Chapter One

Methods of Solving Partial Differential Equations

Contents

Section(1.1): Origin of Partial Differential Equations

(1.1.1) Introduction:

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

(1.1.2) Definition Partial Differential Equations(P.D.E.)

An equation containing one or more partial derivatives of an un known function of two or more independent variables is known as a (P.D.E.).

For examples of partial differential equations we list the following:

1. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy$

2. $(\frac{\partial z}{\partial x})^2+\frac{\partial^3 z}{\partial y^3}=2x(\frac{\partial z}{\partial y})$

3. $z(\frac{\partial z}{\partial x})+\frac{\partial z}{\partial y}=x$

4. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz$

5. $\frac{\partial^2 z}{\partial x^2}=(1+\frac{\partial z}{\partial y})^{\frac{1}{2}}$

6. $y\{(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2=z(\frac{\partial z}{\partial y})$

(1.1.3) <u>Definition: Order of a Partial DifferentialEquation (O.P.D.E.)</u>

The order of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation.

The equations in examples (1),(3),(4) and (6) are of the first order ,(5) is of the second order and (2) is of the third order.

(1.1.4)<u>Definition: Degree of a Partial DifferentialEquation (D.P.D.E.)</u>

The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized, i.e made free from radicals and fractions so for as derivatives are concerned. in (1.1.2), equations (1),(2),(3) and (4) are of first degree while equations(5) and(6) are of second degree.

(1.1.5) <u>Definition: Linear and Non-Linear Partial Differential Equations</u>

A partial differential equation is said to be (Linear) if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied . Apartial differential equation which is not linear is called a(non-linear) partial differential equation.

In (1.1.2), equations (1) and (4) are linear while equation (2),(3),(5) and (6) are non-linear.

(1.1.6) Notations:

When we consider the case of two independent variables we usually assume them to be x and y and assume (z) to be the dependent variable. We adopt the following notations throughout the study of partial differential equations.

$$p=\frac{\partial z}{\partial x}\ ,q=\frac{\partial z}{\partial y}\ ,r=\frac{\partial^2 z}{\partial x^2}\ ,s=\frac{\partial^2 z}{\partial x\,\partial y}\,\text{and}\,t=\frac{\partial^2 z}{\partial y^2}$$

In case there are n independent variables, we take them to be $x_1,x_2,\dots\dots\dots,x_n$ and z is than regarded as the dependent variable. In this case we use the following notations:

$$p_1=\frac{\partial z}{\partial x_1}\ ,p_2=\frac{\partial z}{\partial x_2}\ ,\dots\dots\dots,p_n=\frac{\partial z}{\partial x_n}$$

Sometimes the partial differentiations are also denoted by making use of suffixes. Thus we write :

$$u_x=\frac{\partial u}{\partial x}\ ,u_y=\frac{\partial u}{\partial y}\ ,u_{xx}=\frac{\partial^2 u}{\partial x^2}\ ,u_{yy}=\frac{\partial^2 u}{\partial y^2}$$

and so on.

(1.1.7) <u>Classification of First Order p.d.es into</u>:

linear, semi-linear ,quasi-linear and non-linear equations

*<u>**linear equation:**</u> A first order equation $f(x, y, z, p, q) = 0$

Is known as linear if it is linear in p , q and z , that is ,if given equation is of the form:

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$$

for example:

1. $yx^2p + xy^2q = xyz + x^2y^3$
2. $p + q = z + xy$

are both first order L.P.D.Es

*****<u>**Semi-linear equation:**</u> A first order p.d.e. $f(x, y, z, p, q) = 0$

Is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and $y$only. i.e if the given equation is of the form:

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

for example:

1. $xyp + x^2yq = x^2y^2z^2$
2. $yp + xq = \frac{x^2y^2}{z^2}$

are both semi-linear equations

*****<u>**Quasi-linear equation:**</u> A first order p.d.e.$f(x, y, z, p, q) = 0$

Is known as quasi-linear equation, if it is linear in p and q. i.e if the given equation is of the form:

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

for example:

1. $x^2zp + y^2zq = xy$

2. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

are both quasi-linear equation.

***Non-linear equation:** A first order p.d.e $f(x, y, z, p, q) = 0$ which does not come under the above three types ,is known as a non-linear equation.

for example:

1. $p^2 + q^2 = 1$

2. $pq = z$

3. $x^2p^2 + y^2q^2 = z^2$

are all non-linear p.d.es.

Section(1.2):Derivation of Partial Differential Equation by the Elimination of Arbitrary Constants

For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a and b,the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations $F(x, y, z, a, b) = 0$,$\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$

The equation free from a and b will be the required partial differential equation.

Three situations may arise:

Situation (1):

When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.

Example: Consider $z = ax + y$ …………(1)

where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. x, we get

$\frac{\partial z}{\partial x} = a$ …………(2)

Differentiating (1) partially w.r.t. y, we get

$\frac{\partial z}{\partial y} = 1$ …………(3)

Eliminating abetween (1) and (2) yields

$$z = x\left(\frac{\partial z}{\partial x}\right) + y \qquad \ldots\ldots\ldots\ldots(4)$$

Since (3) does not contain arbitrary constant, so (3) is also partial diff. equation under consideration thus, we get two p.d.es (3) and (4).

Situation (2):

When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial diff. eq. of order one.

Example: Eliminate a and bfrom

$$az + b = a^2x + y \qquad \ldots\ldots\ldots\ldots(1)$$

Differencing (1) partially w.r.t. x and y, we have

$$a\left(\frac{\partial z}{\partial x}\right) = a^2 \qquad \ldots\ldots\ldots\ldots(2)$$

$$a\left(\frac{\partial z}{\partial y}\right) = 1 \qquad \ldots\ldots\ldots\ldots(3)$$

Eliminating a from (2) and (3), we have

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$$

which is the unique p.d.e. of order one.

Situation (3):

When the number of arbitrary constants is greater than the number of independent variables. Then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.

Example: Eliminate a, b and c from

$z = ax + by + cxy$ ………..(1)

Differentiating (1) partially w.r.t. x andy we have

$\frac{\partial z}{\partial x} = a + cy$ ………..(2) $\frac{\partial z}{\partial y} = b + cx$ ………..(3)

from (2) and (3) $\frac{\partial^2 z}{\partial x^2} = 0\, \frac{\partial^2 z}{\partial y^2} = 0$ ………..(4)

$\frac{\partial^2 y}{\partial x\, \partial y} = c$ ………..(5)

Now, (2) and (3)$x\frac{\partial z}{\partial x} = ax + cxy$and

$$y\frac{\partial z}{\partial y} = by + cxy$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \underbrace{ax + by + cxy} + cxy$$

from (1) and (5)

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\frac{\partial^2 y}{\partial x\, \partial y}$ ………..(6)

Thus, we get three p.d.es given by (4) and (6) which are all of order two.

… Examples …

Example1: Find a p.d.e. by eliminating a and bfrom

$$z = ax + by + a^2 + b^2$$

Sol. Given $z = ax + by + a^2 + b^2$ ………..(1)

differentiating (1) partially with respect to x and y,

we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$

substituting these values of a and b in (1) we see thatthe arbitrary constants a and b are eliminated and we obtain

$$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2$$

which is required p.d.e.

Example2: Eliminate arbitrary constants a and b from

$z = (x - a)^2 + (y - b)^2$ to form the p.d.e.

Sol. Given $z = (x - a)^2 + (y - b)^2$ …………(1)

differentiating (1) partially with respect to x and y, to get

$\frac{\partial z}{\partial x} = 2(x - a)$, $\frac{\partial z}{\partial y} = 2(y - b)$

Squatring and adding these equations, we have

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4(x - a)^2 + 4(y - b)^2$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4[(x - a)^2 + (y - b)^2]$$

$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4z$ using (1)

Example 3: from p.d.es by eliminating arbitrary constants a and b from the following relations:

(a) $z = a(x + y) + b$ (b) $z = ax + by + ab$

(c) $z = ax + a^2y^2 + b$ (d) $z = (x + a)(y + b)$

Sol. (a) Given $z = a(x + y) + b$ …………(1)

Differentiating (1) w.r.t.x and y, we get

$\frac{\partial z}{\partial x} = a$,$\frac{\partial z}{\partial y} = a$

eliminating a between these, we get

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ which is the required p.d.e.

(b) Try yourself (c) Try yourself (d) Try yourself

… Exercises …

Ex.(1): Eliminate a and b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ to form the partial differential equation.

Ex.(2): Eliminate h and k from the equation$(x-h)^2 + (y-k)^2 + z^2 = \alpha^2$to form the p.d.e.

Ex.(3): Eliminate a and b from the following equations to form the p.d.es

(a)$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (b)$2z = (ax+y)^2 + b$(c)$\log(az-1) = x + ay + b$

Ex.(4): Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial diff. eqs

(1) $z = Ae^{pt}\sin px$, (p and A)

(2) $z = Ae^{-p^2t}\cos px$, (p and A)

(3) $z = ax^3 + by^3$, (a and b)

(4) $4z = \left[ax + \left(\frac{y}{a}\right) + b\right]^2$,(a and b)

(5) $z = ax^2 + bxy + cy^2$, (a,b,c)

Section (1.3): Methods for solving linear and non-linear partial differential equations of order one

(1.3.1) Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are function of x, y, z.

A quasi-linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are function of x, y, z. Such a partial differential equation is known as (Lagrange equation), for example: * $xyp + yzq = zx$

* $(x - y)p + (y - z)q = z - x$

(1.3.2) Working Rule for solving $Pp + Qq = R$ by Lagrange's method

Step 1. Put the given linear p.d.e. of the first order in the standard form $Pp + Qq = R$ …………(1)

Step 2. Write down Lagrange's auxiliary equations for (1) namely

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{…………(2)}$$

Step 3. Solve (2) by using the method for solving ordinary differential equation of order one. The equation (2) gives three ordinary diff. eqs. every two of them are independent and give a solution.

Let $u(x,y,z) = a$ and $v(x,y,z) = b$, then the (general solution) is $\emptyset(u,v) = 0$, wher $\emptyset$ is an arbitrary function and the complete solution is $u = \alpha v + \beta$ where α, β are arbitrary constant.

<u>Ex.1:</u> Solve $\mathbf{2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x}$

Sol. Given $2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x$ …………(1)

The Lagrange's auxiliary for (1) are

$\frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{2x}$ …………(2)

Taking the first two fractions of (2), we have

$\frac{dx}{2} = \frac{dy}{-3} \rightarrow -3dx - 2dy = 0$ ………..(3)

Integrating (3), $-3x - 2y = a$ …………(4)

a being an arbitrary constant

Next, taking the first and the last fractions of (2), we get

$\frac{dx}{2} = \frac{dz}{2x} \rightarrow xdx = dz \rightarrow xdx - dz = 0$ ……(5)

Integrating (5), $\frac{x^2}{2} - z = b$ …………(6)

b being an arbitrary constant

From (4) and (6) the required general solution is

$$\emptyset(a,b) = 0 \rightarrow \emptyset\left(-3x - 2y, \frac{x^2}{2} - z\right) = 0$$

Where $\emptyset$ is an arbitrary function.

<u>Ex.2</u>: Solve $\mathbf{\left(\frac{y^2 z}{x}\right) p + xzq = y^2}$

Sol. Given$\left(\frac{y^2z}{x}\right)p + xzq = y^2$ ....(1)

The Lagrange's auxiliary equation for (1) are

$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$ ....(2)

Taking the first two fractions of (2), we have

$x^2zdx = y^2zdy \;\; \rightarrow \; x^2dx - y^2dy = 0$ ....(3)

Integrating (3), $\frac{x^3}{3} - \frac{y^3}{3} = a \;\rightarrow\; x^3 - y^3 = a_1$ ......(4)

a_1 being an arbitrary constant.

Next, taking the first and the last fractions of (2), we get

$xy^2dx = y^2zdz \;\;\rightarrow\;\; xdx - zdz = 0$ ....(5)

Integrating (5), $\frac{x^2}{2} - \frac{z^2}{2} = b \;\; \rightarrow x^2 - z^2 = b_1$ ...(6)

$b_1$being an arbitrary constant

From (4) and (6) the general solution is

$$\emptyset(a_1, b_1) = 0 \;\;\rightarrow\;\; \emptyset(x^3 - y^3, x^2 - z^2) = 0$$

<u>Ex.3:</u>Solve $\mathbf{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt}$

Sol. Given $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$ ....(1)

The Lagrange's auxiliary equation for (1) are

$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{xyt}$ ....(2)

Taking the first two fractions of (2), we have

$\frac{dx}{x} = \frac{dy}{y} \;\;\rightarrow\; \frac{dx}{x} - \frac{dy}{y} = 0$ ....(3)

Integrating (3), $\ln x - \ln y = \ln a \;\;\rightarrow\; \frac{x}{y} = a$ ......(4)

Taking the second and the third fractions of (2), we get

$\frac{dy}{y} = \frac{dt}{t} \;\; \rightarrow \;\; \frac{dy}{y} - \frac{dt}{t} = 0$ ....(5)

Integrating (5), $\ln y - \ln t = \ln b \;\; \rightarrow \; \frac{y}{t} = b$ ......(6)

Next, taking the second and the last fractions of (2), we get

$\frac{dy}{y} = \frac{dz}{xyt} \;\; \rightarrow \;\; xtdy - dz = 0$ ....(7)

Substituting (4) and (6) in (7), we get

$\frac{a}{b}y^2dy - dz = 0$ ....(8)

Integrating (8), $\frac{a}{3b}y^3 - z = c$

Using (4) and (6), $\frac{1}{3}xyt - z = c$ ....(9)

Where a, b and c are an arbitrary constant

The general solution is

$$\emptyset(a, b, c) = 0 \;\; \rightarrow \;\; \emptyset\left(\frac{x}{y}, \frac{y}{t}, \frac{1}{3}xyt - z\right) = 0$$

$\emptyset$ being an arbitrary function.

<u>Rule</u>: for any equal fractions, if the sum of the denominators equalto zero,then the sum of the numerators equal to zero also.

Now, Return to the last example depending on the Rule above we will find the constant c.

Multiplying each fraction in Lagrange's auxiliary (2) by $yt, xt, xy, -3$ respectively, we get the sum of the denominators is

$xyt + xyt + xyt - 3xyt = 0$ …………(10)

Then the sum of the numerators equal to zero also:

$ytdx + xtdy + xydt - 3dz = 0 \rightarrow \quad d(xyt) - 3dz = 0$……(11)

Integrating (11), $\quad xyt - 3z = c$ …………(12)

Note that (12) and (9) are the same.

<u>Ex.4:</u> Solve $\quad \mathbf{(y-z)p + (z-x)q = x - y}$

Sol. Given$(y-z)p + (z-x)q = x - y$ …………(1)

The Lagrange's auxiliary equations for (1) are

$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$ …………(2)

The sum of the denominators is

$$y - z + z - x + x - y = 0$$

Then, the sum of the numerators is equal to zero also, (by Rule)

$dx + dy + dz = 0$ …………(3)

Integrating (3), $\quad x + y + z = a$ …………(4)

To find b, multiplying (2) by x, y, z resp. the sum of the denominators is

$$x(y-z) + y(z-x) + z(x-y) = xy - xz + yz - xy + zx - yz = 0$$

Then, the sum of the numerators is equal to zero

$xdx + ydy + zdz = 0$ …………(5)

Integrating (5), $\quad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b$ …………(6)

Where, a and b are arbitrary constants.

