جامعة بغداد

كلية التربية للعلوم الصرفة / ابن الهيثم

التفاضل والتكامل

قسم الرياضيات

المرحلة الاولى

الفصل الخامس

اساتذة المادة

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"chapter five

Trigonometric Functions

we will define six Trigonometric functions in terms of the central angle θ drawn in the center circle $(0,0)$ and radius r.
If the central angle θ with one of its sides is applied to the x axis and the other side is drawn from the origin point and cut the circumference of the circle at The point $P(x,y)$; then :-

Sine :- $\sin\theta = \dfrac{y}{r}$

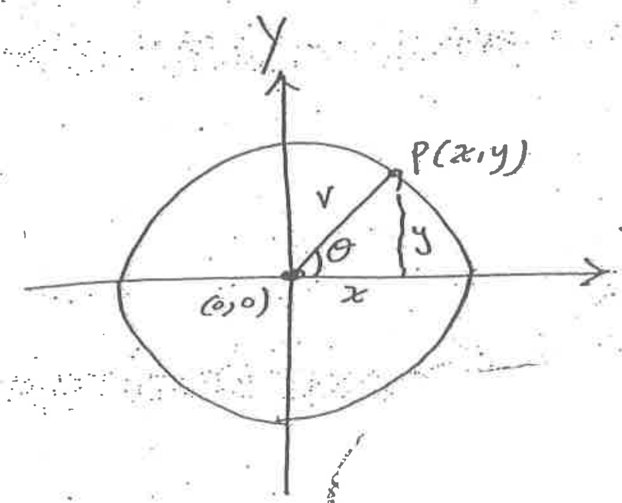
Cosine : $\cos\theta = \dfrac{x}{r}$

tangent : $\tan\theta = \dfrac{y}{x}$

Cosecant : $\csc\theta = \dfrac{r}{y}$

secant : $\sec\theta = \dfrac{r}{x}$

Cotangent : $\cot\theta = \dfrac{x}{y}$



From the previous definitions, a relation can be found between trigonometric functions as follows :-

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{\csc\theta}{\sec\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \quad , \quad \sec\theta = \frac{1}{\cos\theta}$$

And since the equation of the circle center $(0,0)$ and radius r is :

$$x^2+y^2=r^2$$

$\therefore$

$$x = r\cos\theta \qquad y = r\sin\theta$$

(من تعريف الدالتين sin, cos)

$\Rightarrow$

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

$\Rightarrow$

$$r^2(\cos^2\theta+\sin^2\theta) = r^2 \qquad \div r^2$$

$\Rightarrow$

$$\cos^2\theta+\sin^2\theta = 1$$

وهذا أحد أهم قوانين الدوال المثلثية

والذي يمكن من خلاله باشتقاق قوانين أخرى منها :-

* $\sin^2\theta = 1-\cos^2\theta$
* $\cos^2\theta = 1-\sin^2\theta$
* $\tan^2\theta = \sec^2\theta - 1$ or $\sec^2\theta = \tan^2\theta + 1$
* $\cot^2\theta = \csc^2\theta - 1$ or $\csc^2\theta = \cot^2\theta + 1$

If both a , b are two angles, we can find the trigonometric functions of the sum or subtract them as follows :-

* $\cos(A+B) = \cos A\cos B - \sin A\sin B$
* $\cos(A-B) = \cos A\cos B + \sin A\sin B$

* $\sin(A+B) = \sin A \cos B + \sin B \cos A$

* $\sin(A-B) = \sin A \cos B - \sin B \cos A$

* prove that $\tan(A\mp B) = \dfrac{\tan A \mp \tan B}{1 \pm \tan A \tan B}$? :

Proof (للاطلاع)

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} \quad (\text{By def. of } \tan)$$

$$= \frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B - \sin A \sin B}$$ (حسب قانون جمع او طرح زاويتين للدالتين Cos , Sin)

$$= \frac{\dfrac{\sin A \cos B + \sin B \cos A}{\cos A \cos B}}{\dfrac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\dfrac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \dfrac{\sin B \cancel{\cos A}}{\cancel{\cos A} \cos B}}{\cancel{\dfrac{\cos A \cos B}{\cos A \cos B}} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Now using the laws of sum two angles or subtrac two angles in functions sin, cos, The following Laws can be derived :-

* $\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

$\Rightarrow$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

* $\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$

$\Rightarrow$ $\sin(2\theta) = 2\sin\theta\cos\theta$

الآن لو استعملنا القانونين $\cos^2\theta - \sin^2\theta = \cos 2\theta$ و $\cos^2\theta + \sin^2\theta = 1$

فعند جمع المعادلتين نحصل على :-

$2\cos^2\theta = 1 + \cos 2\theta$ or $\cos 2\theta = 2\cos^2\theta - 1$

وعند طرح المعادلتين نحصل على :-

$2\sin^2\theta = 1 - \cos 2\theta$ or $\cos 2\theta = 1 - 2\sin^2\theta$

Trigonometric functions are divided into two types : Odd functions and even function; as follows :-

$$\left.\begin{aligned} \cos(-\theta) &= \cos\theta \\ \sec(-\theta) &= \sec\theta \end{aligned}\right\} \text{even functions}$$

$$\left.\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \csc(-\theta) &= -\csc\theta \\ \tan(-\theta) &= -\tan\theta \\ \cot(-\theta) &= -\cot\theta \end{aligned}\right\} \text{odd functions}$$

هنالك زوايا خاصة يجب حفظ نتيجة دوالها المثلثية وهي :-

θ	0	π	2π	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{2}$
$\sin\theta$	0	0	0	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	-1
$\cos\theta$	1	-1	1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	0	0	∞	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-\infty$

* يمكنك يا أيها الطالب أن تستخدم قوانين جمع أو طرح زاويتين لدالتي sin و cos لتبرهن ان :

* $\cos(\theta + 2\pi) = \cos\theta$
* $\sin(\theta + 2\pi) = \sin\theta$
* $\tan(\theta + 2\pi) = \tan\theta$
* $\cot(\theta + 2\pi) = \cot\theta$
* $\sec(\theta + 2\pi) = \sec\theta$
* $\csc(\theta + 2\pi) = \csc\theta$
* $\cos(\theta + \frac{\pi}{2}) = -\sin\theta$
* $\cos(\theta - \frac{\pi}{2}) = \sin\theta$
* $\sin(\theta + \frac{\pi}{2}) = \cos\theta$
* $\sin(\theta - \frac{\pi}{2}) = -\cos\theta$

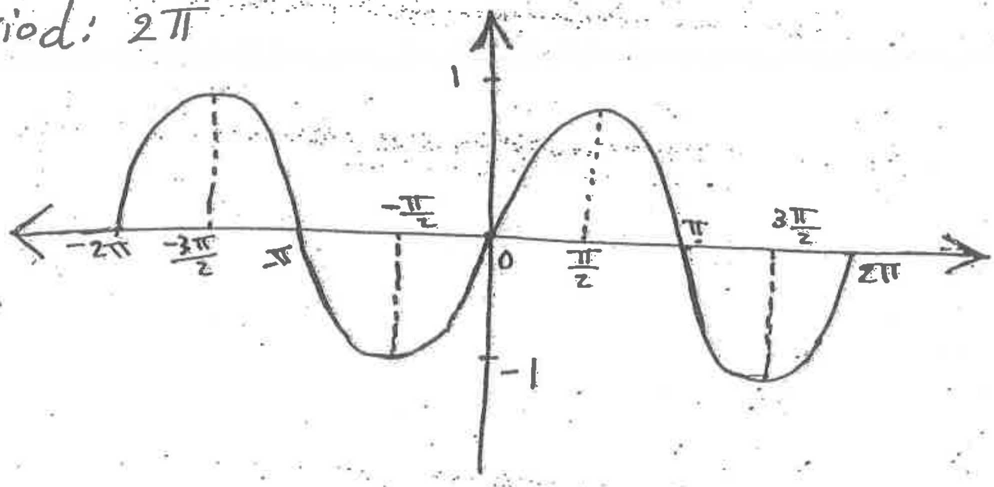
Graphs of Trigonometric Functions

① $y = \sin\theta$

Domain: $\mathbb{R}$ (i.e $-\infty \leq \theta \leq \infty$)

Range: $-1 \leq \sin\theta \leq 1$ (i.e $R_y = [-1, 1]$)

Period: 2π

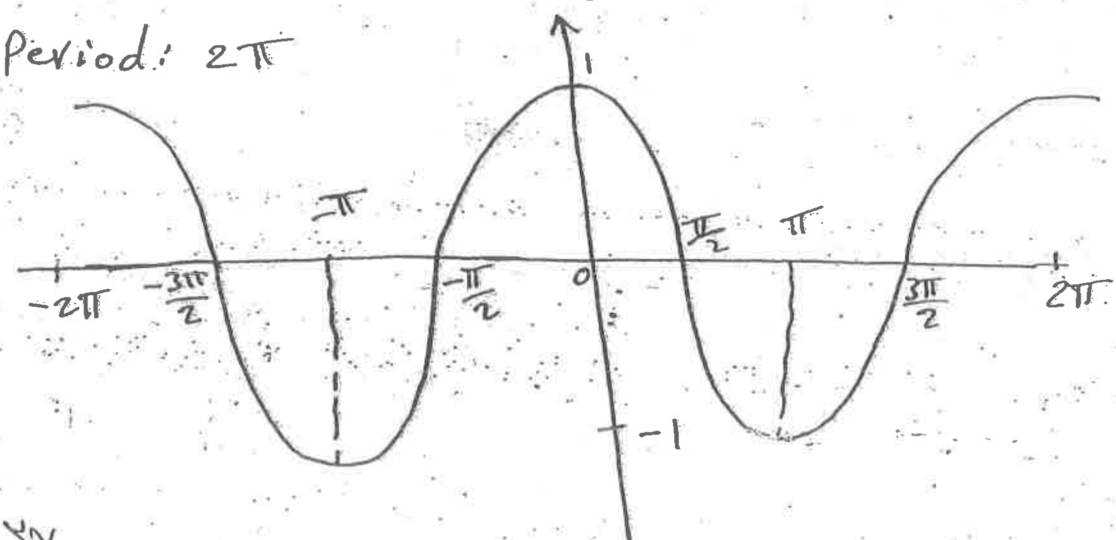


② $y = \cos\theta$

Domain: $\mathbb{R}$ (i.e $-\infty \leq \theta \leq \infty$)

