References المصادر

- Calculus and Analytic geometry by George B- Thomas 7th Edition 1988
- International Edition Thomas calculus part 1

Chapter one

The Real Numbers الاعداد الحقيقية :-

The sub sets of the real numbers are :-

1) Natural Numbers الاعداد الطبيعية : (N)

N = { 1,2,3,4,….}

2) Integer Numbers الاعداد الصحيحة : (I or Ƶ)
   I = Ƶ= { …. , -3 , -2 , -1 , 0 , 1 , 2 , 3 , …..}
3) Rational Numbers الاعداد النسبية : (Q)
   Is any number that can be written in the form $\frac{p}{q}$, where p and q are integer and q ≠ 0
   Q= { ×∈ R ,×= $\frac{p}{q}$, p , q ∈ Ƶ , q ≠ 0 }

Notice

That every integer n is also a rational number since we can write it as the quotient of the $\frac{n}{q}$

4) Irrational Numbers :- (Q) الاعداد الغير نسبية
   Are all those real numbers that cannot be written in the form $\frac{p}{q}$, where p and q are Integers
   For example :- π , $\sqrt{5}$, $\sqrt{3}$, $\sqrt{2}$
   $\sqrt{2}$ = 1- 41421356,…. تعتبر اعداد دورية
   π= 3- 141592,….

Note that :-

مجموعة خالية لا يوجد فيها اي عدد
{ }

Ø C N C I C Q C R
Q U Q^- = R

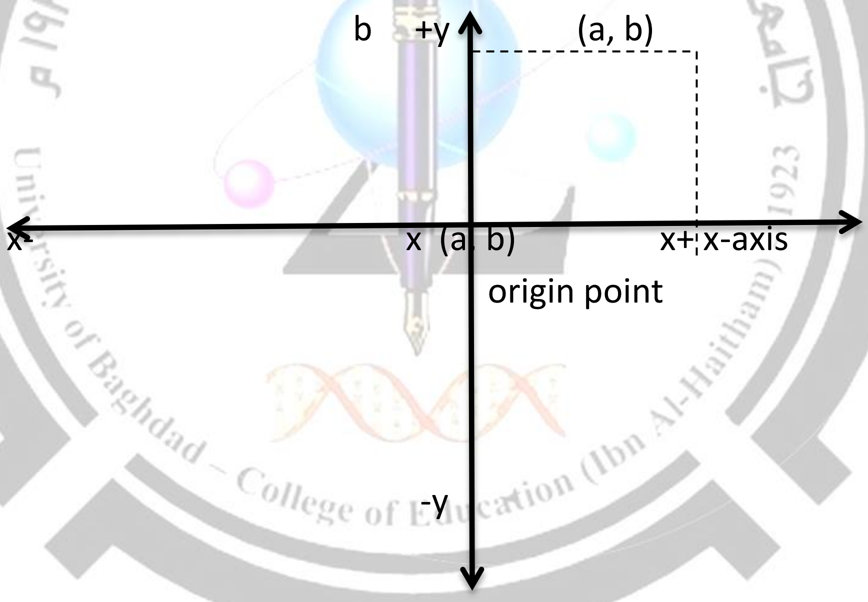
Chapter 1

The rate of change of a function

a) Coordinates for the plane :-

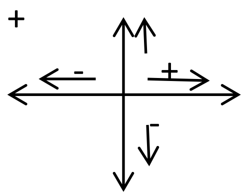
To assign coordinates to points in plane , we start with two number lines that cross at their zero points at right angles the horizontal line is called the x- axis p vertical line the y- axis , the point at which the lines cross is the origin .

- We assign a pair (a, b) of real numbers to each point p in the plane . the number a from the x- axis is the x- coordinate of p .
  The number b from the y- axis is the y- coordinate of p. the pair (a, b) is the ordered pair .



<u>Note :-</u>

1) Every point on the x- axis has y- axis zero& every point on the y- axis has x- axis zero



2) The origin point (0,0)

3) Directions along the axis :- the values of x & y increasing in the positive direction & decreasing in the negative

Ex) plot the points & in which quadrature the following

Points :- (-3 , 5) , (5 , -4) , (4 , -1)

b) The slope of a straight line :-

Definition :- (The Increments التغاير)

When a particle moves from $p_1(x_1, y_1)$ to $p_2(x_2 , y_2)$

Δx = [(x_2 of terminal point)- ($x_1$of initial point)]

Δy = [(y_2 of terminal point) – (y_1 of initial point)]

That is $\Delta x = x_2 - x_1$ & $\Delta y = y_2 - y_1$

Δx is called delta x & Δy delta y

Ex) If a particle moves from A(1,-2) to B (6,7) find Δx & Δy or (find the Increments of them)

Sol:- A (1,-2) Initial point & B (6,7) terminal point

$\Delta x = 6-1=5$

$\Delta y = 7-(-2)=9$

Ex) If particale start at A(-2,3) fits coordinates receive increments $\Delta x=5$, $\Delta y= -6$ what will be its new position ?

Sol:- A (-2,3) & B (?,?) , $\Delta x=5$ & $\Delta y= -6$

$\Delta x = x_2 - x_1 \Longrightarrow 5=x_2+2$ & $\Delta y = y_2-y_1 \Longrightarrow -6 = y_2 -3 \Longrightarrow x_2= 5-2$ & $y_2=-6+3 \Longrightarrow$ $x_2=3$& $y_2 = -3$

B(3,-3) terminal point

Ex) A particle moves along the parabola $y=x_2$ from the point A(1,1) to the point B(x,y) show that $\frac{\Delta y}{\Delta x}$ = x+1 if $\Delta x \neq 0$?

Sol:- $\Delta y = y_2 - y_1$ & $\Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1$

$\boxed{\Delta y = y-1}$ & $\Delta x = x_2 - x_1 \Rightarrow \boxed{\Delta x = x-1}$

$\frac{\Delta y}{\Delta x} = \frac{y-1}{x-1} = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{\cancel{(x-1)}} = x+1$

Note :- The increment Δx & Δy can be any real number , may be positive , negative or zero .

