

References

المصادر

- Calculus and Analytic geometry by George B- Thomas 7th Edition 1988
- International Edition Thomas calculus part 1

Chapter one

The Real Numbers :- الاعداد الحقيقية

The sub sets of the real numbers are :-

1) Natural Numbers : الاعداد الطبيعية (N)

$$N = \{ 1, 2, 3, 4, \dots \}$$

2) Integer Numbers : الاعداد الصحيحة (I or Z)

$$I = Z = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$$

3) Rational Numbers : الاعداد النسبية (Q)

Is any number that can be written in the form $\frac{p}{q}$, where p and q are integer and $q \neq 0$

$$Q = \{ x \in R, x = \frac{p}{q}, p, q \in Z, q \neq 0 \}$$

Notice

That every integer n is also a rational number since we can write it as the quotient of the $\frac{n}{q}$

4) Irrational Numbers :- (Q) الاعداد الغير نسبية

Are all those real numbers that cannot be written in the form $\frac{p}{q}$, where p and q are Integers

For example :- $\pi, \sqrt{5}, \sqrt{3}, \sqrt{2}$

$$\sqrt{2} = 1.41421356, \dots$$

$$\pi = 3.141592, \dots$$

تعتبر اعداد دورية

Note that :-

مجموعة خالية لا يوجد فيها اي عدد

$\downarrow \{ \}$

$$\emptyset \subset N \subset I \subset Q \subset R$$
$$Q \cup Q^c = R$$

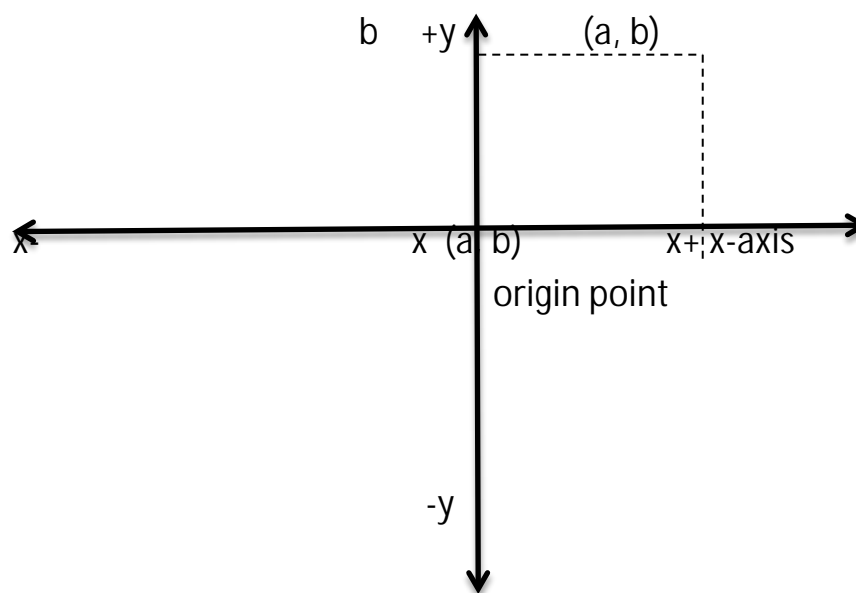
Chapter 1

The rate of change of a function

a) Coordinates for the plane :-

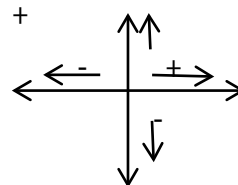
To assign coordinates to points in plane , we start with two number lines that cross at their zero points at right angles the horizontal line is called the x- axis p vertical line the y- axis , the point at which the lines cross is the origin .

- We assign a pair (a, b) of real numbers to each point p in the plane . the number a from the x- axis is the x- coordinate of p . The number b from the y- axis is the y- coordinate of p . the pair (a, b) is the ordered pair .



Note :-

- 1) Every point on the x- axis has y- axis zero& every point on the y- axis has x- axis zero



- 2) The origin point $(0,0)$

3) Directions along the axis :- the values of x & y increasing in the positive direction & decreasing in the negative

Ex) plot the points & in which quadrature the following

Points :- $(-3, 5)$, $(5, -4)$, $(4, -1)$

b) The slope of a straight line :-

Definition :- (The Increments التغيرات)

When a particle moves from $p_1(x_1, y_1)$ to $p_2(x_2, y_2)$

$\Delta x = [(x_2 \text{ of terminal point}) - (x_1 \text{ of initial point})]$

$\Delta y = [(y_2 \text{ of terminal point}) - (y_1 \text{ of initial point})]$

That is $\Delta x = x_2 - x_1$ & $\Delta y = y_2 - y_1$

Δx is called delta x & Δy delta y

Ex) If a particle moves from A(1,-2) to B (6,7) find Δx & Δy or (find the Increments of them)

Sol:- A (1,-2) Initial point & B (6,7) terminal point

$$\Delta x = 6 - 1 = 5$$

$$\Delta y = 7 - (-2) = 9$$

Ex) If particle start at A(-2,3) its coordinates receive increments $\Delta x=5$, $\Delta y= -6$ what will be its new position ?

Sol:- A (-2,3) & B (?,?) , $\Delta x=5$ & $\Delta y= -6$

$$\Delta x = x_2 - x_1 \Rightarrow 5 = x_2 - 2 \text{ \& \> } \Delta y = y_2 - y_1 \Rightarrow -6 = y_2 - 3 \Rightarrow x_2 = 5 - 2 \text{ \& \> } y_2 = -6 + 3 \Rightarrow x_2 = 3 \text{ \& \> } y_2 = -3$$

B(3,-3) terminal point

Ex) A particle moves along the parabola $y=x^2$ from the point A(1,1) to the point B(x,y) show that $\frac{\Delta y}{\Delta x} = x+1$ if $\Delta x \neq 0$?

Sol:- $\Delta y = y_2 - y_1$ & $\Delta x = x_2 - x_1$

$$\Delta y = y_2 - y_1$$

$$\Delta y = y_2 - y_1 \quad \& \quad \Delta x = x_2 - x_1 \Rightarrow \Delta x = x_2 - x_1$$

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x_2 - 1}{x_2 - x_1} = \frac{(x_2 - 1)(x_1 + 1)}{(x_2 - x_1)} = x_1 + 1$$

Note :- The increment Δx & Δy can be any real number , may be positive , negative or zero .

Definition :- (the distance (d) between two points)

$P_1 (x_1, y_1)$ & $P_2 (x_2, y_2)$ is defined by :-

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Ex) particle starts from $P_1 (-1, 2)$ and travels in a straight line to the point $P_2 (2, -2)$ find the distance between P_1 & P_2

Sol:-

$$\Delta x = x_2 - x_1 \Rightarrow \Delta x = 2 - (-1) \Rightarrow \Delta x = 3$$

$$\Delta y = y_2 - y_1 \Rightarrow \Delta y = -2 - 2 \Rightarrow \Delta y = -4$$

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25}$$

$$d = 5$$

the slope الميل

Let L be a straight line which is not parallel to the y-axis . Let $P_1 (x_1, y_1)$ & $P_2 (x_2, y_2)$ be any two distinct points on L then we call $\Delta y = y_2 - y_1$ the rise & $\Delta x = x_2 - x_1$ the run along L from P_1 to P_2 . we define the slope of L as the rate of rise per unit of run :-

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = m$$

$$M = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note :-

- 1) Angle of inclination are measured counter clockwise from the x-axis
- 2) The relationship between the slope m of a non-vertical line and the lines angle of inclination Θ is shown $m = \tan \Theta$
- 3) The slope of a non vertical line is tangent of its angle on inclination

$$M = \frac{\Delta y}{\Delta x} = \tan \Theta = \frac{\sin \Theta}{\cos \Theta}$$

- 4) Lines are perpendicular $\leftrightarrow m \perp = \frac{-1}{m}$ ميل العمود = مقلوب الميل
- 5) Two parallel lines having equal slopes that is $m_1 = \tan \Theta_1$, $m_2 = \tan \Theta_2 \Rightarrow m_1 = m_2$ المستقيمات المتوازية لها نفس الميل
- 6) The slope of $L_1 = m_1$ & $L_2 = m_2$ if $L_1 \perp L_2 \leftrightarrow m_1 m_2 = -1$ شرط تعامد المستقيمين
- 7) The rule of midpoint $p = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Examples :-

Ex1) plot the given points A (-2 , -1) , B (1 , -2) & find the slope of the straight line determined by them & find a perpendicular line?

Sol :- $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - (-1)}{1 - (-2)} = \frac{-1}{3}$

Per , $m \perp = \frac{-1}{m} = \frac{-1}{\frac{-1}{3}} = 3$

Ex2) Let $p_1(2, -1)$ & $p_2(-6, 3)$ find the m & $m \perp$ & find the midpoint of the segment $p_1 p_2$?

Ex3) Let $p_1(1, -1)$ & $p_2(2, 1)$ on the line straight L_1 , & $p_3(-2, 0)$ & $p_4(0, 4)$ on the line straight L_2 , determine the straight lines parallel or vertical ?

Sol :- $m_1 = \frac{1 - (-1)}{2 - 1} = 2$, $m_2 = \frac{4 - 0}{0 - (-2)} = 2 \Rightarrow m_1 = m_2$

$\therefore L_1 \parallel L_2$

C) Equations of a straight line :-

We can write an equation for a non-vertical straight line L if we know its slope m and the coordinates of one $p_1 (x_1, y_1)$ on it. If $p(x, y)$ is any other point on L, then we can use the two points p_1 & p to compute the slopes

$$M = \frac{y - y_1}{x - x_1} \Rightarrow y - y_1 = m (x - x_1)$$

$$\text{Or } y = y_1 + m (x - x_1)$$

The equation $y = y_1 + m (x - x_1) \dots (1)$ is the (point – slope equation) of the line that passes through the point $p_1 (x_1, y_1)$ and has slope.

معادلة مستقيم اذا علم ميله ونقطة من نقاطه

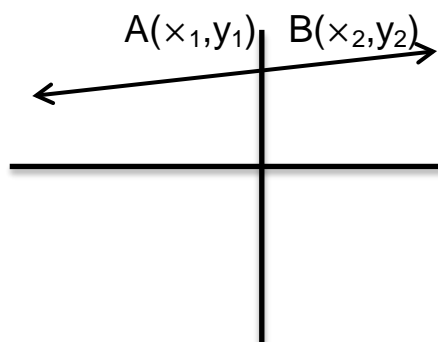
Ex) write an equation for the line through the point (2,3) with slope $\frac{-3}{2}$?

Sol :-

$$y = y_1 + m (x - x_1) \Rightarrow y = 3 - \frac{3}{2} (x - 2) \Rightarrow y = 3 - \frac{3}{2} x + 3 \Rightarrow y = 6 - \frac{3}{2} x .$$

A line through two points

$$M = \frac{y_2 - y_1}{x_2 - x_1} \dots (2)$$

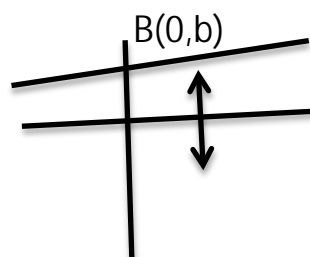


$$\text{From (1) \& (2) } \Rightarrow y = y_1 + \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

معادلة مستقيم الذي يمر من نقطتين

The equation $y = m x + b$ is called the slope – intercept equation of the line with slope m and y – intercept b

معادلة مستقيم بدلالة ميله والجزء الذي يقطعه من محور Y



the equation $Ax + By = c$ (A and B not both 0) is called the general equation in x and y because its graph always represents a line and every line has an equation in this form (including lines with undefined slope).

$$m = \frac{-A}{B}$$

Examples :-

Ex1) Finding the slope and y intercept of the line $8x + 5y = 20$

Sol : solve the equation for y to put it in slope – intercept form

$$Y = m \times + b$$

$$8x + 5y = 20 \Rightarrow 5y = -8x + 20 \Rightarrow y = \frac{-8}{5}x + 4$$

The slope is $m = \frac{-8}{5}$, the y – intercept is $b = 4$ & $A = 8, B = 5, c = 20$

Ex2) write an equation for the line that passes through the point $(2,3)$ with slope $\frac{-3}{2}$

$$\begin{aligned} \text{Sol : } y - y_1 &= m(x - x_1) \Rightarrow y - 3 = \frac{-3}{2}(x - 2) \Rightarrow y = \frac{-3}{2}x + 3 + 3 \\ \Rightarrow y &= \frac{-3}{2}x + 6 \Rightarrow 2y = -3x + 12 \Rightarrow 2y + 3x = 12 \end{aligned}$$

$$A = 3 \varphi B = 2 \varphi c = 12$$

Ex3) find the slope of the line $2x + 3y = 5$

$$\text{Sol : } m = \frac{-A}{B} = \frac{-2}{3}$$

المسافة بين نقطة ومستقيم

Note :- the distance d from the point $p_1 (x_1, y_1)$ to the

line $Ax + By + c = 0$ is $d = \frac{|Ax_1 + By_1 + c|}{\sqrt{A^2 + B^2}}$ المسافة بين نقطة ومستقيم

Ex) find the distance from the points D(-1,0), E(1,1) , and f(-3,5) to the line L : $2x + 3y - 5 = 0$

Sol :-

$$dD = \frac{|2(-1) + 3(0) - 5|}{\sqrt{2^2 + 3^2}} = \frac{7}{\sqrt{13}}$$

$$dE = \frac{|2(1) + 3(1) - 5|}{\sqrt{2^2 + 3^2}} = \frac{0}{\sqrt{13}} = 0$$

$$dF = \frac{|2(-3) + 3(5) - 5|}{\sqrt{2^2 + 3^2}} = \frac{4}{\sqrt{13}}$$

The functions

الدالة

The function Let A and B be two non – empty sets , the – relation that assigns to every element $x \in A$, with a unique value $y \in B$ is called a functions

$$F: A \rightarrow B, \forall x \in A, \exists ! y \in B ! \ni f(x) = y$$

Note :-

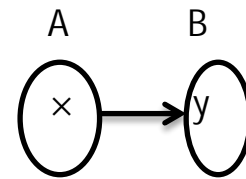
A= Domain = D_f , B = co- domain = $G_o - D_f$

*) the set of all images $f(x) = y, \forall x \in D_f$

Is called the Range of $f = R_f$

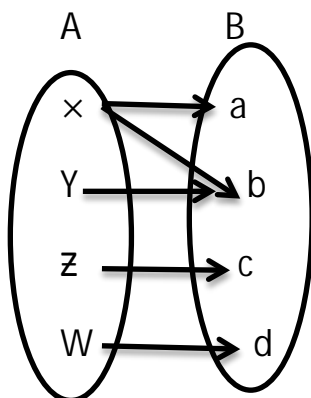
$$R_f = \{ f(x) = y, x \in D_f \}$$

$$Or = \{ y, y = f(x), \forall x \in D_f \}$$

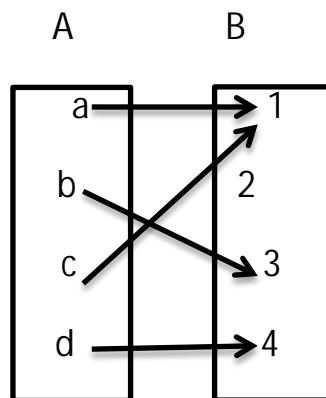


$x \rightarrow f \rightarrow f(y)$
input output

Ex(1) F is not function

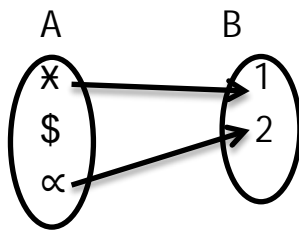


Ex(2) f is a function



Since $f(x) = a$ and $f(x) = b$ Range $f = \{1,3,4\}$

Ex(3)



F is not function

Since $\alpha \in A$ and α has not image

Ex (4)

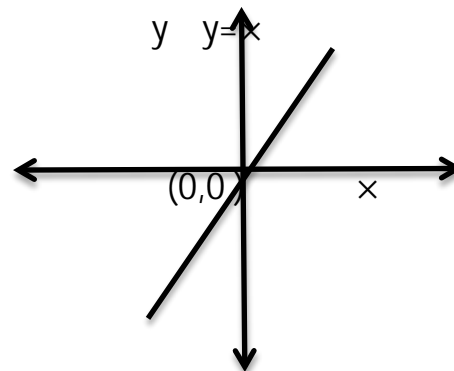
Linear function $y=x$

$$Y=f(x)=x, f: \mathbb{R} \rightarrow \mathbb{R}$$

$$D_f = \mathbb{R} = \{x \in \mathbb{R}, -\infty < x < \infty\}$$

$$R_f = \mathbb{R} = \{y \in \mathbb{R}, -\infty < y < \infty\}$$

Is function



Ex (5)

$$Y = f(x) = x^2$$

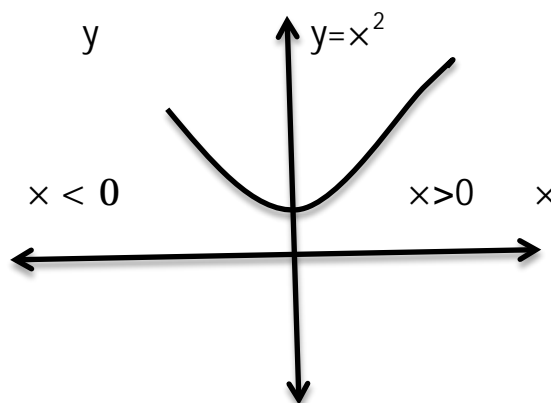
$$D_f = \mathbb{R}$$

$$R_f = \mathbb{R}^+ = \{y \in \mathbb{R} : y \geq 0\}$$

$$= \{y \in \mathbb{R}, 0 \leq \infty\}$$

$$= [0, \infty]$$

Is function

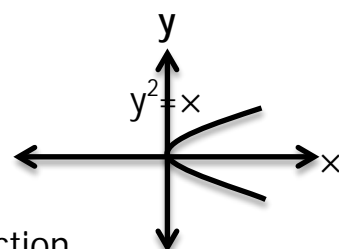


Ex (6)

$y^2=x$, not function

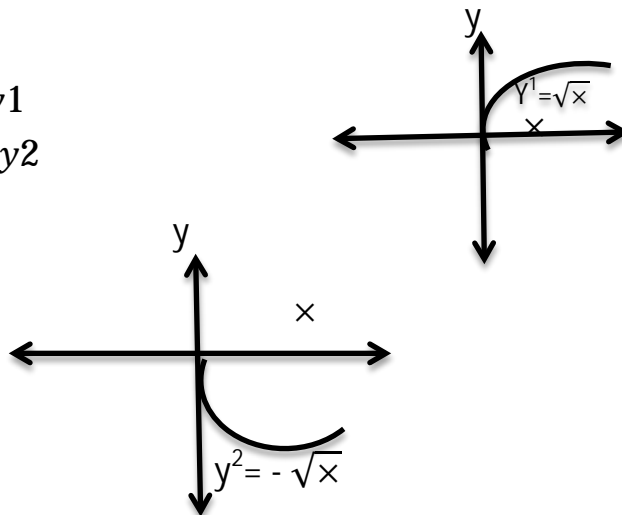
$$= \sqrt{y^2} = \sqrt{x} \Rightarrow |y| = \sqrt{x} \Rightarrow y = \pm \sqrt{x}$$

$\forall x \in D_f, \exists \pm \sqrt{x}$: two image, for every x not function.



But if

$$y = \begin{cases} \sqrt{x} & \text{function } y_1 \\ -\sqrt{x} & \text{function } y_2 \end{cases}$$



Ex(7)

$$2y + 3x = 5$$

$$2y = 5 - 3x \Rightarrow y = \frac{5-3x}{2} \quad \text{function}$$

Ex(8)

$$\frac{y}{x} = 2$$

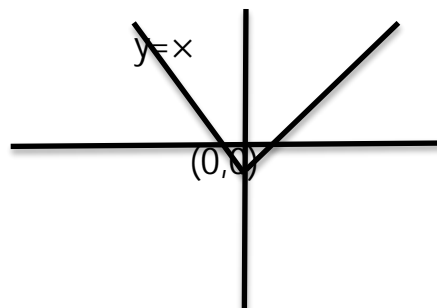
$$y = 2x \quad \text{function}$$

Absolute values :-

Definition :- The absolute value of a real number x is denoted by $|x|$ and is defined by the formula

$$|x| = \sqrt{x^2} \qquad y = -x$$

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$



For example :-

$$|-8| = 8 \quad , \quad |9| = 9 \quad , \quad |0| = 0$$

The absolute value of x is always either positive or zero , but never negative .

Properties of absolute value :-

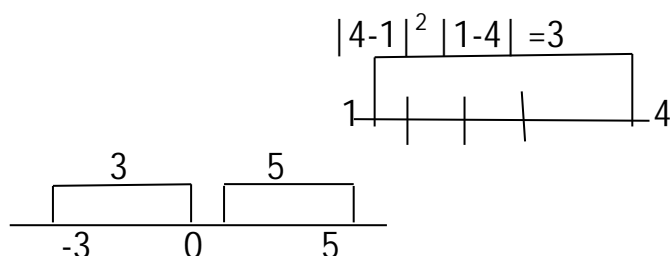
Let a, b be two real numbers then :-

- (1) $|a| = |a|$
- (2) $|a| = ||a||$
- (3) $|a.b| = |a|. |b|$
- (4) $|\frac{a}{b}| = \frac{|a|}{|b|}, b \neq 0$
- (5) $|a+b| \leq |a| + |b|$ (triangle inequality)
- (6) $|x| = a \leftrightarrow x = \pm a$
- (7) $|x| \leq a \leftrightarrow -a \leq x \leq a$ $\varphi |x| < a \leftrightarrow -a < x < a$
- (8) $|x| \geq a \leftrightarrow x \geq a \text{ or } x \leq -a$
- (9) $|x| > a \leftrightarrow x > a \text{ or } x < -a$

Geometrically :- من الناحية الهندسية

The absolute value of x is the distance from x to zero on the real number line since distances are always positive or zero , Also

$|x-y|$ = the distance between x and y on the real line



$$|-3| = 3 \quad |0| = 0 \quad |5| = 5$$

Ex) find the value of x that satisfy the inequality $|x| > 3$?

Sol :- $\{x \in \mathbb{R}, |x| > 3\} = \{x \in \mathbb{R}, x > 3 \text{ or } x < -3\}$

$$= (3, \infty) \cup (-\infty, -3) = \mathbb{R} \setminus [-3, 3]$$

Ex) find the solution set for the following inequality $|7-4x| \geq 1$

Sol :- $\{x \in \mathbb{R}, |7-4x| \geq 1\} = \{x \in \mathbb{R}, 7-4x \geq 1 \text{ or } 7-4x \leq -1\}$

$$= \{x \in \mathbb{R}, -4x \geq -6 \text{ or } -4x \leq -8\}$$

$$= \{ x \in \mathbb{R}, x \leq \frac{3}{2} \text{ or } x \geq 2 \}$$

$$= (-\infty, \frac{3}{2}) \cup (2, \infty) = \mathbb{R} \setminus (\frac{3}{2}, 2)$$

Ex) solving an Equation with absolute values $|2x-3|=7$

Sol:- by property $|x|=a \leftrightarrow x=\pm a$

$$2x-3=7 \quad \& \quad 2x-3=-7$$

$$2x=10 \quad \& \quad 2x=-4$$

$$x=5 \quad \& \quad x=-2$$

Ex) $|5-\frac{2}{x}| < 1$

Sol :-

$$-1 < 5 - \frac{2}{x} < 1 \quad \text{by (7)}$$

$$\leftrightarrow -6 < \frac{-2}{x} < -4 \quad (\text{subtract 5})$$

$$\leftrightarrow 3 > \frac{1}{x} > 2 \quad (\text{multiply by } \frac{1}{2})$$

$$\leftrightarrow \frac{1}{3} < x < \frac{1}{2} \quad (\text{take reciprocals})$$

Ex) $|2-2x| \leq 7$

Sol :-

$$-7 \leq 2-2x \leq 7$$

$$-9 \leq -2x \leq 5$$

$$\frac{9}{2} \geq x \geq \frac{-5}{2}$$

$$-2\frac{1}{2} = -\frac{5}{2} \leq x \leq \frac{9}{2} = 4\frac{1}{2}$$

$$S = [\frac{-5}{2}, \frac{9}{2}]$$

Ex) $|x-1| \geq 6$

Sol :-

$$x-1 \geq 6 \vee x-1 \leq -6$$

$$x \geq 7 \vee x \leq -5$$

$$s = [7, \infty) \vee (-\infty, -5]$$

H.w.

1) $|x| > 5$

2) $|x| + 1$

3) $|\frac{2x}{3}| \leq 1$

4) $|x-4| < 2$

5) $|2x+5| > 4$

6) $|2x-3| \geq 1$

How to find the Domain and the Range of function

- 1) The domain of all polynomial or odd root is all real numbers
(Real number , R)

إذا كانت الدالة على شكل متعددة حدود أو ذات جذر فردي فإن مجالها كل الأعداد الحقيقية

Ex(1) $f(x) = x^3 + 2x^2 + 3x - 5$

$$D_f = R, R_f = R$$

Ex (2) $f(x) = \sqrt[5]{x+1}$

$$D_f = R, R_f = R$$

Ex(3) $f(x) = \sqrt[3]{x^2 - 1}$

$$D_f = R, R_f = R$$

إذا كانت الدالة دالة جذرية والجذر زوجي فإن مجالها كل الأعداد الحقيقية التي تجعل المقدار

- 2) The domain of even root such as square roots is all real numbers that the expression under the radical to greater than or equal to zero

Ex(1) $f(x) = \sqrt{x^2 - 4}$

$$x^2 - 4 \geq 0 \Rightarrow (x-2)(x+2) \geq 0$$

$$(x-2) \geq 0 \wedge (-x+2) \geq 0 \Rightarrow x \geq 2 \wedge x \leq 2 \Rightarrow [2, 2]$$

$$(x-2) \leq 0 \wedge (x+2) \leq 0 \Rightarrow x \leq 2 \wedge x \leq -2 \Rightarrow (-\infty, -2]$$

$$D_f = (-\infty, -2] \cup [2, \infty) = \mathbb{R} \setminus (-2, 2)$$

Ex(2) $\sqrt{2x-1}$

$$2x-1 \geq 0 \Rightarrow 2x \geq 1 \Rightarrow x \geq \frac{1}{2}$$

$$D_f = \left[\frac{1}{2}, \infty\right)$$

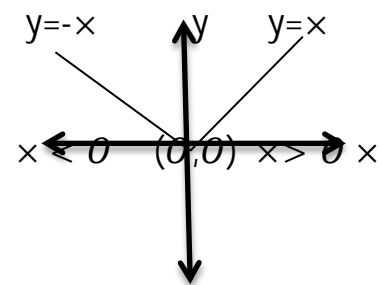
- 3) Piecewise functions :- A function that is defined by more than one formula is called a piecewise function such functions are written using the brace(s), such as signum function, absolute value function, ..., etc.

The domain of these function are the restrictions of the functions.

Ex(1) find the domain of $|x|$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$D_f = \mathbb{R}, R_f = [0, \infty)$$



Ex(2) $f(x) = y = |x+3|$ find D_f

$$|x+3| = \begin{cases} x+3 & \text{if } x+3 \geq 0 \Rightarrow x \geq -3 \\ 0 & \text{if } x+3 = 0 \Rightarrow x = -3 \\ -(x+3) & \text{if } x+3 < 0 \Rightarrow x < -3 \end{cases}$$

$$D_f = \mathbb{R}, R_f = [0, \infty)$$

$$\text{Ex(3)} \quad f(x) = \begin{cases} x & \text{if } x < -2 \\ x+1 & \text{if } -2 \leq x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}, \text{ find } Df$$

$$x < -2 \vee -2 \leq x \leq 1 \vee x > 1$$

$$(-\infty, -2) \cup [-2, 1] \cup (1, \infty) = \mathbb{R}$$

$$D_f = \mathbb{R}$$

Ex(4) find the domain of

$$y=h(x) = \begin{cases} -1 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$$

$$D_f = \mathbb{R}, \quad R_f = \{-1, 3\}$$

4) The domain of Rational function is all not number except the value of x which make the denominator zero

$$\text{Ex)} \quad f(x) = \frac{x}{x^2-1} \qquad x^2-1 \neq 0 \Rightarrow x^2 \neq 1$$

$$Df = \mathbb{R} \setminus \{-1, 1\} \qquad \sqrt{x^2} \neq 1 \Rightarrow |x| \neq 1$$

$$y = \frac{x}{x^2-1} \Rightarrow x = yx^2 - y \Rightarrow yx^2 - x - y = 0 \qquad x = \mp 1$$

$$x = \frac{1 \mp \sqrt{1+4y^2}}{2y}$$

$$2y \neq 0 \Rightarrow y \neq 0$$

$$1+4y^2 \geq 0 \Rightarrow y^2 \geq \frac{-1}{4} \Rightarrow y^2 \geq 0$$

$$y \in \mathbb{R}$$

$$R_f = \mathbb{R} \setminus \{0\}$$

$$\text{Ex)} \quad y = \frac{1}{x} \qquad x = \frac{1}{y}$$

$$D_f = \mathbb{R} \setminus \{0\}$$

$$R_f = \mathbb{R} \setminus \{0\}$$

$$\underline{\text{Ex)}} \quad y = \frac{1}{x-1} \quad y = \frac{1}{x-1}$$

$$D_f = \mathbb{R} \setminus \{1\} \quad yx - y = 1$$

$$R_f = \mathbb{R} \setminus \{0\} \quad yx = 1 + y$$

$$x = \frac{1+y}{y}$$

5) There are special cases of Rational function

a) If the rational function contain a square root in the denominator then domain is all real number the expression under the radical greater than zero

$$\underline{\text{Ex)}} \quad \frac{2-x}{\sqrt{1-x}}$$

$$1-x > 0 \Rightarrow 1 > x, D_f = (-\infty, 1)$$

$$y = \frac{2-x}{\sqrt{1-x}} \Rightarrow y\sqrt{1-x} = 2-x$$

$$y^2(1-x) = (2-x)^2$$

$$y^2 - y^2x = 4 - 4x + x^2$$

$$x^2 - 4x + 4 - y^2 + xy^2 = 0$$

$$x^2 + (-4 + y^2)x + (4 - y^2) = 0$$

$$a=1, b=y+y^2, c=4-y^2, x = -b \mp \frac{\sqrt{b^2+4ac}}{2a}$$

$$x = \frac{-(-4+y^2) \mp \sqrt{(-4+y^2)^2 - 4(1)(4-y^2)}}{2(1)}$$

$$= \frac{4-y^2 \mp \sqrt{16-8y^2+y^4-16+4y^2}}{2}$$

$$= \frac{4-y^2 \mp \sqrt{y^4-4y^2}}{2}$$

$$y^4 - 4y^2 \geq 0 \Rightarrow y^2(y^2 - 4) \geq 0$$

$$y^2 \geq 0 \wedge y^2 - 4 \geq 0 \Rightarrow y^2 \geq 0 \wedge y^2 \geq 4$$

$$\Rightarrow y \in \mathbb{R} \wedge |y| \geq 2$$

$$\therefore R_f = [2, \infty) \cup (-\infty, -2] \Rightarrow y \in \mathbb{R} \wedge y \geq 2 \vee y \leq -2 \text{ or } \mathbb{R} \setminus (-2, 2)$$

6) If the rational function contain odd root in the denominator then domain is all real number except the value of x which make the denominator zero .

$$\underline{\text{Ex}}) f(x) = \sqrt[3]{\frac{x+1}{x-2}}$$

$$y = \sqrt[3]{\frac{x+1}{x-2}} \quad , \quad D_f = \mathbb{R} \setminus \{2\}$$

$$y = \sqrt[3]{\frac{x+1}{x-2}} \Rightarrow y^3 = \frac{x+1}{x-2} \Rightarrow (x-2) y^3 = x+1$$

$$xy^3 - 2y^3 - x - 1 = 0$$

$$(y^3 - 1)x = 2y^3 + 1$$

$$\therefore x = \frac{2y^3 + 1}{y^3 - 1}$$

$$y^3 - 1 \neq 0 \Rightarrow y^3 \neq 1 \Rightarrow y \neq 1$$

$$R_f = \mathbb{R} \setminus \{1\}$$

Exercises

Find the Domains and Ranges for the following functions :-

$$(1) \quad y = f(x) = \frac{1}{x^2 + 1} + 3$$

$$x^2 + 1 \neq 0 \Rightarrow x^2 \neq -1 \quad , \quad D_f = \mathbb{R}$$

$$y = \frac{1}{x^2 + 1} + 3 \Rightarrow y - 3 = \frac{1}{x^2 + 1}$$

$$(y - 3)(x^2 + 1) = 1$$

$$yx^2 + y - 3x^2 - 3 = 1$$

$$(y - 3)x^2 + y - 3 = 1$$

$$(y - 3)x^2 = 4 - y$$

$$x^2 = \frac{4 - y}{y - 3} \Rightarrow |x| = \sqrt{\frac{4 - y}{y - 3}}$$

$$x = \pm \sqrt{\frac{4-y}{y-3}}$$

$$y-4 \geq 0 \wedge y-3 > 0 \quad \vee \quad 4-y \leq 0 \wedge y-3 < 0$$

$$y \leq 4 \wedge y > 3 \quad \vee \quad y \geq 4 \wedge y < 3$$

$$3 < y \leq 4$$

$$\left(\begin{array}{|c|} \hline 3 \\ \hline \end{array} \right] \quad \quad \quad) \quad \begin{array}{|c|} \hline 4 \\ \hline \end{array}$$

$$3 \quad \quad 4 \quad \quad \quad 3 \quad \quad 4$$

$$(3,4] \quad \vee \quad \emptyset = (3,4]$$

$$\therefore R_f = (3,4]$$

Homework

Find the Domains and Ranges for the following functions

$$(1) f(x) = \sqrt{\frac{1}{x} - 2}$$

$$(2) y = f(x) = \frac{\sqrt{x+1}}{x-1}$$

$$(3) f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$$

$$(4) y = f(x) = \frac{x+1}{|x-5|}$$

Graphs of functions :- رسم الدالة

Another way to visualize a function is its graph. If f is a function with domain D , its graph consists of the points in the Cartesian plane whose coordinates are the input – output pair for f . In set notation, the graph is $\{(x, f(x)) \mid x \in D\}$.