Range: $-1 \leq \cos\theta \leq 1$ (i.e $R_y = [-1, 1]$)

Period: 2π

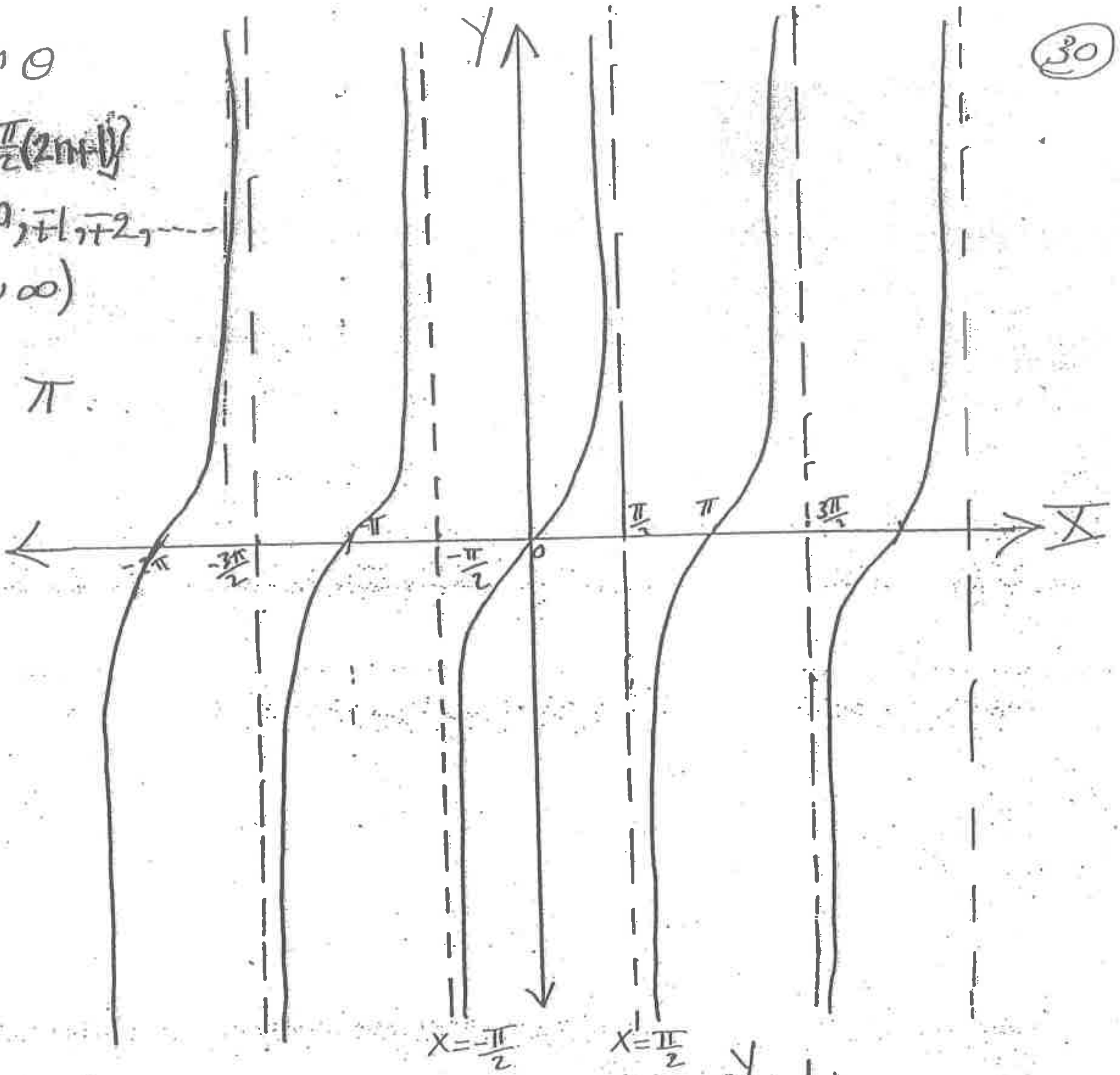


③ $y = \tan\theta$

$D_y: \mathbb{R} \setminus \{\frac{\pi}{2}(2n+1)\}$
$n = 0, \mp 1, \mp 2, \ldots$

$R_y: (-\infty, \infty)$

Period: π.

