جامعة بغداد

كلية التربية للعلوم الصرفة / ابن الهيثم

التفاضل والتكامل

قسم الرياضيات

المرحلة الاولى

الفصل السادس

اساتذة المادة

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The Inverse Trigonometric Functions.

suppose f be a one-one and onto function

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \quad (1-1)$$

or $f(x_1) = f(x_2) \rightarrow x_1 = x_2$

$\forall x \in X \quad \exists y \in Y \quad \ni y = f(x)$ (onto)

$$f : X \longrightarrow Y \ni y = f(x)$$

$$\Leftrightarrow \quad f^{-1} : Y \longrightarrow X \ni x = f^{-1}(y)$$

Let $y = \sin x$

$\sin : \mathbb{R} \longrightarrow [-1, 1]$

$\sin : [-2\pi, 2\pi] \longrightarrow [-1, 1]$

سنعرف دالة جديدة هي معكوس الدالة $\sin$ ونرمز لها

بالرمز $\sin^{-1}$ أو $\arcsin$

$\therefore \sin^{-1} y = \sin^{-1}(\sin x)$

$\sin^{-1} y = x$

$\therefore y = \sin x \Longleftrightarrow x = \sin^{-1} y$

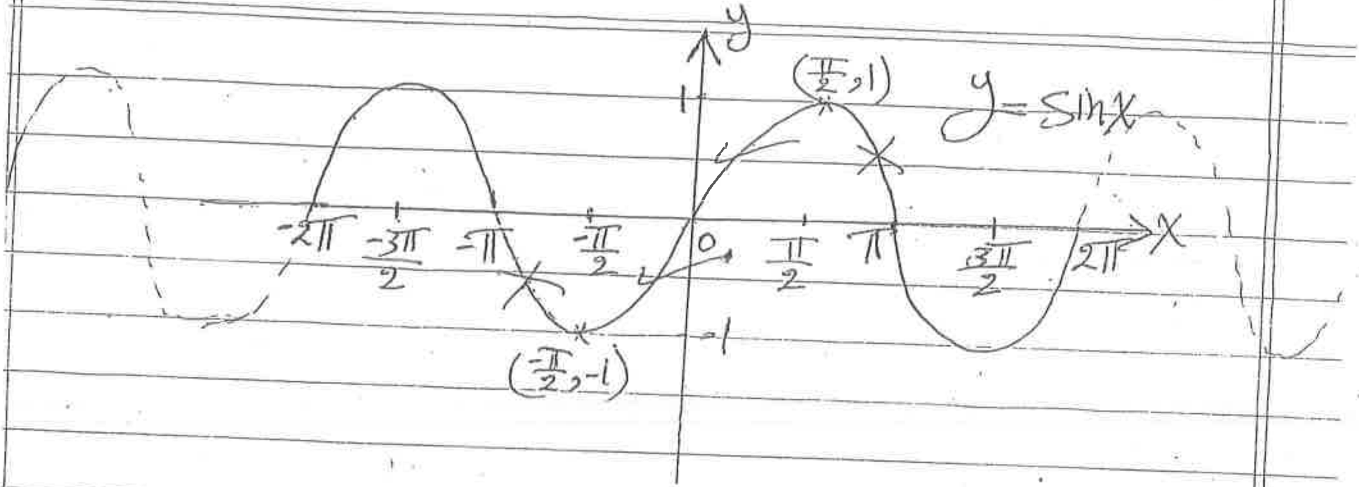
$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$

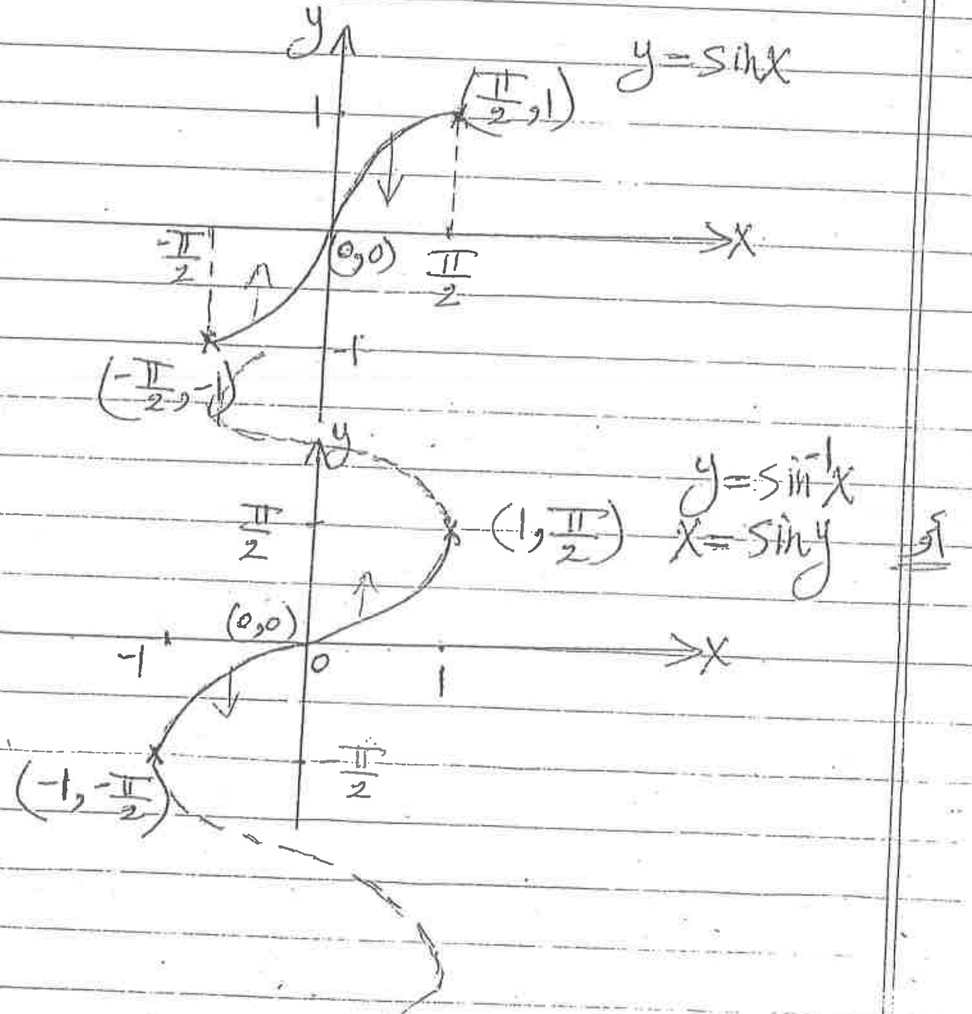
$\sin$ is 1-1 & onto $\therefore \exists \sin^{-1} \ni$

$\sin^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$D_{\sin} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = R_{\sin^{-1}}$

$R_{\sin} = [-1, 1] = D_{\sin^{-1}}$





Def:-

$$\boxed{y = \sin x \iff x = \sin^{-1} y \quad x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]}$$

Note: $\sin^{-1} y \neq \dfrac{1}{\sin y}$

Proposion: $\sin^{-1}(x)$ is odd function :-

$$\boxed{\sin^{-1}(-x) = -\sin^{-1}(x)}$$

Proof:

Let $y = \sin^{-1}(-x) \iff \sin y = -x$

$\iff x = -\sin y$

$\iff x = \sin(-y)$

$\iff \sin^{-1} x = -y$

$\iff y = -\sin^{-1}(x)$

$\therefore \sin^{-1}(-x) = -\sin^{-1}(x)$

Ex:

$$\sin^{-1}(-1) = -\sin^{-1}(1)$$

$$\sin^{-1}(-1) = -\frac{\pi}{2}, \quad \sin^{-1}(1) = \frac{\pi}{2}$$

$y = \cos x$

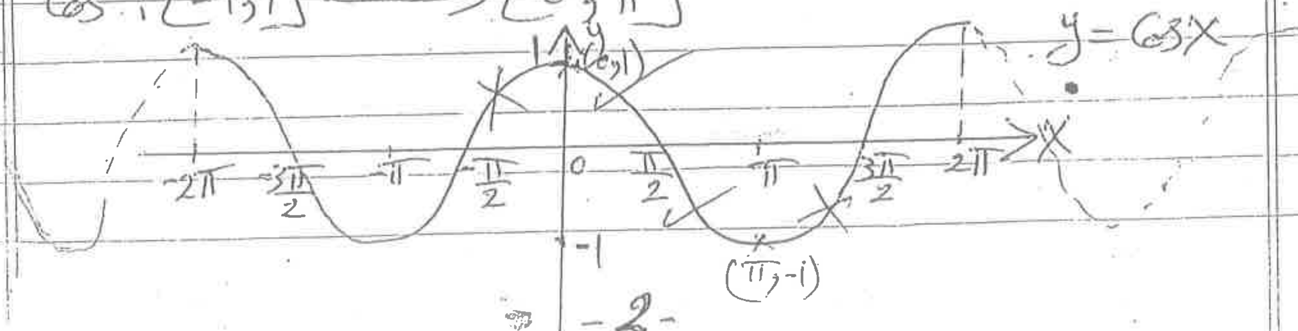
$\cos : \mathbb{R} \longrightarrow [-1, 1]$

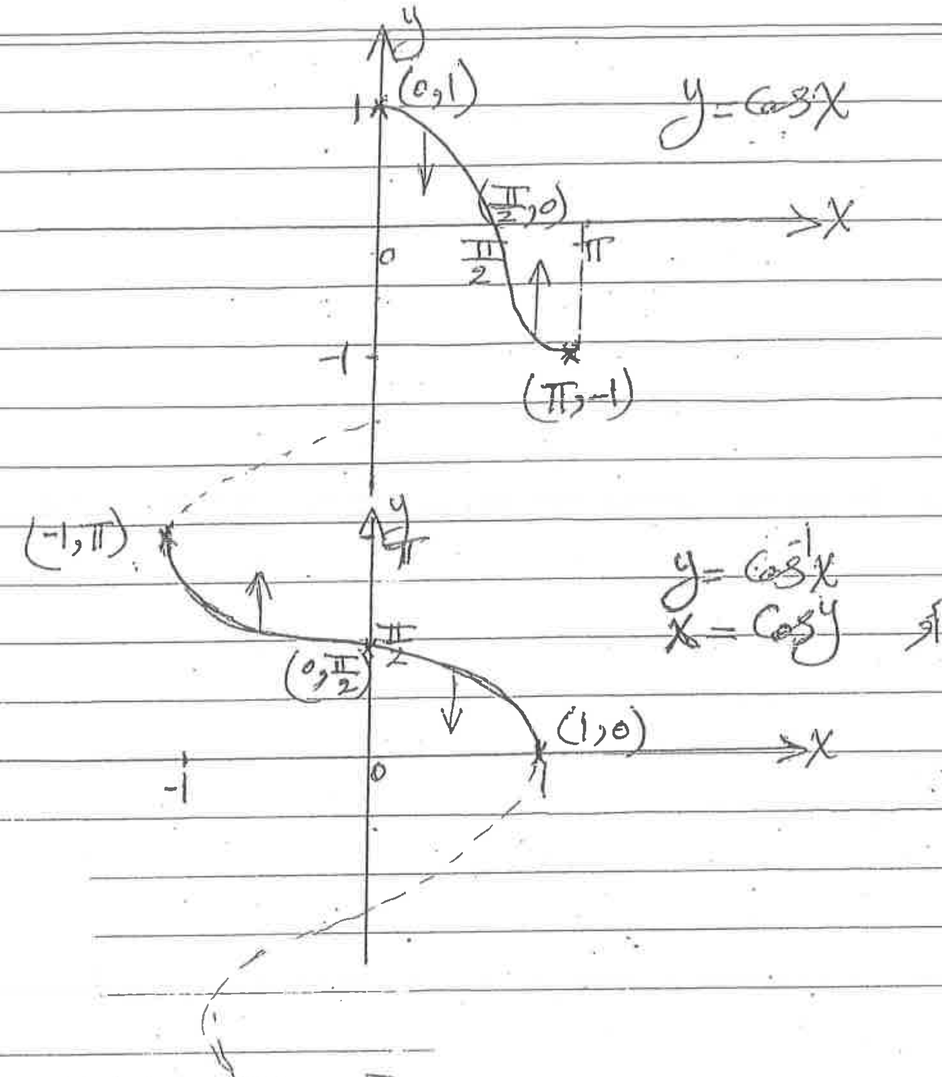
$\cos : [-2\pi, 2\pi] \longrightarrow [-1, 1]$