The general solution is

$$\emptyset(a,b)=0 \;\; \rightarrow \;\; \emptyset\left(x+y+z, \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}\right)=0$$

… Exercises …

Solve the following partial differential equation:

1. $p\,tanx + q\,tany = tanz$.

2. $zp = -x$.

3. $y^2p - xyq = x(z-2y)$.

4. $(x^2+2y^2)p - xyq = xz$.

5. $xp + yq = z$.

6. $(-a+x)p + (-b+y)q = (-c+z)$.

7. $x^2p + y^2q + z^2 = 0$.

8. $yzp + zxq = xy$.

9. $y^2p + x^2q = x^2y^2z^2$.

10. $p - q = \frac{z}{(x+y)}$

(1.3.2) <u>The equation of the form $\mathbf{f(p,q)=0}$</u>

Here we consider equations in which p and q occur other than in the first degree, that is non-linear equations. To solve the equation $f(p,q)=0$ ………(1)

Taking $\quad p = constant = a \quad$ …………(2)

$q = constant = b \quad$ …………(3)

Substituting (2),(3) in (1), we get

$\mathrm{F}(\mathrm{a},\mathrm{b}) = 0 \;\rightarrow\; b = \mathrm{F}_1(a) \quad or \;\; a = \mathrm{F}_2(b)$………..(4)

From $dz = pdx + qdy$ …………(5)

Using (2),(3) $\rightarrow dz = adx + bdy$ ………..(6)

Integrating (6), $z = ax + by + c$ ………..(7)

Where c is an arbitrary constant

Substituting (4) in (7) to obtain the complete integral (complete solution)

$z = ax + \mathrm{F}_1(a)y + c \quad or \quad z = \mathrm{F}_2(b)x + by + c$ ……….(8)

<u>Ex.1:</u> Solve $\boldsymbol{p^2 + p = q^2}$

Sol. $\mathrm{p}^2 + \mathrm{p} - \mathrm{q}^2 = 0$ ………..(1)

The equation (1) of the form $f(p,q) = 0$

Let $\;p = a \;, q = b$

Substituting in (1)

$$a^2 + a - b^2 = 0 \rightarrow \; b^2 = a^2 + a \;\rightarrow b = \pm\sqrt{a^2 + a}$$

The complete integral is

$$z = ax + by + c$$

$$= ax \pm \sqrt{a^2 + a}y + c$$

Where c is an arbitrary constant.

<u>Ex.2</u>: Solve $\mathbf{pq = k}$, where $\mathbf{k}$ is a constant.

Sol. Given that $\;pq = k$ …………(1)

Since (1) is of the form $f(p,q) = 0$, it's solution is

$z = ax + by + c$ …………(2)

Let $p = a$, $q = b$, substituting in (1) , then $ab = k \rightarrow b = \frac{k}{a}$ ...(3)

Putting (3) in (2), to get the complete solution

$z = ax + \frac{k}{a}y + c$; c is an arbitrary constant .

<u>**Ex.3**</u>**: Solve** $\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = (\frac{\partial z}{\partial y})^3$

Sol. Given that $p - 3q = q^3$ ……….. (1)

Since (1) is of the form $f(p,q) = 0$, then

Let $p = a$, $q = b$

Substituting in (1) , $a - 3b = b^3 \rightarrow a = b^3 + 3b$ ………(2)

Putting (2) in the equation $z = ax + by + c$, we get

$$z = (b^3 + 3b)x + by + c$$

Where c is an arbitrary constant

The equation (3) is the complete integral .

(1.3.3) <u>The Equation of the form $z = px + qy + f(p,q)$</u>

A first order partial differential equation is said to be of **Clariaut** form if it can be written in the form

$$z = px + qy + f(p,q) \qquad \ldots(1)$$

to solve this equation taking $p = a$, $q = b$ and substituting in (1), so the complete integral is

$$z = ax + by + f(a,b) \qquad \ldots(2)$$

Example 1: Solve $\boldsymbol{z = px + qy + pq}$

Sol. The given equation is of the form $z = px + qy + f(p,q)$

let $p = a$ and $q = b$ substituting in the given equation, so the complete integral is

$$\boxed{z = ax + by + ab}$$

where a, b being arbitrary constant.

Example 2: Solve $\boldsymbol{x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = z - 5\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}}$

Sol. Rearrange the given equation, we have

$$x\,p + y\,q = z - 5p + pq$$

$$z = x\,p + y\,q + 5p - pq \qquad \ldots(3)$$

Equation (3) is of Clariaut form

let $p = a$ and $q = b$ substituting in (3), then the complete integral is $\boxed{z = ax + by + 5a - ab}$

where a, b being arbitrary constant.

Example 3: Solve $\boldsymbol{px + qy = z - p^3 - q^3}$

Sol. Rearrange the given equation, we have

$$z = px + qy + p^3 + q^3 \qquad \ldots(4)$$

let $p = a$ and $q = b$ substituting in (4)

$\boxed{z = ax + by + a^3 + b^3}$ that is the complete integral and

a, b being arbitrary constants.

(1.3.4) The Equation of the form $f(z,p,q)=0$

To solve the equation of the form

$f(z,p,q)=0 \quad \ldots(1)$

1. Let $\quad u=x+ay \qquad \ldots(2)$

   where a is an arbitrary constant

2. Replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (1) as follows,

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial x}.\frac{\partial u}{\partial u}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}=\frac{dz}{du}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial y}.\frac{\partial u}{\partial u}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}=a\frac{dz}{du} \qquad \ldots(3)$$

   from (2) $\quad \frac{\partial u}{\partial x}=1 \quad$ and $\quad \frac{\partial u}{\partial y}=a$

3. Substituting (3) in (1) and solve the resulting ordinary differential equation of first order by usual methods.

4. Next, replace u by $x+ay$ in the solution obtained in step 3 to get the complete solution.

Example 1: Solve $\boldsymbol{z=p+q}$

Sol. Given equation is $\quad z=p+q \qquad \ldots(4)$

which is of the form $f(z,p,q)=0$. Let $u=x+ay$ where a is an arbitrary constant.

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (4), we get

$$z = \frac{dz}{du} + a\ \frac{dz}{du}$$

$$\Rightarrow z = (1 + a)\ \frac{dz}{du}$$

$$\Rightarrow du = (1 + a)\frac{dz}{z} \qquad \ldots(5)$$

Integrating (5), $\quad u + c = (1 + a)\ln z$

where c is an arbitrary constant

Replacing u,

$$x + ay + c = \ln z^{(1+a)}$$

$$\Rightarrow\ e^{x+ay+c} = z^{(1+a)}$$

$$\Rightarrow\ z = e^{\frac{x+ay+c}{1+a}} \qquad \ldots(6)$$

and that is the complete integral.

Example 2: Solve $\left(\frac{\partial z}{\partial x}\right)^{2} z - \left(\frac{\partial z}{\partial y}\right)^{2} = 1$

Sol. Rearrange the given equation, we have

$p^2 z - q^2 = 1 \ldots(7)$

This equation is of the form $\mathrm{f(z, p, q)} = 0$

Let $u = x + ay$, where a is an arbitrary constant

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (7), we get

$$\left(\frac{dz}{du}\right)^{2} z - \left(a\ \frac{dz}{du}\right)^{2} = 1$$

$$\Rightarrow (z - a^2)\left(\frac{dz}{du}\right)^{2} = 1$$

$\Rightarrow \pm\sqrt{z-a^2}\,\frac{dz}{du} = 1$ by taking the square root

$\Rightarrow \pm\sqrt{z-a^2}\, dz = du$…(8)

Integrating (8),

$\pm\frac{2}{3}(z-a^2)^{3/2} = u + c$…(9)

Replacing u in (9) to get the complete integral

$$\boxed{\pm\frac{2}{3}(z-a^2)^{\frac{3}{2}} = x + ay + c}$$

(1.3.5) <u>The Equation of the form $f_1(x,p) = f_2(y,q) = 0$</u>

In this form z does not appear and the terms containing x and p are on one side and those containing y and q on the other side.

To solve this equation putting

$f_1(x,p) = f_2(y,q) = a$…(1)

where a is an arbitrary constant

$\therefore\ f_1(x,p) = a \quad \Rightarrow \quad p = g_1(x,a)$…(2)

$f_2(y,q) = a \quad \Rightarrow \quad q = g_2(y,a)$…(3)

Substituting (2) and (3) in $dz = pdx + qdy$, we get

$dz = g_1(x,a)dx + g_2(y,a)dy$…(4)

Integrating (4),

$$\boxed{z = \int g_1(x,a)dx + \int g_2(y,a)dy + b}$$

which is a complete integral containing two arbitrary constants$a$ and b.

Example 1: Solve $p = 2xq^2$

Sol. Separating p and x from q and y, the given equation reduces to $\frac{p}{x} = 2q^2$…(5)

Equating each side to an arbitrary constant a, we have

$$\frac{p}{x} = a \qquad \Rightarrow p = ax$$

$$2q^2 = a \quad \Rightarrow q = \pm\sqrt{\frac{a}{2}}$$

Putting these values of p and q in

$dz = pdx + qdy$, we get

$$dz = axdx \pm \sqrt{\frac{a}{2}}dy \qquad \text{…(6)}$$

Integrating (6), $z = \frac{a}{2}x^2 \pm \sqrt{\frac{a}{2}}\ y + b$

where a and b are two arbitrary constants.

Example 2: Solve $xq - y^2p - x^2y^2 = 0$

Sol. Separating p and x from q and y, the given equation reduces to $\frac{p+x^2}{x} = \frac{q}{y^2}$…(7)

Equating each side to an arbitrary constant a, we have

$$\frac{p+x^2}{x} = a \qquad \Rightarrow \qquad p = ax - x^2 \qquad \text{…(8)}$$

$$\frac{q}{y^2} = a \qquad \Rightarrow \qquad q = a\,y^2 \qquad \text{…(9)}$$

Putting (8) and (9) in $dz = pdx + qdy$, we get

$$dz = (ax - x^2)dx + ay^2dy \qquad \text{…(10)}$$

Integrating (10) , $\boxed{z = \frac{ax^2}{2} - \frac{x^3}{3} + a\frac{y^3}{3} + b}$

which is a complete integral containing two arbitrary constants a and b.

Example 3: Solve $p - 3x^2 = q^2 - y$

Sol. Equating each side to an arbitrary constant a, we get

$$p - 3x^2 = a \qquad \Longrightarrow \qquad p = a + 3x^2 \qquad \ldots(11)$$

$$q^2 - y = a \qquad \Longrightarrow \qquad q = \pm\sqrt{a + y} \qquad \ldots(12)$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (a + 3x^2)dx \pm \sqrt{a + y}dy \qquad \ldots(13)$$

Integrating (13) , $\boxed{z = ax + x^3 \pm \frac{2}{3}(a + y)^{3/2} + b}$

which is a complete integral containing two arbitrary constant a and b.

(1.3.6) Charpit's Method (General Method of Solving p.d.es of Order One but of any Degree)

Let the given p.d.e of first order and non- linear in p and q be

$$f(x, y, z, p, q) = 0 \qquad \ldots(1)$$

To solve this equation we will use the following charpit's auxiliary equations.

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

After substituting the partial derivatives in charpit's auxiliary equations select the proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Then, putting p and q in the relation $dz = pdx + qdy$ which on integration gives the complete integral of the given equation.

Example 1: Solve $z = px + qy + p^2 + q^2$ by charpit's method.

Sol. Let $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$…(2)

charpit's auxiliary equation are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

From (2) $f_x = -p$, $f_y = -q$, $f_z = 1$, $f_p = -x - 2p$, $f_q = -y - 2q$

$$\therefore \quad \frac{dp}{-p + p} = \frac{dq}{-q + q} = \frac{dz}{p(x + 2p) + q(y + 2q)} = \frac{dx}{x + 2p}$$

$$= \frac{dy}{y + 2q}$$

Taking the first fraction $dp = 0 \quad \rightarrow \quad p = a$…(3)

Taking the second fraction $dq = 0 \quad \rightarrow \quad q = b$ …(4)

Substituting (3) and (4) in (2) to get the complete integral

$$\boxed{z = ax + by + a^2 + b^2}$$

where a and b are arbitrary constants.

Example 2: Solve $2zx - px^2 - 2qxy + pq = 0$ by charpit's method.

Sol. Let $f(x, y, z, p, q) = 2zx - px^2 - 2qxy + pq = 0$…(5)

$f_x = 2z - 2px - 2qy$, $f_y = -2qx$, $\quad f_z = 2x f_p = -x^2 + q$, $\quad f_q = -2xy + p$

Substituting in charpit's auxiliary equations , we get

$$\frac{dp}{2z-2px-2qy+2px} = \frac{dq}{-2qx+2qx} = \frac{dz}{-p(-x^2+q)-q(-2xy+p)} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} \ldots (6)$$

Taking the second fraction of (6)

$dq = 0 \quad \rightarrow \quad q = c \ldots (7)$

Substituting (7) in (5)

$$2zx - px^2 - 2cxy + cp = 0$$

$$p = \frac{2xz-2cxy}{x^2-c} \quad \rightarrow \quad p = \frac{2x(z-cy)}{x^2-c} \ldots (8)$$

Putting (7) and (8) in $dz = pdx + qdy$

$$dz = \frac{2x(z-cy)}{x^2-c} dx + cdy \Longrightarrow dz - cdy = \frac{2x(z-cy)}{x^2-c} dx$$

$$\frac{dz-cdy}{(z-cy)} = \frac{2x\, dx}{x^2-c} \ldots (9)$$

Integrating (9), $ln|z - cy| = \ln|x^2 - c| + \ln b$

$$z - cy = b(x^2 - c)$$

$$\boxed{z = b(x^2 - c) + cy}$$

which is a complete integral where b and c are two arbitrary constants.

… Exercises …

Solve the following equations:

1. $q = 3p^2$
2. $zpq = p + q$
3. $p^2 - y^2 q = y^2 - x^2$
4. $(y^2 + 4)xpq - (x^2 + 2) = 0$

5. $q - px - p^2 = 0$
6. $px + qy = pq$
7. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{y}{x}$
8. $p^2 - q^2 = z$

(1.3.7) Using Some Hypotheses in the Solution

Sometimes we need some hypotheses to solve the partial differential equation, here we will give three types of hypotheses.

A) When the equation contains the term (px) or its' powers we use the hypothesis $\boxed{X = \ln x}$

as follows

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial X}{\partial X} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}.\frac{1}{x}(\text{since } X = \ln x \implies \frac{\partial X}{\partial x} = \frac{1}{x})$$

$$\implies xp = \frac{\partial z}{\partial X}$$

Then substituting this result in the given equation and solve it by previous methods.