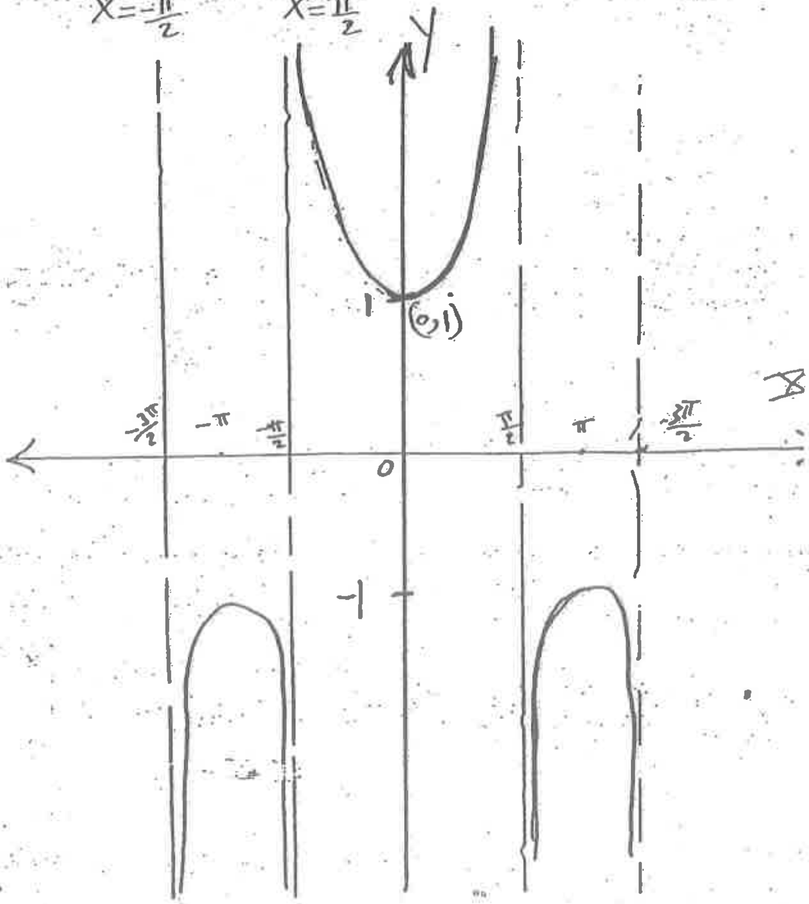


④ $y = \sec\theta$

$D_y: \mathbb{R} \setminus \{\frac{\pi}{2} \mp n\pi\}_{n=0}^{\infty}$

$R_y: \mathbb{R} \setminus (-1, 1)$

period: 2π

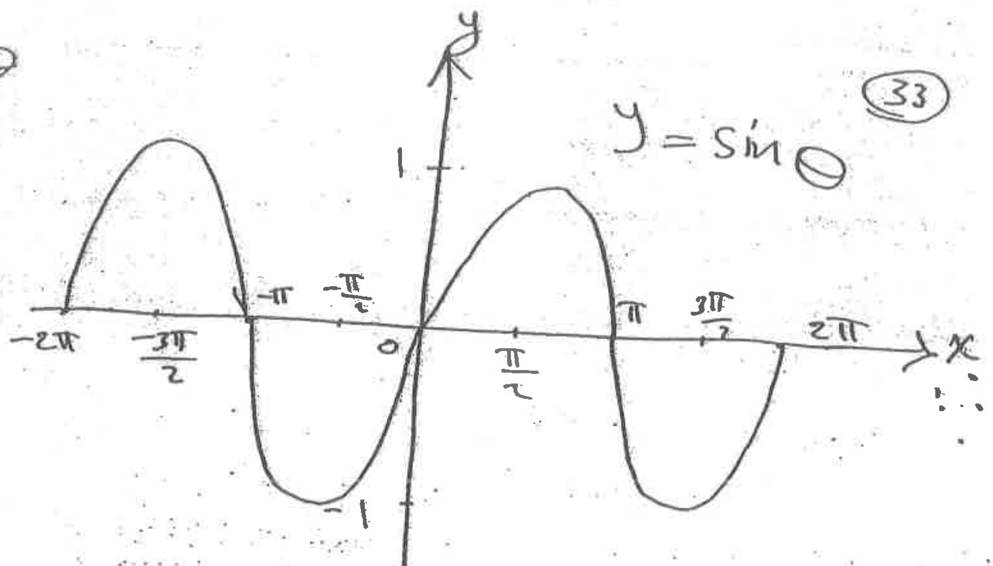


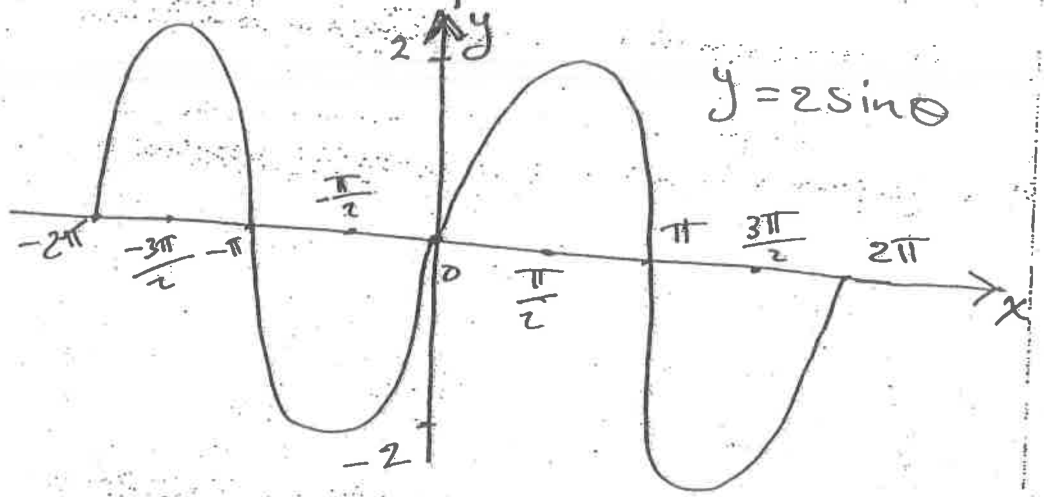
② $y = 2\sin\theta$

$D_y : \mathbb{R}$

$R_{\sin} : [-1, 1]$

$R_y : [-2, 2]$



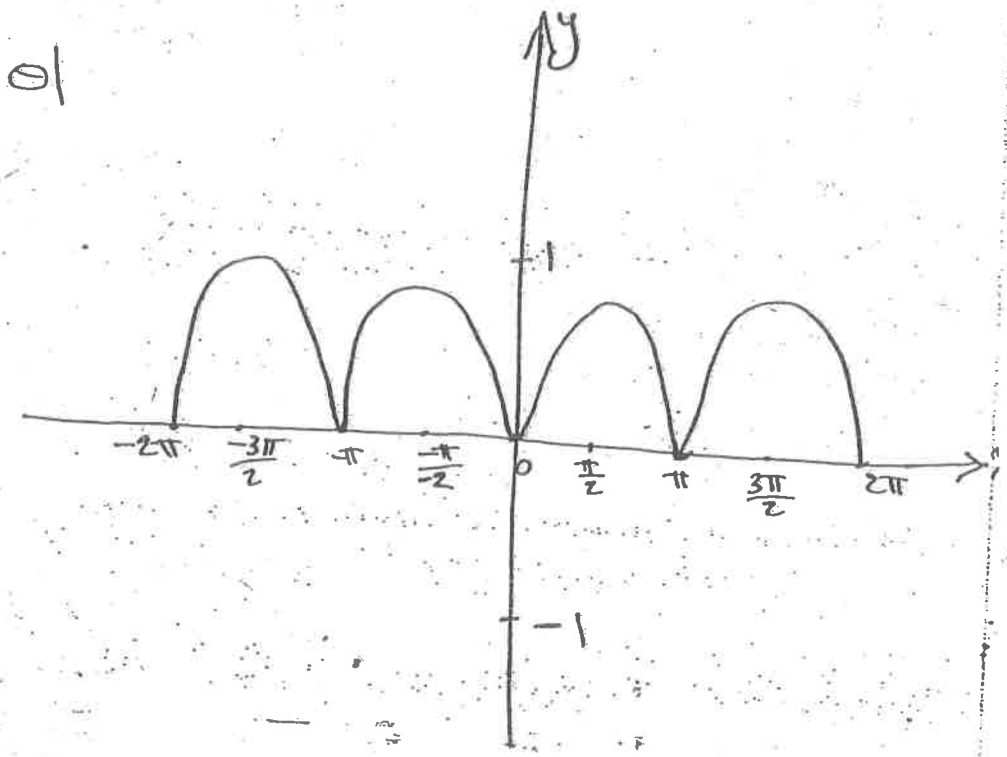


③ $y = |\sin\theta|$

$D_y : \mathbb{R}$

$R_{\sin} : [-1, 1]$

$R_y : [0, 1]$



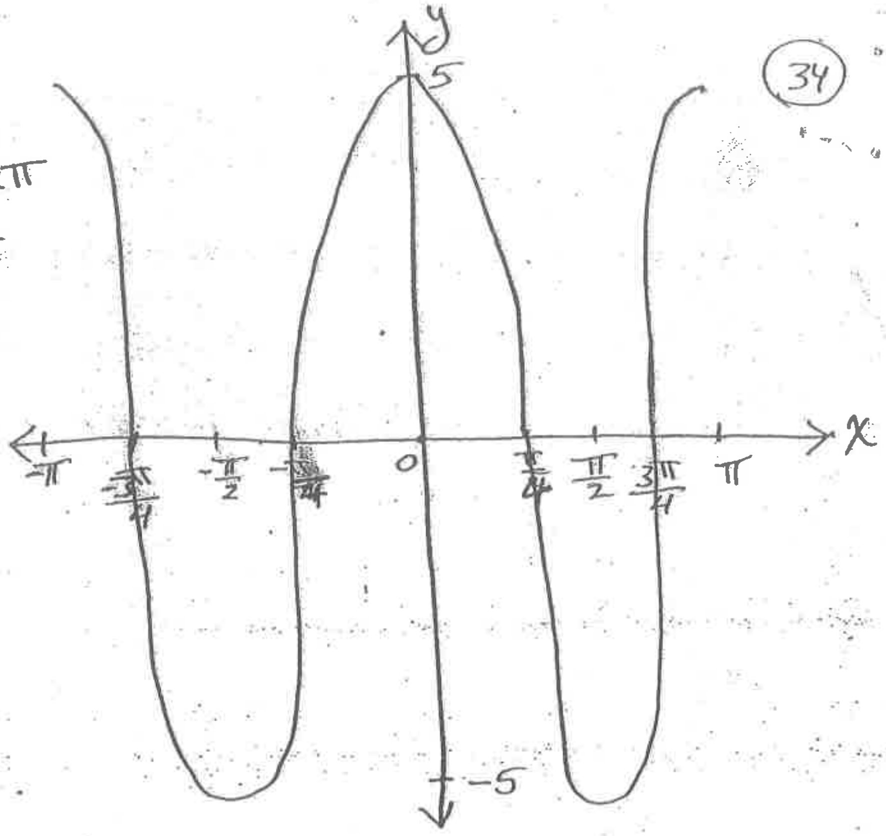
④ $y = 5\cos(2\theta)$

D_y: $-2\pi \leq 2\theta \leq 2\pi$

$-\pi \leq \theta \leq \pi$

$R_{\cos}$: $[-1, 1]$

R_y: $[-5, 5]$



Exercises

Ⓐ Graph the following functions over the given intervals of θ-values

① $y = \sin\left(\frac{\theta}{2}\right)$

② $y = \cos(3\theta)$

③ $y = 1 + \sin\theta$

④ $y = \frac{1 + \cos(2\theta)}{2}$

⑤ $y = |\sin(4\theta)|$

⑥ $2\sin(\theta + \pi)$

Ⓑ which of the following equations have the same graph?

① $y = \sin\theta$ ② $y = \cos\theta$ ③ $y = \sin(-\theta)$ ④ $y = \cos(-\theta)$

⑤ $y = \cos\left(\theta + \frac{\pi}{2}\right)$ ⑥ $y = -\sin\theta$ ⑦ $y = -\cos\theta$ ⑧ شو

8) $y = \sin(\theta + \frac{\pi}{2})$ 11) $y = \cos(\theta - \pi)$ تابع 35

9) $y = \cos(\theta + \pi)$ 12) $y = \sin(\theta - \pi)$

10) $y = \sin(\theta + \pi)$ 13) $\cos(\theta - \frac{\pi}{2}) = y$

* Limits of Trigonometric Functions

* $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

* $\lim_{\theta \to 0} \sin\theta = 0$

* $\lim_{\theta \to 0} \cos\theta = 1$

* $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$

* $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

Examples:- Find the following limits

1) $\lim_{x \to 0} \frac{\sin 3x}{x}$

$= \lim_{x \to 0} \frac{3\sin 3x}{3x}$, when $x \to 0$, then $3x \to 0$

$= \lim_{3x \to 0} \frac{3\sin 3x}{3x} = 3 \lim_{3x \to 0} \frac{\sin 3x}{3x} = 3(1) = 3$

④ $\lim_{x \to \infty} x \sin \frac{1}{x}$

let $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

$x \to \infty \Rightarrow y \to 0$

$= \lim_{y \to 0} \frac{1}{y} \sin y = \lim_{y \to 0} \frac{\sin y}{y} = 1$

H.W

⑤ $\lim_{\theta \to 0} \frac{\sec\theta - \cos\theta}{\theta^2}$

⑥ $\lim_{h \to 2} \frac{\cos \frac{\pi}{h}}{h - 2}$

Exercises

*Find the following limite, if it exist?

① $\lim_{x \to 0} (\tan(2x) \cdot \csc(4x))$

② $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2}$ ✓ $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