$\cos : [0, \pi] \longrightarrow [-1, 1]$

$\cos$ is 1-1 & onto $\therefore \exists \cos^{-1}$ (Arc cos)

$\cos^{-1} : [-1, 1] \longrightarrow [0, \pi]$





$\cos(\cos^{-1}x) = x \quad \forall x \in [0, \pi]$

$\cos(\cos^{-1}y) = y \quad \forall y \in [-1, 1]$

Def:

$\boxed{y = \cos x \iff x = \cos^{-1} y \quad x \in [0, \pi]}$ لكل

Note: $\cos^{-1}(x)$ is neither even nor odd function

since: $\boxed{\cos^{-1}(-x) = \pi - \cos^{-1} x}$

Proof:- Let $y = \pi - \cos^{-1}(x)$

$$\sin(\sin^{-1}x) = \sin\left(\frac{\pi}{2} - \omega\right)$$

$$x = \sin\left(\frac{\pi}{2} - \omega\right)$$
$$x = \cos\omega$$

$\omega \in D_{\cos}$? $\omega \in [0, \pi]$?

$$\because \quad -\frac{\pi}{2} \leqslant \sin^{-1}x \leqslant \frac{\pi}{2}$$

$$-\pi \leqslant -\frac{\pi}{2} + \sin^{-1}x \leqslant 0 \quad \Big] + \left(-\frac{\pi}{2}\right)$$

$$\pi \geqslant \frac{\pi}{2} - \sin^{-1}x \geqslant 0 \quad \Big] \times (-1)$$

$$0 \leqslant \frac{\pi}{2} - \sin^{-1}x \leqslant \pi$$

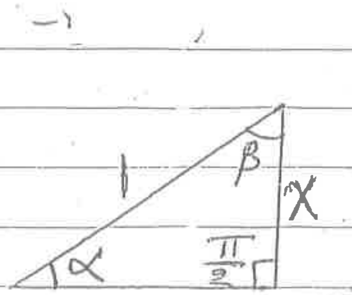
$$0 \leqslant \omega \leqslant \pi$$

$$\therefore \quad \omega = \cos^{-1}(x) \qquad \forall\, x \in [-1, 1]$$

$$\therefore \quad \cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x) \qquad \forall\, x \in [-1, 1]$$

$\sin\alpha = x$, $\cos\beta = x$

$\therefore \alpha = \sin^{-1}x$, $\beta = \cos^{-1}x$



$$\alpha + \beta = \frac{\pi}{2}$$
$$\beta = \frac{\pi}{2} - \alpha$$

$$\left(\alpha + \beta + \frac{\pi}{2} = 180^\circ\right)$$

$$\therefore \cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$$

$\Rightarrow y - \pi = -\cos^{-1}(x)$

$\Rightarrow \cos^{-1}(x) = \pi - y$

$\Rightarrow x = \cos(\pi - y)$

$\Rightarrow x = -\cos y$

$\Rightarrow \cos y = -x$

$\Rightarrow y = \cos^{-1}(-x)$

$\Rightarrow y =$ الطرف الأيسر

$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}(x)$

Let $y = \cos^{-1}(-x)$

$\cos y = -x$ by def. of $\cos^{-1}(x)$

$x = -\cos y$

$x = \cos(\pi - y)$

$\cos^{-1} x = \pi - y$

$\therefore y = \pi - \cos^{-1}(x)$

$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}(x)$

ex:

$\cos^{-1}(-1) = \pi \quad , \quad \cos^{-1}(1) = 0$

Remark:

$$\boxed{\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad \forall \; x \in [-1, 1]}$$

or

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \quad \forall \; x \in [-1, 1]$$

Proof:

$\omega = \frac{\pi}{2} - \sin^{-1}(x) , \quad x \in [-1, 1]$

$\sin^{-1}(x) = \frac{\pi}{2} - \omega$

The inverse of (tan x)

$y = \tan x$

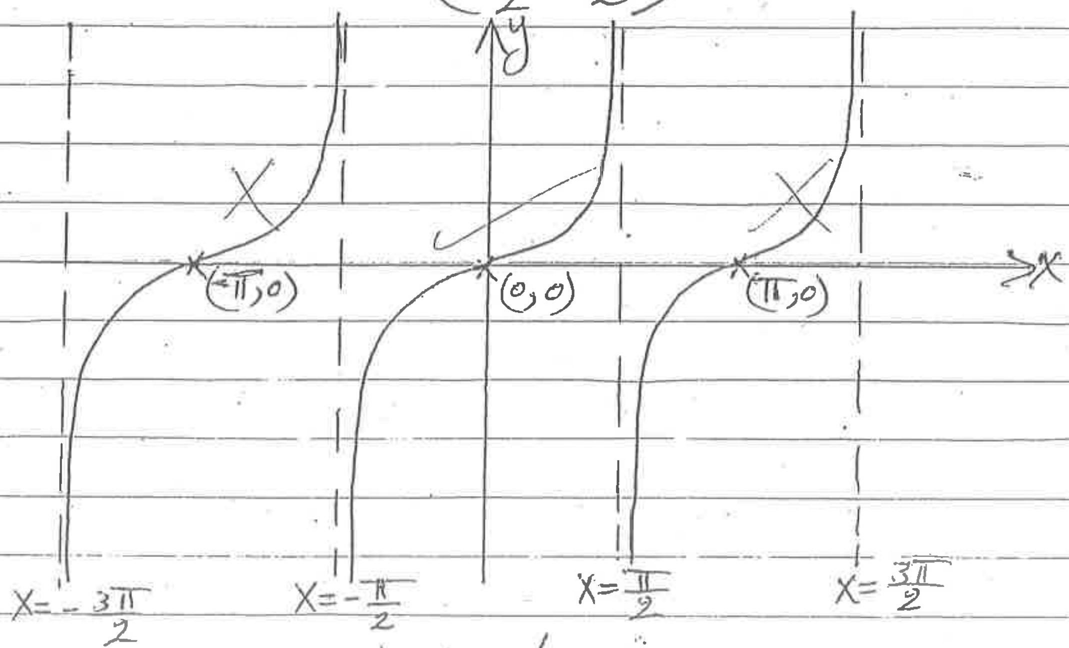
$x = (2n+1)\frac{\pi}{2}$

$\tan : \mathbb{R} \setminus \{x : x = \frac{\pi}{2} + n\pi ; n \in I\} \longrightarrow \mathbb{R}$

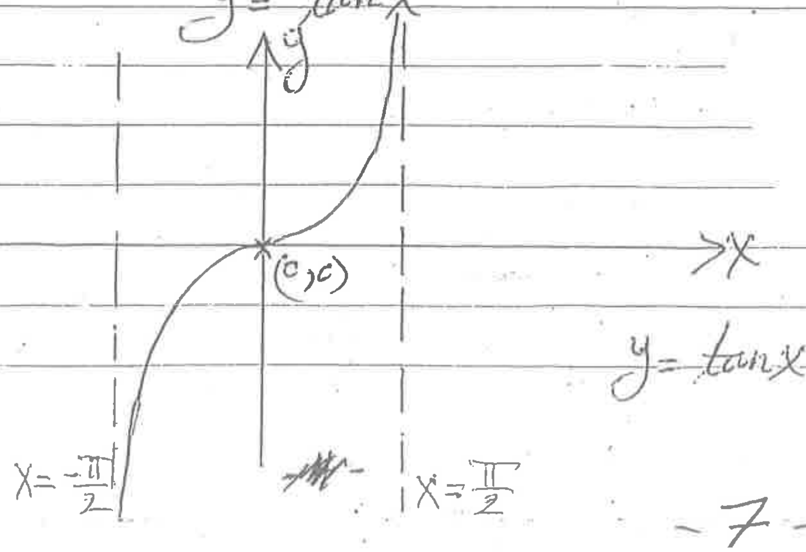
$\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ $\quad x = -\frac{\pi}{2}, x = \frac{\pi}{2}$ are vertical axis

$\tan$ is 1-1 & onto $\therefore \exists \tan^{-1}$ or Arc tan $\ni$

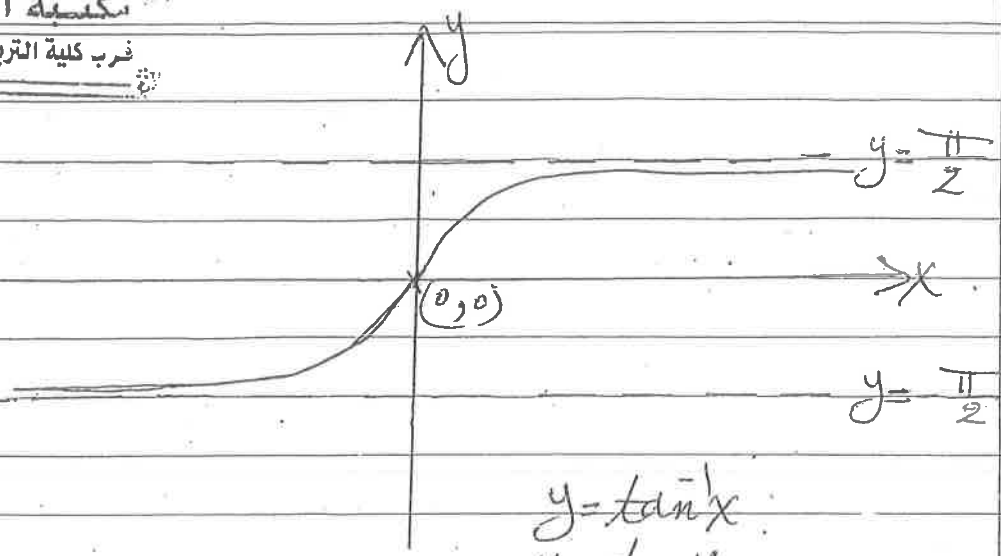
$\tan^{-1} : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ $\quad x = -\frac{\pi}{2}, x = \frac{\pi}{2}$ are Horizontal axis



$y = \tan x$



$y = \tan x$



$y = \tan^{-1} x$
$x = \tan y$

Def:

$$\boxed{y = \tan x \Longleftrightarrow x = \tan^{-1} y \qquad x \in \left(\tfrac{-\pi}{2}, \tfrac{\pi}{2}\right)}$$

$\boxed{\tan^{-1}(-x) = -\tan^{-1}(x)}$ ($\tan^{-1}$ is odd function)

Proof: Let $y = \tan^{-1}(-x)$

$\tan y = -x$

$x = -\tan y$

$x = \tan(-y)$

$-y = \tan^{-1} x$

$y = -\tan^{-1} x$

$\therefore \tan^{-1}(-x) = -\tan^{-1}(x)$

Ex: $\tan^{-1}(1) = \frac{\pi}{4}$, $\tan^{-1}(-1) = -\frac{\pi}{4}$

* The inverse of $(\cot x)$:-

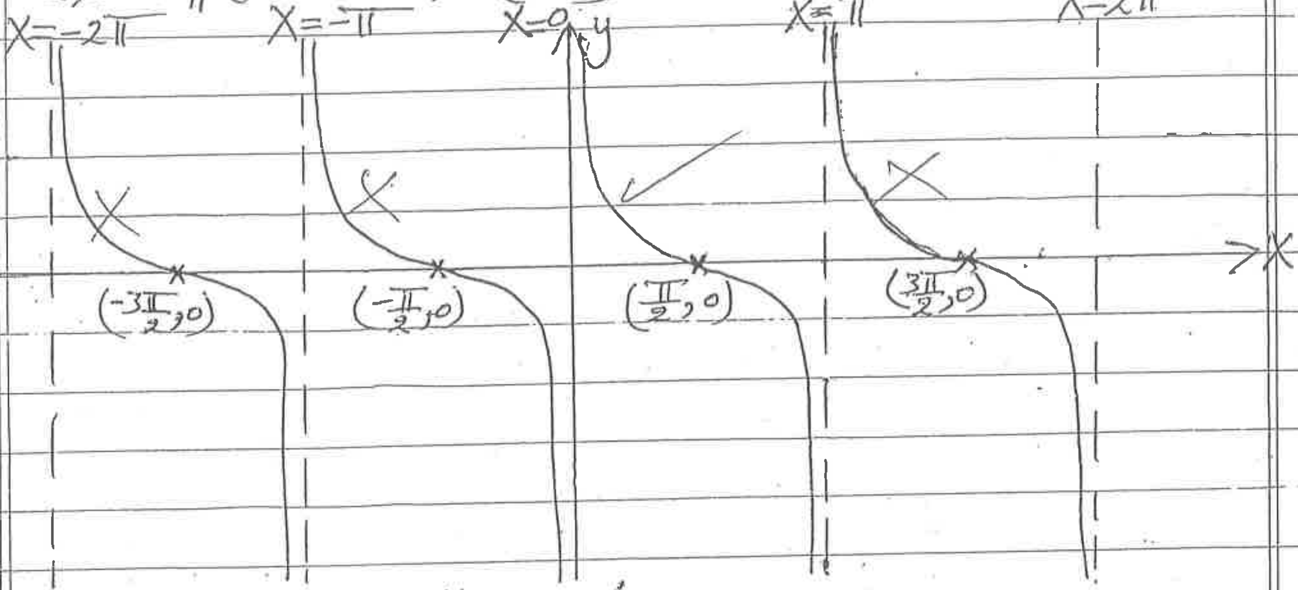
$y = \cot x$

$\cot : \mathbb{R} \setminus \{x ; x = n\pi ; n \in I\} \longrightarrow \mathbb{R}$