Example 1: Solve $z = px$ by hypotheses

Sol. From $X = \ln x$ we have $xp = \frac{\partial z}{\partial X}$ …(1)

Substituting (1) in the given equation, we get

$$z = \frac{\partial z}{\partial X} \implies \partial X = \frac{\partial z}{z} \quad \text{…(2)}$$

Integrating (2), $X = \ln z + \ln \emptyset(y)$ …(3)

where $\emptyset$ is an arbitrary function for y

replacing X in (3) to get the complete integral

$\ln x = \ln \emptyset(y).z$

$\Longrightarrow \boxed{z = \frac{x}{\emptyset(y)}} \ldots(4)$

Example 2: Solve $\boldsymbol{q = px + p^2x^2}$ by hypotheses

Sol. Given that $q = px + (px)^2$ …(5)

from $X = \ln x$ we have $xp = \frac{\partial z}{\partial X}$ …(6)

Substituting (6) in (5), we get

$$q = \frac{\partial z}{\partial X} + \left(\frac{\partial z}{\partial X}\right)^2 \quad \ldots(7)$$

Let $\frac{\partial z}{\partial X} = t$then (7) will be

$$q = t + t^2 \quad \ldots(8)$$

The equation (8) is of the form $f(t, q) = 0$

Then let $t = a$ and $q = b$, putting in (8) $b = a + a^2$

Substituting in $z = aX + by + c$

$\Longrightarrow z = aX + (a + a^2)y + c \ldots(9)$

where c is an arbitrary constant

replacing X in (9) to get the complete integral

$\boxed{z = a\, ln\, x + (a + a^2)y + c}$

B) When the equation contains the term (qy) or its' powers we use the hypothesis $\boxed{Y = \ln y}$

as follows:

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} . \frac{\partial Y}{\partial Y} = \frac{\partial z}{\partial Y} . \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y} . \frac{1}{y} \text{ (since } Y = \ln y \Longrightarrow \frac{\partial Y}{\partial y} = \frac{1}{y})$$

$\Longrightarrow qy = \dfrac{\partial z}{\partial Y}$

Then solving by the same way in (A).

Example 3: Solve $\boldsymbol{2p + qy = 4}$ by hypotheses

Sol. Given that $2p + qy = 4$ …(10)

from $Y = \ln y$ we have $qy = \frac{\partial z}{\partial Y}$ …(11)

Substituting (11) in (10), we get

$$2p + \frac{\partial z}{\partial Y} = 4$$

Let $\frac{\partial z}{\partial Y} = t$then,

$2p + t = 4$ …(12)

The equation (12) is of the form $f(p, t) = 0$

Then let $p = a$ and $t = b$, putting in (12) $2a + b = 4$

$\Longrightarrow b = 4 - 2a$ …(13)

Substituting (13) in $z = ax + bY + c$

$\Longrightarrow z = ax + (4 - 2a)Y + c$…(14)

where c is an arbitrary constant

replacing Y in (14) to get the complete integral

$\boxed{z = a\,x + (4 - 2a)\ln y + c}$

Example 4: Solve $\boldsymbol{p^2x^2 = z^2 + q^2y^2}$ by hypotheses

Sol. Given that $p^2x^2 = z^2 + q^2y^2$ …(15)

from $X = \ln x$ and $Y = \ln y$ we have

$xp = \frac{\partial z}{\partial X}$ and $qy = \frac{\partial z}{\partial Y}$ …(16)

Substituting (16) in (15), we get

$\left(\frac{\partial z}{\partial X}\right)^2 = z^2 + \left(\frac{\partial z}{\partial Y}\right)^2$ …(17)

Let $t = \frac{\partial z}{\partial X}$ and $r = \frac{\partial z}{\partial Y}$putting in (17)

$t^2 - r^2 = z^2$ …(18)

Note that (18) is of the form $f(t, r, z) = 0$

Taking $u = X + aY$ (a is constant)

Then $t = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial X}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial X} = \frac{dz}{du}$

$r = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial Y}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial Y} = a\frac{dz}{du}$ …(19)

because ($\frac{\partial u}{\partial X} = 1$ and $\frac{\partial u}{\partial Y} = a$)

putting (19) in (18)

$$\left(\frac{dz}{du}\right)^2 - a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$(1 - a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$\pm\sqrt{1 - a^2}\,\frac{dz}{du} = z$ (taking the square root)

$\pm\sqrt{1 - a^2}\,\frac{dz}{z} = du$ …(20)

Integrating (20),

$\pm\sqrt{1 - a^2}\ln z = u + c$ (c is constant) …(21)

Now, replacing u in (21) to get the complete integral

$\pm\sqrt{1 - a^2}\ln z = X + aY + \ln c$ …(22)

Next, replacing X and Y in (22) to get the complete integral

$\pm\sqrt{1-a^2}\ln z = \ln x + aln\, y + \ln c$

$\ln z^b = \ln cx\, y^a$ where $b = \pm\sqrt{1-a^2}$

$\Longrightarrow z^b = cx\, y^a$ …(23)

So, (23) is the complete integral.

C) When the equation contains the terms $\frac{p}{z}$ or $\frac{q}{z}$ or its' powers we use the hypothesis $\boxed{Z = \ln z}$

as follows:

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial Z}{\partial Z} = \frac{\partial z}{\partial Z}.\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x}$ (since $\frac{\partial z}{\partial Z} = z$)

hence $\frac{p}{z} = \frac{\partial Z}{\partial x}$

by the same way we have $\frac{q}{z} = \frac{\partial Z}{\partial y}$

then substituting this terms in the given equation and solve it by the same way in (A) and (B).

Example 5: Solve $\boldsymbol{px + qy = z}$ by $\boxed{\boldsymbol{Z = \ln z}}$

Sol. Given that $px + qy = z$ …(24)

Dividing on z, $\frac{p}{z}x + \frac{q}{z}y = 1$ …(25)

using $Z = \ln z$ we have $\frac{p}{z} = \frac{\partial Z}{\partial x}$ and $\frac{q}{z} = \frac{\partial Z}{\partial y}$,substituting in (25)

$x\,\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = 1$ …(26)

Let $t = \frac{\partial Z}{\partial x}$ and $r = \frac{\partial Z}{\partial y}$thus, (26) would be

$xt + yr = 1$ …(27)

Clear that (27) is of the form $f_1(x,t) = f_2(y,r)$

Then putting x, pin one side and y, q in the other side

$xt = 1 - ry = a$ (a is constant)

Then $xt = a \rightarrow t = \frac{a}{x}$ …(28)

$1 - ry = a \rightarrow r = \frac{1-a}{y}$ …(29)

Substituting (28), (29) in $dZ = tdx + rdy$

$\Longrightarrow dZ = \frac{a}{x}dx + \frac{1-a}{y}dy$ …(30)

Integrating (30), we get

$Z = a\ln x + (1-a)\ln y + \ln b$ (where b is constant)

Replacing Z from the hypothesis to get the complete integral

$\therefore \ln z = \ln(bx^a y^{(1-a)})$ (by properties of ln)

$\Longrightarrow \boxed{z = b\, x^a\, y^{(1-a)}}$ …(31)

Then (31) is the complete integral.

Example 6: Solve $p^2 + q^2 = z^2(x+y)$ by hypotheses

Sol. Dividing on z^2,

$\frac{p^2}{z^2} + \frac{q^2}{z^2} = x + y$ …(32)

using $Z = \ln z$ we have $\frac{p}{z} = \frac{\partial Z}{\partial x}$ and $\frac{q}{z} = \frac{\partial Z}{\partial y}$,substituting in (32)

$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y$ …(33)

Let $t = \frac{\partial Z}{\partial x}$ and $r = \frac{\partial Z}{\partial y}$putting in (33)

$t^2 + r^2 = x + y$ …(34)

Then $t^2 - x = a \rightarrow t = \pm\sqrt{a+x}$

$$y - r^2 = a \rightarrow r = \pm\sqrt{y-a}$$

Substituting in $dZ = tdx + rdy$

$$\Rightarrow dZ = \pm\sqrt{a+x}\,dx + \pm\sqrt{y-a}\,dy \qquad \ldots(35)$$

Integrating (35), we get

$$Z = \pm\frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c \qquad \text{(where } c \text{ is constant)}$$

Replacing Z from the hypothesis to get the complete integral

$$\Rightarrow \boxed{\ln z = \pm\frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c}$$

… Exercises …

1. $p^2x^2 = z(z - qy)$
2. $pq = z^2 y \sec x$
3. $p + q = z\, e^{x+y}$
4. $p^2 + zq = z^2(x - y)$
5. $p^2 + zp = z^2(x - y)$
6. $p^2 + q^2 = z^2$
7. $xp + 4q = \cos y$
8. $p^2 + q^2 = z^2 y$

Section(1.4): Homogeneous linear partial differential equations with constant coefficients and higher order

A linear partial differential equation with constant coefficients is called homogeneous if all it's derivatives are of the same order. The general form of such an equation is

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = f(x, y) \quad \ldots\ldots\ldots\ldots(1)$$

Where $A_0, A_1, \ldots, A_n$ are constant coefficients.

For example:

1. $3 \frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 y}{\partial y^2} = o$ homo. of order 2.
2. $2 \frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 \partial y} + 5 \frac{\partial^3 z}{\partial x \partial y^2} - 8 \frac{\partial^3 y}{\partial y^3} = x + y$ homo. of order 3.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will be denoted by $D \ or \ D_x$ and D' or D_y respectively. Then (1) can be rewritten as:

$$(A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n) z = f(x, y) \quad \ldots\ldots\ldots\ldots(2)$$

On the other hand, when all the derivatives in the given equation are not of the same order, then it is called a non-homogenous linear partial differential equation with constant coefficients.

In this section we propose to study the various methods of solving homogeneous linear partial differential equation with constant coefficients, namely (2).

Equation (2) may rewritten as:

$\boxed{F(D_x, D_y)z = f(x,y)}$ ……………(3)

Where $F(D_x, D_y) = A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n$

Equation (3) has a general solution when $f(x,y) = 0$

i.e$F(D_x, D_y)z = 0$

$\rightarrow$ $\boxed{(A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n)z = 0}$……….(4)

And a particular solution (particular integral) when $f(x,y) \neq 0$

❖ Now, we will find the general solution of (4)

Let $z = \emptyset(y + mx)$ be a solution of (4) where $\emptyset$ is an arbitrary function and m is a constant , then

$$D_x z = \emptyset'(y + mx).m$$

$$D_x^2 z = \emptyset''(y + mx).m^2$$

$$\vdots$$

$$D_x^n z = \emptyset^{(n)}(y + mx).m^n$$

$$D_y z = \emptyset'(y + mx)$$

$$D_y^2 z = \emptyset''(y + mx)$$

$$\vdots$$

$$D_y^n z = \emptyset^{(n)}(y + mx)$$

$$D_x D_y z = m\,\emptyset'(y+mx)$$

$$D_x^2 D_y z = m^2 \emptyset^{(3)}(y+mx)$$

$$\vdots$$

$$D_x^r D_y^s z = m^r \emptyset^{(r+s)}(y+mx)$$

$= m^r \emptyset^{(n)}(y+mx)$, where $r+s=n$

Substituting these values in (4) and simplifying, we get :

$(A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n)\emptyset^{(n)}(y+mx) = 0$ …(5)

Which is true if m is a root of the equation

$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0$ ……..(6)

The equation (6) is known as the (characteristic equation) or the (auxiliary equation(A.E.)) and is obtained by putting $D_x = m$ and $D_y = 1$ in $F(D_x, D_y)z = 0$, and it has $n$roots.

Let $m_1, m_2, \dots, m_n$ be n roots of A.E. (6). **Three cases arise**:

Case 1 when the roots are distinct.

If $m_1, m_2, \dots, m_n$ are n distinct roots of A.E. (6) then $\emptyset_1(y+m_1x), \emptyset_2(y+m_2x), \dots\dots\dots, \emptyset_n(y+m_nx)$ are the linear solution corresponding to them and since the sum of any linear solutions is a solution too than the general solution in this case is:

$z = \emptyset_1(y+m_1x) + \emptyset_2(y+m_2x) + \cdots + \emptyset_n(y+m_nx)$ …..(7)

<u>**Ex.1**</u>: **Find the general solution of**

$$(\boldsymbol{D_x^3 + 2D_x^2 D_y - 5D_x D_y^2 - 6D_y^3)z = 0}$$

Sol. The A.E. is$m^3 + 2m^2 - 5m - 6 = 0$

$$\rightarrow \quad (m+1)(m^2 + m - 6 = 0$$

$$\rightarrow \quad (m+1)(m+3)(m-2) = 0$$

$$m_1 = -1, \;\; m_2 = -3, \;\; m_3 = 2$$

Note that m_1, m_2 and m_3 are different roots, then the general solution is

$$z = \emptyset_1(y + m_1x) + \emptyset_2(y + m_2x) + \emptyset_3(y + m_3x)$$

$$\rightarrow \quad z = \emptyset_1(y - x) + \emptyset_2(y - 3x) + \emptyset_3(y + 2x)$$

Where $\emptyset_1, \emptyset_2, \emptyset_3$ are arbitrary functions.

<u>Ex.2</u>: Find the general solution of $m^2 - a^2 = 0$where a is a real number.

Sol. Given that $\mathrm{m}^2 - \mathrm{a}^2 = 0 \;\rightarrow\; \mathrm{m}^2 = \mathrm{a}^2$

$\rightarrow \; \mathrm{m} = \pm \mathrm{a}$ different root

$$m_1 = a \;\;, m_2 = -a$$

The general solution is

$$z = \emptyset_1(y + ax) + \emptyset_2(y - ax)$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Case 2 when the roots are repeated.

If the root m is repeated k times . <u>i.e</u>.$m_1 = m_2 = \cdots = m_k$, then the corresponding solution is :

$$\boxed{z = \emptyset_1(y + m_1x) + x\emptyset_2(y + m_1x) + \cdots + x^{k-1}\emptyset_n(y + m_1x)} \ldots (8)$$

Where $\emptyset_1, \ldots, \emptyset_k$are arbitrary functions.

<u>Note:</u> If some of the roots $m_1, m_2, \ldots, m_n$ are repeated and the other are not . <u>i.e</u>. $m_1 = m_2 = \cdots = m_k \neq m_{k+1} \neq \cdots \neq m_n$ then the general solution is :

$$z = \emptyset_1(y + m_1x) + x\emptyset_2(y + m_1x) + \cdots + x^{k-1}\emptyset_n(y + m_1x) + \emptyset_{k+1}(y + m_{k+1}x) + \cdots + \emptyset_n(y + m_nx) \quad \ldots\ldots\ldots\ldots(9)$$

<u>Ex.3</u>: Solve $(D_x^3 - D_x^2D_y - 8D_xD_y^2 + 12D_y^3)z = 0$

Sol. The A.E. is $\quad m^3 - m^2 - 8m + 12 = 0$

$$\rightarrow \quad (m-2)(m-2)(m+3) = 0$$

$m_1 = m_2 = 2 \quad , \quad m_3 = -3$

Then, the general solution is

$$z = \emptyset_1(y + 2x) + x\emptyset_2(y + 2x) + \emptyset_3(y - 3x)$$

Where $\emptyset_1, \emptyset_2, \emptyset_3$ are arbitrary functions.

<u>Ex.4</u>: Find the general solution of the equation that it's A.E. is :

$$(m-1)^2(m+2)^3(m-3)(m+4) = 0$$

Sol. Given that $(m-1)^2(m+2)^3(m-3)(m+4) = 0$

$$m_1 = m_2 = 1 \quad, m_3 = m_4 = m_5 = -2 \quad, m_6 = 3 \quad, m_7 = -4$$

The general solution is

$$z = \emptyset_1(y + x) + x\emptyset_2(y + x) + \emptyset_3(y - 2x) + x\emptyset_4(y - 2x) + x^2\emptyset_5(y - 2x) + \emptyset_6(y + 3x) + \emptyset_7(y - 4x)$$

Where $\emptyset_1, \ldots, \emptyset_7$ are arbitrary functions.

Case 3 when the roots are complex.

