mathematics

Department of physics level 2

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Chapter One

Sequence:-

A sequence is a function whose domain is the set of positive integer 1,2,3,… . We denote the sequence by $< f(n) >$) . Where $f(n)$ is the function at valued n.

The sequence f is the set $\{(n, f(n)); n \in N\}$. That is ,the set of all pairs $(n, f(n))$, with n appositive integer. We also write a_n to denote the sequence whose ordinate at $y = a_n$.

Examples:-

- $< \frac{1}{2n} > = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots \dots$

The set $\{ (n, \frac{1}{2n}) , n = 1, 2, 3, \dots . . \}$ is a sequence whose value at n is $\frac{1}{2n}$.

- $< \frac{1}{n+1} > = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \dots$

The set $\{ (n, \frac{1}{n+1}) , n = 1, 2, 3, \dots . . \}$ is a sequence whose value at n is $\frac{1}{n+1}$.

- $< 1 > = 1, 1, 1, \dots \dots ..$

The set $\{ (n, 1) , n = 1, 2, 3, \dots . . \}$ is a sequence whose value at n is 1.

Finite sequence :-

The sequence $< f(n) >$ is said to be finite if there exist two numbers a and b such that $a \leq f(n) \leq b$ or $a < f(n) \leq b$ or $a \leq f(n) < b$ or $a < f(n) < b$.

Note:- every finite sequence is bounded.

Infinite sequence:-

The sequence $< f(n) >$ is said to be infinite if both a and b or one of them equal to $\pm\infty$.

Note:- every infinite sequence is unbounded.

Convergent Sequence:-

A sequence is said to be convergent if it approaches some limit .

Formally, a sequence S_n converges to the limit L . A divergent sequence doesn't have a limit. $\lim_{n\to\infty} S_n = L$

Discuss the following sequences:-

- $< \boldsymbol{\frac{1}{2n}} > = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\dots\} . n \in N$

The set $\{ (1, \frac{1}{2}), (2, \frac{1}{4}), (3, \frac{1}{6}), \dots\dots \}$ is a finite and bounded sequence when $n \to \infty$,

$$\lim_{n\to\infty} \frac{1}{2n} = 0$$

$< \frac{1}{2n} > \to 0$ the sequence converge to zero.

$$0 < \frac{1}{2n} \leq \frac{1}{2}$$

- $< \boldsymbol{\frac{1}{n+1}} > = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\dots\}. n \in N$

The set $\{ (1, \frac{1}{2}), (2, \frac{1}{3}), (3, \frac{1}{4}), \dots\dots \}$ is a finite and bounded sequence when $n \to \infty$,

$$\lim_{n\to\infty}\frac{1}{n+1}=\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{n}{n}+\frac{1}{n}}=\frac{0}{1+0}=0$$

$<\frac{1}{n+1}>\to 0$ the sequence is converge to zero.

$$0<\frac{1}{n+1}\le\frac{1}{2}$$

- $<\mathbf{1}>=\{1,1,1,\dots\dots..\}$.

The set $\{\ (1,1\ ),(2,1),(3,1),\dots..\}$ is a finite and bounded sequence when $n\to\infty$,

$$\lim_{n\to\infty} 1=1$$

$<1>\to 1$ the sequence is converge to 1.

- $<\frac{\mathbf{2n+1}}{\mathbf{2n}}>=\left\{\frac{3}{2},\frac{5}{4},\frac{7}{6},\dots\dots\dots\right\}.n\in N$

The set $\{\ (1,\frac{3}{2}),(2,\frac{5}{4}),(3,\frac{7}{6}),\dots..\}$ is a finite and bounded sequence since when $n\to\infty$, there are two Solutions for this sequence:-

First solution:-

$$\lim_{n\to\infty}\frac{2n+1}{2n}=\lim_{n\to\infty}\frac{\frac{2n}{n}+\frac{1}{n}}{\frac{2n}{n}}=\lim_{n\to\infty}\frac{2+\frac{1}{n}}{2}=\frac{2+0}{2}=1$$

$$\therefore <\frac{2n+1}{2n}>\to 1$$

Second solution:-

$$\lim_{n\to\infty}\frac{2n+1}{2n}=\lim_{n\to\infty}\frac{2n}{2n}+\frac{1}{2n}=\lim_{n\to\infty}1+\frac{1}{2n}=1+0=1$$

$$\therefore < \frac{2n+1}{2n} > \to 1$$

The sequence is converge to 1.

$$1 < \frac{2n+1}{2n} \le \frac{3}{2}$$

- $< \boldsymbol{n} > = \{1, 2, 3, \dots \dots\}. n \in N$

The set $\{ (1,1), (2,2), (3,3), \dots . . \}$ is a infinite and unbounded sequence when $n \to \infty, < n > \to \infty$. The sequence is diverged.

Note:- Those sequences are also infinite, unbounded and diverge $< \sqrt{\boldsymbol{n}} >, < \boldsymbol{5n^2} >, < \frac{\boldsymbol{n^3}}{\boldsymbol{6}} >, < \frac{\boldsymbol{n^6}}{\boldsymbol{n^3}} >$ And so on. Have a same solve like $< \boldsymbol{n} >$.

Example:- Find the Domains for the following sequences

- $< \frac{\boldsymbol{1}}{\boldsymbol{n-1}} >$

$n - 1 = 0$

$n = 1$

$n \in N/\{1\}$

Or $n = \{2, 3, 4, \dots .\}$.

- $< \sqrt{\boldsymbol{n-4}} >$

$n - 4 \ge 0$

$n \ge 4$

$n \in N/\{1, 2, 3\}$

Or $n = \{4, 5, 6, \dots.\}$.

- $< \frac{\mathbf{1}}{\sqrt{\boldsymbol{n}+\mathbf{4}}} >$

$n \in N$ since $n + 4 > 0 \quad \rightarrow \quad n > -4$

- $< \frac{\mathbf{1}}{\sqrt{\boldsymbol{n}-\mathbf{3}}} >$

$n - 3 > 0$

$n > 3$

$n \in N/\{1, 2, 3\}$

Or $n = \{4, 5, 6, \dots.\}$.

- $< \frac{\mathbf{1}}{\sqrt{\mathbf{3}-\boldsymbol{n}}} >$

$3 - n > 0$

$3 > n$

$n = \{1, 2\}$.

$\therefore < \frac{1}{\sqrt{3-n}} >$ Is finite sequence having two limits.

- $< \sqrt{\mathbf{7} - \boldsymbol{n}} >$ How ?

Are the sequences converging or diverge?

1. $< \frac{\boldsymbol{n}-\mathbf{2}}{\boldsymbol{n}^{\mathbf{2}}+\mathbf{1}} >$

Solution:-

$$\lim_{n\to\infty}\frac{n-2}{n^2+1}=\lim_{n\to\infty}\frac{\frac{n}{n^2}-\frac{2}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}}=\lim_{n\to\infty}\frac{\frac{1}{n}-\frac{2}{n^2}}{1+\frac{1}{n^2}}=\frac{0-0}{1+0}=0$$

$$\therefore <\frac{n-2}{n^2+1}>\to 0$$

2. $<\frac{n^3+1}{n-2}>$

Sol.:- $\lim_{n\to\infty}\frac{n^3+1}{n-2}=\lim_{n\to\infty}\frac{\frac{n^3}{n^3}+\frac{1}{n^3}}{\frac{n}{n^3}-\frac{2}{n^3}}=\lim_{n\to\infty}\frac{1+\frac{1}{n^3}}{\frac{1}{n^2}-\frac{2}{n^3}}=\frac{1+0}{0-0}=\frac{1}{0}=\infty$

$\therefore <\frac{n^3+1}{n-2}>$ The sequence is diverge.

3. $<(-\mathbf{1})^{\boldsymbol{n}}>, n\in N$

Sol.:- $<(-1)^n>=\{-1,1,-1,1,\dots\}$

$$<(-1)^n>=\begin{Bmatrix}-1 & n\ \text{odd}\\ 1 & n\ \text{even}\end{Bmatrix}$$

$$<(-1)^n>=\begin{Bmatrix}(1,-1),(3,-1),(5,-1),\dots & n\ \text{odd}\\ (2,1),(4,1),(6,1),\dots & n\ \text{even}\end{Bmatrix}$$

$$-1\le(-1)^n\le 1$$

$\therefore <(-1)^n>$ Is bounded.

But $\lim_{n\to\infty}(-1)^n=\begin{Bmatrix}-1 & n\to\infty\ \text{odd}\\ 1 & n\to\infty\ \text{even}\end{Bmatrix}$

$\therefore <(-1)^n>$ Is diverge because it has two limits.

Discuss the following sequences? (Homework)

4. $<(-\mathbf{1})^{\boldsymbol{n+1}}>$

5. $<(-\mathbf{1})^{\boldsymbol{n-1}}>$

6. $<\mathbf{1}-(-\mathbf{1})^{\boldsymbol{n}}>$

7. $< \frac{1}{(-1)^n} >$

8. $< n - (-1)^n >$

9. $< \frac{n}{(-1)^n} >$

10. $< \frac{(-1)^n}{n} >$

11. $<< 5n + 4 >$

12. $< \frac{4}{3n} >$

13. $< \sqrt[5]{n} >$

14. $< \frac{n^2}{n^3+3} >$

Find the values of $a_1,\ a_2,\ a_3$ and a_4 for the following sequence $< a_n >$, were given a_n term ?

1. $a_n = \frac{1-n}{n^2}$

$$a_1 = \frac{1-1}{1^2} = \frac{0}{1} = 1$$

$$a_2 = \frac{1-2}{2^2} = \frac{-1}{4}$$

$$a_3 = \frac{1-3}{3^2} = \frac{-2}{9}$$

$$a_4 = \frac{1-4}{4^2} = \frac{-3}{16}$$

2. $a_n = \frac{(-1)^{n+1}}{2^{n-1}}$

$$a_1 = \frac{(-1)^{1+1}}{2^{1-1}} = \frac{(-1)^2}{2^0} = \frac{1}{1} = 1$$

$$a_2 = \frac{(-1)^{2+1}}{2^{2-1}} = \frac{(-1)^3}{2} = \frac{-1}{2}$$

$$a_2 = \frac{(-1)^{3+1}}{2^{3-1}} = \frac{(-1)^4}{2^2} = \frac{1}{4}$$

$$a_4 = \frac{(-1)^{4+1}}{2^{4-1}} = \frac{(-1)^5}{2^3} = \frac{-1}{8}$$

3. $\boldsymbol{a_n = \frac{(-1)^{n+1}}{2n-1}}$ **Homework?**

4. $\boldsymbol{a_n = (\frac{n+1}{2n})(1-\frac{1}{n})}$ **Homework?**

Are the following sequence converge or diverge find the limited of each convergent sequence?

1. $\boldsymbol{a_n = 2 + (0.1)^n}$

$n = 1, 2, 3, ...$

$a_1 = 2.1\ , a_2 = 2.01\ , a_2 = 2.001, ...$

$n \to \infty\ , \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}(2 + (0.1)^n)$

$$= \lim_{n\to\infty}\left(2 + \frac{1}{10^n}\right)$$

$= 2 + \lim\limits_{n\to\infty} \frac{1}{10^n} = 2 + 0 = 2$

$\therefore a_n \to 2$

$2 \le a_n \le 2.1$

$< a_n >$ is bounded.

2. $\boldsymbol{a_n = 2 + (0.1)^n}$**? Homework**

Not :- All of these sequence are diverge:- $\boldsymbol{< a_n > = \frac{n}{10}}$, $\boldsymbol{< a_n > = (n-1)^2}$ **and** $\boldsymbol{< a_n > = (n + (-1)^n)}$ **.**

when $\boldsymbol{n} \to \infty$ the limit sequences $\lim\limits_{\boldsymbol{n}\to\infty} \boldsymbol{a_n} = \infty$

3. $< \boldsymbol{a_n} > = \frac{\boldsymbol{cos\,n}}{\boldsymbol{n}}$

$-1 \le cos\,n \le 1$

$$\frac{-1}{n} \le \frac{cos\,n}{n} \le \frac{1}{n}$$

But $< \frac{-1}{n} > \to 0$ and $< \frac{1}{n} > \to 0$

$\therefore < \frac{\boldsymbol{cos\,n}}{\boldsymbol{n}} > \to 0$

By theorem if $a_n \le b_n \le c_n\ \forall n\ and\ < a_n > \to L\ and < c_n > \to L\ then < b_n > \to L.$

But this sequence is diverge show that $< \boldsymbol{a_n} > = \boldsymbol{n + cos\,n}$ **?** **(Homework).**

Mathematical Convergent for Sequence:-

A sequence S_n is said to be converge Mathematically for a real number L if and only if for every positive real number $(\epsilon > 0)$, there exist corresponding number $(K \in N)$, such that :

$< S_n > \to L \leftrightarrow \quad |S_n - L| < \epsilon \qquad \forall n > K.$

Example:- prove that

1. $< \frac{\boldsymbol{1}}{\boldsymbol{n}} > \to \boldsymbol{0}$

Proof:- $\left|\frac{1}{n} - 0\right| < \epsilon \qquad \forall n > K$

$$\left|\frac{1}{n}\right| < \epsilon \qquad \forall n > K$$

$$\frac{1}{n} < \epsilon \qquad \forall n > K$$

$$n > \frac{1}{\epsilon} \qquad \forall n > K$$

$\therefore K = \frac{1}{\epsilon}$.

2. $\boldsymbol{< \frac{1}{2^n} > \to 0}$

Proof:- $\left|\frac{1}{2^n} - 0\right| < \epsilon \qquad \forall n > K$

$$\left|\frac{1}{2^{\boldsymbol{n}}}\right| < \epsilon \qquad \forall n > K$$

$$\frac{1}{2^{\boldsymbol{n}}} < \epsilon \qquad \forall n > K$$

$$2^{\boldsymbol{n}} > \frac{1}{\epsilon} \qquad \forall n > K$$

$$\ln 2^{\boldsymbol{n}} > \ln \frac{1}{\epsilon}$$

$$n \ln 2 \ > \ln 1 - \ln \epsilon$$

$$n > \frac{0 - \ln \epsilon}{\ln 2}$$

$\therefore K = \frac{-\ln \epsilon}{\ln 2}$.

3. $\boldsymbol{< \frac{n+1}{n} > \to 1}$

Proof:- $\left|\frac{n+1}{n} - 1\right| < \epsilon \qquad \forall n > K$

$$\left|\frac{n+1-n}{n}\right| < \epsilon \qquad \forall n > K$$

$$\left|\frac{1}{n}\right| < \epsilon \qquad \forall n > K$$

$$\frac{1}{n} < \epsilon \qquad \forall n > K$$

$$n > \frac{1}{\epsilon} \qquad \forall n > K$$

$$\therefore K = \frac{1}{\epsilon}$$

4. **$< \frac{n}{n+1} > \to 1$ prove that ?(Homework)**
5. **$< \frac{2}{n} > \to 0$ prove that ?(Homework)**

Converge of sequence by theorem of continuous

If $s_n \to L$ and f is a continuous function on L ; f is defined on all terms of sequence , then $f(s_n) \to f(L)$.

Example:- prove that

1. **$< \sqrt{\frac{n+1}{n}} > \to 1$**

Proof:- Let $x = \frac{n+1}{n}$

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{n}{n} + \frac{1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1 + 0 = 1$$

$$\therefore < \frac{n+1}{n} > \to 1$$

Let $f(x) = \sqrt{x}$

f is continuous at 1.

Since $f(1) = \lim_{n\to\infty} f(x) = 1$

$\lim_{n\to+1} f(x) = \lim_{n\to-1} f(x) = 1$

The limit from left equal the limit from right equal to 1.

Here , by theorem of converge $f\left(\frac{n+1}{n}\right) \to f(1), < \frac{n+1}{n} > \to 1$.

2. $\boldsymbol{< 2^{\frac{1}{n}} > \to 1}$

Let $x = \frac{1}{n}$, $\lim_{n\to\infty} x = \lim_{n\to\infty} \frac{1}{n} = 0$

Let $f(x) = 2^x$ continuous at 0

$\therefore$ By theorem of converge (if $s_n \to L$ and f is a continuous function on L then $f(s_n) \to f(L)$).

$f(\frac{1}{n}) \to f(0)$

$\to 2^{\frac{1}{n}} \to 2^0 = 1$

$\therefore < 2^{\frac{1}{n}} > \to 1$

3. $\boldsymbol{< \sqrt{\frac{9n+1}{n-2}} > \to 3}$

Let $x = \frac{9n+1}{n-2}$, $\lim_{n\to\infty} x =$ H.W.

Let $f(x) = \sqrt{x}$

f is continuous at 1. H.W.

4. $< \frac{1}{2^{\frac{1}{n}}} > \to 1$

Let $x = \frac{1}{n}$, $\lim_{n\to\infty} x = \lim_{n\to\infty} \frac{1}{n} = 0$

Let $f(x) = \frac{1}{2^x}$ continuous at 0 H.W.

5. $< \sqrt{\frac{1}{n}} > \not\to 0$

Diverge to zero and the function not continuous at zero because the limit from the right doesn't exist.

The series

Definition:-

If $< a_n > = \{a_1, a_2, a_3, \dots\}$ is infinite sequence , then $a_1 + a_2 + a_3 + \dots$ is called an infinite series and it is written as $\sum_{n=1}^{\infty} a_n$.

If series of the form $a_1 + a_2 + a_3 + \dots + a_n$ .Then is called a finite series and it is written as $\sum_{k=1}^{n} a_k = S_n$, which S_n is finite series. The sequence $< S_n >$ is called the sequence of partial sums of $\sum_{k=1}^{n} a_k$, where:-

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

Example :-

$$* \sum_{n=1}^{\infty} 2n = 2(1) + 2(2) + 2(3) + \cdots$$

$$** \sum_{n=1}^{\infty} \frac{(n+1)^2}{n} = \frac{(1+1)^2}{1} + \frac{(2+1)^2}{2} + \frac{(3+1)^2}{3} + \cdots$$

$$= 4 + \frac{9}{2} + \frac{16}{3} + \cdots$$

$$*** \sum_{n=1}^{\infty} \left(\frac{1}{n} + n\right) = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right) + (1 + 2 + 3 + \cdots)$$

Theorem:-

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series, and $c \epsilon R$, then:-

1. $\sum_{n=1}^{\infty} c . a_n = c \sum_{n=1}^{\infty} a_n$
2. $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

Definition:-

The infinite series $\sum_{n=1}^{\infty} a_n$ Is said to be converging to L i.e. $\sum_{n=1}^{\infty} a_n = L$, if the sequence of partial sums $< S_n >$ converge to L.

$$\lim_{n \to \infty} S_n = L, L \epsilon R \ ; \sum_{k=1}^{\infty} a_k = S_n.$$

Example:- Is the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge ? find it's sum if exist?

Sol.:-

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)}$$

$$= \frac{A(n+1) + Bn}{(n+1)}$$

$$= \frac{An + A + Bn}{(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{A}{(n+1)}$$

$$A = 1\,,\, A + B = 0 \Rightarrow 1 + B = 0 \Rightarrow \quad B = -1$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{(n+1)}$$

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$\vdots$$

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{(n+1)}\right) = 1 - \frac{1}{(n+1)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)}\right) = 1 - 0 = 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{(n+1)} = 1$$

Are the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge ? find it's sum if exist?

1. $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ **2.** $\sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)}$ **Homework`**

3. $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ **4.** $\sum_{n=0}^{\infty} (\frac{1}{n+3} - \frac{1}{n+1})$

Geometric Series:-

$$\sum_{n=1}^{\infty} ar^n \;, \sum_{n=0}^{\infty} ar^n \;, \sum_{n=1}^{\infty} ar^{n+1} , \sum_{n=0}^{\infty} ar^{n+1} \; and \sum_{n=1}^{\infty} ar^{n-1} \dots etc.$$

Those are Geometric Series and a is positive constant , r is the base of series .

The sequence of partial some for the series $\sum_{n=1}^{\infty} ar^{n-1}$ is:-

$$S_n = \underset{\text{first term}}{\underline{a}} + ar + ar^2 + \dots + ar^{n-1}$$

1. **If $|r| = 1$**

$$S_n = a + a + a + \dots + a = na$$

$< S_n > = < na >$ **diverge** ➡ $\sum_{n=1}^{\infty} ar^{n-1}$ **diverge**

2. **If $|r| > 1$**

$\sum_{n=1}^{\infty} ar^{n-1}$ **diverge**

3. **If $|r| < 1$**

The series $\sum_{n=1}^{\infty} ar^{n-1}$ is converge and the sum is:-

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{\text{first term}}{1 - \text{base}} = \frac{a}{1 - r}$$

Not:-If the series $\sum_{n=1}^{\infty} ar^n$ and $|r| < 1$ the sum is

$$\sum_{n=1}^{\infty} \boldsymbol{ar^n} = \frac{\boldsymbol{ar}}{\boldsymbol{1-r}}$$

That mean the first term is ar and the partial sequence of the series is

$$\boldsymbol{S_n = ar + ar^2 + ar^3 + \cdots + ar^n}$$

Example:- which series are converge and diverge .if it converge find the sum of series?

$$\mathbf{1.} \sum_{n=1}^{\infty} \left(\frac{\mathbf{7}}{\mathbf{4^n}}\right)$$

$$\text{sol}\,.\sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right) = \frac{7}{4} + \frac{7}{4^2} + \frac{7}{4^3} + \cdots$$

$$\sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right) = \sum_{n=1}^{\infty} 7.\left(\frac{1}{4^n}\right) = \sum_{n=1}^{\infty} ar^n$$

$$a = 7, r = \frac{1}{4}\, and\;\; F.T. = \frac{7}{4}$$

$|r| < 1 \Longrightarrow \sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right)$ converge and the sum is $= \frac{\frac{7}{4}}{1-\frac{1}{4}} = \frac{\frac{7}{4}}{\frac{3}{4}} = \frac{7}{3}$

$$\sum_{n=1}^{\infty} \left(\frac{7}{4^n}\right) \to \frac{7}{3}$$

$$\mathbf{2.} \sum_{n=1}^{\infty} \left(\frac{\mathbf{7}}{\mathbf{3}}\right)^{\boldsymbol{n}}$$

$$\text{sol}\,.\sum_{n=1}^{\infty} \left(\frac{7}{3}\right)^n = \frac{7}{3} + \left(\frac{7}{3}\right)^2 + \left(\frac{7}{3}\right)^3 + \cdots$$

$$\sum_{n=1}^{\infty} \left(\frac{7}{3}\right)^n = \sum_{n=1}^{\infty} 1.\left(\frac{7}{3}\right)^n = \sum_{n=1}^{\infty} ar^n$$

$$a = 1, r = \frac{7}{3} \, and \;\; F.T. = \frac{7}{3}$$

$|r| > 1 \Longrightarrow \sum_{n=1}^{\infty} \left(\frac{7}{3}\right)^n$ and have no summation.

diverge

3. $\sum_{n=1}^{\infty} \frac{1}{2}(1)^n = \sum_{n=1}^{\infty} ar^n$

sol . $\sum_{n=1}^{\infty} \frac{1}{2}(-1)^n = -\frac{1}{2} + \frac{1}{2}.(1)^2 + \frac{1}{2}.(-1)^3 + \cdots$

$a = \frac{1}{2}, |r| = 1 \;\; \therefore \sum_{n=1}^{\infty} \frac{1}{2}(-1)^n$ diverge.