③ $\lim_{x \to 0} \frac{\sin(5x)}{x\cos(5x)}$

④ $\lim_{t \to 0} \frac{5t}{\tan(6t)}$

Sketch the graph of the following functions :-

1- $y = 1 + \cos(-x)$

2- $y = \cos\left(x - \frac{\pi}{2}\right) - 1$

3- $y = \cos\left(\frac{x}{2}\right) + 2$

4- $y = |\cos x| - 1$

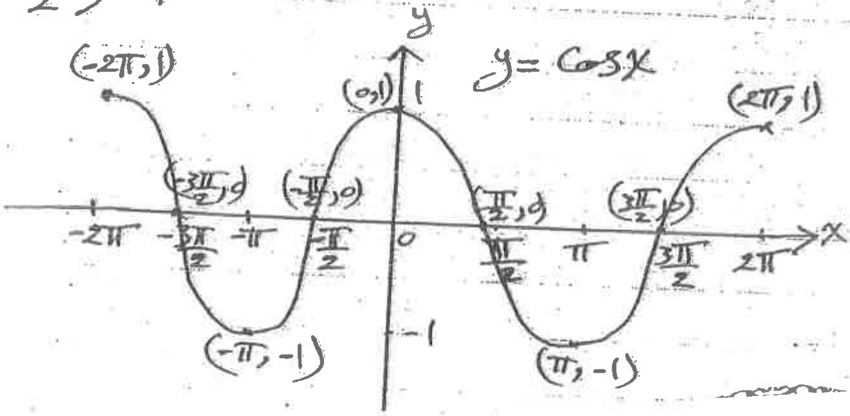
5- $y = -\sin(-x)$

6- $y = 2\sin(x + \pi)$

7- $y = |\sin(4x)|$

sol

2- $y = \cos\left(x - \frac{\pi}{2}\right) - 1$



$D_{\cos} = [-2\pi, 2\pi]$

$R_{\cos} = [-1, 1]$

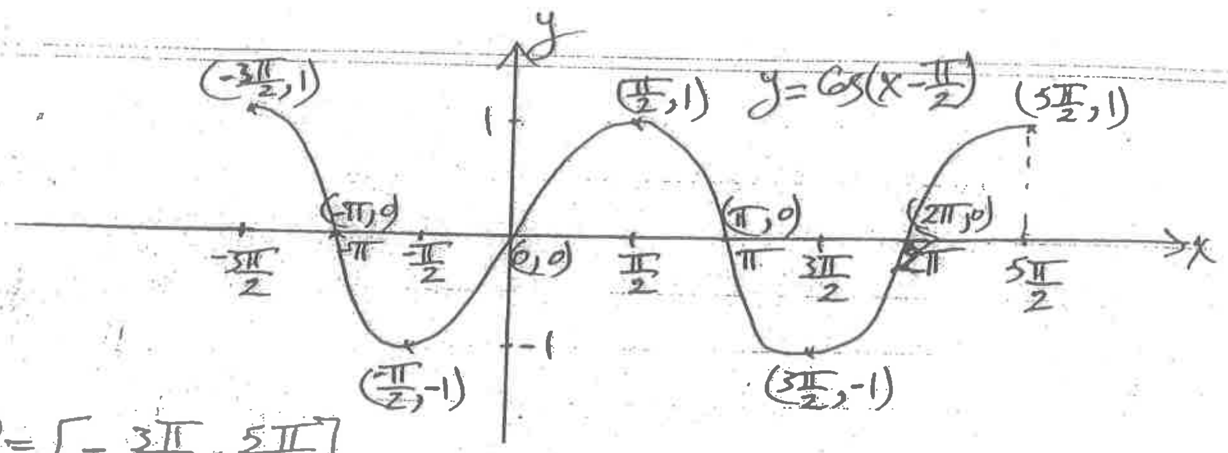
$x - \frac{\pi}{2} \in D_{\cos}$

$x - \frac{\pi}{2} \in [-2\pi, 2\pi]$

$\left(-2\pi \leq x - \frac{\pi}{2} \leq 2\pi\right) + \frac{\pi}{2}$

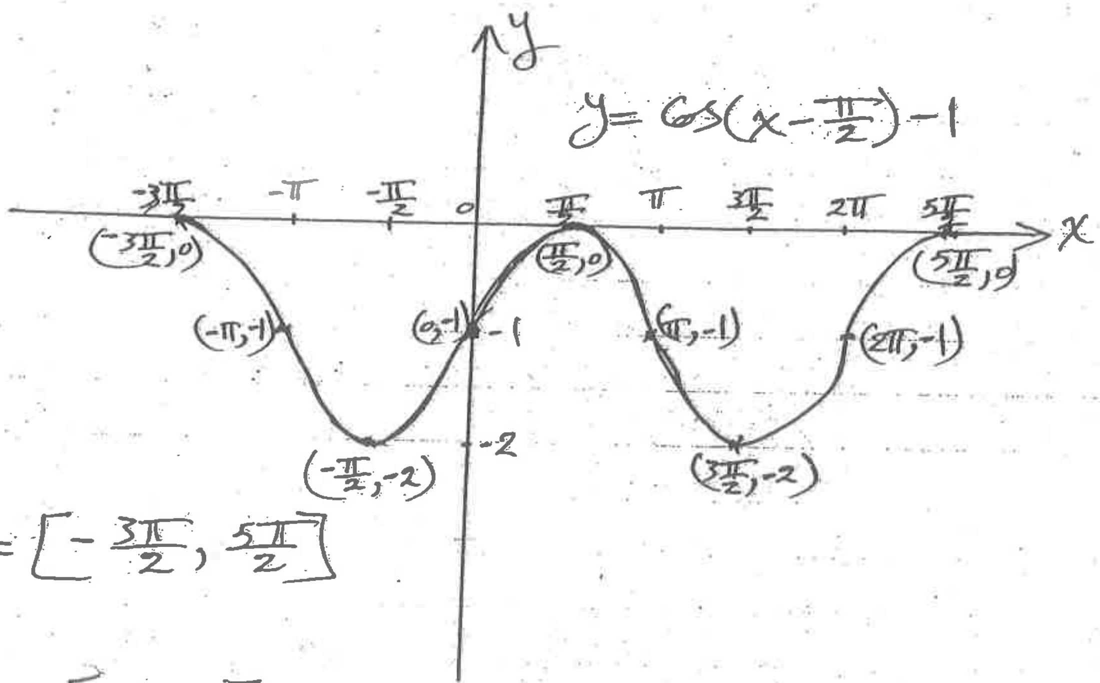
$-2\pi + \frac{\pi}{2} \leq x \leq 2\pi + \frac{\pi}{2}$

$-\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$ الإزاحة نحو اليمين بمقدار $\frac{\pi}{2}$



$D = [-\frac{3\pi}{2}, \frac{5\pi}{2}]$

$R = [-1, 1]$



$D_y = [-\frac{3\pi}{2}, \frac{5\pi}{2}]$

$R_y = [-2, 0]$

3- $y = \cos\left(\frac{x}{2}\right) + 2$

$D_{\cos} = [-2\pi, 2\pi]$

$R_{\cos} = [-1, 1]$

$\frac{x}{2} \in D_{\cos}$

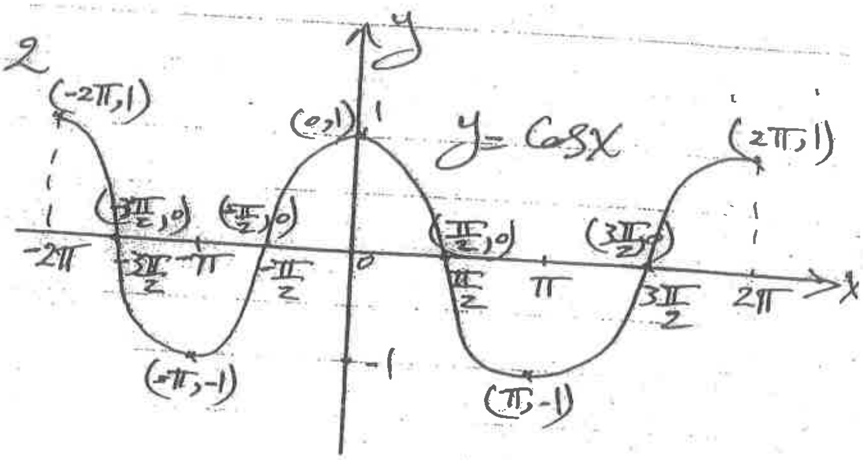
$\frac{x}{2} \in [-2\pi, 2\pi]$

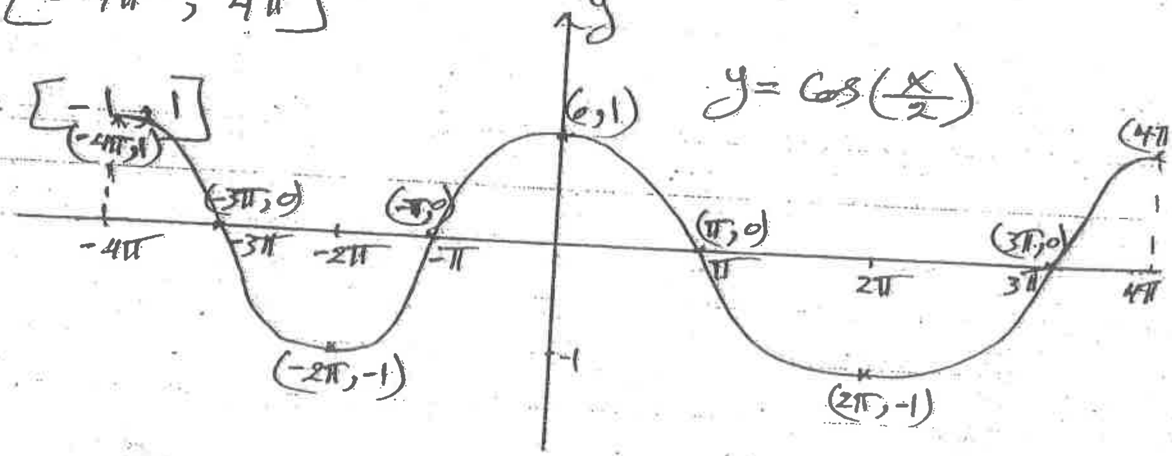
$\left(-2\pi \leq \frac{x}{2} \leq 2\pi\right) * 2$