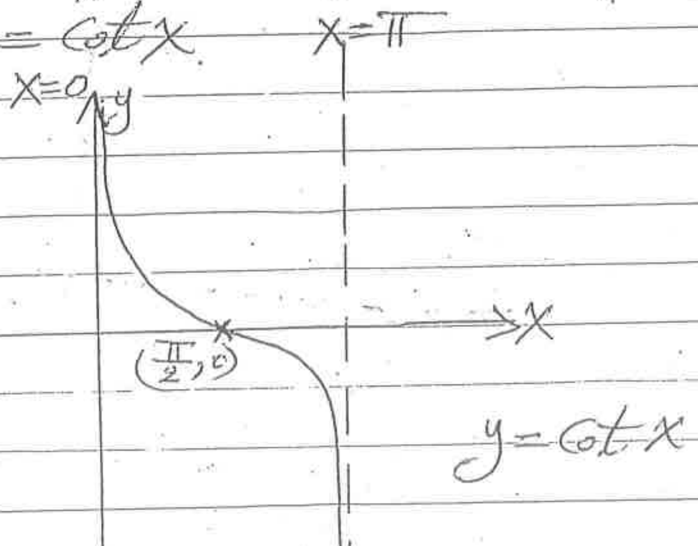
$Cot : (0, \pi) \longrightarrow \mathbb{R}$

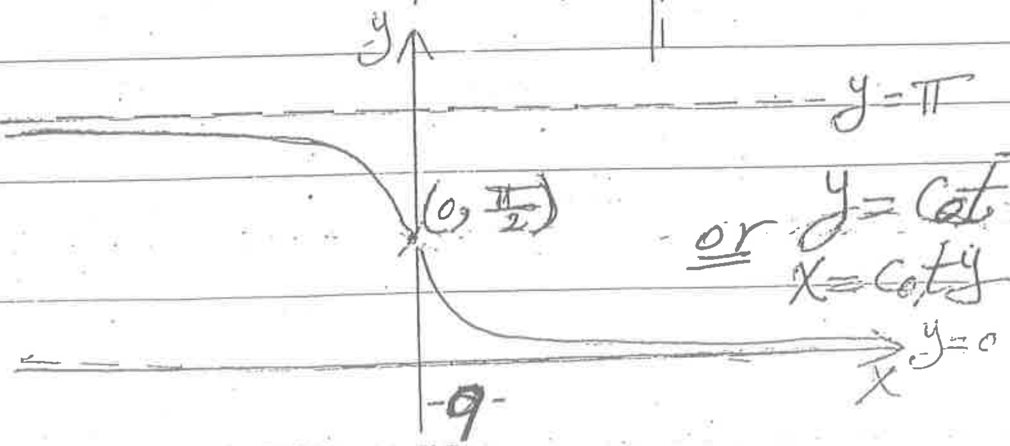
Cot is 1-1 & onto $\therefore \exists\ Cot^{-1}$ or $Arc\,Cot \ni$

$Cot^{-1} : \mathbb{R} \longrightarrow (0, \pi)$



$y = Cot\,x$





$y = Cot^{-1}x$

or $x = Cot\,y$

Def :-

$$\boxed{y = \cot x \Longleftrightarrow x = \cot^{-1} y \quad x \in (0, \pi)}$$

$$\boxed{\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}}$$

* Prove that ?

$$\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$$

Proof: let $w = \frac{\pi}{2} - \tan^{-1}(x)$

$$\tan^{-1}(x) = \frac{\pi}{2} - w$$

$$x = \tan\left(\frac{\pi}{2} - w\right)$$

$$x = \cot w$$

$$w \in D_{\cot} \quad ?$$

$$w \in (0, \pi) \quad ?$$

$$0 < w < \pi \quad ?$$

$$\because \quad -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

$$\frac{\pi}{2} > -\tan^{-1} x > -\frac{\pi}{2} \qquad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \underset{\tan^{-1}}{\overset{\tan}{\rightleftarrows}} \mathbb{R}$$

$$-\frac{\pi}{2} < -\tan^{-1} x < \frac{\pi}{2} \qquad (0, \pi) \underset{\cot^{-1}}{\overset{\cot}{\rightleftarrows}} \mathbb{R}$$

$$0 < \frac{\pi}{2} - \tan^{-1} x < \pi \;\Big]\; + \left(\frac{\pi}{2}\right)$$

$$0 < w < \pi$$

$$\therefore \quad w \in (0, \pi)$$

$$w \in D_{\cot}$$

$$\therefore \quad w = \cot^{-1} x$$

$$\therefore \quad \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

2n - methods:-

$\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$

Let $\tan^{-1}(x) = y$

$\therefore x = \tan y$

$\left(\frac{\pi}{2} - y\right) = z \stackrel{?}{=} \cot^{-1}(x)$

$\cot\left(\frac{\pi}{2} - y\right) = \cot z$

$\tan y = \cot z$

$\tan y = x$

$\therefore x = \cot z$

$\therefore z = \cot^{-1}(x)$

$\therefore \cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$

Notes:

$\tan^{-1}(x) \neq \frac{\sin^{-1}(x)}{\cos^{-1}(x)}$

$\cot^{-1}(x) \neq \frac{\cos^{-1}(x)}{\sin^{-1}(x)}$

* prove that?

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}(x)$

proof:

let $\alpha = \tan^{-1}\left(\frac{1}{x}\right)$ (by Def. of $\tan^{-1}x$)

$\tan\alpha = \frac{1}{x}$

$x = \frac{1}{\tan\alpha}$

$x = \cot\alpha$

$\alpha = \cot^{-1}(x)$

$\therefore \tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}(x)$

The inverse of $(\sec x)$

$y = \sec x$

$\sec : \mathbb{R} \setminus \{x : x = \frac{\pi}{2} + n\pi ; n \in I\} \longrightarrow |y| \geq 1$

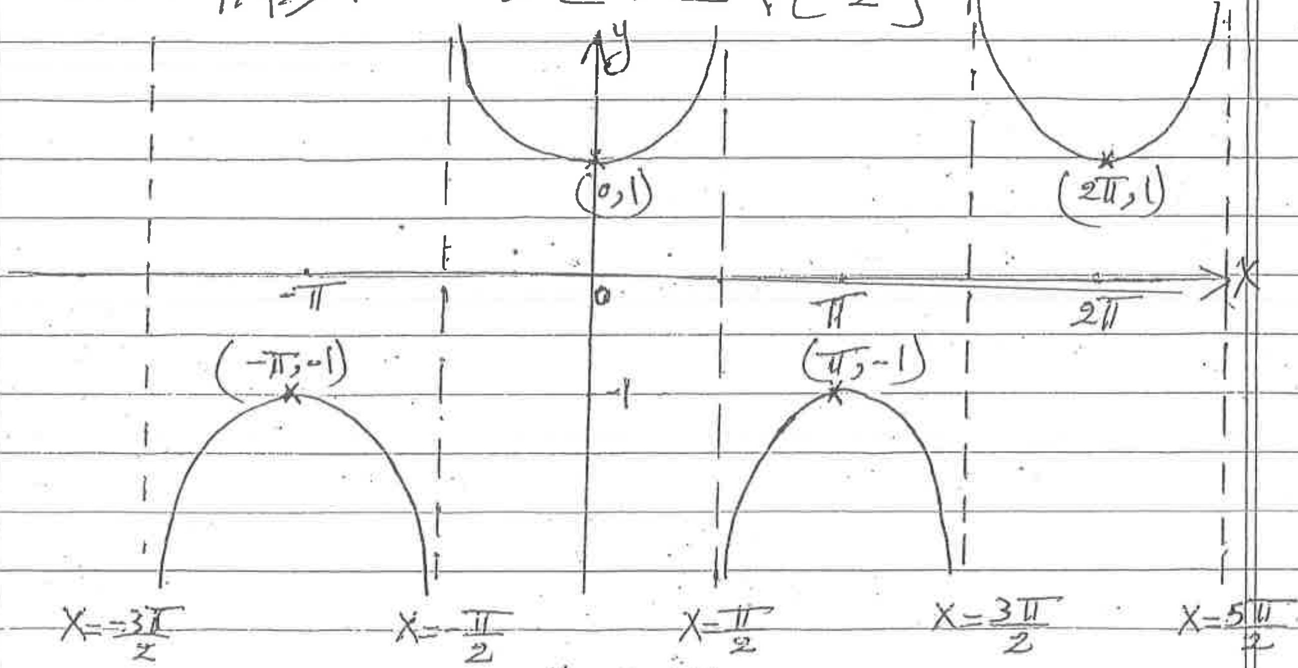
$\sec : [0, \pi] \setminus \{\frac{\pi}{2}\} \longrightarrow |y| \geq 1$