4. $\sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right) = \sum_{n=1}^{\infty} \frac{5}{2^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = \sum_{n=1}^{\infty} 5.\frac{1}{2^n} = \sum_{n=1}^{\infty} ar^n, \quad a = 5, |r| = \frac{1}{2}, \qquad F.T. = \frac{5}{2}$$

The series is converge and the sum is:-

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = \frac{\frac{5}{2}}{1 - \frac{1}{2}} = \frac{\frac{5}{2}}{\frac{1}{2}} = 5 \quad , \therefore \sum_{n=1}^{\infty} \frac{5}{2^n} \to 5$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} 1.\frac{1}{3^n} = \sum_{n=1}^{\infty} ar^n \quad , \quad a = 1, |r| = \frac{1}{3}, \qquad F.T. = \frac{1}{3}$$

The series is converge and the sum is:-

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \quad , \therefore \sum_{n=1}^{\infty} \frac{1}{3^n} \to \frac{1}{2}$$

$$\sum_{n=1}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)=\sum_{n=1}^{\infty}\frac{5}{2^n}+\sum_{n=1}^{\infty}\frac{1}{3^n}=5+\frac{1}{2}=\frac{11}{2}\ ,\therefore \sum_{n=1}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)\to\frac{11}{2}$$

Discuss the following series?

1. $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^{n-1}}+\cdots$

$$=\sum_{n=1}^{\infty}2.\frac{1}{3^{n-1}}=\sum_{n=1}^{\infty}ar^n\ ,a=2,r=\frac{1}{3}\ ,F.T.=2$$

$|r|<1$ ➡ $\sum_{n=1}^{\infty}\left(\frac{2}{3^{n-1}}\right)$ converge and the sum is $=\frac{2}{1-\frac{1}{3}}=\frac{2}{\frac{2}{3}}=3$

$$\therefore\sum_{n=1}^{\infty}\left(\frac{2}{3^{n-1}}\right)\to 3$$

2. $1+\frac{4}{3}+\left(\frac{4}{3}\right)^2+\cdots$

$$\sum_{n=0}^{\infty}\left(\frac{4}{3}\right)^n=\sum_{n=0}^{\infty}1.\left(\frac{4}{3}\right)^n=\sum_{n=0}^{\infty}ar^n$$

$$a=1,r=\frac{4}{3}and\ \ F.T.=1$$

$|r|>1$ ➡ $\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$ diverge.

3. $2-3+8+7^2+7^3+7^4+\cdots$

$$=-1+1+7+7^2+7^3+7^4+\cdots$$

$$=-1+\sum_{n=0}^{\infty}(7)^n=\sum_{n=0}^{\infty}ar^n\ ,a=1,|r|=7>1$$

4. $1-2+4-8+\cdots$

$$=(-2)^0+(-2)^1+(-2)^2+(-2)^3+\cdots$$

$$= \sum_{n=0}^{\infty} (-2)^n \,,, a = 1, |r| = 2 > 1$$

The series is diverge.

5. $3 + 6 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$

$$= 8 + 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$$

$$= 8 + \sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} 1.\frac{1}{3^n} = \sum_{n=0}^{\infty} ar^n \quad , \quad a = 1, |r| = \frac{1}{3} , \qquad F.T. = 1$$

The series is converge and the sum is:-

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2} \quad , \therefore 8 + \sum_{n=1}^{\infty} \frac{1}{3^n} \to 8 + \frac{3}{2} = \frac{19}{2}$$

6. $1 + 2(7) + 2(7)^2 + 2(7)^3 + \cdots$ H.W.
7. $\sum_{n=1}^{\infty} c = c + c + c + \cdots$ is constant series and geometric series with

$a = c, |r| = 1$ where c is constant and The series is diverge.

Write the geometric series with (Homework).

$1. a = \frac{1}{5}, r = \frac{2}{3}$

$2. a = 3, r = \frac{1}{2}$

Method of convergence

1. The P-Series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad , \qquad \boldsymbol{p > 0 \quad \text{is P} - \text{Series}}$$

a) If $P > 1 \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ is converge.

b) If $P < 1$ $\longrightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is diverge.

c) If $P = 1$ $\longrightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is diverge and it's called **Harmonic Series** .

Example:-

1) $\sum_{n=1}^{\infty} \frac{1}{n}$, Harmonic Series and P-Series diverge because p=1.
2) $\sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$, Harmonic Series and P-Series diverge because p=1.
3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, P-Series diverge because $p = \frac{1}{2} < 1$.
4) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ **,** P-Series converge because p = 2 >1.
5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n}$, Harmonic Series and P-Series diverge because p=1.
6) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ **,** P-Series converge because $p = \frac{3}{2} > 1$**.**

2. The n-Ratio n-terms Test:-

Let $\sum_{n=1}^{\infty} a_n$ be a series ; $a_n > 0$,then:-

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \in R$,

a) If $L < 1$ $\longrightarrow$ $\sum_{n=1}^{\infty} a_n$ is converge.
b) If $L > 1$ $\longrightarrow$ $\sum_{n=1}^{\infty} a_n$ is diverge.
c) If $L = 1$ $\longrightarrow$ $\sum_{n=1}^{\infty} a_n$ the series may by converge or diverge.

Examples :- Are the following series converge ?

i. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Sol. $\mathrm{Lim}_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \lim_{n\to\infty} \frac{n+1}{3^{n+1}} . \frac{3^n}{n} = \lim_{n\to\infty} \frac{n+1}{3n}$

$= \frac{1}{3} \lim_{n\to\infty} (\frac{n}{n} + \frac{1}{n}) = \frac{1}{3}(1+0) = \frac{1}{3} < 1$

$$\therefore \sum_{n=1}^{\infty} \frac{\boldsymbol{n}}{\mathbf{3}^{\boldsymbol{n}}} \to \frac{1}{3}$$

ii. $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

Sol. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}} = \lim_{n\to\infty} \frac{2^{n+1}+5}{3^{n+1}} \cdot \frac{3^n}{2^n+5} = \lim_{n\to\infty} \frac{2^{n+1}+5}{3(2^n+5)}$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{2^n.2+5}{2^n+5} = \frac{1}{3}\lim_{n\to\infty} \frac{\frac{2^n.2}{2^n}+\frac{5}{2^n}}{\frac{2^n}{2^n}+\frac{5}{2^n}} = \frac{1}{3}\lim_{n\to\infty} \frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}} = \frac{1}{3}\left(\frac{2+0}{1+0}\right) = \frac{2}{3} < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n+5}{3^n} \to \frac{2}{3}$$

iii. $\sum_{n=1}^{\infty} \frac{4^n\sqrt{n}}{3^n}$

Sol. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{4^{n+1}\sqrt{n+1}}{3^{n+1}}}{\frac{4^n\sqrt{n}}{3^n}} = \lim_{n\to\infty} \frac{4^{n+1}\sqrt{n+1}}{3^{n+1}} \cdot \frac{3^n}{4^n\sqrt{n}} = \lim_{n\to\infty} \frac{4\sqrt{n+1}}{3\sqrt{n}}$

$$= \frac{4}{3}\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \frac{4}{3}\lim_{n\to\infty} \sqrt{\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}}} = \frac{4}{3}\sqrt{\frac{1+0}{1}} = \frac{4}{3} > 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n+5}{3^n} \text{ diverge.}$$

iv. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

Sol. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} = \lim_{n\to\infty} \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^3}$

$$=\frac{1}{3}\lim_{n\to\infty}(\frac{\boldsymbol{n+1}}{\boldsymbol{n}})^3.\frac{(\boldsymbol{-1})^{\boldsymbol{n}}.(\boldsymbol{-1})}{\boldsymbol{3^n.3}}.\frac{\boldsymbol{3^n}}{(\boldsymbol{-1})^{\boldsymbol{n}}}=\frac{-1}{3}\lim_{n\to\infty}(\frac{\boldsymbol{n}}{\boldsymbol{n}}+\frac{1}{n})^3$$

$$=\frac{-1}{3}(1+0)=\frac{-1}{3}<1$$

$$\therefore \sum_{\boldsymbol{n=1}}^{\infty}(\boldsymbol{-1})^{\boldsymbol{n}}\frac{\boldsymbol{n^3}}{\boldsymbol{3^n}}\to\frac{-1}{3}$$

3. The Comparison Ratio Test:-

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two infinite series such that:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=L\neq 0$$

Then:

1) $\sum_{n=1}^{\infty} a_n$ is converge if and only if $\sum_{n=1}^{\infty} b_n$ is converge.
2) $\sum_{n=1}^{\infty} a_n$ is diverge if and only if $\sum_{n=1}^{\infty} b_n$ is diverge.

Examples :- Are the following series converge ?

i. $\sum_{\boldsymbol{n=1}}^{\infty}\frac{\boldsymbol{1}}{\boldsymbol{2^n-1}}$

Sol. $a_n=\frac{\boldsymbol{1}}{\boldsymbol{2^n-1}}$, $b_n=\frac{\boldsymbol{1}}{\boldsymbol{2^n}}$

$$\operatorname{Lim}_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}=\lim_{n\to\infty}\frac{2^n}{2^n-1}=\lim_{n\to\infty}\frac{\frac{2^n}{2^n}}{\frac{2^n}{2^n}-\frac{1}{2^n}}=\frac{1}{1-0}=1\neq 0$$

$\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{1}{2^n}$ geo. Ser. $|r|=\frac{1}{2}<1$

$\therefore \sum_{n=1}^{\infty}\frac{1}{2^n}$ is converge and

$$\operatorname{Lim}_{n\to\infty}\frac{a_n}{b_n}\neq 0$$

$\therefore \sum_{n=1}^{\infty} a_n=\sum_{\boldsymbol{n=1}}^{\infty}\frac{\boldsymbol{1}}{\boldsymbol{2^n-1}}$ is converge

ii. $\sum_{\boldsymbol{n=1}}^{\infty}\frac{\boldsymbol{2n}}{\boldsymbol{3n^2-4n+1}}$

Sol. $a_n = \frac{\frac{2}{n}}{3-\frac{4}{n}+\frac{1}{n^2}}$, $b_n = \frac{2}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\frac{2}{n}}{3-\frac{4}{n}+\frac{1}{n^2}}}{\frac{2}{n}} = \lim_{n\to\infty}\frac{1}{3-\frac{4}{n}+\frac{1}{n^2}} = \frac{1}{3-0+0} = \frac{1}{3} \neq 0$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{2}{n}$ P- Ser. $p = 1$

$\therefore \sum_{n=1}^{\infty}\frac{2}{n}$ is diverge and

$\lim_{n\to\infty}\frac{a_n}{b_n} \neq 0$

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{2n}{3n^2-4n+1}$ is diverge.

iii. $\sum_{n=1}^{\infty}\frac{2n^3+100n^2+1000}{8n^6-n+2}$ Hint let $b_n = \frac{2}{n^3}$

Sol. $a_n = \frac{\frac{2}{n^3}+\frac{100}{n^4}+\frac{1000}{n^6}}{8-\frac{1}{n^5}+\frac{2}{n^6}}$, $b_n = \frac{2}{n^3}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\frac{2}{n^3}+\frac{100}{n^4}+\frac{1000}{n^6}}{8-\frac{1}{n^5}+\frac{2}{n^6}}}{\frac{2}{n^3}} = \lim_{n\to\infty}\frac{1+\frac{50}{n}+\frac{500}{n^3}}{8-\frac{1}{n^5}+\frac{2}{n^6}} = \frac{1+0+0}{8-0+0} = \frac{1}{8} \neq 0$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{2}{n^3}$ P- Ser. $p > 1$

$\therefore \sum_{n=1}^{\infty}\frac{2}{n^3}$ is converge and

$\lim_{n\to\infty}\frac{a_n}{b_n} \neq 0$

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{2n^3+100n^2+1000}{8n^6-n+2}$ is converge.

iv. $\sum_{n=1}^{\infty}\frac{1}{n+1}$ is diverge because $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ (Homework)

4. The n-th Root Test

Let $\sum_{n=1}^{\infty} a_n$ be infinite series , such that :-

$\lim_{n\to\infty} \sqrt[n]{a_n} = L \in R$, then :-

a) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} \boldsymbol{a_n}$ is converge.
b) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} \boldsymbol{a_n}$ is diverge.
c) If $L = 1 \Longrightarrow \sum_{n=1}^{\infty} \boldsymbol{a_n}$ has no information.

Ex. Discuss the convergence for the following series?

i) $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

Sol.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{3^n}} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{3}\right)^n} = \frac{2}{3} < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n}{3^n} \to \frac{2}{3}$$

ii) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Sol.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^n} \to 0$$

iii) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

Sol.

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{2^n}{n^2}}=\lim_{n\to\infty}\frac{2^{\frac{n}{n}}}{n^{\frac{2}{n}}}=2\lim_{n\to\infty}\frac{1}{n^{\frac{2}{n}}}=2.\frac{1}{\infty^{\frac{2}{\infty}}}=2.\frac{1}{\infty^0}=2>1$$

$$\therefore\sum_{n=1}^{\infty}\frac{2^n}{n^2}\text{ the series is diverge.}$$

iv) $\boldsymbol{\sum_{n=1}^{\infty}\frac{5n}{2(3)^n}}$

Sol.

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{5n}{2(3)^n}}=\lim_{n\to\infty}\left(\frac{5}{2}\right)^{\frac{1}{n}}.\frac{n^{\frac{1}{n}}}{3^{\frac{n}{n}}}=\left(\frac{5}{2}\right)^{\frac{1}{\infty}}.\frac{\infty^{\frac{1}{\infty}}}{3}=\left(\frac{5}{2}\right)^{0}.\frac{\infty^0}{3}=\frac{1}{3}<1$$

$$\therefore\sum_{n=1}^{\infty}\frac{5n}{2(3)^n}\text{ the series is convergent .}$$

v) $\boldsymbol{\sum_{n=1}^{\infty}\frac{5^n}{2(3)^n}}$

Sol.

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{5^n}{2(3)^n}}$$

$$=\lim_{n\to\infty}\left(\frac{1}{2}\right)^{\frac{1}{n}}.\left(\frac{5}{3}\right)^{\frac{n}{n}}=\left(\frac{1}{2}\right)^{\frac{1}{\infty}}.\left(\frac{5}{3}\right)=\left(\frac{1}{2}\right)^{0}.\left(\frac{5}{3}\right)=\frac{5}{3}<1$$

$$\therefore\sum_{n=1}^{\infty}\frac{5^n}{2(3)^n}\text{ the series is diverge.}$$

5. The Comparison inequality Test

If $0\le a_n\le b_n\ \forall n\in N$, then:

- If $\sum_{n=1}^{\infty}a_n$ diverge, then $\sum_{n=1}^{\infty}b_n$diverge.
- If $\sum_{n=1}^{\infty}b_n$coverge, then $\sum_{n=1}^{\infty}a_n$converge.

Ex. Are the following series converge?

1. $\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$

Sol.

Is converge since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is converge

$$0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n^2}$$

2. $\sum_{n=1}^{\infty}\frac{1}{n!}$

Sol.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n!}\ ,\sum_{n=1}^{\infty} b_n = \sum_{n=0}^{\infty}\frac{1}{2^n}$$

$$\sum_{n=1}^{\infty}\frac{1}{n!} = \frac{1}{1} + \frac{1}{(2)1} + \frac{1}{(3)(2)1} + \frac{1}{(4)(3)(2)1} + \cdots$$

$$\sum_{n=0}^{\infty}\frac{1}{2^n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$0 \leq \frac{1}{n!} \leq \frac{1}{2^n}$$

$\sum_{n=1}^{\infty} a_n$ converge since $\sum_{n=1}^{\infty} b_n$ is converge .

6. The Absolutely Converge Test:-

- If $\sum_{n=1}^{\infty} a_n$ is Absolutely converge if $\sum_{n=1}^{\infty}|a_n|$ is converge.
- If $\sum_{n=1}^{\infty}|a_n|$ is not converge, then $\sum_{n=1}^{\infty} a_n$ is not Absolutely converge.

Ex.

1. $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}$ Absolutely converge since $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{2^n}\right| = \sum_{n=1}^{\infty}\frac{1}{2^n}$

$\sum_{n=1}^{\infty}\frac{1}{2^n}$ geometric series $|r| = \left|\frac{1}{2}\right| < 1$ converge.

The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}$ is Absolutely converge.

2. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ Absolutely converge since $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ P- series $p = 2 > 1$ converge.

The series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ is Absolutely converge.

3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is not Absolutely converge since

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ the series is diverge.

4. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ is not Absolutely converge since $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ P- series $p = \frac{1}{2} < 1$ diverge.

Conditional Converge :-

If $\sum_{n=1}^{\infty}(-1)^n a_n$ is converge, but $\sum_{n=1}^{\infty}|a_n|$ is not converge.

$\sum_{n=1}^{\infty}(-1)^n a_n$ Is converge if it satisfies the following condition:-

- $a_n > 0 \quad \forall n \in N.$
- $a_n \geq a_{n+1} \ \forall n \in N.$
- $\lim_{n\to\infty} a_n = 0 \ \forall n \in N.$

We said that the series was Conditional Converge not absolutely converge.

Ex. Do the following series absolutely converge or conditional converge?

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not Absolutely converge since

   $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ the series is diverge.

$$a_n > 0 \quad \forall n \in N \rightarrow \frac{1}{n} > 0$$

$$a_n \geq a_{n+1} \quad \rightarrow \frac{1}{n} \geq \frac{1}{n+1}$$

$$\lim_{n\to\infty} a_n = 0 \rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ conditional converge not absolutely converge .

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ Is not Absolutely converge, but conditional converge. (Homework)
3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ Is not Absolutely converge, but conditional converge. (Homework)

Theorem:-

If $\sum_{n=1}^{\infty} a_n$ is converge, then $\lim_{n\to\infty} a_n = 0$.

That means if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is diverge.

4. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{5n+1}$ Is not Absolutely converge

   $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{n}{5n+1}\right| = \sum_{n=1}^{\infty} \frac{n}{5n+1}$

$$\lim_{n\to\infty} a_n = 0 \rightarrow \lim_{n\to\infty} \frac{n}{5n+1} = \lim \frac{1}{5+\frac{1}{n}} = \frac{1}{5} \neq 0$$

By theorem above $\sum_{n=1}^{\infty} \frac{n}{5n+1}$ is diverge.

i.e. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{5n+1}$ Is not Absolutely converge.

Note: - The converse of the theorem above is not true

If $\lim_{n\to\infty} a_n = 0 \to$ this not necessary to be $\sum_{n=1}^{\infty} a_n$ converge.

Ex. Are the series converging or diverging?

1. $\sum_{n=1}^{\infty} \frac{1}{n}$ is diverge

Although $\lim_{n\to\infty} \frac{1}{n} = 0$.

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is diverge

Although $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

3. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is converge

And $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

4. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ Is diverge because

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$. By theorem $\therefore \sum_{n=1}^{\infty} a_n$ is diverge.

5. $\sum_{n=1}^{\infty} n^2$

$\because \lim_{n\to\infty} n^2 = \infty \neq 0 \to$ By theorem $\therefore \sum_{n=1}^{\infty} a_n$ is diverge.

6. $\sum_{n=1}^{\infty} \sqrt{n}$

$\because \lim_{n\to\infty} \sqrt{n} = \infty \neq 0 \to$ By theorem $\therefore \sum_{n=1}^{\infty} a_n$ is diverge.

7. $\sum_{n=1}^{\infty} \frac{n^3}{n\sqrt{n}}$ **(Homework)**

8. $\sum_{n=1}^{\infty} \frac{n^2+1}{n}$

9. $\sum_{n=1}^{\infty} n$

10. $\sum_{n=1}^{\infty} \frac{n^2\sqrt{n}}{n^2+1}$

Ex. Determine whether the given series converge or diverge ?

1. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$
2. $1+\sum_{n=1}^{\infty} \frac{1}{2^n}$
3. $\sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$
4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3}$
5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
6. $\sum_{n=1}^{\infty} \frac{(n+1)!}{3!\, n!\, 3^n}$

Chapter Two

The power series :- are the series of the form:

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n + \cdots$$

In which the center a and the coefficients $C_0, C_1, \ldots$ are constants.

Ex.:- Taking the coefficients to be 1 in the equation of the power series, we get the geometric power series :

$$\sum_{n=0}^{\infty} 1.x^n = 1 + x + x^2 + \cdots + x^n + \cdots, \text{ where } a = 1, r = x$$

when$|x| < 1$,

Ex.:- Find the interval of the convergence of the series $\sum_{n=0}^{\infty} x^n$ and the sum of it ?

$\sum_{n=0}^{\infty} 1.x^n = \sum_{n=0}^{\infty} a.r^n$ geometric series

$a = 1, r = x$, is converge when $|r| = |x| < 1 \rightarrow -1 < x < 1 \rightarrow x \epsilon (-1,1)$.

$$\sum_{n=0}^{\infty} x^n = \frac{F.T.}{1-r} = \frac{1}{1-x} \quad , \qquad \forall x \epsilon (-1,1)$$

Ex. For the following power series: find the interval of convergene and the sum of it?