If one of the roots of the given equation is complex let be m_1 then the conjugate of m_1 is also a root, let be m_2 , so the general solution is:

$$\boxed{z = \emptyset_1(y + m_1x) + \emptyset_2(y + m_2x) + \cdots + \emptyset_n(y + m_nx)}$$

Where $\emptyset_1 , \ldots , \emptyset_n$ are arbitrary functions.

<u>Ex.5</u>: Solve $(D_x^2 + D_y^2)z = 0$

Sol. The A. E. is $\quad m^2 + 1 = 0 \quad \rightarrow \quad m = \pm i$

$$\therefore \quad m_1 = i \quad , \quad m_2 = -i$$

The general solution is

$$z = \emptyset_1(y + ix) + \emptyset_2(y - ix)$$

Where $\emptyset_1 , \emptyset_2$ are arbitrary functions.

<u>Ex.6:</u> Solve $(D_x^2 - 2D_xD_y + 5D_y^2)z = 0$

Sol. The A. E. is $\quad m^2 - 2m + 5 = 0$

$$\rightarrow \quad m = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

$$\therefore \quad m_1 = 1 + 2i \quad , \quad m_2 = 1 - 2i$$

$$z = \emptyset_1(y + (1 + 2i)x) + \emptyset_2(y + (1 - 2i)x)$$

That is the general solution where $\emptyset_1 , \emptyset_2$ are arbitrary functions.

<u>Ex.7</u>: Solve $(D_x^4 - D_x^3D_y + 2D_x^2D_y^2 - 5D_xD_y^3 + 3D_y^4)z = 0$

Sol. The A.E. is $\quad m^4 - m^3 + 2m^2 - 5m + 3 = 0$

$$\rightarrow \quad (m - 1)^2(m^2 + m + 3) = 0$$

$m_1 = m_2 = 1 \quad , \; m = \frac{-1\pm\sqrt{1-12}}{2} = \frac{-1\pm\sqrt{11}i}{2}$

$$\therefore \; m_3 = \frac{-1+\sqrt{11}i}{2} \qquad , m_4 = \frac{-1-\sqrt{11}i}{2}$$

Then, the general solution is

$$z = \emptyset_1(y+x) + x\emptyset_2(y+x) + \emptyset_3\left(y + (\frac{-1+\sqrt{11}i}{2})x\right) + \emptyset_4\left(y + (\frac{-1-\sqrt{11}i}{2})x\right)$$

Where $\emptyset_1 , \dots , \emptyset_4$ are arbitrary functions.

❖ Particular integral (P.I.) of homogeneous linear partial differential equation

When $f(x,y) \neq 0$in the equation (3) which it's$F(D_x, D_y)z = f(x,y)$ multiplying (3) by the inverse operator$\frac{1}{F(D_x,D_y)}$ of the operator $F(D_x, D_y)$ to have

$$\frac{1}{F(D_x,D_y)} \cdot F(D_x, D_y)\, z = \frac{1}{F(D_x,D_y)}\, f(x,y)$$

$$\rightarrow \boxed{z = \frac{1}{F(D_x,D_y)}\, f(x,y)} \quad \dots\dots\dots (11)$$

Which it's the particular integral (P.I.)

The operator $F(D_x, D_y)$ can be written as

$$F(D_x, D_y) = (D_x - m_1 D_y)(D_x - m_2 D_y) \dots (D_x - m_n D_y) \quad \dots.(12)$$

Substituting (12) in (11) :

$z = \frac{1}{(D_x - m_1 D_y)(D_x - m_2 D_y)\ldots(D_x - m_n D_y)} f(x,y)$ ....(13)

Taking $u_1 = \frac{1}{(D_x - m_n D_y)} f(x,y)$

$$\therefore (D_x - m_n D_y)u_1 = f(x,y)$$

This equation can be solved by Lagrange's method .

The Lagrange's auxiliary equations are

$\frac{dx}{1} = \frac{dy}{-m_n} = \frac{du_1}{f(x,y)}$ ....(14)

Taking the first two fractions of (14)

$m_n dx + dy = 0 \quad \rightarrow \boxed{m_n x + y = a}$ ....(15)

Taking the first and third fractions of (14)

$dx = \frac{du_1}{f(x,y)} \quad \rightarrow \boxed{f(x,y)dx = du_1}$ ....(16)

Substituting (15) in (16) we have

$$f(x, a - m_n x)dx = du_1$$

Integrating the last one we have

$$u_1 = \int f(x, a - m_n x)dx + b$$

Let $b = 0$, then we have u_1

By the same way , we take

$$u_2 = \frac{1}{D_x - m_{n-1} D_y} u_1$$

And solve it by Lagrange's method to get u_2 , then continue in this way until we get to

$$z = u_n = \frac{1}{D_x - m_1 D_y} u_{n-1}$$

And by solving this equation we get the particular integral (P.I.)

Ex.1: solve $\left(\boldsymbol{D_x^2 - D_y^2}\right)\boldsymbol{z} = \mathbf{sec^2}(\boldsymbol{x+y})$

Sol. Firstly, we will find the general solution of

$\left(D_x^2 - D_y^2\right)z = 0$ ……….(1)

The A. E. is $m^2 - 1 = 0 \rightarrow m^2 = 1 \rightarrow m = \pm 1$

$$\therefore \quad m_1 = 1\,, \ \ m_2 = -1$$

$\therefore \ \ z = \emptyset_1(y+x) + \emptyset_2(y-x)$ …………(2)

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Second, we will find the particular integral as follows

$$z_2 = \frac{1}{D_x^2 - D_y^2}\sec^2(x+y)$$

$$= \frac{1}{(D_x - D_y)(D_x + D_y)}\sec^2(x+y)$$

Let $u_1 = \frac{1}{(D_x + D_y)}\sec^2(x+y)$

$$\left(D_x + D_y\right)u_1 = \sec^2(x+y)$$

The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du_1}{\sec^2(x+y)}$$

Taking the first two fractions

$dx = dy \rightarrow x - y = a$ ……….(3)

Taking the first and third fractions

$dx = \frac{du_1}{\sec^2(x+y)} \rightarrow \sec^2(x+y)\,dx = du_1$ ……….(4)

Substituting (3) in (4), we have

$\sec^2(2x-a)\,dx = du_1$ ....(5)

Integrating (5), we have

$$u_1 = \frac{1}{2}\tan(2x-a) + b$$

Let $b = 0$ and replacing a , we get

$u_1 = \frac{1}{2}\tan(x+y)$ ....(6)

Putting (6) in z_2

$$z_2 = \frac{1}{(D_x - D_y)} \cdot \frac{1}{2}\tan(x+y)$$

$$\rightarrow \left(D_x - D_y\right)z_2 = \frac{1}{2}\tan(x+y)$$

The Lagrange's auxiliary equation are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz_2}{\frac{1}{2}\tan(x+y)}$$

Taking the first two fractions

$dx = -dy \ \rightarrow \ x + y = a$ ....(7)

Taking the first and third fractions

$$dx = \frac{dz_2}{\frac{1}{2}\tan(x+y)}$$

$\frac{1}{2}\tan(x+y)\,dx = dz_2$ ....(8)

Substituting (7) in (8)

$\frac{1}{2}\tan a \ dx = dz_2$ ....(9)

Integrating (9) , we get

$$\frac{1}{2}x\tan a = z_2 + b$$

Let $b = 0$, and replacing a from (7) we get the particular integral

$z_2 = \frac{1}{2} x \tan(x + y)$ ……………(10)

Hence the required general solution is

$$z = z_1 + z_2$$

$$= \emptyset_1(y + x) + \emptyset_2(y - x) + \frac{x}{2}\tan(x + y) \quad \text{……….(11)}$$

Short methods of finding the P.I. in certain cases :

Case 1 **When $f(x, y) = e^{ax+by}$ where a and b are arbitrary constants**

To find the P.I. when $F(a, b) \neq 0$, we derive $f(x, y)$ for x any y n times:

$$D_x e^{ax+by} = a e^{ax+by}$$

$$D_x^2 e^{ax+by} = a^2 e^{ax+by}$$

$$\vdots$$

$$D_x^n e^{ax+by} = a^n e^{ax+by}$$

$$D_y e^{ax+by} = b e^{ax+by}$$

$$D_y^2 e^{ax+by} = b^2 e^{ax+by}$$

$$\vdots$$

$$D_y^n e^{ax+by} = b^n e^{ax+by}$$

$D_x^r D_y^s e^{ax+by} = a^r b^s e^{ax+by}$ where $r + s = n$

So

$$F(D_x, D_y) e^{ax+by} = F(a, b) e^{ax+by}$$

Multiplying both sides by $\frac{1}{F(D_x,D_y)}$, we get

$$e^{ax+by} = \frac{1}{F(D_x,D_y)} F(a,b) e^{ax+by}$$

Since $F(a,b) \neq 0$, then we can divide on it :

$$\frac{1}{F(a,b)} e^{ax+by} = \frac{1}{F(D_x,D_y)} e^{ax+by} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots *$$

Which it is equal to z , then the P. I. is

$$\boxed{z = \frac{1}{F(D_x,D_y)} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}} \quad \text{, where} \quad F(a,b) \neq 0$$

when $F(a,b) = 0$, then analyze $F(D_x,D_y)$ as follows

$$F(D_x,D_y) = (D_x - \frac{a}{b} D_y)^r G(D_x,D_y)$$

Where $G(a,b) \neq 0$, we get

$$z = \frac{1}{F(D_x,D_y)} e^{ax+by} = \frac{1}{(D_x - \frac{a}{b} D_y)^r G(D_x,D_y)} e^{ax+by}$$

$$= \frac{1}{(D_x - \frac{a}{b} D_y)^r} \cdot \frac{1}{G(a,b)} e^{ax+by} \quad \text{from } *$$

Since $G(a,b) \neq 0$

$$= \frac{1}{G(a,b)} \cdot \frac{1}{(D_x - \frac{a}{b} D_y)^r} e^{ax+by}$$

Then by Lagrange's method r times , we get

$$\boxed{z = \frac{1}{F(D_x,D_y)} e^{ax+by} = \frac{1}{G(a,b)}} \cdot \frac{x^r}{r!} e^{ax+by}$$

Which it's the P.I. where $F(a,b) = 0$, $G(a,b) \neq 0$

<u>Ex.2</u>: Solve $\left(D_x^2 - D_xD_y - 6D_y^2\right)z = e^{2x-3y}$

Sol.

1) To find the general solution

The A.E. of the given equation is

$$m^2 - m - 6 = 0 \rightarrow (m-3)(m+2) = 0$$

$$\therefore \ m_1 = 3 \ , \qquad m_2 = -2$$

$\therefore \ z_1 = \emptyset_1(y+3x) + \emptyset_2(y-2x)$

Where $\emptyset_1$and$\emptyset_2$are arbitrary functions

2) To find the particular Integral (P.I.)

$$a = 2 \ , b = -3$$

$$F(a,b) = a^2 - ab - 6b^2$$

$F(2,-3) = 4 + 6 - 54 = -44 \neq 0$

$$z_2 = \frac{1}{F(a,b)} e^{ax+by} = \frac{1}{-44} e^{2x-3y}$$

$$\therefore \ z = z_1 + z_2$$

$$= \emptyset_1(y+3x) + \emptyset_2(y-2x) - \frac{1}{44} e^{2x-3y}$$

<u>Ex.3</u>: Solve $\left(D_x^2 - D_xD_y - 6D_y^2\right)z = e^{3x+y}$

Sol.

1) The general solution is similar to that in Ex.2

2) To find P.I.

$$a = 3 \ , b = 1$$

$$F(a,b) = a^2 - ab - 6b^2$$

$F(3,1) = 9 - 3 - 6 = 0$,

analyze $F(D_x, D_y)$,$F(D_x, D_y) = D_x^2 - D_x D_y - 6D_y^2$

$$= (D_x - 3D_y)(D_x + 2D_y)$$

$(D_x - \frac{a}{b} D_y)^r \rightarrow \therefore r = 1$, $3 + 2 = 5 \neq 0 = G$

$$z_2 = \frac{1}{G(a,b)} \cdot \frac{x^r}{r!} e^{ax+by} = \frac{1}{5} \cdot \frac{x}{1} e^{3x+y} = \frac{x}{5} e^{3x+y}$$

$$\therefore \quad z = z_1 + z_2$$

$$= \emptyset_1(y + 3x) + \emptyset_2(y - 2x) + \frac{x}{5} e^{3x+y}$$

Where $\emptyset_1$and$\emptyset_2$are arbitrary functions

Case 2 **when $f(x,y) = \sin(ax + by)$ *or* $\cos(ax + by)$**

where a and b are arbitrary constant

Here, we will find the P.I. of (H.L.P.D.E.) of order 2 only, by the same way that in case 1 we will derive $f(x,y)$ for x and y .

Let $f(x,y) = \sin(ax + by)$

$$D_x \sin(ax + by) = a\cos(ax + by)$$

$$D_x^2 \sin(ax + by) = -a^2 \sin(ax + by)$$

$$D_y \sin(ax + by) = b\cos(ax + by)$$

$$D_y^2 \sin(ax + by) = -b^2 \sin(ax + by)$$

$$D_x D_y \sin(ax + by) = D_x[b\cos(ax + by)]$$

$$= -ab \sin(ax + by)$$

$$F\left(D_x^2, D_x D_y, D_y^2\right) \sin(ax + by) = F(-a^2, -ab, -b^2) \sin(ax + by)$$

Multiplying both sides by $\frac{1}{F\left(D_x^2, D_x D_y, D_y^2\right)}$

$$\sin(ax+by) = \frac{1}{F(\mathrm{D}_x^2, D_x D_y, \mathrm{D}_y^2)} F(-a^2, -ab, -b^2)\sin(ax+by)$$

If $\boxed{F(-a^2, -ab, -b^2) \neq 0}$ then we can divide on it

$$\to z = \frac{1}{F(\mathrm{D}_x^2, D_x D_y, \mathrm{D}_y^2)} \sin(ax+by)$$

$$= \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by)$$

Which it is the particular integral.

And if $F(-a^2, -ab, -b^2) = 0$ then we write

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad , \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

And follow the solution of the exponential function in case1.

<u>Ex.4</u>: Solve $\left(D_x^2 - D_x D_y - 6D_y^2\right) z = \sin(2x - 3y)$

Sol.

1) The general solution z_1 is the same in Ex.2
2) The P.I. z_2

$$a = 2 \ , b = -3$$

$$F(-a^2, -ab, -b^2) = -a^2 + ab + 6b^2$$

$$F(-4, 6, -9) = -4 - 6 + 54 = 44 \neq 0$$

$$z_2 = \frac{1}{44} \sin(2x - 3y)$$

The required general solution

$$\therefore \ z = z_1 + z_2$$

$$= \emptyset_1(y + 3x) + \emptyset_2(y - 2x) + \frac{1}{44} \sin(2x - 3y)$$

Where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

<u>Ex. 5</u>: Solve $\left(D_x^2 - 3D_xD_y + D_y^2\right)z = e^{2x+3y} + e^{x+y} + \sin(x - 2y)$

Sol.