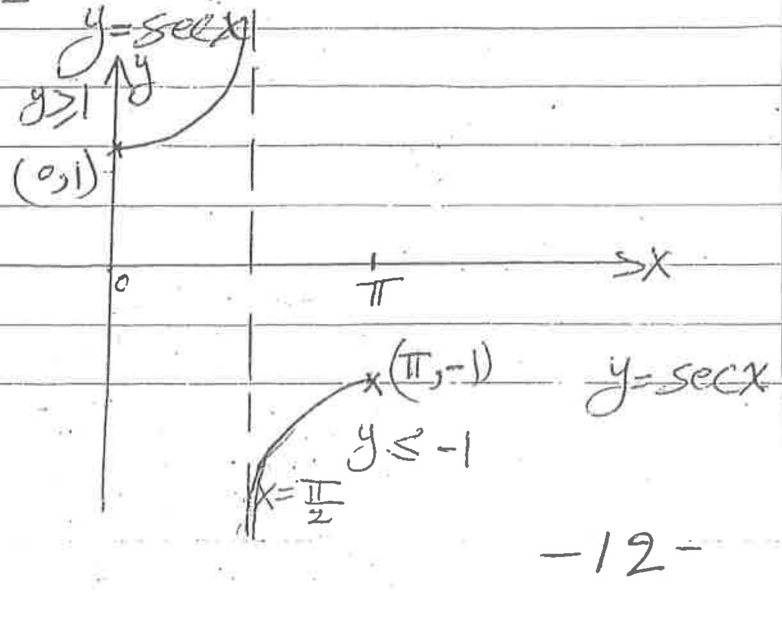
$\equiv \; y \geq 1 \;\vee\; y \leq -1$

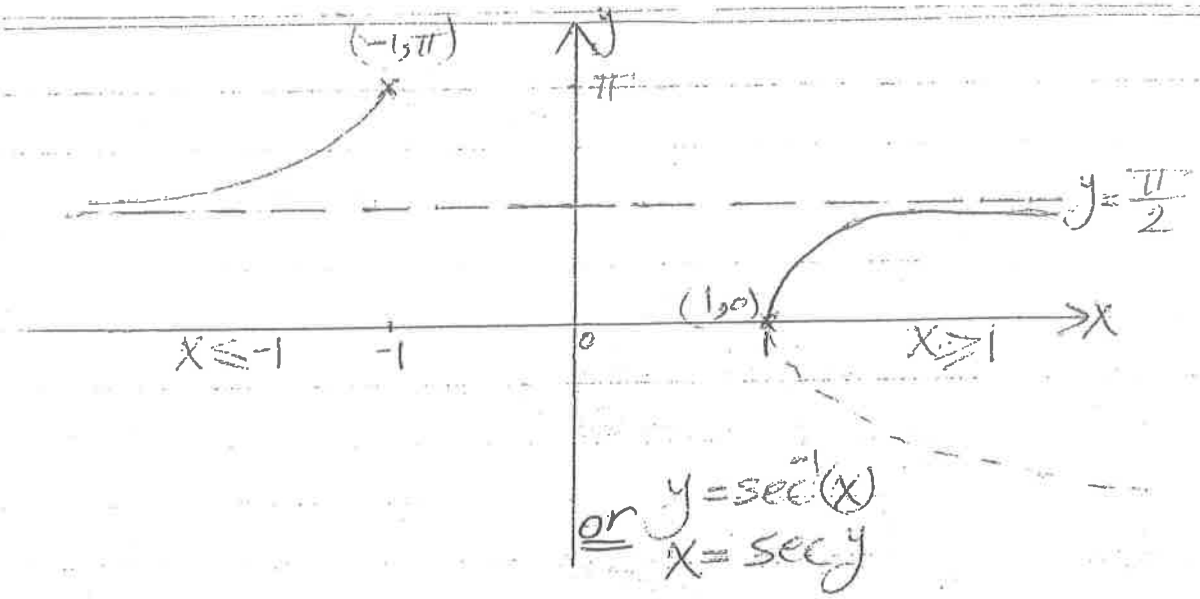
$\equiv \; \mathbb{R} \setminus (-1, 1)$

sec is 1-1 & ont $\therefore \exists \sec^{-1} \equiv \text{Arc sec} \ni$

$\sec^{-1} : |x| \geq 1 \longrightarrow [0, \pi] \setminus \{\frac{\pi}{2}\}$







<u>or</u> $y=\sec^{-1}(x)$

$x=\sec y$

<u>Def:</u> $\sec^{-1}(x)=\cos^{-1}\left(\frac{1}{x}\right) \quad \forall\ |x|\ge 1$

Let $y=\sec^{-1}(x)$

$x=\sec y$

$x=\frac{1}{\cos y}$

$\cos y=\frac{1}{x}$

$\therefore y=\cos^{-1}\left(\frac{1}{x}\right)$

$\therefore \sec^{-1}(x)=\cos^{-1}\left(\frac{1}{x}\right)$

Prove that?

$\sec^{-1}(-x)=\pi-\sec^{-1}(x)$

<u>Proof:</u>

Let $\sec^{-1}(-x)=y$

$\therefore \sec y=-x$

$x=-\sec y$

$x=\frac{1}{-\cos y}$ $\qquad \cos(\pi-y)=-\cos y$

$$X = \frac{1}{\cos(\pi - y)}$$

$$X = \sec(\pi - y)$$

$$\pi - y = \sec^{-1}(X)$$

$$\therefore y = \pi - \sec^{-1}(X)$$

$$\therefore \sec^{-1}(-X) = \pi - \sec^{-1}(X)$$

The inverse of $(\csc x)$ is:

$y = \csc(x)$

$\csc : \mathbb{R} \setminus \{x : x = n\pi ; n \in I\} \longrightarrow |y| \geq 1$

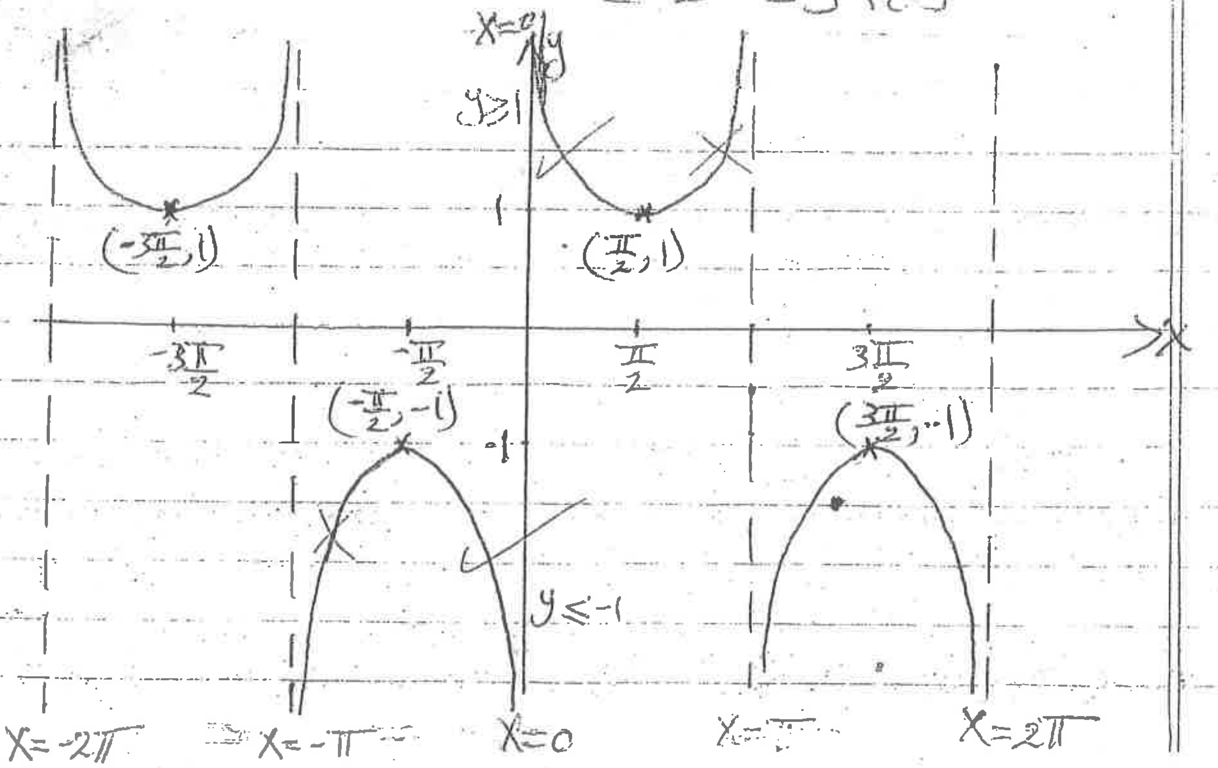
$\csc : [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\} \longrightarrow |y| \geq 1$

$y \geq 1 \vee y \leq -1$

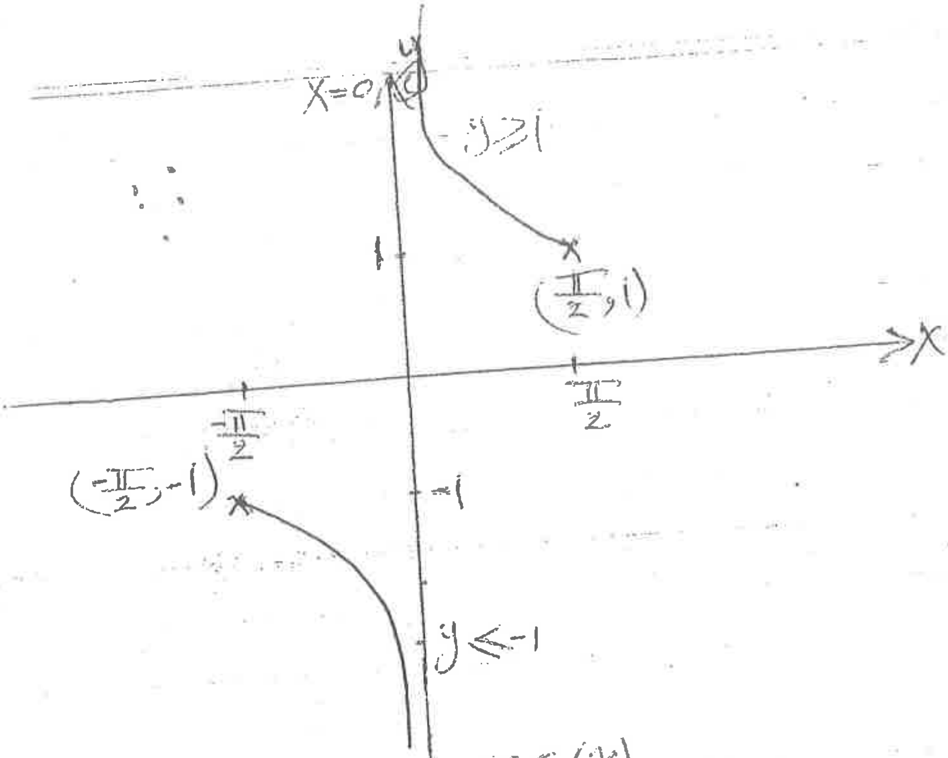
$\mathbb{R} \setminus (-1, 1)$

csc is 1-1 & onto $\therefore \exists \csc^{-1} \ni$

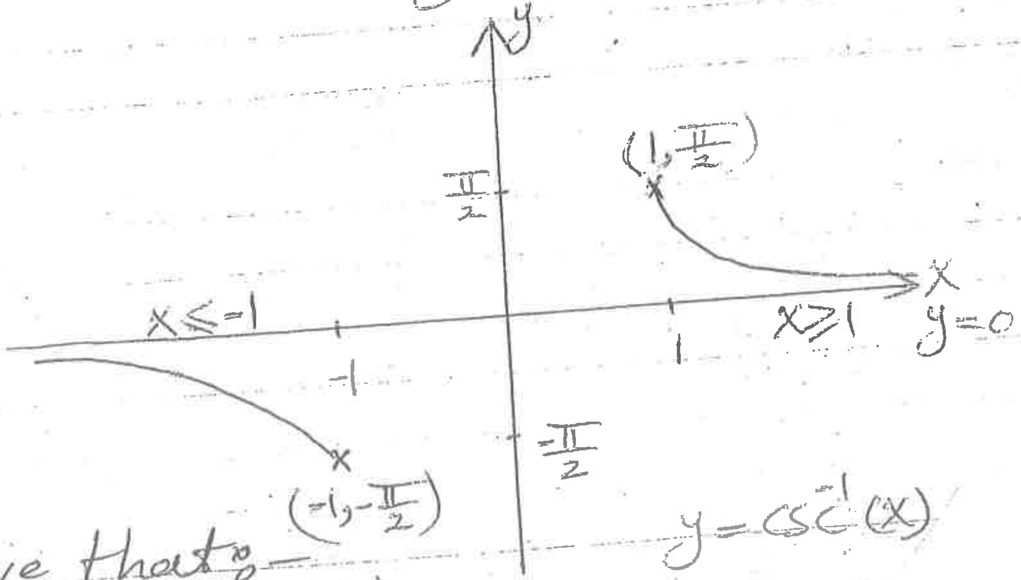
$\csc^{-1} : |x| \geq 1 \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$



$y = \csc(x)$



$y = \csc(x)$



$y = \csc^{-1}(x)$

$|x| \geq 1$

* Prove that:-

$$\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

proof:

Let $y = \csc^{-1}(x)$

$\csc y = x$

$\therefore x = \frac{1}{\sin y}$

$\sin y = \frac{1}{x}$

$\therefore y = \sin^{-1}\left(\frac{1}{x}\right)$

$\therefore \csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$

※ find the value of?