Quick Graphing :- الرسم السريع

A quick way to graph an equation by three steps :-

- (1) find the x - intercept by setting $y=0$
- (2) find the y - intercept by setting $x=0$
- (3) plot the intercepts and draw the line

Ex) sketch the function by using the table ?

$$f(x)=x+2$$

Sol:-

(1) Let $y=0$

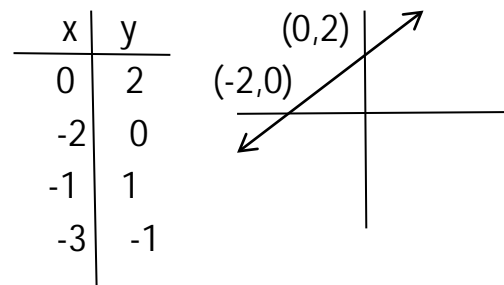
$$y-f(x)=x+2 \Rightarrow x+2=0 \Rightarrow x=-2 \Rightarrow < -2$$

(2) Let $x=0$

$$y=0+2 \Rightarrow y=2 \Rightarrow (0,2)$$

(3) $x=-1 \Rightarrow y=1 \Rightarrow (-1,1)$

$$y=-1 \Rightarrow x=-3 \Rightarrow (-3,-1)$$



Ex) $y=f(x) = x^2$

Sol :- $y=x^2 \Rightarrow$ if $y=0 \Rightarrow x=0$ (0,0) origin point

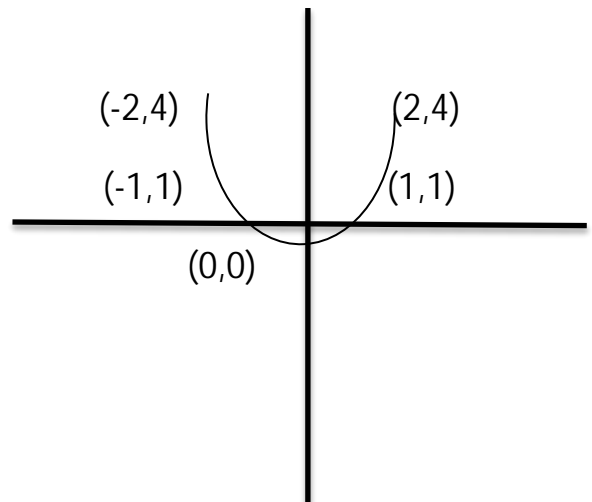
$$y=1 \Rightarrow x^2=1 \Rightarrow x=\pm 1 \Rightarrow (1,1), (-1,1)$$

$$x=1 \Rightarrow y=1 \Rightarrow (1,1)$$

$$x=2 \Rightarrow y=4 \Rightarrow (2,4)$$

$$x=-1 \Rightarrow y=1 \Rightarrow (-1,1)$$

x	y=x ²
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



Ex) $y=f(x)=x$

Sol :- if $x=0 \Rightarrow y=0$ (0,0)

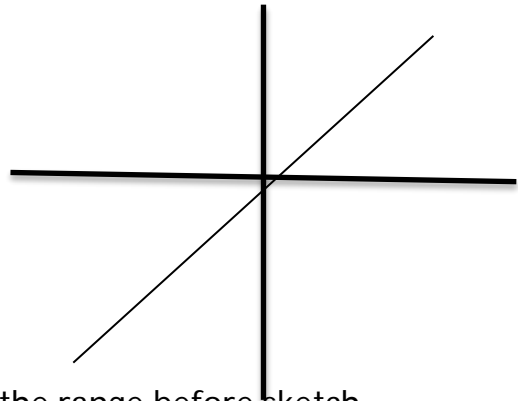
If $y=0 \Rightarrow x=0$ (0,0)

$x=1 \Rightarrow y=1$ (1,1)

$x=2 \Rightarrow y=2$ (2,2)

$x=-1 \Rightarrow y=-1$ (-1,-1)

x	y
-1	1
0	0
1	1
2	2



Note :- we must calculate the domain and the range before sketch .

If the function a straight line we must determine the intercept point and joint it if not we take other point but the point must be in the domain of the function .

There are a number of important types of functions , we identify and briefly summarize them here .

Linear functions :-

A function of the form $f(x) = mx+b$, for constants m and b , is called a-
Linear functions .

Figure 1 shows an array of lines $f(x) = mx$ where $b=0$, so these lines pass through the origin constant functions result when the slope $m=0$

Figure 2

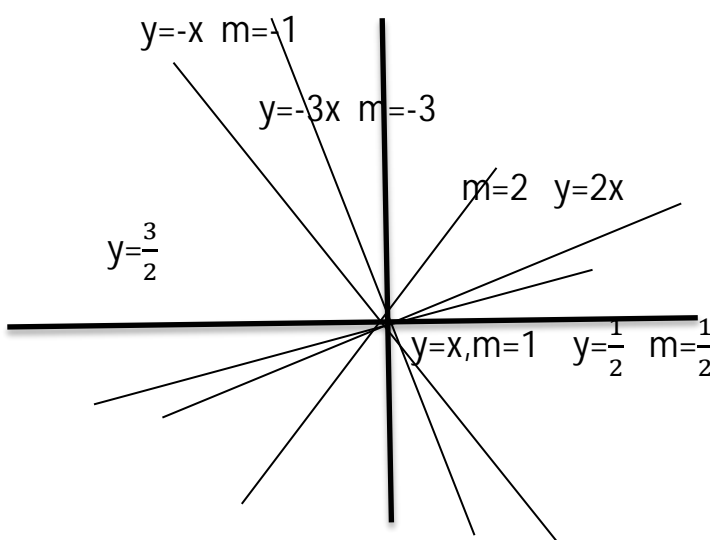


Fig (1)

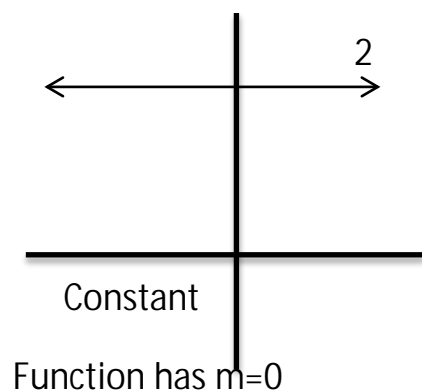
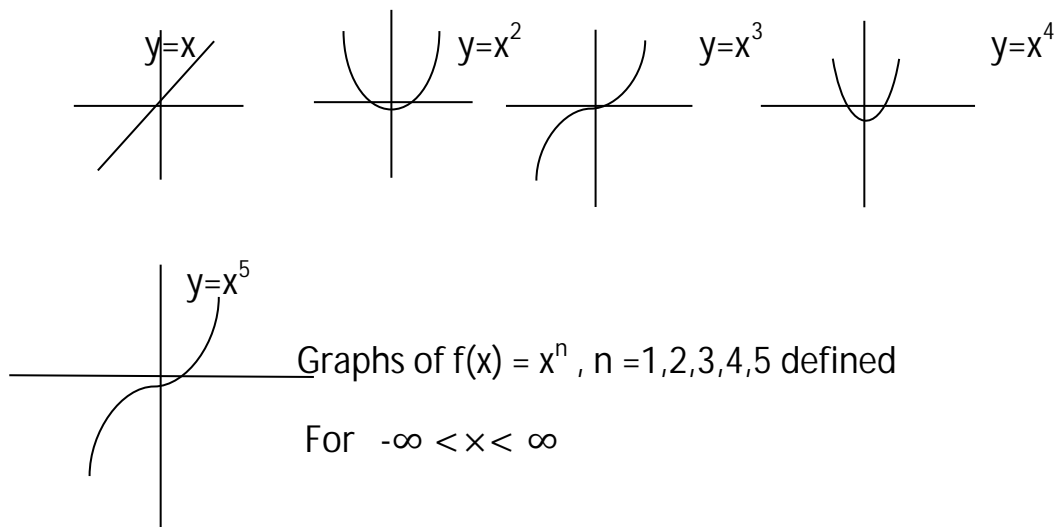


Fig (2)

Power functions:-

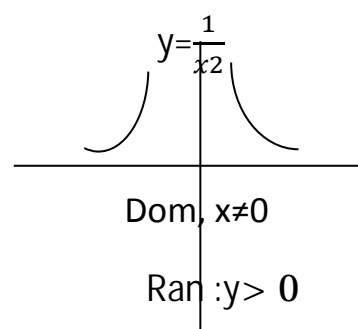
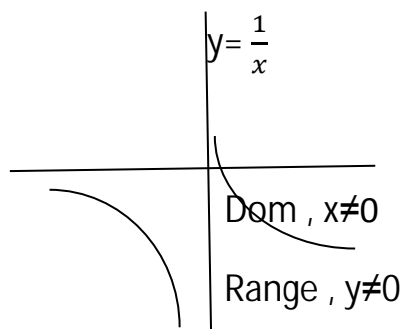
A function $f(x) = x^a$, where a is a constant, is called a power function – there are several important cases to consider

a) $a=n$, n a positive integer



b) $a=-1$ or $a=-2$

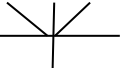
the graphs of the functions $f(x) = x^{-1} = \frac{1}{x}$ and $g(x) = x^{-2} = \frac{1}{x^2}$ are defined for all $x \neq 0$

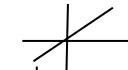



H.w


Sketch the following

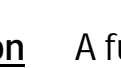
1) $y = \sqrt{x}$, $y = x^{\frac{3}{2}}$, $y = \sqrt[3]{x}$, $y = \frac{1+x}{1-x}$ $D_f = \mathbb{R}$, $R_f = \mathbb{R}^+$

Ex) $y=1x1$ 

2) $y=x+1$ 

3) $y=\frac{1}{x}$ 

4) $y=-x$ 

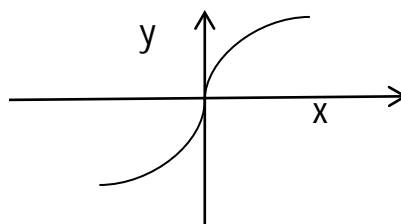
5) $y=\sqrt{x}$ 

Odd function A function $f(x)$ is called odd function if $f(-x)=-f(x)$, for every x in the functions domain

Ex) Let $f(x)=x^3$

$f(-x)=(-x)^3=-x^3=-f(x)$

$\therefore f$ is odd function

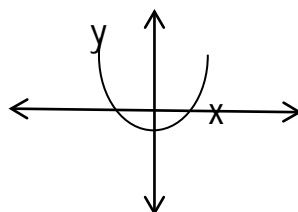


Even function A function $f(x)$ is called even function if $f(-x)=f(x)$ for every x in the functions domain

Ex) Let $f(x)=x^2$

$f(-x)=(-x)^2=x^2=f(x)$

f is even function



Ex) Recognizing Even and functions

(1) $F(x)=(x-1)^2$

(2) $F(x)=x^2+1$

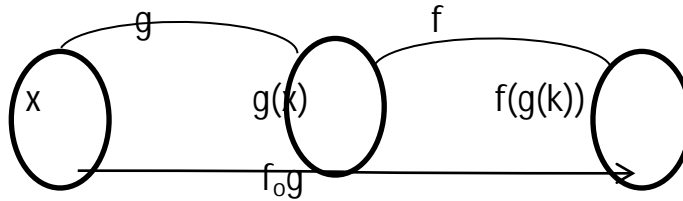
Rcmartc :-

- (1) The graph of an even function is asymmetric about the y -axis ,
appoint (x,y) lies on the graph if and only if the point $(-x,y)$ lies on the graph .
- (2) The graph of an odd function is symmetric about the origin
appoint (x,y) lies on the graph if and only if the point $(-x,-y)$ lies on the graph .

الدوال المركبة The composition of functions

Let f and g be two functions such that $g(x) \in D_f$ and $x \in D_g$, then there exist a function $f \circ g$ which is defined as the following :-

$$(f \circ g)(x) = f(g(x)) \text{ and } \text{Dom}_{(f \circ g)} = \{x : x \in \text{Dom}_g \wedge g(x) \in \text{Dom}_f\}$$



$$(1) R_g \subseteq D_f$$

$$(2) (f \circ g)(x) = f(g(x))$$

$$(3) D_{f \circ g} = \{x, x \in D_g \wedge g(x) \in D_f\}$$

Also, we define $g \circ f$ as :-

$$g \circ f(x) = g(f(x))$$

$$\text{Dom}(g \circ f) = \{x = f(x) \in \text{Dom}_g \wedge x \in \text{Dom}_f\}$$

In general the composition is not commutative $f \circ g \neq g \circ f$

Ex(1) Let $f(x) = \sqrt{x}$, $g(x) = x^2$, then find $f \circ g$, $g \circ f$?

Sol :- $f(x) = \sqrt{x}$, $D_f = \mathbb{R}^+ [0, \infty)$

$$y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow R_f = \mathbb{R}^+ = [0, \infty)$$

$$y(x) = x^2 + 1, D_y = \mathbb{R}$$

$$y = y(x) = x^2 + 1 \Rightarrow y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \pm \sqrt{y - 1}$$

$$y - 1 \geq 0 \Rightarrow y \geq 1, R_g = [1, \infty)$$

Now to find $f \circ g$ is $R_g \subseteq D_f$?

$$[1, \infty) \subseteq [0, \infty)$$

$\therefore f \circ g$ is exist

$$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^2 + 1}$$

$$(f \circ g)(x) = \sqrt{x^2 + 1}$$

$$D_{f \circ g} = \{x: x \in D_g \wedge g(x) \in D_f\}$$

$$= \{x: x \in \mathbb{R} \wedge x^2 + 1 \in \mathbb{R}^+\}$$

$$= \{x: x \in \mathbb{R} \wedge x \in \mathbb{R}\} = \mathbb{R}$$

$$\text{But } x^2 + 1 \geq 0 \Rightarrow x^2 \geq -1 \Rightarrow x^2 \geq 0 \Rightarrow x \in \mathbb{R}$$

To find $g \circ f$ is $R_f \subseteq D_g$?

$$\mathbb{R}^+ \subseteq \mathbb{R} \Rightarrow \therefore g \circ f \text{ exist}$$

$$(g \circ f)(x) = g(f(x)) = (\sqrt{x})^2 + 1 = x + 1$$

$$(g \circ f)(x) = x + 1$$

$$D_{g \circ f} = \{x: x \in D_f \wedge f(x) \in D_g\}$$

$$= \{x: x \in \mathbb{R}^+ \wedge \sqrt{x} \in \mathbb{R}\}$$

$$= \{x: x \in \mathbb{R}^+ \wedge x \in \mathbb{R}^+\} = \mathbb{R}^+$$

$$x \geq 0 \Rightarrow x \in \mathbb{R}^+$$

Ex(2) let $f(x) = \sqrt{x - 4}$, $g(x) = \frac{x+1}{3-x}$ then find $f \circ g$, $g \circ f$

Sol :-

$$f(x) = \sqrt{x - 4}$$

$$x - 4 \geq 0 \Rightarrow x \geq 4, \quad D_f = [4, \infty)$$

$$y = \sqrt{x - 4}, \quad y^2 = x - 4 \Rightarrow x = y^2 + 4 = R_f = \mathbb{R}^+$$

$$g(x) = \frac{x+1}{3-x}$$

$$3 - x \neq 0 \Rightarrow x \neq 3: D_g = \mathbb{R} \setminus \{3\}$$

$$y = \frac{x+1}{3-x} \Rightarrow x+1 = y(3-x) \Rightarrow x+1 = 3y - xy$$

$$x + xy = 3y - 1 \Rightarrow x = \frac{3y-1}{1+y}$$

$$1+y \neq 0 \Rightarrow y \neq -1, R_g = \mathbb{R} \setminus \{-1\}$$

Now, to find $f \circ g$, is $R_g \subseteq D_f$?

$$\mathbb{R} \setminus \{-1\} \not\subseteq [4, \infty)$$

$\therefore f \circ g$ is not exist

Now, to find $g \circ f$ is $R_f \subseteq D_g$

$$\mathbb{R}^+ \not\subseteq \mathbb{R} \setminus \{3\}$$

$\therefore g \circ f$ is not exist

Homework:-

Find $f \circ g$, $g \circ f$ to each of the following functions

$$(1) f(x) = |x|, g(x) = -x$$

$$(2) \text{ If } f(t) = t^2 \text{ find } g, h \text{ such that } (f \circ g)(x) = (f \circ h)(x) = x^2 - 10x + 25$$

Limits :

Definition : Let $f(x)$ be defined on an open interval about x_0 , except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0 , we say that f approaches the Limit L as x approaches x_0 , and we write .

$$\lim_{x \rightarrow x_0} f(x) = L$$

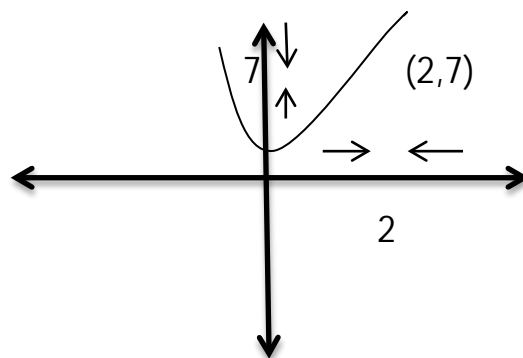
$$x \rightarrow x_0$$

which is read "the Limit of $f(x)$ as x approaches x_0 is L ", let $f(x)$ be a function, we said that the limit of $f(x)$ is L when x tend to x_0 , OR $f(x) \rightarrow L$

$$x \rightarrow x_0$$

Ex: Let $y = f(x) = x^2 + 3$, what happen when $x \rightarrow 2$

X	3	2.5	2.3	2.1	2.01	2.001	2.0001	At the right
F(x)	12	9.25	8.24	7.41	7.040	7.004	7.0004	
X	1	1.2	1.4	1.5	1.9	1.99	1.999	At the left
F(x)	4	4.44	4.96	5.25	5.98	6.96	6.999	



$$L^+ = \lim_{x \rightarrow 2^+} (x^2 + 3) = 7$$

$$D_f = \mathbb{R}$$

$$x \rightarrow 2^+$$

$$L^- = \lim_{x \rightarrow 2^-} (x^2 + 3) = 7$$

$$x \rightarrow 2^-$$

$$L^+ = L^-$$

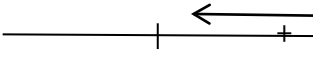
$$L = \lim_{x \rightarrow 2} (x^2 + 3) = 7 \quad \text{limit exist}$$

$$x \rightarrow 2$$

$$F(2) = (2)^2 + 3 = 7 \quad \text{function exist}$$


Note :-

The limit of $f(x)$ as x approaches x_0 from the right is :

$$L^+ = \lim_{x \rightarrow x_0^+} f(x)$$


say that $f(x)$ has right – hand limit L^+

The limit of $f(x)$ as x approaches x_0 from the left is :

$$L^- = \lim_{x \rightarrow x_0^-} f(x)$$


say that $f(x)$ has left – hand limit L^-

function $f(x)$ has a limit as x approaches x_0 if and only if it has left – hand and right hand limits there and these one – sided limit are equal .

$$\lim_{x \rightarrow x_0} f(x) = L \iff \lim_{x \rightarrow x_0^+} f(x) = L \text{ and } \lim_{x \rightarrow x_0^-} f(x) = L$$

The approaches of finding limit :- حالات ايجاد الغاية

To find limit we use some laws of limit

1) For any polynomial $p(x)$ and any real number a , then

$$\lim_{x \rightarrow a} p(x) = p(a) , \text{ where polynomial is}$$

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$$

Ex: finding limits of $f(x) = x^2 + x + 5$ when x is approach 1

$$L = \lim_{x \rightarrow 1} (x^2 + x + 5) = 1 + 1 + 5 = 7$$

$$F(x) \rightarrow 7 \text{ when } x \rightarrow 1$$

2) Quotient

- (a) The limit of a quotient of two functions is the quotient of the limits only when both limits exist and the limit in the denominator is not zero :

Ex: finding the limit of the function $\frac{x^3-5x+4}{x^2-2}$ when x is approach 3?

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^3-5x+4}{x^2-2} &= \frac{\lim(x^3-5x+4)}{\lim x^2-2} \\ &= \frac{\lim x^3 - 5 \lim x + \lim 4}{\lim x^2 - \lim 2} = \frac{3^3 - 5 \cdot 3 + 4}{3^2 - 2} = \frac{16}{7}\end{aligned}$$

In this example we substitute in the limiting value of x

- (b) Now in any case where the limits of both the numerator and denominator are 0 , we should try to algebraically simplify the expression to get cancellation .

Ex: Finding a limit $\lim_{x \rightarrow 1} \frac{x^2-1}{1-x}$

Sol: $\lim_{x \rightarrow 1} \frac{x^2-1}{1-x} \neq \frac{\lim x^2-1}{\lim 1-x}$

the limit in the denominator is zero can resolve this problem by factoring

$$\lim_{x \rightarrow 1} \frac{x^2-1}{1-x} = \lim_{x \rightarrow 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{-(x-1)}} = \lim_{x \rightarrow 1} \frac{x+1}{-1} = -2$$

Ex: finding a limit $\lim_{x \rightarrow 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$

Sol: notice that both the numerator ($\sqrt{x+2} - \sqrt{2}$) and the denominator (x) approach 0 as x approaches 0 , we solve by rationalize

$$\begin{aligned}\text{the numerator } \frac{\sqrt{x+2}-\sqrt{2}}{x} &= \frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})} = \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})} \\ &= \frac{x}{x(\sqrt{x+2}+\sqrt{2})} = \frac{1}{(\sqrt{x+2}+\sqrt{2})}\end{aligned}$$

Now

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \rightarrow 0} \frac{1}{(\sqrt{x+2} + \sqrt{2})} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(c) The limit of an nth root is the nth root of the limit

$$\lim_{x \rightarrow 0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow 0} f(x)} = \sqrt[n]{\ell}$$

if n is an odd positive integer
or
if n is an even positive integer and $\ell > 0$

Ex: Evaluating the limit $\lim_{x \rightarrow 2} \sqrt[5]{3x^2 - 2x}$

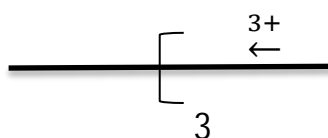
Sol: $D_f = \mathbb{R}$

$$\lim_{x \rightarrow 2} \sqrt[5]{3x^2 - 2x} = \sqrt[5]{\lim_{x \rightarrow 2} (3x^2 - 2x)} = \sqrt[5]{8}$$

3) Find the limit of a function $f(x) = \sqrt{2x - 6}$ when x if is the function contain (a) a square root :- in this case we must find L^+ & L^- and one of them not exist so the final limit not exist approach 3

Sol: $2x - 6 \geq 0 \Rightarrow 2x \geq 6 \Rightarrow x \geq 3$

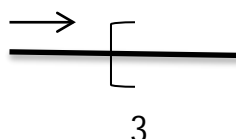
$\therefore D_f = [3, \infty)$



Now, we find L^+, L^-

$$L^+ = \lim_{x \rightarrow 3^+} \sqrt{2x - 6} = \sqrt{6 - 6} = 0$$

$$L^- = \lim_{x \rightarrow 3^-} \sqrt{2x - 6} \text{ not exist}$$



$$\therefore L = \lim_{x \rightarrow 3} \sqrt{2x - 6} \text{ not exist}$$

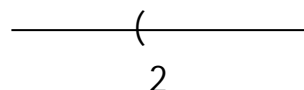
(b) odd root :- in this case the limit is exist since the domain is all real number

Ex:- Find the limit of function $f(x) = \frac{\sqrt{x-2}}{x-2}$ when x is approach 2

$$\therefore x - 2 \geq 0 \Rightarrow x \geq 2$$

$$x - 2 \neq 0 \Rightarrow x \neq 2$$

$$\therefore D_f = (2, \infty)$$



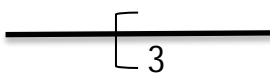
$$L = \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{x-2} = \lim_{x \rightarrow 2} \frac{1}{\sqrt{x-2}}$$

$$L^+ = \lim_{x \rightarrow 2^+} \frac{1}{\sqrt{x-2}} = +\infty$$

$$L^- = \lim_{x \rightarrow 2^-} \frac{1}{\sqrt{x-2}} \text{ not exist}$$

$$\therefore L = \lim_{x \rightarrow 2} \frac{1}{\sqrt{x-2}} \text{ not exist}$$

Ex : find the limit of function $f(x) = \sqrt{2x-6}$ when x is approach 5

Sol : $D_f = [3, \infty)$ 

$$5 \in D_f = [3, \infty)$$

$$L = \lim_{x \rightarrow 5} \sqrt{2x-6} = \sqrt{10-6} = \sqrt{4} = 2$$

Now , sometimes a function is described by using different for mulas on different parts of its domain and we illustrate such a function in the next example .

Ex : Evaluate $\lim_{x \rightarrow 0} f(x)$, where f is defined by

$$F(x) = \begin{cases} x^2 + 2 \cos x + 1, & \text{for } x < 0 \\ e^x - 4, & \text{for } x \geq 0 \end{cases}$$

Sol : since f is defined by different expressions for $x < 0$ and for $x \geq 0$, we must consider one – sided limits we have

$$= \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x^2 + 2 \cos x + 1) = 2 \cos 0 + 1 = 3$$

And

$$L^+ = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (e^x - 4) = e^0 - 4 = 1 - 4 = -3$$

Since the one – sided limits are different , we have that

$$\lim_{x \rightarrow 0} f(x) \text{ does not exist .}$$

Ex: Evaluate $\lim_{y \rightarrow 0} f(y)$, where f is defined by :

$$F(y) = \begin{cases} y + 2 & y \geq 0 \\ 2 & y < 0 \end{cases}$$

Sol : since f is defined by different expressions for $y < 0$ and $y \geq 0$, we must consider one – sided limits, we have

$$L_{+} = \lim_{y \rightarrow 0^{+}} f(y) = \lim_{y \rightarrow 0^{+}} (y+2) = 0+2=2$$

$$L_{-} = \lim_{y \rightarrow 0^{-}} f(y) = \lim_{y \rightarrow 0^{-}} 2=2$$

Since the one – sided limits are equal, we have that $\lim_{y \rightarrow 0} f(y)$ exist .

Ex: Evaluate $\lim_{y \rightarrow 1} f(y)$, where f is defined in the above Example

$$L = \lim_{y \rightarrow 1} f(y) = \lim_{y \rightarrow 1} y+2 = 1+2=3$$

Note : The limit value does not depend on how the function is defined at x_0

The limit Laws :

The next theorem tells how to calculate limits of functions

Theorem (1) limit laws

If L , M , and k are real numbers and $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} g(x) = M$, then

(1) Sum Rule : $\lim_{x \rightarrow x_0} (f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits

(2) Difference Rule : $\lim_{x \rightarrow x_0} (f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits

(3) Product Rule : $\lim_{x \rightarrow x_0} (f(x)g(x)) = L.M$

The limit of a product of two functions is the product of their limits

(4) Constant multiple Rule : $\lim_{x \rightarrow x_0} (k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function

(5) Quotient Rule : $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero

(6) Power Rule : If r and s are integers with no common factor and $s \neq 0$, then $\lim_{x \rightarrow x_0} (f(x))^{r/s} = L^{r/s}$ provided that $L^{r/s}$ is a real number (If s is even, we assume that $L > 0$)

Theorem (2) let $p(x)$ is a polynomial function then $\lim_{x \rightarrow x_0} p(x) = p(x_0)$

$$\lim_{x \rightarrow x_0} (c_0 + c_1x + c_2x^2 + \dots + c_nx^n) = c_0 + c_1x_0 + c_2x_0^2 + \dots + c_nx_0^n = p(x_0)$$

Infinity limits :

$$\lim_{x \rightarrow x_0} f(x) = +\infty$$

$f(x)$ tends to $+\infty$

Or

x approach to 0

$$\lim_{x \rightarrow x_0} f(x) = -\infty$$

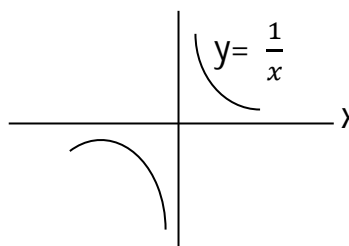
Ex(1) : find

$$\lim_{x \rightarrow 0} \frac{1}{x}, \quad Df = \mathbb{R} \setminus \{0\}$$

$$L^+ = \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty, \quad L^- = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$\therefore L^+ \neq L^-$$

$$L = \lim_{x \rightarrow 0} \frac{1}{x} \text{ does not exist}$$



Ex(2) : find

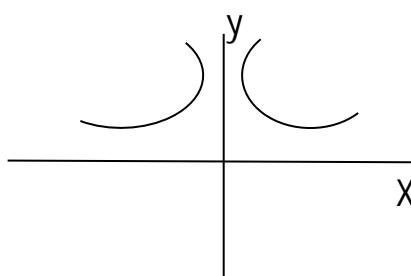
$$\lim_{x \rightarrow 0} \frac{1}{x^2}, \quad Df = \mathbb{R} \setminus \{0\}$$

$$L^+ = \lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty$$

$$L^- = \lim_{x \rightarrow 0^-} \frac{1}{x^2} = \infty$$

$$\therefore L = \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

$$\therefore L^+ = L^-$$



Ex: find $\lim_{x \rightarrow 1} \frac{-1}{(x-1)^3}$

$$L^+ = \lim_{x \rightarrow 1^+} \frac{-1}{(x-1)^3} = \frac{-1}{0^+} = -\infty$$

$$L^- = \lim_{x \rightarrow 1^-} \frac{-1}{(x-1)^3} = \frac{-1}{0^-} = -\infty$$

$$\therefore \lim_{x \rightarrow 1} \frac{-1}{(x-1)^3} \text{ not exist.}$$

Finite limits as $x \rightarrow \pm \infty$:-

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0, \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

We say that $f(x)$ has the limit L as x approaches infinity and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

We say that $f(x)$ has the limit L as x approaches minus infinity and write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

Ex1) $\lim_{x \rightarrow \infty} \frac{2x^2+1}{x+1}$

Sol : $\lim_{x \rightarrow \infty} \frac{2\frac{x^2}{x^2} + \frac{1}{x^2}}{\frac{x}{x^2} + \frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{\lim 2 + \lim \frac{1}{x^2}}{\lim \frac{1}{x} + \lim \frac{1}{x^2}}$
 $\therefore \frac{2+0}{0+0} = \frac{2}{0} = \infty$

Ex2) $\lim_{x \rightarrow \infty} \frac{-x}{7x+4} = \lim_{x \rightarrow \infty} \frac{-\frac{x}{x}}{7\frac{x}{x} + \frac{4}{x}} = \lim_{x \rightarrow \infty} \frac{-1}{7 + \frac{4}{x}}$
 $= \frac{-1}{7+0} = \frac{-1}{7}$

Ex3) $\lim_{x \rightarrow \infty} \frac{5x+2}{2x^3-1} = \lim_{x \rightarrow \infty} \frac{\frac{5x}{x^3} + \frac{2}{x^3}}{\frac{2x^3}{x^3} - \frac{1}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{5}{x^2} + \frac{2}{x^3}}{2 + \frac{1}{x^3}}$
 $= \frac{0+0}{2+0} = 0$

Note Important : limits at infinity have properties similar to those of finite limits

Exercises : find the limit if it exist :

- 1) $\lim_{n \rightarrow \infty} (\sqrt{n^2 + 1} - n)$
- 2) $\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}}$
- 3) $\lim_{x \rightarrow \infty} \sqrt{\frac{1}{x}}$
- 4) $\lim_{x \rightarrow \infty} \frac{x^2 + 2x + 3}{x^4 + 4x + 4}$
- 5) $\lim_{x \rightarrow 0} \frac{1}{x^3}$
- 6) $\lim_{x \rightarrow \infty} \frac{x^3 - 2x - 5}{3x^2 + 5x + 1}$
- 7) $\lim_{x \rightarrow 3} \frac{4}{(x-3)^4}$
- 8) $\lim_{x \rightarrow \infty} \left(\frac{2x^2 + 3}{x^3 + x - 5} \operatorname{sgn}(x) \right)$
- 9) $\lim_{x \rightarrow \infty} \frac{\sqrt{x} + 2}{x + 3}$
- 10) $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - \sqrt{n^2 + 10})$
- 11) $\lim_{x \rightarrow 3} \frac{[x]^2 - 9}{x^2 - 4}$
- 12) $\lim_{x \rightarrow -4} |x + 4|$
- 13) $\lim_{x \rightarrow 5} \frac{1}{(x-5)^3}$
- 14) $\lim_{x \rightarrow -1} \frac{2}{\sqrt{x+1}}$
- 15) $\lim_{x \rightarrow \infty} \frac{1}{(x-2)^2}$
- 16) $\lim_{x \rightarrow 0} \frac{1}{x^4 + x^2 + 2}$
- 17) $\lim_{x \rightarrow 3} \frac{x^2 - 16}{x^2 - 9}$
- 18) $\lim_{x \rightarrow 0} f(x)$

$$f(x) = \begin{cases} x^2 + 7 & x \leq 0 \\ x - 4 & x > 0 \end{cases}$$
- 19) $\lim_{x \rightarrow \infty} \frac{1+x}{1-x}$

Continuity

Let f be a function , then we say that f is continuous at x_0 if :-

- (1) $f(x_0)$ is exist
- (2) $\lim_{x \rightarrow x_0} f(x)$ is exist
- (3) $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

Ex) $f(x) = x^2$ if the function is continuous at $x=2$?