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

$a = 1, r = -\frac{1}{2}(x-2)$, the series is converge when

$$|r| = \left|-\frac{1}{2}(x-2)\right| < 1 \rightarrow \left|\frac{x-2}{2}\right| < 1 \rightarrow -1 < \frac{x-2}{2} < 1$$

$-2 < x - 2 < 2 \quad \to 0 < x < 4 \qquad x \epsilon\ (0,4).$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = \frac{F.T.}{1-r} = \frac{1}{1-(-\frac{1}{2}(x-2))}$$

$$= \frac{1}{1+\frac{x-2}{2}} = \frac{1}{\frac{2+x-2}{2}} = \frac{2}{x}, \quad \forall\, x \in (0,4)$$

Ex. Find the power series if $C_n = 3$ *and* $a = 2$? And find the interval of convergence and the sum of it? Homework

Sol.

$$\sum_{n=0}^{\infty} 3(x-2)^n$$

The Taylor Series Expansion of $f(x)$ about a or near $x = a$

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then Taylor series generated by f at a is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Remark:-

When a=0 , then the series said Maclaurin Series.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Ex. : - Find Taylor Series

Expansion of $f(x) = \frac{1}{x}$ $at\ a = 2$, and find the convergence sum? or Find Taylor Series Expansion generated by $f(x) = \frac{1}{x}$ $near\ \ a = 2$, and find the convergence sum?

Sol.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots (*)$$

Now we found the derivatives of the function

$$f(x) = \frac{1}{x} = x^{-1}, f'(x) = -x^{-2}, f''(x) = 2x^{-3},$$

$$f'''(x) = -6x^{-4}, f^{(4)}(x) = 24x^{-5}, \dots, f^{(n)}(x) = (-1)^n . n! . x^{-(n+1)}, \dots$$

Now we put a=2

$$f(2) = \frac{1}{2} = 2^{-1}, f'(2) = -2^{-2}, f''(2) = 2.2^{-3}, f'''(2) = -6.2^{-4}$$

$$f^{(4)}(2) = 24(2)^{-5}, \dots, f^{(n)}(2) = (-1)^n . n! . 2^{-(n+1)}, \dots$$

$$\frac{f^{(n)}(2)}{n!} = (-1)^n . 2^{-(n+1)} = \frac{(-1)^n}{2^{(n+1)}}$$

Now we put those values in series * and we get

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$f(x) = \frac{1}{2} + -2^{-2}(x-2) + \frac{2.2^{-3}}{2!}(x-2)^2 + \cdots + \frac{(-1)^n}{2^{(n+1)}}(x-2)^n + \cdots$$

$$f(x)=\frac{1}{2}-\frac{1}{2^2}(x-2)+\frac{1}{2^3}(x-2)^2-\cdots+\frac{(-1)^n}{2^{(n+1)}}(x-2)^n+\cdots$$

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{(n+1)}}(x-2)^n$$

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{2}(-1)^n\left(\frac{1}{2}\right)^n(x-2)^n$$

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{2}\left(-\frac{1}{2}(x-2)\right)^n$$

This is a geometric series $a=\frac{1}{2}$, $r=-\frac{1}{2}(x-2)$ and it's converge at the intervals: $|r|<1\rightarrow\left|-\frac{1}{2}(x-2)\right|<1\rightarrow\left|\frac{1}{2}(x-2)\right|<1$

$$-1<\frac{1}{2}(x-2)<1\rightarrow-2<(x-2)<2\rightarrow 0<x<4\quad x\,\epsilon\,(0,4)$$

$$\sum_{n=0}^{\infty}ar^n=\frac{F.T.}{1-r}=\frac{\frac{1}{2}}{1+\frac{x-2}{2}}=\frac{\frac{1}{2}}{\frac{2+x-2}{2}}=\frac{1}{x}$$

Ex. Find the Taylor Series for $f(x)=\frac{1}{x}$ $near\ a=1$?(Homework)

Ex. Find the converge interval and the sum if exist?

1. $\sum_{n=1}^{\infty}\frac{x^n}{n}$

Sol. We solve this series by using The n-Ratio n-terms test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right|=\lim_{n\to\infty}\frac{|x|^{n+1}}{n+1}.\frac{n}{|x|^n}=\lim_{n\to\infty}\frac{|x|.n}{n+1}$$

$$=|x|.1<1\rightarrow-1<x<1\rightarrow x\,\epsilon(-1,1).$$

2. $\sum_{n=0}^{\infty}2x^n$

   Sol.

$$\sum_{n=0}^{\infty} \boldsymbol{2x^n} = \sum_{n=0}^{\infty} ar^n = \frac{F.T.}{1-r} = \frac{2}{1-x}$$

$|r| = |x| < 1 \rightarrow -1 < x < 1, x \in (-1,1).$

3. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$

Sol.

$$\sum_{n=0}^{\infty} \mathbf{1}.\frac{(\boldsymbol{x}+\mathbf{2})^{\boldsymbol{n}}}{\mathbf{3}^{\boldsymbol{n}}} = \sum_{n=0}^{\infty} ar^n = \frac{F.T.}{1-r} = \frac{1}{1-\frac{x+2}{3}} = \frac{1}{\frac{3-x-2}{3}} = \frac{3}{1-x}$$

$$|r| = \left|\frac{x+2}{3}\right| < 1 \rightarrow -1 < \frac{x+2}{3} < 1 \rightarrow -3 < x+2 < 3$$

$-1 < x < 5$, converge interval $x \in (-5,1)$.

4. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}}$

Sol.

$$\sum_{n=0}^{\infty} \frac{\mathbf{1}}{\mathbf{3}}.\frac{(\boldsymbol{x}+\mathbf{2})^{\boldsymbol{n}}}{\mathbf{3}^{\boldsymbol{n}}} = \sum_{n=0}^{\infty} ar^n = \frac{F.T.}{1-r} = \frac{\frac{1}{3}}{1-\frac{x+2}{3}} = \frac{\frac{1}{3}}{\frac{3-x-2}{3}} = \frac{1}{1-x}$$

$$|r| = \left|\frac{x+2}{3}\right| < 1 \rightarrow -1 < \frac{x+2}{3} < 1 \rightarrow -3 < x+2 < 3$$

$-1 < x < 5$, converge interval $x \in (-5,1)$.

5. $\sum_{n=0}^{\infty} \frac{(2x-5)^n}{n^2}$

Sol.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2x-5)^n}{n^2}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{|(2x-5)^n|}{n^2}}$$

$$= |2x-5| \lim_{n\to\infty} \frac{1}{n^{\frac{2}{n}}}$$

$$= |2x-5| < 1 \rightarrow -1 < 2x-5 < 1$$

$4 < 2x < 6 \rightarrow 2 < x < 3$, $\quad x \in (2,3)$ converge interval

6. $\sum_{n=0}^{\infty} nx^n$ **(Homework)**
7. $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$
8. $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$
9. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

Ex. Find the Taylor polynomial generated by $f(x) = e^x$ $near$ $x = 0$?

Sol.

$f(x) = e^x$, $a = 0$

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$f(x) = e^x \to f(0) = e^0 = 1.$

$f'(x) = e^x \to f'(0) = e^0 = 1.$

$f''(x) = e^x \to f''(0) = e^0 = 1$

$\vdots$

$f^{(n)}(x) = e^x \to f^{(n)}(0) = e^0 = 1$

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = e^0 + e^0x + \frac{e^0}{2!}x^2 + \cdots + \frac{e^0}{n!}x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Ex. Find the Taylor polynomial generated by

$f(x) = e^{-x}$ $near$ $x = 0$?(Homework).

Ex. Find the Maclaurin Series generated by $f(x) = cos\ x$?

Sol.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$f(x) = \cos x \to f(0) = \cos 0 = 1$

$f'(x) = -\sin x \to f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \to f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \ \to f'''(0) = \sin 0 = 0$

$\vdots$

$f^{(2n)}(x) = (-1)^n \cos x \quad n \text{ even} \to f^{(2n)}(0) = (-1)^n$

$f^{(2n+1)}(x) = (-1)^n \sin x \quad n\ odd \to f^{(2n+1)}(0) = 0$

$$f(x) = \cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots$$

$$cos\ x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

Ex. Find the Maclaurin polynomial for $\boldsymbol{f(x) = sin\ x}$ **?**

Sol.

$$f(x)\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

$$f(x) = \sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$\sin x = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} + \cdots$$

$$\sin\ x = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!}x^{2k-1}$$

Computation of Logarithms:-

Frome Taylor series expansion, we get

$$\mathrm{Ln}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^n}{n} + \cdots$$

And

$$\mathrm{Ln}(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + \cdots$$

$$\mathrm{Ln}\frac{(1+x)}{(1-x)} = \mathrm{Ln}(1+x) - \mathrm{Ln}(1-x)$$

$$\mathrm{Ln}\frac{(1+x)}{(1-x)} = 2(x + \frac{x^3}{3} + \frac{x^5}{5} \ldots + \frac{x^{2n-1}}{2n-1} + \cdots)$$

Now put $\frac{N+1}{N} = \frac{(1+x)}{(1-x)}$

$$N(1+x) = (N+1)(1-x)$$

$$N + Nx = N - Nx + 1 - x$$

$$2Nx + x = 1 \rightarrow x(2N+1) = 1 \rightarrow x = \frac{1}{(2N+1)}$$

$$\mathrm{Ln}\frac{N+1}{N} = 2(\frac{1}{(2N+1)} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots)$$

Ex. By approximate find Ln2 ?

Sol. $\mathrm{Ln}2 = \mathrm{Ln}\frac{1+1}{1} = 2(\frac{1}{(2(1)+1)} + \frac{1}{3(2(1)+1)^3} + \frac{1}{5(2(1)+1)^5} + \cdots)$

$$\mathrm{Ln}2 = 2\left(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5(3)^5} + \cdots\right)$$

$$= 2(0.333 + 0.0123 + 0.0008 + \cdots)$$

$$\cong 2(0.3464) \cong 0.69314$$

$$\mathrm{Ln}2 \cong 0.69314$$

The logarithms of the number $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots, \frac{N+1}{N}$ are ordinarily computed first informing a table of natural logarithms of the integers.

Then it is a matter of simple arithmetic to compute:-

$$\mathrm{Ln}3 = \mathrm{Ln}\frac{2}{1} + \mathrm{Ln}\frac{3}{2}$$

$$\mathrm{Ln}4 = \mathrm{Ln}3 + \mathrm{Ln}\frac{4}{3}$$

$$\mathrm{Ln}5 = \mathrm{Ln}4 + \mathrm{Ln}\frac{5}{4}$$

$$\vdots$$

$$\mathrm{Ln}(N+1) = \mathrm{Ln}N + \mathrm{Ln}\frac{N+1}{N}$$

Ex. Construct a table of natural logarithms in $N\ for\ N = 1,2,3, \dots 10$ by the method discussed in connection with equation

$$\mathrm{Ln}\frac{N+1}{N} = 2\left(\frac{1}{(2N+1)} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right)$$

But taking advantage of the relations ships:-

$$Ln4 = 2Ln2\,, \qquad Ln6 = Ln2 + Ln3\,, \qquad Ln\,8 = 3Ln2$$

$$Ln9 = 2Ln\,3\,, \qquad Ln10 = Ln2 + Ln5$$

Properties :-

$$Ln(a.b) = Lna + Lnb$$

$$Ln\frac{a}{b} = Lna - Lnb$$

$$Lna^k = kLna$$

Fourier Series

A Fourier series is an infinite series expansion in terms of trigonometric functions of $f(x)$ on the interval $(0, 2\pi)$ is

$$f(x) = a_o + \sum_{k=1}^{\infty}[a_k \cos kx + b_k \sin kx]$$

Where

$$a_o = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$

and

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos kx \ dx \qquad for\ k = 01,2,3, \dots, n$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin kx \ dx$$

Ex. Find the Fourier series for

$$f(x) = \begin{Bmatrix} 1 & 0 \le x \le \pi \\ 2 & \pi < x \le 2\pi \end{Bmatrix}$$

Sol.

$$f(x) = a_o + \sum_{k=1}^{\infty}[a_k \cos kx + b_k \sin kx] \qquad (1)$$

$$a_o = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$

$$a_o = \frac{1}{2\pi}(\int_0^{\pi} 1\, dx + \int_{\pi}^{2\pi} 2\, dx)$$

$$= \frac{1}{2\pi}\left[x|_0^{\pi} + 2x|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\pi - 0 + 2(2\pi - \pi)\right]$$

$$= \frac{1}{2\pi}[\pi + 2\pi] = \frac{3\pi}{2\pi}$$

$$\therefore a_o = \frac{3}{2}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx \; dx$$

$$a_k = \frac{1}{\pi}\left(\int_0^{\pi} cos\, kx\, dx + \int_{\pi}^{2\pi} 2\, cos\, kx\, dx\right)$$

$$= \frac{1}{\pi}\left[\frac{\sin kx}{k}|_0^{\pi} + 2\frac{sin\, kx}{k}|_{\pi}^{2\pi}\right] \quad ; k = 1,2,3, \dots, n$$

$$= \frac{1}{\pi}\left[\frac{1}{k}(sin\, k\, \pi - sin\, 0) + \frac{2}{k}(sin\, k\, 2\pi - sin\, k\, \pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{k}(0-0) + \frac{2}{k}(0-0)\right] = \frac{1}{\pi}(0) = 0$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx \; dx$$

$$b_k = \frac{1}{\pi}\left(\int_0^{\pi} sin\, kx\, dx + \int_{\pi}^{2\pi} 2\, sin\, kx\, dx\right)$$

$$= \frac{1}{\pi}\left[-\frac{\cos kx}{k}|_0^{\pi} - 2\frac{cos\, kx}{k}|_{\pi}^{2\pi}\right] \quad ; k = 1,2,3, \dots, n$$

$$= \frac{1}{\pi}\left[\frac{1}{k}(-\, cos\, k\, \pi + cos\, 0) + \frac{2}{k}(-\cos k\, 2\pi + cos\, k\, \pi)\right]$$

$$= \frac{1}{\pi k}[(-\cos k\pi + 1) + 2(-1 + \cos k\pi)]$$

$$= \frac{1}{\pi k}(\cos k\pi - 1) \quad ; k = 1,2,3, \dots, n$$

$$\therefore b_k = \frac{(-1)^k - 1}{k\pi}, \qquad b_k = \begin{cases} \frac{-2}{k\pi} & if k\ odd \\ 0 & if\ k\ even \end{cases}$$

$$b_1 = \frac{-2}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{-2}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{-2}{5\pi}$$

Now we put the$a_0,\ a_k,\ b_k$ in the series (1), we get:-

$$f(x) = \frac{3}{2} + \sum_{k=1}^{\infty}[0(\cos kx) + b_k \sin kx]$$

$$f(x) = \frac{3}{2} + \sum_{k=1}^{\infty}[b_k \sin kx]$$

$$= \frac{3}{2} + [b_1 \sin x + b_3\, sin\, 3x + b_5\, sin\, 5x + \cdots]$$

$$\therefore f(x) = \frac{3}{2} - \frac{2}{\pi}\left[\sin x + \frac{sin\, 3x}{3} + \frac{sin\, 5x}{5} + \cdots\right]$$

Ex2. $F(x) = 3,\ \ 0 \le x \le 2\pi$ find the Fourier series?

Sol.

$$f(x) = a_o + \sum_{k=1}^{\infty}[a_k \cos kx + b_k \sin kx] \qquad (*)$$

$$a_o = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} 3\, dx$$

$$= \frac{1}{2\pi}[3x|_0^{2\pi}]$$

$$= \frac{1}{2\pi} 3(2\pi - 0)$$

$$= \frac{6\pi}{2\pi}$$

$$\therefore a_o = 3$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \; dx$$

$$a_k = \frac{1}{\pi} (\int_0^{2\pi} 3 \, cos \, kx \, dx)$$

$$= \frac{1}{\pi} \left[3 \frac{sin \, kx}{k} |_0^{2\pi} \right] \quad ; k = 1,2,3, \dots, n$$

$$= \frac{1}{\pi} [\frac{3}{k} (sin \, k \, 2\pi - sin \, k \, 0)]$$

$$= \frac{1}{\pi} \left[\frac{3}{k} (0 - 0) \right] = \frac{1}{\pi} (0) = 0$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \; dx$$

$$b_k = \frac{1}{\pi} (\int_0^{2\pi} 3 \, sin \, kx \, dx)$$

$$= \frac{1}{\pi} \left[-3 \frac{cos \, kx}{k} |_0^{2\pi} \right] \quad ; k = 1,2,3, \dots, n$$

$$= \frac{3}{\pi k} (- \cos k \, 2\pi + cos \, k \, 0)$$

$$= \frac{3}{\pi k} (-1 + 1) = \frac{3}{\pi k} (0)$$

Now we put thea_0, a_k, b_k in the series (*), we get:-

$$f(x) = 3 + \sum_{k=1}^{\infty}[0(\cos kx) + 0(\sin kx)]$$

$\therefore\ f(x) = 3$

Ex.3:- $f(x) = x,\ \ \forall\ 0 \leq x \leq 2\pi$ find the Fourier series?

Sol.

$$f(x) = a_o + \sum_{k=1}^{\infty}[a_k \cos kx + b_k \sin kx] \qquad (*)$$

$$a_o = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx$$

$$= \frac{1}{2\pi}[\frac{x^2}{2}|_0^{2\pi}$$

$$= \frac{1}{4\pi}(x^2|_0^{2\pi})$$

$$= \frac{4\pi^2}{4\pi}$$

$\therefore a_o = \pi$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\ \ dx$$

$$a_k = \frac{1}{\pi}(\int_0^{2\pi} x\,cos\,kx\,dx)$$

By using $udv = v.u - \int vdu$

$$u = x, \qquad du = dx, \qquad dv = cos\,kx\,dx, \qquad v = \frac{\sin kx}{k}$$

$$= \frac{1}{\pi}\left[x.\frac{sin\,kx}{k}\Big|_{0}^{2\pi} - \int_{0}^{2\pi}\frac{sin\,kx}{k}\;dx\right]$$

$$= \frac{1}{\pi}\left[\left(2\pi.\frac{sin\,k2\pi}{k} - 0\right) + \frac{cos\,kx\,\Big|_{0}^{2\pi}}{k^2}\right]$$

$$= \frac{1}{\pi}\left[\left(2\pi.\frac{0}{k} - 0\right) + \frac{cos\,k2\pi - cos\,k0}{k^2}\right]$$

$$a_k = 0$$

$$b_k = \frac{1}{\pi}\left(\int_{0}^{2\pi} x\,sin\,kx\,dx\right)$$

By using $udv = v.u - \int vdu$

$$u = x, \qquad du = dx, \qquad dv = sin\,kx\,dx, \qquad v = \frac{-\cos kx}{k}$$

$$= \frac{1}{\pi}\left[-x.\frac{cos\,kx}{k}\Big|_{0}^{2\pi} + \int_{0}^{2\pi}\frac{cos\,kx}{k}\;dx\right]$$

$$= \frac{1}{\pi}\left[\left(-2\pi.\frac{cos\,k2\pi}{k} + 0\right) + \frac{sin\,kx\,\Big|_{0}^{2\pi}}{k^2}\right]$$

$$= \frac{1}{\pi}\left[\left(-2\pi.\frac{1}{k}\right) + \frac{sin\,k2\pi - sin\,k0}{k^2}\right] = \frac{1}{\pi}\left(-2\pi.\frac{1}{k}\right) + \frac{0-0}{k^2}$$

$$\therefore\, b_k = \frac{-2}{k}$$

Now we put the $a_0,\ a_k,\ b_k$ in the series (*), we get:-

$$f(x) = \pi + \sum_{k=1}^{\infty}\left[0(\cos kx) - \frac{2}{k}(\sin kx)\right]$$

$$\therefore \; f(x) = \pi \; - \frac{2}{k} \sum_{k=1}^{\infty} (\sin kx)]$$

Ex. Find the Fourier series associated with the following functions. Sketch each functions?

