Sol, $f'(x) = -2x$

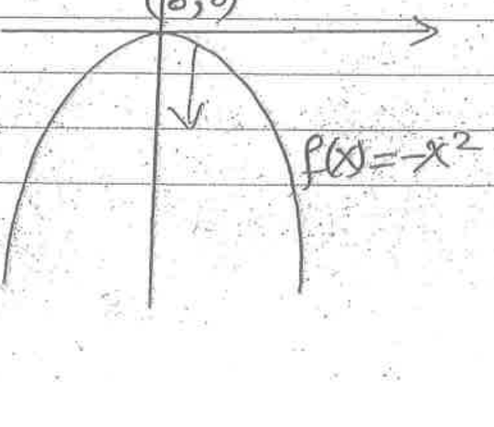
$f'(x) = 0 \Longrightarrow x = 0$



$f''(x) = -2$

$x = 0 \Longrightarrow y = 0$

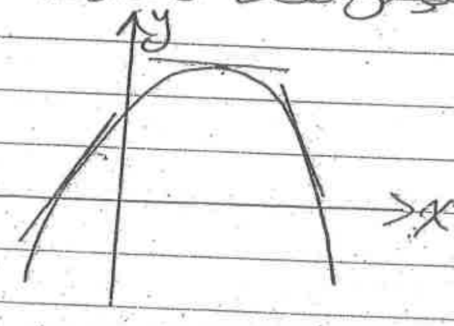
(0,0) نقطة نهاية عظمى (محلية)



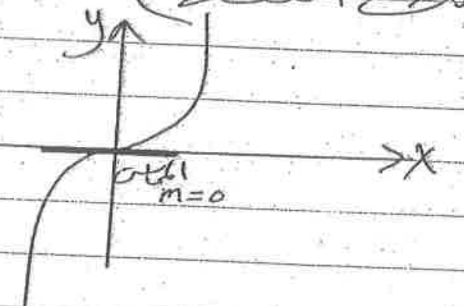
ملاحظة

لتكن f دالة قابلة للاشتقاق عند كل نقطة في الفترة (a,b) فان f مقعرة نحو الأعلى عند هذه الفترة اذا كان المستقيم المماس عند كل نقطة من هذه الفترة يقع أسفل مخطط f.

ويقال للدالة f بأنها مقعرة نحو الأسفل على هذه الفترة اذا كان المستقيم المماس عند كل نقطة من هذه الفترة يقع أعلى مخطط الدالة f.



وان المماس سوف يقطع مخطط f عند نقطة الانقلاب.
(يجب أن يكون للمنحني عند نقطة الانقلاب مماس وغير يخترق المنحني عند تلك النقطة)



Theorem مبرهنة

لتكن f دالة قابلة للاشتقاق في الفترة المفتوحة الحاوية على x_0 فان:

أ) اذا كانت $f'(x_0)=0$ و $f''(x_0)>0$ فان f لها نهاية صغرى عند النقطة x_0

ب) " " " $f''(x_0)<0$ " " " عظمى " " "

ملاحظة

اذا كانت $f'(x)=0$ و $f''(x)=0$ فأن للدالة f نهاية عظمى أو صغرى أو نقطة تحول.

أرسم مخطط الدوال الآتية :-

EX(1) Let $f(x)=\sqrt[3]{x}=x^{\frac{1}{3}}$

$D_f=\mathbb{R}$

لا توجد محاذيات شاقولية وأفقية.

$x=0 \Rightarrow y=0$

$y=0 \Rightarrow \sqrt[3]{x}=0$ بالتكعيب $\Rightarrow x=0$ $\Rightarrow$ نقطة التقاطع $(0,0)$ مع المحورين.

$f(-x)=\sqrt[3]{-x}=-\sqrt[3]{x}=-f(x)$ المنحني متناظر مع نقطة الأصل

$f(-x)\neq f(x)$ المنحني غير متناظر مع المحور y.