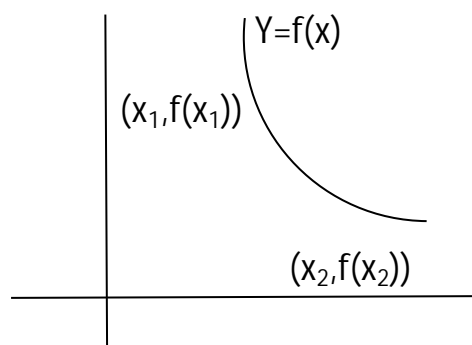
Sol : $Df = R$

- (1) $F(2) = 2^2 = 4$
- (2) $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} x^2 = 2^2 = 4$
- (3) $\lim_{x \rightarrow 2} x^2 = f(2) = 2^2 = 4$

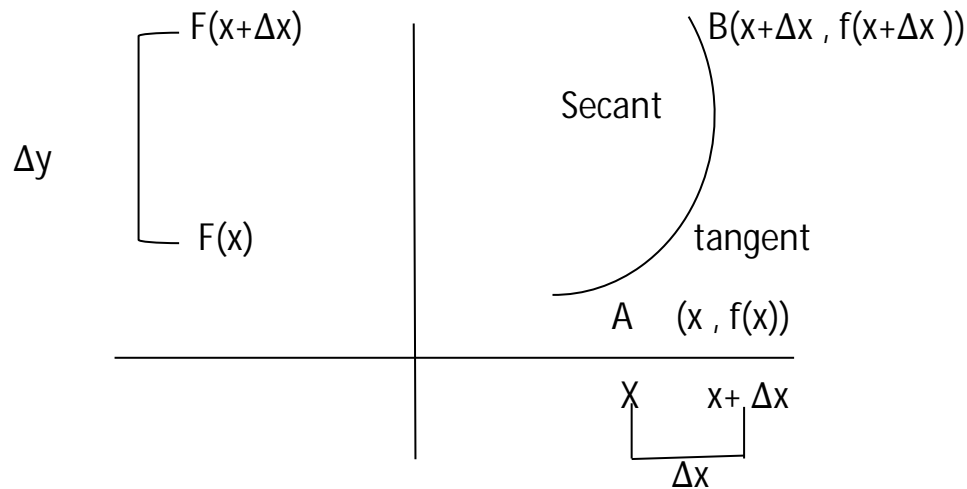
$\therefore f$ is cont at $x = 2$

Differentiation

Each point on the curve $y = f(x)$ there is a single straight tangent at that point , the slope of straight tangent of the curve $y=f(x)$ at the point $(x,f(x))$ it represents a derivative at that point .



Let A (x,f(x)) be a fixed point on the curve , and B (x+Δx,f(x+Δx)) is a nother point therefore Δy=f(x+Δx)-f(x)



Note : that at Δx decreasing length (close to zero) the straight secant AB more and more a pliability begins on the straight tangent at the point (x,f(x)) , this means that slop straight secant AB be equal to slop straight tangent at the point (x,f(x)) , that's when (Δx→0) , knowing that slop straight tangent at the point (x,f(x)) represents a derived function at that point .

$$m_{tan} = \lim_{\Delta x \rightarrow 0} m_{sec} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$\hat{f}(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\dot{y}, \hat{f}(x), \frac{dy}{dx}, \frac{df(x)}{dx}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

Note : when the value of the limit exist then the function called differentiable function , and f called the derivative of f at x

Ex(1) let $f(x) = 4x - 2$, find $f'(x)$ by definition

Sol: $y' = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$f(x) = 4x - 2$, $f(x + \Delta x) = 4(x + \Delta x) - 2$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{4(x+\Delta x) - 2 - [4x - 2]}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cancel{4x} + 4\Delta x - 2 - \cancel{4x} + 2}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{4\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 4 = 4$$

Ex(2) let $f(x) = \sqrt{x}$, find the equation of the tangent line and normal line at the point (4,2) by definition.

Sol: $m_{tan} \left] \begin{array}{l} f(x) \\ (4, 2) \end{array} \right]$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta x \neq 0}} \frac{\Delta x}{\Delta x (\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$m_{tan} \left] (4, 2) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \right]$$

$$(y - y_1) = m_{tan} (x - x_1)$$

$$y - 2 = \frac{1}{4} (x - 4)$$

$$y = \frac{1}{4}x + 1$$

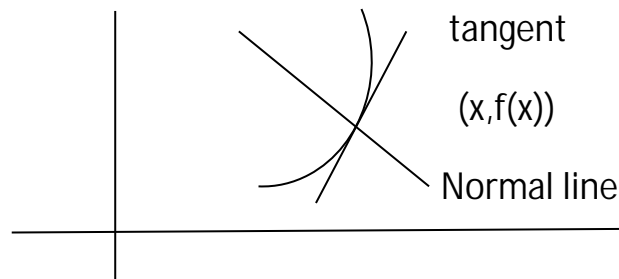
$$m_{\perp} = \frac{-1}{m_{tan}} = \frac{-1}{\frac{1}{4}} = -4$$

$$(y - y_1) = m_{\perp} (x - x_1)$$

$$y - 2 = -4(x - 4)$$

$$y = -4x + 18$$

Definition : the normal line to a curve is the line that is perpendicular to the tangent of the curve at a particular point



Exc: find $\hat{f}(x)$ by definition :

1- $f(x) = x^3$

2- $f(x) = x^2 + \frac{1}{x}$

3- Let $f(x) = x^2$, find the equation of the tangent line and normal line at the point (3,9) by definition

4- Using definition to prove that $\hat{f}(x) = m$ for $f(x) = y = mx + b$

5- Find the tangent line at (6,3) for $y = \sqrt{x + 3}$

Theorem : Every function is differentiable at x , then _____ f is continuous at x_0 .

Proof : to prove $\lim_{x \rightarrow x_0} f(x) \rightarrow f(x_0)$

i.e $\lim_{x \rightarrow x_0} [f(x) - f(x_0)] \rightarrow 0$

suppose that $\Delta x = x - x_0 \Rightarrow x = x_0 + \Delta x$

$$f(x) = f(x_0 + \Delta x)$$

when $x \rightarrow x_0$, then $\Delta x \rightarrow 0$

$$\lim_{x \rightarrow x_0} [f(x) - f(x_0)] = \lim_{x \rightarrow x_0} [f(x_0 + \Delta x) - f(x_0)]$$

$$= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cdot \Delta x \right]$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \lim_{\Delta x \rightarrow 0} \Delta x$$

$$= f(x_0) \cdot 0$$

$$= 0$$

Note : The inverse of the above theorem is not true ____ if this function f continuous at the point, it is not necessary to be differentiable at that point as in the example :

Let $f(x) = |x|$, $x_0 = 0$

$$F(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$F(x) = \lim_{\Delta x \rightarrow 0} \frac{|x + \Delta x| - |x|}{\Delta x} \quad |\Delta x| = \begin{cases} \Delta x & \text{if } \Delta x \geq 0 \\ -\Delta x & \text{if } \Delta x < 0 \end{cases}$$

$$F(0) = \lim_{\Delta x \rightarrow 0} \frac{|0 + \Delta x| - |0|}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{|\Delta x|}{\Delta x} \begin{cases} L_+ = \lim_{\Delta x \rightarrow 0^+} \frac{\Delta x}{\Delta x} = 1 \\ L_- = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{\Delta x} = -1 \end{cases}$$

$L_+ \neq L_- \Rightarrow$ limit is not exist

$\therefore f$ is not differentiable function at $x_0 = 0$

$$(7) \left(\frac{f}{g} \right)'(x) = \frac{f'(x)g(x) - f(x) \cdot g'(x)}{(g(x))^2}, g(x) \neq 0$$

Ex :- $f(x) = \frac{x+1}{x}$, find $f'(x)$

Sol : $f'(x) = \frac{x \cdot 1 - (x+1) \cdot 1}{x^2} = \frac{x - x - 1}{x^2} = \frac{-1}{x^2}$

Corollary

Let $y = f(x) = x^{-n}$, for $n \in \mathbb{Z}_+$, $x \neq 0$, then $y = f(x) = \frac{dx^{-n}}{dx} = -nx^{-n-1}$

Ex: let $f(x) = -5x^{-3}$, find $f'(x)$

$$f'(x) = -5(-3)x^{-3-1} = 15x^{-4}$$

$$\frac{d}{dx}(x^r) = rx^{r-1} \forall r \in \mathbb{R}$$

$$(8) h(x) = f(g^{(x)}) \cdot g(x), h = f \circ g(x)$$

$$h = f \circ g(x) = f(g^{(x)})$$

$$h' = (f \circ g)'(x) =$$

$$f'(g^{(x)}) \cdot g'(x)$$

Corollary :-

Let $y = (f(x))^n$, $n \in \mathbb{Z}$

$$\frac{dy}{dx} = n(f(x))^{n-1} \cdot f'(x)$$

Derivation properties :-

$$(1) f(x) = c, c \text{ is constant, then } f'(x) = 0$$

Ex: let $f(x) = -5$, $f'(x) = 0$

$$(2) (c \cdot f)'(x) = c \cdot f'(x), c \text{ is constant}$$

Ex: let $f(x) = 3x \Rightarrow f'(x) = 3$

$$(3) (f+g)'(x) = f'(x) + g'(x)$$

Ex: let $f(x) = 2x$, $g(x) = 1$

Sol: $(f+g)'(x) = f'(x) + g'(x) = (2x)' + (1)' = 2(1) + 0 = 2$

Rem: $(f_1 \mp f_2 \mp f_3 \mp \dots \mp f_n)'(x) = f_1'(x) \mp f_2'(x) \mp \dots + f_n'(x)$

$$(4) f'(x) = nx^{n-1}, n \text{ is positive integer}$$

Ex: $f(x) = x^6 \Rightarrow f'(x) = 6x^{6-1} = 6x^5$

$$(5) (f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Ex: $f(x) = (x^3+2)(1-x^2)$, find $f'(x)$

Sol: $f'(x) = (x^3+2) \frac{d}{dx}(1-x^2) + (1-x^2) \frac{d}{dx}(x^3+2)$

$$= (x^3+2) \cdot (-2x) + (1-x^3) \cdot (3x^2)$$

Rem : $(f \cdot g \cdot h)'(x) = f(x) \cdot g(x) \cdot h'(x) + f(x) \cdot h(x) \cdot g'(x) + g(x) \cdot h(x) \cdot f'(x)$

Ex(2) : let $y=t^2-1$, $x=2t+3$, find $\frac{dy}{dx}$ at $t=1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2t}{2}$$

$$= t \quad \left. \frac{dy}{dx} \right]_{t=1}$$

$$= 1$$

Exc :

1- Let $y = x^3 - 3x^2 + 5x - 4$, $x = t^2 + t$, find $\frac{dy}{dx}$

Find $\frac{dy}{dx}$:

2- Let $y = u^3 + 1$, $u = x^2 + 3$

3- $Y = 3t^2 - 1$, $x = 6t - 1$

4- $Y = \frac{t^2}{1+t}$, $x = \frac{t}{2+t}$

5- $Y = t^2$, $x = \frac{t}{1-t}$

6- $Y = Z^{2/3}$, $Z = x^2 + 1$

7- $Y = w^2 - w^{-1}$, $w = 3x$

8- $Y = 2v^3 + \frac{2}{v^3}$, $v = (2x+2)^{2/3}$

9- $Y = \frac{u^2}{u^2+1}$, $u = \sqrt{2x+1}$

Chain Rule

Let $y=f(x)$, $x=g(t)$, find $\frac{dy}{dt}$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Let $y=f(t)$, $t=g(x)$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Let $y=f(t)$, $x=g(t)$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Ex(1): let $y=3x-1$, $x=2t$, find $\frac{dy}{dt}$

Sol : $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$$= (3) \cdot (2)$$

$$= 6$$

$$y = 3x-1$$

$$= 3(2t) - 1$$

$$Y = 6t - 1$$

$$\frac{dy}{dt} = 6$$

Implicit differentiation الاشتقاق الضمني

Ex : $x^2 + xy + y^5 = 0$ نشتق ضمناً بالنسبة لـ x باعتبار y دالة ضمنية لـ x

$$2x \frac{dx}{dx} + (x \frac{dy}{dx} + y \frac{dx}{dx}) + 5y^4 \frac{dy}{dx} = 0$$

$$2x + xy' + y + 5y^4 y' = 0$$

$$xy' + 5y^4 y' = -2x - y$$

$$(x + 5y^4) y' = -2x - y$$

$$y' = \frac{-2x - y}{x + 5y^4}$$

نشتق ضمناً بالنسبة لـ y (باعتبار x دالة ضمنية لـ y)

$$2x \frac{dx}{dy} + (x \frac{dy}{dy} + y \frac{dx}{dy}) + 5y^4 \frac{dy}{dy} = 0$$

$$2x \cdot x' + x + y \cdot x' + 5y^4 = 0$$

$$x + 5y^4 = -2x x' - y x'$$

$$x + 5y^4 = (-2x - y) x'$$

$$x' = \frac{x+5y^4}{-2x-y}$$

$$x' = \frac{1}{y'}$$

Ex : find the equation of the tangent line and normal line of the curve $x^2+y^2=2$ at the point (1,1)

Sol: $2x \frac{dx}{dx} + 2y \frac{dy}{dx} = 0$

$$2x + 2y y' = 0$$

$$2y y' = -2x$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$

$$y' = m]_{(1,1)} = \frac{-1}{1} = -1$$

$$(y-y_1) = m(x-x_1)$$

$$y-1 = -1(x-1)$$

$$y-1 = -x+1$$

$y = -x+2$ the equation of the tangent line

$$m_{\perp}]_{(1,1)} = \frac{-1}{m]_{(1,1)}}$$

$$m_{\perp}]_{(1,1)} = \frac{-1}{-1} = 1$$

$$(y-y_1) = m(x-x_1)$$

$$y-1 = 1(x-1)$$

$y=x$ the equation of the normal line

Higher - order derivatives

المشتقات من مرتبة عليا

Let $y = f(x)$

$$\bar{f}(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx} = \bar{y} = y^{(1)} \quad \text{المشتقة الاولى للدالة}$$

$$\bar{\bar{f}}(x) = \lim_{\Delta x \rightarrow 0} \frac{\bar{f}(x+\Delta x) - \bar{f}(x)}{\Delta x} = \frac{d^2y}{dx^2} = \bar{\bar{y}} = y^{(2)} \quad \text{المشتقة الثانية للدالة}$$

$$\bar{\bar{\bar{f}}}(x) = \lim_{\Delta x \rightarrow 0} \frac{\bar{\bar{f}}(x+\Delta x) - \bar{\bar{f}}(x)}{\Delta x} = \frac{d^3y}{dx^3} = \bar{\bar{\bar{y}}} = y^{(3)} \quad \text{المشتقة الثالثة للدالة}$$

$$(n) \quad (n-1) \quad (n-1) \quad n \quad (n)$$

$$f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx^n} = y, n \in \mathbb{N} \quad \text{المشتقة النونية للدالة}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) \quad \text{second derivative}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) \quad \text{third derivative}$$

ملاحظة :

Ex : let $y = 2x^3 + x^2 - 1$

$$y' = 6x^2 + 2x$$

$$y'' = 12x + 2$$

$$y''' = 12$$

$$y^{(4)} = 0, \dots$$

Chapter three

رسم المخطط البياني للدالة Applications of Differentiations Graphing function

Ex(1): let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$D_f = \mathbb{R}$

لا توجد محاذيات شاقوليه وافقية

$x=0 \Rightarrow y=0$

$y=0 \Rightarrow \sqrt[3]{x} = 0 \Rightarrow \text{نقطة التقاطع مع محاورين} \Rightarrow \text{بالتكعيب}$

$\Rightarrow x=0$

$f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$ المنحني متناظر مع نقطة الاصل

$f(-x) \neq f(x)$

المنحني غير متناظر مع المحور y

$\bar{f}(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

+++++ معرف +++++

اشارة f^-

$\bar{f}=0 \Rightarrow 1=0$ وهذا غير ممكن $x < 0$ $x=0$ $x > 0$

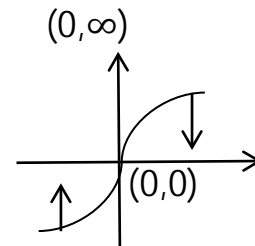
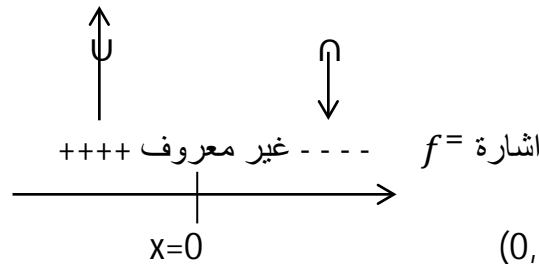
$\bar{\bar{f}}(x) = \frac{1}{3} \cdot \frac{-2}{3} \cdot x^{-\frac{5}{3}}$

$= \frac{-2}{9} \frac{1}{\sqrt[3]{x^5}}$

$= \frac{-2}{9x\sqrt[3]{x^2}}$

$\bar{\bar{f}}=0 \Rightarrow -2=0$

$R_f = \mathbb{R}$



Ex(2): $f(x) = \frac{1}{x^2-1}$

$x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow \sqrt{x^2} = \sqrt{1} \Rightarrow |x| = 1 \Rightarrow x = \pm 1$

$D_f = \mathbb{R} \setminus \{1, -1\}$

المحاذيات الشاقولية

$$x=1, x=-1$$

$$\frac{y}{1} = \frac{1}{x^2-1}$$

$$1=yx^2-y$$

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \frac{1}{x^2-1}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$

$$1+y=yx^2$$

$$x^2 = \frac{1+y}{y}$$

$$\sqrt{x^2} = \sqrt{\frac{1+y}{y}}$$

$$|x| = \sqrt{\frac{1+y}{y}}$$

$$x = \pm \sqrt{\frac{1+y}{y}}$$

$$= \frac{\left(\lim_{x \rightarrow \infty} \frac{1}{x}\right)^2}{\lim_{x \rightarrow \infty} 1 - \left(\lim_{x \rightarrow \infty} \frac{1}{x}\right)^2}$$

$$= \frac{(0)^2}{1-(0)^2}$$

$$= \frac{0}{1}$$

$$= 0$$

$$y = 0 \quad \text{محاذي الافقي}$$

$$\text{If } x=0 \Rightarrow y = \frac{1}{0^2-1} = -1$$

$$\text{If } y=0 \Rightarrow 0 = \frac{1}{x^2-1} \Rightarrow 0=1$$

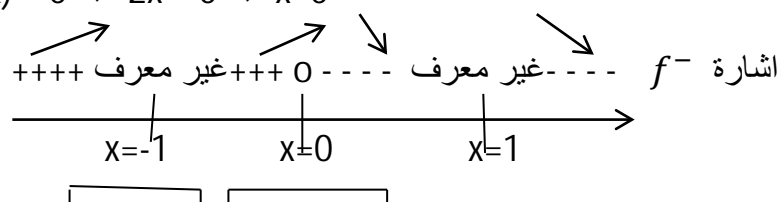
$$f(-x) = \frac{1}{(-x)^2-1} = \frac{1}{x^2-1} = f(x)$$

$$f(-x) \neq -f(x)$$

$$\bar{f}(x) = \frac{0-2x}{(x^2-1)^2}$$

$$\bar{f}(x) = \frac{-2x}{(x^2-1)^2}$$

$$\bar{f}(x) = 0 \Rightarrow -2x = 0 \Rightarrow x=0$$



$$\bar{f}(x) = \frac{(x^2-1)^2 \cdot -2 + 2x \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4}$$

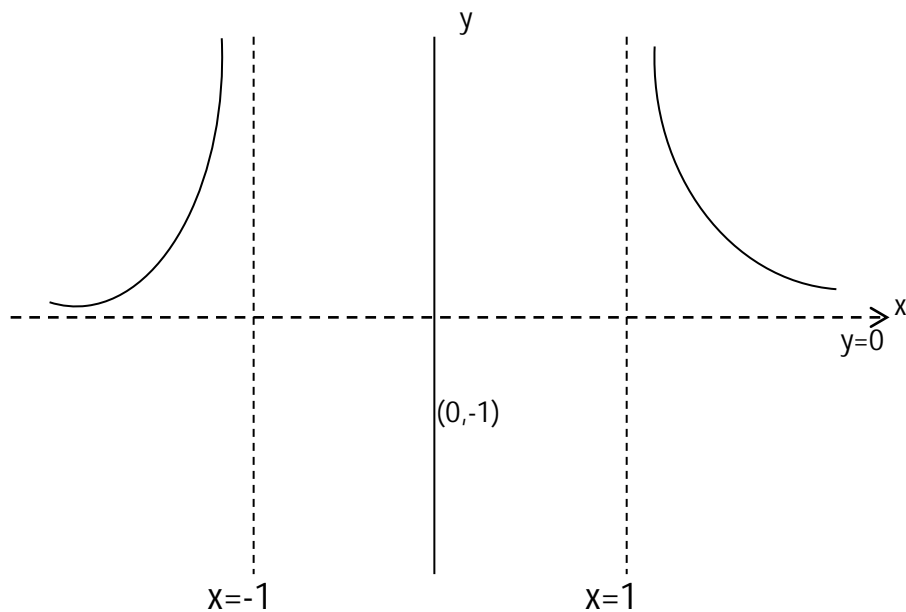
$$= \frac{-2 \cancel{(x^2-1)} \{x^2-1-4x^2\}}{(x^2-1)^3}$$

$$= \frac{-2\{-3x^2-1\}}{(x^2-1)^3}$$

$$f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}$$

$$f''(x) = 0 \Rightarrow 3x^2+1=0 \Rightarrow 3x^2=-1 \Rightarrow x^2 = -\frac{1}{3}$$

$\updownarrow \quad \quad \quad \updownarrow \quad \quad \quad \updownarrow$
 $+++++ \text{ غير معرف } - - - - - \text{ غير معرف } +++++ \quad f'' \text{ إشارة}$
 $\xrightarrow{\quad \quad \quad x=-1 \quad \quad \quad 0 \quad \quad \quad x=1 \quad \quad \quad}$



$$R_f = (-\infty, -1] \cup (0, \infty) - R \mid (1, 0]$$

Ex(3): $y = |x^2 - 4|$

$$= \begin{cases} x^2 - 4 & \text{if } x^2 - 4 > 0 \Rightarrow x^2 > 4 \Rightarrow |x| > 2 \Rightarrow x > 2 \vee x < -2 \\ 0 & \text{if } x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow |x| = 2 \Rightarrow x = 2, -2 \\ -(x^2 - 4) & \text{if } x^2 - 4 < 0 \Rightarrow x^2 < 4 \Rightarrow |x| < 2 \Rightarrow -2 < x < 2 \end{cases}$$

$D_f = R$ لا يوجد محاذيات شاقولية وافقية

$$x=0 \Rightarrow y = |-4| = 4$$

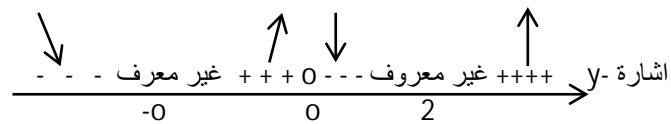
$$y=0 \Rightarrow |x^2 - 4| = 0 \Rightarrow x=2, -2$$

$$f(-x) = |(-x)^2 - 4| = |x^2 - 4| = f(x)$$

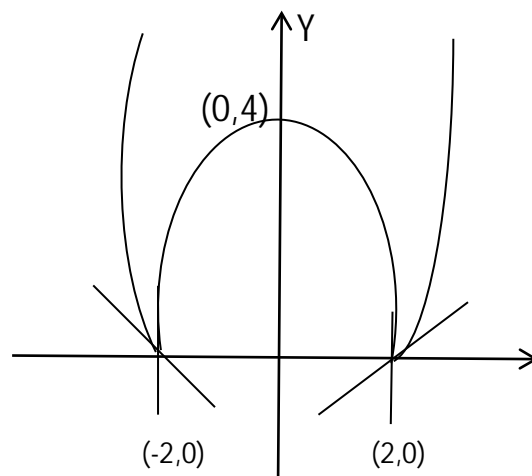
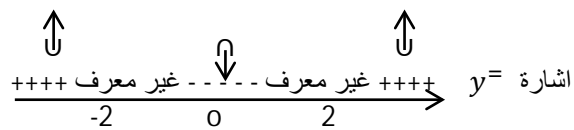
$$f(-x) \neq -f(x)$$

$$y = \begin{cases} x^2 - 4 & \text{if } x > 2 \vee x < -2 \\ 0 & \text{if } x = 2, -2 \\ 4 - x^2 & \text{if } -2 < x < 2 \end{cases}$$

$$y = \begin{cases} 2x & \text{if } x > 2 \vee x < -2 \\ \text{غير معرف} & \text{if } x = 2, -2 \\ -2x & \text{if } -2 < x < 2 \end{cases}$$



$$y = \begin{cases} 2 & \text{if } x > 2 \vee x < -2 \\ \text{غير معرف} & \text{if } x = 2, -2 \\ -2 & \text{if } -2 < x < 2 \end{cases}$$



$$R_f = \mathbb{R}^+$$

Ex(4) : Graph of the following functions :-

1- $y = x^2 - 2x + 4$

2- $y = x^3 - 12x + 10$

3- $y = x^5$

4- $y = \frac{1}{x^2+3}$

5- $y = \frac{x}{x-1}$

6- $y = \frac{x^2}{x^2-1}$

7- $y = \frac{x^2+1}{x}$

8- $y = \frac{x+3}{x+2}$

9- $y = \sqrt{x^2 - 1}$

10- $y = x^4 - 1$

11- $y = \sqrt{x + 1}$

12- $y = \sqrt[3]{x^2 - 1}$

$\mathbb{R} \setminus \{-1\} \not\subseteq [4, \infty)$

$\therefore f_0g$ is not exist

Now , to find g_0f is $R_f \subseteq D_g$

$\mathbb{R}^+ \not\subseteq \mathbb{R} \setminus \{3\}$

$\therefore g_0f$ is not exist

Homework :-

Find fog , gof to each the following functions

(1) $f(x) = |x|$, $g(x) = -x$

(2) If $f(t) = t^2$ find g, h such that $(f_0g)(x) = (f_0h)(x) = x^2 - 10x + 25$

chapter four

The inverse trigonometric functions معكوس الدوال المثلثية

إذا كانت الدالة متباينة bijective f (onto & 1-1)

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

$$\forall y \in Y \quad \exists x \in X \quad \exists y = f(x)$$

$$\Leftrightarrow \therefore \exists \quad f: X \rightarrow Y \exists \quad y = f(x) \quad , f \text{ is 1-1 \& onto}$$

$$f^{-1}: Y \rightarrow X \exists \quad x = f^{-1}(y)$$

معكوس دالة sin :

Let $y = \sin x$

$$\sin : \mathbb{R} \rightarrow [-1, 1]$$

$$\sin : [-2\pi, 2\pi] \rightarrow [-1, 1]$$

سنعرف دالة جديدة هي معكوس الدالة \sin ويرمز لها بالرمز \sin^{-1} او \arcsin

$$\therefore \sin^{-1} y = \sin^{-1}(\sin x)$$

$$\sin^{-1} y = x$$

$$\therefore y = \sin x \Leftrightarrow x = \sin^{-1} y$$

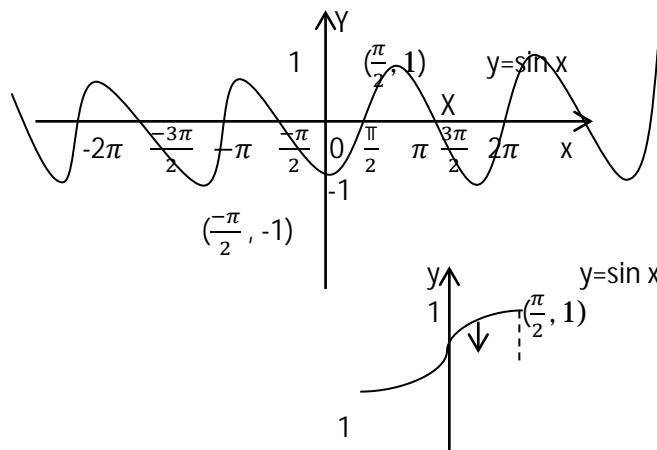
$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$$

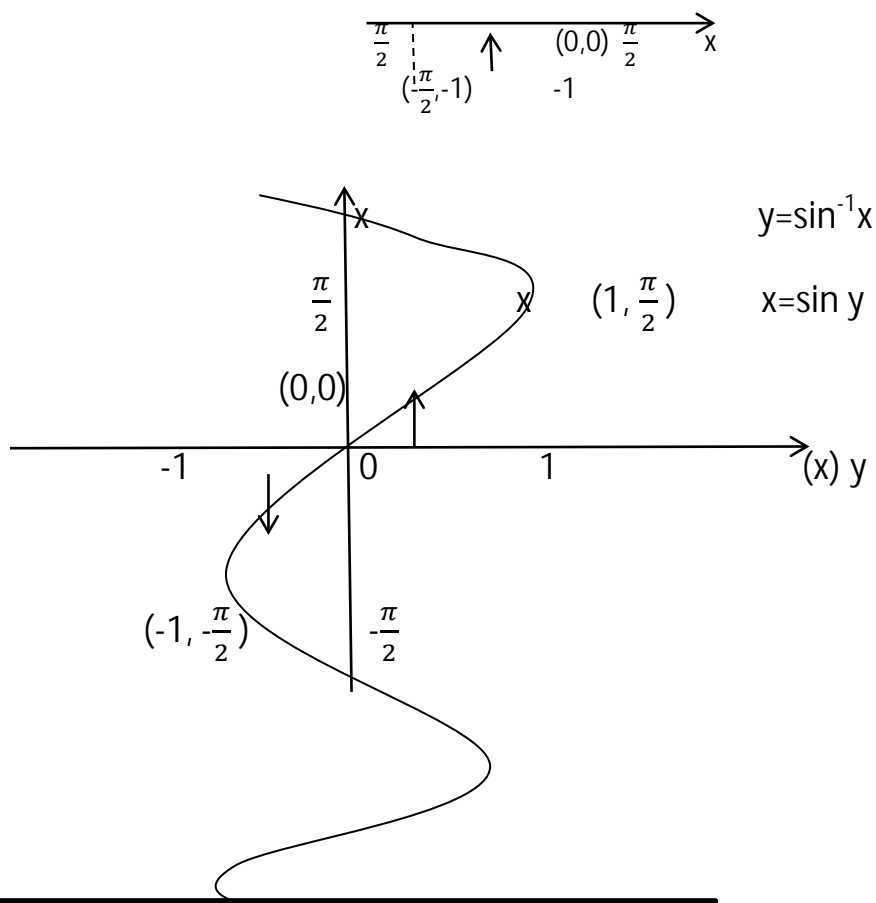
$$\sin \text{ is 1-1 \& onto } \therefore \exists \sin^{-1}$$

$$\sin^{-1}: [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$D_{\sin} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = R_{\sin^{-1}}$$

$$R_{\sin} = [-1, 1] = D_{\sin^{-1}}$$





تعريف : لكل $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $y = \sin x \Leftrightarrow x = \sin^{-1} y$

$$\sin^{-1} y \neq \frac{1}{\sin y}$$

ملاحظة :