1- $f(x) = \begin{Bmatrix} 0 & 0 \le x \le \pi \\ 1 & \pi < x \le 2\pi \end{Bmatrix}$

2- $f(x) = \begin{Bmatrix} -1 & 0 \le x \le \pi \\ 1 & \pi < x \le 2\pi \end{Bmatrix}$

3- $f(x) = 3x, \;\; 0 \le x \le 2\pi$

4- $f(x) = \begin{Bmatrix} 2 & 0 \le x \le \pi \\ x & \pi < x \le 2\pi \end{Bmatrix}$

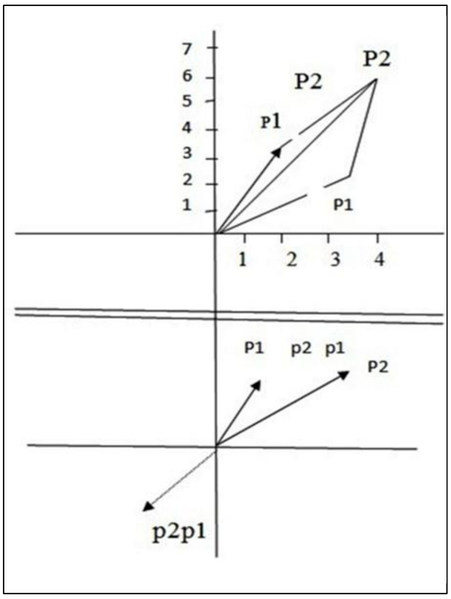
5- $f(x) = \begin{Bmatrix} x & 0 \le x \le \pi \\ -x & \pi < x \le 2\pi \end{Bmatrix}$

Chapter Three

The vectors

The vectors

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in RXR, then $\overrightarrow{P_1 P_2}$ or $\overrightarrow{P_2 P_1}$ are vectors where :-



$\overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1)$

$\overrightarrow{P_2 P_1} = (x_1 - x_2, y_1 - y_2)$

Ex. $\boldsymbol{P_1 = (2, 3)}$ **,** $\boldsymbol{P_2 = (4, 7)}$

$\overrightarrow{P_1 P_2} = (4 - 2, 7 - 3)$

$= (2,4)$

$\overrightarrow{P_2 P_1} = (2 - 4, 2 - 7)$

$= (-2, -4)$

Ex. $\boldsymbol{P_1 = (4, 2)}$ **,** $\boldsymbol{P_2 = (-3, 5)}$ find $\overrightarrow{P_1 P_2}$ and

$\overrightarrow{P_2 P_1}$?(Homework)

- If $\boldsymbol{P = (x, y)}$ **,** $\boldsymbol{O = (0, 0)}$, then

$\overrightarrow{O P} = (x - 0, y - 0) = (x, y)$

$\overrightarrow{PO} = (0 - x, 0 - y) = (-x, -y)$

$\overrightarrow{V} = (a, b) = (a, 0) + (0, b)$

$= a(1,0) + b(0,1)$

$\therefore \overrightarrow{V} = ai + bj$

$slop\ m = \dfrac{b}{a} = \tan\theta$

$(3,4) = 3i + 4j$

The Length of $\overrightarrow{\boldsymbol{V}}$:-

We denoted for The Length of $\overrightarrow{\boldsymbol{V}}$ by $\left|\overrightarrow{\boldsymbol{V}}\right|$ for the vector

$\overrightarrow{\boldsymbol{V}} = \boldsymbol{xi} + \boldsymbol{yj}$ **which equal :-**

$\left|\overrightarrow{\boldsymbol{V}}\right| = \sqrt{\boldsymbol{x^2} + \boldsymbol{y^2}}$

Ex. Find The Length of

1. **1-** $\overrightarrow{\boldsymbol{V}} = \boldsymbol{5i} + \boldsymbol{7j}$
2. **2-** $\overrightarrow{\boldsymbol{V}} = \boldsymbol{4j}$
3. **3-** $\overrightarrow{\boldsymbol{V}} = -\boldsymbol{3i}$ **(Homework)**
4. **4-** $\overrightarrow{\boldsymbol{V}} = -\boldsymbol{i} - \boldsymbol{j}$ **(Homework)**

Sol. 1-

$\left|\overrightarrow{V}\right| = \sqrt{5^2 + 7^2} = \sqrt{25 + 49} = \sqrt{74}$

Sol. 2-

$\left|\overrightarrow{V}\right| = \sqrt{0^2 + 4^2} = \sqrt{0 + 16} = \sqrt{16} = 4$

The Zero Vector:-

$\overrightarrow{\mathbf{0}} = \mathbf{0}\boldsymbol{i} + \mathbf{0}\boldsymbol{j} = (\mathbf{0}, \mathbf{0})$

$\left|\overrightarrow{\mathbf{0}}\right| = \sqrt{\mathbf{0}^2 + \mathbf{0}^2} = \mathbf{0}$

The Unit Vector:-

$u = xi + yj$ **w**hen $|u| = 1$, then u is said to be a unit vector.

- $\boldsymbol{i}(\mathbf{1}, \mathbf{0}) \Rightarrow |\boldsymbol{i}| = \sqrt{\mathbf{1}^2 + \mathbf{0}^2} = \mathbf{1}$**,** $\boldsymbol{i}$ **is a unit vector.**
- $\boldsymbol{j}(\mathbf{0}, \mathbf{1}) \Rightarrow |\boldsymbol{j}| = \sqrt{\mathbf{0}^2 + \mathbf{1}^2} = \mathbf{1},$ $\boldsymbol{j}$ **is a unit vector.**
- $\boldsymbol{u} = \mathbf{cos}\,\boldsymbol{\theta} + \mathbf{sin}\,\boldsymbol{\theta}$

  $\Rightarrow |\boldsymbol{u}| = \sqrt{(\boldsymbol{cos}\,\boldsymbol{\theta})^2 + (\boldsymbol{sin}\,\boldsymbol{\theta})^2} = \mathbf{1}, \boldsymbol{u}$ **is a unit vector.**

To find a unit vector for any vector

We must find the direction of vector $\boldsymbol{V}$ **is** $\frac{\boldsymbol{V}}{|\boldsymbol{V}|}$ **which is the unit vector such that** $\left|\frac{\boldsymbol{V}}{|\boldsymbol{V}|}\right| = \mathbf{1}.$

Ex. $\overrightarrow{\boldsymbol{V}} = \mathbf{3}\boldsymbol{i} - \mathbf{4}\boldsymbol{j}$ **find a unit vector for** $\overrightarrow{\boldsymbol{V}}$ **?**

Sol.

$$u = \frac{V}{|V|} \quad ; \quad |u| = 1$$

$$|V| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$u = \frac{V}{|V|} = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$\therefore u$ is the unit vector such that:

$$|u| = \left|\frac{V}{|V|}\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

Propositions:-

If $\boldsymbol{V_1 = a_1 i + b_1 j}$ & $\boldsymbol{V_2 = a_2 i + b_2 j}$ are two vectors :

1- $V_1 + V_2 = (a_1 + a_2)i + (b_1 + b_2)j$

$V_2 + V_1 = (a_2 + a_1)i + (b_2 + b_1)j$

Where $V_1 + V_2 = V_2 + V_1$

2- $V_1 - V_2 = (a_1 - a_2)i + (b_1 - b_2)j$

$V_2 - V_1 = (a_2 - a_1)i + (b_2 - b_1)j$

$V_1 - V_2 = -(V_2 - V_1)$

3- $cV_1 = ca_1 i + cb_1 j = c(a_1 i + b_1 j)$

4- $c(V_1 \pm V_2) = cV_1 \pm cV_2$

Ex. If $\boldsymbol{V_1 = 3i + 5j}$ & $\boldsymbol{V_2 = 3i - 2j}$, find ?

$V_1 + V_2 = 6i + 3j$

$V_2 + V_1 =$

$V_1 + V_1 =$

$V_1 - V_2 =$

$3V_1 =$

$2(5V_1 - V_2) - V_1 =$

Def.:- Two vectors $\boldsymbol{V_1 = a_1 i + b_1 j}$ & $\boldsymbol{V_2 = a_2 i + b_2 j}$ are equal iff $\boldsymbol{V_1 = V_2 \Rightarrow a_1 i + b_1 j = a_2 i + b_2 j}$

$\boldsymbol{a_1 = a_2 \ and \ b_1 = b_2}$

Ex. Show that $\boldsymbol{V_1 = 3i + 7j}$ & $\boldsymbol{V_2 = \frac{15}{5} i + \left(\frac{12}{2} + 1\right) j}$ are equal ?

Sol.

$$3 = \frac{15}{5} \quad \& \quad 7 = \frac{12}{2} + 1$$

$\therefore V_1 = V_2$

Ex. If $\boldsymbol{P_1}$ is the point (1,3) and $\boldsymbol{P_2}$(2,-1) find$\boldsymbol{\overrightarrow{P_1P_2}}$, $\boldsymbol{\overrightarrow{P_2P_1}}$, $\boldsymbol{5\overrightarrow{P_1P_2} - 3\overrightarrow{P_2P_1}}$?(Homework).

Ex. Find $\boldsymbol{\overrightarrow{OP_3}}$ if $\boldsymbol{O}$ is the origin point and $\boldsymbol{P_3}$ is the mid point of the vector $\boldsymbol{\overrightarrow{P_1P_2}}$ joining $\boldsymbol{P_1}$(2,-1) and $\boldsymbol{P_2}$(-4,3) ?

Sol.

$$P_3 = \frac{P_1 + P_2}{2}$$

P_1(2,-1) , P_2(-4,3)

$$P_3 = \left(\frac{2-4}{2}, \frac{-1+3}{2}\right)$$

$P_3 = (-1,1)$ & $O = (0,0)$

$\overrightarrow{OP_3} = (-1-0, 1-0) = (-1,1)$

Ex. Find the vector from the point A (2,3) to the origin point?

Sol.

$\overrightarrow{AO} = (0-2, 0-3) = (-2,-3)$

Ex. Find the sum of the vector $\overrightarrow{AB}$ and $\overrightarrow{CD}$,from the four given points A(1,-1) , B(2,0) , C(-1,3) , D(-2,2)?

Sol.

$\overrightarrow{AB} = (2-1, 0+1) = (1,1)$

$\overrightarrow{CD} = (-2+1, 2-3) = (-1,-1)$

$\overrightarrow{AB} + \overrightarrow{CD} = (1,1) + (-1,-1) = (0,0)$

Ex. Find the length and direction of the following vectors:-

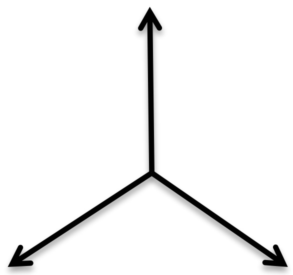
1- $\boldsymbol{i} + \boldsymbol{j} \Longrightarrow |\boldsymbol{i} + \boldsymbol{j}| = \sqrt{\mathbf{1^2 + 1^2}} = \sqrt{\mathbf{2}}$
2- $\sqrt{\mathbf{3}}\boldsymbol{i} + \boldsymbol{j} \Longrightarrow$
3- $\mathbf{2}\boldsymbol{i} - \mathbf{3}\boldsymbol{j} \Longrightarrow$
4- $\mathbf{5}\boldsymbol{i} + \mathbf{12}\boldsymbol{j} \Longrightarrow$
5- $-\mathbf{2}\boldsymbol{i} + \mathbf{3}\boldsymbol{j} \Longrightarrow$

Space Co-ordinates Vectors In Space

The vector in the3D is $\boldsymbol{V = (a, b, c) or\ V = (x, y, z)}$

Let V be a vector from the origin point to the point $\boldsymbol{P(x, y, z)}$.

$\overrightarrow{\boldsymbol{OP}} = \vec{\boldsymbol{V}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$



Ex. Locate (plot) the following points:-

$\boldsymbol{P_1(1,2,3), P_2(1,2,-3), P_3(1,-2,3), P_4(-1,2,3), P_5(1,-2,-3)}$

$\boldsymbol{, P_6(-1,2,-3), P_7(-1,-2,3), P_8(-1,-2,-3)}$.

The vector between two points in space

Let $\boldsymbol{P_1(x_1, y_1, z_1)}$ & $\boldsymbol{P_2(x_2, y_2, z_2)}$ **be two points**

$\overrightarrow{\boldsymbol{P_1P_2}} = (\boldsymbol{x_2 - x_1, y_2 - y_1, z_2 - z_1})$

$$\overrightarrow{P_2P_1} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$$

$$\overrightarrow{0P} = (x - 0, y - 0, z - 0) = (x, y, z)$$

$$\vec{V} = (a, b, c) \leftrightarrow V = ai + bj + ck$$

$$|V| = \sqrt{a^2 + b^2 + c^2},\ \left|\overrightarrow{0P}\right| = \sqrt{x^2 + y^2 + z^2}$$

The Direct of Vector in Space

$$dir(V) = \frac{V}{|V|}$$

The Distant Between Two Point in Space

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2, (y_2 - y_1)^2, (z_2 - z_1)^2}$$

$$V_1 = (x_1, y_1, z_1),\ V_2 = (x_2, y_2, z_2),$$

Prove that $V_1+V_2= V_2 +V_1$

Sol.

1) $V_1+V_2 = (x_1+x_2, y_1+y_2, z_1+z_2)$

$= (x_2+x_1, y_2+y_1, z_2+z_1)$

$= (x_2,y_2,z_2) + (x_1, y_1, z_1)$

$= V_2 +V_1$

$\therefore V_1+V_2 = V_2+V_1$

2) $V_1+V_2 = (x_1-x_2, y_1-y_2, z_1-z_2)$

$\mathbf{V_2 - V_1 = (x_2-x_1, y_2-y_1, z_2-z_1)}$

3) $\mathbf{CV = (cx , cy , cz)}$

4) $\mathbf{\vec{V}_1 \parallel \vec{V}_2 \rightarrow \exists\, C \in R}$ **s.t.** توازي متجهين

$\mathbf{\vec{V}_1 = C\vec{V}_2}$ **OR** $\mathbf{\vec{V}_2 = C\vec{V}_1}$

Such that $c = \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$

5) If V = (a , b , c) = O (0, 0,0) = oi +oj +ok

$\rightarrow a = b = c = 0$

6) $\vec{V}_1 \perp \vec{V}_2$, $\theta = \frac{\pi}{2}$ & $\vec{V}_1 . \vec{V}_2 = 0$ تعامد متجهين

(Scalar) Dot Product الضرب النقطي للمتجهات

$A = a_1 i + b_1 j + c_1 k$ & $B = a_2 i + b_2 j + c_2 k$

$A.B = a_1.a_2 + b_1.b_2 + c_1.c_2$

$A.B = |A| . |B| \cos\theta$

$\cos\theta = \frac{A.B}{|A||B|}$

(Ex.) A = 2i – 3j +5k & B= 8i – 12j +20 k , find A.B ?

Sol. :

A.B = 2(8) + (-3) (-12) +5 (20)

= 16 + 36 +100 = 152

(Ex.) A= ai –j+k & B = i +j +2k & A.B = 3 , find a = ?

Sol :

$A.B = a(1) + (-1)(1) + 1(2)$

$3 = a+1$

$\therefore a = 2$

(Ex.) If A and B are unit vectors and $\theta = 60^\circ$ find A.B = ??

Sol .

$|A| = 1, |B| = 1$, $\cos\theta = \frac{1}{2}$

$A.B = |A||B|\cos\theta = 1(1)(\frac{1}{2}) = \frac{1}{2}$

Propositions

1) $A.B = B.A$ s.t $A = a_1 i + b_1 j + c_1$, $B = a_2 i + b_2 j + c_2 k$

Pf : $A.B = a_1 a_2 + b_1 b_2 + c_1 c_2$

$= a_2 a_1 + b_2 b_1 + c_2 c_1$

$= B.A$

Another proof : برهان اخر

$A.B = |A||B|\cos\theta$ $\left(\cos\theta = \cos(-\theta) \right)$

لانه الـ Cos دالة زوجية

$B.A = |B||A|\cos(-\theta)$

$= |A||B| \cos(\theta)$

2) $i.i = 1$, $j.j = 1$, $k.k = 1$

pf. $i.i = (1,0,0).\ (1,0,0)$

$= 1(1)+0(0)+0(0) = 1$

$j.j = (0,1,0).(0,1,0)$

$= 0(0)+1(1)+0(0) = 1$

$k.k = (0,0,1).\ (0,0,1)$

$= 0(0)+0(0)+1(1) = 1$

3) $A.A = |A|^2$

Pf. : $A = (a,b) = ai + bj$ Or $A = (a,b,c) = ai+bj+ck$

$A.A = ??$ $|A|^2 = ??$

$A.A = (ai+bj).(ai+bj)$

$= a(a) + b(b)$

$= a^2 + b^2$

$|A| = \sqrt{a^2 + b^2}$

$|A|^2 = a^2 + b^2$

$\therefore A.A = |A|^2$

4) $A(B+C) = A.B + A.C$

$A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ & $C = (c_1, c_2, c_3)$

Pf : $(B+C) = (b_1 + c_1, b_2+c_2, b_3+c_3)$

$A.(B+C) = (a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3))$

$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$

A.B+A.C = a_1b$_1$ +a_2b$_2$+a_1c$_1$ +a_2c$_2$ + a_3b$_3$+ a_3c$_3$

(5) (B +C).A = B.A+C.A

(6) A. (B.C) = (A.B).C

(7) i.j = j.k = k.i=0

(Ex.) If A= i+j , B =i find A.B = ? θ = ??

A.B = 1(1) + 1(0)= 1 |A| = $\sqrt{2}$, |B| = 1

Cos θ = $\frac{A.B}{|A||B|} = \frac{1}{\sqrt{2}\sqrt{1}} = \frac{1}{\sqrt{2}}$

$\therefore \theta = \frac{\pi}{4}{}^\circ$

(Ex.) If A= i+ j – k , B= j – k & C = 2i compute A. (B. C) ?

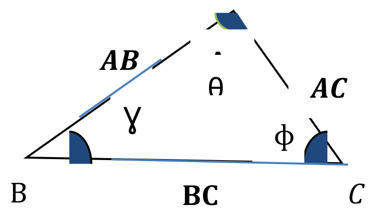
Sol. B. C = 1(2) + (-1) (0) = 2

A.(B.C) = 2(I +j –k)

= 2i +2j-2k

(Ex.) Find the angle of the triangle that A (-1,0,2) , B(2,1,-1) , C(1,-2,2) Are Vertices of it ??

Sol . Cos θ = $\frac{AB.AC}{|AB||AC|}$



AB = (2-(-1) , 1-0, -1-2)

متجه بين نقطتين AB = (3,1,-3)

AC= (1- (-1) , -2-0 , 2-2)

AC = (2,-2,0)

AB. AC=3(2)+1(-2)+(-3)(0) = 4

$|AB| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19}$

$|AC| = \sqrt{2^2 + (-2)^2 + (-3)^2} = \sqrt{8}$

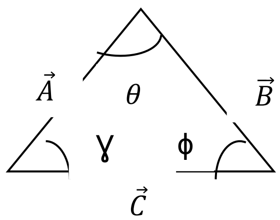
Cos $\theta = \frac{4}{\sqrt{19}\sqrt{8}} \longrightarrow \theta = \text{Cos}^{-1} \frac{4}{\sqrt{76}}$

Cos $\gamma = \frac{AB.BC}{|AB||AC|}$, Cos $\phi = \frac{AC\,.BC}{|AC||BC|}$

Ex. : If $\vec{A}$= i+j & $\vec{B}$ = k , $\vec{C}$ = 2j +k Find the angles of Δ ABC

<u>Sol : (Homework)</u>

هنا الاضلاع معطاة
فنحل مباشرة



Cos $\theta = \frac{A.B}{|A||B|} =$

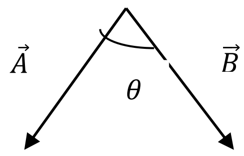
Cos $\gamma = \frac{A.C}{|A||C|} =$

Cos $\phi = \frac{B.C}{|B||C|} =$

(Ex.): Let $\vec{A}$ = 2i – j + k , $\vec{B} =$ i + j + 2k Find θ between $\vec{A}$ and $\vec{B}$?

<u>Sol</u> : A.B = 2(1) + (-1) (1) + 1(2)

= 2-1+2 = 3



|A | = $\sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

|B | = $\sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

Cos $\theta = \frac{A.B}{|A||B|}$

= $\frac{3}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$

∴ θ = Cos $^{-1}$ ($\frac{1}{2}$) → ∴ θ = 60° = $\frac{\pi}{3}$°

(Ex.): Let $\vec{A}$ = 2i + j - k , $\vec{B} =$ 4 j - k find A.B?

Sol.:- A.B = 2(0)+1(4)+(-1)(-1) =4+1=5

(Ex.): If $\vec{A}$ = 2i + aj + k , $\vec{B} =$ 2i+ j -2k ; and A⊥B find a ?

Sol.:- A⊥B → $\theta = \frac{\pi}{2} \rightarrow cos\theta = 0$

$A.B = |A|\,|B|\;Cos\;\theta = 0$

$A.B = 2(2) + a(1) + 1(-2) = 0$

$4 + a - 2 = 0 \rightarrow a = -2$

∴ $A = 2i - 2j + k$ & $B = 2i + j - 2k$

A⊥B = $A.B = 0$

الضرب الاتجاهي **Vector Product**

الضرب الاتجاهي بين متجهتين هو ناتج المحدد التالي :

$$\begin{vmatrix} i & j & k \\ a1 & b1 & c1 \\ a2 & b2 & c2 \end{vmatrix}$$

$\vec{A}$ = a1 i+ b1 j + c1 k
$\vec{B}$ = a2 i+ b2 j + c2 k

$\vec{A} \times \vec{B} =$

$$= i \begin{vmatrix} b1 & c1 \\ b2 & c2 \end{vmatrix} - j \begin{vmatrix} a1 & c1 \\ a2 & c2 \end{vmatrix} + K \begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$$

= i (b1 c2 – c1b2) – j (a2c2-a2c1) + k (a1b2-b1a2)

(Ex.) : $\vec{A}$ = 3i +7j – k , $\vec{B}$ = i+j+3k

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 3 & 7 & -1 \\ 1 & 1 & 3 \end{vmatrix}$$

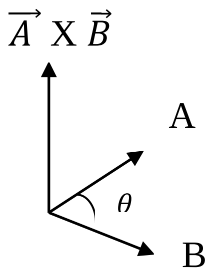
= i(7(3)-(-1)(1)) –j (3(3) -1) (+1) + k (3(1) -7(+1))

= 22i - 10j -4k

تعريف آخر للضرب الاتجاهي

A nother Diffenition for $\vec{A} \times \vec{B}$:-

$\vec{A} \times \vec{B} = \vec{n}$ |A| |B| Sin θ



$$\vec{n} = \frac{\vec{A} \times \vec{B}}{|A||B|\text{Sin } \theta}$$

Unit Vector متجه وحدة

(Ex) Find the unit vector $\vec{n}$ If $\vec{A}$ = i+j & $\vec{B}$ = j – k ??

<u>Sol .</u> $\vec{A} \times \vec{B} \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$

$$= i \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} -j \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} +k \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$= i\ (1(-1)\ -0(1)\ )\ –j\ (1(-1)-0(0))\ +k\ (1(1)\ -1(0))$

$\overrightarrow{A}$ X $\vec{B}$ = - i +j +k

A . B = 1 (0) +1(+1) 0(-1) = 1

COS $\theta = \frac{A\ .B}{|A|\ \ |B|} = \frac{1}{\sqrt{2\sqrt{2}}} = \frac{1}{2}$

$|A| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$|B| = \sqrt{0^2+1^2+(-1)^2} = \sqrt{2}$

$\therefore \theta = \text{Cos}^{-1}\ (\frac{1}{2}) \rightarrow \theta = \frac{\pi}{3}{}^{\circ}$ sin $\theta = \frac{\sqrt{3}}{2}$

$\vec{n} = \frac{\overrightarrow{A} \text{ X } \vec{B}}{|A||B|Sin\ \theta}$

$= \frac{(-i+j+k)}{\cancel{\sqrt{2}}\ \cancel{\sqrt{2}}\ \frac{\sqrt{3}}{\cancel{2}}} = \frac{-1}{\sqrt{3}}$ i+$\frac{1}{\sqrt{3}}$ j + $\frac{1}{\sqrt{3}}$ k

$|\overleftrightarrow{n}| = \sqrt{(\frac{-1}{\sqrt{3}})^2 + (\frac{1}{\sqrt{3}})^2 + (\frac{1}{\sqrt{3}})^2}$

$= \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = \sqrt{\frac{3}{3}}$ =(1)

$\therefore \vec{n}$ is unit vector

(Ex.) If $\vec{A}$ = i + 4j +k , B= -2i+j-2k? Is A $\perp$ B ?