$\sec\left(\sin^{-1}\left(\frac{-2}{3}\right)\right)$

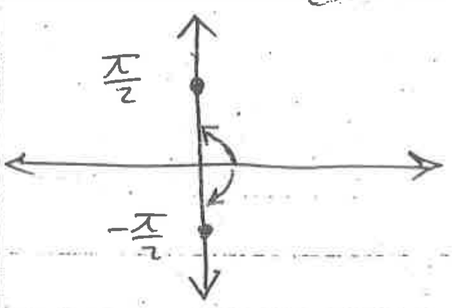
$\theta = \sin^{-1}\left(\frac{-2}{3}\right)$

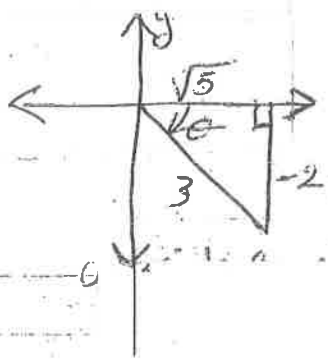
$\sin\theta = \frac{-2}{3}$

$\sin x < 0$ in 4th and 3-ed quarter.

$\theta \in D_{\sin}$

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$





by pythagoras property

$(3)^2 = (-2)^2 + x^2$

$\therefore x^2 = 9 - 4 \Rightarrow x^2 = 5 \Rightarrow x = \mp\sqrt{5}$

$\sec\left(\sin^{-1}\left(\frac{-2}{3}\right)\right) = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{5}/3} = \frac{3}{\sqrt{5}} > 1$

$\sec : \mathbb{R} \setminus \{x : x = \frac{\pi}{2} + n\pi ; n \in I\} \longrightarrow |y| \geq 1$

$\sec x \geq 1 \vee \sec x \leq -1$

※ find the value of :-

$\tan\left(\tan^{-1}\frac{3}{4} - \sin^{-1}\frac{1}{2}\right)$

let $\alpha = \tan^{-1}\frac{3}{4}$

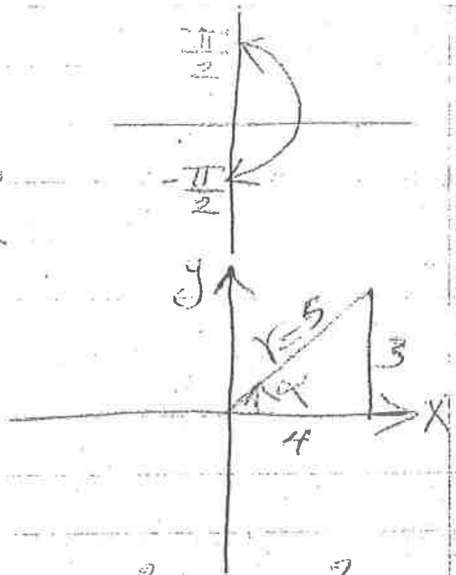
$\tan\alpha = \frac{3}{4}$

$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$

α في الربعين (1) و (4)

$\tan\alpha > 0$ موجب في الربعين (1) و (3)

∴ الزاوية α تقع في الربع الأول



$r^2 = 3^2 + 4^2$

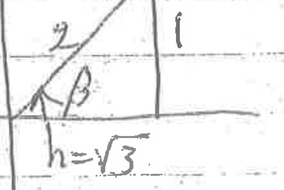
$= 9 + 16 = 25$

$r = 5$

$\beta = \sin^{-1}\frac{1}{2}$

$\sin\beta = \frac{1}{2}$

$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$



$\sin(x) > 0$ in 1st and 2nd quarter

∴ $\sin(\beta)$ in 1st quarter

$(2)^2 = (1)^2 + h^2$

$4 - 1 = h^2$

$h^2 = 3$

$h = \mp\sqrt{3}$

∴ $h = \sqrt{3}$

$$\tan\left(\tan^{-1}\frac{3}{4}-\sin^{-1}\frac{1}{2}\right)=\tan(\alpha-\beta)$$

$$=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$$

$$=\frac{\frac{3}{4}-\frac{1}{\sqrt{3}}}{1+\frac{3}{4}\times\frac{1}{\sqrt{3}}}=\frac{\frac{3\sqrt{3}-4}{4\sqrt{3}}}{1+\frac{\sqrt{3}}{4}}$$

$$=\frac{\frac{3\sqrt{3}-4}{\cancel{4}\sqrt{3}}}{\frac{4+\sqrt{3}}{\cancel{4}}}=\frac{3\sqrt{3}-4}{\sqrt{3}}\cdot\frac{1}{4+\sqrt{3}}$$

$$=\frac{3\sqrt{3}-4}{\sqrt{3}(4+\sqrt{3})}$$

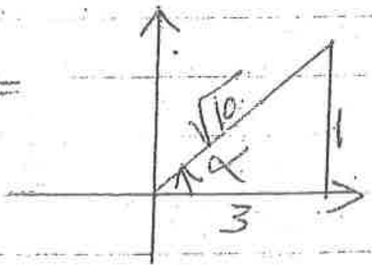
※ Prove that ?

$$\sin^{-1}\frac{1}{\sqrt{10}}+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}$$

Proof:

Let $\alpha=\sin^{-1}\frac{1}{\sqrt{10}} \Longleftrightarrow \sin\alpha=\frac{1}{\sqrt{10}}$

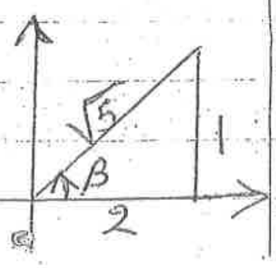
Let $\beta=\sin^{-1}\frac{1}{\sqrt{5}} \Longleftrightarrow \sin\beta=\frac{1}{\sqrt{5}}$



$$\alpha+\beta\overset{?}{=}\frac{\pi}{4}$$

$$\sin(\alpha+\beta)\overset{?}{=}\sin\frac{\pi}{4}$$

$$\sin\alpha\cos\beta+\cos\alpha\sin\beta\overset{?}{=}\frac{1}{\sqrt{2}}$$



$$\frac{1}{\sqrt{10}}\cdot\frac{2}{\sqrt{5}}+\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{5}}\overset{?}{=}\frac{1}{\sqrt{2}}$$

$$\frac{2}{\sqrt{50}} + \frac{3}{\sqrt{50}} \stackrel{?}{=} \frac{1}{\sqrt{2}}$$

$$\frac{5}{\sqrt{50}} \stackrel{?}{=} \frac{1}{\sqrt{2}}$$

$$\frac{5}{\sqrt{25\times 2}} \stackrel{?}{=} \frac{1}{\sqrt{2}}$$

$$\frac{5}{\sqrt{25}\,\sqrt{2}} \stackrel{?}{=} \frac{1}{\sqrt{2}}$$

$$\frac{\cancel{5}}{\cancel{5}\,\sqrt{2}} \stackrel{?}{=} \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

* The derivative of inverse trigonometric functions:

$$\frac{d(\sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\frac{d(\cos^{-1}u)}{dx} = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\frac{d(\tan^{-1}u)}{dx} = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d(\cot^{-1}u)}{dx} = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d(\sec^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$\frac{d(\csc^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Examples: find the derivative of :-

1- $f(x) = \sin^{-1}(x^2)$

$$f'(x) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$$

2- $y = \cos^{-1}\sqrt{x}$

$$y' = -\frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{-1}{2\sqrt{x}\,\sqrt{1-x}}$$

3- $y = \tan^{-1}(e^x)$

$$y' = \frac{1}{1+(e^x)^2} \cdot e^x \cdot \underbrace{\ln e}_{1} \cdot 1 = \frac{e^x}{1+e^{2x}}$$

4- $y = \sin^{-1}\sqrt{1-\sqrt{x}}$

$$y' = \frac{1}{\sqrt{1-(\sqrt{1-\sqrt{x}})^2}} \cdot \frac{1}{2}(1-\sqrt{x})^{-\frac{1}{2}} \cdot -\frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{-1}{4\sqrt{x}\,\sqrt{1-\sqrt{x}}\,\sqrt{1-(1-\sqrt{x})}} = \frac{-1}{4\sqrt{x}\sqrt{1-\sqrt{x}}\,\sqrt{\cancel{1}-\cancel{1}+\sqrt{x}}}$$

$$= \frac{-1}{4\sqrt{x}\,\sqrt{1-\sqrt{x}}\;\sqrt[4]{x}}$$

$$f(x) = \cot^{-1}\left(\frac{1-x}{1+x}\right)$$

$$f'(x) = -\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \cdot \frac{(1+x)\cdot -1 - (1-x)\cdot 1}{(1+x)^2}$$

$$= -\frac{1}{\frac{(1+x)^2+(1-x)^2}{(1+x)^2}} \cdot \frac{-1-\cancel{x}-1+\cancel{x}}{(1+x)^2}$$

$$= -\frac{\cancel{(1+x)^2}}{1+\cancel{2x}+x^2+1-\cancel{2x}+x^2} \cdot \frac{-2}{\cancel{(1+x)^2}}$$

$$= -\frac{1}{2+2x^2} \cdot -2 = \frac{\cancel{2}}{\cancel{2}(1+x^2)}$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f(x) = \sec^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

$$f'(x) = \frac{1}{\left|\frac{\sqrt{1+x^2}}{x}\right|\sqrt{\left(\frac{\sqrt{1+x^2}}{x}\right)^2-1}} \cdot \frac{x\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x - \sqrt{1+x^2}\cdot 1}{x^2}$$

$$= \frac{1}{\frac{\sqrt{1+x^2}}{|x|}\sqrt{\frac{1+x^2}{x^2}-1}} \cdot \frac{\frac{x}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}x - \sqrt{1+x^2}}{x^2}$$

$$= \frac{1}{\frac{\sqrt{1+x^2}}{|x|}\sqrt{\frac{1+\cancel{x^2}-\cancel{x^2}}{x^2}}} \cdot \frac{\frac{x^2-(1+x^2)}{\sqrt{1+x^2}}}{\frac{x^2}{1}}$$

$$= \frac{1}{\frac{\sqrt{1+x^2}}{\sqrt{x^2}} \cdot \frac{1}{\sqrt{x^2}}} \cdot \frac{\cancel{x^2} - 1 - \cancel{x^2}}{\sqrt{1+x^2}} \cdot \frac{1}{x^2}$$

$$= \frac{\cancel{x^2}}{\sqrt{1+x^2}} \cdot \frac{-1}{\sqrt{1+x^2}} \cdot \frac{1}{\cancel{x^2}}$$

$$f'(x) = \frac{-1}{1+x^2}$$

$$\therefore\ y = x \cdot \csc^{-1}\frac{1}{x} + \sqrt{1-x^2}$$

$$y' = x \cdot \frac{-1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}} \cdot \frac{x \cdot 0 - 1 \cdot 1}{x^2} + \csc^{-1}\frac{1}{x} \cdot 1 + \frac{1}{\cancel{2}}(1-x^2)^{-\frac{1}{2}} \cdot -\cancel{2}x$$

$$= \frac{-x}{\frac{1}{\sqrt{x^2}}\sqrt{\frac{1-x^2}{x^2}}} \cdot \frac{-1}{x^2} + \csc^{-1}\frac{1}{x} - \frac{x}{\sqrt{1-x^2}}$$

$$= \frac{1}{x \frac{1}{\sqrt{x^2}} \frac{\sqrt{1-x^2}}{\sqrt{x^2}}} + \csc^{-1}\frac{1}{x} - \frac{x}{\sqrt{1-x^2}}$$

$$= \frac{1}{\cancel{x} \cdot \frac{1}{\cancel{x^2}} \sqrt{1-x^2}} + \csc^{-1}\frac{1}{x} - \frac{x}{\sqrt{1-x^2}}$$

$$= \cancel{\frac{x}{\sqrt{1-x^2}}} + \csc^{-1}\frac{1}{x} - \cancel{\frac{x}{\sqrt{1-x^2}}}$$

$$y' = \csc^{-1}\frac{1}{x}$$

※ Prove that ?