1) Finding the general solution z_1

The A.E. is

$$m^2 - 3m + 2 = 0 \implies (m-2)(m-1) = 0$$

$$\therefore m_1 = 2\,, m_2 = 1$$

$\therefore z_1 = \emptyset_1(y + 2x) + \emptyset_2(y + x)$

where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

2) The P.I. of the given equation is

P.I. $= z_2 = \frac{1}{F(D_x,D_y)}\, e^{2x+3y} + \frac{1}{F(D_x,D_y)}\, e^{x+y} + \frac{1}{F(D_x,D_y)} \sin(x - 2y)$

Let $u_1 = \frac{1}{F(D_x,D_y)}\, e^{2x+3y}$ $\quad ,a = 2, b = 3$

$F\left(D_x, D_y\right) = a^2 - 3ab + 2b^2$

$F(1,1) = 4 - 18 + 18 = 4 \neq 0$

$\boxed{u_1 = \frac{1}{4} e^{2x+3y}}$

$u_2 = \frac{1}{F(D_x,D_y)}\, e^{x+y}$ $\quad ,a = 1, b = 1$

$F\left(D_x, D_y\right) = a^2 - 3ab + 2b^2$

$F(1,1) = 1 - 3 + 2 = 0$

Analyze $\quad F\left(D_x, D_y\right)$,

$F\left(D_x, D_y\right) = \left(D_x - 2D_y\right)(D_x - D_y)$

$u_2 = \frac{1}{G(a,b)} \frac{x^r}{r!}\, e^{ax+by}$

$$= \frac{1}{-1}\frac{x}{1}e^{x+y}$$

$\boxed{u_2 = -x\,e^{x+y}}$

$$u_3 = \frac{1}{F(D_x, D_y)}\sin(x-2y)$$

$F(-a^2, -ab, -b^2) = -a^2 + 3ab - 2b^2$

$F(-1,2,-4) = -1 - 6 - 8 = -15 \neq 0$

$\boxed{u_3 = \frac{1}{-15}\sin(x-2y)}$

Then, the required general solution is

$$z = z_1 + z_2 = \emptyset_1(y+2x) + \emptyset_2(y+x) + \frac{1}{4}e^{2x+3y} - xe^{x+y}$$

$$-\frac{1}{15}\sin(x-2y)$$

where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

Ex. 6: Find the P.I. of the equation

$$\left(D_x^2 - 4D_xD_y + 3D_y^2\right)z = \cos(x+y)$$

Sol. $a = 1, b = 1$

$F(-a^2, -ab, -b^2) = -a^2 + 4ab - 3b^2$

$F(-1,-1,-1) = -1 + 4 - 3 = 0$

Taking $\cos(x+y) = \frac{e^{ix+iy}+e^{-ix-iy}}{2}$

$$z = \frac{1}{2}\left[\frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{ix+iy} + \frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{-ix-iy}\right]$$

Let $u_1 = \frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{ix+iy}$

To find u_1 $a = i, b = i$

$F(a,b) = a^2 - 4ab + 3b^2$

$F(i,i) = i^2 - 4i^2 + 3i^2 = 0$

Analyze $F(D_x, D_y)$,

$F(D_x, D_y) = (D_x - D_y)(D_x - 3D_y)$

$u_1 = \frac{1}{-2i} x e^{ix+iy}$

By the same way $u_2 = \frac{1}{2i} x e^{-ix-iy}$

$$\therefore z = \frac{1}{2}\left[\frac{1}{-2i} x e^{ix+iy} + \frac{1}{2i} x e^{-ix-iy}\right]$$

$= \frac{-x}{2}\left[\frac{e^{ix+iy} - e^{-ix-iy}}{2i}\right] = \frac{-x}{2} \sin(x+y)$ which is the P.I.

Case 3 When $f(x,y) = x^a y^b$ where a and b are Non- Negative Integer Number

The particular integral (P.I.) is evaluated by expanding the function $\frac{1}{F(D_x,D_y)}$ in an infinite series of ascending powers of D_x or D_y (i.e.) by transfer the function $\frac{1}{F(D_x,D_y)}$ according to the following

$$\frac{1}{1-\theta} = 1 + \theta + \theta^2 + \cdots$$

<u>Ex.7</u>: Find P.I. of the equation $(D_x^2 - 2D_x D_y) z = x^3 y$

Sol.P.I. $= \frac{1}{D_x^2 - 2D_x D_y} x^3 y$

$= \frac{1}{D_x^2(1-2\frac{D_y}{D_x})} x^3 y D_y^n y^m = 0 \; if \; n > m$

$= \frac{1}{D_x^2}\left[1 + 2\frac{D_y}{D_x} + \frac{4D_y^2}{D_x^2} + \cdots\right] x^3 y \qquad , \; \frac{4D_y^2}{D_x^2} = 0$

$$= \frac{1}{D_x^2}\left[x^3 y + \frac{1}{2}x^4\right]$$

$$= \frac{1}{D_x}\left[\frac{x^4 y}{4} + \frac{x^5}{10}\right] = \frac{x^5 y}{20} + \frac{x^6}{60}$$

<u>Ex.8</u>: Find P.I. of the equation $(D_x^3 - 7D_x D_y^2 - 6D_y^3)z = x^2 y$

Sol. P.I. $= \frac{1}{D_x^3 - 7D_x D_y^2 - 6D_y^3} x^2 y$

$= \frac{1}{D_x^3\left[1 - \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)\right]} x^2 y$

$$= \frac{1}{D_x^3}\left[1 + \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right) + \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)^2 + \cdots\right] x^2 y$$

$= \frac{1}{D_x^3}[x^2 y]$ since $\left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right) = 0 \, , \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)^2 = 0$

$$= \frac{1}{D_x^2}\frac{x^3 y}{3} = \frac{1}{D_x}\frac{x^4 y}{12} = \frac{x^5 y}{60}$$

<u>Ex.9</u>: Solve $(D_x^3 - a^2 D_x D_y^2)z = x$, where $a \in R$

Sol.

1) the general solution z_1

The A.E. of the given equation is

$m^3 - a^2 m = 0 \;\; \Rightarrow \;\; m(m^2 - a^2) = 0$

$$\Rightarrow m(m-a)(m+a) = 0$$

$\therefore m_1 = 0\,, m_2 = a \;\;, m_3 = -a$ (different roots) $\therefore z_1 = \emptyset_1(y) + \emptyset_2(y + ax) + \emptyset_3(y - ax)$

where $\emptyset_1, \emptyset_2$ and $\emptyset_3$ are arbitrary functions.

2) The P.I. of the given equation is

P.I. $= z_2 = \frac{1}{D_x^3 - a^2 D_x D_y^2}\, x$

$= \frac{1}{D_x^3\left[1 - \frac{a^2 D_y^2}{D_x^2}\right]}\, x$

$= \frac{1}{D_x^3}\left[1 + \frac{a^2 D_y^2}{D_x^2} + \left(\frac{a^2 D_y^2}{D_x^2}\right)^2 + \cdots\right] x \left(\frac{a^2 D_y^2}{D_x^2} = 0\,, \left(\frac{a^2 D_y^2}{D_x^2}\right)^2 = 0\right)$

$$= \frac{1}{D_x^3}[x]$$

$$= \frac{1}{D_x^2}\left[\frac{x^2}{2}\right]$$

$$= \frac{1}{D_x}\left[\frac{x^3}{6}\right] = \frac{x^4}{24}$$

then, the required general solution is

$z = z_1 + z_2 = \emptyset_1(y) + \emptyset_2(y + ax) + \emptyset_3(y - ax) + \frac{x^4}{24}$

Case 4 When $f(x, y) = e^{ax+by} V$ where V is a function of x and y

The P.I. in this case is $z = \frac{1}{F(D_x, D_y)} e^{ax+by}\, V$

$$= e^{ax+by} \frac{1}{F(D_x + a\,,\; D_y + b)}\, V$$

and solving this equation depending on the type of V can get the particular integral (P.I.), as follows:

<u>Ex.10</u>: Find P.I. of the equation $D_xD_yz = e^{2x+3y}x^2y$

Sol. P.I. $=\frac{1}{D_xD_y}e^{2x+3y}x^2y a = 2, b = 3 \; and \; V = x^2y$

$=e^{2x+3y}\frac{1}{(D_x+2)(D_y+3)}x^2y$

$$= e^{2x+3y}\frac{1}{3(D_x+2)(1+\frac{D_y}{3})}x^2y$$

$$= e^{2x+3y}\frac{1}{3(D_x+2)}\left[1-\frac{D_y}{3}+\frac{D_y^2}{9}-\cdots\right]x^2y$$

$$= e^{2x+3y}\frac{1}{3(D_x+2)}\left[x^2y-\frac{x^2}{3}\right]$$

$$= e^{2x+3y}\frac{1}{6(1+\frac{D_x}{2})}\left[x^2y-\frac{x^2}{3}\right]$$

$=\frac{1}{6}e^{2x+3y}\left[1-\frac{D_x}{2}+\frac{D_x^2}{4}-\frac{D_x^3}{8}+\cdots\right]\left[x^2y-\frac{x^2}{3}\right], (\frac{D_x^3}{8}=0)$

$$= \frac{1}{6}e^{2x+3y}\left[x^2y-\frac{x^2}{3}-xy+\frac{x}{3}+\frac{y}{2}-\frac{1}{6}\right]$$

$$= e^{2x+3y}\left[\frac{1}{6}x^2y-\frac{x^2}{18}-\frac{1}{6}xy+\frac{x}{18}+\frac{y}{12}-\frac{1}{36}\right]$$

<u>Ex.11</u>: Find P.I. of the equation $\left(D_x^2-D_xD_y\right)z = e^{x+y}\,xy^2$

Sol.

P.I. $=\frac{1}{D_x^2-D_xD_y}e^{x+y}\,xy^2 a = 1, b = 1 \; and \; V = xy^2$

$=e^{x+y}\frac{1}{(D_x+1)(D_x-D_y)}xy^2$since $D_x^2-D_xD_y = D_x(D_x-D_y)$

$$= e^{x+y} \frac{1}{(D_x+1)D_x(1-\frac{D_y}{D_x})} xy^2$$

$$= e^{x+y} \frac{1}{(D_x+1)D_x}\left[1+\frac{D_y}{D_x}+\frac{D_y^2}{D_x^2}+\cdots\right]xy^2$$

$$= e^{x+y} \frac{1}{(D_x+1)D_x}\left[xy^2+\frac{2xy}{D_x}+\frac{2x}{D_x^2}\right]$$

$$= e^{x+y} \frac{1}{(D_x+1)D_x}\left[xy^2+x^2y+\frac{x^3}{3}\right]$$

$$= e^{x+y} \frac{1}{(D_x+1)}\left[\frac{x^2y^2}{2}+\frac{x^3y}{3}+\frac{x^4}{12}\right]$$

$$= e^{x+y}[1-D_x+D_x^2-D_x^3+D_x^4-D_x^5+\cdots]\left[\frac{x^2y^2}{2}+\frac{x^3y}{3}+\frac{x^4}{12}\right]$$

where $D_x^5 = 0$

$$= e^{x+y}\left[\frac{x^2y^2}{2}+\frac{x^3y}{3}+\frac{x^4}{12}-xy^2-x^2y-\frac{x^3}{3}+y^2+2xy+x^2-2y-2x+2\right]$$

<u>Ex.12</u>: Find P.I. of the equation $\left(D_x-D_y\right)^2 z = e^{x+y}\sin(x+2y)$

Sol. P.I. $= \frac{1}{\left(D_x-D_y\right)^2} e^{x+y}\sin(x+2y)$, $a_1 = 1, b_1 = 1$

$=e^{x+y} \frac{1}{\left(D_x+1-D_y-\cancel{1}\right)^2}\sin(x+2y)$

$$= e^{x+y} \frac{1}{\left(D_x-D_y\right)^2}\sin(x+2y)$$

$= e^{x+y} \frac{1}{D_x^2-2D_xD_y+D_y^2}\sin(x+2y)$, $a_2 = 1, b_2 = 2$

$$F(-a_2^2\,,-a_2b_2\,,-b_2^2) = -a_2^2+2a_2b_2-b_2^2$$

$$F(-1\,,-2\,,-4) = -1+4-4 = -1 \neq 0$$

$$\therefore z = e^{x+y}.\frac{1}{-1}\sin(x+y) \quad \Longrightarrow \quad z = -e^{x+y}\sin(x+y)$$

Case 5 **When $\boldsymbol{f(x,y) = g(ax+by)}$ where $\boldsymbol{F(a,b) \neq 0}$**

The particular integral of H.L.P.D.E. of order n is

$$z = \frac{1}{F(a,b)}\underbrace{\int \cdots \int}_{n-times} g(ax+by)\,\underbrace{d(ax+by)\ldots d(ax+by)}_{n-times}$$

<u>Ex.13</u>: Find P.I. of $\boldsymbol{(D_x^2 + 2D_xD_y - 8D_y^2)z = \sqrt{2x+3y}}$

Sol.

$a = 2, b = 3 \quad , g(2x+3y) = \sqrt{2x+3y}$

$F(a,b) = a^2 + 2ab - 8b^2$

$F(2,3) = 4 + 12 - 72 = -56 \neq 0$, integrating g twice

$$\therefore \text{P.I.} = z = \frac{1}{-56}\iint \sqrt{2x+3y}\, d(2x+3y)d(2x+3y)$$

$$= \frac{1}{-56}\int \frac{2}{3}(2x+3y)^{3/2} d(2x+3y)$$

$$= \frac{4}{-56\,(15)}(2x+3y)^{5/2}$$

$$= \frac{-1}{210}(2x+3y)^{5/2}$$

Case 6 **When $\boldsymbol{f(x,y) = g(ax+by)}$ where $\boldsymbol{F(a,b) = 0}$**

If $F(a,b) = 0$, then $F(D_x, D_y)$ can be written as

$$F(D_x, D_y) = (bD_x - aD_y)^n$$

and the particular solution is $\boxed{Z = \frac{x^n}{n!}\frac{g(ax+by)}{b^n}}$

<u>Ex.14:</u> Find P.I. of $\left(\boldsymbol{D_x^2 - 6D_xD_y + 9D_y^2}\right)\boldsymbol{z = 3x + y}$

Sol. $a = 3, b = 1 \quad , g(3x + y) = 3x + y$

$F(a, b) = a^2 - 6ab + 9b^2$

$$F(3,1) = 9 - 18 + 9 = 0$$

Then $F(D_x, D_x) = D_x^2 - 6D_xD_y + 9D_y^2 = (D_x - 3D_y)^2$, so $n = 2$

$\therefore$ P.I. $= z = \frac{x^2}{2!}\frac{3x+y}{1^2} = \frac{1}{2}x^2(3x + y)$

<u>Ex.15</u>: Find P.I. of $\left(\boldsymbol{D_x^2 - 4D_xD_y + 4D_y^2}\right)\boldsymbol{z} = \mathbf{tan}(\boldsymbol{2x + y})$

Sol. $a = 2, b = 1 \quad , g(2x + y) = \tan(2x + y)$

$F(a, b) = a^2 - 4ab + 4b^2$

$$F(2,1) = 4 - 8 + 4 = 0$$

Then $F\left(D_x, D_y\right) = D_x^2 - 4D_xD_y + 4D_y^2 = (D_x - 2D_y)^2$, so $n = 2$

$\therefore$ P.I. $= z = \frac{x^2}{2!}\frac{\tan(2x+y)}{1^2} = \frac{1}{2}x^2 tan(2x + y)$

<u>Ex.16</u>: Find P.I. of $\left(\boldsymbol{D_x^2 - D_y^2}\right)\boldsymbol{z} = \mathbf{sec}^2(\boldsymbol{x + y})$