$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{x^2}}$ موجب

إشارة f': $+ + + + +$ ($x<0$) | غير معرف ($x=0$) | $+ + + + +$ ($x>0$)

$f'=0 \Rightarrow 1=0$ وهذا غير ممكن

$(0,0)$ نقطة حرجة لأن $f'(0)$ غير معرفة

مناطق التزايد هي $(0,\infty)$, $(-\infty,0)$

$f''(x)=\frac{1}{3}\cdot\frac{-2}{3}\cdot x^{-\frac{5}{3}}$

$=\frac{-2}{9}\cdot\frac{1}{\sqrt[3]{x^5}}$

$=\frac{-2}{9x\sqrt[3]{x^2}}$ سالب / موجب

إشارة f'': $+ + + + +$ | غير معرف ($x=0$) | $- - - - -$

$(0,0)$ نقطة انقلاب

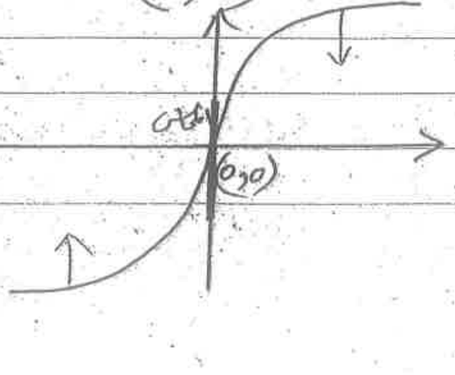
$f''(0)$ غير معرفة

مناطق التقعر $(-\infty,0)$

ومناطق التحدب $(0,\infty)$

$f''=0 \Rightarrow -2=0$ غير ممكن

$R_f=\mathbb{R}$



EX(2)

$$f(x)=y=\sqrt[3]{x^2}=x^{2/3}$$

$$D_f=\mathbb{R}$$

لا توجد محاذيات شاقولية وأفقية

$x=0 \Longrightarrow y=\sqrt[3]{0^2}=0$

$y=0 \Longrightarrow 0=\sqrt[3]{x^2} \Longrightarrow x=0$ $\Big\} \Longrightarrow$ (0,0) نقطة التقاطع مع المحورين .

$f(-x)=\sqrt[3]{(-x)^2}=\sqrt[3]{x^2}=f(x)$ المنحني متناظر مع المحور y

$-f(x)=-\sqrt[3]{x^2}\neq f(-x)$ المنحني غير متناظر مع نقطة الأصل

$$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3\sqrt[3]{x}}$$

$f'(x)=0 \Longrightarrow 2=0$ غير ممكن

$- \ - \ -$ غير معرف $+ + + + + +$ إشارة f'

$x=0$

(0,0) نقطة حرجة لأن $f'(0)$ غير معرفة

لا توجد نقاط زوايا

مناطق التناقص $(-\infty,0)$ ومناطق التزايد $(0,\infty)$

$$f''(x)=\frac{-2}{9}x^{-4/3}=\frac{-2^{\text{سالب}}}{9\sqrt[3]{x^4}}=\frac{-2}{9x\sqrt[3]{x}}$$

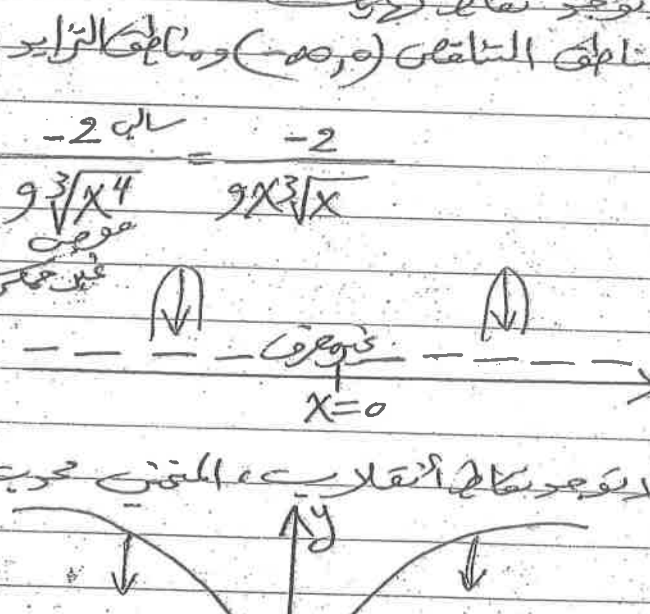
$f''(x)=0 \Longrightarrow -2=0$ غير ممكن