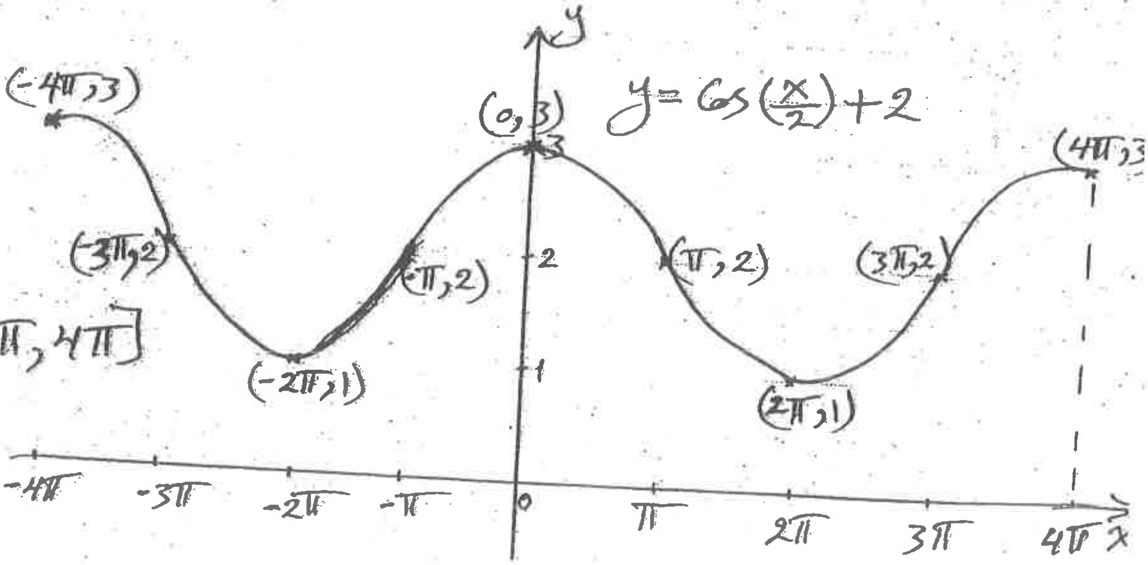
$-4\pi \leq x \leq 4\pi$

$D_{\cos\frac{x}{2}} = [-4\pi, 4\pi]$

$R_{\cos\frac{x}{2}} = [-1, 1]$







$D_y = [-4\pi, 4\pi]$

$R_y = [1, 3]$

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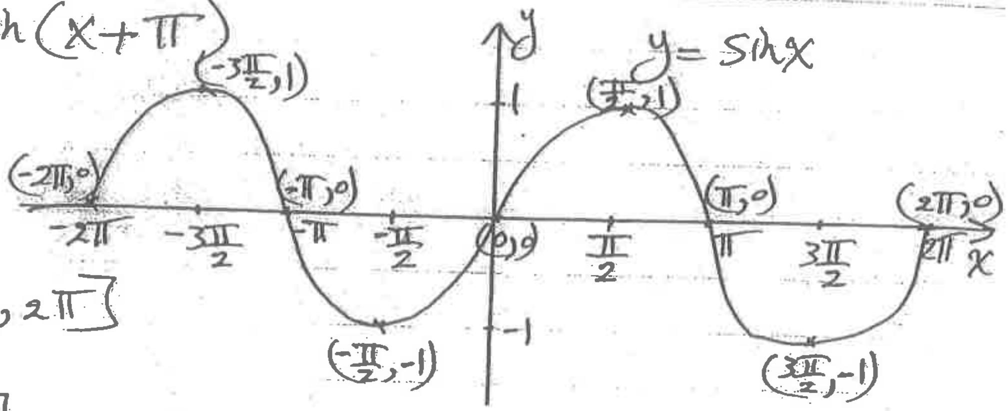
6- $y = 2\sin(x+\pi)$



$D_{\sin} = [-2\pi, 2\pi]$

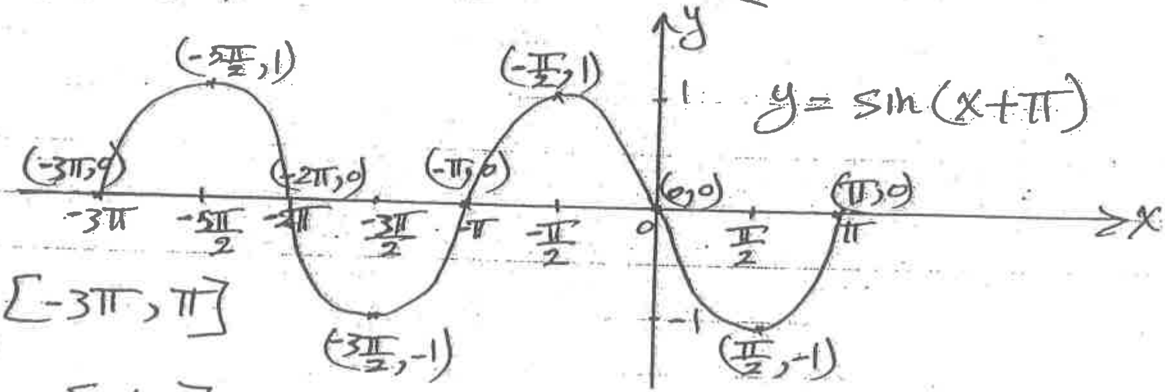
$R_{\sin} = [-1, 1]$

$x+\pi \in D_{\sin}$

$x+\pi \in [-2\pi, 2\pi]$

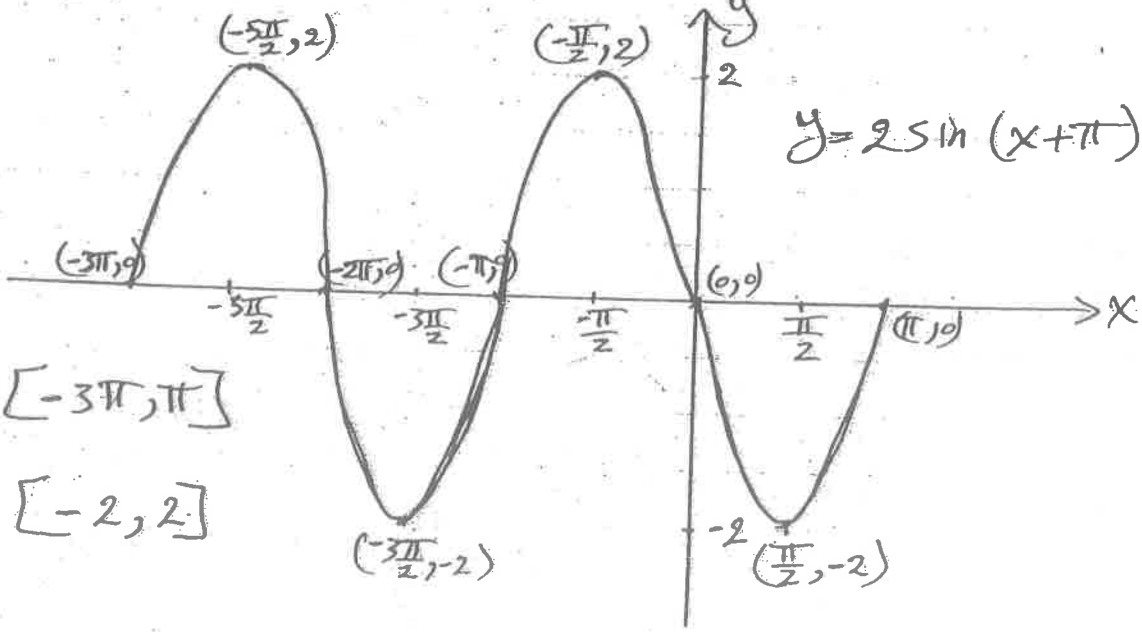
$(-2\pi \leq x+\pi \leq 2\pi) - \pi$

$-3\pi \leq x \leq \pi$ السحب نحو اليسار بمقدار π



$D_{\sin(x+\pi)} = [-3\pi, \pi]$

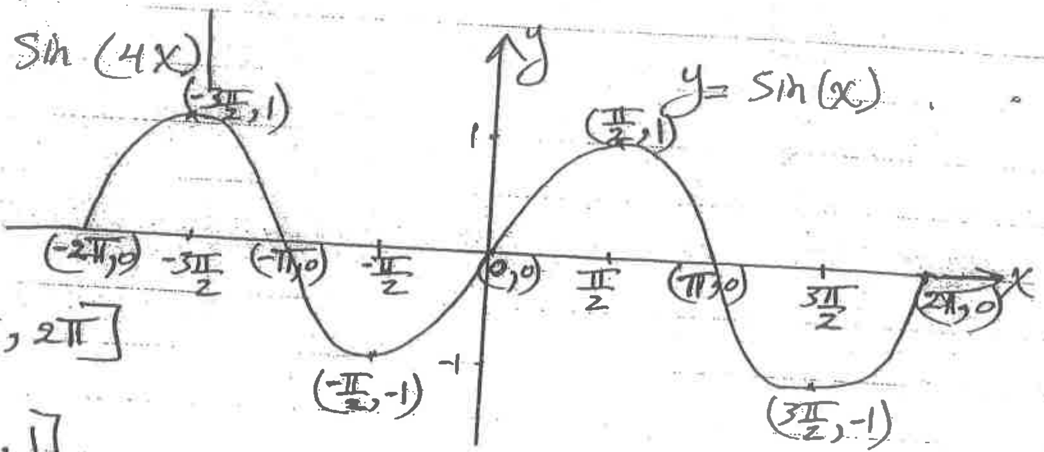
$R_{\sin(x+\pi)} = [-1, 1]$



$D_y = [-3\pi, \pi]$

$R_y = [-2, 2]$

7- $y = |\sin(4x)|$



$D_{\sin} = [-2\pi, 2\pi]$

$R_{\sin} = [-1, 1]$

$4x \in D_{\sin}$

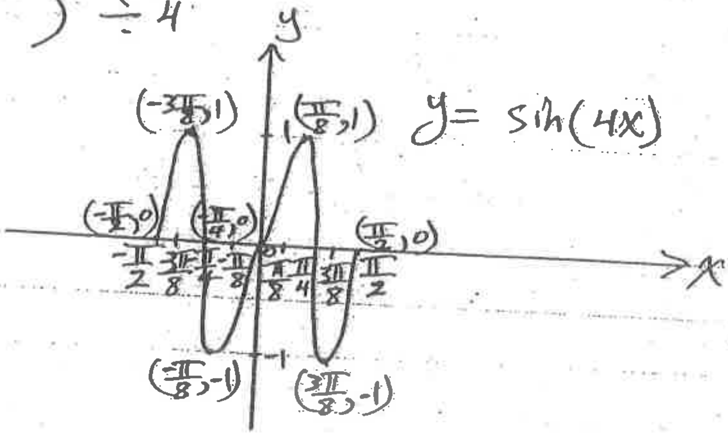
$4x \in [-2\pi, 2\pi]$

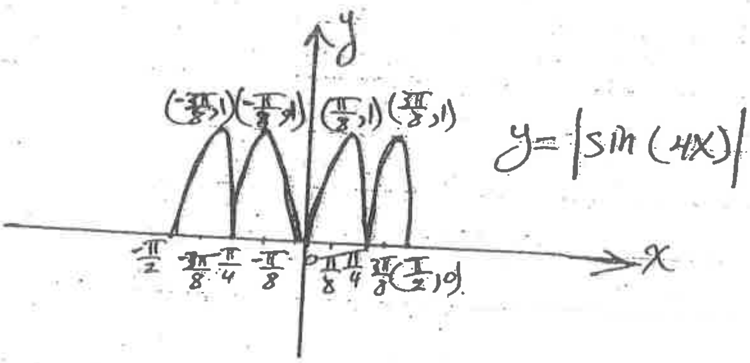
$(-2\pi \leq 4x \leq 2\pi) \div 4$

$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$D_{\sin(4x)} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$R_{\sin(4x)} = [-1, 1]$