دالة \sin^{-1} هي دالة فردية حيث ان

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

البرهان :

من تعريف المعكوس $\text{Let } y = \sin^{-1}(-x) \Leftrightarrow \sin y = -x$

$$\Leftrightarrow x = -\sin y$$

$$\Leftrightarrow x = \sin(-y) \quad \text{لان دالة } \sin \text{ فردية}$$

$$\Leftrightarrow \sin^{-1} x = -y$$

$$\Leftrightarrow y = -\sin^{-1}(x)$$

$$\therefore \sin^{-1}(-x) = -\sin^{-1}(x)$$

مثال :-

$$\sin^{-1}(-1) = \sin^{-1}(1)$$

$$\sin^{-1}(-1) = -\frac{\pi}{2}, \quad \sin^{-1}(1) = \frac{\pi}{2}$$

معكوس دالة (cos) :

$$Y = \cos x$$

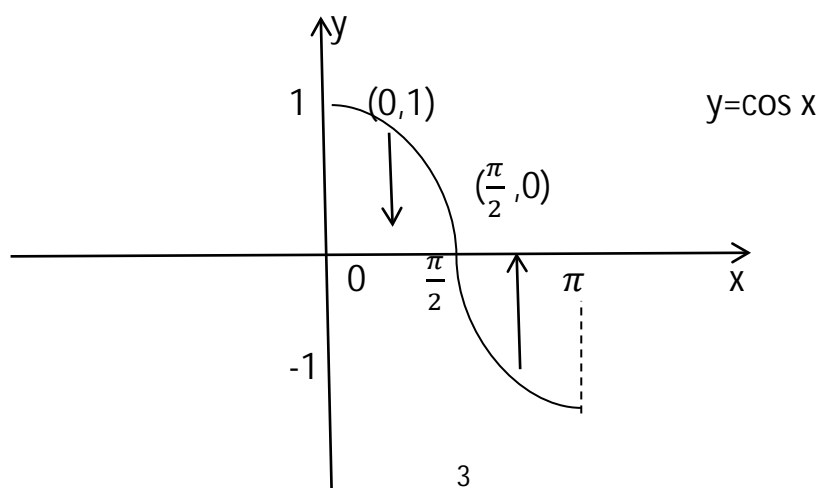
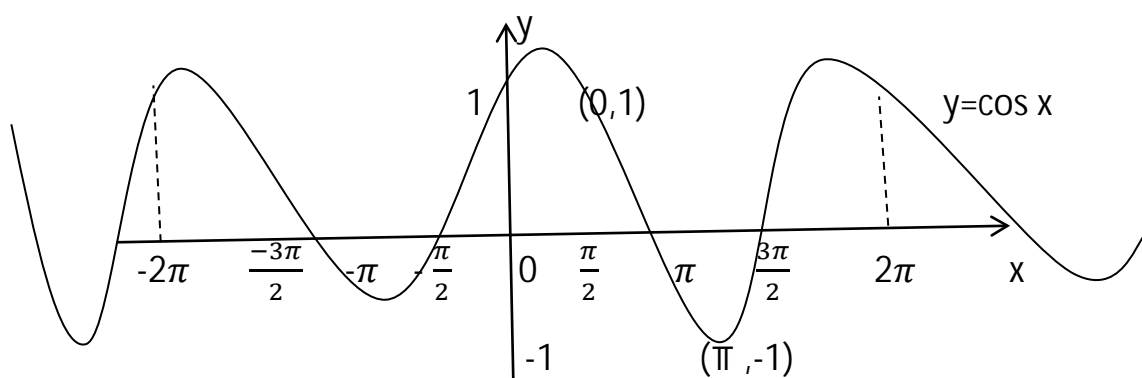
$$\cos : \mathbb{R} \rightarrow [-1, 1]$$

$$\cos : [-2\pi, 2\pi] \rightarrow [-1, 1]$$

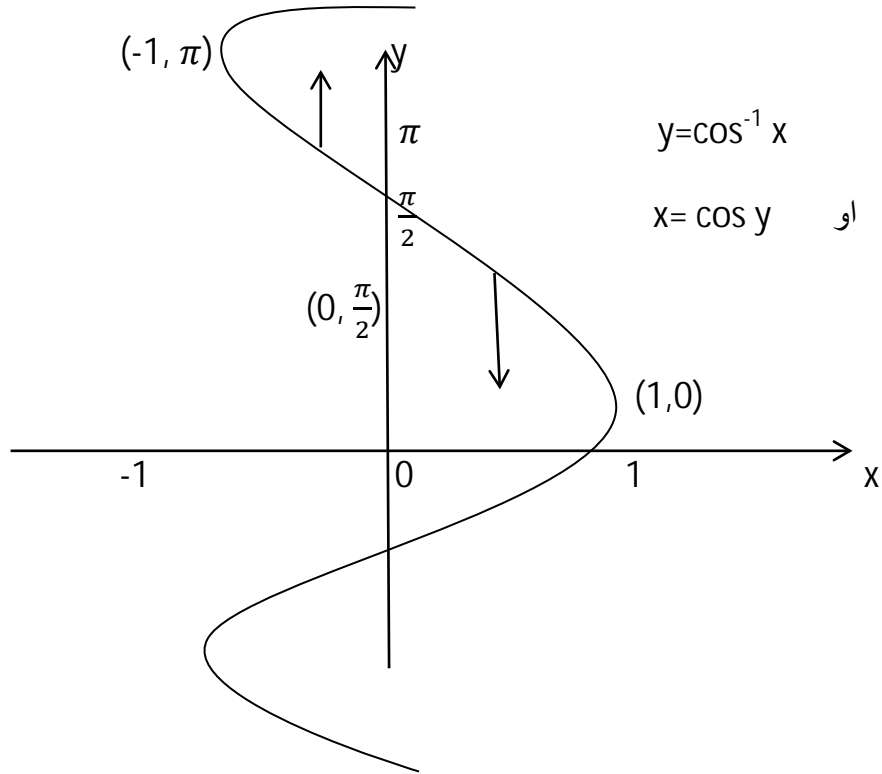
$$\cos : [0, \pi] \rightarrow [-1, 1]$$

\cos is 1-1 & onto $\therefore \exists \cos^{-1}$ أو $\text{Arc cos } \ni$

$$\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$$



$(\pi, -1)$



$$x \in [0, \pi] \text{ لكل } \cos^{-1}(\cos x) = x$$

$$y \in [-1, 1] \text{ لكل } \cos(\cos^{-1} y) = y$$

تعريف -: $y = \cos x \Leftrightarrow x = \cos^{-1} y \quad x \in [0, \pi]$ لكل

ملاحظة : دالة \cos^{-1} هي دالة ليست زوجية ولا فردية $\cos^{-1}(-x) = \pi - \cos^{-1} x$

البرهان : Let $y = \pi - \cos^{-1}(x)$ الطرف الايمن

$$\Rightarrow y - \pi = -\cos^{-1}(x)$$

$$\Rightarrow \cos^{-1}(x) = \pi - y$$

$$\Rightarrow x = \cos(\pi - y)$$

$$\Rightarrow x = -\cos y$$

$$\Rightarrow \cos y = -x$$

$$\Rightarrow y = \cos^{-1}(-x)$$

الطرف الايسر $\Rightarrow y =$

$$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

طريقة اخرى :-

$$\text{Let } y = \cos^{-1}(-x)$$

من تعريف المعكوس $\cos y = -x$

$$x = -\cos y$$

$$x = \cos(\pi - y)$$

$$\cos^{-1}x = \pi - y$$

$$\therefore y = \pi - \cos^{-1}(x)$$

$$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

$$\cos^{-1}(-1) = \pi, \quad \cos^{-1}(1) = 0$$

مثال :-

نسنتج علاقة مهمة تجمع بين الدالتين \cos^{-1}, \sin^{-1}

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \quad \text{لكل } x \in [-1, 1]$$

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x \quad \text{لكل } x \in [-1, 1]$$

برهن أن :-

$$w = \frac{\pi}{2} - \sin^{-1}(x), \quad x \in [-1, 1]$$

البرهان :- لنفرض أن

$$\sin^{-1}(x) = \frac{\pi}{2} - w$$

$$\sin(\sin^{-1}x) = \sin\left(\frac{\pi}{2} - w\right)$$

$$x = \sin\left(\frac{\pi}{2} - w\right)$$

$$x = \cos w$$

$$w \in D_{\cos}? \quad w \in [0, \pi] ?$$

$$-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$$

وبما ان :-

بأضافة $-\frac{\pi}{2}$ الى جميع الاطراف نضرب المتراجعة في -1

$$-\pi \leq \frac{\pi}{2} + \sin^{-1}x \leq 0$$

$$\pi \geq \frac{\pi}{2} - \sin^{-1}x \geq 0$$

$$0 \leq \frac{\pi}{2} - \sin^{-1}x \leq \pi$$

$$0 \leq w \leq \pi$$

$$\therefore w = \cos^{-1}(x) \quad \text{لكل } x \in [-1, 1]$$

$$\therefore \cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x) \quad \text{لكل } x \in [-1, 1]$$

طريقة اخرى :-

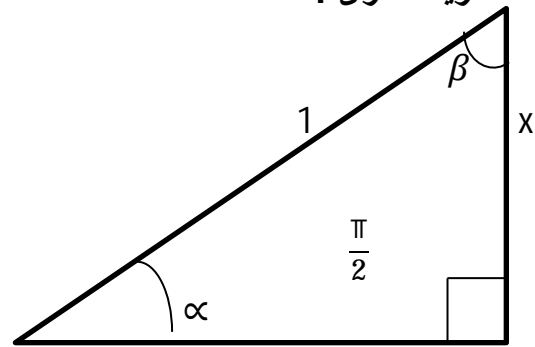
$$\sin \alpha = x, \quad \cos \beta = x$$

$$\therefore \alpha = \sin^{-1}x, \quad \beta = \cos^{-1}x$$

$$\alpha + \beta = \frac{\pi}{2}$$

$$\beta = \frac{\pi}{2} - \alpha$$

$$\therefore \cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$$



لان مجموع زوايا المثلث 180

معكوس دالة (tan)

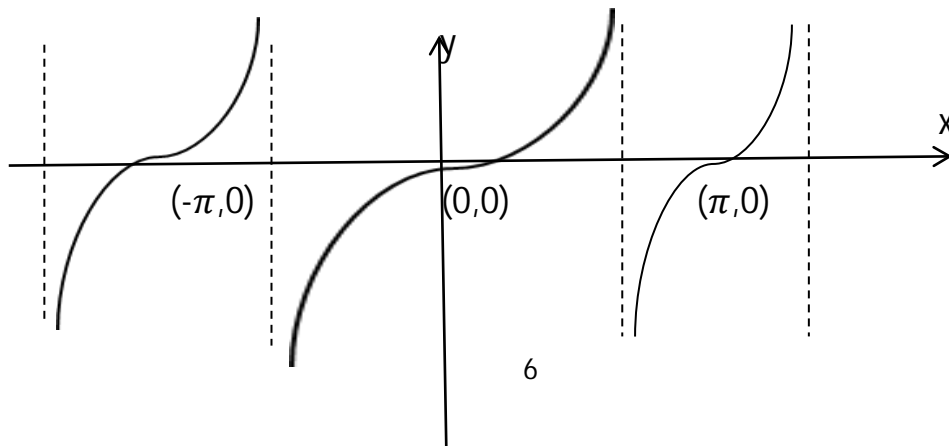
$$y = \tan x$$

$$\tan: \mathbb{R} \setminus \{x: x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\} \rightarrow \mathbb{R}$$

$$\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R} \quad \text{محاذيات شاقولية } x = -\frac{\pi}{2}, x = \frac{\pi}{2}$$

$$\tan \text{ is 1-1 \& onto } \therefore \exists \tan^{-1} \text{ او } \text{Arc tan } \exists$$

$$\tan^{-1}: \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{محاذيات افقية } y = -\frac{\pi}{2}, y = \frac{\pi}{2}$$



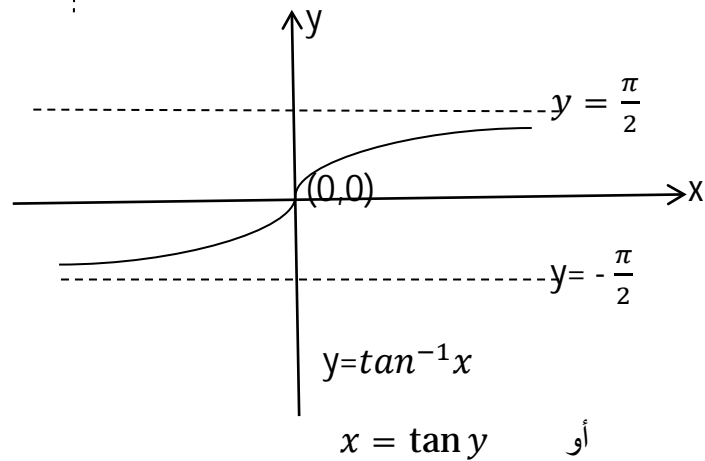
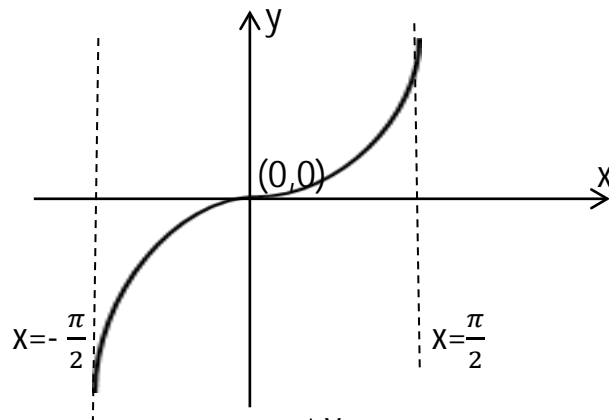
$$x = -\frac{3\pi}{2}$$

$$x = -\frac{\pi}{2}$$

$$x = \frac{\pi}{2}$$

$$x = \frac{3\pi}{2}$$

$$y = \tan x$$



$$y = \tan x \Leftrightarrow x = \tan^{-1} y$$

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ لكل}$$

تعريف :

دالة \tan^{-1} هي دالة فردية

$$\tan^{-1}(-x) = -\tan^{-1}(x)$$

$$\text{Let } y = \tan^{-1}(-x)$$

البرهان :

$$\tan y = -x$$

$$x = -\tan y$$

$$x = \tan(-y)$$

$$-y = \tan^{-1} x$$

$$y = -\tan^{-1} x$$

$$\therefore \tan^{-1}(-x) = -\tan^{-1}(x)$$

$\tan^{-1}(1) = \frac{\pi}{4}, \tan^{-1}(-1) = -\frac{\pi}{4}$ مثال :

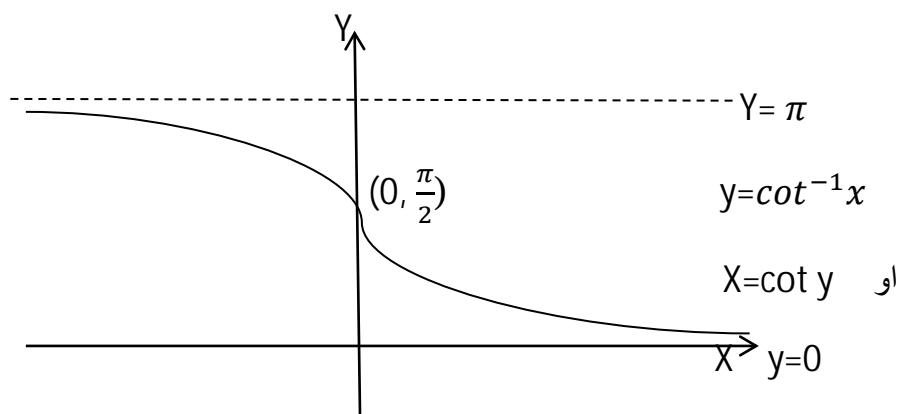
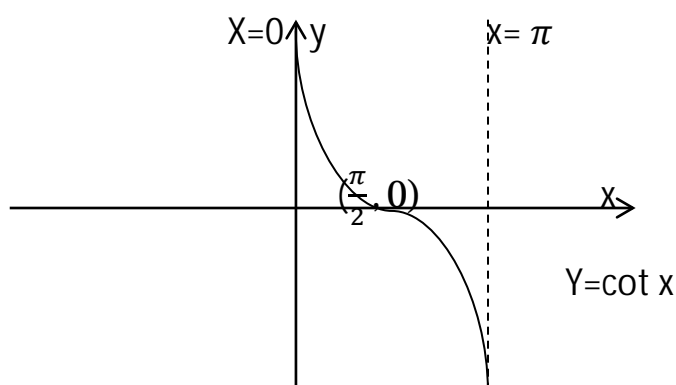
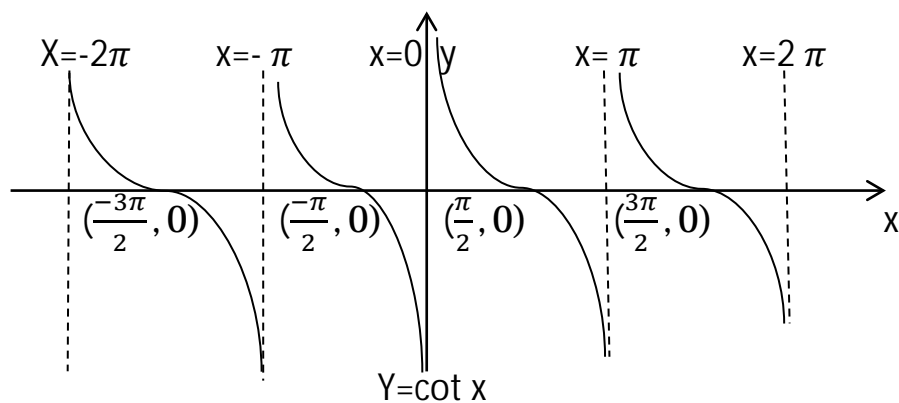
معكوس دالة (cot)

$Y = \cot x$

$\cot : \mathbb{R} \setminus \{x: x = n\pi, n \in \mathbb{I}\} \rightarrow \mathbb{R}$

$\cot: (0, \pi) \rightarrow \mathbb{R}$

\cot is 1-1 & onto $\therefore \exists \cot^{-1}$ أو $\text{Arc cot} \ni \cot^{-1}: \mathbb{R} \rightarrow (0, \pi)$



للكل $x \in (0, \pi)$ $y = \cot x \Leftrightarrow x = \cot^{-1} y$ تعريف :

$$\cot^{-1}x + \tan^{-1}x = \frac{\pi}{2}$$

$$\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x) \quad \text{برهن أن :}$$

$$w = \frac{\pi}{2} - \tan^{-1}(x) \quad \text{البرهان :} \quad \text{لتكن}$$

$$\tan^{-1}(x) = \frac{\pi}{2} - w$$

$$x = \tan\left(\frac{\pi}{2} - w\right) = \frac{\sin(\pi/2 - w)}{\cos(\pi/2 - w)} = \frac{\cos w}{\sin w} = \cot w$$

$$x = \cot w$$

$$w \in D_{\cot}?$$

$$w \in (0, \pi)?$$

$$0 < w < \pi?$$

$$\frac{-\pi}{2} < \tan^{-1}x < \frac{\pi}{2} \quad \text{بما ان}$$

$$\frac{\pi}{2} > -\tan^{-1}x > -\frac{\pi}{2} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow[\leftarrow]{\tan} R$$

$$-\frac{\pi}{2} < -\tan^{-1}x < \frac{\pi}{2} \quad (0, \pi) \xrightarrow[\leftarrow]{\cot} R$$

$$0 < \frac{\pi}{2} - \tan^{-1}x < \pi \quad \text{بإضافة } \frac{\pi}{2} \text{ لجميع اطراف المتراجحة}$$

$$0 < w < \pi$$

$$\therefore w \in (0, \pi)$$

$$w \in D_{\cot}$$

$$\therefore w = \cot^{-1}x$$

$$\therefore \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

$$\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x) \quad \text{طريقة أخرى :}$$

$$\text{Let } \tan^{-1}(x) = y$$

$$\therefore x = \tan y$$

$$\left(\frac{\pi}{2} - y\right) = Z = \cot^{-1}(x)$$

$$\cot\left(\frac{\pi}{2} - y\right) = \cot Z$$

$$\tan y = \cot Z$$

$$\tan y = x \quad \text{لكن}$$

$$\therefore x = \cot Z$$

$$\therefore \cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$$

$$\tan^{-1}(x) \neq \frac{\sin^{-1}(x)}{\cos^{-1}(x)} \quad \text{ملاحظة :}$$

$$\cot^{-1}(x) \neq \frac{\cos^{-1}(x)}{\sin^{-1}(x)}$$

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}(x) \quad \text{اثبت ان :}$$

$$\alpha = \tan^{-1}\left(\frac{1}{x}\right) \quad \text{لتكن البرهان :}$$

$$\tan \alpha = \frac{1}{x} \quad \text{من تعريف المعكوس}$$

$$x = \frac{1}{\tan \alpha}$$

$$x = \cot \alpha$$

$$\alpha = \cot^{-1}(x)$$

$$\therefore \tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}(x)$$

معكوس دالة (sec) :

$$y = \sec x$$

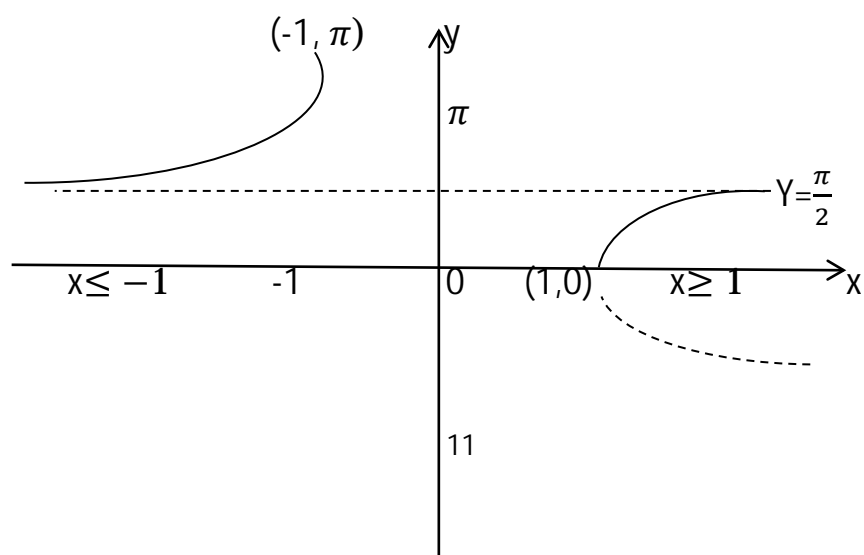
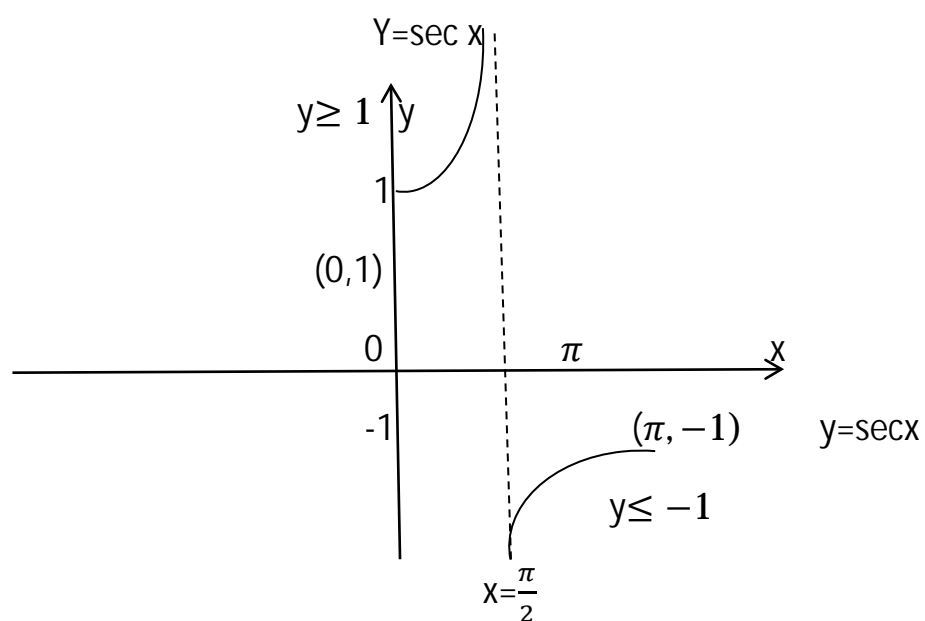
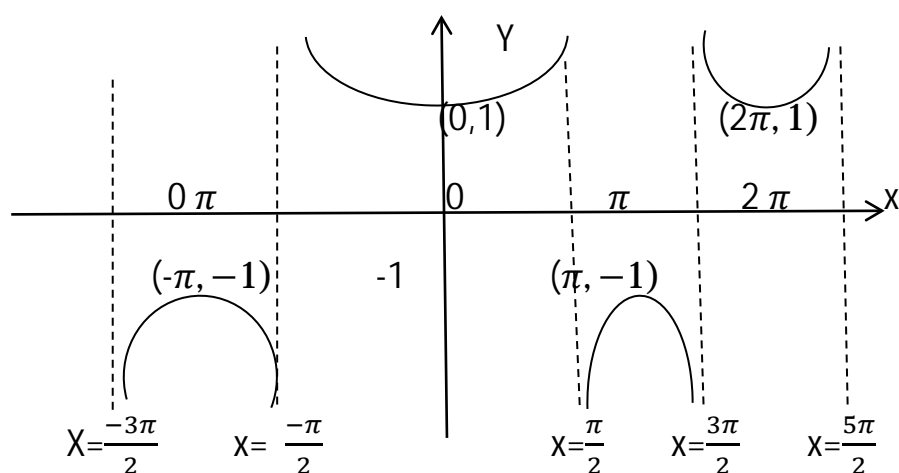
$$\sec : R \left\{ x : x = \frac{\pi}{2} + n\pi, n \in \mathbb{I} \right\} \rightarrow |y| \geq 1$$

$$\sec : [0, \pi] \setminus \left\{ \frac{\pi}{2} \right\} \rightarrow |y| \geq 1$$

$$y \geq 1 \vee y \leq -1$$

أو $\mathbb{R} \setminus (-1, 1)$

Sec is 1-1 & onto $\therefore \exists \sec^{-1}$ و $\text{Arc sec} \ni \sec^{-1}: |x| \geq 1 \rightarrow [0, \pi] \setminus \{\frac{\pi}{2}\}$



$$y = \sec^{-1}(x)$$

$$x = \sec y \text{ أو } x = \sec y$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right) \quad \text{لكل } |x| \geq 1 \quad \text{تعريف:}$$

$$\text{Let } y = \sec^{-1}(x)$$

$$x = \sec y$$

$$x = \frac{1}{\cos y}$$

$$\cos y = \frac{1}{x}$$

$$\therefore y = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\therefore \sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1}(-x) = \pi - \sec^{-1}(x) \quad \text{اثبت ان:}$$

$$\text{Let } \sec^{-1}(-x) = y \quad \text{البرهان:}$$

$$\therefore \sec y = -x \quad \text{من تعريف المعكوس}$$

$$x = \sec y$$

$$x = -\cos y \quad \cos(\pi - y) = -\cos y$$

$$x = \frac{1}{\cos(\pi - y)}$$

$$x = \sec(\pi - y)$$

$$\pi - y = \sec^{-1}(x)$$

$$\therefore y = \pi - \sec^{-1}(x)$$

$$\therefore \sec^{-1}(-x) = \pi - \sec^{-1}(x)$$

معكوس دالة (csc):

$$Y = \csc(x)$$

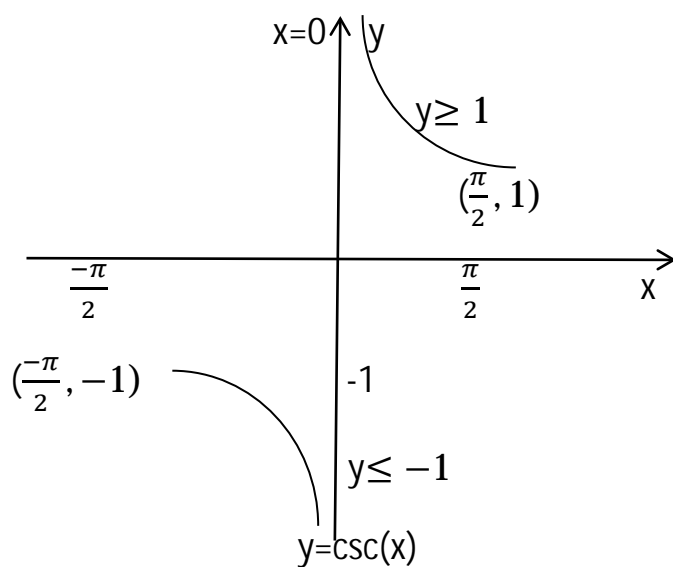
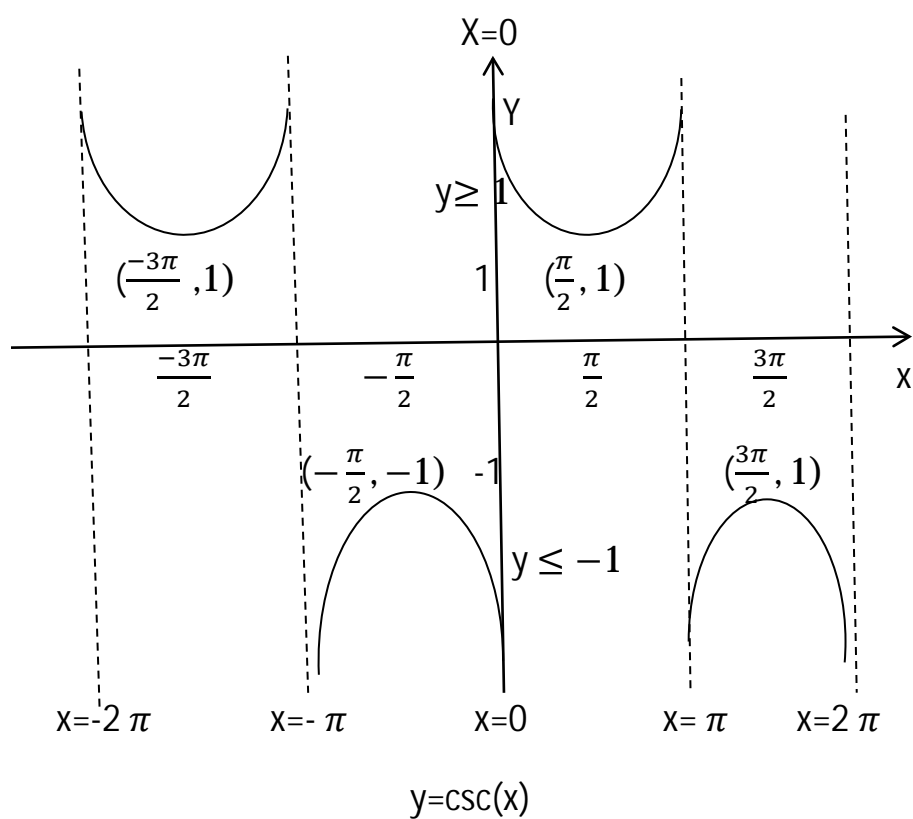
$$\text{Csc}: \mathbb{R} \setminus \{x: x = n\pi, n \in \mathbb{I}\} \rightarrow |y| \geq 1$$

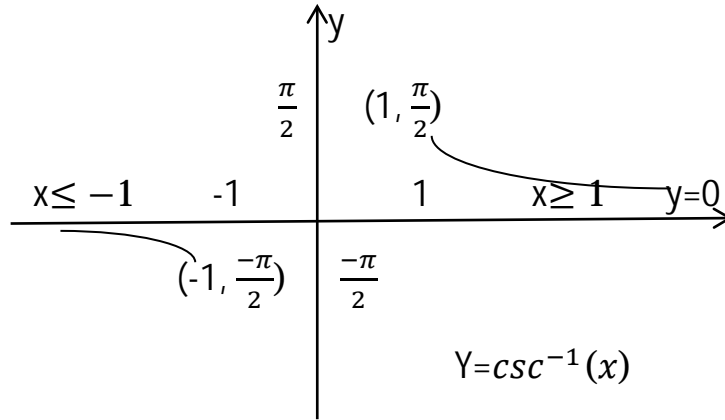
$$\text{Csc}: [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\} \rightarrow |y| \geq 1$$

$$\text{و } y \geq 1 \vee y \leq -1$$

$$\text{و } \mathbb{R} \setminus (-1, 1)$$

$$\text{Csc is 1-1 \& onto} \quad \therefore \exists \text{csc}^{-1} \ni \text{csc}^{-1}: |x| \geq 1 \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$$