Sol. A . B ? (If A.B=0 Then A $\perp$ B)

= 1(-2) +4(1) +1(-2)

= -2+4 -2= 0

$\therefore$ A.B = 0 → i.e   A $\perp$ B

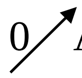
Remark : If $\vec{A} \,||\, \vec{B} \;\rightarrow\; \theta$ = Zero = 0°

Sin 0 $= 0$ → $\therefore \vec{A}$ X $\vec{B}$ = Zero → $\vec{A}$ X $\vec{B}$ = $\vec{n}$ |A| |B| Sin $\theta = 0$

Propositions :-

1) $\vec{A}$ X $\vec{B}$ = $\vec{B}$ X $\vec{A}$

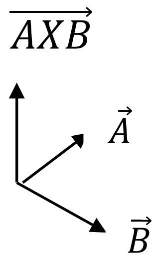
2) A X (C +C) = (A X B)+ (A X C)

3) A X A = 0   Since θ = 0 θ = 0  A

4) ixi = j x j = k x k = 0

5) A.($\vec{A}$ X $\vec{B}$)=B .($\vec{B}$ X $\vec{A}$)

= Zero

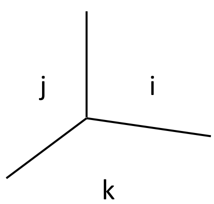


عمودي

Since $\vec{A}$ X $\vec{B}$ are $\left(\begin{array}{c}\text{Normal}\\ \text{Vertica}\\ \text{perpendicular}\end{array}\right)$ on $\vec{A}$ and on $\vec{B}$

6) i x j = k & j x k = i & k x i=j

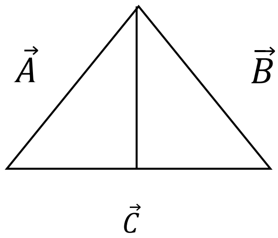
because i$\perp$ j $\perp$ k



7) Area of triangle A B C is

$= \frac{1}{2}|\overrightarrow{A} \text{ X } \overrightarrow{B}|$

الاضلاع $\vec{C}\ ,\ \vec{B}\ ,\ \vec{A}$



(Ex.) Let $\vec{\boldsymbol{A}}$ =i+j & $\vec{B}$ = i+k Find the Area of atriangle ABC

Where $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$ are two sides of it ??

<u>**Sol**</u> . Area = $\frac{1}{2}\,|\vec{\boldsymbol{A}} \text{ X } \vec{B}|$

$$\vec{\boldsymbol{A}} \text{ X } \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= i - j - k$$

$$|\vec{\boldsymbol{A}} \text{ X } \vec{\boldsymbol{B}}| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}$$

:. Area $= \frac{1}{2}\sqrt{3} = \frac{\sqrt{3}}{2}$

(Ex.) : Find the Area of triangle ABC where are vertices

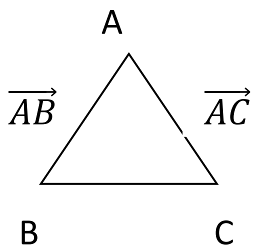
A (1, -1 , 0) , B (2, 1, -1) , (-1, 1, 2)

ملاحظه :- يُفضل للرؤوس احرف كبيرة والمتجهات احرف صغيرة كذلك الرؤوس نأخذها بشكل احداثيات والمتجهات نأخذها بشكل مجموع ضرب

A (a, b, c)   **&**   **$\vec{V}$ = ai+bj +ck**

نقطة   ***متجه***

(لكي نُميز بينهما)



<u>**Sol:**</u> $\overrightarrow{\boldsymbol{AB}}$ **= (2 -1 , 1-(-1), (-1)-0)**

= (1, 2, -1)

$\overrightarrow{\boldsymbol{AB}}$ = i+2j+k

$\overrightarrow{\boldsymbol{AC}}$ = (-1-1, 1-(-1),2-0)

= (-2, 2, 2)

$\overrightarrow{\boldsymbol{AC}}$ = -2i+2j +2k

$$\overrightarrow{\boldsymbol{AB}}\ X\ \overrightarrow{\boldsymbol{AC}} = \begin{matrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{matrix}$$

$$= i \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \quad -j \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}$$

= i(4+2) –j (2-2)+k(2+4)

= 6i+6k

$|\overrightarrow{\boldsymbol{AB}}\ X\ \overrightarrow{\boldsymbol{AC}}| = \sqrt{6^2 + 0^2 + 6^2}$

$= \sqrt{36 + 36} \quad = \sqrt{72}$

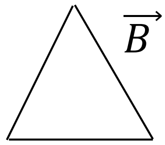
$$\begin{array}{c|c} 2 & 72 \\ 6 & 36 \\ 6 & 6 \\ & 1 \end{array}$$

$\therefore$ Area = $\frac{1}{2}\,|\overrightarrow{\boldsymbol{AB}}\ X\ \overrightarrow{\boldsymbol{AC}}|$

$= \frac{1}{2}\sqrt{72} \qquad = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

(Ex.) Find Area of triangle where $\overrightarrow{A} = i + j$ & $\overrightarrow{B} = i + k$ are two sides of it ?(Homework)

Sol . $\overrightarrow{\boldsymbol{A}}\ X\ \overrightarrow{\boldsymbol{B}}$ = $\begin{vmatrix} i & \text{j} & \text{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

$\overrightarrow{A}$  $\overrightarrow{B}$

:. $\overrightarrow{\boldsymbol{A}}\ X\ \overrightarrow{\boldsymbol{B}} =$

| $\overrightarrow{\boldsymbol{A}}\ X\ \overrightarrow{\boldsymbol{B}}$ | = $\sqrt{\quad}$

Area = $\frac{1}{2}$ | $\overrightarrow{\boldsymbol{A}}\ X\ \overrightarrow{\boldsymbol{B}}$| = $\frac{1}{2}$ (   )

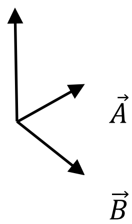
Orthogonal = vertical = Normal = Perpndicule

تعني هذه الكلمات مرادفات كلمة عمودي

(Ex .) Find the normal Vector from $\vec{N} = \vec{A}X\ \overrightarrow{B} = ?\, Where$

$\vec{A}$ = (2, 1, -1)   & $\overrightarrow{B}$ = (-2, 4, -1)

$\vec{N} = \overrightarrow{A}X\ \overrightarrow{B}$



$\vec{A}$ = 2i+j-k $\vec{B}$ = -2i+4j –k

Sol . N = $\vec{A}\ X\ \vec{B}$ $\begin{array}{ccc} i & \text{j} & \text{k} \\ 2 & 1 & -1 \\ -2 & 4 & -1 \end{array}$ =

Direction (Homework)

(Ex) find the direction of $\vec{A}$ and $\vec{B}$, $\vec{A} = i$, $\vec{B} = j$?

هنا كلمة Dir *هو متجه الوحدة* unit vector, Dir *نفس المعنى اي الاتجاه* *المطلوب.*

Sol. Dir = $\frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$; $\vec{N} = \vec{A} \times \vec{B}$

Dir = $\vec{u} = \frac{\vec{N}}{|\vec{N}|}$

Dir = $\frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i(0)+j(0)+k(1)$

$= \vec{K}$

$|\vec{A} \times \vec{B}| = \sqrt{0^2 + 0^2 + 1^2} = (1)$

$\therefore \vec{u} = Dir = \frac{\vec{k}}{1} = \vec{K}$

Where $|\vec{K}| = 1$

هنا متجهة الوحدة الناتج او (Dir) الاتجاه بين المتجهتين نتج المتجه $\vec{K}$ والذي هو متجه وحده وطول (1) و ($0,0,1$) $\vec{K}$ وهذا تطبيق للخاصيه $i \times j = k$

(Ex) : Find the Dir (direction) between $\vec{A} = 2i - 2j + k$ &

$\vec{B} = i + j + k$

Sol. (Homework)

(Ex) : If $\vec{V} = 2i - j$ & $\vec{w} = i + 3j - 2k, \vec{u}$ = 3i-4j+2k ??

Compute أحسب

(1) $(\vec{V}\ X\ \vec{W}) . \vec{U} = ??$ (Homework)

Sol(1) $\vec{V}\ X\ \vec{W} = ?$

$(\vec{V}\ X\ \vec{W}) . \vec{U} = ?$

(2) $(\vec{W} X\ \vec{U}) . \vec{V}$

Sol(2) $\vec{W} X\ \vec{U}$ =?

$(\vec{W} X\ \vec{U}) . \vec{V}$ =?

(3) $(\vec{U} X\ \vec{V}) . \vec{W}$

Sol(2) $\vec{U} X\ \vec{V} =$?

$(\vec{U} X\ \vec{V}) . \vec{W}$ =?

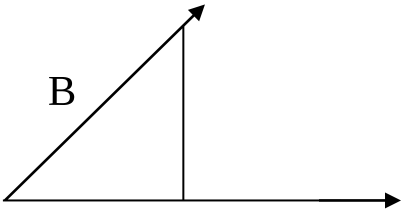
Ex.: If $\vec{A}$ = 4i -8j +k & $\vec{B}$ = 2i+j -2k

Are $(\vec{A}\ x\ \vec{B}). \vec{A} = \vec{B}\ (\vec{A}\ x\ \vec{B})$??هل تحقق المساواة

Vector projection الاسقاط الاتجاهي

وهو يعرف حسب القيمة التالية

$$\text{Proj}\,\frac{B}{A} = \frac{B.A}{A.A}\,.\overrightarrow{A}$$



Praposition:

$[A]^2 = A.A$; $\vec{A} = ai + bj + ck$

Proof: $A.A = a(a) + b(b) + c(c)$

$= a^2 + b^2 + c^2$ …………..*

$|A| = \sqrt{a^2 + b^2 + c^2}$

$|A|^2 = a^2 + b^2 + c^2$ ………….*

$\therefore \quad A.A = |A|^2$

$$\therefore\ \text{Proj}\,\frac{B}{A} = \frac{B.A}{|A|^2}\,.\,\vec{A}$$

Scalar projection الاسقاط الدوري

هو يمثل طول الاسقاط الاتجاه اي:-

$$\begin{bmatrix} sc.pro. \\ for\vec{B}\ on\ \vec{A} \end{bmatrix} = |Proj_{A}^{B}\,|$$

Ex:- a) Find vertor projection for $\vec{B}$ on $\vec{A}$ and scalar projection of it ?

b) Find vertor projection for $\vec{B}$ on $\vec{A}$ and scalar projection $|Proj\ _A^B| = ?$

Where $\vec{A}$ = 1Oi +11j -2k

$\vec{B}$ = 2i +3j+4k

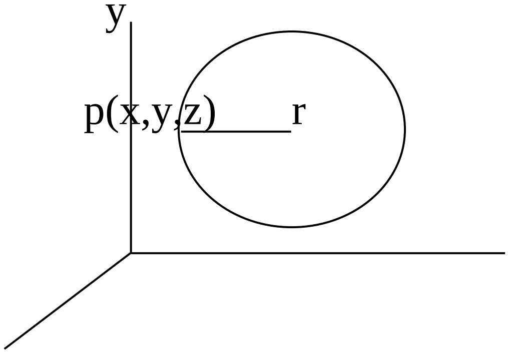
Sol :- a)-$|Proj\ _A^B|$ =? & $Proj\ _A^B = \frac{B.A}{|A|^2}\vec{A}$=?

b)- $Proj\ _B^A = \frac{A.B}{|B|^2}\vec{B}$= & $\left|Proj\ _B^A\right|$ =?

Equation of sphere معادلة الكرة

$(x - x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$

حيث ان $C(z_0 , y_0 , x_0)$ نقطة مركز الكرة و r نصف القطر حيث p اية نقطة على محيط الكرة $p(x,y,z)$



$r = |p_o p|$

$r = \sqrt{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2}$

Ex.:- find the center c and the radins r for the following sphere

$(x - 3)^2 + (y + 7)^2 + (z-5)^2 = 4$?

Sol. :- c (3, -7 , 5) , r = 2

Exc:- find c = 2 and r = 2 where $x^2 + (y-2)^2 + (z+2) = 36$

Sol.:-(Homework)

Ex.:- find the center and radius from the following sphere equation $x^2 + y^2 + z^2 + 4x - 3z = 4$?

Sol.:-

$x^2 + y^2 + z^2 + 4x - 3z = 4$

لاكمال المربع نضيف(١/٢معامل x)

$x^2 + 4x + 4 + y^2 + z^2 - 3z + (1/2(3))^2 = 4 + 4 + \frac{9}{4}$

$(x + 2)^2 + y^2 + (z - \frac{3}{2})^2 = \frac{16+16+4}{4}$

$\therefore$ C(-2 , o, $\frac{3}{2}$) , r = $\frac{\sqrt{41}}{2}$

Ex.:- find the center and radius from sphere eq.

$x^2 + y^2 + -4x + z^2 + 3z = 4$? (Homework)

Ex.:- find the center and radius from sphere eq.

$x^2 + y^2 + z^2 - 2az = 0$?

Sol :- $(x-0)^2 + (y-0)^2 + z^2 - 2az + \left(\frac{-2a}{2}\right)^2 = a^2$

$$(x-0)^2 + (y-0)^2 + (z-a)^2 = a^2$$

$C(0,0,a)$ $r = a$

Ex.:- find the center and radius from $x^2 - 4x + y^2 + 2ay = z^2$?

(Homework).

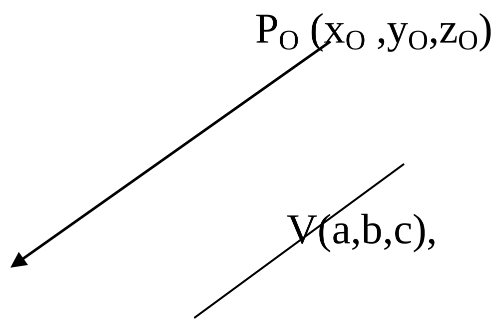
معادلة المستقيم   Equation of the line

$V(a,b,c)$, $P_O(x_O, y_O, z_O)$

$x = x_O + ta$

$y = y_o + tb$

$z = z_o + tc$



هذه المعادلات لوسيطة للمستقيم

$t = \frac{X-X_O}{a} = \frac{Y-Y_O}{b} = \frac{Z-Z_O}{c}$ هذه معادلة المستقيم

اشتقاق القانون اعلاه حسب :-

هنا $L = P_oP$ حيث P اي نقطة على L

$P(x, y, z)$, $P_O(x_O, y_O, z_O)$

ويكون L موازي للمتجه $V(a, b, c)$

$V \,//\, PP_O \rightarrow PP_O = tV$

$(x-x_O)i + (y-y_O)j + (z-z_O)k = t(ai+bj+ck)$

ومنها وجدنا المعادلات اعلاه

$\rightarrow$ $at = x-x_O$, $bt = y-y_O$, $ct = z-z_O$

EX:- Find the equation of line L , which pass through $P_O(1,2,3)$ And Parallel to the line $\overleftrightarrow{V}(4,7,-1) = \overleftrightarrow{v} = 4i + 7j - k$??

Sol:- $L = t = \frac{X-X_O}{a} = \frac{Y-Y_O}{b} = \frac{Z-Z_O}{c}$

$x = x_O + at$ $\qquad$ $P_O(x_O, y_O, z_O)$

$y = y_O + bt$ $\qquad$ $V(a,b,c)$

$z = z_O + ct$

$L = \frac{X-1}{4} = \frac{Y-2}{7} = \frac{Z-4}{-1}$

$x = 1 + t(4)$ $\qquad$ $x = 1+4t$

$y = 2+t(7)$ $\qquad$ $y = 2+7t$

$z = 4 + t(-1)$ $\qquad$ $z = 4-t$

Ex.:- find line equation for $p_o(0\,,1\,,-2)$ $v = (7,6,2)$?(Homework)

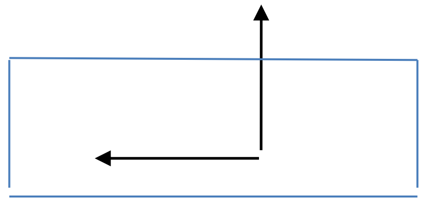
معادلات المستوي plane equations

$A x + By + Cz = D$ معادلة المستوي $D = Ax_o + By_o + Cz_o$

$\vec{N}(A,B,C)$

$P_O\ (x_O\ , y_O, z_O)$

$\vec{N}(A,B,C)$ &



$p\ (x,y,z)$ $P_O\ (x_O\ , y_O, z_O)$

اشتقاق القانون :- هنا لدينا نقطة $P_O\ (x_O\ , y_O\ , z_O)$

ومستقيم N عمودي على المستوى على P_O لذا P اية نقطة المستو المتكون منها

حيث سيكون التعامد $PP_O \perp N$ هنا $N . PP_O = 0$ اي التعامد يعني

$$\cos\frac{\pi}{2} = 0 \quad \emptyset = \frac{\pi}{2}$$

$$\overrightarrow{P_\circ P} = (x - x_O)\, i + (y - y_O) j + (z - z_O) k$$

$$0 = \vec{N} \,.\, \overrightarrow{P_\circ P} = A\,(x - x_O) + B(y - y_O) + (z - z_O)$$

$$Ax + By + Cz = Ax_O + By_O + Cz_O$$

$$\therefore\ Ax + By + Cz = D$$

Ex :- find plan eqution for p_o (1,4.0) and $\vec{N} = 4i + 7i - 2k$?

p_o $(x_o,y_o.z_o)$ and $\vec{N}$= Ai+Bi +Ck ??

$Ax + By + Cz = Ax_O + By_O + Cz_O = D$

$4X + 7Y - 2Z = 4(1) + 7(4) + (-2)(0)$

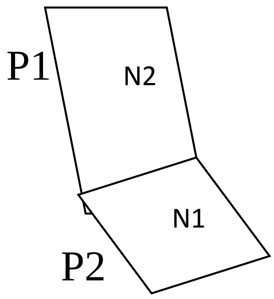
<u>$4X + 7Y + 2Z = 32$</u>

EXE : find plane eqution for $\vec{N}$ (3,5,7) and $p_o(-3,4,5)$??

تعامد مستويات

$P_1 \perp p_2$ ------ $N_2 \perp N_1$

$N_2 . N_1 = 0$



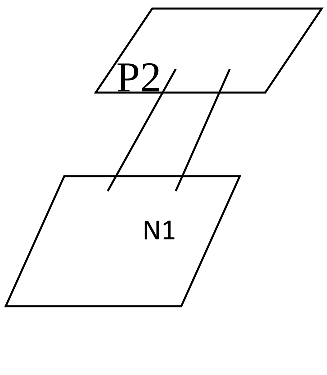
حيث p1 & p2 مستويات و $\overrightarrow{N_1}$, $\overrightarrow{N_2}$ *المستقيمات العموديات عليها على التوالي*

$\overrightarrow{N_1}$ (A_1, B_1, C_1) & $\overrightarrow{N_2}$ $(A2, B_2, C_2)$

توازي المستويات :-

P1 // P2 $\rightarrow$ $\overrightarrow{N_2}$ // $\overrightarrow{N_1}$ $\longrightarrow$ $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

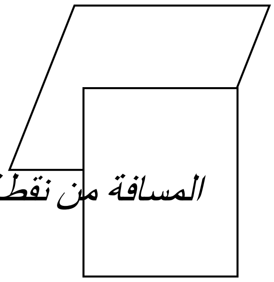
$\overrightarrow{N_1}$ (A_1, B_1, C_1) & $\overrightarrow{N_2}$ $(A2, B_2, C_2)$



تقاطع المستويات

$$P1 \cap P2 \;\rightarrow\; \overrightarrow{N_2} \cap \overrightarrow{N_1} = \theta$$

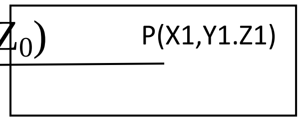
$$\cos\theta = \frac{\overrightarrow{N_1}.\overrightarrow{N_2}}{|\overrightarrow{N_1}||\overrightarrow{N_2}|}$$



المسافة من نقطة الى مستوى حيث P مستوى Q نقطة

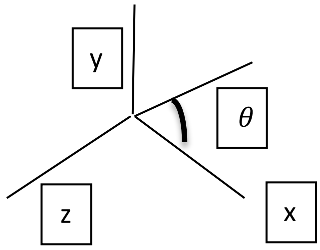
$$|PQ| = \sqrt{(X_1 - X_O)^2 + (Y_1 - Y_O)^2 + (Z_1 - Z_O)^2}$$

$Q\ (X_0.Y_0.Z_0)$



حيث p مستو و Q نفطة خارج المستو و P1 اية نقطة داخل المستو فتكون المسافة من المستو P الى النقطة Q هي نفسها المسافة من النقطة Q الى اية نقطة P1 داخل المستو .

CYLINDRICAL COORDINATES ↔ CARTESIAN COORDIN ATES



$(r, \theta, z) \rightarrow (x, y, z)$

$$\tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

$x = r\cos\theta$

$y = r\sin\theta$

Ex. :- $(r, \theta, z) = (1, \frac{\pi}{2}, 1) \rightarrow (x, y, z) \rightarrow x_{=} \cos\frac{\pi}{2} = 0$

$Y = 1\sin\frac{\pi}{2} = 2$

Z = 1

$\therefore (x , y , z) = (0 ,1,1)$

Ex.:-(x,y,z)=(1,0,1) find (r, θ,z)?

Sol.:- r = $\sqrt{x^2 + y^2} = 1$

Tan $\theta = \frac{0}{1} = 0$

$\theta = 0$

$\therefore (r, \theta, z) = (1,0,1)$

Chapter four

Partial Differential

Partial Differential

1- Functions of two and more variables

- $y = f(x)$ Two variables x and y.