Definition :- (the distance (d) between two points)

P1 (x_1, y_1) & $p_2(x_2, y_2)$ is defined by :-

$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Ex) particle starts from p1 (-1,2) and travels in a straight line to the point $p_2(2,-2)$ find the distance between p_1 & p_2

Sol:-

$\Delta x = x_2 - x_1 \Rightarrow \Delta x = 2-(-1) \Rightarrow \Delta x = \boxed{3}$

$\Delta y = y_2 - y_1 \Rightarrow \Delta y = -2-2 \Rightarrow \Delta y = \boxed{-4}$

$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25}$

$\boxed{d=5}$

the slope الميل

Let $\underline{L}$ be a straight line which is not parallel to the y-axis . Let $p_1(x_1, y_1)$ & $p_2(x_2, y_2)$ be any two distinct points on $\underline{L}$ then we call $\Delta y = y_2 - y_1$ the rise & $\Delta x = x_2 - x_1$ the run along L from p_1 to p_2 . we define the slope of $\underline{L}$ as the rate of rise per unit of run :-

Slope $= \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = m$

$M = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Note :-

1) Angle of inclination are measured counter clockwise from the x-axis
2) The relationship between the slope m of a non-vertical line and the lines angle of inclination $\ominus$ is shown m= tan $\ominus$
3) The slope of a non vertical line is tangent of its angle on inclination .

   M = $\frac{\Delta y}{\Delta \times}$ = tan $\ominus$ = $\frac{\sin\ominus}{sos\ominus}$
4) Lines are perpendicular $\leftrightarrow$ m $\perp$ = $\frac{-1}{m}$ ميل العمود = مقلوب الميل
5) Two parallel lines having equal slops that is m_1 = tan $\ominus_1$, m_2 = tan $\ominus_2 \Longrightarrow m_1 = m_2$ المستقيمات المتوازية لها نفس الميل
6) The slope of $L_1 = m_1$ & $L_2 = m_2$ if $L_1 \perp L_2 \leftrightarrow m_1 m_2 = 1$ شرط تعامد المستقيمين
7) The rule of midpoint p= $(\frac{x1+\times2}{2}, \frac{y1+y2}{2})$

Examples :-

Ex1) plot the given points A (-2 , -1) , B (1 , -2) & find the slope of the straight line determined by them & find a perpendicular line?

Sol :- m= $\frac{y2-y1}{\times2-\times1} = \frac{-2+1}{1+2} = \frac{-1}{3}$

Per , m$\perp$ = $\frac{-1}{m} = \frac{-1}{\frac{-1}{3}}$ = 3

Ex2) Let $p_1(2,-1)$ & $p_2(-6,3)$find the m & m $\perp$ & find the midpoint of the segment p_1 p_2 ?

Ex3) Let p_1 (1,-1) & p_2 (2,1) on the line straight L_1 , & p_3 (-2 , 0) & p_4 (0,4) on the line straight L_2, determined the straights lines parallel or vertical ?

Sol :- $m_1 = \frac{1+1}{2-1} = 2$, $m_2 = \frac{4-0}{0+2} = 2 \Longrightarrow m_1 = m_2$

$\therefore L1 \setminus\setminus L2$

C) Equations of a straight line :-

We can write an equation for a non-vertical straight line L if we know its slope m and the coordinates of one p_1 (x_1,y_1) on it If $p(x,y)$ is any other point on L , then we can use the two points p_1 & p to compute the slopes

$M= \frac{y-y1}{x-x1} \Longrightarrow y-y_1 = m\,(x-x_1)$

Or $y= y_1+m\,(x-x_1)$

The equation $y = y_1 + m\,(x-x_1)$ .....(1) is the (point – slope equation)

of the line that passes through the point p_1 (x_1,y_1) and has slope.

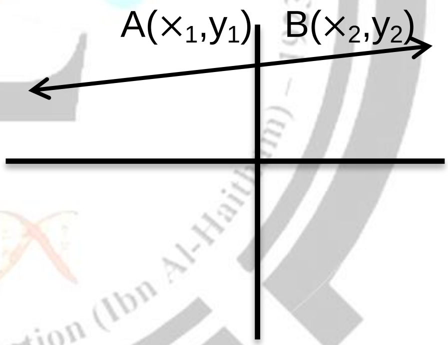
معادلة مستقيم اذا علم ميله ونقطة من نقاطه

Ex) write an equation for the line through the point (2,3) with slope $\frac{-3}{2}$?

<u>**Sol :-**</u>

$y=y_1 +m\,(x- x_1) \Longrightarrow y= 3 - \frac{3}{2}\,(x-2) \Longrightarrow y= 3 - \frac{3}{2}\,x + 3 \qquad \Longrightarrow y= 6 - \frac{3}{2}\,x$.

A line through two points



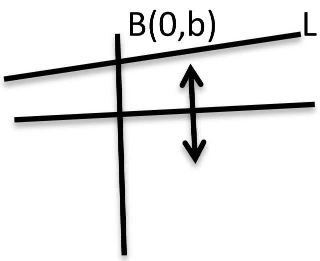
$M= \frac{y2-y2}{x2-x1}$ ..... (2)

From (1) &(2) $\Longrightarrow y = y_1 + \frac{y2-y2}{x2-x1}\,(x-x_1)$

معادلة مستقيم الذي يمر من نقطتين

The equation $y = m\,x +b$ is called the slope – intercept equation of the line with slope m and y – intercept b

معادلة مستقيم بدلالة ميله والجزء الذي يقطعه من محور Y



the equation A× + By = c (A and B not both 0) is called the general equation in x and y because it graph always represents a line and every line has an equation in this form (including lines with undefined slope) .