$- \ - \ - \ -$ غير معرف $- \ - \ - \ -$ إشارة f''

$x=0$

لا توجد نقاط انقلاب ، المنحني محدب على الفترتين $(-\infty,0)$ و $(0,\infty)$

$$R_f=[0,\infty)$$



Ex (3)

$$f(x) = \frac{1}{x^2-1}$$

$$x^2-1=0 \Rightarrow x^2=1 \Rightarrow \sqrt{x^2}=\sqrt{1} \Rightarrow |x|=1 \Rightarrow x=\mp 1$$

$$D_f = \mathbb{R} \setminus \{1,-1\}$$

$x=1 \,,\; x=-1$ المحاذيات الشاقولية

$$\cancel{y = \frac{1}{x^2-1}} \qquad \text{or} \quad \lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x^2-1}$$

$$1 = yx^2 - y \qquad = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}}$$

$$1+y = yx^2 \qquad = \frac{\left(\lim_{x\to\infty}\frac{1}{x}\right)^2}{\lim_{x\to\infty} 1 - \left(\lim_{x\to\infty}\frac{1}{x}\right)^2}$$

$$x^2 = \frac{1+y}{y}$$

$$\sqrt{x^2} = \sqrt{\frac{1+y}{y}} \qquad = \frac{(0)^2}{1-(0)^2}$$

$$|x| = \sqrt{\frac{1+y}{y}} \qquad = \frac{0}{1}$$

$$x = \mp\sqrt{\frac{1+y}{y}} \qquad = 0$$

$y=0$ (المحور x) محاذي أفقي

if $x=0 \Rightarrow y = \frac{1}{0^2-1} = -1$ نقطة التقاطع مع محور y $(0,-1)$

if $y=0 \Rightarrow 0 = \frac{1}{x^2-1} \Rightarrow 0=1$ غير ممكن

∴ لا توجد نقطة تقاطع مع محور x.

$f(-x) = \frac{1}{(-x)^2-1} = \frac{1}{x^2-1} = f(x)$ المنحني متناظر مع المحور y.

$f(-x) \neq -f(x)$ المنحني غير متناظر مع نقطة الأصل.

$$f'(x) = \frac{0 - 2x}{(x^2-1)^2}$$

$$f'(x) = \frac{-2x}{\underbrace{(x^2-1)^2}_{\text{موجب}}}$$

$$f'(x) = 0 \Longrightarrow -2x = 0 \Longrightarrow x = 0$$

$$+ + + + \ \underset{x=-1}{\text{غير معرف}} \ + + + + \ \underset{x=0}{0} \ - - - \ \underset{x=1}{\text{غير معرف}} \ - - - - \longrightarrow \text{إشارة } f'$$

(0,1) نقطة نهاية عظمى (محلية)