برهن على ان : لكل $|x| \geq 1$ $csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$

البرهان :

$$\csc y = x$$

$$\therefore x = \frac{1}{\sin y}$$

$$\sin y = \frac{1}{x}$$

$$\therefore y = \sin^{-1}\left(\frac{1}{x}\right)$$

$$\therefore csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

بسط التعبير الاتي :- $\sec \sin^{-1}\left(\frac{-2}{3}\right)$

لتكن $\sigma = \sin^{-1}\left(\frac{-2}{3}\right)$

$$\sin \sigma = \frac{-2}{3}$$

$\sin \sigma < 0$ سالب في الربعين (3) و (4)

$$\sigma \in D_{\sin}$$

$$\sigma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$$

σ في الربعين (1) و (4)

\therefore الزاوية σ تقع في الربع الرابع

σ in the fourth quarter

حسب مبرهنة فيثاغورس

$$(3)^2 = x^2 + 4$$

$$9 = x^2 + 4$$

$$x^2 = 9 - 4 = 5$$

$$x = \pm\sqrt{5}$$

$$x = +\sqrt{5}$$

$$\sec\left(\sin^{-1}\left(\frac{-2}{3}\right)\right) = \sec\sigma = \frac{1}{\cos\sigma} = \frac{\text{الوتر}}{\text{المجاور}} = \frac{3}{\sqrt{5}} > 1$$

$$\text{Sec}:\mathbb{R}\setminus\{x: x=\frac{\pi}{2}+n\pi, n\in\mathbb{I}\} \rightarrow |y| \geq 1$$

$$\text{Sec } x \geq 1 \vee \text{sec } x \leq -1$$

$$\text{Ex) Find the value } \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\text{Sol:- } \because \sin^{-1}: [-1,1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sigma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin^{-1}\left(\frac{-1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) \quad [\text{since } \sin^{-1} \text{ odd func.}]$$

$$\text{Let } y = \sin^{-1}\left(\frac{-1}{2}\right)$$

$$\sin y = \frac{-1}{2} \Rightarrow y = \frac{-\pi}{6}$$

$\therefore \sigma$ in the four th quarter

$$\text{Ex) Find the value } \sec^{-1}(2)$$

$$\text{Sol: } \because \sec^{-1}: |x| \geq 1 \rightarrow \{0, \pi\} \setminus \left\{\frac{\pi}{2}\right\}$$

$$\therefore \sigma \in [0, \pi] \setminus \{\frac{\pi}{2}\}$$

$$\text{Let } y = \sec^{-1}(2) \Rightarrow \sec y = 2$$

$$\frac{1}{\cos y} = 2 \Rightarrow \cos y = \frac{1}{2} > 0$$

$$\Rightarrow y = \frac{\pi}{3}$$

$\therefore \sigma$ in the first quarter

مشتقات الدوال المثلثية العكسية :-

$$1- \frac{d(\sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

تعطى بعد البرهان

$$2- \frac{d(\cos^{-1}u)}{dx} = - \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$3- \frac{d(\tan^{-1}u)}{dx} = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$4- \frac{d(\cot^{-1}u)}{dx} = - \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$5- \frac{d(\sec^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$6- \frac{d(\csc^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

امثلة : جد المشتقة

$$1- f(x) = \sin^{-1}(x^2)$$

$$\dot{f}(x) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$$

$$2- y = \cos^{-1}\sqrt{x}$$

$$\dot{y} = - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2} x^{\frac{1}{2}} = \frac{-1}{2\sqrt{x}\sqrt{1-x}}$$

$$3- y = \tan^{-1}(e^x)$$

$$\dot{y} = \frac{1}{1+(e^x)^2} \cdot e^x \cdot Lne \cdot 1 = \frac{e^x}{1+e^{2x}}$$

$$4- y = \sin^{-1}\sqrt{1-\sqrt{x}}$$

$$\dot{y} = \frac{1}{\sqrt{1-(\sqrt{1-\sqrt{x}})^2}} \cdot \frac{1}{2} (1-\sqrt{x})^{-\frac{1}{2}} \cdot -\frac{1}{2} x^{-\frac{1}{2}} =$$

$$-\frac{1}{4} \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1-1+\sqrt{x}}} \frac{1}{\sqrt{(1-\sqrt{x})}}$$

$$\begin{aligned}
&= \frac{-1}{4\sqrt{x}\sqrt{1-\sqrt{x}}\sqrt{1-(1-\sqrt{x})}} = \frac{-1}{4\sqrt{x}\sqrt{1-\sqrt{x}}\sqrt{x-x+\sqrt{x}}} \\
&= \frac{-1}{4\sqrt{x}\sqrt{1-\sqrt{x}}^4\sqrt{x}} \\
5- \quad &f(x) \cot^{-1}\left(\frac{1-x}{1+x}\right) \\
&\dot{f}(x) = \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2} \cdot \frac{(1+x) \cdot -1 - (1-x)1}{(1+x)^2} \\
&= -\frac{1}{\frac{(1+x)^2+(1-x)^2}{(1+x)^2}} \cdot \frac{-1-x-1+x}{(1+x)^2} \\
&= -\frac{(1+x)^2}{1+2x+x^2+1-2x+x^2} \cdot \frac{-2}{(1+x)^2} \\
&= -\frac{1}{2+2x^2} - 2 = \frac{2}{2(1+x^2)} = \tan x \\
&\dot{f}(x) = \frac{1}{1+x^2} \\
6- \quad &f(x) = \sec^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)
\end{aligned}$$

$$\begin{aligned}
&\dot{f}(x) = \frac{1}{\left|\frac{\sqrt{1+x^2}}{x}\right| \sqrt{\left(\frac{\sqrt{1+x^2}}{x}\right)^2 - 1}} \cdot \frac{x \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x - \sqrt{1+x^2} - 1}{x^2} \\
&= \frac{1}{\frac{\sqrt{1+x^2}}{|x|} \sqrt{\frac{1+x^2}{x^2} - 1}} \cdot \frac{\frac{x}{2\sqrt{1+x^2}} \cdot 2x - \sqrt{1+x^2}}{x^2} \\
&= \frac{1}{\frac{\sqrt{1+x^2}}{|x|} \sqrt{\frac{1+x^2-x^2}{x^2}}} \cdot \frac{\frac{x^2-(1+x^2)}{\sqrt{1+x^2}}}{\frac{x^2}{1}} = \frac{1}{\frac{\sqrt{1+x^2}}{\sqrt{x^2}} \cdot \frac{1}{\sqrt{x^2}}} \cdot \frac{x-1-x^2}{\sqrt{1+x^2}} \cdot \frac{1}{x^2} \\
&= \frac{1}{\frac{\sqrt{1+x^2}}{x^2}} \cdot \frac{-1}{\sqrt{1+x^2}} \cdot \frac{1}{x^2} = \frac{x^2}{\sqrt{1+x^2}} \cdot \frac{-1}{\sqrt{1+x^2}} \cdot \frac{1}{x^2} \\
&\dot{f} = \frac{-1}{1+x^2}
\end{aligned}$$

تمارين : احسب قيمة :-

- 1- $\sin^{-1}(1) - \sin^{-1}(-1)$
- 2- $\tan^{-1}(1) - \tan^{-1}(-1)$
- 3- $\sec^{-1}(2) - \sec^{-1}(-2)$

بسط التعبيرات التالية :-

- 1- $\cos(\sin^{-1} 0.8)$
- 2- $\sin(2 \sin^{-1} 0.8)$
- 3- $\cos^{-1}(-\sin \frac{\pi}{6})$

4- $\sec^{-1}(\sec(-30^\circ))$

جد قيمة كل مما يأتي :-

1- $\sin(\cos^{-1} \frac{\sqrt{2}}{2})$

2- $\sec(\cos^{-1} \frac{1}{2})$

3- $\cos(\cos^{-1} \frac{1}{2})$

4- $\csc(\sec^{-1} 2)$

5- $\cos(\cot^{-1} 1)$

6- $\tan(\sin^{-1} (-\frac{1}{2}))$

7- $\cot(\sin^{-1} (-\frac{1}{2}))$

8- $\cot(\tan^{-1} (-\sqrt{3}))$

9- $\csc(\sin^{-1} (\frac{-\sqrt{2}}{2}))$

10- $\tan(\sec^{-1}(1))$

11- $\cot(\cos^{-1} 0)$

دالة اللوغارتم الطبيعي (Natural Logarithm function) :

$y = \ln x, x > 0$

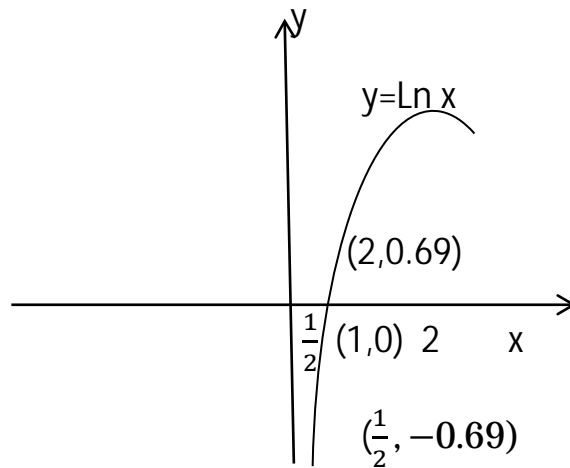
$\ln : (0, \infty) \rightarrow \mathbb{R}$

$\ln e = 1$

$\ln 1 = 0$

$\ln 2 = 0.69$

$\ln 10 = x = 1$



$y = \ln(u(x))$ تعميم

حيث $u(x)$ دالة موجبة $x \in \mathbb{R}$

مثال: $y = \ln(2x)$

$y = \ln(\cos x)$

$y = \ln(e^x) = x$

خواص الدالة اللوغارتمية :-

$$1- \ln (x.y) = \ln x + \ln y \quad , x > 0 , y > 0$$

$$2- \ln \left(\frac{x}{y}\right) = \ln x - \ln y$$

$$3- \ln\left(\frac{1}{x}\right) = -\ln x \Rightarrow \ln \left(\frac{1}{x}\right) = \ln 1 - \ln x = -\ln x$$

$$4- \ln x^a = a \ln x$$

امثلة : جد الاتي : (اذا علمت ان $\ln 2 = 0.69$)

$$\ln 16 = \ln 2^4 = 4 \ln 2 = 4 (0.69) = 2.76$$

$$\ln \sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2} \ln 2 = (0.5)(0.69) = 0.345$$

$$\ln 8 = \ln 2^3 = 3 \ln 2 = 3 (0.69) = 2.07$$

$$\ln \frac{1}{2} = -\ln 2 = -0.69$$

امثلة : برهن كلا مما يأتي :-

$$1- 2 \ln \left(\cos \frac{\sigma}{2}\right) = \ln \frac{1+\cos \sigma}{2}$$

$$\begin{aligned} \text{الطرف الايمن} &= \ln \frac{1+\cos \sigma}{2} \\ &= \ln \frac{2 \cos^2 \left(\frac{\sigma}{2}\right)}{2} \\ &= \ln \cos^2 \left(\frac{\sigma}{2}\right) \\ &= 2 \ln \cos \left(\frac{\sigma}{2}\right) \\ &= \text{الطرف الايسر} \end{aligned}$$

$$2- \ln (x + \sqrt{x^2 - 1}) = -\ln (x - \sqrt{x^2 - 1})$$

$$\begin{aligned} \text{الطرف الايسر} &= \ln (x + \sqrt{x^2 - 1}) \\ &= \ln \frac{(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})}{(x-\sqrt{x^2-1})} \\ &= \ln \frac{x^2-(x^2-1)}{x-\sqrt{x^2-1}} \\ &= \ln \frac{x^2-x^2+1}{x-\sqrt{x^2-1}} \\ &= \ln \frac{1}{x-\sqrt{x^2-1}} = \ln 1 - \ln (x - \sqrt{x^2 - 1}) \\ &= -\ln (x - \sqrt{x^2 - 1}) = \text{الطرف الايمن} \end{aligned}$$

$$3- 2 \ln \sin \sigma = \ln (1 - \cos \sigma) + \ln (1 + \cos \sigma)$$

$$\text{الطرف الايمن} = \ln (1 - \cos \sigma) + \ln (1 + \cos \sigma)$$

$$\begin{aligned}
&= \text{Ln} ((1 - \cos \sigma) (1 + \cos \sigma)) \\
&= \text{Ln} (1 - \cos^2 \sigma) \\
&= \text{Ln} \sin^2 \sigma \\
&= 2 \text{Ln} \sin \sigma = \text{الطرف الايسر}
\end{aligned}$$

$$Y=r \sin \theta, x = r \cos \theta$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2$$

$$x^2 + y^2 = 1$$

$$\frac{(x)^2}{r} + \frac{(y)^2}{r} = \frac{(r)^2}{r}, r = 1$$

$$U=u(x)$$

$$Y=a^u$$

$$\text{Ln } y = \text{Ln } a^u = u \text{Ln } a$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{du}{dx} \cdot \text{Ln } a$$

$$\dot{y} = \frac{dy}{dx} = y \cdot \text{Ln } a \frac{du}{dx}$$

$$\dot{y} = \text{Ln } a a^u \frac{du}{dx}$$

$$\log_a = \frac{\text{Ln } a}{\text{Ln } b}$$

$$\log_{10} a = \frac{\text{Ln } a}{\text{Ln } 10} = \frac{\text{Ln } a}{1} = \text{Ln } a$$

مشتقة الدالة اللوغارتمية : حيث u دالة موجبة بالنسبة الى x

$$\frac{d \text{Ln } u}{dx} = \frac{d \text{Ln } u}{du} \cdot \frac{du}{dx} \quad \text{حسب قانون السلسلة}$$

$$= \frac{1}{u} \cdot \frac{du}{dx}$$

$$\frac{d \text{Ln } u}{dx} = \frac{\frac{du}{dx}}{u}$$

دخول الدالة اللوغارتمية في الغاية

$$\lim_{x \rightarrow \infty} \text{Ln } x = + \infty$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

امثلة : جد الغاية ان وجدت :

$$1- \lim_{x \rightarrow 1} (x - \ln x) = 1 - \ln 1 = 1 - 0 = 1$$

$$2- \lim_{x \rightarrow 1} \cos(\ln x) = \cos(\ln 1) = \cos(0) = 1$$

$$3- \lim_{x \rightarrow 1} \ln x^{(x+1)} = \lim_{x \rightarrow 1} \{(x+1) \cdot \ln x\}$$

$$= \lim_{x \rightarrow 1} (x+1) \lim_{x \rightarrow 1} \ln x$$

$$= (1+1) \cdot \ln 1 = 2(0) = 0$$

$$4- \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} \quad \frac{\ln(1)}{\sigma} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{x+1}}{1} = \lim_{x \rightarrow 0} \frac{1}{x+1} = \frac{1}{0+1} = \frac{1}{1} = 1$$

امثلة : جد y'

$$1- y = \ln(x^2 + 2x)$$

$$y' = \frac{2x+2}{x^2+2x}$$

$$2- y = \ln(\tan x + \sec x)$$

$$y' = \frac{\sec^2 x + \sec x \cdot \tan x}{\tan x + \sec x} = \frac{\sec x(\sec x + \tan x)}{\tan x + \sec x}$$

$$y' = \sec x$$

$$3- y = (\ln x)^3$$

$$y' = 3(\ln x)^2 \cdot \frac{1}{x}$$

واجب (H.W.) : جد y'

$$1- y = \ln(x\sqrt{x^2 + 1})$$

$$2- y = \ln(3x\sqrt{x+2})$$

$$3- y = x \ln x - x$$

$$4- y = x^3 \ln(2x)$$

$$5- y = \frac{1}{2} \ln \frac{1+x}{1-x}$$

$$6- y = \frac{1}{3} \ln \frac{x^3}{1+x^3}$$

$$7- y = \ln \frac{x}{2+3x}$$

$$8- y = \ln(x^2 + 4) - x \tan^{-1} \frac{x}{2}$$

$$9- y = x(\ln x)^3$$

الدالة الأسسية (The exponential function)

$$y = e^x$$

$$e^x : \mathbb{R} \rightarrow (0, \infty)$$

$$e^x = \ln^{-1}(x)$$

$$e = 2.718$$

$$e^0 = (2.7)^0 = 1 > 0$$

$$e^2 = (2.7)^2 = 7.29 > 0$$

$$e^{-1} = (2.7)^{-1} = \frac{1}{2.7} > 0$$

$$e^x = \exp(x) = \text{Exp}(x) \quad \text{رمزها :}$$

$$y = e^{u(x)} \quad \text{تعميم :}$$

$$y = e^{\sin x} \quad \text{امثلة :}$$

$$y = e^{\tan^{-1} x}$$

$$y = e^{\sqrt{x}}$$

$$y = e^{3x}$$

$$y = e^{x^2}$$

$$y = \frac{1}{e^x}$$

العلاقة بين الدالة الأسسية والدالة اللوغارتمية :

$$y = e^x \Leftrightarrow x = \ln y$$

خواص الدالة الأسسية :-

$$1- e^0 = 1$$

$$2- e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$$

$$3- \frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$$

$$4- (e^x)^r = e^{rx} \quad \forall r \in R$$

$$5- e^{-x} = \frac{1}{e^x}$$

$$6- e^{Ln x} = x = Ln e^x$$

امثلة : احسب قيمة كل مما يأتي :-

$$1- Ln (e^{-x^2}) = -x^2$$

$$2- Ln \left(e^{\frac{1}{x}} \right) = \frac{1}{x}$$

$$3- e^{Ln \frac{1}{x}} = \frac{1}{x}$$

$$4- e^{2Ln x} = e^{Ln x^2} = x^2$$

$$5- \exp(Ln x - 2Ln y) \\ = \exp (Ln x - Ln y^2) = \exp \left(Ln \frac{x}{y^2} \right) = \frac{x}{y^2}$$

$$6- e^{x+Ln x} = e^x \cdot e^{Ln x} \\ = e^x \cdot x \\ = x e^x$$

دخول الدالة الاسسية في الغاية :

$$\lim_{x \rightarrow +\infty} e^x = +\infty$$

$$\lim_{x \rightarrow -\infty} e^x = 0$$

مشتقة الدالة الاسسية :

حيث $u=f(x)$ دالة قابلة الاشتقاق $y = e^u$

$$\frac{d e^u}{dx} = e^u \cdot \frac{du}{dx} \quad e=2.718$$

امثلة : جد المشتقة

$$1- y = e^{\tan^{-1} x}$$

$$y' = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

$$2- y = Ln \frac{e^x}{1+e^x}$$

$$y' = \frac{1}{\frac{e^x}{1+e^x}} \cdot \frac{(1+e^x) \cdot e^x \cdot 1 - e^x \cdot e^x \cdot 1}{(1+e^x)^2}$$

$$= \frac{1}{e^x} \cdot \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)}$$

$$= \frac{1}{e^x} \cdot \frac{e^x}{(1+e^x)}$$

$$= \frac{1}{(1+e^x)}$$

$$\tan y = e^x + \ln x$$

$$\sec^2 y \cdot \dot{y} = e^x \cdot 1 + \frac{1}{x}$$

$$\dot{y} = \frac{e^x + \frac{1}{x}}{\sec^2 y}$$

واجب بيتي (H.W.) جد \dot{y}

$$1- y = x^2 \cdot e^x$$

$$2- y = \frac{1}{2} (e^x + e^{-x})$$

$$3- y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$4- y = e^{\sin^{-1} \sqrt{x}}$$

$$5- y = (1 + 2x) \cdot e^{-2x}$$

$$6- y = (9x^2 - 6x + 2)e^{3x}$$

$$7- y = \frac{ax-1}{a^2} \cdot e^{ax} \quad a = \text{constant}$$

$$8- y = e^{-x^2}$$

$$9- y = x^2 - e^{-x^2}$$

$$10- y = e^x \cdot \ln x$$

اللوغارتم الاعتيادي $y = \log_a x$

$$\log_a x = \frac{\ln x}{\ln a}, x > 0, a > 0, a \neq 1$$

$$\log_a u = \frac{\ln u}{\ln a} \quad \text{تعميم :}$$

$$u > 0, a > 0, a \neq 1$$

الخواص :

$$1- \log_a (x \cdot y) = \log_a x + \log_a y$$

$$2- \log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$$

$$3- \log_a x^y = y \cdot \log_a x$$

$$4- \log_a a = 1 \leftrightarrow \ln e = 1$$

$$5- \log_a 1 = 0 \leftrightarrow \ln 1 = 0$$

$$\log_2 16 = \frac{\ln 16}{\ln 2} = \frac{\ln 2^4}{\ln 2} = \frac{4 \ln 2}{\ln 2} = 4 \quad \text{امثلة :}$$

$$6- \log_{\frac{1}{7}} 49 = \frac{\ln 49}{\ln \frac{1}{7}} = \frac{\ln 7^2}{-\ln 7} = \frac{2 \ln 7}{-\ln 7} = -2$$

$$\log_{10} 10 = 1$$

$$\log_{10} 100 = \log_{10} 10^2 = 2 \log_{10} 10 = 2(1) = 2$$

$$\log_{10} \frac{1}{1000} = \log_{10} 10^{-3} = -3 \log_{10=1} 10 = 1 = -3(1) = -3$$

مشتقة الدالة اللوغارتمية العامة :-

$$\log_a u = \frac{\ln u}{\ln a} \quad \text{حسب التعريف}$$

$$\frac{d}{dx} \log_a u = \frac{d}{dx} \left(\frac{\ln u}{\ln a} \right)$$

$$= \frac{1}{\ln a} \cdot \frac{d}{dx} \ln u$$

$$= \frac{1}{\ln a} \cdot \frac{\frac{du}{dx}}{u}$$

$$\frac{d \log_a u}{dx} = \frac{du | dx}{u \cdot \ln a}$$

$$u > 0, a > 0, a \neq 1$$

امثلة : جد y'

$$1- y = \log_2 (x^2 + 3x)$$

$$y' = \frac{2x+3}{(x^2+3x) \cdot \ln 2}$$

$$2- y = \log_7 (\tan x + \sin x)$$

$$y' = \frac{\sec^2 x + \cos x}{(\tan x + \sin x) \cdot \ln 7}$$

$$3- y = \ln x \cdot \log_{10} x$$

$$y' = \ln x \cdot \frac{1}{x \cdot \ln 10} + \log_{10} x \cdot \frac{1}{x}$$

$$4- y = \log_a \sin^{-1} x + \frac{x}{e^x} \quad a = \text{constant}$$

$$y' = \frac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1} x \cdot \ln a} + \frac{e^x \cdot 1 - x \cdot e^x \cdot 1}{e^{2x}}$$

واجب : جد y'

$$1- y = \log_4 \sin x$$

$$2- y = \log_a (e^x + \sin x)$$

الدالة الاسسية العامة :

$$\because x = e^{Lnx}$$

$$a = e^{Lna}$$

$$a^u = (e^{Lna})^u$$

$$a^u = e^{u Lna}$$

حيث u عدد حقيقي $-\infty < u < \infty$ والاساس $a > 0$

خواص الدالة الاسسية العامة :-

$$1- a^1 = a \quad (a > 0)$$

$$2- a^0 = 1$$

$$3- a^u \cdot a^v = a^{u+v}$$

$$4- (a^{m/n})^n = a^m$$

$$5- (a \cdot b)^u = a^u \cdot b^u \quad (a > 0, b > 0 \text{ حيث})$$

مشتقة الدالة الاسسية العامة :-

$$a^u = e^{uLna} \text{ حسب التعريف}$$

$$\frac{da^u}{dx} = \frac{d}{dx} (e^{uLna})$$

$$\frac{da^u}{dx} = e^{uLna} \cdot Lna \cdot \frac{du}{dx}$$

$$\frac{da^u}{dx} = a^u \cdot Lna \cdot \frac{du}{dx}$$

امثلة : جد المشتقة :-

$$1- y = 2^{(x^2+secx)}$$

$$\dot{y} = 2^{(x^2+secx)} \cdot Ln2(2x + secx \cdot tanx)$$

$$2- y = 4^{\sin^{-1}x}$$

$$\dot{y} = 4^{\sin^{-1}x} \cdot Ln4 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$3- y = x^\pi \cdot \pi^x = a^u$$

$$\dot{y} = x^\pi \cdot \pi^x \cdot Ln\pi \cdot 1 + \pi^x \cdot \pi x^{\pi-1} \cdot 1$$

واجب : جد المشتقة :-

- 1- $y = 5^{x^2+x-1}$
- 2- $y = 6^{\sin x + \ln x + 3}$
- 3- $y = 2^{\sec x}$
- 4- $y = 3^{\tan x}$
- 5- $y = \ln\left(\frac{x^4}{1+x^3}\right) + 7x^{2/3}$
- 6- $S = 2^{-t^2}$

امثلة :-

- 1- Find x if $3^x = 2^{x+1}$
 $\ln 3^x = \ln 2^{x+1}$
 $x \ln 3 = (x + 1) \ln 2$
 $x \ln 3 = x \ln 2 + \ln 2$
 $x \ln 3 - x \ln 2 = \ln 2$
 $(\ln 3 - \ln 2)x = \ln 2$
 $x = \frac{\ln 2}{\ln 3 - \ln 2} = \frac{\ln 2}{\ln \frac{3}{2}} = \frac{\ln 2}{\ln 1.5}$
- 2- $3^{\log_3 7} + 2^{\log_2 5} = 5^{\log_5 x}$ Find x
 $e^{\log_3 7 \cdot \ln 3} + e^{\log_2 5 \cdot \ln 2} = e^{\log_5 x \cdot \ln 5}$
 $e^{\frac{\ln 7}{\ln 3} \cdot \ln 3} + e^{\frac{\ln 5}{\ln 2} \cdot \ln 2} = e^{\frac{\ln x}{\ln 5} \cdot \ln 5}$
 $e^{\ln 7} + e^{\ln 5} = e^{\ln x}$
 $7 + 5 = x$
 $x = 12$

العلاقة بين الدالة الاسسية العامة والدالة اللوغارتمية العامة :-

$$\log_a y = x \Leftrightarrow y = a^x, a > 0, a \neq 1$$

$$\log_{10} y = x \Leftrightarrow y = 10^x$$

$$\log_e y = \ln y = x \Leftrightarrow y = e^x$$

$$\log_2 32 = 5 \Leftrightarrow 32 = 2^5$$

الدالة الاسسية واللوغارتمية هي دوال عكسية لبعضها البعض ولكن بنفس الاساس

طريقة الاشتقاق اللوغارتمي (Logarithmic differentiation)

اذا كان الاسس والاساس متغيرات

امثلة : جد المشتقة :-

$$1- y = x^{x^2} \cdot v = u, x > 0$$

$$\ln y = \ln x^{x^2}$$

$$\ln y = x^2 \cdot \ln x$$

$$\frac{\dot{y}}{y} = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x$$

$$\dot{y} = y(x + 2x \ln x)$$

$$\dot{y} = x^{x^2} (x + 2x \ln x)$$

$$2- y = (x^2 + 1)$$

$$\ln y = \ln (x^2 + 1)^{\ln x}$$

$$\ln y = \ln x \cdot \ln (x^2 + 1)$$

$$\frac{\dot{y}}{y} = \ln x \cdot \frac{2x}{x^2+1} + \ln (x^2 + 1) \cdot \frac{1}{x}$$

$$\dot{y} = y \left(\frac{2x \ln x}{x^2+1} + \frac{\ln (x^2+1)}{x} \right)$$

$$\dot{y} = (x^2 + 1)^{\ln x} \left(\frac{2x \ln x}{x^2+1} + \frac{\ln (x^2+1)}{x} \right)$$

$$3- y = (\sin x)^{\tan x}, \sin x > 0$$

$$\ln y = \ln (\sin x)^{\tan x}$$

$$\ln y = \tan x \cdot \ln (\sin x)$$

$$\frac{\dot{y}}{y} = \tan x \cdot \frac{\cos x}{\sin x} + \ln (\sin x) \cdot \sec^2 x$$

$$\dot{y} = y \left(\frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x} + \sec^2 x \cdot \ln (\sin x) \right)$$

$$\dot{y} = (\sin x)^{\tan x} (1 + \sec^2 x \cdot \ln (\sin x))$$

H.W.

$$1) y = x^x, x > 0$$

$$2) y = \sqrt{e^{x^2} + e^x}$$

$$3) x^{(\sqrt{5} + \ln x)}$$

$$4) y = x^{\ln x}$$

Chapter Five

Hyperbolic functions

الدوال الزائدية

1- $\sinh(x) = \frac{e^x - e^{-x}}{2}$

دالة الجيب الزائدي

2- $\cosh(x) = \frac{e^x + e^{-x}}{2}$

دالة الجيب تمام الزائدي

3- $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

4- $\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$\coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)}$

5- $\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}$

$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$

6- $\operatorname{csch}(x) = \frac{2}{e^x - e^{-x}}$

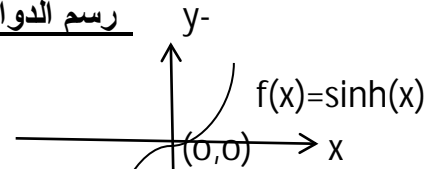
$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$

Graph hyperbolic functions :-

رسم الدوال الزائدية

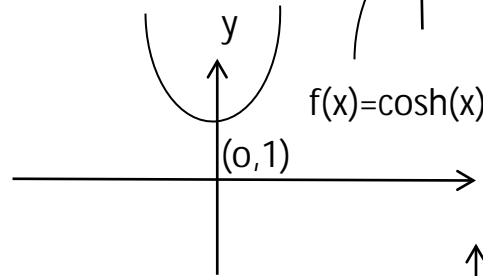
1- $f(x) = \sinh(x)$

$D_f = \mathbb{R}, R_f = \mathbb{R}$



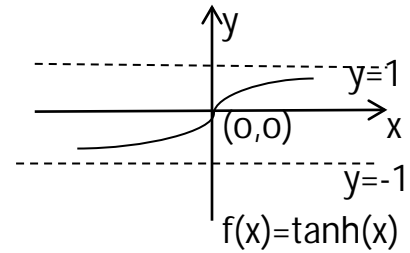
2- $f(x) = \cosh(x)$

$D_f = \mathbb{R}, R_f = [1, \infty)$



3- $f(x) = \tanh(x)$

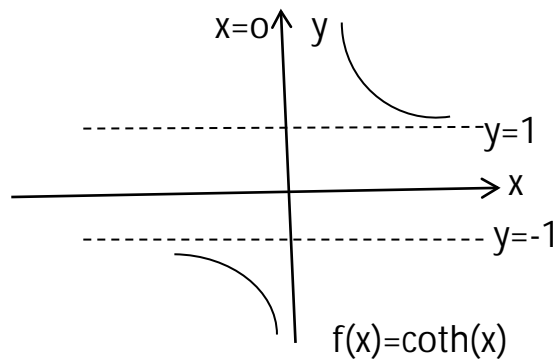
$D_f = \mathbb{R}, R_f = (-1, 1)$



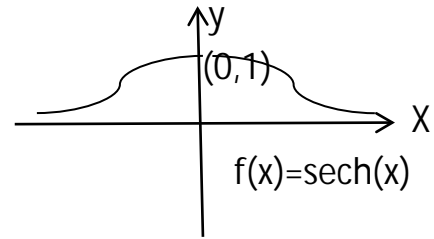
4- $f(x) = \coth(x)$

$D_f = (-\infty, 0) \cup (0, \infty)$

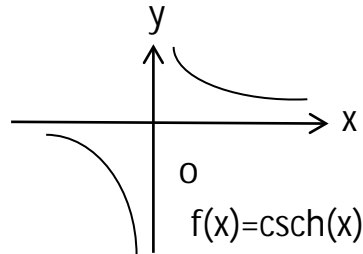
$R_f = (-\infty, -1) \cup (1, \infty)$



5- $f(x) = \operatorname{sech}(x)$
 $D_f = \mathbb{R}$, $R_f = (0, 1]$



6- $f(x) = \operatorname{csch}(x)$
 $D_f = (-\infty, 0) \cup (0, \infty)$
 $R_f = (-\infty, 0) \cup (0, \infty)$



Identities for hyperbolic functions

العلاقات بين الدوال المثلثية الزائدية

$$\left. \begin{aligned} \cosh^2(x) - \sinh^2(x) &= 1 \\ 1 - \tanh^2(x) &= \operatorname{sech}^2(x) \\ \coth^2(x) - 1 &= \operatorname{csch}^2(x) \end{aligned} \right\} \begin{aligned} \tanh^2(x) + \operatorname{sech}^2(x) &= 1 \\ \coth^2(x) - \operatorname{csch}^2(x) &= 1 \end{aligned}$$

$$\left. \begin{aligned} \cosh(-x) &= \cosh(x) \text{ زوجية} \\ \sinh(-x) &= -\sinh(x) \text{ فردية} \\ \tanh(-x) &= -\tanh(x) \text{ فردية} \end{aligned} \right\}$$

$$\left. \begin{aligned} \cosh(x) + \sinh(x) &= e^x \\ \cosh(x) - \sinh(x) &= e^{-x} \end{aligned} \right\}$$

$$\left. \begin{aligned} \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\ \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \end{aligned} \right\}$$