m = $\frac{-A}{B}$

Examples :-

Ex1) Finding the slope and b intercept of the line 8× + 5y = 20

Sol : solve the equation for y to put it in slope – intercept form

$$Y = m \times + b$$

$8\times + 5y = 20 \Longrightarrow 5y = -8\times + 20 \Longrightarrow y = \frac{-8}{5} \times + 4$

The slope is m = $\frac{-8}{5}$, the y – intercept is b = 4 & A= 8,B= 5,c = 20

Ex2) write an equation for the line that passes through the point (2,3) with slope $\frac{-3}{2}$

Sol : $y-y1 = m(\times-\times1) \Longrightarrow y-3 = \frac{-3}{2}(\times-2) \Longrightarrow y= \frac{-3}{2} \times + 3 + 3$

$\Longrightarrow y= \frac{-3}{2} \times + 6 \Longrightarrow 2y = -3x + 12 \Longrightarrow 2y + 3\times = 12$

A = 3 φ B = 2 φ c = 12

Ex3) find the slope of the line 2× + 3y = 5

Sol : m = $\frac{-A}{B} = \frac{-2}{3}$

المسافة بين نقطة ومستقيم

Note :- the distance d from the point p1 ($\times_1$,y_1) to the line A× + By + c = 0 is d = $\frac{|A\times1+By1+c|}{\sqrt{A^2+B^2}}$ المسافة بين نقطة ومستقيم

Ex) find the distance from the points D(-1,0), E(1,1) , and f(-3,5) to the line L : $2x + 3y - 5 = 0$

<u>Sol :-</u>

$$dD = \frac{|\,2(-1)+3(0)-5|}{\sqrt{2^2+3^2}} = \frac{7}{\sqrt{13}}$$

$$dE = \frac{|\,2(1)+3(1)-5|}{\sqrt{2^2+3^2}} = \frac{0}{\sqrt{13}} = 0$$

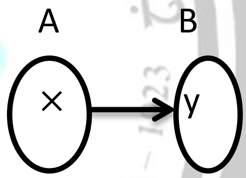
$$dF = \frac{|\,2(-3)+3(5)-5|}{\sqrt{2^2+3^2}} = \frac{4}{\sqrt{13}}$$

The functions الدالة

The function Let A and B be two non – empty sets , the – relation that assigns to every element $x \in A$, with aunique value $y \in B$ is called a functions

$F: A \rightarrow B , \forall x \in A , \exists ! y \in B ! \ni f(x) = y$

<u>Note :-</u>



A= Domain = D_f , B = co- domain = Go – D_f

*) the set of all images $f(x) = y , \forall x \in Df$

Is called the Range of f= R_f

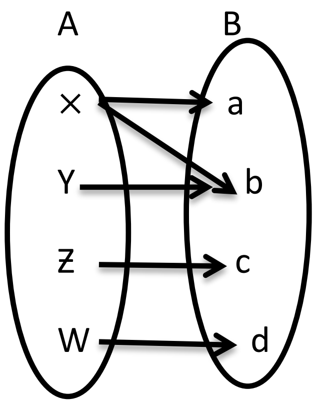
$x \rightarrow f \rightarrow f(y)$

input output

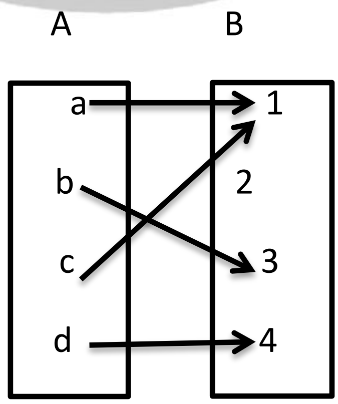
$R_f = \{ f(x) = y , x \in D_f \}$

Or = $\{y, y=f(x) , \forall x \in Df \}$

Ex(1) F is not function

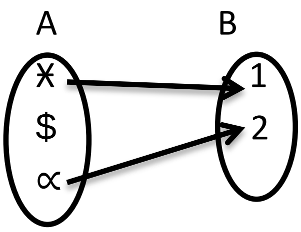


Ex(2) f is a function



Since f (x) = a and f(x) = b   Range f = {1,3,4}

Ex(3)

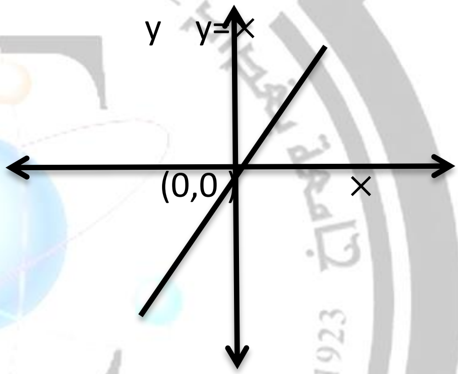


F is not function

Since ∝ ∈ A and ∝ has not image

Ex (4)

Linear function y=x



Y=f(x)=x , f : R → R

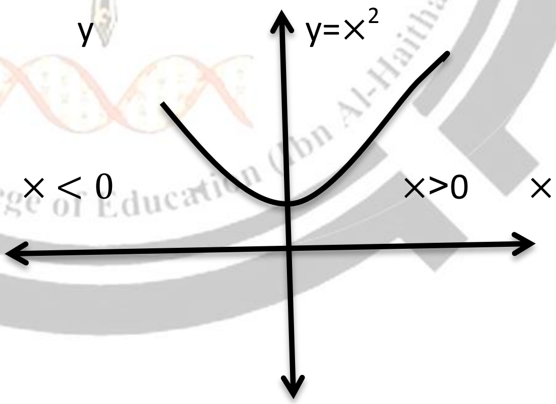
Df = R= { x∈R,-∞ < x < ∞}

Rf=R={y∈R, ,-∞ < y < ∞}

Is function

Ex (5)

Y= f(x)= x^2



D_f=R

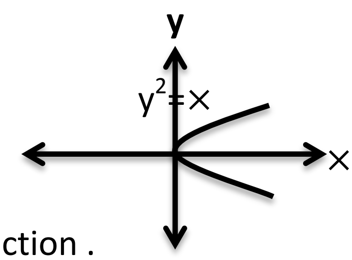
R_f=R+= { y∈R : y≥ 0 }

= { y∈R , 0≤ ∞}

= [0, ∞]