Sol. $a = 1, b = 1 \quad , g(x + y) = \sec^2(x + y)$

$$F(a, b) = a^2 - b^2$$

$$F(1,1) = 1 - 1 = 0$$

Then $F\left(D_x, D_y\right) = D_x^2 - D_y^2 = \left(D_x - D_y\right)(D_x + D_y)$

$$\therefore z = \frac{1}{\left(D_x - D_y\right)(D_x + D_y)}\sec^2(x + y)$$

Let $u_1 = \frac{1}{(D_x + D_y)}\sec^2(x + y)$ by case (5) we have

$u_1 = \frac{1}{F(a,b)} \int g(ax+by)d(ax+by)$, $F(1,1) = 1+1 = 2$

$$= \frac{1}{2} \int \sec^2(x+y)d(x+y)$$

$$= \frac{1}{2} \tan(x+y)$$

$$\Rightarrow z = \frac{1}{(D_x - D_y)} \frac{1}{2} \tan(x+y)$$

$F(D_x, D_y) = D_x - D_y$

$F(1,1) = 1 - 1 = 0$ where $n = 1$

$$\therefore z = \frac{x^1}{1!} \frac{1}{2} \frac{\tan(x+y)}{1}$$

$= \frac{x}{2} \tan(x+y)$ which its' the particular integral

…General Exercises …

1- $(D_x^4 - D_y^4)z = 0$

2- $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \cos(x-y) + x^2 + xy^2 + y^2$

3- $(D_x - 2D_y)z = e^{3x}(y+1)$

4- $(D_x^2 + 3D_xD_y + 2D_y^2)z = x + y$

5- $(D_x^2 - 5D_xD_y + 4D_y^2)z = \sin(4x+y)$

6- $(2D_x^2 - D_xD_y - 3D_y^2)z = \frac{5\, e^x}{e^y}$

7- $(D_x^2 - 3D_xD_y + 2D_y^2)z = e^{2x-y} + \cos(x+2y)$

8- $(D_x^2 - D_xD_y)z = \ln y$

9- $(D_x + D_y)z = \sec(x+y)$

10- $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

11- $(y^2 + z^2 - x^2)p - 2xyq = -2xz$

12- $pq + 2y(x+1)q + x(x+2)q - 2(x+1) = 0$

13- $(x^2 + 2x)p + (x+1)qy = 0$

14- $\left(D_x^3 - 3D_xD_y^2 + 2D_y^3\right)z = \frac{1}{\sqrt{3x-y}}$

15- $\left(D_x^3 + 2D_x^2D_y - D_xD_y^2 - 2D_y^3\right)z = (y+2)e^x$

16- $\left(4D_x^2 - 4D_xD_y + D_y^2\right)z = (x+2y)^{3/2}$

17- $D_xD_yz = e^{x-y}xy^2$

18- $\left(D_x - D_y\right)z = \tan(x+2y)$

19- $2\left(D_x^3 - 9D_x^2D_y + 27D_xD_y^2 - 27D_y^3\right)z = \tan^{-1}(3x+y)$

20- $(y^3x - 2x^4)\frac{\partial z}{\partial x} + (2y^4 - x^3y)\frac{\partial z}{\partial y} = x^3 - y^3$

Chapter Two

Non-homogeneous Linear Partial Differential Equations

Contents

Section(2.1):Non-homogeneous linear partial differential equations with constant coefficients

Definition: A linear partial differential equation with constant coefficients is known as non-homogeneous l.p.d.e. with constant coefficients if the order of all the partial derivatives involved in the equation are not all equal.

For example:

1) $\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial y}+z=x+y$

2) $\frac{\partial^3 z}{\partial x^3}+\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial y}=e^{x+y}$

Definition: A linear differential operator $F(D_x, D_y)$ is known as (reducible), if it can be written as the product of linear factors of the form aD_x+bD_y+c with a, b and c as constants. $F(D_x, D_y)$ is known as (irreducible), if it is not reducible.

For example:

The operator $D_x^2-D_y^2$ which can be written in the form $(D_x-D_y)(D_x+D_y)$ isreducible, whereas the operator $D_x^2-D_y^3$ which cannot be decomposed into linear factors is irreducible.

Note: A l.p.d.e with constant coefficient $F\big(D_x, D_y\big)z=f(x, y)$ is known as reducible, if $F(D_x, D_y)$ reducible, and is known as irreducible, if $F(D_x, D_y)$ is irreducible.

(2.1.1)Determination of Complementary function (C.F.)(the general solution) of a reducible non-homo.l.p.d.e. with constant coefficients

(A) let $F(D_x, D_y) = (aD_x + bD_y + c)^k$,where a, b, c are constants and k is anatural number

then the equation $F(D_x, D_y)z = 0$ will by

$(aD_x + bD_y + c)^k z = 0$ and the solution is

$$z = e^{\frac{-c}{a}x}\emptyset(ay - bx) \qquad ; \quad a \neq 0 \;\; , k = 1$$

Or

$$z = e^{\frac{-c}{b}y}\emptyset(ay - bx) \qquad ; \quad b \neq 0 \;\; , k = 1$$

For any $k > 1$, the solution is

$$z = e^{\frac{-c}{b}y}[\emptyset_1(ay - bx) + x\emptyset_2(ay - bx) + \cdots + x^{k-1}\emptyset_k(ay - bx); \; b \neq 0$$

Or

$$z = e^{\frac{-c}{a}x}[\emptyset_1(ay - bx) + x\emptyset_2(ay - bx) + \cdots + x^{k-1}\emptyset_k(ay - bx); \; a \neq 0$$

Where $\emptyset_1, \dots, \emptyset_n$ are arbitrary functions.

<u>Ex.1</u>: Solve $(2D_x - 3D_y - 5)z = 0$

Sol. The given equation is linear in $F(D_x, D_y)$

Then $a = 2, b = -3, c = -5, k = 1$

The general solution is

$$z = e^{\frac{5}{2}x}\emptyset(2y + 3x)$$

Where $\emptyset$ is an arbitrary function.

Ex.2: Solve $(D_x - 5)z = e^{x+y}$

Sol. To find the general solution of $(D_x - 5)z = 0$

We have $a = 1 , b = 0 , c = -5 , k = 1$

$\therefore z_1 = e^{5x}\emptyset(y)$, Where $\emptyset$ is an arbitrary function.

To find the P.I. z_2 , we have $a = 1 , b = 1$

$$F(a,b) = a - 5 \;\; \rightarrow \;\; F(1,1) = 1 - 5 = -4 \neq 0$$

$$\therefore z_2 = \frac{1}{-4} e^{x+y}$$

Then the required general solution of the given equation is

$$z = z_1 + z_2 \;\; \rightarrow \;\; z = e^{5x}\emptyset(y) - \frac{1}{4} e^{x+y}$$

Ex.3: Solve $(2D_y + 5)^2 z = 0$

Sol. The given equation is reducible, then

$$a = 0 , b = 2 , c = 5 , k = 2 .$$

The general solution is

$$z = e^{\frac{-5}{2}y}[\emptyset_1(-2x) + x\emptyset_2(-2x)]$$

Where $\emptyset_1$ *and* $\emptyset_2$ are arbitrary functions

Ex.4: Solve $(D_x - 2D_y + 1)^4 z = 0$

Sol. We have $a = 1, b = -2, c = 1, k = 4$

then

$$z = e^{\frac{1}{2}y}[\emptyset_1(y + 2x) + x\emptyset_2(y + 2x) + x^2\emptyset_3(y + 2x) + x^3\emptyset_4(y + 2x)]$$

Where $\emptyset_1 , \dots , \emptyset_4$ are arbitrary functions

(B) when $F(D_x, D_y)$ can be written as the product of linear factors of the form $(aD_x + bD_y + c)$, i.e. $F(D_x, D_y)$ is reducible , then the general solution is the sum of the solutions corresponding to each factor.

Ex.5: solve $\underbrace{\left(\boldsymbol{2D_x - 3D_y + 1}\right)}_{\text{linear}}\underbrace{\left(\boldsymbol{D_x + 2D_y - 2}\right)}_{\text{linear}}\boldsymbol{z = 0}$

Sol. The given equation is reducible, then we have

$$a_1 = 2 \; , b_1 = -3 \quad , c_1 = 1 \quad , k_1 = 1$$

$$z_1 = e^{\frac{-1}{2}x} \emptyset_1(2y + 3x)$$

$$a_2 = 1 \; , b_2 = 2 \quad , c_2 = -2 \quad , k_2 = 1$$

$$z_2 = e^{2x} \emptyset_2(y - 2x)$$

The general solution is

$$\mathrm{z} = \mathrm{z}_1 + \mathrm{z}_2 \quad \rightarrow \quad \mathrm{z} = e^{\frac{-1}{2}x} \emptyset_1(2y + 3x) + e^{2x} \emptyset_2(y - 2x)$$

Where $\emptyset_1 , \emptyset_2$ are two arbitrary functions.

Ex.6: solve $\boldsymbol{D_x(D_x + D_y + 1)(D_x + 3D_y - 2)z = 0}$

Sol. We have

$$a_1 = 1 \; , b_1 = 0 \quad , c_1 = 0 \quad , k_1 = 1$$

$$a_2 = 1 \; , b_2 = 1 \quad , c_2 = 1 \quad , k_2 = 1$$

$$a_3 = 1 \; , b_3 = 3 \quad , c_3 = -2 \quad , k_3 = 1$$

Then the general solution is

$$\mathrm{z} = \emptyset_1(y) + e^{-x} \emptyset_2(y - x) + e^{2x} \emptyset_3(y - 3x)$$

Where $\emptyset_1, \dots, \emptyset_3$ are arbitrary functions.

<u>Ex.7</u>: solve $\left(D_x^3 - D_x D_y^2 - D_x^2 + D_x D_y\right)z = 0$

Sol. We have , $\left(D_x^3 - D_x D_y^2 - D_x^2 + D_x D_y\right)z = 0$

$$D_x\left(D_x^2 - D_y^2 - D_x + D_y\right)z = 0$$

$$D_x\left[\left(D_x - D_y\right)\left(D_x + D_y\right) - \left(D_x - D_y\right)\right] = 0$$

$$D_x\left(D_x - D_y\right)\left(D_x + D_y - 1\right)z = 0$$

Then , $a_1 = 1\ , b_1 = 0\quad , c_1 = 0\quad , k_1 = 1$

$$a_2 = 1\ , b_2 = -1\quad , c_2 = 0\quad , k_2 = 1$$

$$a_3 = 1\ , b_3 = 1\quad , c_3 = -1\quad , k_3 = 1$$

Then the general solution is

$$\mathrm{z} = \emptyset_1(y) + \emptyset_2(y + x) + e^x \emptyset_3(y - x)$$

Where $\emptyset_1, \dots, \emptyset_3$ are arbitrary functions.

(C) When $F(D_x, D_y)$ is irreducible then the general solution is

$$z = \sum_{i=1}^{\infty} A_i\, e^{a_i x + b_i y}$$

Where $F(a_i, b_i) = 0$,A_i , a_i, b_i are all constants.

<u>Ex.8</u>: Solve $\left(D_x - D_y^3\right)z = 0$

Sol. The given equation is irreducible, then

$$F(a, b) = 0 \quad \rightarrow \quad F(a_i, b_i) = 0$$

$$a - b^3 = 0 \quad \rightarrow \quad a_i - {b_i}^3 = 0 \quad \rightarrow \quad a_i = {b_i}^3$$

The general solution is

$$z = \sum_{i=1}^{\infty} A_i\, e^{a_i x + b_i y} = \sum_{i=1}^{\infty} A_i\, e^{{b_i}^3 x + b_i y}$$

Where A_i , $b_i$are constants.

<u>Ex.9</u>: Solve $\left(\boldsymbol{D_x^2+D_x+D_y}\right)\boldsymbol{z}=\mathbf{0}$

Sol. The given equation is irreducible, then

$$F(a,b)=a^2+a+b=0 \quad \rightarrow\; {a_i}^2+\;a_i+b_i=0$$

$$\rightarrow \quad b_i=-{a_i}^2-\;a_i$$

The general solution is

$$z=\sum_{i=1}^{\infty} A_i\, e^{a_ix+b_iy}=\sum_{i=1}^{\infty} A_i\, e^{a_ix+(-{a_i}^2-\,a_i)y}$$

Where A_i , $a_i$are constants.

<u>Ex.10</u>: Solve $\left(\boldsymbol{D_x-D_y^2}\right)\boldsymbol{z}=\boldsymbol{e^{2x+3y}}$

Sol. (1) we find the general solution of the irreducible eqution

$$\left(\mathrm{D_x-D_y^2}\right)\mathrm{z}=0$$

$$F(a,b)=a-b^2=0 \;\rightarrow\; F(a_i,b_i)=\;a_i-{b_i}^2=0\rightarrow\; a_i$$
$$={b_i}^2$$

Then

$$z_1=\sum_{i=1}^{\infty} A_i\, e^{a_ix+b_iy}=\sum_{i=1}^{\infty} A_i\, e^{{b_i}^2x+b_iy}$$

Where A_i , $b_i$are constants.

(2) The P.I. is

$$F(a,b)=a-b^2$$

$$\therefore F(2,3)=2-9=-7\neq 0$$

$$\therefore z_2=\frac{1}{-7}e^{2x+3y}$$

And the required general solution is

$$z = z_1 + z_2 = \sum_{i=1}^{\infty} A_i\, e^{{b_i}^2 x + b_i y} - \frac{1}{7} e^{2x+3y}$$

(D) When $F(D_x, D_y)$ can be written as the product of linear and non-linear factors the general solution is the sum of the solutions corresponding to each factor.

<u>Ex.11</u>: Solve $\left(D_x + 2D_y\right)\left(D_x - 2D_y + 1\right)\left(D_x - D_y^2\right)z = 0$

Sol:

Factor 1, $a_1 = 1\ , b_1 = 2\quad , c_1 = 0\quad , k_1 = 1$

Factor 2, $a_2 = 1\ , b_2 = -2\quad , c_2 = 1\quad , k_2 = 1$

Factor 3, $F(a, b) = a - b^2 = 0 \quad \to a = b^2 \quad \to \quad a_i = b_i^2$

$$\therefore \mathrm{z} = \emptyset_1(y - 2x) + \mathrm{e}^{\frac{1}{2}\mathrm{y}} \emptyset_2(y + 2x) + \sum_{i=1}^{\infty} A_i\, e^{{b_i}^2 x + b_i y}$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions and A_i , $b_i$are constants.