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 - \tanh(x)\tanh(y)}$$

$$\cosh(2x) = 2\cosh^2(x) - 1$$

$$\begin{aligned} \cosh^2(x) &= \frac{1}{2} (1 + \cosh(2x)) \\ \sinh^2(x) &= \frac{1}{2} (\cosh(2x) - 1) \end{aligned} \quad \left. \begin{array}{l} \cosh(2x) + 1 = 2\cosh^2(x) \\ \cosh(2x) - 1 = 2\sinh^2(x) \end{array} \right\}$$

$$\cosh(2x) = 2\sinh^2(x) + 1$$

$$\begin{aligned} \cosh(2x) &= \cosh^2(x) + \sinh^2(x) \\ \sinh(2x) &= 2\sinh(x)\cosh(x) \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\}$$

Derivatives of hyperbolic functions :- مشتقات الدوال الزائدية

$$\begin{aligned} 1- \frac{d \sinh(u)}{dx} &= \cosh(u) \frac{du}{dx} \\ 2- \frac{d \cosh(u)}{dx} &= \sinh(u) \frac{du}{dx} \\ 3- \frac{d \tanh(u)}{dx} &= \operatorname{sech}^2(u) \frac{du}{dx} \\ 4- \frac{d \coth(u)}{dx} &= -\operatorname{csch}^2(u) \frac{du}{dx} \\ 5- \frac{d \operatorname{sech}(u)}{dx} &= -\operatorname{sech}(u) \cdot \tanh(u) \frac{du}{dx} \\ 6- \frac{d \operatorname{csch}(u)}{dx} &= -\operatorname{csch}(u) \cdot \coth(u) \frac{du}{dx} \end{aligned} \quad \left. \begin{array}{l} \left. \begin{array}{l} 1- \\ 2- \\ 3- \end{array} \right\} \\ \left. \begin{array}{l} 4- \\ 5- \\ 6- \end{array} \right\} \end{array} \right\}$$

Examples :- finding Derivatives

- 1- $Y = \sinh(3x)$
 $y' = 3\cosh(3x)$
- 2- $Y = \cosh(5x)$
 $y' = 2\cosh(5x) \cdot \sinh(5x) \cdot 5 = 5\sinh(10x)$
- 3- $y = \tanh(2x)$
 $y' = 2\operatorname{sech}^2(2x)$
- 4- $y = \coth(\tan x)$

$$y' = -\operatorname{csch}^2(\tan x) \cdot \sec^2 x \cdot 1$$

5- $y = \operatorname{sech}^3 x$

$$y' = 3\operatorname{sech}^2(x) \cdot \operatorname{sech}(x) \cdot \tanh(x) \cdot 1$$

$$y' = -3\operatorname{sech}^3(x) \cdot \tanh(x)$$

6- $y = 4\operatorname{csch}\left(\frac{x}{4}\right)$

$$y' = 4 \cdot \operatorname{csch}\left(\frac{x}{4}\right) \cdot \coth\left(\frac{x}{4}\right) \cdot \frac{1}{4}$$

$$y' = -\operatorname{csch}\left(\frac{x}{4}\right) \cdot \coth\left(\frac{x}{4}\right)$$

Exercises:- finding derivatives :-

- 1- $f(x) = \frac{\cosh(x)}{x}$
- 2- $f(x) = e^x \cdot \cosh(x)$
- 3- $f(x) = \ln(\sinh(x^2))$
- 4- $f(x) = \tanh\left(\frac{4x+1}{5}\right)$
- 5- $f(x) = \ln(\tanh(x))$
- 6- $f(x) = e^x \cdot \tanh(2x)$
- 7- $f(x) = \tan^{-1}(\sinh^2 x)$
- 8- $f(x) = \coth\left(\frac{1}{x}\right)$
- 9- $y = \cosh^2(5x) - \sinh^2(5x)$
- 10- $\sinh y = \tan x$
- 11- $y = \sinh^2(3x)$
- 12- $\sin^{-1} x = \operatorname{sech} y$
- 13- $x = \cosh(\ln y)$
- 14- $\tan x = \tanh^2 y$
- 15- $\sinh y = \sec x$
- 16- $y = \tanh(\ln x)$
- 17- $y = \sinh(\tan^{-1} e^{3x})$
- 18- $y^2 + x \cosh y + \sinh^2 x = 50$
- 19- $y = \cot(\operatorname{csch}(e^x))$
- 20- $f(x) = \operatorname{csch}^3(\sqrt{2x})$

Inverse Hyperbolic Functions :-

الدوال الزائدية العكسية

$$x = \sinh y \Leftrightarrow y = \sinh^{-1}(x)$$

$$-\infty < x < \infty, -\infty < y < \infty$$

$$x = \cosh y \Leftrightarrow y = \cosh^{-1}(x)$$

$$x \geq 1, y \geq 0$$

$$x = \tanh y \Leftrightarrow y = \tanh^{-1}(x)$$

$$x = \coth y \Leftrightarrow y = \coth^{-1}(x)$$

$$x = \operatorname{sech} y \Leftrightarrow y = \operatorname{sech}^{-1}(x)$$

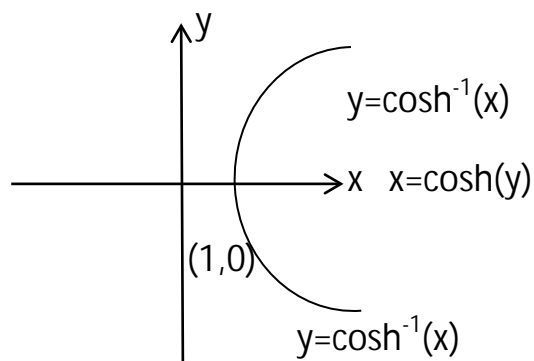
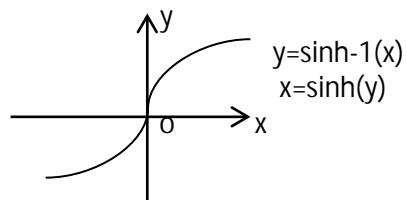
$$0 < x \leq 1, y > 0$$

$$x = \operatorname{csch} y \Leftrightarrow y = \operatorname{csch}^{-1}(x)$$

$$\operatorname{sech}^{-1}(x) = \cosh^{-1}\left(\frac{1}{x}\right)$$

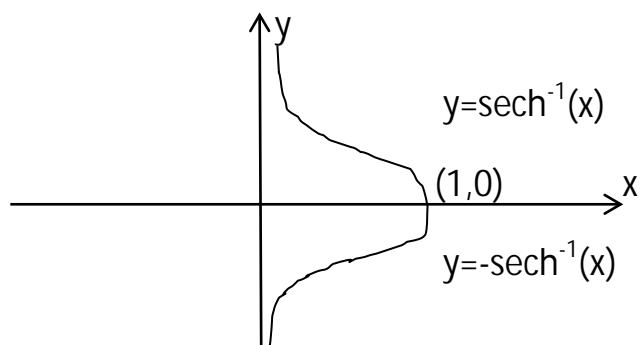
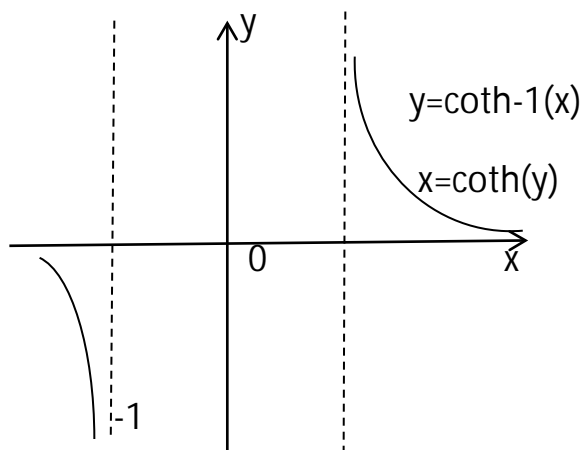
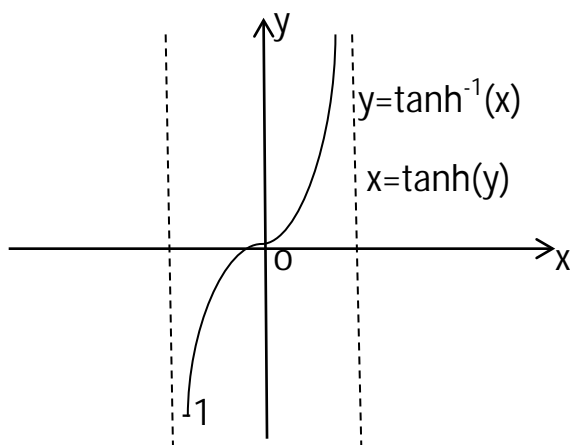
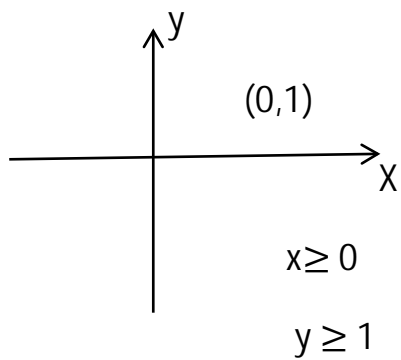
$$\operatorname{csch}^{-1}(x) = \sinh^{-1}\left(\frac{1}{x}\right)$$

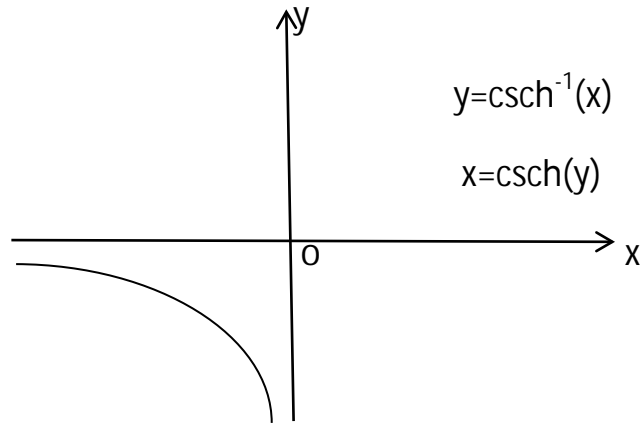
رسم الدوال الزائدية العكسية :- Graph Inverse hyperbolic functions



1-1&onto







$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) , -\infty < x < \infty , x \in \mathbb{R} \text{ لكل}$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}) , x \geq 1 \text{ لكل}$$

$$\tanh^{-1}(x) = \frac{1}{2} \ln \frac{1+x}{1-x} \quad |x| < 1 \text{ لكل}$$

$$\coth^{-1}(x) = \frac{1}{2} \ln \frac{x+1}{x-1} = \tanh^{-1}\left(\frac{1}{x}\right) , |x| > 1 \text{ لكل}$$

$$\operatorname{sech}^{-1}(x) = \ln \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) = \cosh^{-1}\left(\frac{1}{x}\right) , 0 < x \leq 1$$

$$\operatorname{csch}^{-1}(x) = \ln \left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|} \right) = \sinh^{-1}\left(\frac{1}{x}\right) , x \neq 0$$

$$\operatorname{csch}^{-1}(x) = \begin{cases} \ln \frac{1 + \sqrt{1 + x^2}}{x} , & x > 0 \\ -\ln \frac{1 + \sqrt{1 + x^2}}{-x} , & x < 0 \end{cases}$$

Chapter Six

التكامل Integration

التكامل الغير محدد The indefinite integral

تعريف يعرف التكامل الغير محدد للدالة f بـ :

$$\int f(x) dx = F(x) + c$$

حيث c هو ثابت التكامل و $F(x)$ هو عكس مشتقة الدالة $f(x)$ حيث :

$$F'(x) = f(x)$$

ويمكن تعريف التكامل على انه عكس المشتقة

Theorems :-

(1) For any rational power $r \neq -1$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c$$

(2) For any constant a

$$\int a dx = ax + c$$

(3) For any constant a

$$\int a f(x) dx = a \int f(x) dx$$

(4) $\int [f(x) \mp g(x)] dx = \int f(x) dx \mp \int g(x) dx$

\therefore from (3) and (4) we have :

$$\int [a f(x) \mp b g(x)] dx = a \int f(x) dx \mp b \int g(x) dx$$

s.t a, b constants

Examples :-

$$(1) \int x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$$

$$(2) \int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + c = \frac{x^{-2}}{-2} + c = \frac{-1}{2x^2} + c$$

$$(3) \int 3x^5 dx = 3 \int x^5 dx = 3 \cdot \frac{x^{5+1}}{5+1} + c = \cancel{3}^1 \cdot \frac{x^6}{\cancel{2}_2} + c = \frac{x^6}{2} + c$$

$$(4) \int \sqrt{x} dx = \int x^{1/2} dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} + c = \frac{x^{3/2}}{\frac{3}{2}} + c = \frac{2}{3} \sqrt{x^3} + c$$

$$(5) \int (3 + 6x^2) dx = \int 3 dx + \int 6x^2 dx = 3 \int dx + 6 \int x^2 dx$$

$$\begin{aligned}
&= 3x + \cancel{6}^2 \cdot \frac{x^3}{\cancel{3}_1} + c = 3x + 2x^2 + c \\
(6) \int (5x^4 - \frac{10}{x^2}) dx &= \int 5x^4 dx - \int \frac{10}{x^2} dx \\
&= 5 \int x^4 dx - 10 \int x^{-2} dx \\
&= \cancel{5}^1 \cdot \frac{x^5}{\cancel{5}_1} - 10 \cdot \frac{x^{-1}}{-1} + c \\
&= x^5 + \frac{10}{x} + c
\end{aligned}$$

تكامل الدالة $[f(x)]^n$ Integration of the power functions $[f(x)]^n$

$$\int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

s.t $n \neq -1$ and c any constant

Examples :-

$$\cdot \int_{[f(x)]^n}^{[5+6x]^2} \cdot 6 dx = \frac{[5+6x]^3}{3} + c$$

$$\cdot \int [1 + 5x^2]^9 \cdot x dx$$

$$\begin{aligned}
f(x) &= 1 + 5x^2 \\
f'(x) &= 10x
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{10}{10} \int [1 + 5x^2]^9 \cdot x dx &= \frac{1}{10} \int \frac{[1 + 5x^2]^9}{[f(x)]^4} \cdot f'(x) dx \\
&= \frac{1}{10} \cdot \frac{[1+5x^2]^{10}}{10} + c = \frac{[1+5x^2]^{10}}{100} + c
\end{aligned}$$

تكاملات الدوال المثلثية :-

$$1) \int \sin(x) dx = -\cos(x) + c$$

$$\int \sin(u) \frac{du}{dx} = -\cos(u) + c$$

$$2) \int \cos(x) dx = \sin(x) + c$$

$$\int \cos(u) \frac{du}{dx} = \sin(u) + c$$

$$3) \int \sec^2(x) dx = \tan(x) + c$$

$$\int \sec^2(u) \frac{du}{dx} = \tan(u) + c$$

$$\begin{aligned}
4) \int \csc^2(x) dx &= -\cot(x) + c \\
\int \csc^2(u) \frac{du}{dx} &= -\cot(u) + c \\
5) \int [\sec(x) \cdot \tan(x)] dx &= \sec(x) + c \\
\int [\sec(u) \cdot \tan(u)] \frac{du}{dx} &= \sec(u) + c \\
6) \int [\csc(x) \cdot \cot(x)] dx &= -\csc(x) + c \\
\int [\csc(u) \cdot \cot(u)] \frac{du}{dx} &= -\csc(u) + c
\end{aligned}$$

Examples :-

$$\begin{aligned}
(1) \int \cos(2x) dx &= \frac{2}{2} \int \cos(2x) dx \\
&= \frac{1}{2} \int \cos(2x) \cdot 2 dx \\
&= \frac{1}{2} \sin(2x) + c \\
(2) \int x \sin(2x^2) dx &= \frac{4}{4} \int x \cdot \sin(2x^2) dx \\
&= \frac{1}{4} \int \sin(2x^2) \cdot 4x dx \\
&= \frac{1}{4} \cdot -\cos(2x^2) + c \\
&= -\frac{1}{4} \cos(2x^2) + c \\
(3) \int (\sec^2(x) + 4x^8) dx &= \int \sec^2(x) dx + 4 \int x^8 dx \\
&= \tan(x) + 4 \cdot \frac{x^9}{9} + c \\
(4) \int 3 \csc(x) \cdot \cot(x) dx &= 3 \int [\csc(x) \cdot \cot(x)] dx \\
&= -3 \csc(x) + c \\
(5) \int \csc^2(\sqrt{x}) \cdot \frac{dx}{\sqrt{x}} & \quad (\sqrt{x})^{-} = (x^{\frac{1}{2}})^{-} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \\
\frac{2}{2} \int \csc^2(\sqrt{x}) \cdot \frac{dx}{\sqrt{x}} &= 2 \int \csc^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} dx \\
&= -2 \cot(\sqrt{x}) + c
\end{aligned}$$

التكامل المحدد :-

إذا كانت a, b واقعة ضمن الفترة $[\alpha, \beta]$ وكانت F عكس تفاضل للدالة f عليها فإن $\int_a^b f$ يعرف

كما يلي :

$$\int_a^b f = \int_a^b f(x) dx$$

$$= F(x) + c \Big|_a^b$$

التعويض بالحد الأدنى - التعويض بالحد الأعلى =

$$= F(b) + c - (F(a) + c)$$

$$= F(b) - F(a)$$

يسمى العدد $\int_a^b f$ بالتكامل المحدد للدالة f من a إلى b

وتسمى a بالحد الأدنى للتكامل و b بالحد الأعلى للتكامل كما يقال

ان للدالة f تكون قابلة للتكامل على الفترة $[a, b]$ اذ وجد $\int_a^b f$

خواص التكامل المحدد :-

$$(1) \int_a^b (k_1 f \mp k_2 g) = k_1 \int_a^b f \mp k_2 \int_a^b g$$

$$(2) \int_a^b f = \int_a^c f + \int_c^b f \exists c \in [a, b]$$

$$(3) \int_a^b f = F(b) - F(a) = -(F(a) - F(b)) = -\int_a^b f$$

$$(4) \int_a^b f = F(a) - F(a) = 0$$

$$(5) \int_{-a}^a f = 2 \int_0^a f$$

$$\text{او } \int_{-a}^0 f = 2 \int_{-a}^0 f$$

$$\int_{-a}^a f = 0$$

$$\text{Ex: } \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{\cos x}{\sin^2 x} + (4x + 1)^{1/2} \right] dx$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin^{-2} x \cos x}{u} \frac{dx}{du} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(4x + 1)^{1/2}}{u^n} \frac{dx}{du} \frac{4}{4} du = 4dx$$

$$\frac{\sin^{-1} x}{-1} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{1}{4_2} \frac{(4x+1)^{3/2}}{3/2} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$- \frac{1}{\sin x} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{1}{6} \sqrt{(4x + 1)^3} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\int_a^b f = F(b) - F(a)$$

$$-\frac{1}{\sin \frac{\pi}{2}} - \left(-\frac{1}{\sin \frac{\pi}{4}}\right) + \frac{1}{6} \sqrt{\left(4\frac{\pi}{2} + 1\right)^3} - \frac{1}{6} \sqrt{\left(4\frac{\pi}{4} + 1\right)^3}$$

$$-\frac{1}{1} + \frac{1}{\frac{1}{\sqrt{2}}} + \frac{1}{6} \sqrt{(2\pi + 1)^3} - \frac{1}{6} \sqrt{(\pi + 1)^3}$$

تطبيقات على التكامل المحدد :-

المساحة تحت المنحني : (Area under a curve)

لتكن A المساحة المحددة بالمنحني $y=f(x)$ ومحور x ($y=0$) والمستقيمان $x=a$ و $x=b$ فإن :-

$$A_a^b = \left| \int_{x=a}^{x=b} f(x) dx \right|$$

وإذا كانت المساحة A محددة بالمنحني $x=g(y)$ ومحور y ($x=0$) والمستقيمتين $y=c$ و $y=d$ فإن :-

$$A_c^d = \left| \int_{y=c}^{y=d} g(y) dy \right|$$

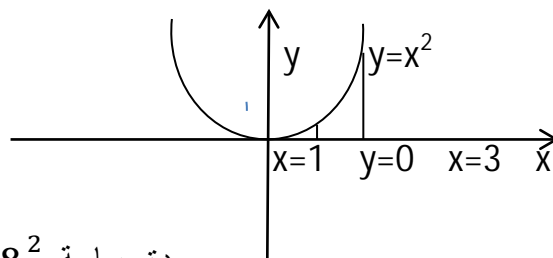
Ex1) Find the area bounded by the x- axis and the curve $y=x^2$ & $x=1$ & $x=3$

احسب مساحة المنطقة المحددة بالمنحني والمستقيمان $x=3$ و $x=1$

$$A_1^3 = \left| \int_{x=1}^{x=3} f(x) dx \right|$$

$$= \left| \int_1^3 x^2 dx \right| = \left| \left[\frac{x^3}{3} \right]_1^3 \right|$$

$$= \left| \frac{(3)^3}{3} - \frac{(1)^3}{3} \right| = \left| 9 - \frac{1}{3} \right| = \left| 8\frac{2}{3} \right| = 8\frac{2}{3} \text{ وحدة مساحة}$$



Ex2) Find the area bounded by the x-axis and the curve $y=6-x-x^2$

المحور $x \Leftarrow y=0$

$$y_1=y_2$$

$$(6-x-x^2=0) \quad x-1$$

$$x^2+x-6=0 \Rightarrow (x+3)(x-2)=0 \Rightarrow x=-3, x=2$$

نقاط التقاطع مع المحور x

$$A_{-3}^2 = \left| \int_{-3}^2 f(x) dx \right| = \left| \int_{-3}^2 (6 - x - x^2) dx \right|$$

$$= \left| \int_{-3}^2 6 dx - \int_{-3}^2 x dx - \int_{-3}^2 x^2 dx \right|$$

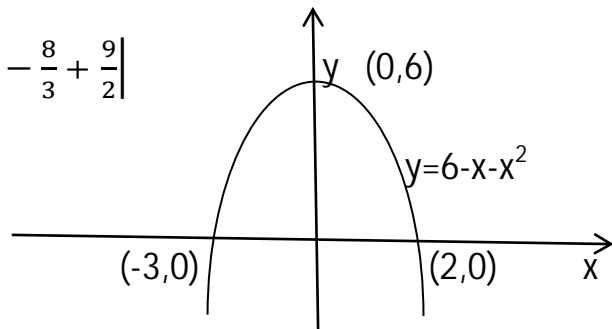
$$= \left| 6x - \frac{x^2}{2} - \frac{x^3}{3} \right|_{-3}^2$$

$$= \left| \left(12 - 2 - \frac{8}{3} \right) - \left(-18 - \frac{9}{2} + \frac{27}{3} \right) \right|$$

$$= \left| 10 - \frac{8}{3} + 18 + \frac{9}{2} - 9 \right| = \left| 19 - \frac{8}{3} + \frac{9}{2} \right|$$

$$= \left| \frac{114 - 16 + 27}{6} \right|$$

$$= \left| \frac{114 + 11}{6} \right| = \left| \frac{125}{6} \right|$$



Ex3) Find the area bounded by the line $x+y=1$ and the coordinate axes

$$f(x)=y=1-x$$

$$x=0 \Rightarrow y=1 \quad (0,1)$$

$$y=0 \Rightarrow x=1 \quad (1,0)$$

$$y_1=1-x, y_2=0$$

$$y_1=y_2$$

$$1-x=0 \Rightarrow x=1$$

$$A_0^1 = \left| \int_{x=0}^{x=1} f(x) dx \right| = \left| \int_0^1 (1-x) dx \right| = \left| \int_0^1 dx - \int_0^1 x dx \right| = \left| x - \frac{x^2}{2} \right|_0^1$$

$$= \left| \left(1 - \frac{1}{2} \right) - 0 \right| = \left| \frac{1}{2} \right| = \frac{1}{2} \quad \text{وحدة مساحة}$$

$$x_1=1-y, x_2=0$$

$$x_1=x_2$$

$$1-y=0 \Rightarrow y=1$$

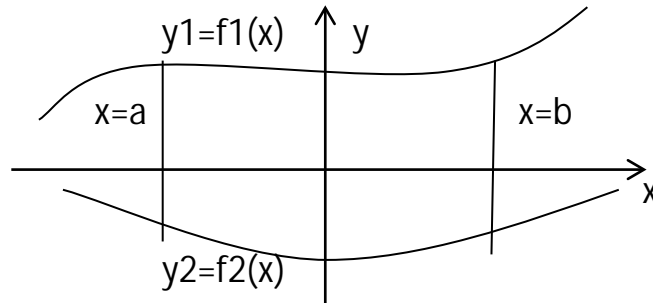
$$x=g(y)=1-y$$

$$A_0^1 = \left| \int_0^1 g(y) dy \right| = \left| \int_0^1 (1-y) dy \right| = \left| \int_0^1 dy - \int_0^1 y dy \right|$$

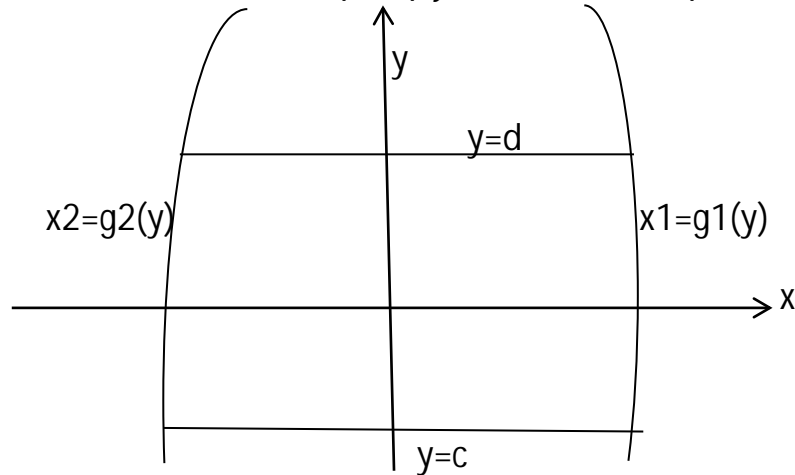
$$= \left| y - \frac{y^2}{2} \right|_0^1 = \left| \left(1 - \frac{1}{2} \right) - 0 \right| = \left| \frac{1}{2} \right| = \frac{1}{2} \quad \text{وحدة مساحة}$$

المساحة المحصورة بين منحنين (Area between two curves)

$$A_a^b = \left| \int_{x=a}^{x=b} (f_1(x) - f_2(x)) dx \right| = \left| \int_{x=a}^{x=b} (y_1 - y_2) dx \right|$$



$$A_c^d = \left| \int_{y=c}^{y=d} (g_1(y) - g_2(y)) dy \right| = \left| \int_{y=c}^{y=d} (x_1 - x_2) dy \right|$$



Ex4) find the area bounded by the curve $y=2-x^2$ and the line $y=-x$

$$y_1=y_2$$

$$2-x^2=-x$$

$$x^2-x-2=0 \Rightarrow (x-2)(x+1)=0$$

$$x=2, x=-1$$

$$y=-x \quad \text{أو} \quad y=2-x^2$$

$$x=-1 \Rightarrow y=1 \quad x=-1 \Rightarrow y=2-1=1$$

$$x=2 \Rightarrow y=-2 \quad x=2 \Rightarrow y=2-4=-2$$

$$\begin{aligned}
A_{-1}^2 &= \left| \int_{x=-1}^{x=2} (f_1(x) - f_2(x)) dx \right| = \left| \int_{-1}^2 (y_1 - y_2) dx \right| \\
&= \left| \int_{-1}^2 (2 - x^2 - (-x)) dx \right| = \left| \int_{-1}^2 (2 - x^2 + x) dx \right| \\
&= \left| \int_{-1}^2 2 dx - \int_{-1}^2 x^2 dx + \int_{-1}^2 x dx \right| = \left| 2x - \frac{x^3}{3} + \frac{x^2}{2} \right|_{-1}^2 \\
&= \left| \left(4 - \frac{8}{3} + \frac{4}{2} \right) - \left(-2 + \frac{1}{3} + \frac{1}{2} \right) \right| \\
&= \left| \left(6 - \frac{8}{3} \right) - \left(-\frac{12+2+3}{6} \right) \right| \\
&= \left| \frac{18-8}{3} - \frac{-7}{6} \right| = \left| \frac{10}{3} + \frac{7}{6} \right| = \left| \frac{20+7}{6} \right| = \left| \frac{27}{6} \right| = 4\frac{3}{2} \\
&= 4\frac{1}{20}
\end{aligned}$$

Ex(2)

أوجد مساحة المنطقة المثلثية الشكل في الربع الاول والمحددة بالمحور y والمنحنيين $y=\sin x$ و $y=\cos x$ ؟

$$y_1=y_2$$

$$\cos x = \sin x$$

$$x = \frac{\pi}{4}$$

$$A_0^{\frac{\pi}{4}} = \left| \int_0^{\frac{\pi}{4}} (f_1(x) - f_2(x)) dx \right|$$

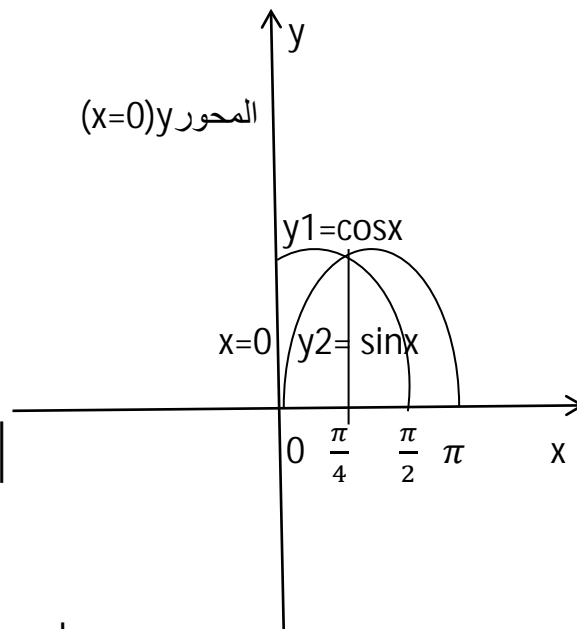
$$= \left| \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx \right|$$

$$= \left| \int_0^{\pi/4} \cos x dx - \int_0^{\pi/4} \sin x dx \right|$$

$$= \left| \sin x \Big|_0^{\pi/4} + \cos x \Big|_0^{\pi/4} \right|$$

$$= \left| \left(\sin \frac{\pi}{4} - \sin 0 \right) + \left(\cos \frac{\pi}{4} - \cos 0 \right) \right|$$

$$= \left| \left(\frac{1}{\sqrt{2}} - 0 \right) + \left(\frac{1}{\sqrt{2}} - 1 \right) \right| = \left| \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right| = \left| \frac{2}{\sqrt{2}} - 1 \right|$$



$$= \sqrt{2} - 1 = |1.4 - 1| = |0.4| = 0.4$$

(1) التكامل بطريقة التجزئة :

$$d(u.v)=u.dv+v.du$$

$$\int u . dv = \int [d(u . v) - v . du] \quad \text{نوزع التكامل}$$

$$\int u . dv = u . v - \int v . du$$

$$\underline{\text{Ex:-}} \int \frac{x^3}{(x^2+1)^{3/2}} dx$$

$$\int x^2 \frac{x}{(x^2+1)^{3/2}} dx$$

$$\text{Let } u=x^2 \Rightarrow du = 2x dx$$

$$dv = \frac{x}{(x^2+1)^{3/2}} dx \Rightarrow v = \frac{1}{2} \int (x^2 + 1)^{-3/2} . 2x dx$$

$$= \frac{1}{2} \frac{(x^2+1)^{-1/2}}{-\frac{1}{2}}$$

$$= \frac{-1}{(x^2+1)^{1/2}}$$

$$\int u . dv = u . v - \int v . du$$

$$\int \frac{x^3 dx}{(x^2+1)^{1/2}} = x^2 . \frac{-1}{(x^2+1)^{1/2}} - \int \frac{-1}{(x^2+1)^{1/2}} . 2x dx$$

$$= \frac{-x^2}{(x^2+1)^{1/2}} + \int (x^2 + 1)^{-1/2} . 2x dx$$

$$= \frac{-x^2}{(x^2+1)^{1/2}} + \frac{(x^2+1)^{1/2}}{\frac{1}{2}} + c$$

$$= \frac{-x^2}{\sqrt{x^2+1}} + 2\sqrt{x^2 + 1} + c$$

(2) التكامل بطريقة تجزئة الكسور :-

إذا كان البسط اكبر او يساوي المقام فأننا نقوم بعملية القسمة الطويلة ثم نكامل كما موضح في المثال التالي :

$$\text{Ex: } \int \frac{4+x^2}{8+x}$$

نرتب حدود x من الاعلى الى الادنى

$$\int \frac{x^2+4}{x+8} dx$$

$$\int \left[(x-8) + \frac{68}{x+8} \right] dx$$

$$\int \frac{(x-8)dx}{u} + 68 \int \frac{1}{x+8} \frac{dx}{u}$$

$$\frac{(x-8)^2}{2} + 68 \ln|x+8| + c$$

$$\begin{array}{r} x-8 \\ \hline x+8 \sqrt{x^2+4} \\ \sqrt{x^2+8x} \\ \hline -8x+4 \\ \hline \pm 8x \pm 64 \\ \hline \text{الباقى } 68 \end{array}$$

اما اذا كان المقام اكبر من البسط فأننا نقوم بتجزئة الكسور الى مجموعة من الكسور يسهل علينا ايجاد تكامله .