Is function

Ex (6)



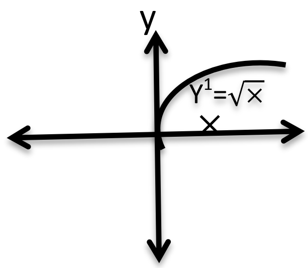
y^2=x , not function

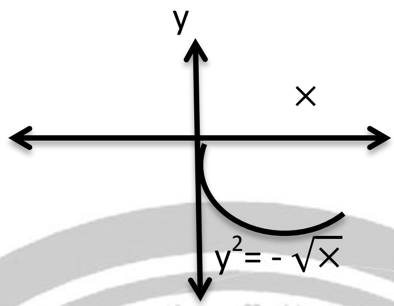
$= \sqrt{y^2} = \sqrt{x} \Longrightarrow |y| = \sqrt{x} \Longrightarrow y = \mp\sqrt{x}$

∀ x∈Df, ∃ ∓ √∴ two image , for every x not function .

But if

$$y=\begin{cases} \sqrt{\times}\ \ function\ y1 \\ -\sqrt{\times}\ \ function\ y2 \end{cases}$$





Ex(7)

2y+3×=5

2y=5-3×⟹ y= $\frac{5-3\times}{2}$ function

Ex(8)

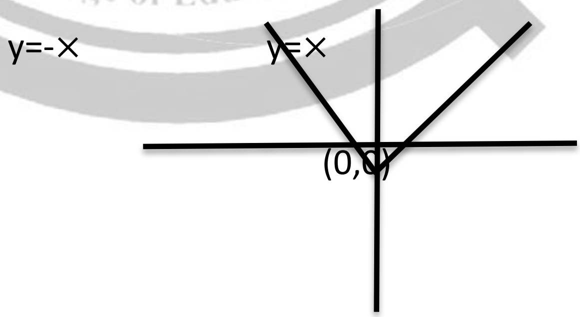
$\frac{y}{x}$ = 2

y=2× function

<u>Absolute values :-</u>

<u>Definition :-</u> The absolute value of a real number x is denoted by |×| and is defined by the formula

$|\times| = \sqrt{x^2}$



$$|\times|=\begin{cases} \times & \times > 0 \\ o & \times = 0 \\ -\times & \times < 0 \end{cases}$$

For example :-

|-8|=8 , |9| =9 , |0|=0

The absolute value of x is always either positive or zero , but never negative .

Properties of absolute value :-

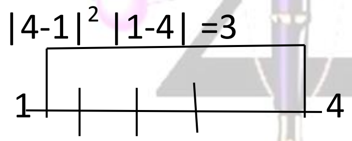
Let a, b be two real numbers then :-

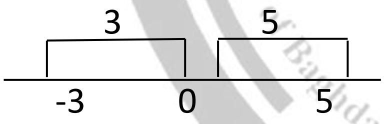
(1) $|a| = |a|$
(2) $|a| = ||a||$
(3) $|a.b| = |a|.\,|b|$
(4) $|\frac{a}{b}| = \frac{|a\,|}{|b|}$, $b \neq 0$
(5) $|a+b| \leq |a| + |b|$ (triangle inequality)
(6) $|\times| = a \leftrightarrow \times = \neq a$
(7) $|\times| \leq a \leftrightarrow -a \leq x \leq a$ φ $|\times| < a \in y - a < \times < a$
(8) $|\times| \geq a \leftrightarrow \times \geq a$ or $\times \leq -a$
(9) $|\times| > a \leftrightarrow \times > a$ or $\times < -a$

<u>Geometrically :-   من الناحية الهندسية</u>

The absolute value of × is the distance from × to zero on the real number line since distances are always positive or zero , Also

$|\times - y|$ = the distance between x and y on the real line





$|-3| = 3$ $|0| = 0$ $|5| = 5$

Ex) find the value of x that satisfy the inequality $|\times| > 3$?

<u>Sol :-</u> $\{\times \in R ,\ |\times| > 3\} = \{\times \in R ,\ \times > 3 \text{or } \times < 3\}$

$= (3, \infty) \cup (-\infty, -3) = R \backslash [-3,3]$

Ex) find the solution set for the following inequality $|7-4\times| \geq 1$

<u>Sol :-</u> $\{\times \in R ,\ |7-4\times| \geq 1\} = \{\times \in R ,\ 7-4\times| \geq 1 \text{or } 7-4\times| \leq 1\}$

$= \{\times \in R, -4\times \geq -6$ or $-4\times \leq -8$

$= \{ x \in R , x \le \frac{3}{2}$ or $x \ge 2\}$

$= (-\infty , \frac{3}{2}) \cup (2 , \infty) = R \setminus (\frac{3}{2} , 2)$

Ex) solving an Equation with absolute values $|2x-3|=7$

Sol:- by property $|x|=a \leftrightarrow x=\mp a$

$2x-3=7 \quad \& \quad 2x-3=-7$

$2x=10 \quad \& \quad 2x=-4$

$x=5 \quad \& \quad x=-2$

Ex) $|5=\frac{2}{x}| < 1$

Sol :-

$-1 < 5-\frac{2}{x} < 1$ by (7)

$\leftrightarrow -6 < \frac{-2}{x} < -4$ (subtract 5)

$\leftrightarrow 3 > \frac{1}{x} > 2$ (multiply by $\frac{1}{2}$)

$\leftrightarrow \frac{1}{3} < x < \frac{1}{2}$ (take reciprocals)

Ex) $|2-2x| \le 7$

Sol :-

$-7 \le 2-2x \le 7$

$-9 \le -2x \le 5$

$\frac{9}{2} \ge x \ge \frac{-5}{2}$

$-2\frac{1}{2} = -\frac{5}{2} \le x \le \frac{9}{2} = a\,\frac{1}{2}$

$S= [\frac{-5}{2} , \frac{9}{2}]$

Ex) $|x-1| \geq 6$

Sol :-

$x-1 \geq 6 \vee x-1 \leq -6$

$x \geq 7 \vee x \leq -5$

$s = [7, \infty) \vee (-\infty, -5]$

H.w.