<u>Ex.12</u>: Solve $\left(D_x^2 - D_y^2 + D_x\right)z = x^2 + 2y$

Sol: (1) The general solution of $\left(\mathrm{D}_\mathrm{x}^2 - \mathrm{D}_\mathrm{y}^2 + \mathrm{D}_\mathrm{x}\right)\mathrm{z} = 0$ is

$F(a, b) = a^2 - b^2 + a = 0 \ \to b = \pm\sqrt{a^2 + a} \to b_i = \pm\sqrt{a_i^2 + a_i}$

Then

$$z_1 = \sum_{i=1}^{\infty} A_i\, e^{a_i x \pm \sqrt{a_i^2 + a_i}\, y}$$

(2) The P.I. is

$$z_1 = \frac{1}{D_x^2 - D_y^2 + D_x}(x^2 + 2y)$$

$$= \frac{1}{D_x(1 + D_x - \frac{D_y^2}{D_x})}(x^2 + 2y)$$

$$= \frac{1}{D_x[1 - \left(\frac{D_y^2}{D_x} - D_x\right)]}(x^2 + 2y)$$

$$= \frac{1}{D_x}[1 + \underbrace{\frac{D_y^2}{D_x}}_{=0} - D_x + \left(\frac{D_y^2}{D_x} - D_x\right)^2 + \cdots](x^2 + 2y)$$

$$= \frac{1}{D_x}[x^2 + 2y - 2x + 2] = \frac{x^3}{3} + 2xy - x^2 + 2x$$

The required general solution is

$$z = z_1 + z_2 = \sum_{i=1}^{\infty} A_i\, e^{a_i x \pm \sqrt{a_i^2 + a_i y}} + \frac{x^3}{3} + 2xy - x^2 + 2x$$

<u>Ex.13:</u>Solve $\boldsymbol{(2D_x + 3D_y)(3D_x - 4D_y + 5)(3D_x - D_y^2)z = 0}$

Sol:

Factor 1, $a_1 = 2$, $b_1 = 3$, $c_1 = 0$, $k_1 = 1$

Factor 2, $a_2 = 3$, $b_2 = -4$, $c_2 = 5$, $k_2 = 1$

Factor 3, $F(a, b) = 3a - b^2 = 0 \quad \rightarrow \; a = \frac{b^2}{3} \; \rightarrow \; a_i = \frac{b_i^2}{3}$

The general solution is

$$\therefore z = \emptyset_1(2y - 3x) + e^{\frac{5}{4}y}\emptyset_2(3y + 4x) + \sum_{i=1}^{\infty} A_i\, e^{\frac{b_i^2}{3}x + b_i y}$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions and A_i , $b_i$are constants.

NoteTo determine the P.I. of non-homo.p.d.e. when

$f(x, y) = \sin(ax + by)$ $\quad or \quad$ $\cos(ax + by)$ we put $D^2 = -a^2$, $D_y^2 = -b^2$, $D_x D_y = -ab$, which provided the denominator is non-zero, as follows.

Ex.14: Solve $\left(D_x^2 - D_y\right)z = \sin(x - 2y)$

Sol: (1)The general solution z_1 of $\left(D_x^2 - D_y\right)z = 0$ is

$$F(a, b) = a^2 - b = 0 \quad \rightarrow \; a_i^2 = b_i$$

$$z_1 = \sum_{i=1}^{\infty} A_i \, e^{a_i x + a_i^2 y}$$

(2) To find the P.I. of the given equation

$$P.I. = z = \frac{1}{D_x^2 - D_y} \sin(x - 2y)$$

$$a = 1, \qquad b = -2 \quad \rightarrow D_x^2 = -a^2 = -1$$

$$= \frac{1}{-1 - D_y} \sin(x - 2y)$$

Multiplying by $\frac{1}{-1 + D_y}$

$$= \frac{-1 + D_y}{1 - D_y^2} \sin(x - 2y)$$

$$D_y^2 = -b^2 = -4$$

$$= \frac{-1 + D_y}{1 + 4} \sin(x - 2y)$$

$$= \frac{1}{5}[-\sin(x-2y) - 2\cos(x-2y)]$$

…Exercises…

Solve the following equations:

1. $\left(D_x^2 + D_x D_y + D_y - 1\right)z = 0$
2. $(D_x + 1)\left(D_x - D_y + 1\right)z = 0$
3. $(D_x^2 + D_x D_y + D_x)z = 0$
4. $\left(D_x^2 + D_y + 4\right)z = e^{4x-y}$
5. $\left(D_x^2 + D_x D_y + D_y - 1\right)z = \sin(x+2y)$
6. $\left(D_x - D_y - 1\right)\left(D_x - D_y - 2\right)z = x$
7. $\left(D_x^2 - D_y^2 + D_x + 3D_y - 2\right)z = x^2 y$
8. $(D_x + 3D_y - 2)^2 z = 2e^{2x}\sin(y+3x)$

Section(2.2): Partial differential equations of order two with variable coefficients

In the present section, we propose to discuss partial differential equations of order two with variable coefficients. An equation is said to be of order two, if it involves at least one of the differential coefficients $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$, but none of higher order, the quantities p and q may also inter into the equation. Thus the general from of a second order partial differential equation is

$$\boldsymbol{R(x,y)\frac{\partial^2 z}{\partial x^2} + S(x,y)\frac{\partial^2 z}{\partial x \partial y} + T(x,y)\frac{\partial^2 z}{\partial y^2} + P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} + V(x,y)z = f(x,y)}$$

...(1)

Or $\quad Rr + Ss + Tt + Pp + Qq + Vz = f$ ...(2)

Where R, S, T, P, Q, V, f are functions of x and y only and not all R, S, T are zero.

We will discuss three cases of the equation (2):

Case 1when one of R, S, T not equal to zero and P, Q, V are equal to zero ,then the solution can be obtained by integrating both sides of the equation directly.

<u>Ex.15:</u>Solve $\boldsymbol{y\frac{\partial^2 z}{\partial x^2} + 5y - x^2 y^2 = 0}$

Sol: Given equation can be written

$\frac{\partial^2 z}{\partial x^2} = yx^2 - 5$...(3)

Integrating (3) w.r.t. x

$\frac{\partial z}{\partial x} = \frac{yx^3}{3} - 5x + \emptyset_1(y)$...(4)

Integrating (4) w.r.t. x

$$z = \frac{yx^4}{12} - \frac{5}{2}x^2 + x\emptyset_1(y) + \emptyset_2(y)$$

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

<u>Ex.16:</u> Solve $\boldsymbol{xy\frac{\partial^2 z}{\partial x \partial y} - y^2 x = 0}$

Sol: Given equation can be written

$\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}\, \partial \mathrm{y}} = y$ ...(5)

Integrating (5) w.r.t. x

$\frac{\partial z}{\partial y} = xy + \emptyset_1(y)$...(6)

Integrating (6) w.r.t. y

$$z = \frac{xy^2}{2} + \int \emptyset_1(y)\partial y + \emptyset_2(x)$$

$$= \frac{xy^2}{2} + \varphi(y) + \emptyset_2(x)$$

Where $\varphi$and $\emptyset_2$ are two arbitrary functions.

Case2 When all the derivatives in the equation for one independent variable i.e the equation is of the form

$\boldsymbol{Rr + Pp + Vz = f(x, y)}$ **or** $\boldsymbol{Tt + Qq + Vz = f(x, y)}$

Some of these coefficients may be Zeros.

These equations will be treated as a ordinary linear differential equations, a follows:

<u>Ex.17</u>: Solve $\boldsymbol{y\frac{\partial^2 z}{\partial y^2} + 3\frac{\partial z}{\partial y} = 2x + 3}$

Sol: let $\frac{\partial z}{\partial y} = q \;\; \rightarrow \;\; \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}$

Substituting in the given equation, we get

$y\frac{\partial q}{\partial y} + 3q = 2x + 3 \;\;\; \rightarrow \;\; \frac{\partial q}{\partial y} + \frac{3}{y}q = \frac{2x+3}{y}$ ...(7)

Which it's linear diff. eq. in variables q and y , regarding x as a constant.

Integrating factor (I.F.)of (7) $= e^{\int \frac{3}{y}\partial y} = e^{3lny} = y^3$

And solution of (7) is

$$y^3 q = \int \frac{2x+3}{y} y^3 \partial y + \emptyset_1(x)$$

$$y^3 q = (2x+3)\frac{y^3}{3} + \emptyset_1(x)$$

$$q = \frac{2x+3}{3} + y^{-3}\emptyset_1(x)$$

$\frac{\partial z}{\partial y} = \frac{2x+3}{3} + y^{-3}\emptyset_1(x)$, integrating w.r.t. y

$$z = \frac{2x+3}{3}y - \frac{1}{2y^2}\emptyset_1(x) + \emptyset_2(x)$$

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

<u>Ex.18:</u> Solve $\frac{\partial^2 z}{\partial x^2} - 2y\frac{\partial z}{\partial x} + y^2 z = (y-3)e^{2x+3y}$

Sol: The given equation can be written as

$$D_x^2 - 2yD_x + y^2 z = (y-3)e^{2x+3y}$$

$\rightarrow (D_x - y)^2 z = (y-3)e^{2x+3y}$...(8)

The A.E. of the equation $(D_x - y)^2 z = 0$ is

$$(m-y)^2 = 0 \rightarrow m_1 = m_2 = y$$

$\therefore z_1 = \emptyset_1(y)e^{yx} + x\emptyset_2(y)e^{yx}$...(9)

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

The P.I. (z_2) is

$$z_2 = \frac{1}{(D_x - y)^2}(y-3)e^{2x+3y} = (y-3)\frac{1}{(2-y)^2}e^{2x+3y}$$

$$\therefore z = z_1 + z_2$$

$$= \emptyset_1(y)e^{yx} + x\emptyset_2(y)e^{yx} + (y-3)\frac{1}{(2-y)^2}e^{2x+3y}$$

Case3 under this type, we consider equations of the form

$$\boxed{\boldsymbol{Rr + Ss + Pp = f(x,y) \rightarrow R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x \partial y} + P\frac{\partial z}{\partial x} = f(x,y)}}$$

And $\boxed{\boldsymbol{Ss + Tt + Qq = f(x,y) \rightarrow S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} + Q\frac{\partial z}{\partial y} = f(x,y)}}$

These can be transform to a linear, p.d.es of order one with p or q as dependent variable and x, y as independent variables. In such situations we shall apply well known Lagrange's method.

Ex.19: Solve $\boldsymbol{x\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} = 0}$

Sol: let $p = \frac{\partial z}{\partial x} \to \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}, \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$

Substituting in the given equation , we get

$x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y} - p = 0$...(10)

Which it is in Lagrange's form, the Lagrange's auxiliary equations are : $\frac{dx}{x} = \frac{dy}{-y} = \frac{dp}{p}$ ...(11)

Taking the first and second fractions of (11)

: $\frac{dx}{x} = \frac{dy}{-y} \to lnx = -lny + lna \to \boxed{xy = a}$...(12)

Taking the first and the third fractions of (11)

$\frac{dx}{x} = \frac{dp}{p} \to lnx = lnp + lnb \to \boxed{\frac{x}{p} = b}$...(13)

From (12) &(13) , the general solution is

$$\emptyset(a, b) = 0 \to \emptyset\left(xy, \frac{x}{p}\right) = 0 \to \frac{x}{p} = g(xy)$$

$$\to p = \frac{x}{g(xy)}$$

$\to \frac{\partial z}{\partial x} = \frac{x}{g(xy)}$...(14)

Integrating (14) w.r.t. x , we get

$z = \int \frac{x}{g(xy)} \partial x + \varphi(y)$...(15)

Where g and φ are two arbitrary functions.

Then (15) is the required solution of the given equation.

…Exercises…

Solve the following equations:

1)) $\ln\left(\frac{\partial^2 z}{\partial x \partial y}\right) = x + y$ 2)) $\frac{\partial^2 z}{\partial y^2} - x\frac{\partial z}{\partial y} = x^2$

3)) $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = \frac{x}{y}$ 4)) $y^2\frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} = 1$

Section2.3: Partial differential equations reducible to equations with constant coefficients

In this section, we propose to discuss the method of solving the partial differential equation, which is also called Euler-Cauchy type partial differential equations of the form :

$$a_0 x^n \frac{\partial^n z}{\partial x^n} + a_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + a_n y^n \frac{\partial^n z}{\partial y^n} + \cdots = f(x,y) \ldots (1)$$

i.e. all the terms of the equation of the formula $a_r x^n y^m \frac{\partial^{n+m} z}{\partial x^n \partial y^m}$

To solve this equation ,define two new variables u and v by

$x = e^u$ and $y = e^v$ so that $u = lnx$ and $v = lny$…(2)

Let $D_u = \frac{\partial}{\partial u}$and $D_v = \frac{\partial}{\partial v}$

Now, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{1}{x} \cdot \frac{\partial z}{\partial u}$, using (2)

$\therefore \frac{\partial z}{\partial u} = x \cdot \frac{\partial z}{\partial x} \rightarrow \boxed{D_u z = x D_x z}$…(3)

Again $x^2 \cdot \frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$

$= x^2 \frac{\partial}{\partial x}(\frac{1}{x} \cdot \frac{\partial z}{\partial u})$ from (3)

$$= x^2 \cdot \frac{1}{x} \cdot \frac{\partial^2 z}{\partial x \partial u} - x^2 \cdot \frac{\partial z}{\partial u} \cdot x^{-2}$$

$$= x \cdot \frac{\partial^2 z}{\partial x \partial u} - \frac{\partial z}{\partial u}$$

$$= x \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial z}{\partial u}$$

$$= x \frac{\partial}{\partial u}\left(\frac{1}{x} \cdot \frac{\partial z}{\partial u}\right) - \frac{\partial z}{\partial u}$$

$$= x \cdot \frac{1}{x} \cdot \frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u}$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u}$$

$$\therefore x^2 D_x^2 z = D_u(D_u - 1)z$$

And so on similarly, we have

$yD_y z = D_v z$,$y^2 D_y^2 z = D_v(D_v - 1)z$,…

Hence

$x^n \frac{\partial^n z}{\partial x^n} z = D_u(D_u - 1)(D_u - 2) \dots (D_u - n + 1)z$…(4)

$y^m \frac{\partial^m z}{\partial y^m} z = D_v(D_v - 1)(D_v - 2) \dots (D_v - m + 1)z$…(5)

$$x^n y^m \frac{\partial^{n+m} z}{\partial x^n \partial y^m} z = D_u(D_u - 1) \dots (D_u - n + 1) D_v(D_v - 1) \dots (D_v - m + 1) z \quad \dots(6)$$

Substituting (4),(5),(6) in (1) to get an equation having constant coefficients can easily be solved by the methods of solving homo. And non-homo. Partial differential equations with constant coefficients, Finally , with help of (2), the solution is obtained in terms of old variables x and y.

<u>Ex.20</u>: Solve $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = 0$

Sol: let $x = e^u$, $y = e^v$ then $u = lnx$ and $v = lny$

and
$$\left.\begin{array}{lll} x \frac{\partial z}{\partial x} = D_u z & , & x^2 \cdot \frac{\partial^2 z}{\partial x^2} = D_u(D_u - 1) z \\ y \frac{\partial z}{\partial y} = D_v z & , & y^2 \cdot \frac{\partial^2 z}{\partial y^2} = D_v(D_v - 1) z \end{array}\right\} \dots(7)$$

Substituting (7) in the given equation,

$$(D_u^2 - D_u - D_v^2 + D_v - D_v + D_u) z = 0$$

$$(D_u^2 - D_v^2) z = 0 \rightarrow (D_u - D_v)(D_u + D_v) z = 0$$

The A.E. is $\underbrace{(m - 1)}_{m_1 = 1} \underbrace{(m + 1)}_{m_2 = -1} = 0$

Then the general solution is

$$z = \emptyset_1(v + u) + \emptyset_2(v - u)$$

$$= \emptyset_1(lny + lnx) + \emptyset_2(lny - lnx)$$

$$= \emptyset_1(lnxy) + \emptyset_2\left(ln\frac{y}{x}\right)$$

$$= h_1(xy) + h_2\left(\frac{y}{x}\right)$$

Where h_1 and h_2 are two arbitrary functions.

…Exercises…

Solve the following equations:

1)) $(x^2D_x^2 - y^2D_y^2 - yD_y + xD_x)z = xy$

2)) $(x^2D_x^2 - 2xyD_xD_y + y^2D_y^2 + yD_y + xD_x)z = 0$

3)) $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} = lnxy$

Classification of partial differential equations of second order:

Consider a general partial differential equation of second order for a function of two independent variables x and y in the form

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ ...(*)

Where A, B, C, D, E, F, G are function of x, y or constants.