مناطق التزايد $(-\infty,-1)$ و $(-1,0)$

مناطق التناقص $(0,1)$ و $(1,\infty)$

$$f''(x) = \frac{(x^2-1)^2 \cdot -2 + 2x \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4}$$

$$= \frac{-2\cancel{(x^2-1)}\{x^2-1-4x^2\}}{(x^2-1)^{\cancel{4}\,3}}$$

$$= \frac{-2\{-3x^2-1\}}{(x^2-1)^3}$$

$$f''(x) = \frac{2\overbrace{(3x^2+1)}^{\text{موجب}}}{(x^2-1)^3}$$

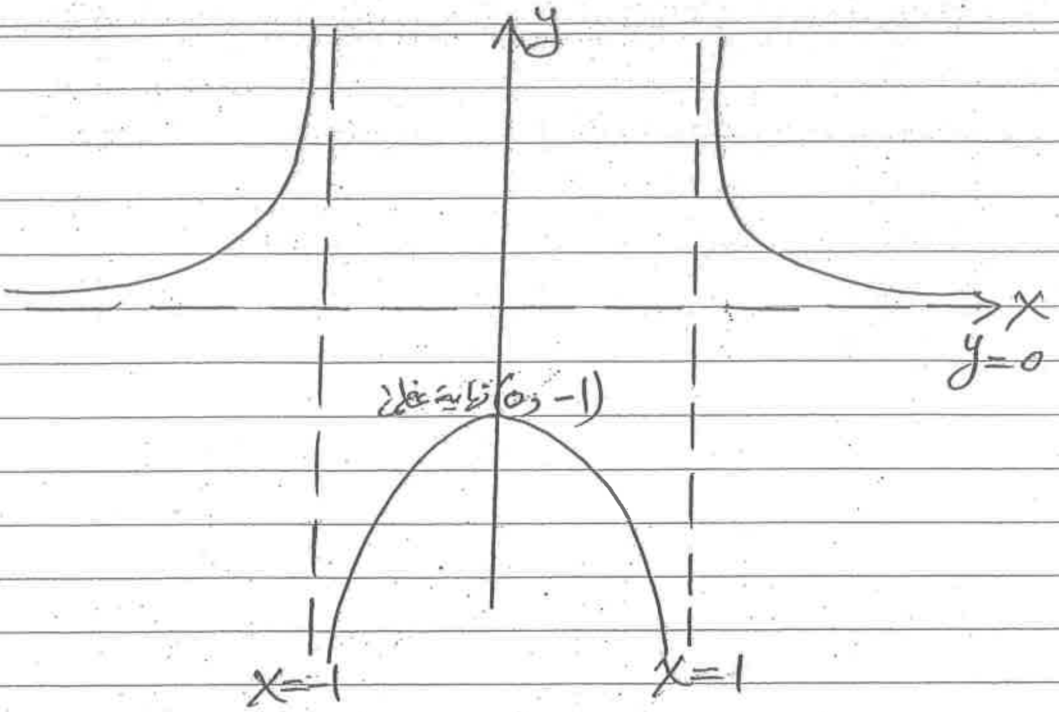
$$f''(x) = 0 \Longrightarrow 3x^2+1 = 0 \Longrightarrow 3x^2 = -1 \longrightarrow x^2 = -\frac{1}{3} \text{ يهمل}$$

$$+ + + + + \ \underset{x=-1}{\text{غير معرف}} \ - - \ \underset{0}{|} \ - - \ \underset{x=1}{\text{غير معرف}} \ + + + + + \longrightarrow \text{إشارة } f''$$

لا توجد نقاط انقلاب

المنحنى مقعر على الفترتين $(-\infty,-1)$ و $(1,\infty)$

" محدب على الفترة $(-1,1)$



$R_f = (-\infty, -1] \cup (0, \infty) = \mathbb{R} \setminus (-1, 0]$

EX(4): $y = |x^2 - 4|$

$$= \begin{cases} x^2-4 & \text{if } x^2-4>0 \Rightarrow x^2>4 \Rightarrow |x|>2 \Rightarrow x>2 \vee x<-2 \\ 0 & \text{if } x^2-4=0 \Rightarrow x^2=4 \Rightarrow |x|=2 \Rightarrow x=2,-2 \\ -(x^2-4) & \text{if } x^2-4<0 \Rightarrow x^2<4 \Rightarrow |x|<2 \Rightarrow -2<x<2 \end{cases}$$