1) $|x| > 5$

2) $|x| + 1$

3) $|\frac{2x}{3}| \leq 1$

4) $|x-4| < 2$

5) $|2x+5| > 4$

6) $|2x-3| \geq 1$

How to find the Domain and the Range of function

1) The domain of all polynomial or odd root is all real numbers (Real number , R)
   اذا كانت الدالة على شكل متعددة حدود او ذات جذر فردي فأن مجالها كل الاعداد الحقيقية

Ex(1) $f(x)\ \ x3 + 2x^2 + 3x - 5$

Df = R , Rf = R

Ex (2) $f(x) = \sqrt[5]{x+1}$

Df = R , Rf = R

Ex(3) $f(x) = \sqrt[3]{x\,7 - 1}$

$D_f = R$, $R_f = R$

اذا كانت الدالة دالة جذرية والجذر زوجي فأن مجالها كل الاعداد الحقيقية التي تجعل المقدار

2) The domain of even root such as square roots is all real numbers that the expression under the radical to greater than or equal to zero

Ex(1) f(×) = $\sqrt{\times 2 - 4}$

×2-4 $\geq$ 0 $\Longrightarrow$ (×-2) (×+2) $\geq$ 0

(×-2) $\geq$ 0 $\wedge$ (-× +2) $\geq$ 0 $\Longrightarrow$ ×$\geq$2 $\wedge$ ×$\geq$ -2$\Longrightarrow$[2, ∞]

(×-2) $\leq$ 0 $\wedge$ (×+2) $\leq$ 0 $\Longrightarrow$ ×$\leq$2 $\wedge$ ×$\leq$-2 $\Longrightarrow$ [-∞ , -2]

D_f (-∞ , -2) ∪ (2, ∞) = R\(-2,2)

Ex(2) $\sqrt{2 \times -1}$

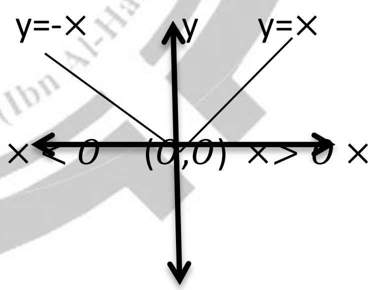
2x-1$\geq$0 $\Longrightarrow$ 2x$\geq$1$\Longrightarrow$ x $\geq \frac{1}{2}$

Df = [$\frac{1}{2}$, ∞]

3) Piecewise functions :- A function that is defined by more than one formula is called a piecewise function such functions are written using the brace(s) , such as signum function , absolute value function , … , etc .

The domain of these function are the restrictions of the functions .

Ex(1) find the domain of |×|



$$F(x) = |x| = \begin{cases} \times & if \quad \times > 0 \\ 0 & if \quad \times = 0 \\ -\times & if \quad \times < 0 \end{cases}$$

D_f = R , R_f = |R+ = [0, ∞)

Ex(2) f(×) = y = |×+3| find D_f

$$|x+3| = \begin{cases} \times +3 & if \ \times +3 > 0 \Rightarrow \times > 3 \\ 0 & if \ \times +3 = 0 \Rightarrow \times = 3 \\ -(\times +3) & if \ \times +3 < 0 \Rightarrow \times < -3 \end{cases}$$

D_f = R , R_f = R^+

Ex(3) f(x) = $\begin{cases} \times & if\ x < -2 \quad , find\ Df \\ \times +1 & if -2 \le \times \le 1 \\ \times 2 & if \quad \times > 1 \end{cases}$

$\times < -2 \vee -2 \le \times \le 1 \vee \times > 1$

$(-\infty , -2) \cup [-2,1] \cup (1, \infty) = R$

$D_f = R$

Ex(4) find the domain of

y=h(x)= $\begin{cases} -1 & if \times \le 2 \\ 3 & if \times > 2 \end{cases}$

$D_f = R$, $R_f = \{-1 , 3\}$

4) The domain of Rational function is all not number except the value of × which make the denominator zero

Ex) f(x) = $\frac{\times}{x^2-1}$ $\qquad x^2-1 \neq 0 \Rightarrow x^2 \neq 1$

Df = R \ {-1,1} $\qquad \sqrt{x^2} \neq 1 \Rightarrow |x| \neq 1$

y=$\frac{\times}{x^2-1}$ $\Rightarrow x=yx^2-y \Rightarrow yx^2 - x - y = 0$ $\qquad x=\mp 1$

x=$\frac{1\mp\sqrt{1+4y^2}}{2y}$

$2y \neq 0 \Rightarrow y \neq 0$

$1+4y^2 \geq 0 \Rightarrow y^2 \geq \frac{-1}{4} \Rightarrow y^2 \geq 0$

$y \in R$

$R_f = |R \setminus \{0\}$

Ex) y=$\frac{1}{x}$ $\qquad$ x= $\frac{1}{y}$

$D_f = R \setminus \{0\}$ $\qquad R_f = R \setminus \{0\}$

Ex) $y=\frac{1}{x-1}$ $\qquad$ $y=\frac{1}{x-1}$

$D_f = R\backslash \{1\}$ $\qquad$ $yx-y=1$

$R_f = R\backslash \{0\}$ $\qquad$ $yx=1+y$

$X=\frac{1+y}{y}$

5) There are special cases of Rational function
a) I f the rational function contain a sequare root in the denominator then domain is all real number the expression under the radical greater than zero