The equation (*) is said to be

(i) Hyperbolic at a point (x, y)in domain Dif $B^2 - 4AC > 0$.
(ii) Parabolic at a point (x, y)in domain Dif $B^2 - 4AC = 0$.
(iii)Elliptic at a point (x, y)in domain Dif $B^2 - 4AC < 0$.

Ex.21: Classify the following partial differential equation $\boldsymbol{2u_{xx} + 3u_{xy} = 0}$

Sol:

Comparing the given equation with (*),we get $A = 2, B = 3, C = 0$

$$B^2 - 4AC = 9 - 4(2)(0) = 9 > 0$$

Showing that the given equation is hyperbolic at all points.

Ex.22: Classify the following p.d.eqs.

(1) $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

(2) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol. (1) Re-writing the given equation, we get

$$\alpha^2 u_{xx} - u_t = 0$$

Comparing with (*), we get $A = \alpha^2, B = 0, C = 0$

$$B^2 - 4AC = 0 - 4(\alpha^2)(0) = 0$$

Showing that the given equation is Parabolic at all points.

Sol. (2) Re-writing the given equation, we get

$$c^2 u_{xx} - u_{tt} = 0$$

Comparing with (*), we get $A = c^2, B = 0, C = -1$

$$B^2 - 4AC = 0 - 4(c^2)(-1) = 4c^2 > 0$$

Showing that the given equation is hyperbolic at all points.

Sol. (3) Comparing with (*), we get $A = 1, B = 0, C = 1$

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

Then the equation is an Elliptic at all points.

…Exercises…

Classify the following equations:

1)) $u_x - u_{xy} - u_y = 0$

2)) $u_{rr} - ru_{r\theta} + r^2 u_{\theta\theta} = 0 \qquad ; u(r, \theta)$

3)) $z_{xx} + z_{xy} + z_y = 2x$

4))$xyz_{xx} - (x^2 - y^2)z_{xy} - xyz_{yy} + yz_x - xz_y = 2(x^2 - y^2)$

5)) $x^2(y - 1)\frac{\partial^2 z}{\partial x^2} - x(y^2 - 1)\frac{\partial^2 z}{\partial x \partial y} + 4(y - 1)\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

6))$u_r - u_{\theta\theta} = 5$

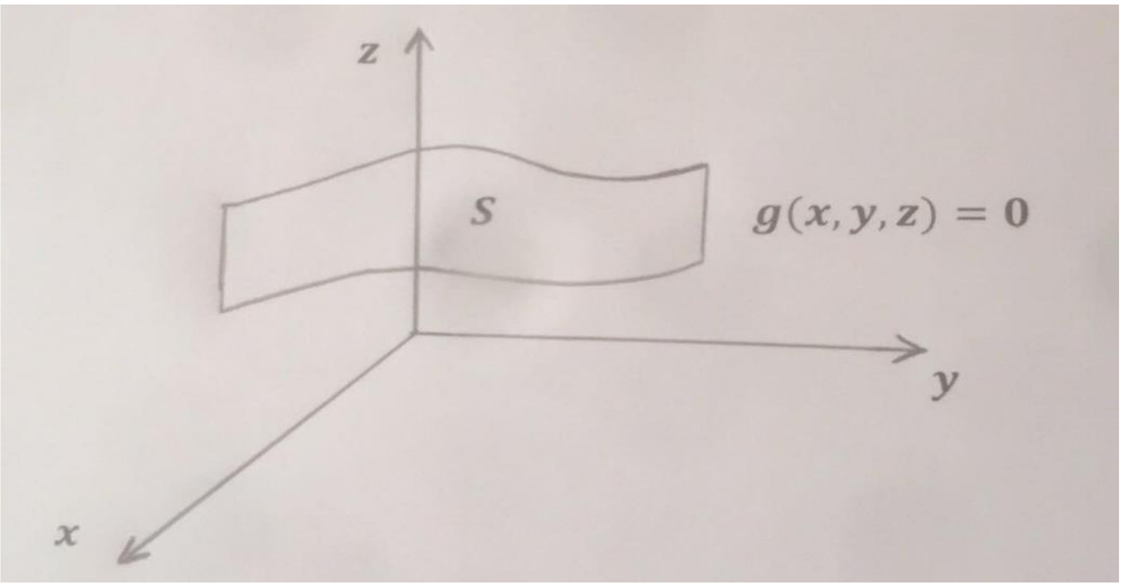
7)) $2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 2$

Section 2.4: Method of Lagrange multipliers

This method applies to minimize (or maximize) a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, construct the auxiliary function

Discussion of the method

Suppose we want to find the minimum (maximum) value of the function $f(x, y, z)$ which represents the distance between the required plane $g(x, y, z) = 0$ and the origin and suppose that f and g having continuous first partial derivatives and ending of f is at the point (x_0, y_0, z_0) which it's on the surface S that defined by $g(x, y, z) = 0$



We said that f has minimum (maximum)value at the point (x_0, y_0, z_0) if it satisfies the following condition

$\boxed{\nabla f(x,y,z) = \lambda \nabla g(x,y,z)}$ ...(1)

Where λ is Lagrange's multiplier , ∇ denote to the partial derivatives of f and g w.r.t. x, y and z.

<u>Ex.23</u>: by using $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, find the point on the straight $y = 3 - 2x$ that is nearest the origin.

Sol. Let $f(x,y) = x^2 + y^2 \rightarrow \nabla f(x,y) = < 2x, 2y >$...(2)

$g(x,y) = y + 2x - 3 = 0 \rightarrow \nabla g(x,y) = < 2,1 >$...(3)

Substituting (2) & (3) in $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$,we get

$$< 2x, 2y > = \lambda < 2,1 >$$

$\therefore 2x = 2\lambda \,\&\, 2y = \lambda \quad \rightarrow \quad x = \lambda = 2y$ ...(4)

Substituting (4) in $g(x,y)$, we have

$$y = 3 - 4y \quad \rightarrow \quad 5y = 3 \quad \rightarrow \quad y = \frac{3}{5}$$

Then form (4),we have $x = \frac{6}{5}$

$$\therefore (x, y) = (\frac{6}{5}, \frac{3}{5})$$

Which it's the point on $y = 3 - 2x$ that is nearest the origin.

Note **The distance between the point (x, y)on a straight and the origin is**

$$\boldsymbol{w = \sqrt{(x - x_0)^2 + (y - y_0)^2}} \quad \boldsymbol{, (x_0, y_0) = (0, 0)}$$

$$\boldsymbol{= \sqrt{x^2 + y^2}}$$

Squaring both sides, we get

$$\boldsymbol{w^2 = x^2 + y^2 = f(x, y)}$$

<u>Ex.24</u>: Find the point on the plane $2x - 3y + 5z = 19$ that is nearest the origin, using the method of Lagrange multiplier.

Sol. As before, let

Let $f(x, y, z) = x^2 + y^2 + z^2 \rightarrow \nabla f(x, y, z) =< 2x, 2y, 2z >$...(5)
$g(x, y, z) = 2x - 3y + 5z - 19 = 0 \rightarrow \nabla g(x, y) =< 2, -3, 5 >$...(6)

From the relation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ ...(7)

$$\rightarrow < 2x, 2y, 2z >= \lambda < 2, -3, 5 >$$

$$\therefore 2x = 2\lambda \quad , \quad 2y = -3\lambda \quad , \quad 2z = 5\lambda$$

$\rightarrow x = \lambda \quad , \quad y = \frac{-3\lambda}{2} \quad , \quad z = \frac{5\lambda}{2}$ ...(8)

Substituting this values in g , we get

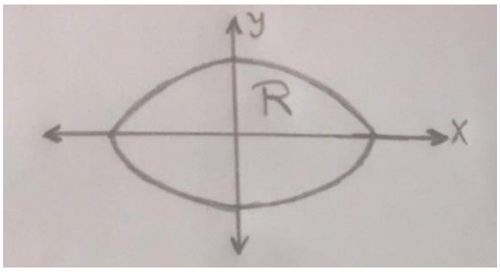
$$2\lambda+\frac{9}{2}\lambda+\frac{25}{2}\lambda=19\ \rightarrow 38\lambda=38\quad\rightarrow\ \lambda=1$$

Substituting ($\lambda=1$) in (8) , we have

$$x=1\quad,\quad y=\frac{-3}{2}\quad,\quad z=\frac{5}{2}$$

$$\therefore p(x,y,z)=(1,\frac{-3}{2},\frac{5}{2})$$

<u>Ex.25</u>: Suppose that the temperature of metal plate is given by $\boldsymbol{T(x,y)=x^2+2x+y^2}$. For the points (x,y) on a plate ellipse defined by $\boldsymbol{x^2+4y^2\leq 24}$. Find minimum and maximum temperature on the plate.



Sol. For the plate in the figure

Firstly, we will find the critical

Points of $T(x,y)$ in R

$$T(x,y)=x^2+2x+y^2\ \ \rightarrow\nabla T(x,y)=<2x+2,2y>=<0,0>$$

$$\therefore 2x+2=0\ \ \&\ 2y=0\ \ \rightarrow\ \ x=-1\ \&\ y=0$$

$\therefore\ (x,y)=(-1,0)$ is in R

Now, using the relation $\nabla f(x,y)=\lambda\nabla g(x,y)$

$$f(x,y)=T(x,y)=x^2+2x+y^2\ \ \rightarrow\nabla T(x,y)=<2x+2,2y>$$

$$g(x,y)=x^2+4y^2-24\ \ \rightarrow\nabla g(x,y)=<2x,8y>$$

$$\nabla T(x,y)=\lambda\nabla g(x,y)$$

$$<2x+2,2y>=\lambda<2x,8y>$$

$\therefore\ 2x+2=2\lambda x$…(9) $\quad$ &$2y=8\lambda y$…(10)

$y(2-8\lambda)=0$

From (10) $\boxed{y=0}$ or $2-8\lambda=0 \rightarrow \boxed{\lambda=\frac{1}{4}}$

*if $y=0 \rightarrow x^2+4(0)=24 \rightarrow x=\pm\sqrt{24}$

$$\therefore (x,y)=(\sqrt{24},0) \; or \; (-\sqrt{24},0)$$

*if $\lambda=\frac{1}{4} \rightarrow 2x+2=\frac{1}{2}x$ from (9)

$$\rightarrow x=\frac{-4}{3}$$

Substituting in g , we have

$\frac{16}{9}+4y^2=24 \rightarrow y=\pm\frac{\sqrt{50}}{3}$

$$\therefore \boxed{(x,y)=\left(\frac{-4}{3},\frac{\sqrt{50}}{3}\right) \quad or \quad \left(\frac{-4}{3},\frac{-\sqrt{50}}{3}\right)}$$

Now , to find the minimum and maximum temperature T substituting all points in T

$$T(-1,0)=-1$$

$$T(\sqrt{24},0)=24+2\sqrt{24}\cong 33.8$$

$$T(-\sqrt{24},0)=24-2\sqrt{24}\cong 14.2$$

$$T\left(\frac{-4}{3},\frac{\sqrt{50}}{3}\right)=\frac{14}{3}\cong 4.7$$

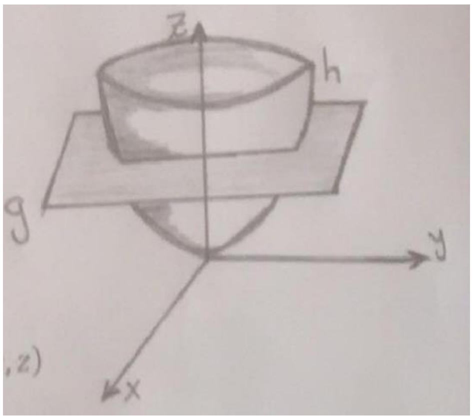
$$T\left(\frac{-4}{3},\frac{-\sqrt{50}}{3}\right)=\frac{14}{3}\cong 4.7$$

Note that the minimum temperature is (-1) at the point $(-1,0)$ and the maximum temperature is (33.8)at the point $(\sqrt{24},0)$.

Remark if there are two constraints intersecting ,say $\boldsymbol{g(x,y,z)=0}$ **and** $\boldsymbol{h(x,y,z)=0}$**, we introduce two Lagrange's multipliers** $\boldsymbol{\lambda}$**and** $\boldsymbol{\mu}$ **and the relation will be**

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)$$

<u>Ex.26</u>: the plane $\boldsymbol{x+y+z=12}$ **intersects with the cone** $\boldsymbol{z=x^2+y^2}$**by an ellipse. Find the point on the intersection that is nearest to the origin.**



Sol. $f(x,y,z)=x^2+y^2+z^2$

$g(x,y,z)=x+y+z-12=0$

$h(x,y,z)=x^2+y^2-z=0$

$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)$

$<2x,2y.2z>=\lambda<1,1,1>+\mu<2x,2y,-1>$

$\therefore 2x=\lambda+2\mu x$ ...(11)

$2y=\lambda+2\mu y$...(12)

$2z=\lambda-\mu$ ...(13)

From (11) and (12)

$$\left.\begin{array}{l}\lambda=2x(1-\mu)\\ \lambda=2y(1-\mu)\end{array}\right\}\rightarrow 2x(1-\mu)=2y(1-\mu)\rightarrow(2x-2y)(1-\mu)=0$$

Then $1 - \mu = 0 \rightarrow \mu = 1 \rightarrow \lambda = 0$ from (11) & (12)

Substituting in (13) we have $z = \frac{-1}{2}$...(14)

Substituting (14) in g and h , we have

$$x + y - \frac{1}{2} - 12 = 0$$

$x^2 + y^2 = -\frac{1}{2}$ (Contradiction)

Or $2x - 2y = 0 \rightarrow x = y$, in this case (Substitutingin h and g) we get

In $h x^2 + y^2 - z = 0 \rightarrow z = 2x^2$

In $g 2x + 2x^2 - 12 = 0 \rightarrow x^2 + x - 6 = 0$

$$(x + 3)(x - 2) = 0 \rightarrow x = -3 \; or \; x = 2$$

$x = y$ & $z = 2x^2 \rightarrow (x, y, z) = (2,2,8)$ or $(-3, -3,18)$

When $(x, y, z) = (2,2,8) \rightarrow f(2,2,8) = 72$

When $(x, y, z) = (-3, -3,18) \rightarrow f(-3, -3,18) = 342$

Then $(2,2,8)$ is the nearest to the origin.

… Exercises …

1)) Find the point on the curve $y = x^2 + 3$ that is nearest the origin, using the method of Lagrange multipliers.

2)) Find the minimum distance from the surface$x^2 + y^2 - z^2 = 1$ to the origin.

3)) Find the point on the surface $z = xy + 1$ nearest the origin.

4)) Find the maximum and minimum values of $f(x, y, z) = x - 2y + 5z$on the sphere $x^2 + y^2 + z^2 = 30$.

5)) Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

6)) The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin what are the highest and lowest temperatures encountered by the ant?

7)) Factory produces three types of product x, y, z , the factory's profit (calculated in thousands of dollars) can be formulated in equation $p(x, y, z) = 4x + 8y + 2z$,where the account is bounded by $x^2 + 4y^2 + 2z^2 \leq 800$, find highest profit for the factory.

8)) find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse$\frac{x^2}{8} + \frac{y^2}{2} = 1$.

9)) Find the point on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values .

10)) Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.