ملاحظة (1) اذا كان العامل بالصورة $\frac{1}{(x-a)^n}$ حيث $n \in I^+$ فان تجزئته تكون بالشكل :

$$\frac{1}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$$

(2) اما اذا كان العامل بالصورة ax^2+bx+c ولا يمكن تحليله فان بسطه يكون بالشكل $Ax+B$ اي تقليل درجة واحدة .
كما موضح في المثال التالي :

$$\text{Ex: } \int \frac{4-2x}{(x^2+1)(x-1)^2} dx$$

$$\begin{aligned} \frac{4-2x}{(x^2+1)(x-1)^2} &= \frac{Ax+B}{x^2+1} + \frac{c}{x-1} + \frac{D}{(x-1)^2} \\ &= \frac{(Ax+B)(x-1)^2 + c(x^2+1)(x-1) + D(x^2+1)}{(x^2+1)(x-1)^2} \\ &= \frac{(Ax+B)(x^2-2x+1) + (cx^2+c)(x-1) + Dx^2+D}{(x^2+1)(x-1)^2} \\ &= \frac{Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+cx^3-cx^2+cx-c+Dx^2+D}{(x^2+1)(x-1)^2} \\ &= \frac{(A+c)x^3 + (-2A+B-c+D)x^2 + (A-2B+c)x + (B-c+D)}{(x^2+1)(x-1)^2} \end{aligned}$$

$$A+c=0 \dots (1)$$

$$-2A + B - c + D = 0$$

$$2A+B-c+D=0 \dots (2) \quad \begin{array}{l} \hline \mp B \pm c \mp D = \mp 4 \\ \hline -2A = -4 \end{array}$$

$$\therefore A = 2$$

$$A-2B+c=-2 \dots (3)$$

$$B-c+D=4 \dots (4)$$

$$C=-2 \Leftarrow c = -A \dots (1) \text{ من معادلة}$$

$$B=1 \Leftarrow 2-2B+(-2) = -2 \dots (3) \text{ من معادلة}$$

$$D=1 \Leftarrow 1-(-2)+D=4 \dots (4) \text{ من معادلة}$$

$$\begin{aligned} \therefore \frac{4-2x}{(x^2+1)(x-1)^2} &= \frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2} \\ \int \frac{4-2x}{(x^2+1)(x-1)^2} dx &= \int \left[\frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2} \right] dx \\ &= \int \frac{2x+1}{x^2+1} dx - 2 \int \frac{dx}{x-1} + \int \frac{1}{(x-1)^2} dx \\ &= \int \frac{2x}{x^2+1} \frac{dx}{u} + \int \frac{dx}{x^2+1} - 2 \int \frac{dx}{x-1} + \int \frac{(x-1)^{-2} dx}{u} \\ &= \ln |x^2 + 1| + \tan^{-1}(x) - 2 \ln |x - 1| + \frac{(x-1)^{-1}}{-1} + c \\ &\text{أو } \ln |x^2 + 1| - \cot^{-1}(x) - 2 \ln |x - 1| + \frac{(x-1)^{-1}}{-1} + c \end{aligned}$$

(3) التكامل باستخدام التعويضات المثلثية : $a^2 + u^2, \sqrt{u^2 - a^2}, \sqrt{a^2 - u^2}$

- (1) $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + c$ أو $-\cos^{-1} \frac{u}{a} + c$
- (2) $\int \frac{du}{\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + c$ أو $\frac{-1}{a} \csc^{-1} \frac{u}{a} + c$
- (3) $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c$ أو $-\frac{1}{a} \cot^{-1} \frac{u}{a} + c$

يتم تعويض عن u بالاعتماد على شكل الحد :-

(1) إذا كان $\sqrt{a^2 - u^2}$ نستخدم التعويض $u = a \sin \theta$.

$$\text{Ex: } \int \frac{dx}{\sqrt{9-x^2}} \quad \sqrt{9-x^2} = \sqrt{a^2 - u^2} \Rightarrow a = 3, u = x$$

$$\text{Let } u = a \sin \theta$$

$$x = 3 \sin \theta \Rightarrow dx = 3 \cos \theta d\theta$$

$$\int \frac{3 \cos \theta d\theta}{3 \cos \theta} \quad \sqrt{9-x^2} = \sqrt{9-9\sin^2 \theta}$$

$$= \sqrt{9 \cos^2 \theta}$$

$$\int d\theta = 3 \cos \theta$$

$$\theta + c$$

$$\sin^{-1} \frac{x}{3} + c$$

$$x = 3 \sin \theta \Rightarrow \sin \theta = \frac{x}{3} \Rightarrow \theta = \sin^{-1} \frac{x}{3}$$

(2) إذا كان $a^2 + u^2$ نستخدم التعويض $u = a \tan \theta$.

Ex: $\int \frac{dx}{\sqrt{25+x^2}}$

$$\sqrt{25+x^2} = \sqrt{a^2+u^2} \Rightarrow a=5, u=x$$

$$\text{Let } u = a \tan \theta$$

$$x = 3 \sin \theta \Rightarrow dx = 3 \cos \theta d\theta$$

$$\int \frac{5 \sec^2 \theta d\theta}{5 \sec \theta}$$

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2 \theta} = \sqrt{9(1-\sin^2 \theta)}$$

$$= \sqrt{9\cos^2 \theta}$$

$$\int d\theta$$

$$= 3 \cos \theta$$

$$\theta + c$$

$$x = 3 \sin \theta \Rightarrow \sin \theta = \frac{x}{3} \Rightarrow \theta = \sin^{-1} \frac{x}{3}$$

$$\sin^{-1} \frac{x}{3} + c$$

(2) إذا كان $a^2 + u^2$ نستخدم التعويض $u = a \tan \theta$.

Ex: $\int \frac{dx}{\sqrt{25+x^2}}$

$$\sqrt{25+x^2} = \sqrt{a^2+u^2} \Rightarrow a=5, u=x$$

$$\text{Let } u = a \tan \theta$$

$$x = 5 \tan \theta \Rightarrow dx = 5 \sec^2 \theta d\theta$$

$$\int \frac{5 \sec^2 \theta d\theta}{5 \sec \theta}$$

$$\sqrt{25+x^2} = \sqrt{25+25\tan^2 \theta}$$

$$\int \sec \theta d\theta$$

$$= \sqrt{25(1+\tan^2 \theta)} = \sqrt{25 \sec^2 \theta}$$

$$\ln |\sec \theta + \tan \theta| + c$$

$$= 5 \sec \theta$$

$$\ln \left| \frac{\sqrt{25+x^2}}{5} + \frac{x}{5} \right| + c$$

$$\sec \theta = \frac{\sqrt{25+x^2}}{5}, \tan \theta = \frac{x}{5}$$

(3) إذا كان $\sqrt{a^2 + u^2}$ نستخدم التعويض $u = a \sec \theta$.

Ex: $\int \frac{dx}{x\sqrt{x^2-16}}$

$$\sqrt{x^2-16} = \sqrt{u^2-a^2}, u=x, a=4$$

$$\text{Let } u = a \sec \theta$$

$$x = 4 \sec \theta \Rightarrow dx = 4 \sec \theta \tan \theta d\theta$$

$$\begin{aligned}
&= \int \frac{4 \sec \theta \tan \theta d\theta}{4 \sec \theta \cdot 4 \tan \theta} & \sqrt{x^2 - 16} &= \sqrt{16 \sec^2 \theta - 16} \\
&= \frac{1}{4} \int d\theta & &= \sqrt{16 (\sec^2 \theta - 1)} \\
&= \frac{1}{4} \theta + c & &= \sqrt{16 \tan^2 \theta} = 4 \tan \theta \\
&= \frac{1}{4} \sec^{-1} \frac{x}{4} + c & x = 4 \sec \theta \Rightarrow \sec \theta &= \frac{x}{4} \Rightarrow \theta = \sec^{-1} \frac{x}{4}
\end{aligned}$$

(4) التكامل بطريقة اكمال المربع :-

إذا كان من الممكن تحويل الدالة $f(x) = ax^2 + bx + c$ حيث $a \neq 0$ الى الشكل $au^2 + B$ باستخدام طريقة اكمال المربع وكالاتي :-

$$ax^2 + bx + c = a \left(x^2 + \frac{b}{a}x \right) + c$$

بأضافة وطرح $(\text{نصف معامل } x) \text{ للقوس اعلاه}$

$$\begin{aligned}
&= a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) + c - \frac{b^2}{4a} \\
&= a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a} \\
&= a u^2 + B
\end{aligned}$$

$$B = c - \frac{b^2}{4a} \text{ \& } u = x + \frac{b}{2a} \quad \text{حيث ان :}$$

ملاحظة :- ان استخدام هذا الاسلوب يكون عندما :

- (1) لا يمكن تحليل الحد $ax^2 + bx + c$
- (2) عند اختفاء الثابت c
- (3) عندما يكون الحد $ax^2 + bx + c$ واقعا تحت جذر اي (مرفوع لقوى كسرية) .

والسبب في تحويل الحد $ax^2 + bx + c$ الى الصيغة $au^2 + B$ وذلك لغرض استخدام التكامل بالتعويضات المثلثية او التكامل المباشر ان امكن .

Ex: $\int \frac{dx}{\sqrt{2x - x^2}}$ جد التكامل باستخدام طريقة اكمال المربع

$$\begin{aligned}
\sqrt{2x - x^2} &= \sqrt{-(x^2 - 2x)} = \sqrt{-(x^2 - 2x + 1 - 1)} \\
&= \sqrt{-(x^2 - 2x - 1) - 1} \\
&= \sqrt{-(x - 1)^2 - 1}
\end{aligned}$$

$$= \sqrt{1 - (x - 1)^2}$$

$$\sqrt{a^2 - u^2}$$

$$a=1, u=x-1 \Rightarrow du = dx$$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

$$= \sin^{-1}(x-1) + c$$

$$= -\cos^{-1}(x-1) + c$$

$$\int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + c$$

$$= -\cos^{-1} u + c$$

Double Integral 1-A

$$(1) \int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y \, dx \, dy$$

$$= \int_0^{\sqrt{2}} yx \Big|_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} dy$$

$$= \int_0^{\sqrt{2}} [(y \cdot \sqrt{4-2y^2}) - (y \cdot -\sqrt{4-2y^2})] dy$$

$$= \int_0^{\sqrt{2}} 2y \sqrt{4-2y^2} dy = \frac{2}{-2} \int_0^{\sqrt{2}} -2y \cdot \sqrt{4-2y^2} dy$$

$$= 2 \cdot \frac{1}{-4} \left[\frac{(4-2y^2)^{3/2}}{\frac{3}{2}} \right]_0^{\sqrt{2}} = \frac{-1}{2} \int_0^{\sqrt{2}} \sqrt{4-2y^2} \cdot 4y dy$$

$$= -\frac{1}{3} (4-2y^2)^{3/2} \Big|_0^{\sqrt{2}}$$

$$= -\frac{1}{3} [0 - (4)^{3/2}] = \frac{8}{3}$$

$$(2) \int_0^1 \int_0^{e^x} dy \, dx$$

$$= \int_0^1 y \Big|_0^{e^x} dx = \int_0^1 e^x dx = e^x \Big|_0^1 = e^1 - e^0 = e - 1$$

$$(3) \int_{-1}^1 \int_0^2 (1 - 6x^2y) dx \, dy$$

$$\int_{-1}^1 [\int_0^2 (1 - 6x^2y) dx] dy = \int_{-1}^1 (x - 2x^3y) \Big|_0^2 dy = \int_{-1}^1 (2 - 16y) dy$$

$$= -2y - 8y^2 \Big|_{-1}^1 = 2^{-6} - 8 - (-2^{-10} - 8) = -2^{-6+10} - 8 + 2 + 8 = 4$$

Triple Integral – Volume

$$\begin{aligned} 1- \int_0^2 \int_0^{4-y^2} \int_0^y dz dx dy &= \int_0^2 \int_0^{4-y^2} z \Big|_0^y dx dy = \int_0^2 \int_0^{4-y^2} (y - 0) dx dy \\ &= \int_0^2 yx \Big|_0^{4-y^2} dy = \int_0^2 (y * (4 - y^2) - 0) dy = \int_0^2 (4y - y^3) dy \\ &= 2y^2 - \frac{y^4}{4} \Big|_0^2 = 8 - 4 = 4 \end{aligned}$$

$$\begin{aligned} 2- \int_{-2}^1 \int_0^{2x} \int_0^{2-x} dz dy dx &= \int_{-2}^1 \int_0^{2x} z \Big|_0^{2-x} dy dx = \int_{-2}^1 \int_0^{2x} (2 - x) dy dx \\ &= \int_{-2}^1 (2y - xy) \Big|_0^{2x} dx = \int_{-2}^1 (4x - 2x^2) dx \\ &= 2x^2 - \frac{2}{3} x^3 \Big|_{-2}^1 = (2 - \frac{2}{3}) - (8 - \frac{-8*2}{3}) \\ &= 2 - \frac{2}{3} - (8 - \frac{16}{3}) \quad \quad \quad -(8 + \frac{16}{3}) \\ &= -6 - \frac{18}{3} = -6 - 6 = -12 \\ &= \frac{-18-18}{3} = \frac{-36}{3} = -12 \end{aligned}$$

$$\begin{aligned} 3- \int_{-1}^1 \int_0^{1-x} \int_{4x^2}^{5-x^2} dz dy dx &= \int_{-1}^1 \int_0^{1-x} z^{5-x^2} dy dx \\ &= \int_{-1}^1 \int_0^{1-x} (5 - x^2 - 4x^2) dy dx = \int_{-1}^1 \int_0^{1-x} (5 - 5x^2) dy dx \\ &= \int_{-1}^1 (5y - 5x^2y) \Big|_0^{1-x} dx \\ &= \int_{-1}^1 (5(1-x) - 5x^2(1-x)) dx = \int_{-1}^1 (5 - 5x - 5x^2 + 5x^3) dx \\ &= 5x - \frac{5x^2}{2} - \frac{5}{3} x^3 + \frac{5}{4} x^4 \Big|_{-1}^1 \end{aligned}$$

$$\begin{aligned}
& 4- \int_0^1 \int_{-\frac{1}{2}}^0 \int_{x-y}^{x+y+1} dz \, dy \, dx \\
& \int_0^1 \int_{-\frac{1}{2}}^0 z \Big|_{x-y}^{x+y+1} dy \, dx = \int_0^1 \int_{-\frac{1}{2}}^0 [(x+y+1) - (x-y)] dy \, dx \\
& = \int_0^1 \int_{-\frac{1}{2}}^0 (1+2y) dy \, dx = \int_0^1 (y+y^2) \Big|_{-\frac{1}{2}}^0 dx \\
& = \int_0^1 0 - \left(-\frac{1}{2} + \frac{1}{4} \right) dx = \int_0^1 \frac{1}{4} dx = \frac{1}{4} x \Big|_0^1 \\
& = \frac{1}{4}
\end{aligned}$$

H.W

$$(1) \int_0^A \int_0^x x \sin y \, dy \, dx$$

$$(2) \int_{-2}^1 \int_{-y^2}^{y-2} dx \, dy$$

$$(3) \int_{-2}^0 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} y \, dy \, dx$$

$$(4) \int_{-1}^1 \int_{2y\sqrt{1-y^2}}^{3y\sqrt{1-y^2}} \int_{8+x^2+9y^2}^{x^2+9y^2} dZ \, dx \, dy$$

$$(5) \int_{-3}^3 \int_{-\frac{9-x^2}{3}}^{\frac{9-x^2}{3}} \int_{18-x^2-9y^2}^{8+x^2+9y^2} dZ \, dx \, dy$$

Chapter Eight

Conic Sections

The distance between two points $p_1 (x_1, y_1)$ and $p_2 (x_2, y_2)$ is :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance between point $p(x,y)$ and line $Ax+By+c=0$ is :

$$d = \frac{|Ax+By+c|}{\sqrt{A^2+B^2}}$$

1- The circle

The set of points in a plane whose distance from some fixed center point is constant radius value .

The general equation for circle of second degree is :

$$(x-h)^2 + (y-k)^2 = r^2$$

Let $c(h,k)$ and r are the center and the radius of circle respectively . if $p(x,y)$ be a point on this circle then the standard equation for circle is :

$$cp=r \text{ where } r > 0$$

and
$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

Note : if the center of circle is $(0,0)$ then $x^2 + y^2 = r^2$

Example 1 : Find the center and radius of the circle has the following equation : $x^2 + y^2 - 4x + 6y - 3 = 0$

Solution : $(x^2 - 4x + 4) + (y^2 + 6y + 9) = 4 + 9 + 3$

$$(x-2)^2 + (y+3)^2 = 16$$

The center is $(2,-3)$ and the radius equal 4 .

Example 2 : Find the equation of circle if the center $(-1,1)$ and through tangent the line $x+2y=4$

Solution : $d = \frac{|Ax+By+c|}{\sqrt{A^2+B^2}} = \frac{|1(-1)+2(1)-4|}{\sqrt{1^2+2^2}} = \frac{3}{\sqrt{5}}$

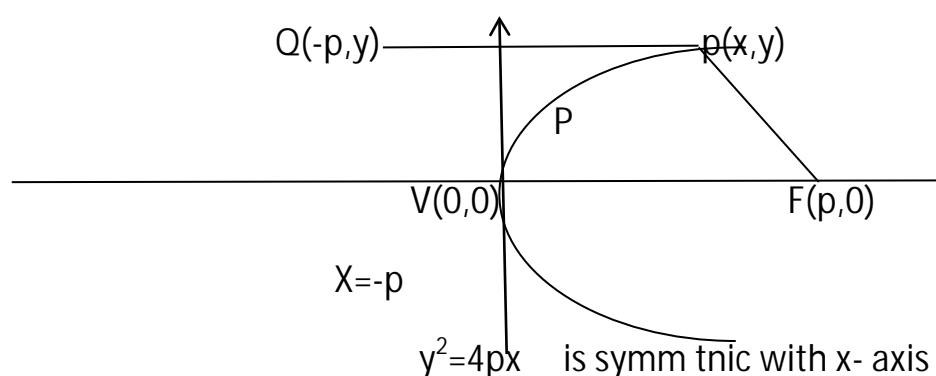
$$(x+1)^2 + (y-1)^2 = \frac{9}{5}$$

Exercises :

- 1- Prove that the loci of the point $p(x,y)$ is circle . if the distance between p and $A(6,0)$ equal twice the distance between p and $B(0,3)$?
- 2- Find in equation for the circle through the points $(1,0)$, $(0,1)$ and $(2,2)$?
- 3- Find an equation for the circle through the points $(2,3)$, $(3,2)$ and $(-4,3)$?
- 4- Find an equation for the circle centered at $(-2,1)$ that passes through the point $(1,3)$ Is the point $(1.1 , 2.8)$ inside , outside , or on the circle ?
- 5- Find equations for the tangents to the circle $(x-2)^2 + (y-1)^2 = 5$, at the points where the circle crosses the coordinate axis?

2- Parabola

Parabola is a set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane the fixed point is called focus (F) and the fixed line is called directrix (L)

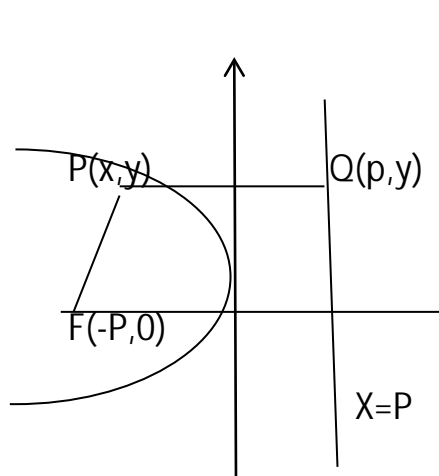


$$pF=pQ$$

$$\sqrt{(x-p)^2 + (y-0)^2} = \sqrt{(x+p)^2 + (y-y)^2}$$

$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$$

$$y^2 = 4px \quad , x \geq 0 \quad \text{is symmetric with } x\text{-axis}$$

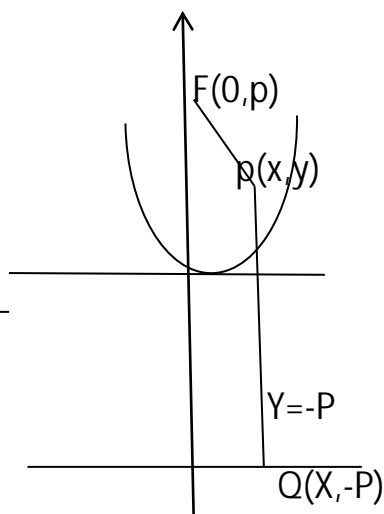


$$y^2 = -4px$$

Symmetric with x-axis

& open to left

$$V(h,k) : (y,0)$$

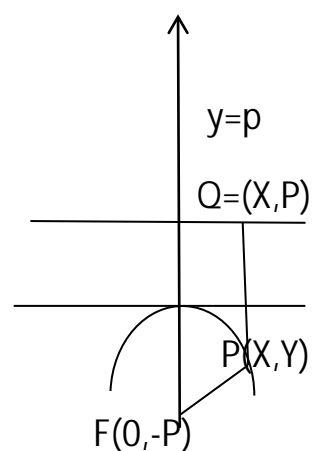


$$x^2 = 4py$$

sym – with – axis

& open to up

$$v(h,k)= (0,0)$$



$$x^2 = -4py$$

sym – with y – axis

& open to down

$$v(u,k)=(0,0)$$

	Equation	Focus(F)	Directrix	Axis
1	$y^2 = 4px$ $(y-k)^2 = 4p(x-h)$	$(p,0)$ $(h+p,k)$	$X=-p$ $X=h-p$	x-axis
2	$y^2 = -4px$ $(y-k)^2 = -4p(x-h)$	$(-p,0)$ $(h-p,k)$	$X=p$ $X=h+p$	x-axis
3	$x^2 = 4py$ $(x-h)^2 = -4p(y-k)$	$(0,p)$ $(h,k+p)$	$Y=-p$ $Y=k-p$	y-axis
4	$x^2 = -4py$ $(x-h)^2 = -4p(y-k)$	$(0,-p)$ $(h,k-p)$	$Y=p$ $Y=k+p$	y-axis

Example1 : Discuss and sketch the following equation :

$$y^2 + 6y + 2x + 5 = 0$$

Solution :

$$(y+3)^2 + 2x - 4 = 0$$

$$(y+3)^2 = -2x - 4$$

$$(y+3)^2 = -2(x - 2)$$

$$2=4p$$

$$p = \frac{1}{2}$$

$$V(h,k) = (2, -3), F(h-p, k) = (2 - \frac{1}{2}, -3) = (\frac{3}{2}, -3)$$

$$X = h+p = 2 + \frac{1}{2} = \frac{5}{2}$$

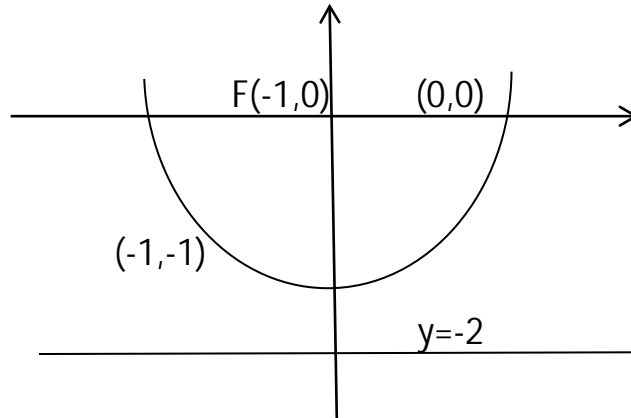
Example2 : find the focus and vertex and of the following parabola :

$$x^2 + 2x + 1 - 4y - 4 = 0$$

Solution : $(x + 1)^2 = 4(y + 1)$

$$V(h,k) = (-1, -1), p = 1, F(h, k+p) = (-1, -1+1) = (-1, 0)$$

$$y = k - p = -1 - 1 = -2$$



Exercises :

- 1) Find an equation , focus and directory of each of the following parabola , then sketch each one :

$$y^2 = 12x, y^2 = -2x, x^2 = 6y, x^2 = -8y, y = 4x^2$$

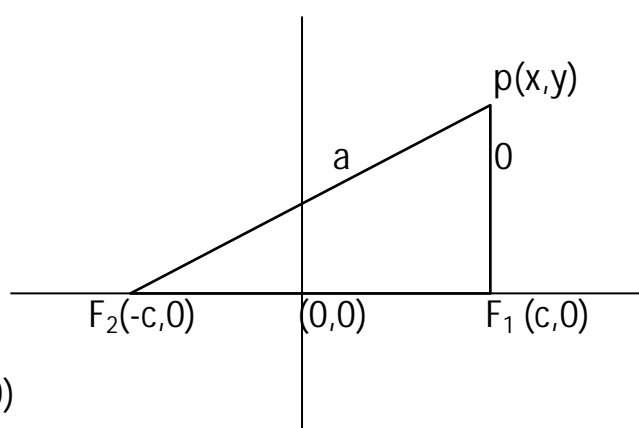
$$y = -8x^2, x = 2y^2, x = -3y^2$$

- 2) Find focus , vertex and directory , and sketch of each of the following parabola :

$$a.(x+1)^2 = -4(y-3), b.(y+2)^2 = 8(x-1)$$

3- Ellipse

Ellipse is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are called foci (F_1, F_2) of the Ellipse



$$F_1(c,0), F_2(-c,0)$$

$$PF_1 + PF_2 = \text{constant}$$

$$\sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x+c)^2 + (y-0)^2} = 2a \text{ where } a > 0$$

$$\sqrt{x^2 - 2cx + c^2 + y^2} = 2a - \sqrt{x^2 + 2cx + c^2 + y^2}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{x^2 + 2cx + c^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$4cx + 4a^2 = 4a\sqrt{x^2 + 2cx + c^2 + y^2}$$

$$cx + a^2 = a\sqrt{x^2 + 2cx + c^2 + y^2}$$

$$(cx + a^2)^2 = a^2(x^2 + 2cx + c^2 + y^2)$$

$$c^2x^2 + 2ca^2x + a^4 = a^2x^2 + 2ca^2x + a^2c^2 + a^2y^2 \text{ where } a > c$$

$$(a^2 - c^2)x^2 + a^2y^2 = (a^2 - c^2)a^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad \text{where } a > c$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow c^2 = a^2 - b^2 \text{ where } a > b, b^2 = a^2 - c^2$$

The Ellipse equation of x-axis with center (0,0) and foci ($\pm c, 0$)

$$e = \frac{c}{a}, c < a \rightarrow e < 1 \quad (\text{Eccentricity})$$

The Directrix L_1, L_2 are :

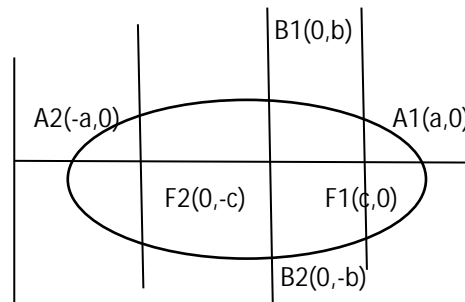
$$x = \pm \frac{a}{e} \quad \text{or} \quad x = \pm \frac{a^2}{c}$$

if $x=0$

$(0, -b)$, $(0, b)$

If $y=0$

$(-a, 0)$, $(a, 0)$



But the Ellipse equation of y-axis with center $(0,0)$ and foci $(0, \pm c)$

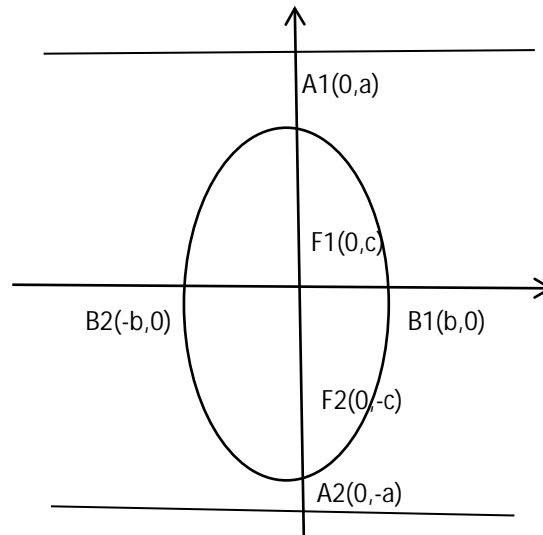
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

If $x=0$

$(0, -a)$, $(0, a)$

If $y=0$

$(-b, 0)$, $(b, 0)$



The Directrix L_1 , L_2 are :

$$y = \pm \frac{a}{e} \quad \text{or} \quad y = \pm \frac{a^2}{c}$$

x-axis	y-axis
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $F_1(c,0)$, $F_2(-c,0)$ Centre $C(0,0)$ $A_1(a,0)$, $A_2(-a,0)$ $B_1(0,b)$, $B_2(0,-b)$ $x = \pm \frac{a}{e}$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ $F_1(0,c)$, $F_2(0,-c)$ Centre $C(0,0)$ $A_1(0,a)$, $A_2(0,-a)$ $B_1(b,0)$, $B_2(-b,0)$ $y = \pm \frac{a}{e}$

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $F_1(h+c, k), F_2(h-c, k)$ <p>Centre C(h, k)</p> $A_1(h+a, k), A_2(h-a, k)$ $B_1(h, k+b), B_2(h, k-b)$ $x = h \pm \frac{a}{e}$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ $F_1(h, k+c), F_2(h, k-c)$ <p>Centre C(h, k)</p> $A_1(h, k+a), A_2(h, k-a)$ $B_1(h+b, k), B_2(h-b, k)$ $y = k \pm \frac{a}{e}$
--	--

Example 1 : Find and sketch the equation of Ellipse that the focus F(0,0) and a=3 , c(0,2) ?