Ex) $\frac{2-x}{\sqrt{1-x}}$

$1-x>0 \Rightarrow 1>x$, $Df = (-\infty ,1)$

$y=\frac{2-x}{\sqrt{1-x}} \Rightarrow y\sqrt{1-x}= 2-x$

$y^2(1-x) = (2-x)^2$

$y^2- y^2x = 4-4x+x^2$

$x^2- 4x+4-y^2+xy^2=0$

$x^2+(-4+y^2) x+(4-y^2)=0$

$a=1,\ b=y+y^2\ ,\ c=4-y^2\ ,\ x= -b\mp \frac{\sqrt{b^2+4ac}}{2a}$

$x=\frac{-(-4+y2)\mp\sqrt{(-4+y^2\ )2-4(1)(4-y2)}}{2(1)}$

$=\frac{4-y2\mp\sqrt{16-8y2+y4-16+4y2}}{2}$

$=\frac{4-y2\mp\sqrt{y4-4y2}}{2}$

$y^4-4y^2\geq 0 \Rightarrow y^2\ (y^2-4) \geq 0$

$y^2\geq 0 \wedge y^2 - 4\geq 0 \Rightarrow y^2\geq 0 \wedge y^2\geq 4$

$\Rightarrow\ y\in R \wedge |y|\geq 2$

$\therefore R_f = [\ 2, \infty) \cup (\ -\infty\ , -2] \Rightarrow y\in R\ \wedge y\geq 2 \vee y \leq -2$ or $R\backslash (-2,2)$

6) I f the rational function contain odd root in the denominator then domain is all real number except the value of x which make the denominator zero .

Ex) f(x)= $\sqrt[3]{\frac{x+1}{x-2}}$

y=$\sqrt[3]{\frac{x+1}{x-2}}$, $D_f=R\backslash\{2\}$

y=$\sqrt[3]{\frac{x+1}{x-2}} \Longrightarrow y^3 = \frac{x+1}{x-2} \Longrightarrow (x-2)\, y^3=x+1$

$xy^3-2y^3-x-1=0$

$(y^3-1)x=2y^3+1$

$\therefore x=\frac{2y3+1}{y3-1}$

$y^3 -1 \neq 0 \Longrightarrow y^3 \neq 1 \Longrightarrow y\neq 1$

$R_f=R\backslash\{1\}$

Exercises

Find the Domains and Ranges for the following functions :-

(1) y=f(x)= $\frac{1}{x2+1}$ +3

$X^2+1\neq 0 \Longrightarrow X^2\neq -1$, $D_f=R$

y=$\frac{1}{x2+1}$ +3 $\Longrightarrow$ y-3 = $\frac{1}{x2+1}$

$(y-3)\,(x^2+1) =1$

$yx^2+y-3x^2-3=1$

$(y-3)\, x^2+y-3=1$

$(y-3)x^2=4-y$

$X^2=\frac{4-y}{y-3} \Longrightarrow |x| = \sqrt{\frac{4-y}{y-3}}$

$X = \mp\sqrt{\frac{4-y}{y-3}}$

$y-4 \geq 0 \;\wedge\; y-3 > 0 \quad \vee \quad 4-y \leq 0 \;\wedge\; y-3 < 0$

$y \leq 4 \wedge y > 3 \quad \vee\; y \geq 4 \wedge y < 3$

$3 < y \leq 4$

(———]——   ) ——— [———

3   4   3   4

(3,4]   ∨   Ø = (3,4]

$\therefore R_f = (3,4]$

Homework

Find the Domains and Rangers for the following functions

(1) $f(x) = \sqrt{\frac{1}{x} - 2}$

(2) $y = f(x) = \frac{\sqrt{x+1}}{x-1}$

(3) $f(x) = \begin{cases} 1 & if \quad x \geq 0 \\ 2 & if\; x < 0 \end{cases}$

(4) $y = f(x) = \frac{x+1}{|x-5)}$

Graphs of functions :- رسم الدالة

A nother way to visualize a function is its graph If f is a function with domain D , its graph . consists of the points in the cartesian plane whose coordinates are the input – output pair for f In set notation , the graph is {(x)f(x) | x∈D}

Quick Graphing :- الرسم السريع

A quick way to graph an equation by three steps :-

(1) find the x- intercept by setting y=0
(2) find the y- intercept by setting x=0
(3) plot the intercepts and draw the line

Ex) sketch the function by using the table ?

f(x)=x_1+2

Sol:-

(1) Let y=0

y-f(x)=x+2 $\Longrightarrow$ x+2=0 $\Longrightarrow$ x= -2 $\Longrightarrow$ < -2

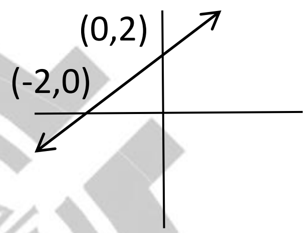
(2) Let x=0

y=0+2 $\Longrightarrow$ y=2 $\Longrightarrow$ (0,2)

(3) X=-1 $\Longrightarrow$ y=1 $\Longrightarrow$ (-1,1)

y=-1 $\Longrightarrow$ x=-3 $\Longrightarrow$ (-3,-1)

x	y
0	2
-2	0
-1	1
-3	-1



Ex) y=f(x) = x^2

Sol :- y=x^2 $\Longrightarrow$ if y=0 $\Longrightarrow$ x=0 (0,0) origin point

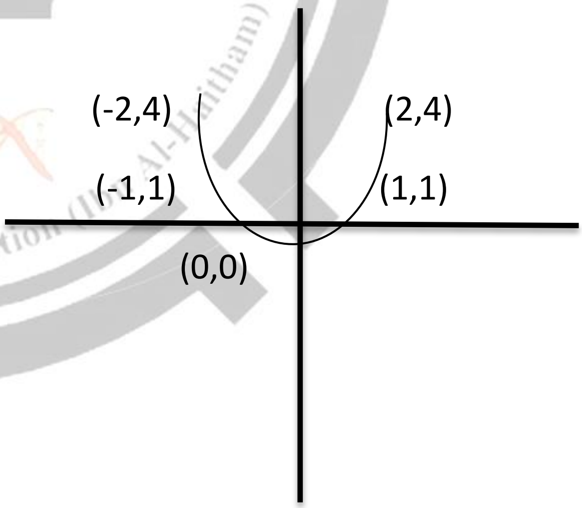
y=1 $\Longrightarrow$ x2=1 $\Longrightarrow$x=$\mp$1 $\Longrightarrow$ (1,1) , (-1,1)