Ex1. $y = 3x^2$

$$y = f(x) = 3x^2$$

$$\frac{dy}{dx} = 3(2x) = 6x$$

- $Z = F(x, y)$ Three variables x, y and z

Ex2. $z = 3x^2 + y^2$

$$\frac{\partial z}{\partial x} = 3(2x) + 0$$

$$\frac{\partial z}{\partial x} = 6x$$

$$\frac{\partial z}{\partial y} = 2y$$

- $W = f(x, y, z)$ Four variables x, y, z and w

  $\vdots$

- $W = f(x_1, x_2, x_3, \dots, x_n)$

Ex3. $W = \frac{x^2 - z^2}{y^2}$

There are two ways to solve that example:-

First way

$$\frac{\partial w}{\partial x} = \frac{(2x - 0).\, y^2 - (x^2 - z^2).\,(0)}{(y^2)^2}$$

$$\frac{\partial w}{\partial x} = \frac{2xy^2}{y^4} = \frac{2x}{y^2}$$

Second way

$$W = \frac{1}{y^2}(x^2 - z^2)$$

$$W = \frac{x^2}{y^2} - \frac{z^2}{y^2}$$

$$\frac{\partial w}{\partial x} = \frac{1}{y^2}(2x) - 0 = \frac{2x}{y^2}$$

$$\frac{\partial w}{\partial y} = \frac{(0).\, y^2 - 2y.\,(x^2 - z^2)}{(y^2)^2}$$

$$= \frac{-2y(x^2 - z^2)}{y^4} = \frac{-2(x^2 - z^2)}{y^3}$$

$$\frac{\partial w}{\partial z} = \frac{(-2z).y^2 - 0}{y^4} = \frac{-2z}{y^2}$$

Ex4. $W = e^x Ln(x^2 + y^2 + 1)$

$$\frac{\partial w}{\partial x} = e^x.\frac{1}{x^2 + y^2 + 1}.(2x) + e^x Ln(x^2 + y^2 + 1)$$

$$\frac{\partial w}{\partial y} = e^x.\frac{1}{x^2 + y^2 + 1}.(2y)$$

Ex5. $z = f(x, y) = Ln\sqrt{x^2 + y^2}$

$$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}}.\frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}.2x$$

$$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}}.\frac{1}{2\sqrt{x^2 + y^2}}.2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial w}{\partial y} = \frac{1}{\sqrt{x^2 + y^2}}.\frac{1}{2\sqrt{x^2 + y^2}}.2y = \frac{y}{x^2 + y^2}$$

Ex6. $W = \frac{(x^2+y^2)}{(u^2+v^2)}$

$$\frac{\partial w}{\partial x} = \frac{2x(u^2 + v^2) - (0)(x^2 + y^2)}{(u^2 + v^{2s})^2} = \frac{2x(u^2 + v^2)}{(u^2 + v^2)^2}$$

$$= \frac{2x}{(u^2 + v^2)}$$

$$\frac{\partial w}{\partial y}=\frac{2y(u^2+v^2)-(0)(x^2+y^2)}{(u^2+v^2)^2}=\frac{2y}{(u^2+v^2)}$$

$$\frac{\partial w}{\partial u}=\frac{(0)(u^2+v^2)-2u(x^2+y^2)}{(u^2+v^2)^2}=-\frac{2u(x^2+y^2)}{(u^2+v^2)^2}$$

$$\frac{\partial w}{\partial v}=\frac{(0)(u^2+v^2)-2v(x^2+y^2)}{(u^2+v^2)^2}=-\frac{2v(x^2+y^2)}{(u^2+v^2)^2}$$

Ex7. Find all Partial Differential of the following blow :-

1. $Z=\frac{(u^2+v^2)}{2x}$
2. $f(x,y)=x^2-y$
3. $f(x,y)\,\frac{3y}{x}$
4. $W=\frac{z+x}{y}$

Differential by definition

$Z=f(x,y)$ and the partial differential of this function are:

Z_x , Z_y , f_x *and* f_y .

$$f_x=\frac{\partial f}{\partial x}=\lim_{Dx\to 0}\frac{f(x_\circ+Dx,y_\circ)-f(x_\circ,y_\circ)}{Dx}$$

$$f_y=\frac{\partial f}{\partial y}=\lim_{Dy\to 0}\frac{f(x_\circ,y_\circ+Dy)-f(x_\circ,y_\circ)}{Dy}$$

Ex. Find f_x and f_y for the following function:-

$$f(x,y) = 100 - x^2 - y^2$$

Sol.

$$f_x = \frac{\partial f}{\partial x} = \lim_{Dx \to 0} \frac{f(x_\circ + Dx, y_\circ) - f(x_\circ, y_\circ)}{Dx}$$

$$f(x_\circ, y_\circ) = 100 - x_\circ{}^2 - y_\circ{}^2$$

$$f(x_\circ + Dx, y_\circ) = 100 - (x_\circ + Dx)^2 - y_\circ{}^2$$

$$f_x = \frac{\partial f}{\partial x} = \lim_{Dx \to 0} \frac{100 - (x_\circ + Dx)^2 - y_\circ{}^2 - (100 - x_\circ{}^2 - y_\circ{}^2)}{Dx}$$

$$= \lim_{Dx \to 0} \frac{100 - x_\circ{}^2 - 2x_\circ Dx - Dx^2 - y_\circ{}^2 - 100 + x_\circ{}^2 + y_\circ{}^2}{Dx}$$

$$= \lim_{Dx \to 0} \frac{-2x_\circ Dx - Dx^2}{Dx} = \lim_{Dx \to 0} \frac{(-2x_\circ - Dx)Dx}{Dx}$$

$$= \lim_{Dx \to 0}(-2x_\circ - Dx) = -2x_\circ - 0 = -2x_\circ$$

$$f_x = \frac{\partial f}{\partial x} = -2x_\circ$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{Dy \to 0} \frac{f(x_\circ, y_\circ + Dy) - f(x_\circ, y_\circ)}{Dy}$$

$$f(x_\circ, y_\circ + Dy) = 100 - x_\circ{}^2 - (y_\circ + Dx)^2$$

$$f_y \;= \frac{\partial f}{\partial y} = \lim_{Dy\to 0} \frac{100 - x_\circ^2 - (y_\circ + Dy)^2 - (100 - x_\circ^2 - y_\circ^2)}{Dy}$$

$$= \lim_{Dy\to 0} \frac{100 - x_\circ^2 - y_\circ^2 - 2y_\circ Dy - Dy^2 - 100 + x_\circ^2 + y_\circ^2)}{Dy}$$

$$= \lim_{Dy\to 0} \frac{-2y_\circ Dy - Dy^2}{Dy} = \lim_{Dy\to 0} \frac{(-2y_\circ - Dy)Dy}{Dy}$$

$$= \lim_{Dy\to 0}(-2y_\circ - Dy) = -2y_\circ - 0 = -2y_\circ$$

$$f_y \;= \frac{\partial f}{\partial y} = -2y_\circ$$

Ex. Find f_x *and* f_y for the function $f(x,y) = x + 3y$ by using definition?(Homework).

The Chair Rule

1. $w = f(x,y)\,,\;\; x = x(t)\;\;\&\; y = y(t)$

$$\frac{\partial w}{\partial x}\,,\;\; \frac{\partial w}{\partial y}\,,\;\; \frac{dx}{dt}\,,\;\; \frac{dy}{dt}$$

$$\therefore \frac{dw}{dt} = \frac{\partial w}{\partial x}\,.\frac{dx}{dt} + \;\frac{\partial w}{\partial y}\,.\frac{dy}{dt}$$

Ex. Find $\frac{dw}{dt}$ for the function:-

$w = x^2 + 2y^2 \quad , x = \sin\, t$ and $y = \cos t$

$$\frac{\partial w}{\partial x} = 2x + 0\;, \qquad \frac{dx}{dt} = \cos t$$

$$\frac{\partial w}{\partial y} = 0 + 4y\;, \qquad \frac{dy}{dt} = -\sin t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2x \cos t - 4y \sin t$$

$$= 2 \sin t \cos t - 4 \cos t \sin t$$

$$\frac{dw}{dt} = -2 \sin t \cos t$$

2. $w = f(x, y)$, $x = x(t, z)$ & $y = y(t, z)$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial z}$$

Ex.

$z = f(v, w)$, $\quad v = v(n, u)$ & $w = w(n, u)$

$z = vw$, $\quad v = un$, $\quad w = n^2 + u^2$

$$\frac{\partial z}{\partial n} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial n} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial n}$$

$$= wu + 2vn$$

$$\frac{\partial z}{\partial n} = (n^2 + u^2)u + 2un^2$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial u} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial u}$$

$$= wn + 2vu$$

$$\frac{\partial z}{\partial u} = (n^2 + u^2)u + 2nu^2$$

Ex. Find $\frac{dz}{dt}$ for the functions:- (Homework)

1. $z = f(x, y)\ ,\ x = x(t)\ \ and\ y = y(t)$
2. $z = \sin(x.y)\ ,\ x = \frac{1}{t}\ \ and\ y = 3t$

Ex. Find $\frac{\partial z}{\partial u}\ and\ \frac{\partial z}{\partial w}$ for the functions:- (Homework)

$$z = f(x, y) = y\sin x\ \ , x = w - u^2\ and\ y = \frac{w}{u}$$

The gradate & vector differential

We denoted for gradate function $f(x, y, z)$ at the point $p_\circ(x_\circ, y_\circ, z_\circ)$ by:-

$$\Delta f|p_\circ = \frac{\partial f}{\partial x}|_{p_\circ}.i + \frac{\partial f}{\partial y}|_{p_\circ}.j + \frac{\partial f}{\partial z}|_{p_\circ}.k$$

Ex. Find the gradate for $f(x, y, z)$ where

$f(x, y, z) = x^3 - xy^2 - z$ at $p_\circ(1,1,0)$?

Sol.

$$\Delta f|p_\circ = \frac{\partial f}{\partial x}|_{p_\circ}.i + \frac{\partial f}{\partial y}|_{p_\circ}.j + \frac{\partial f}{\partial z}|_{p_\circ}.k$$

$$\frac{\partial f}{\partial x} = 3x^2 - y^2\ \ , \qquad \frac{\partial f}{\partial y} = -2xy\ \ , \quad \frac{\partial f}{\partial z} = -1$$

$$\frac{\partial f}{\partial x}|_{p_\circ(1,1,0)} = 3.(1)^2 - (1)^2 = 2$$

$$\frac{\partial f}{\partial y}|_{p_\circ(1,1,0)} = -2.(1).(1) = -2$$

$$\frac{\partial f}{\partial z}|_{p_\circ(1,1,0)} = -1$$

$$\Delta f|p_\circ = 2i - 2j - k$$

Ex. Find Find the gradate for $f(x, y, z)$ where

$f(x, y, z) = \sin xyz$ at $p_\circ(0,0,0)$?(Homework)

Vector differential

We denoted for vector differential by $Du\Delta f|p_\circ$ which equals the gradate multiplying by unit vector.

$$Du\Delta f|p_\circ = \Delta f|p_\circ . \vec{u}$$

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|}$$

Ex. Find vector differential for the previous example

$f(x, y, z) = x^3 - xy^2 - z$ at $p_\circ(1,1,0)$in the direction of $\vec{A} = 2i - 3j + 6k$

Sol.

$\because \Delta f|p_\circ = 2i - 2j - k \,, \qquad \vec{A} = 2i - 3j + 6k$

$|\vec{A}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k$

$Du\Delta f|p_\circ = \Delta f|p_\circ . \vec{u}$

$= (2i - 2j - k).\left(\frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k\right)$

$= (2).\left(\frac{2}{7}\right) + (-2).\left(-\frac{3}{7}\right) + (-1).\left(\frac{6}{7}\right)$

$\therefore Du\Delta f|p_\circ = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$

Higher order differentials

$$z = f(x, y)$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xxx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3}$$

$$\vdots$$

$$f_{\underset{n\ times}{x \dots x}} = \underset{n\ times}{\frac{\partial}{\partial x}(\frac{\partial}{\partial x}(\frac{\partial}{\partial x}\left(\cdots \frac{\partial f}{\partial x}\right)} = \frac{\partial^n f}{\partial x^n}$$

$$f_y = \frac{\partial f}{\partial y} = z_y$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{yyy} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^3}$$

$$\vdots$$

$$f_{\underset{n-\ times}{y \dots y}} = \underset{n-\ times}{\frac{\partial}{\partial y}(\frac{\partial}{\partial y}(\frac{\partial}{\partial y}\left(\cdots \frac{\partial f}{\partial y}\right)} = \frac{\partial^n f}{\partial y^n}$$

Now, if we have

$$f_x = \frac{\partial f}{\partial x}$$

$$f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}$$

$$f_{xyx}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)=\frac{\partial^3 f}{\partial x\partial y\partial x}$$

$\vdots$

And so on .

Ex. $f(x,y)=x^5-7x+y^2$ Find $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$$\frac{\partial f}{\partial x}=5x^4-7=f_x$$

$$\frac{\partial f}{\partial y}=2y=f_y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=0=f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=0=f_{yx}$$

$$f_{xy}=f_{yx}$$

Ex. Prove that $f_{xy}=f_{yx}$ for ? (Homework)

1. $f(x,y)=\sin(xy)$
2. $f(x,y)=\frac{x}{y}$
3. $z=x^2+2y^2$

Ex. Prove that $f_{xxy}=f_{xyx}=f_{yxx}$ for $f(x,y)=x\,cos\,y+ye^x$

$$\frac{\partial f}{\partial x}=\cos y+ye^x=f_x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right) = e^x = f_{xxy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = e^x = f_{xyx}$$

$$\frac{\partial f}{\partial y} = -x\sin y + e^x = f_y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right) = e^x = f_{yxx}$$

$$\therefore f_{xxy} = f_{xyx} = f_{yxx} = e^x$$

Ex. Prove that $f_{xxy} = f_{xyx} = f_{yxx}$ for $f(x, y) = \sin xy - \frac{y}{x}$?

(Homework).

Definition: Local Extrema

We call $f(a, b)$ a **local maximum** of f if there is an open disk R cantered at point (a, b) for which $f(a, b) \geq f(x, y)$ for all $(x, y) \in R$. Similarly, we call $f(a, b)$ a **local minimum** of f if there is an open disk R cantered at point (a, b) for which $f(a, b) \leq f(x, y)$ for all $(x, y) \in R$.

Definition: Critical Point

The point (a, b) is a critical point of the function $f(x, y)$ if (a, b) is in the domain of f and *either* $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ *or* one or both of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not exist at (a, b).

Saddle Point

The point $P(a, b, f(a, b))$ is a saddle point of $z = f(x, y)$ if (a, b) is a critical point of f and if every open disk centered at (a, b) contains points (x, y) in the domain of f for which $f(x, y) < f(a, b)$ and points (x, y) in the domain of f for which $f(x, y) > f(a, b)$.

Maximum and Local Minmum Points

To find the function if it has a Local Maximum or Local Minmum Points by using some steps which are :-

1. We find the differential $f_x = ?\, , f_y = ?$
2. $f_x = 0\ ,\ f_y = 0$ and solve it to find the critical points.
3. We find the differential $f_{xx} = ?\, , f_{xy} = ?\, , f_{yy} = ?$ And put the critical points in it .
4. We put it into the relation

$$D = f_{xx}|_c . f_{yy}|_c - f_{xy}|_c{}^2$$

5. $D > 0\ or\ D < 0\ or\ D = 0.$

Here compared with zero .

a. If $D > 0\ , f_{xx} > 0 \Longrightarrow$ Local Min.
b. If $D > 0\ , f_{xx} < 0 \Longrightarrow$ Local Max.
c. If $D < 0\ \Longrightarrow$ Saddle Points.
d. If $D = 0\ \Longrightarrow$ No Conclution.

Ex. Find Local max or min if exist for

1- $\boldsymbol{f(x,y) = 6xy - x^2 - 3y^2}$

Sol. $f_x = \cancel{6y} - 2x = 0$

$f_y = 6x - \cancel{6y} = 0$

$4x = 0 \Longrightarrow x = 0$

$6y = 0 \Longrightarrow y = 0$

$\therefore c(0,0)$

$f_{xx} = -2 \Longrightarrow f_{xx}|_{(0,0)} = -2$

$f_{yy} = -6 \Longrightarrow f_{yy}|_{(0,0)} = -6$

$f_{xy} = 6 \Longrightarrow f_{xy}|_{(0,0)} = 6$

$D = f_{xx}|_{(0,0)} . f_{yy}|_{(0,0)} - f_{xy}|_{(0,0)}{}^2$

$= (-2).(-6) - 36 = -24$

$\therefore D = -24 < 0$

$\therefore c(0,0)$ is Saddle point.

2- $\boldsymbol{f(x,y) = 3 - 6x - 6y + x^2 + y^2}$

Sol. $f_x = -6 + 2x = 0 \Longrightarrow x = 3$

$f_y = -6 + 2y = 0 \Longrightarrow y = 3$

$\therefore c(3,3)$ is critical point.

$f_{xx} = 2 \Longrightarrow f_{xx}|_{(3,3)} = 2$

$f_{yy} = 2 \Longrightarrow f_{yy}|_{(3,3)} = 2$

$f_{xy} = 0 \Longrightarrow f_{xy}|_{(3,3)} = 0$

$D = f_{xx}|_{(3,3)}.f_{yy}|_{(3,3)} - f_{xy}|_{(3,3)}{}^2$

$= (2).(2) - 0 = 4$

$\therefore D = 4 > 0,\ f_{xx} = 2 > 0$

$\therefore c(3,3)$ is Local Min. Point.

3- $\boldsymbol{f(x,y) = 6x + 3y - x^2 - y^3}$ **(Homework)**.

Chapter Five

The Differential Equation

chapter five الفصل الخامس

the Differantial Equation المعادلات التفاضلية

المعادلة التفاضلية :- هي اي علاقة بين متغيرات وتفاضلاتها او مشتقاتها (Differential Equation)هناك نوعان منها :-

١- **المعادلة التفاضلية الاعتيادية** Ordinary Differential Equation

هي المعادلة التي تحتوي على متغيير مستقل واحد فقط مع تفاضلة ومشتقة.

٢- **المعادلة التفاضلية الجزئية** Partial Differential Equation

هي المعادلة التي تحتوي على متغييرين مستقلين أو أكثر .

رتبة المعادلة التفاضلية The Order of Differential Equation

هي اعلى مشتقة (تفاضل) يظهر في المعادلة .

درجة المعادلة التفاضلية The Degree of Differential Equation

هي اس (قوة) اعلى مشتقة تظهر في المعادلة وترمز ب ∂ لتفاضل او المشتقة الجزئية وترمز ب d للتفاضل او المشتقة الاعتيادية .

Examples:-

Ex1 :- Qy '' +3y = x^2

- معادلة تفاضلية اعتيادية من الرتبة الثانية والدرجة الاولى

Ex2 :- $(3x+2y)^2 \cdot y^{2} = 1$

- معادلة تفاضلية اعتيادية من الرتبة الاولى والدرجة الاولى

Ex3 :- $\left(\frac{\partial^2 P}{\partial X^2}\right)^4 + \left(\frac{\partial^2 P}{\partial y^2}\right)^3 + \frac{\partial^2 P}{\partial z^2} = 0$

- معادلة تفاضلية جزئية من الرتبة الثانية والدرجة الرابعة

Ex4:- $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

- معادلة تفاضلية جزئية من الرتبة الاولى والدرجة الاولى

Ex 5 :- $\left(\frac{dy}{dx}\right)^4 - x^5 = 0$

- معادلة تفاضلية اعتيادية من الرتبة الاولى والدرجة الرابعة

Ex 6 :- $y'' - y^2 = 0$

- معادلة تفاضلية اعتيادية من الرتبة الثانية والدرجة الاولى

Ex 7 :- $y' + y = 2$

- معادلة تفاضلية اعتيادية من الرتبة الاولى والدرجة الاولى

The Solution of The Differential Equation حل المعادلة التفاضلية

هي اي علاقة بين متغيرات المعادلة التفاضلية وتكون خالية من المشتقات او التفاضلات وتحقق المعادلة التفاضلية .

Ex:- show that $y = x \, Lnx - x$

Is a solution for the following diff . eq. $xy' = x+y$ ……. ***

Sol. :- $y = x \, Lux - x$ …. *

$y' = 1 \, . \, Lnx + x \, \frac{1}{x} - 1$

$y' = Lnx$ …….**

Put (*) and (**) in (***)

$xy' = x + y$

$x(Lnx) = x + (xLnx - x)$

$\therefore \; xLnx = xLnx$

The general solution الحل العام

هو مجموعه كل الحلول للمعادلة حيث ان عدد الثوابت الاختيارية تكون بقدر رتبة المعادلة .

Ex :- prove that $y = x^4 + c_1 x + c_2$ …… * Is ageneral solution for the following differentail equation $y'' = 12 x^2$ …………***

Sol. :- $y' = 4x^3 + c_1$……**

$y'' = 12 x^2$ …….***

∴ * is a general sol. For ***

The special solution الحل الخاص

هو حل واحد فقط او اكثر ينتج من فرض وتعويض قيم ثوابت الاختيارية

Ex :- show that $y = x^4 +5x +4$ is a special sol. for the differentail equation $y'' = 12x^2$?

Sol. y = x4 +5x =4 …………….*

$y' = 4x^2 +5$ …………………**

$y'' = 12 x^2$ ………………….***

∴ * is a sp.sol . for ***

$y= x^4 + c_1x +c_2$ هنا كان الحل عام

حيث الثوابت (هي حل للمعادلة y'') وتم فرض $c_1 =5$, $c_2=4$ فنتج حل خاص للمعادلة y'' .

Ex1:- find the ordred and degree for the following differential equation ?

1. $(y')2 + x\, y' = y^2$

2. $y'' = (3\,y' + x)2$
3. $(y'')^2 = (1 + y')3$

Ex2 :- prove that the following values are a general sdution for the equations bested them ??

a- y'' + 4 y = 0 , $y = c_1 \sin 2x + c_2 \cos 2x$

b- y'' + 3y' +2y = 0 , $y = c^1 e^{-x} + c^2 e^{-2x}$

solution for the حلول المعادلات التفاضلية الرتبة الاولى والدرجة الاولى first order and first degree diff. eq.