Solution :

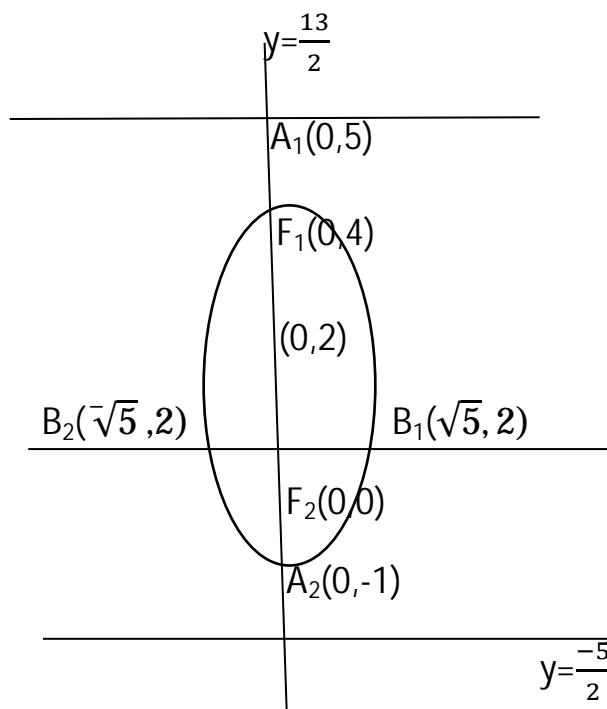
$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

$$b^2 = a^2 - c^2 \rightarrow b^2 = 9 - 4 \rightarrow b = \pm\sqrt{5}$$

$$\frac{(y-2)^2}{9} + \frac{(x-0)^2}{5} = 1$$

$$e = \frac{c}{a} = \frac{2}{3}$$

$$y = k \pm \frac{a}{e} = 2 \pm \frac{3}{\frac{2}{3}} = 2 \pm \frac{9}{2}$$



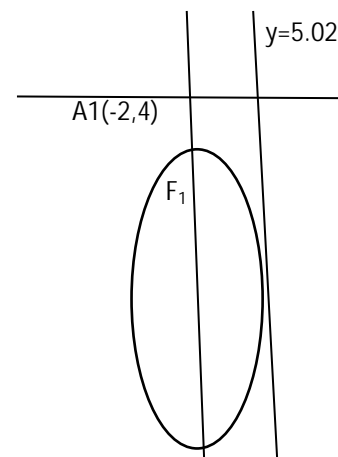
Example 2: Discuss and sketch the following equation :

$$9x^2 + 4y^2 + 36x - 8y + 4 = 0$$

Solution :

$$(9x^2 + 36x + 36) + (4y^2 - 8y + 4) = 36$$

$$9(x^2 + 4x + 4) + 4(y^2 - 2y + 1) = 36$$



$$9(x+2)^2 + 4(y-1)^2 = 36$$

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = 1$$

$$a^2 = 9 \rightarrow a = \mp 3, b^2 = 4 \rightarrow b = \mp 2$$

$$c(-2,1)$$

$$B2(-4,1)$$

$$B1(0,1)$$

$$F_2$$

$$A2(-2,-2)$$

$$y=-3.02$$

$$c^2 = a^2 - b^2 \rightarrow c^2 = 9 - 4 = 5 \rightarrow c = \pm\sqrt{5}$$

$$h=-2, k=1 \rightarrow c(-2,1) \text{ centre of Ellipse}$$

$$F_1(h, k+c), F_2(h, k-c) \rightarrow F_1(-2, 1 + \sqrt{5}), F_2(-2, 1 - \sqrt{5})$$

$$A_1(h, k+a), A_2(h, k-a) \rightarrow A_1(-2, 1 + 3), A_2(-2, 1 - 3) \rightarrow A_1(-2, 4), A_2(-2, -2)$$

$$B_1(h+b, k), B_2(h-b, k) \rightarrow B_1(-2 + 2, 1), B_2(-2 - 2, 1) \rightarrow B_1(0, 1), B_2(-4, 1)$$

$$y=k \pm \frac{a^2}{c} \rightarrow y = 1 \pm \frac{9}{\sqrt{5}}$$

Exercises :

- 1- Find an equation of Ellipse if the vertices $A_1(1,1)$, $A_2(1,7)$, $B_1(3,4)$ and $B_2(-1,4)$?
- 2- Prove that the tangent of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $p(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$?
- 3- Find the equation of an Ellipse of eccentricity $\frac{2}{3}$ if the line $x=9$ is one directrix and the focus is at $(4,0)$?
- 4- Find the equation of Ellipse having the center C, focus F and sketch graph :
 - 1) $C(0,0)$, $F(0,2)$, $a=4$
 - 2) $C(0,0)$, $F(-3,0)$, $a=5$
 - 3) $C(0,2)$, $F(0,0)$, $a=4$
 - 4) $C(-3,0)$, $F(-3,-2)$, $a=4$
- 5- Find and sketch the center , vertices and foci of the Ellipse :
 - 1) $25x^2 + 9y^2 - 100x + 54y - 44 = 0$
 - 2) $9x^2 + 4y^2 = 36$

$$3) 4x^2 + 9y^2 = 144$$

4- Hyperbole

Hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference, The two fixed points are called foci (F_1, F_2) of the Hyperbola.

$$F_1(-c,0), F_2(c,0)$$

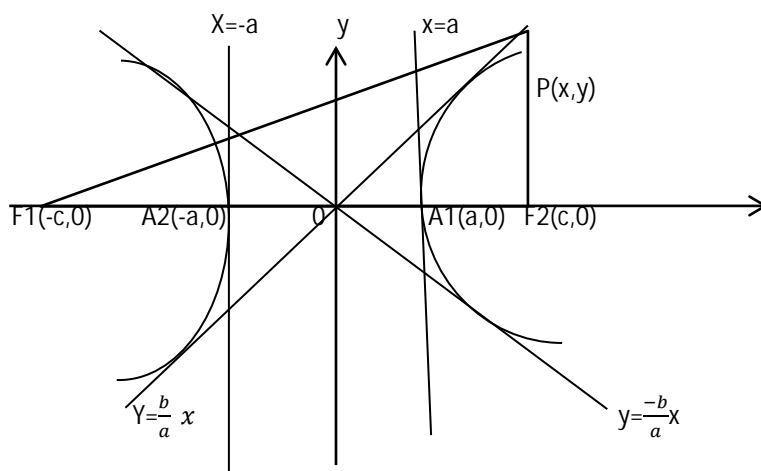
$$PF_1 - PF_2 = 2a$$

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a \text{ where } a > 0$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \quad \text{where } c > a, b^2 = c^2 - a^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \rightarrow c^2 = a^2 + b^2$$

The Hyperbola equation the foci on x-axis with center (0,0), intersection with x-axis at $(\pm a, 0)$ and its asymptotes $y = \pm \frac{bx}{a}$



Eccentricity: $e = \frac{c}{a}, c > a$

Directrix: $x = \pm \frac{a}{e}$

x-axis	y-axis
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $F_1(c,0), F_2(-c,0)$ Centre $C(0,0)$ $A_1(a,0), A_2(-a,0)$ $B_1(0,b), B_2(0,-b)$ $y = \pm \frac{b}{a} x$ $x = \pm \frac{a}{e} = \pm \frac{a^2}{c}$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ $F_1(0,c), F_2(0,-c)$ Centre $C(0,0)$ $A_1(0,a), A_2(0,-a)$ $B_1(b,0), B_2(-b,0)$ $y = \pm \frac{a}{b} x$ $y = \pm \frac{a}{e} = \pm \frac{a^2}{c}$
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $F_1(h+c,k), F_2(h-c,k)$ Centre $C(h,k)$ $A_1(h+a,k), A_2(h-a,k)$ $B_1(h,k+b), B_2(h,k-b)$ $(y-k) = \pm \frac{b}{a} (x-h)$ $x = h \pm \frac{a}{e}$ $x = h \pm \frac{a^2}{c}$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ $F_1(h,k+c), F_2(h,k-c)$ Centre $C(h,k)$ $A_1(h,k+a), A_2(h,k-a)$ $B_1(h+b,k), B_2(h-b,k)$ $(y-k) = \pm \frac{b}{a} (x-h)$ $y = k \pm \frac{a}{e}$ $y = k \pm \frac{a^2}{c}$

Example 1: sketch the following hyperbolas :

1. $\frac{y^2}{9} - \frac{x^2}{16} = 1$

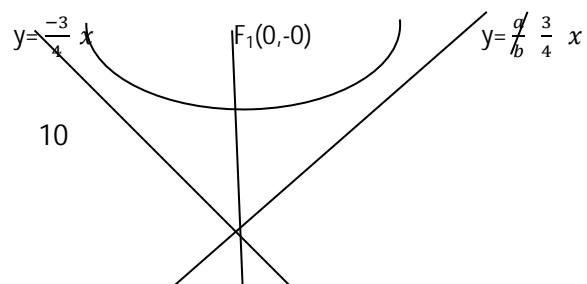
2. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Solution : 1. $C(0,0)$

$a^2 = 9 \rightarrow a = 3, b^2 = 16 \rightarrow b = 4$

$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2} = \pm 5$

$F_1(0,5), F_2(0,-5)$

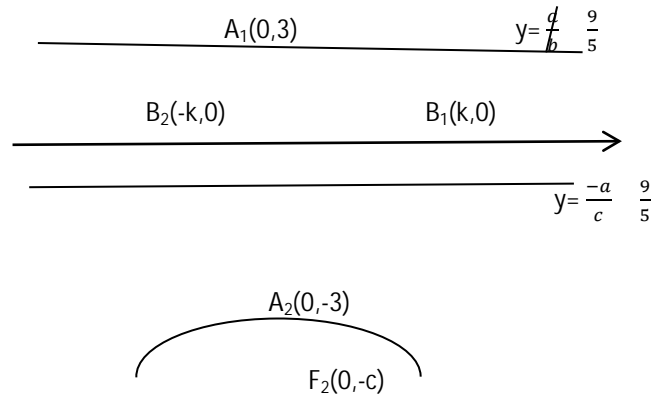


$$A_1(0,3) , A_1(0,-3)$$

$$B_1(4,0) , B_2(-4,0)$$

$$y = \pm \frac{a}{b} x \rightarrow y = \pm \frac{3}{4} x$$

$$y = \pm \frac{a^2}{c} \rightarrow y = \pm \frac{9}{5}$$



Example 2: Sketch the following hyperbolas :

1. $9(x-2)^2 - 4(y+3)^2 = 36$
2. $4(x-2)^2 - 9(y+3)^2 = 36$
3. $4x^2 - 5y^2 - 16x + 10y + 31 = 0$

Solution : 1. $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

$$C(2,-3)$$

$$a^2 = 4 \rightarrow a = 2, b^2 = 9 \rightarrow b = \pm 3$$

$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2} = \pm \sqrt{13}$$

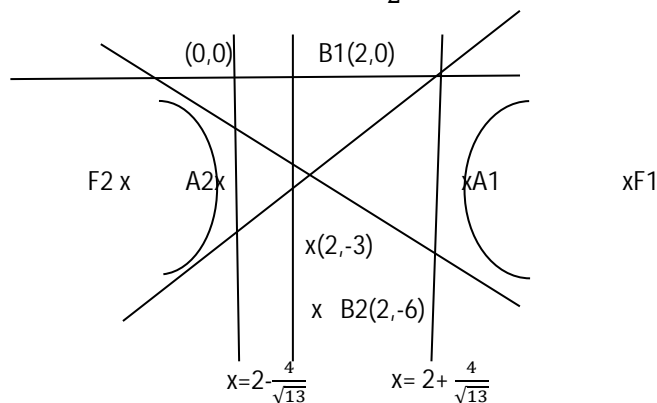
$$F_1(2+\sqrt{13}, -3) F_2(2-\sqrt{13}, -3)$$

$$A_1(2+2,-3) , A_2(2-2,-3) \rightarrow A_1(4,-3) , A_2(0,-3)$$

$$B_1(2,-3+3) , B_2(2,-3-3) \rightarrow B_1(2,0) , B_2(2,-6)$$

$$x = h \pm \frac{a^2}{c} \rightarrow x = 2 \pm \frac{4}{\sqrt{13}}$$

$$(y-k) = \pm \frac{b}{a} (x-h) \rightarrow (y+3) = \pm \frac{3}{2} (x-2)$$



Exercises

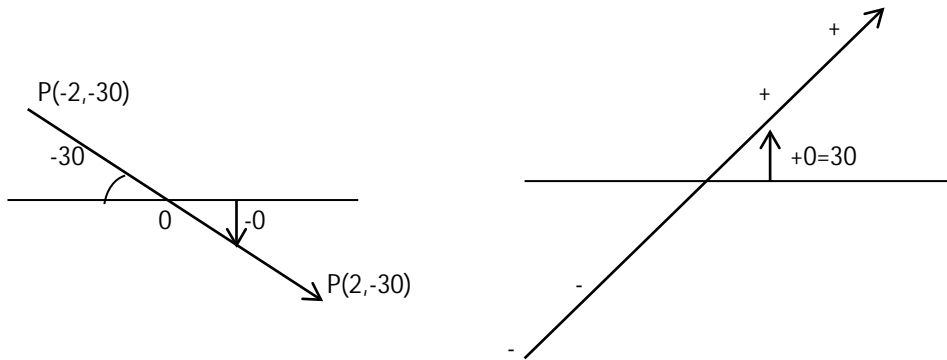
- 1- Prove that tangent of Hyperbole $b^2x^2 - a^2y^2 = a^2b^2$ at the points $p_1 (x_1, y_1)$ is $b^2xx_1 - a^2yy_1 = a^2b^2$?
- 2- Find an equation of Hyperbola from the given information ,
 $F_1(0,0)$, $F_2(0,4)$ and pass through the point $(12,9)$?
- 3- Find an equation of Hyperbola that the eccentricity $\sqrt{2} = \frac{c}{a}$ and the vertices $(0, \pm 2)$?

Chapter Nine

Polar Coordinates

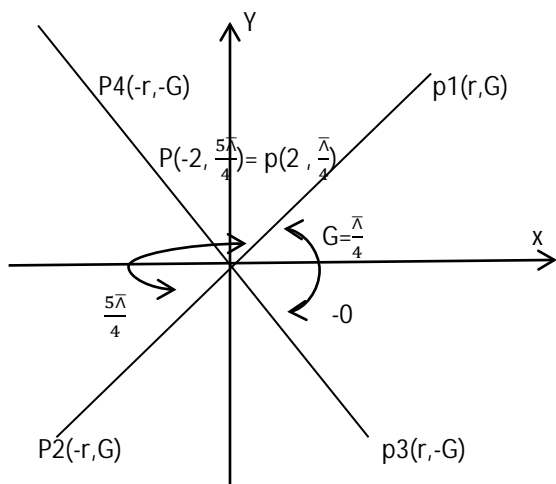
Cartesian Coordinates : any point in the plane can be represented as a pair of real numbers (x,y) .

Polar Coordinates : in the plane fix an origin O (called the pole) and an initial ray from O . then each point P can be located by assigning to it a polar coordinate (r,θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP .



So r is positive when measured on ray and negative when going backward (opposite direction) but θ is positive when measured counterclockwise and negative when measured clockwise .

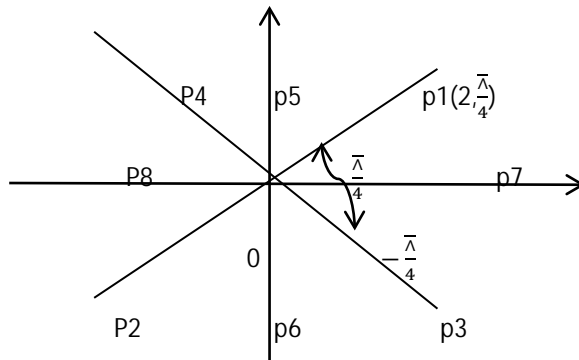
Polar Coordinates are not unique , for example , $(2, \frac{7\pi}{6}) = (-2, \frac{\pi}{6})$ and $(2, \frac{\pi}{6}) = (2, \frac{-11\pi}{6})$.



Example 1: plot the following points ?

$$p_1 \left(2, \frac{\pi}{4} \right), p_2 \left(-2, \frac{\pi}{4} \right), p_3 \left(2, \frac{-\pi}{4} \right), p_4 \left(-2, \frac{-\pi}{4} \right), p_5 \left(2, \frac{\pi}{2} \right), p_6 \left(-2, \frac{\pi}{2} \right),$$

$$p_7 (2,0), p_8 (-2,0)$$



The Relation between Cartesian Coordinate (x,y) and Polar Coordinate (r,θ) :

$$\frac{x}{r} = \cos \theta \rightarrow x = r \cos \theta$$

$$\frac{y}{r} = \sin \theta \rightarrow y = r \sin \theta$$

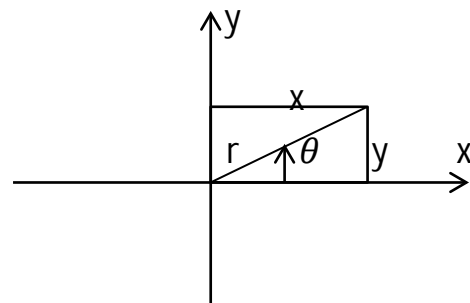
$$x^2 + y^2 = r^2 \rightarrow r = \sqrt{x^2 + y^2}$$

$$\frac{r \sin \theta}{r \cos \theta} = \frac{y}{x} \rightarrow \frac{\sin \theta}{\cos \theta} = \frac{y}{x} \rightarrow \tan \theta = \frac{y}{x} \rightarrow \theta = \tan^{-1} \frac{y}{x}$$

Example 2: converting the following Cartesian coordinate to polar coordinate :

a) $x^2 + y^2 - 6x = 0$

b) $x^2 + (y - 3)^2 = 0$ (H.W.)?



Solution :

a) $x = r \cos \theta, y = r \sin \theta$

$$(r \cos \theta)^2 + (r \sin \theta)^2 - 6r \cos \theta = 0$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) - 6r \cos \theta = 0$$

$$r^2 - 6r \cos \theta = 0$$

$$r = 6r \cos \theta$$

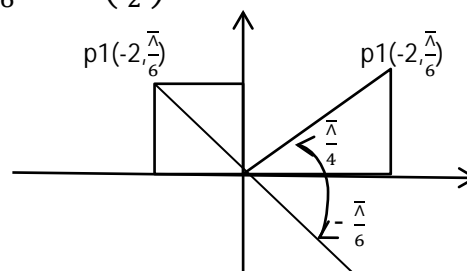
Example 3 : plot and transfer the points

$p_1 \left(-2, \frac{-\pi}{6} \right)$ and $p_2 \left(2, \frac{\pi}{4} \right)$ (H.W.) to Cartesian coordinate ?

$$\text{Solution : } x = r \cos \theta \rightarrow x = -2 \cos \frac{-\pi}{6} = -2 \left(\frac{\sqrt{3}}{2} \right) = -\sqrt{3}$$

$$y = r \sin \theta \rightarrow y = -2 \sin \frac{-\pi}{6} = 2 \sin \frac{\pi}{6} = 2 \left(\frac{1}{2} \right) = 1$$

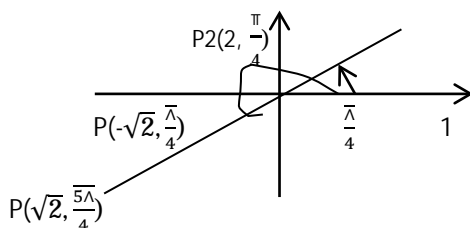
$$p_1 \left(-2, \frac{-\pi}{6} \right) \rightarrow p \left(-\sqrt{3}, 1 \right)$$



Example 4 : plot and transfer the points $p_1 (1,1), p_2 (-1, -\sqrt{3})$ (H.W.) to polar coordinate ?

$$\text{Solution : } x^2 + y^2 = r^2 \rightarrow 1 + 1 = r^2 \rightarrow 2 = r^2 \rightarrow r = \pm\sqrt{2}$$

$$\theta = \tan^{-1} \frac{y}{x} \rightarrow \theta = \tan^{-1} 1 \rightarrow 1 = \tan \theta \rightarrow \theta = \frac{\pi}{4}$$



Note :

- 1- If $r=0$ for any value of θ then $(0, \theta)$ is the origin point .
- 2- If $r=a$ (a is a constant) for all value of θ then these points are represented a circle of center $(0,0)$ and radius a .
- 3- $\theta = A$ (A is an angle) for all value of r the these points are represented a line through origin point .

Example 5: Find Cartesian equation to polar equation

a) $R \cos \left(\theta - \frac{\pi}{3} \right) = 3$

b) $r^2 = 2r \cos \theta$

c) $r^2 = 2 \sin 2\theta$ (H.W.)

Solution :

$$\begin{aligned} \text{a) } r \cos\left(\theta - \frac{\pi}{3}\right) &= 3 \rightarrow r \left(\cos \theta \cos \frac{\pi}{3} + \sin \theta \sin \frac{\pi}{3}\right) = \\ 3 &\rightarrow r\left(\frac{1}{2} r \cos \theta + \frac{\sqrt{3}}{2} r \sin \theta\right) = 3 \rightarrow \frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3 \rightarrow x + \\ \sqrt{3}y &= 6 \end{aligned}$$

Example 6: Find intersection points of the following two curves constant

$$\text{a) } r=a(1+\cos\theta), r = a(1 - \sin \theta)$$

Solution : $a(1+\cos\theta) = a(1 - \sin \theta)$

$$1+\cos\theta = 1 - \sin \theta \rightarrow \frac{\sin \theta}{\cos \theta} = -1 \rightarrow \tan \theta = -1$$

$$\theta = \pi - \frac{\pi}{4} \rightarrow \theta = 135^\circ$$

$$\theta = 2\pi - \frac{\pi}{4} \rightarrow \theta = 315^\circ$$

$$r=a(1+\cos 135^\circ) \rightarrow r = a\left(1 + \frac{1}{\sqrt{2}}\right)$$

$$\left(a\left(1-\frac{1}{\sqrt{2}}\right), 135^\circ\right), \left(a\left(1 + \frac{1}{\sqrt{2}}\right), 315^\circ\right)$$

Exercises : 1- plot and converting the following points to Cartesian coordinates :

$$\text{a. } \left(3, \frac{\pi}{4}\right) \text{ b. } \left(-3, \frac{\pi}{4}\right), \text{ c. } \left(3, \frac{-\pi}{4}\right), \text{ d. } \left(-3, \frac{-\pi}{4}\right)$$

2- Graph the locus of point $p(r, \theta)$ whose polar coordinate satisfy the given equation inequality or inequalities .

$$\begin{aligned} \text{a) } r=2, \text{ b) } r>2, \text{ c) } r<2, \text{ d) } 1<r<2, \text{ e) } 0\leq\theta\leq \\ 30^\circ, r\geq 0, \text{ f) } \theta=120^\circ, r\leq -2, \text{ g) } \theta=60^\circ, 1\leq r\leq 3, \\ \text{h) } \theta=495^\circ, r\geq -1 \end{aligned}$$

3-Graph the loci :

$$\begin{aligned} \text{a) } r \cos \theta = 2, \text{ b) } r \sin \theta = -1, \text{ c) } r \cos(\theta - 60^\circ) = 3 \\ \text{d) } r \cos(30^\circ - \theta) = 0 \end{aligned}$$

4-Show that $(-2, \frac{3\pi}{4})$ is on the curve $r = 2 \sin 2\theta$ and $(\frac{1}{2}, \frac{3\pi}{2})$ is on the curve $r = -\sin(\frac{\theta}{3})$

5-Show that equations represent the same curve :

$$r = \cos \theta + 1, r = \cos \theta - 1$$

6-Find some intersections of the following curves (a constant)

$$a) r = a(1 + \cos \theta), r = a(1 - \cos \theta) \quad b) r = a(1 + \sin \theta), r = 2a \cos \theta$$

Draw Curve in polar Coordinate :

$$F(\theta) = r \quad \text{or} \quad F(r, \theta) = 0$$

- Symmetry about the x-axis (polar axis)

$$F(r, \theta) = F(r, -\theta)$$

$$\text{For example, } r = 3 \cos \theta = 3 \cos(-\theta)$$

- Symmetry about the origin :

$$F(r, \theta) = F(-r, \theta)$$

$$\text{For example, } r^2 = 3 \cos \theta, \rightarrow (-r)^2 = 3 \cos \theta$$

- Symmetry about the y-axis

$$F(r, \theta) = F(r, \pi - \theta)$$

$$\text{For example, } r = 3 \sin \theta = 3 \sin(\pi - \theta)$$

Example 7 : Discuss and sketch the following curves :

$$1) r = a(1 - \cos \theta)$$

$$2) r = a(1 - \sin \theta)$$

$$3) r^2 = 2a^2 \cos 2\theta$$

$$4) r = a \sin 2\theta$$

$$5) r = \sin 3\theta$$

$$6) r = \theta \text{ and}$$

$$7) r = -\theta$$

Solution 1) : Cardioid curve

- $0 \leq \theta \leq 2\pi$
- $F(r, \theta) = F(r, -\theta)$
- *symmetry about the polar axis*
- $-1 \leq \cos \theta \leq 1 \rightarrow 0 \leq 1 - \cos \theta \leq 2 \rightarrow 0 \leq a(1 - \cos \theta) \leq 2a \rightarrow 0 \leq r \leq 2a$
- **Intersection points**
 - If $\theta = \frac{\pi}{2} \rightarrow r = a \quad (a, \frac{\pi}{2})$
 - If $\theta = \frac{3\pi}{2} \rightarrow r = a \quad (a, \frac{3\pi}{2})$
 - If $\theta = \pi \rightarrow r = 2a \quad (2a, \pi)$
 - If $\theta = 0 \rightarrow r = 0 \quad (0, 0)$
- $\frac{dr}{d\theta} = a \sin \theta \rightarrow \frac{dr}{d\theta} > 0$ when $\sin \theta > 0$

($\sin \theta > 0$ in the interval $[0, \pi]$) \rightarrow increases when $\theta \in [0, \pi]$

	P ₁	P ₂	P ₃	P ₄	P ₅
θ	30°	60°	90°	120°	150°
r	0.15a	0.5a	a	1.5a	1.85a

2) Cardioid curve

* $0 \leq \theta \leq 2\pi$

* $a(1 - \sin(\pi, \theta)) = a(1 - \sin \theta)$

$F(r, \theta) = F(r, \pi - \theta)$ (*symmetry about the y - axis*)

- $-1 \leq \sin \theta \leq 1 \rightarrow 0 \leq 1 - \sin \theta \leq 2 \rightarrow 0 \leq a(1 - \sin \theta) \leq 2a \rightarrow 0 \leq r \leq 2a$
- *Intersection points*

If $\theta = \frac{\pi}{2} \rightarrow r = 0 \quad (0, \frac{\pi}{2})$

If $\theta = \frac{3\pi}{2} \rightarrow r = 2a \quad (2a, \frac{3\pi}{2})$

If $\theta = \pi \rightarrow r = a \quad (a, \pi)$

If $\theta = 0 \rightarrow r = a \quad (a, 0)$

• $\frac{dr}{d\theta} = -a \cos \theta \rightarrow \frac{dr}{d\theta} \leq 0$ when $\cos \theta \geq 0$

$(\cos \theta \geq 0 \text{ in the interval } [\frac{-\pi}{2}, \frac{\pi}{2}]) \rightarrow \text{decreases when } \theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

	P ₁	P ₂	P ₃	P ₄	P ₅
θ	30°	45°	60°	-30°	-45°
r	0.5a	0.3a	0.1a	1.5a	1.7a

3) Lemniscate Curve

* $\frac{-\pi}{2} \leq 2\theta \leq \frac{\pi}{2} \rightarrow \frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$

* $F(r, \theta) = F(r, -\theta), F(r, \theta) = F(-r, \theta), F(r, \theta) = F(-r, -\theta)$

(Symmetry about the polar axis, y-axis and origin)

• Intersection points

If $\theta = 0 \rightarrow r = \pm\sqrt{2}a \quad (\pm\sqrt{2}a, 0)$

If $\theta = \pi \rightarrow r = \pm\sqrt{2}a \quad (\pm\sqrt{2}a, \pi)$

• $2r \frac{dr}{d\theta} = -4a^2 \sin 2\theta \rightarrow \frac{dr}{d\theta} = \frac{-2a^2}{r} \sin 2\theta$

$\frac{dr}{d\theta} < 0$ when $\sin 2\theta > 0$

$0 \leq 2\theta \leq \pi \rightarrow 0 \leq \theta \leq \frac{\pi}{2} \rightarrow \text{decreases when } \theta \in [0, \frac{\pi}{4}]$

	P ₀	P ₁	P ₂
θ	0	30°	45°
r	$\pm\sqrt{2}a$	$\pm a$	0

H.W.

$r = a(1 + \cos \sigma)$

$r = a(1 + \sin \sigma)$

$r^2 = 2a^2 \sin 2\sigma$

لرسم منحنى المعادلة بالصيغة القطبية سوف نتبع الخطوات التالية :-

1- مجال الرسم (domain of graphs) : من تعريف المنحنى سوف نأخذ القيم الحقيقية

التي تحقق المعادلة لـ r و θ

2- التناظر (symmetric) : هناك ثلاثة انواع للتناظر

أ- التناظر حول المحور القطبي (المحور x) : يكون المنحنى متناظر حول x اذا كانت

$$F(r, \theta) = 0 \text{ لا تتغير عند تبديل } (\theta \text{ بـ } -\theta) \text{ اي ان } F(r, \theta) = F(r, -\theta)$$

ب- التناظر حول المحور العمودي (محور y) : يكون المنحنى متناظر حول y اذا كانت

$$F(r, \theta) = 0 \text{ لا تتغير عند تبديل } (\theta \text{ بـ } \pi - \theta) \text{ اي ان } F(r, \theta) = F(r, \pi - \theta)$$

ت- التناظر حول نقطة الاصل : يكون المنحنى متناظر حول نقطة الاصل اذا كانت

$$F(r, \theta) = 0 \text{ لا تتغير عند تبديل } (r \text{ بـ } -r) \text{ اي ان } F(r, \theta) = F(-r, \theta)$$

3- نقاط التقاطع (intercept point) :

أ- لإيجاد نقاط التقاطع مع محور x سوف نضع $(\theta = 0 \text{ أو } \theta = \pi)$ ثم نجد قيمة r

ب- لإيجاد نقاط التقاطع مع محور y سوف نضع $(\theta = \frac{\pi}{2} \text{ أو } \theta = \frac{3\pi}{2})$ ثم نجد قيمة r

4- مناطق التزايد والتناقص (increasing & decreasing region) :

نجد $\frac{dr}{d\theta}$ المشتقة ثم نجد

أ- الفترات التي تكون فيها المشتقة $\frac{dr}{d\theta} \geq 0$ يكون المنحنى متزايد اي ان (كلما تزداد

θ يزداد r)

ب- الفترات التي تكون فيها المشتقة $\frac{dr}{d\theta} \leq 0$ يكون المنحنى متناقص اي ان (كلما

تزداد θ يقل r)

5- لضبط الشكل سوف نستخدم الجدول التالي :-

θ						
r						

Integrals in polar Coordinates :

If $f(r, \theta)$ is a function defined on region R and bounded by two curves

$\theta = \alpha, \theta = \beta$ and two curves $r = r_1(\theta), r = r_2(\theta)$. Then Double

Integral in polar Coordinates as follows :

$$\int_A \int f(r, \theta) dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r dr d\theta \quad \text{where } A \text{ inside } R$$

The Area in polar Coordinates :

By using Jaction transformation to transform Double Integral from Cartesian plane to polar Coordinates as follows :

$$\int \int f(x, y) dx dy = \int \int f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$dx dy = r dr d\theta$$

$$A = \int \int dx dy \text{ [the area in Cartesian plane]}$$

$$A = \int \int r dr d\theta \text{ [the area in polar Cooordinates)}$$

Example 10 : Find the value of Double Integral in the polar coordinates

$$\int_{-1}^1 \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$$

Solution :

$$y = \pm \sqrt{1-x^2} \rightarrow y^2 = 1-x^2 \rightarrow x^2 + y^2 = 1 \rightarrow r^2 = 1$$

$$\rightarrow r = \pm 1 \rightarrow r = 0 \text{ and } r = 1, \theta = 0 \text{ and } \theta = 2\pi$$

$$\int_0^{2\pi} \int_0^1 r^2 r dr d\theta = \int_0^{2\pi} \int_0^1 r^3 dr d\theta = \int_0^{2\pi} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} d\theta = \left[\frac{\theta}{4} \right]_{\theta=0}^{\theta=2\pi} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Example 11: Find an equivalent Double Integral in term of polar

$$\text{coordinates } \int_0^2 \int_0^x y dy dx$$

Solution :

$$x = r \cos \theta, y = r \sin \theta$$

$$y = x \rightarrow r \sin \theta = r \cos \theta \rightarrow \cos \theta = \sin \theta \rightarrow \theta = \frac{\pi}{4}$$

$$x = 2 \rightarrow r \cos \theta = 2 \rightarrow r = \frac{2}{\cos \theta} \rightarrow r = 2 \sec \theta$$

$$r = 0 \text{ and } r = 2 \sec \theta, \theta = 0 \text{ and } \theta = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}} \int_0^{2 \sec \theta} r \sin \theta \cdot r dr d\theta =$$

$$\int_0^{\frac{\pi}{4}} \int_0^{2 \sec \theta} r^2 \sin \theta \cdot dr d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^3}{3} \right]_{r=0}^{r=2 \sec \theta} \sin \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{8}{3} \sec^3 \theta \sin \theta d\theta = \frac{8}{3} \int_0^{\frac{\pi}{4}} \frac{\sin \theta}{\cos \theta} \sec^2 \theta d\theta = \frac{8}{3} \int_0^{\frac{\pi}{4}} \tan \theta \sec^2 \theta d\theta =$$

$$\frac{8}{3} \left[\frac{\tan^2 \theta}{2} \right]_{\theta=0}^{\theta=\frac{\pi}{4}} = \frac{4}{3} \left[\tan^2 \frac{\pi}{4} - \tan^2 0 \right] = \frac{4}{3} [1 - 0] = \frac{4}{3}$$

Example 12 : Find the area inside the Cardioid $r=a(1 - \cos \theta)$

Solution : $0 \leq \theta \leq 2\pi$

$$\theta = 0 \rightarrow r = a(1 - \cos \theta) \rightarrow r = 0$$

$$r = 0 \text{ and } r = a(1 - \cos \theta), \theta = 0 \text{ and } \theta = 2\pi$$

$$A = \int_0^{2\pi} \int_0^{a(1-\cos\theta)} r \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} \right]_{r=0}^{r=a(1-\cos\theta)} d\theta =$$

$$\frac{a^2}{2} \int_0^{2\pi} (1 - \cos\theta)^2 d\theta = \frac{a^2}{2} \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta =$$

$$\frac{a^2}{2} \int_0^{2\pi} (1 - 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)) d\theta =$$

$$\frac{a^2}{2} \left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta \right]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{a^2}{2} \left[2\pi - 2\sin 2\pi + \frac{2\pi}{2} + \frac{1}{4}\sin 4\pi - 0 + 2\sin 0 - 0 - \frac{1}{4}\sin 0 \right]$$

$$= \frac{a^2}{2} [2\pi + \pi] = \frac{3a^2\pi}{2}$$