$D_f = R$

لا يوجد محاذيات شاقولية وأفقية

$x=0 \Longrightarrow y = |-4| = 4$ نقطة التقاطع مع المحور y $(0,4)$

$y=0 \Longrightarrow |x^2-4| = 0 \Longrightarrow x = 2, -2$

نقاط التقاطع مع المحور x $(-2,0), (2,0)$

$f(-x) = |(-x)^2 - 4| = |x^2 - 4| = f(x)$ المنحني متناظر مع المحور y

$f(-x) \neq -f(x)$ المنحني غير متناظر مع نقطة الأصل

$$y = \begin{cases} x^2 - 4 & \text{if } x > 2 \lor x < -2 \\ 0 & \text{if } x = 2, -2 \\ 4 - x^2 & \text{if } -2 < x < 2 \end{cases}$$

$$y' = \begin{cases} 2x & \text{if } x > 2 \lor x < -2 \\ \text{غير معرف} & \text{if } x = 2, -2 \\ -2x & \text{if } -2 < x < 2 \end{cases}$$

إشارة y': $- - -$ غير معرف $+ + +$ 0 $- - -$ غير معرف $+ + +$

-2 حرجة، 0 عظمى، 2 حرجة

نقاط حرجة $(-2,0)$, $(2,0)$

لأن $f'(-2)$ و $f'(2)$ غير معرفة

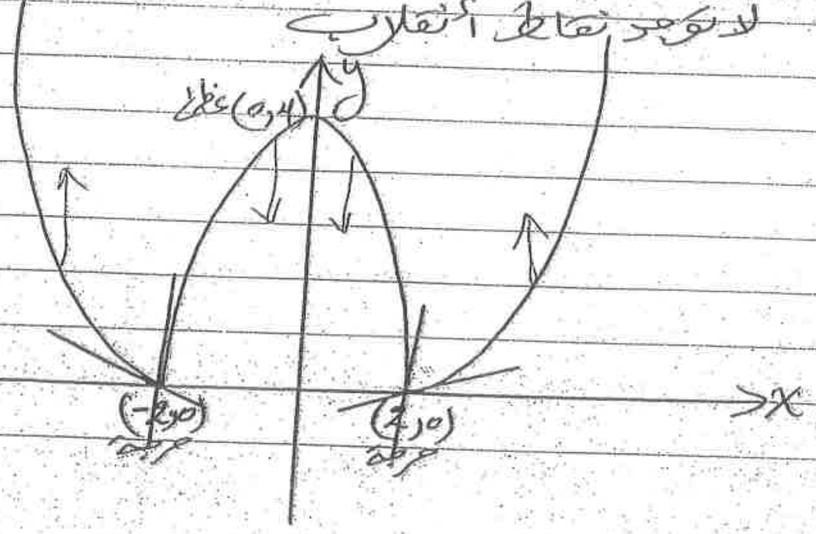
نقطة زاوية عظمى (حرجة) $(0,4)$

$$y'' = \begin{cases} 2 & \text{if } x > 2 \lor x < -2 \\ \text{غير معرف} & \text{if } x = 2, -2 \\ -2 & \text{if } -2 < x < 2 \end{cases}$$

إشارة y'': $+ + + +$ غير معرف $- - - -$ 0 $- - - -$ غير معرف $+ + + +$

-2 , 0 , 2

لا توجد نقاط انقلاب



$R_f = \mathbb{R}^+$

EXC

: Graph of the following functions :-

1- $y = x^2 - 2x + 4$

2- $y = x^3 - 12x + 10$

3- $y = x^5$

4- $y = \frac{1}{x^2+3}$

5- $y = \frac{x}{x-1}$

6- $y = \frac{x^2}{x^2-1}$

7- $y = \frac{x^2+1}{x}$

8- $y = \frac{x+3}{x+2}$

9- $y = \sqrt{x^2-1}$

10- $y = x^4 - 1$

11- $y = \sqrt{x+1}$

12- $y = \sqrt[3]{x^2-1}$