$D_y = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$R_y = [0, 1]$

prove that : برهن المتطابقات الآتية

1- $\sin^2\theta + \cos^2\theta + \tan^2\theta = \sec^2\theta$

الطرف الأيسر $= \underbrace{\sin^2\theta + \cos^2\theta} + \tan^2\theta$

$= 1 + \tan^2\theta$ $\boxed{\sin^2\theta + \cos^2\theta = 1}$

$= \sec^2\theta$

$=$ الطرف الأيمن

2- $\tan^2\theta - \sin^2\theta = \tan^2\theta \cdot \sin^2\theta$

الطرف الأيسر $= \tan^2\theta - \sin^2\theta$

$= \dfrac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta$

$= \dfrac{\sin^2\theta - \sin^2\theta \cdot \cos^2\theta}{\cos^2\theta}$

$= \dfrac{\sin^2\theta(1 - \cos^2\theta)}{\cos^2\theta}$

$= \dfrac{\sin^2\theta \cdot \sin^2\theta}{\cos^2\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta} \cdot \sin^2\theta$

$= \tan^2\theta \cdot \sin^2\theta$

$=$ الطرف الأيمن

3. $\sqrt{\dfrac{1+\tan^2\theta}{1+\cot^2\theta}} = \tan\theta$

الطرف الأيسر $= \sqrt{\dfrac{1+\tan^2\theta}{1+\cot^2\theta}}$

$= \sqrt{\dfrac{\sec^2\theta}{\csc^2\theta}}$

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$$= \sqrt{\frac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}}} = \sqrt{\frac{1}{\cos^2\theta}\cdot\frac{\sin^2\theta}{1}} = \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{الطرف الأيمن}$$

4- $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \csc\theta + \cot\theta$

$$\text{الطرف الأيسر} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{(1+\cos\theta)(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \csc\theta + \cot\theta = \text{الطرف الأيسر}$$

5- $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$$\text{الطرف الأيسر} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{\frac{\cos^2\theta-\sin^2\theta}{\cancel{\cos^2\theta}}}{\frac{\cos^2\theta+\sin^2\theta}{\cancel{\cos^2\theta}}} = \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$

$$= \frac{\cos 2\theta}{1}$$

$$\boxed{\begin{array}{l}\cos 2\theta = \cos^2\theta - \sin^2\theta \\ \cos^2\theta + \sin^2\theta = 1\end{array}}$$

$$= \cos 2\theta = \text{الطرف الأيمن}$$

Differentiation of trigonometric functions

1- $\frac{d\sin\theta}{dx} = \cos\theta \cdot \frac{d\theta}{dx}$

2- $\frac{d\cos\theta}{dx} = -\sin\theta \cdot \frac{d\theta}{dx}$

3- $\frac{d\tan\theta}{dx} = \sec^2\theta \cdot \frac{d\theta}{dx}$

4- $\frac{d\cot\theta}{dx} = -\csc^2\theta \cdot \frac{d\theta}{dx}$

5- $\frac{d\sec\theta}{dx} = \sec\theta \cdot \tan\theta \cdot \frac{d\theta}{dx}$

6. $\frac{d\csc\theta}{dx} = -\csc\theta \cdot \cot\theta \cdot \frac{d\theta}{dx}$

proof -1-

$$\frac{d\sin\theta}{dx} = \frac{d\sin\theta}{d\theta} \cdot \frac{d\theta}{dx}$$ by chain Rule

$$= \lim_{\Delta\theta \to 0} \frac{\sin(\theta+\Delta\theta) - \sin\theta}{\Delta\theta} \cdot \frac{d\theta}{dx}$$

$$= \lim_{\Delta\theta \to 0} \left(\frac{\sin\theta \cdot \cos\Delta\theta + \cos\theta \sin\Delta\theta - \sin\theta}{\Delta\theta}\right) \frac{d\theta}{dx}$$

$$= \lim_{\Delta\theta \to 0} \left(\frac{\sin\theta(\cos\Delta\theta - 1) + \cos\theta \cdot \sin\Delta\theta}{\Delta\theta}\right) \frac{d\theta}{dx}$$

$$= \left(\lim_{\Delta\theta \to 0} \frac{\sin\theta(\cos\Delta\theta - 1)}{\Delta\theta} + \lim_{\Delta\theta \to 0} \frac{\sin\Delta\theta \cdot \cos\theta}{\Delta\theta}\right) \frac{d\theta}{dx}$$

$$= \left(\sin\theta \underbrace{\left(\lim_{\Delta\theta \to 0} \frac{\cos\Delta\theta - 1}{\Delta\theta}\right)}_{0} + \cos\theta \underbrace{\left(\lim_{\Delta\theta \to 0} \frac{\sin\Delta\theta}{\Delta\theta}\right)}_{1}\right) \cdot \frac{d\theta}{dx}$$

$$= (\sin\theta \cdot 0 + \cos\theta \cdot 1) \cdot \frac{d\theta}{dx}$$

$$= \cos\theta \cdot \frac{d\theta}{dx}$$

Proof -2-

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\frac{d\cos\theta}{dx} = \frac{d\sin\left(\frac{\pi}{2} - \theta\right)}{dx}$$

$$= \cos\left(\frac{\pi}{2} - \theta\right) \cdot \left(0 - \frac{d\theta}{dx}\right)$$

$$= \sin\theta \cdot \frac{-d\theta}{dx}$$

$$\frac{d\cos\theta}{dx} = -\sin\theta \cdot \frac{d\theta}{dx}$$

Proof -3-

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\frac{d\tan\theta}{dx} = \frac{\cos\theta \cdot \frac{d\sin\theta}{dx} - \sin\theta \cdot \frac{d\cos\theta}{dx}}{(\cos\theta)^2}$$

$$= \frac{\cos\theta \cdot \cos\theta \cdot \frac{d\theta}{dx} - \sin\theta \cdot -\sin\theta \cdot \frac{d\theta}{dx}}{\cos^2\theta}$$

$$= \frac{(\cos^2\theta + \sin^2\theta) \cdot \frac{d\theta}{dx}}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{d\tan\theta}{dx} = \sec^2\theta \cdot \frac{d\theta}{dx}$$

proof -4- : Similarly $\frac{d\cot\theta}{dx} = -\csc^2\theta \frac{d\theta}{dx}$

proof -5- : $\sec\theta = \frac{1}{\cos\theta}$

$$\frac{d\sec\theta}{dx} = \frac{\cos\theta \cdot 0 - 1 \cdot -\sin\theta \frac{d\theta}{dx}}{\cos^2\theta}$$

$$= \frac{\sin\theta}{\cos^2\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{d\sec\theta}{dx} = \sec\theta \cdot \tan\theta \cdot \frac{d\theta}{dx}$$

proof -6- : Similarly $\frac{d\csc\theta}{dx} = -\csc\theta \cdot \cot\theta \cdot \frac{d\theta}{dx}$

Examples : Find $\frac{dy}{dx}$

1- $y = \frac{\sin x}{x}$

$$y' = \frac{x \cdot \cos x - \sin x \cdot 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

2- $y = \frac{2}{\cos(3x)}$

$$y' = \frac{\cos(3x) \cdot 0 - 2 \cdot -\sin(3x) \cdot 3}{\cos^2(3x)} = \frac{6\sin(3x)}{\cos^2(3x)} = 6\sec(3x)\tan(3x)$$

3- $y = \cot(x^2)$

$$y' = -\csc^2(x^2) \cdot 2x$$

4- $y = \sec^2(5x)$

$y' = 2\sec(5x) \cdot \sec(5x) \cdot \tan(5x) \cdot 5$

$= 10 \sec^2(5x) \cdot \tan(5x)$.