1) $\dfrac{d(\sin^{-1}u)}{dx} = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$

Let $y = \sin^{-1}u \iff u = \sin y$

$\therefore u \in (-1, 1)$ and $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\frac{du}{dx} = \cos y \cdot \frac{dy}{dx}$$

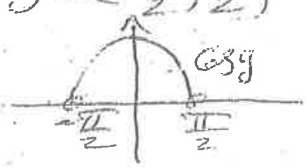
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} \cdot \frac{du}{dx}$$

$\cos y = \mp\sqrt{1-\sin^2 y}$

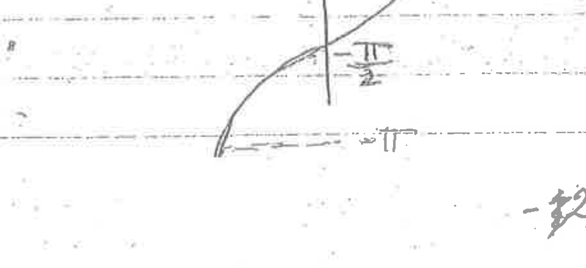
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} \cdot \frac{du}{dx}$$

$\cos y > 0$

$y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

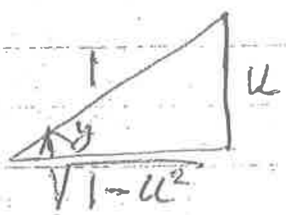


$$\frac{d(\sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$



$u = \sin y$

$\Rightarrow \sin y = \dfrac{u}{1}$



$\cos y = \dfrac{1}{\sqrt{1-u^2}}$

$\therefore \cos y = \sqrt{1-u^2}$

) $\frac{d(\cos^{-1}u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$

roof:

Let $y = \cos^{-1}u \Longleftrightarrow u = \cos y$

لكل $u \in (-1,1)$

ولكل $y \in (0,\pi)$ نشتق طرفي المعادلة بالنسبة الى x

$$\frac{du}{dx} = -\sin y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}\,\frac{du}{dx}$$

$\sin y = \overset{\text{تهمل}}{\mp}\sqrt{1-\cos^2 y}$

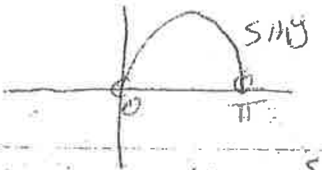
$$\frac{d(\cos^{-1}u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

بما أن $\sin y > 0$

لكل $y \in (0,\pi)$

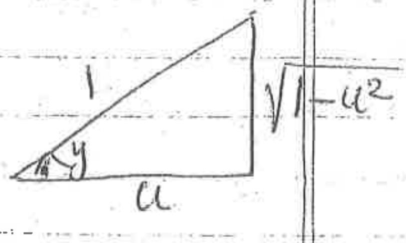
∴ الإشارة السالبة تهمل

توضيح



أو $\cos y = u = \frac{u}{1} = \frac{\text{المجاور}}{\text{الوتر}}$

$\sin y = \frac{\text{المقابل}}{\text{الوتر}}$



$\sin y = \frac{\sqrt{1-u^2}}{1}$

$\therefore \sin y = \sqrt{1-u^2}$

) $\frac{d(\tan^{-1}u)}{dx} = \frac{1}{1+u^2} \cdot \frac{du}{dx}$

proof:

Let $y = \tan^{-1}u \iff u = \tan y$

نشتق طرفي المعادلة بالنسبة الى X

$$\frac{du}{dx} = \sec^2 y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y} \cdot \frac{du}{dx}$$

$$\frac{d(\tan^{-1}u)}{dx} = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d(\cot^{-1}u)}{dx} = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$$

البرهان

Let $y = \cot^{-1}u \iff u = \cot y$

نشتق الطرفين

$$\frac{du}{dx} = -\csc^2 y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\csc^2 y} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{1+\cot^2 y} \cdot \frac{du}{dx}$$

$$\frac{d(\cot^{-1}u)}{dx} = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$$

) $\frac{d(\sec^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

Proof:

$\because \quad \sec^{-1}u = \cos^{-1}\frac{1}{u}$

$$\frac{d(\sec^{-1}u)}{dx} = \frac{d\cos^{-1}(\frac{1}{u})}{dx}$$

$$= -\frac{1}{\sqrt{1-(\frac{1}{u})^2}} \cdot -\frac{1}{u^2} \cdot \frac{du}{dx}$$

$$= \frac{1}{u^2\sqrt{\frac{u^2-1}{u^2}}} \cdot \frac{du}{dx}$$

$$= \frac{1}{\frac{u^2\sqrt{u^2-1}}{\cancel{\sqrt{u^2}}}} \cdot \frac{du}{dx}$$

$\sqrt{u^2}$

$$= \frac{1}{\sqrt{u^2}\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$\frac{d\sec^{-1}u}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

) $\frac{d(\csc^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

roof: $\because \csc^{-1}u = \sin^{-1}\frac{1}{u}$

$$\frac{d(\csc^{-1}u)}{dx} = \frac{d(\sin^{-1}\frac{1}{u})}{dx} = \frac{1}{\sqrt{1-(\frac{1}{u})^2}} \cdot \frac{-1}{u^2} \cdot \frac{du}{dx}$$

$$= \frac{1\cdot \cancel{u}}{\sqrt{u^2-1}} \cdot \frac{-1}{u^2} \cdot \frac{du}{dx}$$

1 ...

find the value of the following ?

تمارين : أحسب قيمة :

$\sin^{-1}(1) - \sin^{-1}(-1)$

$\tan^{-1}(1) - \tan^{-1}(-1)$

$\sec^{-1}(2) - \sec^{-1}(-2)$

compute of the following ?

$\cos(\sin^{-1} 0.8)$

$\sin(2\sin^{-1} 0.8)$

$\cos^{-1}\left(-\sin\frac{\pi}{6}\right)$

$\sec^{-1}(\sec(-30^\circ))$

find the value of the following ?

$\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

$\sec\left(\cos^{-1}\frac{1}{2}\right)$

$\cos\left(\cos^{-1}\frac{1}{2}\right)$

$\csc(\sec^{-1} 2)$

$\cos(\cot^{-1} 1)$

$\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

$\cot\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

$\cot(\tan^{-1}(-\sqrt{3}))$

$\csc\left(\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)\right)$

$\tan(\sec^{-1}(1))$

$\cot(\cos^{-1} 0)$

* find the value of the following

$\cos\left(\sin^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{2}\right)$

$\tan\left(\tan^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\frac{1}{\sqrt{3}}\right)$

$\cos\left(\cos^{-1}\frac{3}{4} - \cot^{-1}\frac{1}{4}\right)$

find $\frac{dy}{dx}$ for the following?

$y = \sin^{-1}\frac{x-1}{x+1}$

$y = x\left(\sin^{-1}x\right)^2 - 2x + 2\sqrt{1-x^2}\cdot\sin^{-1}x$

$y = x\cdot\cos^{-1}(2x) - \frac{1}{2}\sqrt{1-4x^2}$

$y = \frac{\cos^{-1}(2x)}{\sqrt{1+4x^2}}$

$y = \sin^{-1}x + \cos^{-1}x$

$y = \cos^{-1}x + \frac{x}{1-x^2}$

$y = \sec^{-1}x + \csc^{-1}x$

$y = \sec^{-1}\sqrt{x^2+4}$

$y = x\cdot\sin^{-1}(2x)$

$y = a\cdot\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$, a is constant

$y = e^{\tan^{-1}x}$

[illegible]

) $y = e^{\sin^{-1} x}$

) $y = \sec^{-1}\left(e^{2x}\right)$

) $y = e^{\tan^{-1}(\cos 3x)}$

$y = x \cdot \sec^{-1} x - \ln\left(x + \sqrt{x^2 - 1}\right) \quad , \quad x > 1$

$y = e^{\sin^{-1}(\ln x) + \tan x}$

$f(x) = x^3 \cdot \cot^{-1}\left(\frac{x}{3}\right)$

$f(x) = \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$ حيث $L = a$

$f(x) = \sin\left(\tan^{-1} x\right)$

$f(x) = \tan^{-1}\left(3 \tan x\right)$

$f(x) = \tan^{-1} \frac{x-3}{1+3x}$

$y = \sec^{-1}(5x)$

$y = \cot^{-1}\left(\frac{2}{x}\right) + \tan^{-1}\left(\frac{x}{2}\right)$

) $y = \tan^{-1} \frac{x-1}{x+1}$

$f(x) = \sqrt{x^2 - 4} - 2\tan^{-1}\left(\frac{1}{2}\sqrt{x^2 - 4}\right)$

$$f(x) = \tan^{-1}\left(\frac{3\sin x}{4+5\cos x}\right)$$

$$f(x) = \frac{\cot^{-1}(3x)}{1+x^2}$$

$$y = \frac{\sqrt{x^2-4}}{x^2} + \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)$$

$$x \cdot \sin y + x^3 = \tan^{-1} y$$

$$\sin^{-1}(xy) = \cos^{-1}(x+y)$$

(Hyperbolic functions) :-

1) $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

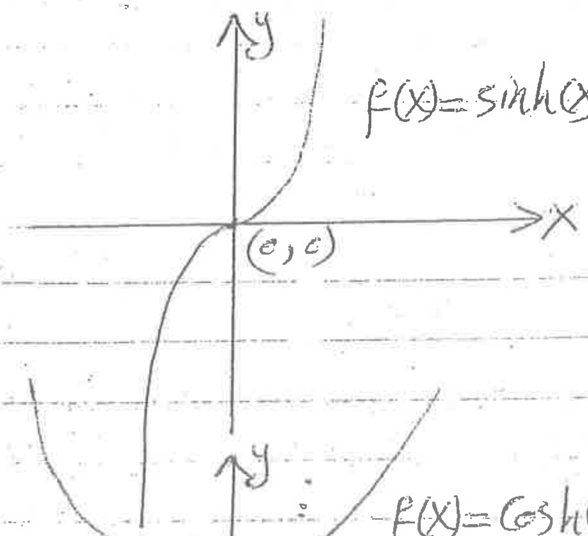
2) $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$

- $\tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ $\qquad \tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$
- $\coth(x) = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$ $\qquad \coth(x) = \dfrac{1}{\tanh(x)} = \dfrac{\cosh(x)}{\sinh(x)}$
- $\operatorname{sech}(x) = \dfrac{2}{e^{x} + e^{-x}}$ $\qquad \operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$
- $\operatorname{csch}(x) = \dfrac{2}{e^{x} - e^{-x}}$ $\qquad \operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$