x=1 $\Longrightarrow$ y=1 $\Longrightarrow$ (1,1)

x=2 $\Longrightarrow$ y=4 $\Longrightarrow$ (2,4)

x=-1 $\Longrightarrow$ y=1 $\Longrightarrow$ (-1,1)

x	y=x2
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



<u>Ex)</u> y=f(x)=x

<u>Sol :-</u> if x=0 $\Longrightarrow$ y=0 (0,0)

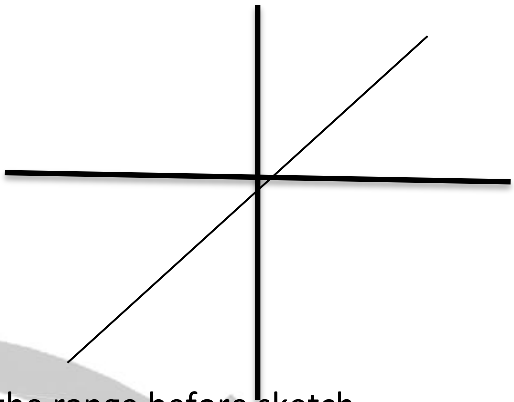
If y=0$\Longrightarrow$ x=0 (0,0)

x=1$\Longrightarrow$y=1 (1,1)

x=2 $\Longrightarrow$y=2 (2,2)

x=-1$\Longrightarrow$ y=-1 (-1,-1)

x	y
-1	1
0	0
1	1
2	2



<u>Note :-</u> we must calculate the domain and the range before sketch .

If the function a straight line we must determine the intercept point and joint it if not we take other point but the point must be in the domain of the function .

There are a number of important types of functions , we identify and briefly summarize them here .

Linear functions :-

A function of the form $\boxed{f(x) = mx+b}$, for constants m and b , is called a- Linear functions .

<u>Figure 1</u> shows an array of lines f(x) = mx where b=0 , so these lines pass through the origin constant functions result when the slope m=0 <u>Figure 2</u>

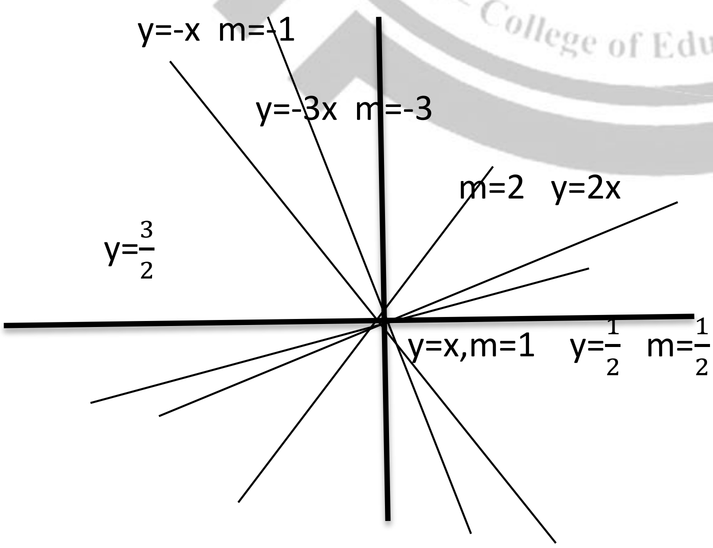


Fig (1)

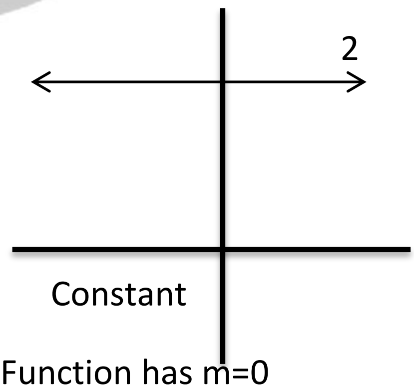
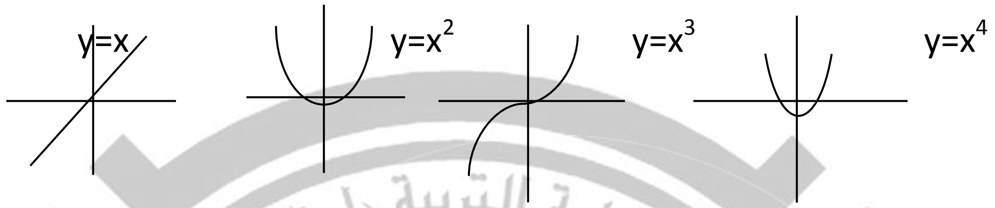


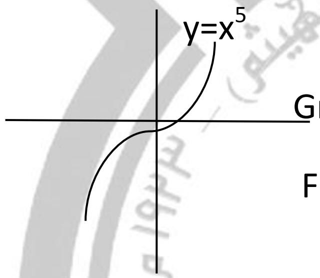
Fig (2)