5- $y = \sin(\cos x)$

$y' = \cos(\cos x) \cdot -\sin x \cdot 1$

EXC

Find y'

1. $y = \left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right)^3$
2. $y = 4\cos^2(-3x)$
3. $y = \sin^2 x + \cos^2 x$
4. $y = \sqrt{2 + \cos(2x)}$
5. $y = \frac{\sqrt{x}}{\cos(3x)}$
6. $y = x^3 \cdot \sin(2x^2 + 3)$
7. $y = \left(\cos^2(1+x) + \sqrt{x+5}\right)^5$
8. $y = 2\sqrt{\sec x}$
9. $y = 2\sin\left(\frac{1}{2}x\right) - x\cos\left(\frac{1}{2}x\right)$
10. $y = \sin(3x) \cdot \cos(3x)$

Example: Find y'

1- $x \cdot \sin(2y) = y \cdot \cos(2x)$

$$x \cdot \cos(2y) \cdot 2y' + \sin(2y) \cdot 1 = y \cdot -\sin(2x) \cdot 2 + \cos(2x) \cdot y'$$

$$\sin(2y) + 2y\sin(2x) = \left(\cos(2x) - 2x\cos(2y)\right)y'$$

$$y' = \frac{\sin(2y) + 2y\sin(2x)}{\cos(2x) - 2x\cos(2y)}$$

2- $\cot(xy) + xy = 0$

$$-\csc^2(xy)\,(xy' + y \cdot 1) + xy' + y \cdot 1 = 0$$

$$-x\csc^2(xy)\,y' - y\csc^2(xy) + xy' + y = 0$$

$$x\left(1 - \csc^2(xy)\right)y' = y\left(\csc^2(xy) - 1\right)$$

$$x \cdot -\cancel{\cot^2(xy)}\,y' = y\,\cancel{\cot^2(xy)}$$

$\boxed{1 + \cot^2\theta = \csc^2\theta}$

$$y' = -\frac{y}{x}$$

EXC: Find y'

1- $\sec^2 x + \csc^2 y = 4$

2- $y = \tan(x + y)$

3- $y^2 = \sin^4(2x) + \cos^4(2x)$

4- $\cos(x^2y^2) = x$

5- $x^2y = \frac{\cot y}{1+\csc y}$

6- $\sqrt{xy} + \csc(-xy) = y$

7- $y(3+\tan y)^{\frac{1}{3}} = x+5$

8- $y = \tan y + \sec^2(xy) + \cot(x^2+y^2)$

9- $x^2 = \sin y + \sin(2y)$

Examples

: Find the Limit if it exist: (By L'Hopitals Rul

1. $\lim\limits_{x\to 0} \frac{x^2+2x}{\sin(2x)}$ $\qquad \frac{0}{0}$

$= \lim\limits_{x\to 0} \frac{2x+2}{2\cos(2x)} = \lim\limits_{x\to 0} \frac{\cancel{2}(x+1)}{\cancel{2}\cos(2x)} = \lim\limits_{x\to 0} \frac{x+1}{\cos(2x)}$

$= \frac{\lim\limits_{x\to 0}(x+1)}{\lim\limits_{x\to 0}\cos(2x)} = \frac{0+1}{\cos(0)} = \frac{1}{1} = 1$

2- $\lim\limits_{h\to 2} \frac{\cos\left(\frac{\pi}{h}\right)}{h-2}$ $\qquad \frac{\cos\frac{\pi}{2}}{2-2} = \frac{0}{0}$

$= \lim\limits_{h\to 2} \frac{-\sin\left(\frac{\pi}{h}\right)\cdot\frac{h\cdot 0-\pi\cdot 1}{h^2}}{1-0} = -\lim\limits_{h\to 2}\left(\sin\left(\frac{\pi}{h}\right)\cdot\frac{-\pi}{h^2}\right)$

$= -\sin\left(\frac{\pi}{2}\right)\cdot\frac{-\pi}{4}$

$= -(1)\cdot\frac{-\pi}{4}$

$= \frac{\pi}{4}$

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طريقة أخرى، $\lim\limits_{h \to 2} \dfrac{\cos\frac{\pi}{h}}{h-2}$

Let $x = \dfrac{\pi}{h}$

when $h \to 2 \Rightarrow x \to \dfrac{\pi}{2}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{x} - 2}$$

$$\frac{x}{1} = \frac{\pi}{h}$$

$$x \cdot h = \pi$$

$$h = \frac{\pi}{x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\frac{\pi - 2x}{x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x \cdot x}{\pi - 2x}$$

Let $\theta = \pi - 2x$

when $x \to \frac{\pi}{2} \Rightarrow \theta \to 0$

$$\theta = \pi - 2x \Rightarrow 2x = \pi - \theta$$

$$\Rightarrow x = \frac{\pi}{2} - \frac{\theta}{2}$$

$$= \lim_{\theta \to 0} \frac{\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\left(\frac{\theta}{2}\right) \cdot \left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\left(\frac{\theta}{2}\right)}{2\frac{\theta}{2}} \cdot \lim_{\theta \to 0} \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= \frac{1}{2} \lim_{\frac{\theta}{2} \to 0} \frac{\sin\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)} \cdot \left\{ \lim_{\theta \to 0} \frac{\pi}{2} - \lim_{\theta \to 0} \left(\frac{\theta}{2}\right) \right\}$$

$$= \frac{1}{2}(1) \cdot \left\{ \frac{\pi}{2} - 0 \right\}$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

3- $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$ $\qquad \frac{1-\cos 0}{(0)^2} = \frac{1-1}{0} = \frac{0}{0}$

$= \lim\limits_{x\to 0} \frac{-\cdot-\sin x}{2x} = \frac{1}{2} \lim\limits_{x\to 0} \frac{\sin x}{x}$ $\qquad \frac{\sin 0}{0} = \frac{0}{0}$

$= \frac{1}{2} \lim\limits_{x\to 0} \frac{\cos x}{1} = \frac{1}{2} \lim\limits_{x\to 0} \cos x$

$= \frac{1}{2}\cos(0) = \frac{1}{2}(1) = \frac{1}{2}$

أو $= \frac{1}{2} \underbrace{\lim\limits_{x\to 0} \frac{\sin x}{x}}_{=1} = \frac{1}{2}(1) = \frac{1}{2}$

طريقة ثانية: $\cos(2x) = 1 - 2\sin^2 x$

$\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\left(1-2\sin^2\left(\frac{x}{2}\right)\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{1}-\cancel{1}+2\sin^2\left(\frac{x}{2}\right)}{x^2}$$

$$= 2\lim_{x\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{x} \cdot \frac{\sin\left(\frac{x}{2}\right)}{x}\right)$$

$$= 2\lim_{x\to 0} \frac{\sin\left(\frac{x}{2}\right)}{2\frac{x}{2}} \cdot \lim_{x\to 0} \frac{\sin\left(\frac{x}{2}\right)}{2\cdot\frac{x}{2}}$$

$$= 2\cdot\frac{1}{2} \underbrace{\lim_{\frac{x}{2}\to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}}_{=1} \cdot \frac{1}{2} \underbrace{\lim_{\frac{x}{2}\to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}}_{=1}$$

$$= \cancel{2}\cdot\cancel{\tfrac{1}{2}}(1)\cdot\frac{1}{2}(1)$$

$$= \frac{1}{2}$$

4- $\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{1-\sin\theta}{1+\cos 2\theta}$ $\qquad \dfrac{1-\sin\frac{\pi}{2}}{1+\cos\pi} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$

$= \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{-\cos\theta}{-2\sin 2\theta} = \dfrac{1}{2} \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\cos\theta}{\sin 2\theta}$ $\qquad \dfrac{\cos\frac{\pi}{2}}{\sin\pi} = \dfrac{0}{0}$

$= \dfrac{1}{2} \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{-\sin\theta}{2\cos 2\theta}$

$= -\dfrac{1}{4} \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\sin\theta}{\cos 2\theta}$

$= -\dfrac{1}{4} \dfrac{\sin\frac{\pi}{2}}{\cos\pi}$

$= -\dfrac{1}{4} \dfrac{1}{-1}$

$= \dfrac{1}{4}$

<u>طريقة أخرى</u>

$\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{1-\sin\theta}{1+\cos 2\theta}$ $\qquad \boxed{\cos 2\theta = 1 - 2\sin^2\theta}$

$= \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{1-\sin\theta}{1+1-2\sin^2\theta}$

$= \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{1-\sin\theta}{2-2\sin^2\theta}$

$= \dfrac{1}{2} \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{1-\sin\theta}{1-\sin^2\theta}$

$= \dfrac{1}{2} \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\cancel{(1-\sin\theta)}}{\cancel{(1-\sin\theta)}(1+\sin\theta)}$

$= \dfrac{1}{2} \lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{1}{1+\sin\theta}$

$= \dfrac{1}{2} \dfrac{1}{1+\sin\frac{\pi}{2}} = \dfrac{1}{2} \dfrac{1}{1+1} = \dfrac{1}{2} \dfrac{1}{2} = \dfrac{1}{4}$

EXC

—: Find the limit if it exists:

1. $\lim_{x \to 0} \frac{1-\cos^3 x}{2x^2}$

2. $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta \cdot \sin\theta}$

3. $\lim_{\theta \to 0} \frac{\sec\theta - \cos\theta}{\theta^2}$

4. $\lim_{x \to 0} \frac{1-\cos x}{\sin x}$

Sketch the graph of the function :

$$y = \cot x$$

$$y = \frac{\cos x}{\sin x}$$

$\cot x$ غير معرفة عندما $\sin x = 0$

أي عند $x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \ldots$

بمعنى عندما $x = n\pi$ حيث $\{n = 0, \mp 1, \mp 2, \ldots\}$

وتعتبر كل منها محاذيات شاقولية

ولا توجد محاذيات أفقية

$$D_{\cot} = \mathbb{R} / \{x \in \mathbb{R} ; x = n\pi ; n \in I\}$$

نضع $y = 0 \Rightarrow \cot x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \ldots$

i.e. $x = \frac{\pi}{2} + n\pi$; $n = 0, \mp 1, \mp 2, \mp 3, \ldots$

∴ نقاط التقاطع مع محور x هي : $(\frac{\pi}{2}, 0)$, $(-\frac{\pi}{2}, 0)$, $(\frac{3\pi}{2}, 0)$,
$(-\frac{3\pi}{2}, 0)$, $(\frac{5\pi}{2}, 0)$, $(-\frac{5\pi}{2}, 0)$, ...

نضع $x = 0 \Rightarrow y = \frac{\cos 0}{\sin 0} =$ غير معرف

∴ لا توجد نقاط تقاطع مع المحور y (لأن $x = 0$ المحور y محاذي شاقولي)

$$f(x) = \cot(x)$$

$$f(-x) = \cot(-x) = -\cot(x) = -f(x)$$ الدالة فردية

∴ المنحني متناظر مع نقطة الأصل .