The Graph of hyperbolic functions :-

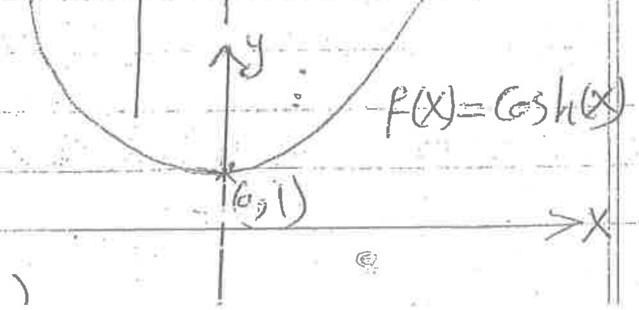
$f(x) = \sinh(x)$

$D_f = \mathbb{R}$, $R_f = \mathbb{R}$



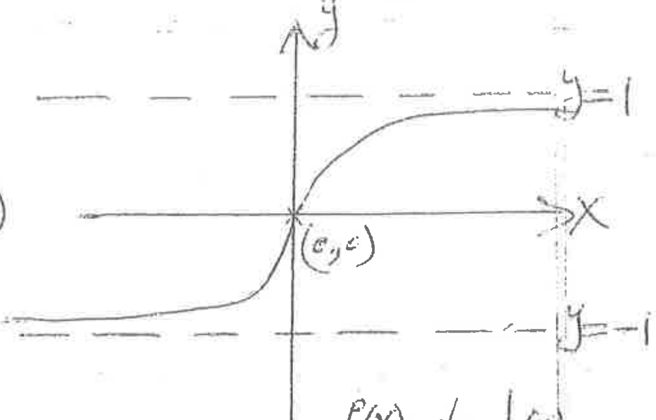
$f(x) = \cosh(x)$

$D_f = \mathbb{R}$, $R_f = [1, \infty)$



3) $f(x) = \tanh(x)$

$D_f = \mathbb{R}$, $R_f = (-1, 1)$

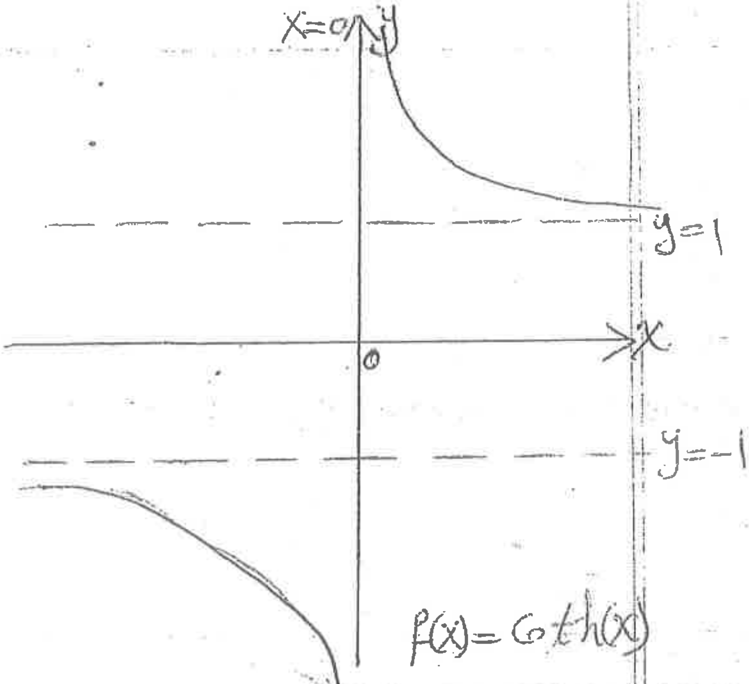


$f(x) = \tanh(x)$

4) $f(x) = \coth(x)$

$D_f = (-\infty, 0) \cup (0, \infty)$

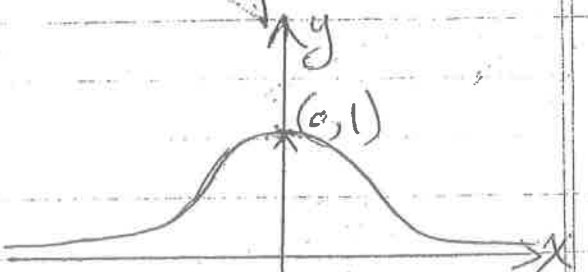
$R_f = (-\infty, -1) \cup (1, \infty)$



$f(x) = \coth(x)$

5) $f(x) = \operatorname{sech}(x)$

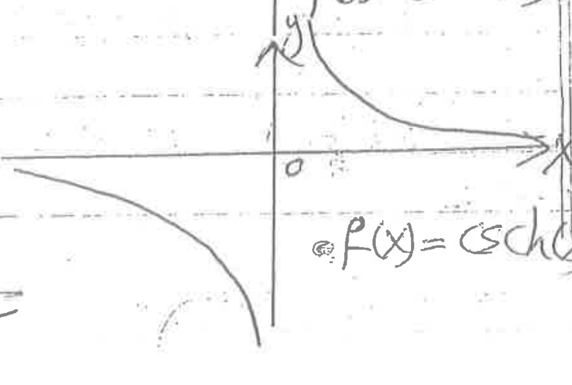
$D_f = \mathbb{R}$, $R_f = (0, 1]$



$f(x) = \operatorname{sech}(x)$

6) $f(x) = \operatorname{csch}(x)$

$D_f = (-\infty, 0) \cup (0, \infty)$
$R_f = (-\infty, 0) \cup (0, \infty)$



$f(x) = \operatorname{csch}(x)$

※ Some facts about hyperbolic functions:

1) $\cosh^2(x) - \sinh^2(x) = 1$

2) $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

3) $\coth^2(x) - 1 = \operatorname{csch}^2(x)$

4)
* $\cosh(-x) = \cosh(x)$ (even function)
* $\sinh(-x) = -\sinh(x)$ (odd function)
* $\tanh(-x) = -\tanh(x)$ (odd function)

5)
* $\cosh(x) + \sinh(x) = e^{x}$
* $\cosh(x) - \sinh(x) = e^{-x}$

6)
* $\cosh(x+y) = \cosh(x)\cosh(y) \overset{\text{نفس الإشارة}}{+} \sinh(x)\sinh(y)$
* $\sinh(x+y) = \sinh(x)\cosh(y) \overset{\text{نفس الإشارة}}{+} \cosh(x)\sinh(y)$
* $\tanh(x+y) = \dfrac{\tanh(x) + \tanh(y)}{1 - \tanh(x)\cdot\tanh(y)}$

7)
* $\cosh^2(x) = \frac{1}{2}(1 + \cosh(2x))$ — $\cosh(2x) + 1 = 2\cosh^2(x)$
* $\sinh^2(x) = \frac{1}{2}(\cosh(2x) - 1)$ — $\cosh(2x) - 1 = 2\sinh^2(x)$

8)
* $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$
* $\sinh(2x) = 2\sinh(x)\cosh(x)$

※ The derivatives of hyperbolic functions

1) $\frac{d \sinh(u)}{dx} = \cosh(u) \frac{du}{dx}$

2) $\frac{d \cosh(u)}{dx} = \sinh(u) \frac{du}{dx}$

3) $\frac{d \tanh(u)}{dx} = \operatorname{sech}^2(u) \frac{du}{dx}$

4) $\frac{d \coth(u)}{dx} = -\operatorname{csch}^2(u) \frac{du}{dx}$

5) $\frac{d \operatorname{sech}(u)}{dx} = -\operatorname{sech}(u) \cdot \tanh(u) \frac{du}{dx}$

6) $\frac{d \operatorname{csch}(u)}{dx} = -\operatorname{csch}(u) \cdot \coth(u) \frac{du}{dx}$

※ Examples:- find $\frac{dy}{dx}$ or y' for the following?

1) $y = \sinh(3X)$

$y' = 3 \cosh(3X)$

2) $y = \cosh^2(5X)$

$y' = 2 \cosh(5X) \cdot \sinh(5X) \cdot 5 = 5 \sinh(10X)$

3) $y = \tanh(2X)$

$y' = 2 \operatorname{sech}^2(2X)$

4) $y = \coth(\tan x)$

$y' = -\operatorname{csch}^2(\tan x) \cdot \sec^2 x \cdot 1$

5) $y = \operatorname{sech}^3 x$

$y' = 3\ \text{sech}^2(x) \cdot -\text{sech}(x) \cdot \tanh(x) \cdot 1$
$y' = -3\ \text{sech}^3(x) \cdot \tanh(x)$

6) $y = 4\ \text{csch}\left(\frac{x}{4}\right)$

$y' = 4 \cdot -\text{csch}\left(\frac{x}{4}\right) \cdot \coth\left(\frac{x}{4}\right) \cdot \frac{1}{4}$

$y' = -\text{csch}\left(\frac{x}{4}\right) \cdot \coth\left(\frac{x}{4}\right)$

※ Exercises: find y' or $f'(x)$?

1) $f(x) = \frac{\cosh(x)}{x}$

2) $f(x) = e^x \cdot \cosh(x)$

3) $f(x) = \ln(\sinh(x^2))$

) $f(x) = \tanh\left(\frac{4x+1}{5}\right)$

) $f(x) = \ln(\tanh(x))$

) $f(x) = e^x\ \tanh(2x)$

7) $f(x) = \tan^{-1}(\sinh^2 x)$

) $f(x) = \coth\left(\frac{1}{x}\right)$

) $y = \cosh^{12}(5x) - \sinh^2(5x)$

10- $\sinh y = \tan x$

11- $y = \sinh^2(3x)$

12- $\sin^{-1} x = \text{sech}\, y$

13- $x = \cosh(\ln y)$

14- $\tan x = \tanh^2 y$

15- $\sinh y = \sec x$

16- $y = \tanh(\ln x)$

17- $y = \sinh(\tan^{-1} e^{3x})$

18- $y^2 + x\cosh y + \sinh^2 x = 50$

19 $y = \cot(\text{csch}(e^x))$

20- $f(x) = \text{csch}^3(\sqrt{2x})$

(35)

* The hyperbolic inverse functions :

* $x = \sinh y \Longleftrightarrow y = \sinh^{-1}(x)$

$-\infty < x < \infty \quad , \quad -\infty < y < \infty$

* $x = \cosh y \Longleftrightarrow y = \cosh^{-1}(x)$

$x \geq 1 \quad , \quad y \geq 0$

* $x = \tanh y \Longleftrightarrow y = \tanh^{-1}(x)$

* $x = \coth y \Longleftrightarrow y = \coth^{-1}(x)$

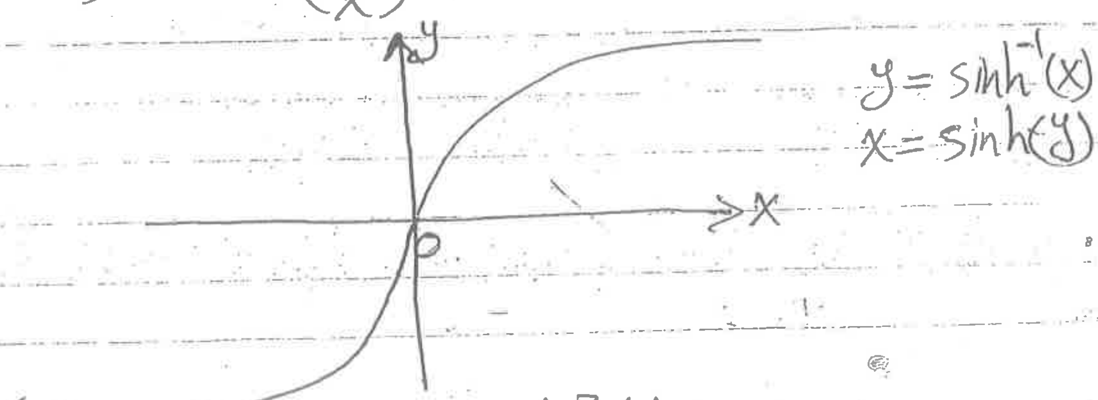
$x = \operatorname{sech} y \Longleftrightarrow y = \operatorname{sech}^{-1}(x)$

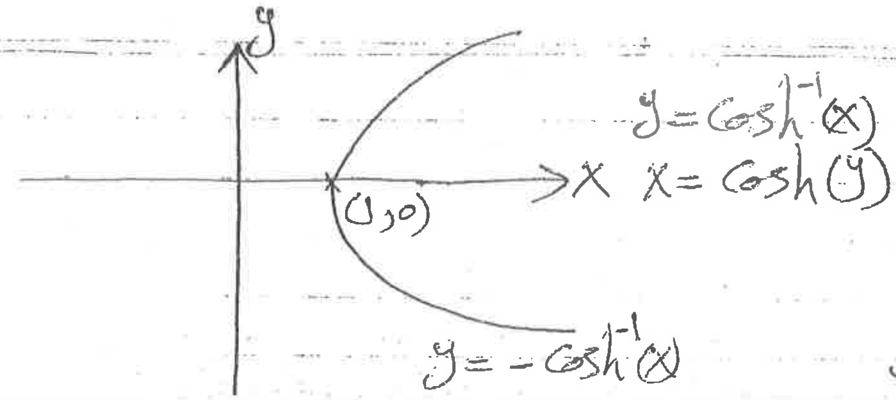
$0 < x \leq 1 \quad , \quad y > 0$

* $x = \operatorname{csch} y \Longleftrightarrow y = \operatorname{csch}^{-1}(x)$

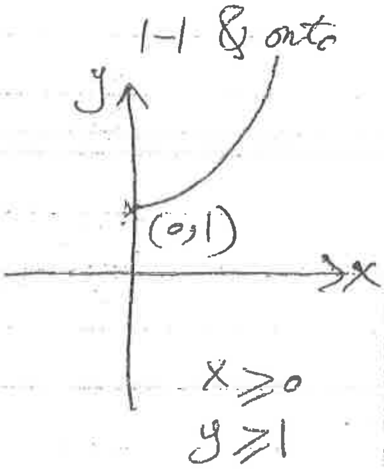
$\operatorname{sech}^{-1}(x) = \cosh^{-1}\left(\frac{1}{x}\right)$