$\frac{dy}{dx}$ =f (x,y) تكون المعادلة التفاضلية من الرتبة الاولى والدرجة الاولى بالشكل

او بالشكل m(x,y)dx + n(x,y)dy = 0

وتكون على عدة انواع :-

١- المعادلة التفاضلية القابله للفصل Seperable Differential Equation

وهي المعادلة التفاضلية التي نستطيع فصل متغييراتها مع تفاضلها ونجد التكامل لها فنحصل على حلها اى اولا نضغطها بالشكل A(x)dx + B(y)dy =0 ثم نكامل الاطراف فنحصل على f (x,y)=c وهي مجموعه الحل للمعادلة.

Ex:- solve the following diff. eq. ?

$$x^2\,(1 - y^2)\,dx + y(1 - x^2)dy = 0$$

Sol. :- $\frac{x^2}{1+x^2}$ dx + $\frac{y}{1-y^2}$ dx =0

$$\int (1 - \frac{1}{1 - x^2}\; dx + \left(\frac{1}{-2}\right) \int \frac{-2dy}{1-y^2} = c$$

$x^2+1 \overline{\big)\, x^2}$

بالطرح $\frac{x^2+1}{-1}$

$$\int dx - \int \frac{dx}{1+x^2} - \frac{1}{2} \ln|1 - y^2| = c$$

$$x - \tan^{-1} x - \frac{1}{2} \ln|1 - y^2| = c$$

$\therefore$ f (x,y) =c general sol.

Ex : solve the following diff.eq. xy dy + (2yx^2 +4x^2 –y -2) dx =0

Sol. :- xy dy + 2x^2 (y+2) – (y +2)) dx =0

xy dy + (2x^2 -1) (y+2)dx =0

$\frac{2x^2-1}{x}$ dx + $\frac{y}{y+2}$ dx = 0

(2x - $\frac{1}{x}$) dx + (1 - $\frac{2}{y+2}$)dy =0

$\int 2x\,dx - \int \frac{dx}{x} + \int dy - 2\int \frac{dy}{y+2}$ = c

$$\boxed{\begin{array}{c} y+2\,\overline{\big)\,y} \\ \frac{y+2}{-2} \end{array}}$$ بالطرح

X^2 - ln$|x|$ +y – 2ln$|y+2|$ =c

$\therefore$ f (x,y) =c g . sol . for the diff. eq.

<u>Ex : Homework</u> solve the following diff. eq. ??

xydy +(cx^2 -2yx^2 +4-8y)dx =0

٢- المعادلات التفاضلية المتجانسة **Homgeneous diff. eq.**

الدالة المتجانسة هي الدالة التي تحقق الشرط التالي $f(tx,ty) = t^n f(x,y)$

Ex :- $f(x,y)= x^3 -2xy^2 +x^2y +y^3$

$f(tx, ty) = (tx)^3 - 2(tx)(ty)^2 + (tx)^2(ty) + (ty)^3$

$= t^3x^3 -2t^3xy^2 +t^3 x^2 y +t^3y^3$

$= t^3 (x^3 -2xy^2 +x^2y +y^3)$

$= t^3 f(x,y)$

$\therefore f$ متجانسة من الدرجة الثالثة

Ex : - $f\left(\frac{y}{x}\right) = e^{\frac{y}{x}} + \sin\frac{y}{x} - \frac{1}{\left(\frac{y}{x}\right)} + \left(\frac{y}{x}\right)^2$

$f\left(\frac{ty}{tx}\right) = e^{\frac{ty}{tx}} + \sin\frac{ty}{tx} - \frac{1}{\left(\frac{ty}{tx}\right)} + \left(\frac{ty}{tx}\right)^2$

$= e^{\frac{y}{x}} + \sin\frac{y}{x} - \frac{1}{\frac{y}{x}} + \left(\frac{y}{x}\right)^2$

$= t^0 \left(f\left(\frac{y}{x}\right)\right)$

Ex. Is $f(x,y) = \frac{x^3+y^3}{xy^2}$ Homo. Function?(Homework)

Homogeneous differential equation **المعادلة التفاضلية المتجانسة**

هي المعادلة التي تحقق فيها كون كل من M & N دالة متجانسة من نفس الدرجة وطريقة حلها كالاتي:-

$M(x,y)dx+N(x,y)dy=0$

$M(tx,ty) = t^n M(x,y)$

$N(tx,ty) = t^n N(x,y)$

وكل من M و N دالة متجانسة من نفس الدرجة.

خطوات حل هذا النوع من المعادلات:-

١. نضع المعادلة بالشكل $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{f}(\mathrm{x}, \mathrm{y})$

٢. ونضعها بالشكل $\frac{\mathrm{dy}}{\mathrm{dx}} = f\left(\frac{y}{x}\right)$

٣. نفرض $v = \frac{x}{y} \rightarrow \frac{\mathrm{dy}}{\mathrm{dx}} = f(v)$

٤. اصبحت لدينا معادلة تفاضلية قابلة للفصل بالنسبة ل x , v

$$\frac{dx}{x} + \frac{dv}{v - f(v)} = 0$$

٥. نحل الصيغة اعلاه ونعوض ب$\frac{x}{y}$ بدل v .

Ex. Solve the following differential equation

$$xdy - ydx = \sqrt{x^2 + y^2}\, dx?$$

Sol.

$$\left(y + \sqrt{x^2 + y^2}\right)dx - xdy = 0$$

$$M(x, y) = \left(y + \sqrt{x^2 + y^2}\right)$$

$$M(tx, ty) = \left(ty + \sqrt{(tx)^2 + (ty)^2}\right)$$

$$M(tx, ty) = \left(ty + \sqrt{t^2(x^2 + y^2)}\right)$$

$$M(tx, ty) = \left(ty + t\sqrt{(x^2 + y^2)}\right)$$

$$M(tx, ty) = t\left(y + \sqrt{(x^2 + y^2)}\right)$$

$\therefore\ M\,(tx, ty) = t\,M(x, y)$ دالة متجانسة من الدرجة الأولى

$$N(x, y) = -x$$

$$N(tx, ty) = -(tx)$$

$= t\,(-x)$

$= t\,N(x,y)$

$\therefore$ N is homo . function of degree one

1- we put $\frac{dy}{dx} = f(x,y)$

$$\frac{dy}{dx} = \frac{\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}}{\frac{x}{x}} = \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}}$$

$$= \frac{y}{x} + \sqrt{\frac{x^2}{x^2} + \frac{y^2}{x^2}}$$

$$= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

$$\therefore \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right)$$

Put $v = \frac{y}{x} \rightarrow f(v) = v + \sqrt{1+v^2}$ -

Solve $\frac{dx}{x} + \frac{dv}{v - f(v)} = 0$ -

$$\frac{dx}{x} + \frac{dv}{v - (v + \sqrt{1+v^2})} = 0$$

$$\frac{dx}{x} + \frac{dv}{-\sqrt{1+v^2}} = 0$$

اصبحت لدينا معادلة تفاضلية قابلة للفصل بالنسبة ل x , y نكامل للطرفين فنحصل على

$$\int \frac{dx}{x} - \int \frac{dv}{\sqrt{1+v^2}} = \int o$$

$$\ln|x| - \int \frac{dv}{\sqrt{1+v^2}} = c \quad \text{……..*}$$

$\int \frac{dv}{\sqrt{1+v^2}}$??

هنا نحل التكامل التالي بطريقة التعويضات المثلثية

$\int \frac{dv}{\sqrt{1+v^2}}$

Let v = tanθ

dv = sec $\theta \;\; d\theta$

sin$^2\theta$ + los^2 θ = 1 $\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{\sec\theta \;\; d\theta}{\sqrt{\tan\theta+1}}$`

$= \int \frac{\sec\theta \;\; d\theta}{\sin\theta}$ tan^2 θ + 1 = sec$^2\theta$

$= \int sec\theta \; d\theta$

= Ln $|sec\theta + \tan\theta|$

= Ln $\left|\sqrt{1+\tan\theta} + \tan\theta\right|$

= Ln $\left|\sqrt{1+v^2} + v\right|$

= ln $\left|\sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x}\right|$ ……………..**

Ln $|x|$ - ln $\left|\sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x}\right|$ = c نعوض في

F(x,y)=c

Ex :- Homework :- Solve xy dx + (cx^2 – 2y^2)dy =0

المعادلات التفاضلية الغير المتجانسة non – homogeneous differential eq.

المعادلة التفاضلية من الرتبة الاولى والدرجة الاولى تكون غير متجانسة اذا لم تحقق شرط التجانس M(tx, ty) & N(tx,ty)

هنا ستكون بالشكل التالي

$$(ax + by + c)\,dx + (\propto x + \beta Y + \gamma)dy = 0$$

نضفها بالشكل التالي $\frac{dy}{dx} = \frac{ax+by+c}{\propto x+\beta Y+\gamma}$

الان ناخذ الحالتان

الحالة الاولى :- في حالة المشتقات متقاطعات نحلها انيا ونجد نقطة التقاطع

$$ax + by + c = 0$$

$$ax + \beta Y + \gamma = 0$$

$\frac{a}{\propto} \neq \frac{d}{\beta}$ ولتكن

نفرض الفرضية التالية $x = x_1 + h$ & $y = y_1 + k$

ونعوض في المعادلة اعلاه & $\frac{dy}{dx} = \frac{dy_1}{dx_1}$

فنحصل على معادلة متجانسة بدلالة x_1, y_1 ونحلها حسب الطريقة في الحالة الثانية (من نوع الثاني) بعدها نعوض بدلالة x,y ونحصل على الحل العام

Ex :- solve

$$(2x - 3y + 4)\,dx + (3x - 2y + 1)dy = 0$$

Sol:- $\frac{dy}{dx} = \frac{-2x+3y-4}{3x-2y+1}$

$$\frac{a}{\propto} = \frac{-2}{3} + \frac{d}{\beta} = \frac{3}{-2}$$

المستقيمان متقاطعان نحلها انيا

$-2x+3y-4=0$

$3x-2y\ +1\ =0$

-6x + 9y -12 =0

6x -4y +2 =0

(h , k) = ?(1,2)

5 y = -10

Y = - 2

3x -2(-2)+1 =0

3x -3 =0 → x=1

Let $x = x_1 + h$ & $y= y_1 + k$

$x = x_1 + 1$ & $y = y_1 +2$

$$\frac{dy_1}{dx_1} = \frac{-2(x_1+1)3(y_1+2\ )-4}{3(x_1+1\ )-2(y_1+2)+1}$$

$$\frac{dy_1}{dx_1} = \frac{-2x_1-2+3y_1+6-4}{3x_1+3-2y_1-4+1}$$

$$\frac{dy_1}{dx_1} = \frac{-2x_1+3y_1}{3x_1-2y_1}$$

$$\frac{dy_1}{dx_1} = \frac{-\frac{2x_1}{x_1}+\frac{3y_1}{x_1}}{\frac{3x_1}{x_1}-\frac{2y_1}{x_1}}$$

$$\frac{dy_1}{dx_1} = \frac{-2+3(\frac{y_1}{x_1})}{3-2(\frac{y_1}{x_1})}$$

Let $v = \frac{y_1}{x_1} \longrightarrow \frac{dy_1}{dx_1} = \frac{-2+3v}{3-2v}$ =f(v)

$$\frac{dx_1}{x_1} + \frac{dv}{v-f(v)} = 0$$

$$\frac{dx_1}{x_1} + \frac{dv}{v-\frac{-2+3v}{3-2v}} = 0$$

$$\frac{dx_1}{x_1} + \frac{dv}{\frac{3v-2v^2+2-3v}{3-2v}}$$

$$\frac{dx_1}{x_1} + \frac{dv}{\frac{-2\left(v^2-1\right)}{(3-2v)}}$$

$$\frac{dx_1}{x_1} - \frac{(3-2v)dv}{2(v^2-1)} = 0$$

$$\frac{dx_1}{x_1} - \frac{3}{2}\frac{dv}{v2-1} + \frac{2v\,dv}{2(v^2-1)} = 0$$

$$\int \frac{dx_1}{x_1} - \frac{3}{2} \int \frac{dv}{v^2-1} + \frac{1}{2}\int \frac{2vdv}{v^2-1} = \int 0$$

$$\text{Ln}\,|x_1| - \frac{3}{2}\int \frac{dv}{(v-1)(v+1)} + \frac{1}{2}\ \text{Ln}|v^2 - 1| = c$$

هنا تجزئة كسور

$$\int \frac{dv}{(v-1)(v+1)} \overset{?}{=} \int \frac{\frac{1}{2}}{v-1} - \frac{\frac{1}{2}}{v+1}\,dv$$

$$= \frac{1}{2}\int \frac{dv}{v-1} - \frac{1}{2}\int \frac{dv}{v+1}$$

$$= \frac{1}{2}\,\text{Ln}\,|v-1| - \frac{1}{2}\,\text{Ln}|v+1|$$

$$\text{Ln}\,|x_1| - \frac{3}{2}\left(\frac{1}{2}\,\text{Ln}|v-1| - \frac{1}{2}\,\text{Ln}|v+1|\right) + \frac{1}{2}\ \text{Ln}|v^2-1| = c$$

$$\text{Ln}\,|x_1| - \frac{3}{4}\,\text{Ln}\left|\frac{y_1}{x_1}-1\right| + \frac{3}{4}\,\text{Ln}\left|\frac{y_1}{x_1}+1\right| + 1 + \frac{1}{2}\,\text{Ln}\left\|\left(\frac{y_1}{x_1}\right)^2-1\right\|$$

$$\text{Ln}\,|x-1| - \frac{3}{4}\,\text{Ln}\left|\frac{y-2}{x-1}-1\right| + \frac{3}{4}\,\text{Ln}\left|\frac{y-2}{x-1}+1\right| + \frac{1}{2}\,\text{Ln}\left|\frac{(y-2)^2}{(x-1)^2}-1\right| = c$$

$x = x_1 + 1 \rightarrow x_1 = x - 1$, $y = y_1 + 2 \rightarrow y_1 = y - 2 \quad \therefore\ f(x,y) = c$

الحالة الثانية :- في حالة المستقيمان متوازيان

$$\frac{dy}{dx} = \frac{ax + by + c}{\alpha x + \beta Y + \gamma}$$

هنا نختبر $\frac{a}{\propto} \overset{?}{=} \frac{b}{\beta}$ في حالة التساوي نستنتج ن مستقيمان متوازيان.

ونحلها باستخدام الفرضية التالية الطريقة $dz = adx + bdy, z = ax + by$ حيث تصبح قابلة للفصل بعدها نعوض بدلالة x, z فنحصل على معادلة متجانسة بدلالة z

Ex :- solve

$$(2x + y)dx - (4x + 2y - 1)dy = 0$$

Sol:- $\frac{dy}{dx} = \frac{2x+y}{4x+2y-1}$

$\frac{a}{\propto} = \frac{2}{4} + \frac{b}{\beta} = \frac{1}{2}$ المستقيمان متوازيان

$$z = 2x + y$$

$$dz = 2dx + dy$$

$$dy = dz - 2dx$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 2$$

$$\frac{dy}{dx} = \frac{2x+y}{4x+2y-1}$$

$$\frac{dz}{dx} - 2 = \frac{2x+y}{2(2x+y)-1}$$

$$\frac{dz}{dx} = \frac{z}{2z-1} + 2$$

$$\frac{dz}{dx} = \frac{z+4z-2}{2z-1}$$

$$\frac{dz}{dx} = \frac{5z-2}{2z-1}$$

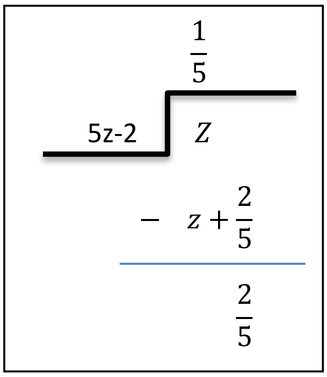
$$\int \frac{2z-1}{5z-2} dz - \int dx = \int 0$$

$$2\int \frac{z}{5z-2} dz - \int \frac{1}{5z-2} dz - \int dx \ = \int 0$$

$$2\int \frac{z}{5z-2} dz - \frac{1}{5} \ \mathrm{Ln}|5z - \ 2\,| - x = c \ \ldots\ldots\ldots\ldots .1)$$

$$2\int \frac{z}{5z-2} dz = 2\left(\frac{1}{5}\int dz + \frac{2}{5}\int \frac{1}{5z-2} dz\right)$$

$$= 2\left(\frac{1}{5}z + \frac{2}{25} \ \ln|5z-2|\right) \ldots\ldots\ldots..*$$



Put * in equation 1 we get:-

$$2\left(\frac{1}{5}z + \frac{2}{25} \ \ln|5z-2|\right) - \frac{1}{5} \ \mathrm{Ln}|5z - \ 2\,| - x = c \ \ldots\ldots\ldots\ldots .1)$$

$$\frac{2}{5}z + \frac{4}{25} \ \ln|5z-2| - \frac{1}{5} \ \mathrm{Ln}|5z - \ 2\,| - x = c$$

$$\frac{2}{5}z + \frac{4}{25} \ ln|5z-2| - \frac{1}{5} \ Ln|5z - \ 2\,| - x = c$$

$$\frac{2}{5}z + \frac{4-5}{25} \ ln|5z-2| - x = c$$

$$\frac{2}{5}z - \frac{1}{25} \ ln|5z-2| - x = c$$

Put z =2x+y we get

$$\frac{2}{5}(2x+y) - \frac{1}{25} \ ln|5(2x+y\,) - 2| - x = c$$

$\therefore \ f(x,y) = c$

Exact differential equation (٤) المعادلة التفاضلية تامة

The differential equation $M(x,y)dx + N(x,y)dy = 0$
become Exact if it satisfy the condition :-

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and the general solution is $f(x,y) = c$

To find the function f ,we used the relation

$$f(x,y) = \int M(x,y)dx + \emptyset(y) - - - - - - - - - - - - - *$$

حيث $\emptyset(y)$ دالة اختيارية بالنسبة ل y ونجدها عن طريق اشتقاق الدالة f_y ومساواتها بدالة $N(x,y)$

$$f_y = \frac{\partial f}{\partial y} = N(x,y) - - - - - - - - - - - - - - - - - - - **$$

نجد $\emptyset(y)$ ونعوضها في* ونجد بذلك الحل العام للمعادلة التفاضلية التامة.

<u>Ex :</u> *solve* $(3x^2 + 3xy^2)\,dx + (3x^2 y - 3y^2 + 2y)dy = 0$

<u>Sol.</u>

$$M(x,y) = 3x^2 + 3xy^2\,,\quad N(x,y) = 3x^2 y - 3y^2 + 2y$$

$$\frac{\partial M}{\partial y} = 6xy \quad = \quad \frac{\partial N}{\partial x} = 6x\,y$$

Exact diff. eq.

$$f(x,y) = \int M(x,y)dx + \emptyset(y)$$

$$f(x,y) = \int (3x^2 + 3xy^2)dx + \emptyset(y)$$

$$f(x,y) = x^3 + \frac{3}{2}x^2y^2 + \emptyset(y)$$

$$Now\, f_y = \frac{\partial f}{\partial y} = N(x,y)$$

$$0 + \frac{3}{2}x^2(2y) + \emptyset'(y) = 3x^2\, y - 3y^2 + 2y$$

$$\emptyset'(y) = \frac{d\emptyset}{dy} = -3y^2 + 2y$$

$$\int d\emptyset = \int (-3y^2 + 2y)dy$$

$$\emptyset(y) = -y^3 + y^2 - c$$

$$\therefore F(x,y) = x^3 + \frac{3}{2}x^2y^2 + -y^3 + y^2 = c$$

$$\therefore F(x,y) = c$$

Ex. Solve $3x(xy-2)dx + (x^3 + 2y)dy = 0$ *(Homework)*

Non – Exact diff. eq. (٥) المعادلة التفاضلية الغير تامة

هنا في هذه الحالة تكون المشتقة غير متساوية أي انه

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

ولكي نجعلها تساوي تجري العمليات التالية : -

١- نجد عامل التكامل للمعادلة وذلك كالأتي :

نختبر

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial M}{\partial x}\right) = f(x)$$

اذا كان الناتج دالة لـــ x معامل التكامل سيكون

$$\boldsymbol{I\ (x\ ,y)\ =\ e^{\int f(x)dx}}$$

٢- نضرب طرفي المعادلة به فتصبح المعادلة تامة ونحلها حسب الحالة السابقة

Remark :

$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial M}{\partial x}\right)$ = g (y)   في حالة كون الاختبار حقق تساوي دالة (y)

يكون عامل التكامل

$$\boldsymbol{I\ (x\ ,y)\ =\ e^{-\int g(y)dy}}$$

<u>**Ex. :**</u> solve $(3x\ y^3+4y\ )\ dx+(3x^2\ y^2+2x)dy=0$

<u>Sol.</u>

$\frac{\partial M}{\partial y}=q\ x\ y^2+4 \quad \neq \quad \frac{\partial N}{\partial x}=6\ xy^2+2$

Not Exact

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=?\,f(x)$$

$$\frac{1}{3x^2y^2+2x}\left(\ (qxy^2+4)-(6xy^2+2)\ \right)=\frac{1}{x(\cancel{3y}^2+2)}(\cancel{3x}y^2+2)=\frac{1}{x}=f(x)$$

$$I\ (x\ ,y)\ =\ e^{\int f(x)dx}$$

$$I\ (x\ ,y)=\ e^{\int\frac{dx}{x}}=e^{ln\ |x|}=x$$

$x(3x\ y^3+4y\ )\ dx+\ x\ (3x^2y^2+2x)dy=0$

$(3x^2y^3+4xy\ )\ dx+(3x^3y^2+2x^2)dy=0$

M   N

$\frac{\partial M}{\partial y}=9\ x^2y^2+4x=\frac{\partial N}{\partial x}=9\ x^2y^2+4x$

:. Exact diff. eq.