∴ أخذ الفترة بين $x = 0$, $x = \pi$ المحاذيات الشاقولية

$$0 < x < \pi$$

$$y = \cot x$$

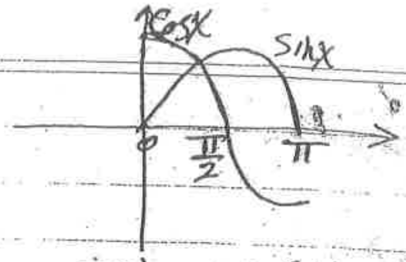
$$y' = -\csc^2 x$$ موجبة

إشارة y' : غير معرف ($x = 0$) — — — — غير معرف ($x = \pi$)

الدالة متناقصة على الفترة $(0, \pi)$

لا توجد نقاط نهايات عظمى أو صغرى (حرجة)

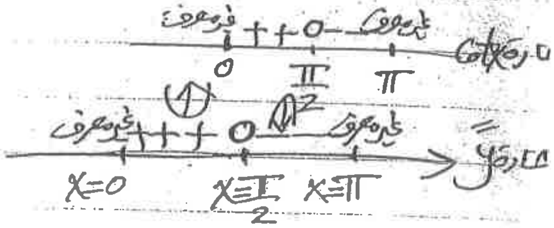
$$y'' = -2\csc x \cdot -\csc x \cdot \cot x \cdot 1$$



$$y' = \underbrace{2\csc^2 x}_{\text{موجب}} \cdot \cot x$$

$$x = \frac{\pi}{2} \Rightarrow y = \cot\frac{\pi}{2} = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{0}{1} = 0$$

$\therefore$ النقطة $(\frac{\pi}{2}, 0)$ نقطة انقلاب للدالة



$$\cot(x+\pi) = \cot x$$

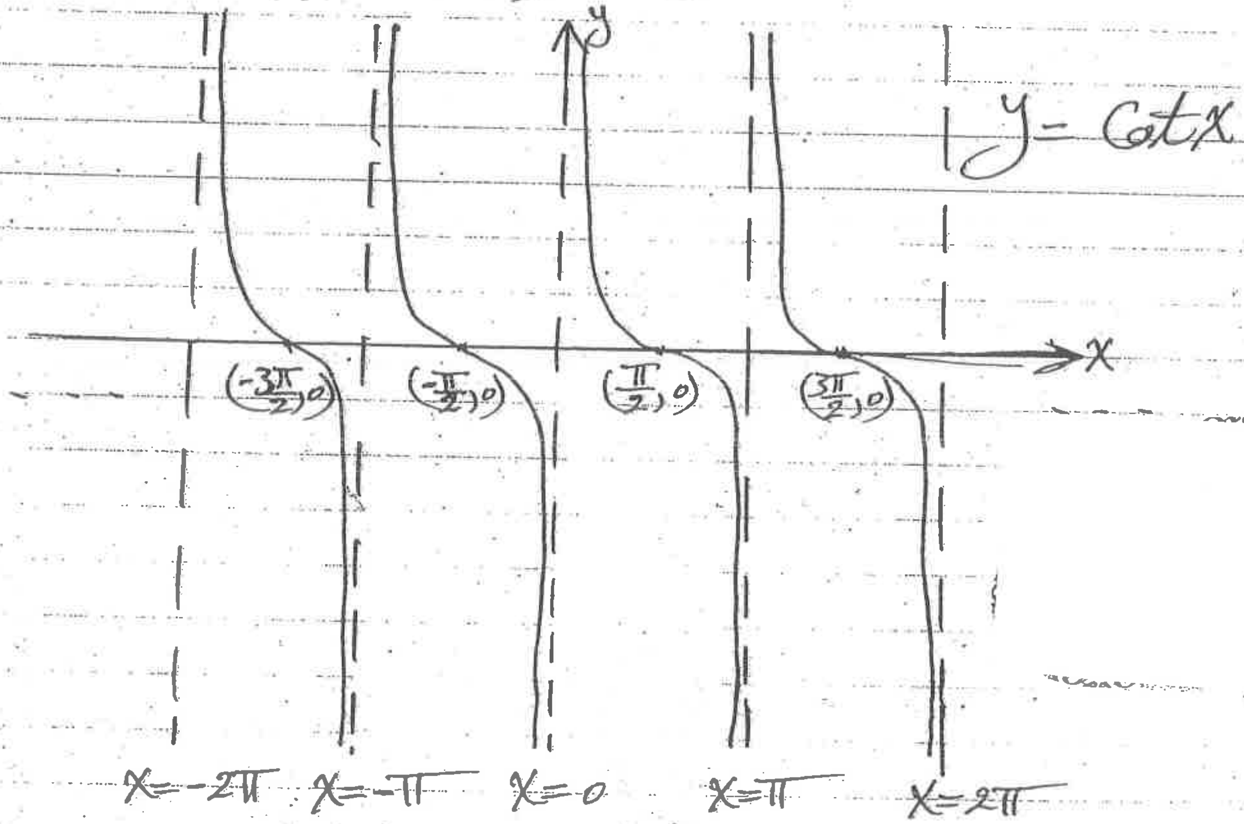
$\therefore$ دالة $\cot$ دورية ومقدار دورتها π

توجد ما لانهاية من نقاط الانقلاب وهي نفس نقاط التقاطع مع محور x.

الدالة مستمرة على $\mathbb{R} \setminus \{x ; x = n\pi\}$

أي أنها مستمرة على الفترات $(0,\pi), (\pi,2\pi), (2\pi,3\pi), \ldots$

و $(-\pi,0), (-2\pi,-\pi), (-3\pi,-2\pi), \ldots$



$$R_{\cot} = \mathbb{R}$$

$y = \csc x = \frac{1}{\sin x}$

$\csc x$ غير معرف عندما $\sin x = 0$ أي عندما $x = 0, \pi, -\pi, 2\pi, -2\pi, \dots$

$x = n\pi \; ; \; n = 0, \mp 1, \mp 2, \dots$

∴ المحاذيات الشاقولية هي $x = n\pi \; ; \; n \in I$

لا توجد محاذيات أفقية $D_{csc} = R \setminus \{x : x = n\pi \; ; \; n = 0, \mp 1, \mp 2, \dots\}$

نضع $y = 0 \Longrightarrow \csc x = 0 \Longrightarrow 1 = 0$ غير ممكن

∴ لا توجد نقاط تقاطع مع محور x.

نضع $x = 0 \Longrightarrow y = \frac{1}{\sin 0} =$ غير معرف

∴ لا توجد نقاط تقاطع مع محور y.

$-1 \le \sin x \le 1$

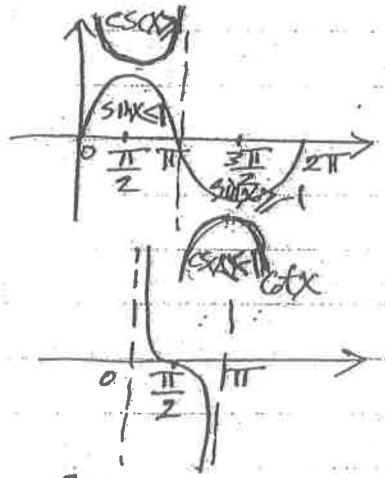
$\sin x \ne 0$

$0 < |\sin x| \le 1$

$-1 \le \sin x < 0 \quad or \quad 0 < \sin x \le 1$

$\csc x \le -1 \quad or \quad \csc x \ge 1$

$(-\infty, -1] \cup [1, \infty)$



$0 < x < \pi \qquad \sin x \le 1$

$y = \csc x \qquad \csc x \ge 1$ موجب

$y' = \underbrace{-\csc x}_{\text{سالب}} \cot x$

إشارة y': غير معرف (0) − − − 0 ($\frac{\pi}{2}$) + + + + غير معرف (π)

$x = \frac{\pi}{2} \Longrightarrow y = \csc\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = \frac{1}{1} = 1$

∴ النقطة $(\frac{\pi}{2}, 1)$ نقطة نهاية صغرى محلية (نقطة حرجة)

الدالة متناقصة على الفترة $(0, \frac{\pi}{2})$

= متزايدة = = $(\frac{\pi}{2}, \pi)$

$y'' = -[\csc x(-\csc^2 x) + \cot x \cdot -\csc x \cot x]$

$= \csc^3 x + \csc x \cot^2 x$

$= \underbrace{\csc x}_{\text{موجب}} \underbrace{(\csc^2 x + \cot^2 x)}_{\text{موجب}}$

إشارة y'': غير معرف (0) + + + + + ∪ ($\frac{\pi}{2}$) + + + + غير معرف (π)

لا توجد نقاط انقلاب

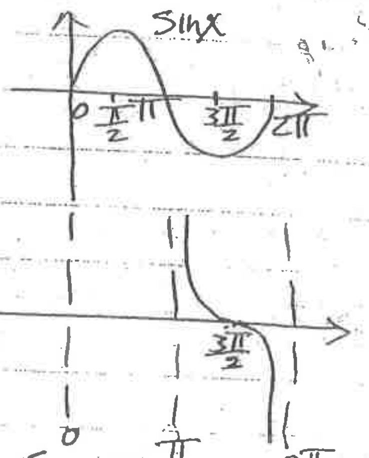
المنحني مقعر على الفترة $(0, \pi)$

$\pi < x < 2\pi$

$y' = \underbrace{-\csc x \cot x}_{\text{موجب}}$

$\sin x \geq -1$

$\csc x \leq -1$



غير معرف $+ + + + + + 0 - - - -$ غير معرف

$\pi \qquad \frac{3\pi}{2} \qquad 2\pi$ إشارة y'

$x = \frac{3\pi}{2} \Rightarrow y = \csc\left(\frac{3\pi}{2}\right) = \frac{1}{\sin\frac{3\pi}{2}} = \frac{1}{-1} = -1$

$\therefore \left(\frac{3\pi}{2}, -1\right)$ نقطة نهاية عظمى (نقطة حرجة)

$y'' = \underbrace{\csc x}_{\text{سالب}}\left(\csc^2 x + \cot^2 x\right)$

غير معرف $- - - - - - \downarrow - - - - -$ غير معرف

$\pi \qquad \frac{3\pi}{2} \qquad 2\pi$ إشارة y''

لا توجد نقاط انقلاب

المنحني محدب على الفترة $(\pi, 2\pi)$

$R_{\csc} = \mathbb{R} / (-1, 1)$

$= (-\infty, -1] \cup [1, \infty)$