$\operatorname{csch}^{-1}(x) = \sinh^{-1}\left(\frac{1}{x}\right)$

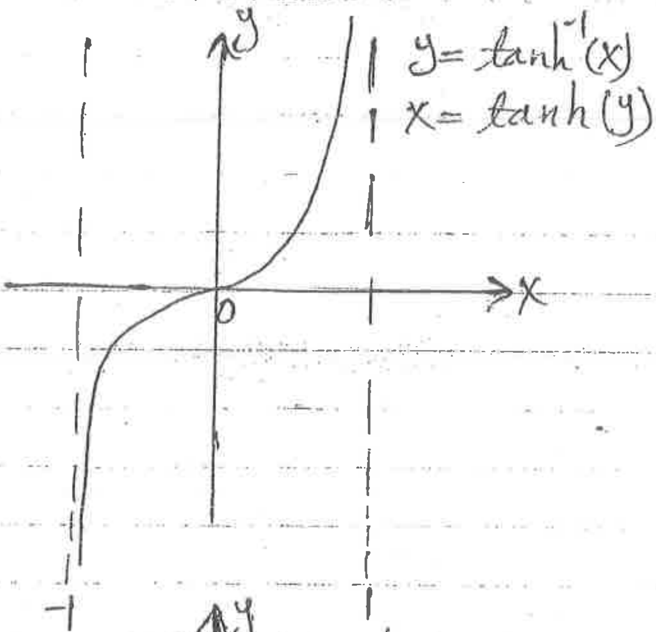




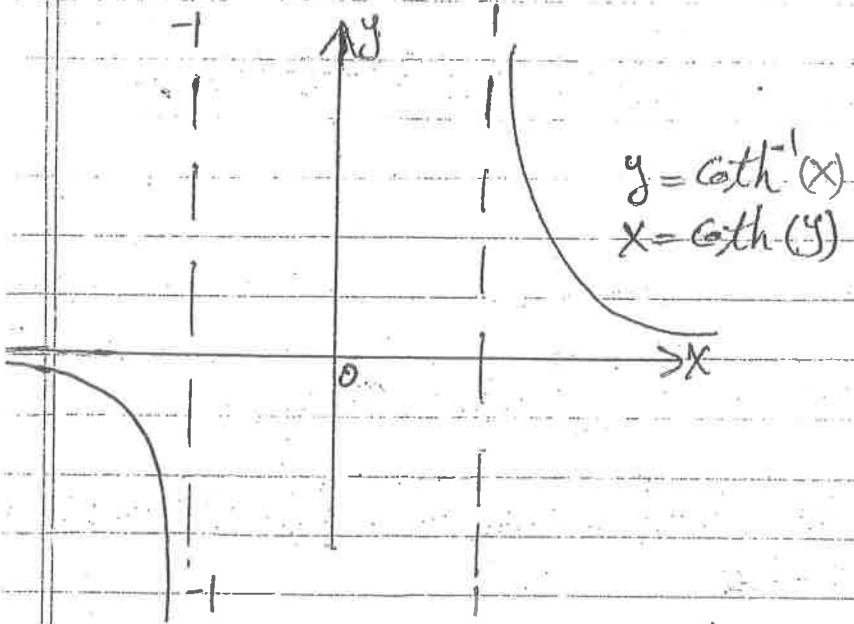
y
x
(1,0)
$y = \cosh^{-1}(x)$
$x = \cosh(y)$
$y = -\cosh^{-1}(x)$



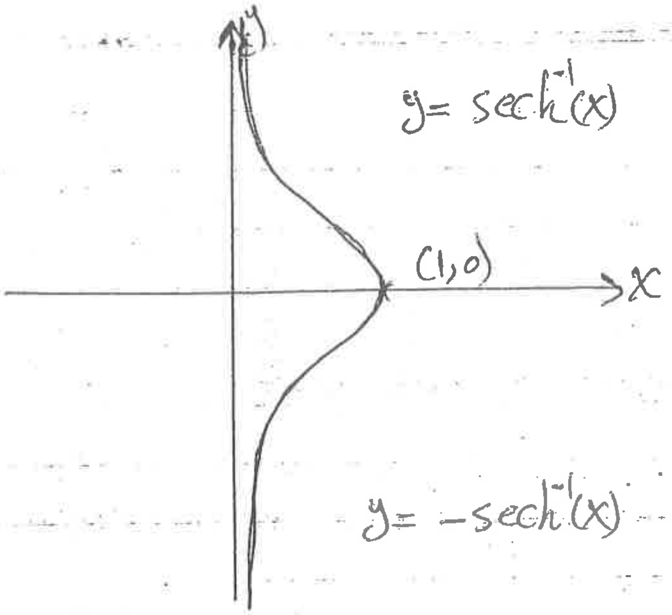
1-1 & onto
y
x
(0,1)
$x \geq 0$
$y \geq 1$

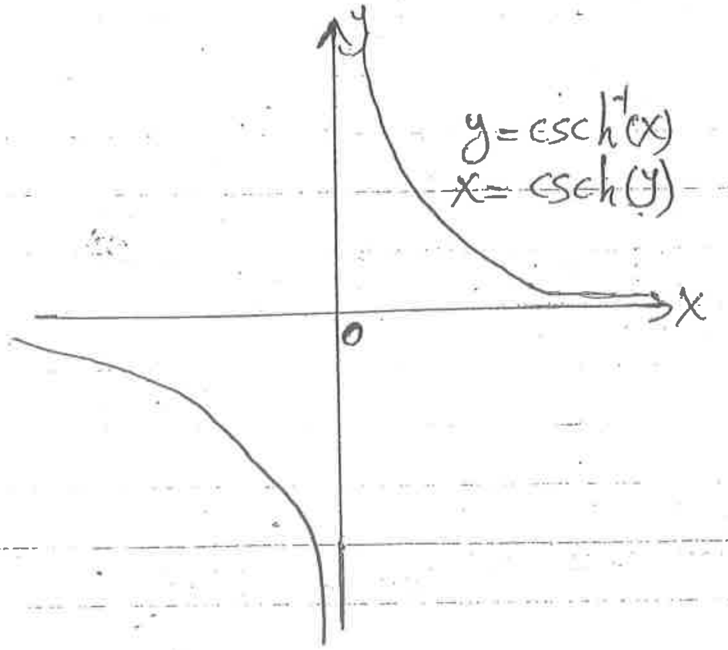


y
x
0
-1
$y = \tanh^{-1}(x)$
$x = \tanh(y)$



y
x
0
-1
$y = \coth^{-1}(x)$
$x = \coth(y)$


y
y= sech⁻¹(x)
(1,0)
x
y= -sech⁻¹(x)


y
y=csch⁻¹(x)
x= csch(y)
x
0

$$\sinh^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right), \quad -\infty < x < \infty \quad ; \ x \in \mathbb{R}$$ لكل

$$\cosh^{-1}(x) = \ln\left(x+\sqrt{x^2-1}\right), \quad x \geq 1$$ لكل

$$\tanh^{-1}(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1$$ لكل

$$\coth^{-1}(x) = \frac{1}{2}\ln\frac{x+1}{x-1} = \tanh^{-1}\left(\frac{1}{x}\right), \quad |x| > 1$$ لكل

$$\operatorname{sech}^{-1}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = \cosh^{-1}\left(\frac{1}{x}\right), \quad 0 < x \leq 1$$

$$\operatorname{csch}^{-1}(x) = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) = \sinh^{-1}\left(\frac{1}{x}\right) \quad ; \quad x \neq 0$$

$$\operatorname{csch}^{-1}(x) = \begin{cases} \ln\dfrac{1+\sqrt{1+x^2}}{x}, & x > 0 \\ -\ln\dfrac{1+\sqrt{1+x^2}}{-x}, & x < 0 \end{cases}$$

※ The derivatives of hyperbolic inverse functions.

1) $$\frac{d(\sinh^{-1}u)}{dx} = \frac{1}{\sqrt{1+u^2}} \cdot \frac{du}{dx}$$

2) $$\frac{d(\cosh^{-1}u)}{dx} = \frac{1}{\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

3) $$\frac{d(\tanh^{-1}u)}{dx} = \frac{1}{1-u^2} \cdot \frac{du}{dx}, \quad |u| < 1$$

4) $\dfrac{d(\coth^{-1}u)}{dx} = \dfrac{1}{1-u^2} \cdot \dfrac{du}{dx}$, $|u|>1$

5) $\dfrac{d(\operatorname{sech}^{-1}u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$

6) $\dfrac{d(\operatorname{csch}^{-1}u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}} \cdot \dfrac{du}{dx}$

* Examples: find $\acute{y}$ for the following ?

1- $y = \sinh^{-1}(2X)$

$$\acute{y} = \frac{1}{\sqrt{1+(2X)^2}} \cdot 2 = \frac{2}{\sqrt{1+4X^2}}$$

2- $y = \cosh^{-1}(\sec X)$

$1+\tan^2 x = \sec^2 x$
$\tan^2 x = \sec^2 x - 1$

$$\acute{y} = \frac{1}{\sqrt{\sec^2 x - 1}} \cdot \sec x \cdot \tan x \cdot 1 = \frac{\sec x \cdot \tan x}{\sqrt{\tan^2 x}} = \frac{\sec x \cdot \cancel{\tan x}}{\cancel{\tan x}} = \sec x$$

3- $y = \tanh^{-1}(\cos(2X))$

$$\acute{y} = \frac{1}{1-(\cos(2X))^2} \cdot -\sin(2X) \cdot 2 = \frac{-2\sin(2X)}{1-\cos^2(2X)} = \frac{-2\cancel{\sin(2X)}}{\sin^2(2X)}$$

$\sin^2 x + \cos^2 x = 1$
$\sin^2 x = 1 - \cos^2 x$

$$= \frac{-2}{\sin(2X)}$$

$$= -2\csc(2X)$$

$y = \coth^{-1}(\sec x)$

$y' = \frac{1}{1-\sec^2 x} \cdot \sec x \cdot \tan x \cdot 1$

$= \frac{\sec x \cdot \tan x}{-\tan^2 x} = \frac{\sec x}{-\tan x} = -\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = -\frac{1}{\sin x} = -\csc x$

$y = \operatorname{sech}^{-1}(\sin(2x))$

$y' = \frac{-1}{\sin(2x)\sqrt{1-\sin^2(2x)}} \cdot \cos(2x) \cdot 2$

$= \frac{-2\cos(2x)}{\sin(2x)\sqrt{\cos^2(2x)}} = \frac{-2\cancel{\cos(2x)}}{\sin(2x)\cdot\cancel{\cos(2x)}} = -2\csc(2x)$

$y = \operatorname{csch}^{-1}(5x^2)$

$\boxed{\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}} \frac{du}{dx}}$

$y' = \frac{-1}{|5x^2|\sqrt{1+25x^4}} \cdot 10x$

$= \frac{-\overset{2}{\cancel{10}}x}{\underset{1}{\cancel{5}}x^2\sqrt{1+25x^4}} = \frac{-2}{x\sqrt{1+25x^4}}$

$\tan^{-1} y = \tanh^{-1} x$

$\frac{1}{1+y^2} y' = \frac{1}{1-x^2} \cdot 1$

$(1-x^2)y' = 1+y^2 \Rightarrow y' = \frac{1+y^2}{1-x^2}$

(8) (41)

※ Exercises :- find $f'(x)$ or y' for the following

1) $f(x) = \sinh^{-1}(\tan x)$

2) $f(x) = \sinh^{-1}(\tan^{-1} e^{3x})$

3) $f(x) = x \cdot \sinh^{-1} x - \sqrt{1+x^2}$

4) $f(x) = x^2 \cdot \cosh^{-1} x^2$

5) $f(x) = x \cdot \cosh^{-1} x - \sqrt{x^2-1}$

6) $f(x) = \tanh^{-1}(\sin 3x)$

7) $f(x) = \ln\sqrt{x^2-1} - x \cdot \tanh^{-1} x$

8) $f(x) = \coth^{-1}(3x+1)$

9) $f(x) = \coth^{-1}(\cosh x)$

10) $f(x) = (\operatorname{sech}^{-1} x)^2$

11) $y = \operatorname{csch}^{-1}(\operatorname{sech}(\ln x))$

تمارين

أكتب كلاً مما يلي بدلالة اللوغاريتم الطبيعي :

1- $\tanh^{-1}\left(\frac{-4}{5}\right)$; 3- $\sinh^{-1}\frac{1}{4}$; 5- $\cosh^{-1} 3$

2- $\sinh^{-1} 2$; 4- $\tanh^{-1}\frac{1}{2}$; 6- $\coth^{-1}(-2)$