The Sol. Is :

$f(x,y)= c$ where

$f(x,y) = \int M(x,y)\, dx + \varphi (y)$

$f(x,y) = \int (3x^2y^3 + 4xy)\, dx + \varphi (y)$

$f(x,y) = x^3y^3+2x^2y+ \varphi (y)$

$f_y = \frac{-\partial f}{\partial y} = N (x,y)$

$\frac{\partial f}{\partial y} = 3x^3y^2+2x^2+ \varphi'(y) = N(x,y)$

$\rightarrow \cancel{3x^2}y^2+\cancel{2x^2}+\frac{d\,\varphi}{dy} = \cancel{3x^3}y2+\cancel{2x^2}$

$\rightarrow \frac{d\,\varphi}{dy}= 0 \quad \rightarrow \int d\,\varphi = \int 0\, dy$

$\rightarrow \varphi = c\, y^{\circ}$ ← constant

$\therefore \varphi (y) = 0$

$\therefore F(x,y) = x^3y^3+2x^2y+c$

$\therefore F(x,y) = c$ g. s . for diff eq.

Linear diff.eq **(٦) المعادلة التفاضلية الخطيه**

تكون على الصيغة $\frac{dy}{dx} + p(x)\, y = q (x)$

حيث p و q دوال لــ x وقد تكون ثوابت خطوات حل هكذا معادلة :-

١- نجد عامل التكامل كالاتي

$$\boldsymbol{I\,(x\,,y) \;=\; e^{\int P(x)dx}}$$

٢- نجد الحل بالخطوة التالية $f(x\,,\,y) = c$

$y.I = \int q(x)\, .\, I\, dx+c$

Ex. Solve $(1+x^2)\, y' + x\, y = x^{-1} \sqrt{x^2+1}$

Sol. $\frac{dy}{dx} + \frac{x}{x^2+1} . y = \frac{\sqrt{x^2+1}}{x(x^2+1)}$

$\frac{dy}{dx} + \frac{x}{x^2+1} . y = \frac{1}{x\sqrt{x^2+1}}$

$I(x, y) = e^{\int \frac{x}{x^2+1} dx}$

$= e^{\frac{1}{2} \ln|x^2+1|}$

$I = e^{\ln\sqrt{x^2+1}} = \sqrt{x^2+1}$

g.s. f(x,y) =c

$y.\, I = \int q(x) . I\, dx + c$

$y. \sqrt{x^2+1} = \int \frac{1}{x\sqrt{x^2+1}} \sqrt{x^2+1}\, dx + c$

$y. \sqrt{x^2+1} = \int \frac{1}{x}\, dx + c$

$y \sqrt{x^2+1} = \ln|x| + c$

f (x,y) = c

ملاحظه : هنا اذا كان y بالنسبة لــ x اما اذا كان بالعكس x بالنسبة لــ y

$\frac{dx}{dy} + p(y) . x = q(y)$

$\boldsymbol{I(x, y) = e^{\int P(y) dy}}$

$x.I = \int q(y) . I\, dy + c$

Ex . $y \frac{dx}{dy} + 2x = y^3$

Sol . $y \frac{dx}{dy} + 2x = y^3$

$\frac{dx}{dy} + \frac{2}{y}$ x = y^2

$\frac{dx}{dy} + p(y).x = q(y)$

$\boldsymbol{I\ (x\ , y) = \ e^{\int P(y)dy}}$

= $e^{\int \frac{2}{y}dy}$ = e^{2Lny} = e^{Lny2} = y^2

x.I =∫ q(y) . I dy + c

x.y^2 = ∫ y2 . y2dy +c

x.y^2 = $\frac{1}{5}$ y^5 + c

:. f(x,y) = c

Special Types of Second order eq.

حالات خاصة من المعادلات ذات الرتبة الثانية

هنا تخفيض الرتبة الثانية للرتبة الاولى له حالتان :-

(1) f (x, y' , y'') = 0 *اذا كانت المعادلة بالشكل*

نفرض الفرضية التالية

P= y' → p' = $\frac{dp}{dx} = \frac{d}{dx}$ (y')

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ = y''

:. P= y' → p' = y''

اما اذا كانت بالشكل التالي :

(2) F (y,y', y'') = 0

فتكون الفرضية بالاشتقاق بالنسبة لـ y

P = y' → $\frac{dp}{dy} = \frac{d}{dy}$ (y')

$= \frac{d}{dy} (\frac{dy}{dy}) = \frac{d^2y}{dy^2} = y''$

P = y' → $p' = y''$

Ex : $x^2 y'' - (y')^2 2x\, y' = o$ معادلة تخفيض رتبة

Sol . (1) F $(x, y' . y'') = 0$

$P = y' \to p' = \frac{dp}{dx} = y''$

$x^2\frac{dp}{dx} - p^2 - {}^2xp = 0$

$\frac{dp}{dx} - \frac{2x}{x^2} . p = \frac{1}{x^2} p^2$

Bernolly eq.

$\frac{dy}{dx} + p(x) .y = q(x) .y^n \quad n \in R/\{0.1\}$

معادلة برنولي

$\frac{dy}{dx} + p(x)\, y = q(x) \quad \leftarrow$ هذه خطية عادية

الآن وجود y^n مضروب في q(x) تمثل حالة آخرى من المعادلات الخطية وتسمى معادلة برنولي (Bernolly eq)

* هنا اذا كانت (n = 0) تصبح خطية و (1-n) تكون عامل مشترك مع الطرف الايسر ولاتصبح خطية لذا (n = 2 فما فوق) وهي تدعى معادلة برنولي

Berolly Equation

$\frac{dy}{dx} + p(x) .y = q(x) .y^n$

(1) Let $Z = y^{1-n}$ (١) نفرض الفرضية التالية

(2) $\frac{dz}{dx} + (1-n)\, p(x) .Z= (1-n)\, q\,(x)$

(٢) نحل المعادلة بالنسبة لــ z و x حيث ستكون خطية

نعود الان لحل السؤال

$\frac{dy}{dx} - \frac{2}{x}p = \frac{1}{x^2}p^2$, n=2

(1) Let $Z = p^{1-n} = p^{1-2} = p^{-1} = \frac{1}{p}$

(2) $\frac{dz}{dx} + (1-n)\, p(x).z = (1-n)q\,(x)$

$\frac{dz}{dx} + (1-2)\left(-\frac{2}{x}\right) . Z = (1-2)\left(\frac{1}{x^2}\right)$

$\frac{dz}{dx} + \frac{2}{x} . z = \frac{-1}{x^2}$

$I\,(x,z) = e^{\int \frac{2}{x} dx} = x^2$

$z.\, I\,(x, z) = \int q(x), I\,(x, z)\, dx + c$

$z\, .\, x^2 = \int - \frac{1}{X^2} . x2\, dx + c$

$Z\, .\, x^2 = -x + c$

$Z = \frac{-x+c}{x^2}$, $Z = \frac{1}{P}$

$\frac{1}{P} = \frac{-X+C}{X^2}$

$P = \frac{x^2}{-x+c}$, $p = \frac{dy}{dx} = y'$

$\frac{dy}{dx} = \frac{x2}{-x+c}$

$-x$ -c

$\int dy = \int \frac{x^2}{-x+c} dx$

$\sqrt[-x+c]{x^2}$

بالطرح $\frac{+x^2 - c}{c\,x}$

$\frac{cx - c^2}{c^2}$

$y= \int (- x - c + \frac{c^2}{-x+c}) \, dx$

$y = -x^2 - cx - c^2 Ln |-x+c|$

$\therefore f (x, y) = c$

المعادلات التفاضلية الخطية من الرتبة الثانية

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \ldots + a' y' + a_0 y = 1$

معادلة تفاضلية خطية من الرتبة n ومعاملاتها دوال لـ x والطرف الايمن دالة لــ f (x)

في حالة a_n , — a_0, تكون ثوابت تسمى المعادلة تفاضلية خطية من الرتبة n ذات معاملات ثابته واذا كان الطرف الايمن 0 = f (x)

سُميت معادلة تفاضلية خطية متجانسة

$$y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, \ldots, y^{(n)} = \frac{d^ny}{dx^n}$$

هنا في حالة n=2 تكون بالشكل $y'' + p(x)y' + q\,(x)y = f(x)$

عندما f (x) =0 تكون معادلة متجانسة بالشكل التالي

$$y'' + p(x)y' + q\,(x)y = 0$$

الآن سنأخذ المعادلة الخطية المتجانسة من الرتبة الثانية ذات المعاملات الثانية اي a_0, a_1, a_2 تكون ثوابت وهنا تمثل (ثوابت p,q)

$$y'' + p\,y' + q\,y = 0$$

الحل العام لها بطريقة جبرية حيث سنفرض

$y'' = m^2 \,\&\, y' = m \,\&\, y = 1$ ونعوضها في المعادلة

$m^2 + pm + q\,(1) = 0$

نحلها ونجد جذور المعادلة والتي ستأخذ ثلاث حالات

١) عندما $m_1 \,\&\, m_2$ جذور حقيقية مختلفة سيكون الحل للمعادلة التفاضلية

$y= c_1e^{m1x}+ c_2e^{m2x}$

٢) عندما تكون $(m_1=m_2)$ حقيقية ومتساوية

$y=c_1e^{mx}+x.c_2\, e^{mx}$

٣) ثالثاً عندما تكون m1.m2 جذور خيالية الحل للمعادلة سيكون $m=a \mp ib$

$y=e^{ax}(c_1 \cos bx \mp c_2 \text{ Sin } bx)$

Ex . solve (find the generd solution)

$\frac{d^2y}{dx^2}+\frac{dy}{dx}- 6y=0$

هنا نلاحظة من الرتبة الثانية وطرفها الايمن صفر تعني حالة الــ m اي نفرض

$y = 1, \frac{dy}{dx} = m, \frac{d^2y}{dx^2} = m^2$

$m^2+m-6=0$

$(m+3)\ (m-2)=0$

$m_1=-3,\ m_2=2$

g.s. $y=c_1e^{m1x} + c_2e^{\ m2x}$

$y=c_1e^{-3x}+c_2e^{2x}$

*هنا للتحقق من صحة الحل نشتق y' و y'' وتفوق في الطرف الايسر سيكون الناتج = صفر اي الطرف الايمن

Ex. $y'' - 6y' = -9y$

Sol. $y'' + 6y' + 99 = 0$

$m^2+6m+9=0$

$(m-3)^2 = 0$

$m_1 = m_2 = m = 3$

$\therefore\ y=c_1e^{3x} +c_2xe^{3x}$

Ex. : $y'' + qy = 0??$

Sol. $m^2 + q = 0$

$m^2 = -9$

$m1,2 = \mp\sqrt{-9}$

$= \mp 3\sqrt{-1}$

$= \mp 3i$

$m_1, m_2 = a \mp bi \rightarrow a = 0\,, b = 3$

$y = e^{ax}(c_1 \cos bx \mp c_2 \sin bx)$

$y = c_1 \cos 3x \mp c_2 \sin 3x$

Exc. (1) $y'' - 3y' + 2y = 0$ (3) $y'' + qy = 0$

(2) $y'' = 4y' + 4y = 0$ (Homework)

Laplace Trans for motions تحويلات لابلاس

تحويل لابلاس : يعُرف تحويل لابلاس للدالة f (x) بانه التكامل التالي :-

$L\{f(x)\} = \int_0^\infty f(x)e^{-px}\,dx;\quad p$ عدد حقيقي

خواص تحويل لابلاس :-

(1) $L\{ f(x) \mp g(x) \} = L\{f(x)\} \mp \{g(x)\}$

(2) $L\{A f(x)\} = AL\{f(x)\}$

(3) $L\{A f(x) \mp B g(x)\} = AL\{f(x)\} \mp BL\{g(x)\}$

هنا خاصية رقم واحد تمثل توزيع تحويل لابلاس على الجمع والطرح لانه يمثل متكامل لذا يحقق هذه الخاصية للتكامل اي التوزيع على الجمع والطرح .

والخاصية الثانية تمثل الثابت يخرج خارج التحويل حيث ان التكامل بحاصل ضرب ثابت في دالة يساوي الثابت في تامل الدالة .

اما الشرط الثالث فهو جامع للشرطين السابقين ويسمى الخاصية الخطية (اي انه التكامل يحقق الخاصية الخطية) وبالتالي فأن تحويل لابلاس يحقق الخاصية الخطية .

Proof :

(1) L i{f (x) ∓ g (x) } ??= L{f (x) } ∓ L{g(x) }

Pf: L{ f(x) ∓ g(x) } = $\int_0^\infty (f(x) \mp g(x))e^{-px}dx$

= $\int_0^\infty f(x).e^{-px}dx \mp \int_0^\infty g(x)e^{-px}dx$

= L {f(x) ∓ L{g(x)}

(2) L{Af (x) } = AL{f (x) }

Df: L{Af(x) }= $\int_0^\infty (Af(x)e^{-px}\, dx$

= $\int_0^\infty A(f(x)e^{-px})\, dx$

= A $\int_0^\infty f(x)e^{-px}\, dx$

= A L{f (x)}

(3) L{A f (x) ∓ B g (x) }??

= $\int_0^\infty (Af(x)e^{-px})\, dx \mp \int_0^\infty (B\, g(x)e^{-px})dx$

= A $\int_0^\infty f(x)e^{-px}\, dx \mp B\int_0^\infty g\ (x)e^{-px}\, dx$

= A { f(x) }∓ B L {g (x)}

Pf (3) H.W . برهن خاصية (٣) بالاعتماد على خاصية (١) و& (٢)

L { A f(x) ∓ Bg (x) } ??

from propo. (1)

from propo. (2)

NOW if (1) f(x) =1 → find L {f(x) }=?

(1) L {1} = $\int_0^\infty 1.e^{-p}\,dx = \frac{-1}{p}\,e^{-px}\Big|_0^\infty$

$= \frac{-1}{p}\left(e^{-p(\infty)} - e^{-p(0)}\right)$

$= -\frac{1}{p}\left(\frac{1}{e\infty} - e^{0}\right)$

$= -\frac{1}{p}(0-1)$

:. L { 1} = $+\frac{1}{p}$ ← تحويل لابلاس للواحد هو

(2) If f(x) $= e^{ax}$

(2) L $\{e^{ax}\} = \int_0^\infty e^{ax}\ e^{-px}dx$

$= \int_0^\infty e^{(a-p)x}dx = \frac{1}{(a-p)}\int_0^\infty (a-p)e^{(a-p)x}\,dx$

$= \frac{1}{a-p}\ e^{(a-p)}\Big|_0^\infty = \frac{1}{a-p}\left(e^{(a-p)\infty} - e^{0}\right)$

$= \frac{1}{a-p}\ \left(e^{a\infty}\left(e^{-p\infty}\right) - 1\right) = \frac{1}{a-p}\left(\infty(0) - 1\right)$

وهكذا نجد تحويلات لابلاس لكل دالة تعطى لنا

Ex. : (**Homework**) Find Laplace transformation for the following function:-

1.$f(x) = 2x^4$

2. $f(x) = \sin 3x\ e^{-x}$

3. $f(x) = \frac{1}{2}\sinh 5x$

4. $f(x) = \frac{1}{3}\cos 2x\ e^{3x}$

5. $f(x) = 4\cosh\ 7x$

6. $f(x) = 3\,x^2\ e^{x}$

F(t)	F(s)=L(f(t))	
1		
t^n	$\frac{n!}{s^{n+1}}$	s>0, n=0, 1, 2…
e^{at}	$\frac{1}{s-a}$	s>a
Sinh(at)	$\frac{a}{s^2-a^2}$	S>|a|
Cosh(at)	$\frac{s}{s^2-a^2}$	s>|a|
Sin(at)	$\frac{a}{s^2+a^2}$	s>0
Cos(at)	$\frac{s}{s^2+a^2}$	s>0
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	
f'(t)	sF(s)-f(0)	
$\frac{1}{t}f(t)$	∫F(u)du	
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	n=1, 2, 3…

جدول تحويلات لابلاس لبعض الدوال

ملاحظه :- الجدول ليس للحفظ حيث يعطى التحويل المطلوب استعماله في السؤال.

Inverse Laplace transformation معكوس تحويل لابلاس

1. $L^{-1}\left\{\frac{1}{P}\right\} = 1$
2. $L^{-1}\left\{\frac{1}{P-a}\right\} = e^{ax}$

اي انه معكوس تحويلات لابلاس للمقدار هو ارجاع الدالة الاصلية التي كان تحويل لابلاس لها هو المقدار المراد ايجاد معكوس التحويل له.

Ex. Find $L^{-1}\left\{\frac{3}{P(P+3)}\right\} =?$

Sol. $L^{-1}\left\{\frac{3}{P(P+3)}\right\} = L^{-1}\left\{\frac{1}{P} - \frac{1}{P+3}\right\}$

$= L^{-1}\left\{\frac{1}{P}\right\} - L^{-1}\left\{\frac{1}{P+3}\right\} \quad by\ prop. 1)$

$= 1 - e^{-3x}$ حسب جدول معكوس

Ex. Find Laplace inverse transformation for the following:-

1. $L^{-1}\left\{\frac{3P+2}{P^2+4}\right\}$

Sol. $= L^{-1}\left\{\frac{3P}{P^2+4} + \frac{2}{P^2+4}\right\}$

$= L^{-1}\left\{\frac{3P}{P^2+4}\right\} + L^{-1}\left\{\frac{2}{P^2+4}\right\}$

$= 3L^{-1}\left\{\frac{P}{P^2+2^2}\right\} + 2L^{-1}\left\{\frac{1}{P^2+2^2}\right\}$

$= 3\cos 2x + 2\sin 2x$ من الجدول

2. $L^{-1}\left\{\frac{P+5}{(P+2)^2+9}\right\}$

$Sol. = L^{-1}\left\{\frac{P+2}{(P+2)^2+3^2}\right\} + L^{-1}\left\{\frac{3}{(P+2)^2+3^2}\right\}$

$= \cos 3x\ e^{-2x} + \sin 3x\ e^{-2x}$ من الجدول

Solution of differential Equation by using Laplace transformation and inverse

Let $a_2y'' + a_1y' + a_0y = f(x)$

معادلة تفاضية خطية من الرتبة الثانية وتكون ذات معاملات ثابتة وغير متجانسة.

ولحل المعادلة نتبع الخطوات الاتية:-

١- ناخذ تحويل لابلاس لطرفي المعادلة المطلوب حلها.

٢- بعد ترتيب المعادلة حسب التحويل $L\{y\}$ ناخذ معكوس تحويل لابلاس للطرفين فنحصل على الحل العام للمعادلة.

لحل المعادلة نحتاج الى تحويلات لابلاس للمشتقة الاولى والثانية وكالاتي:-

$$L\{y'(x)\} = P\ L\{y(x)\} - y(0)$$

$$L\{y''(x)\} = P^2\ L\{y(x)\} - Py(0) - y'(0)$$

Pf.1:-

$$L\{y'(x)\} = \int_0^\infty y'(x)e^{-px}\,dx$$

$$= \int_0^\infty e^{-px}.y'^{(x)}\,dx = y(x)e^{-px}\,\Big|_0^\infty + \int_0^\infty y(x)pe^{-px}\,dx$$

$$= (y(\infty)e^{-p\infty} - y(0)e^{-p0}) + p\int_0^\infty y(x)e^{-px}\,dx$$

$$= \big(0 - 1.y(0)\big) + pL\{y(x)\}$$

$$= pL\{y(x)\} - \mathrm{y}(0)$$

Pf 2:-

$$L\{y''(x)\} = P^2\ L\{y(x)\} - Py(0) - y'(0)$$

$$L\{y''(x)\} = \int_0^\infty y''(x)e^{-px}\,dx$$

Ex:- Solve $y' - y = 1$; $y(0) = 0$, $L\ (1) = \frac{1}{P}$

$$, L(e^{ax}) = \frac{1}{p-a}$$

Sol. L{y'}-L{y}=L{1}

p L{y}- y(0) - L{y}= $\frac{1}{P}$

(1-p) L{y}- 0 = $\frac{1}{P}$

(1-p) L{y} = $\frac{1}{P}$

L{y} = $\frac{1}{(1-\mathrm{p})P} = \frac{1}{(1-p)} - \frac{1}{P}$

$L^{-1}\{L\{y\}\} = L^{-1}\{\frac{1}{(1-p)} - \frac{1}{P}\}$

$= L^{-1}\{\frac{1}{(1-p)}\} - L^{-1}\{\frac{1}{P}\}$

$\therefore y = e^x - 1$

Ex. :- Solve $y'' + 2y' + 5y = 0$; $y(0) = 1$, $y'^{(0)} = 5$

$$L\ \{\sin bx\ e^{ax}\} = \frac{b}{(P+a)^2 + b^2}$$

$$L\ \{cos\ bx\ e^{ax}\} = \frac{P+a}{(P+a)^2 + b^2}$$

Sol. L{$y'' + 2y' + 5y$ }=L{0}

L{y''}+2 L{y'}+5 L{y} = 0

P^2 L{y}- p y(0)- y'(0) +2(p L{y}- y(0)) +5 L{y} = 0

(P^2 +2p+5) L{y} – p - 5 - 2 = 0

(P^2 +2p+5) L{y} = $p + 7$

$$L\{y\} = \frac{p+7}{(P^2+2p+5)}$$

$$L^{-1}\{L\{y\}\} = L^{-1}\{\frac{p+7}{(P^2+2p+5)}\}$$

$$y = L^{-1}\{\frac{p+1}{(P^2+2p+1)+4} + \frac{6}{(P^2+2p+1)+4}\}$$

$$y = L^{-1}\{\frac{p+1}{(P^2+2p+1)+4}\} + L^{-1}\{\frac{6}{(P^2+2p+1)+4}\}$$

$$y = L^{-1}\{\frac{p+1}{(p+1)^2+4}\} + 3L^{-1}\{\frac{2}{(p+1)^2+4}\}$$

$$\therefore y = \cos 2x \,.\, e^{-x} + 3\sin 2x \,.\, e^{-x}$$

Ex. Solve (Homework)

1. $\boldsymbol{y' + 2y = e^{x};\ \ y(0) = 0}$
2. $\boldsymbol{\frac{dy}{dx} + 2y = cosx\ ;\ y(0) = 1}$
3. $\boldsymbol{y' - 4y = 2\ ;\ \ y(0) = 2}$
4. $\boldsymbol{y'' - 4y' + 3y = 6e^{4x}\ ;\ y(0) = 2\ ,\ \ y'(0) = 6}$
5. **y''- 9 y =6 cos 3x ; y(0) = 0 , y'(0) = 3**