Power functions:-

A function $f(x)= x^a$, where a is a constant , is called a power function – there are several important cases to consider

a) a=n , a appositive integer



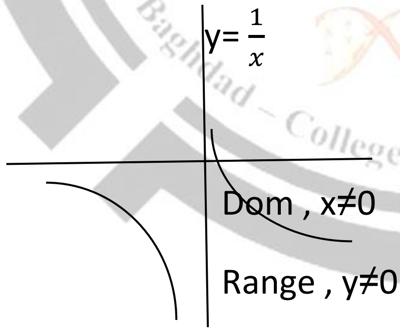


Graphs of $f(x) = x^n$, n =1,2,3,4,5 defined

For $-\infty < x < \infty$

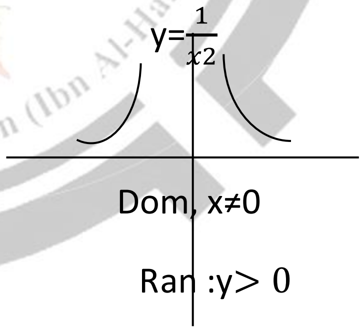
b) a=-1 or a=-2

the graphs of the functions $f(x)=x^{-1}=\frac{1}{x}$ and $g(x)= x^{-2}= \frac{1}{x2}$ are defined for all $x\neq0$



Dom , $x\neq0$

Range , $y\neq0$



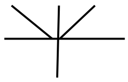
Dom, $x\neq0$

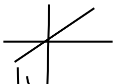
Ran :$y> 0$

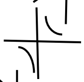
H.w

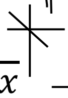
Sketch the following

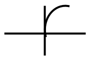
1) $y=\sqrt{x}$, $y= x\frac{3}{2}$, $y = \sqrt[3]{x}$, $y= \frac{1+x}{1-x}$ $D_f=R$, $R_f=R^+$

Ex) y=1x1 

2) y=x+1 

3) y= $\frac{1}{x}$ 

4) y=-x 

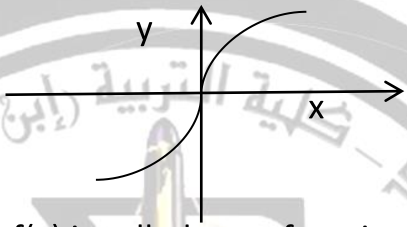
5) y= $\sqrt{x}$ 

Odd function A function f(x) is called odd function if f(-x)=-f(x), for every x in the functions domain

Ex) Let $f(x)=x^3$

$F(-x)=(-x)^3=-x^3=-f(x)$

∴ f is odd function

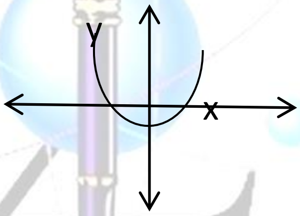


Even function A function f(x) is called even function if f(-x)=f(x) for every x in the functions domain

Ex) Let $f(x)=x^2$

$f(-x)= (-x)^2 =x^2= f(x)$

f is even function



Ex) Recognizing Even and functions

(1) $F(x)=(x-1)^2$

(2) $F(x)=x^2+1$

Rcmartc :-

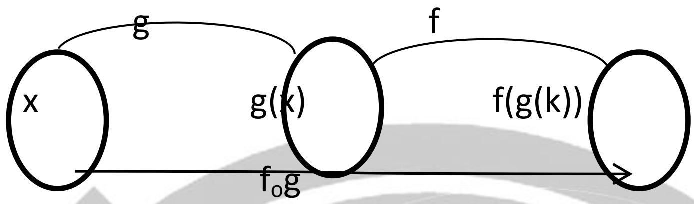
(1) The graph of an even function is asymmetric about the y-axis , appoint (x,y) lies on the graph if and only if the point (-x,y) lies on the graph .

(2) The graph of an odd function is symmetric about the origin appoint (x,y) lies on the graph if and only if the point (-x,-y) lies on the graph .

The composition of functions الدوال المركبة

Let f and g be two functions such that $g(x) \in D_f$ and $x \in D_g$, then there exist a function $f_o g$ which is defined as the following :-

$(f_o g)_{(x)} = f(g_{(x)})$ and $Dom_{(fog)} = \{x : x \in Dom_g \wedge g^{(x)} \in Dom_f\}$



(1) $R_g \subseteq D_f$
(2) $(f_o g)_{(x)} = f(g_{(x)})$
(3) $D\ f_o g(x) = \{x , x \in D_g \wedge g(x) \in D_f\}$

Also , we define go f as :-

$g_o f(x) = g(f(x))$

Dom $(g_o f) = \{ x = f(x) = \in Domg \wedge x \in Dom\ f \}$

In general the composition is not commutative $fog \neq g_o f$

Ex(1) Let $f(x) = \sqrt{x}$, g(x) , x^{1} , then find $f_o g$, $g_o f$?

Sol :- $f(x) = \sqrt{x}$, $D_f = R^+ [0, \infty)$

$y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow R_f = R^+ = [0, \infty)$

$y(x) = x^2 + 1$, $D_y = R$

$y = y(x) = x^2 + 1 \Rightarrow y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \mp\sqrt{y-1}$

$y - 1 \geq 0 \Rightarrow y \geq 1$, $R_g = [1, \infty)$

Now to find fog is $R_g \subseteq D_f$?

$[1, \infty) \subseteq [0, \infty)$

$\therefore$fog is exist

$(fog)\ (x) = f(g^{(x)}) = \sqrt{g(x)} = \sqrt{x2 + 1}$

$(f_o g)(x) = \sqrt{x2+1}$

$D_{fog} = \{x: x \in D_g \wedge g(x) \in D_f\}$

$= \{x= x \in R \wedge x^2+1 \in R^+\}$

$= \{x: x \in R \wedge x \in R\} = R$

But $x^2+1 \geq 0 \Longrightarrow x^2 \geq -1 \Longrightarrow x^2 \geq 0 \Longrightarrow x \in R$

To find $g_o f$ is $R_f \subseteq D_g$?

$R^+ \subseteq R \Longrightarrow \quad \therefore \quad g_o f$ exist

$(g_o f)(x) = g(f(a)) = (\sqrt{x})^2 + 1 = x+1$

$(g_o f)(x) = x+1$

$D\ g_o f = \{x: x \in D_f \wedge f(x) \in D_g\}$

$= \{x: x \in R^+ \wedge \sqrt{x} \in R\}$

$= \{ x: x \in R^+ \wedge x \in R^+\} - R^+$

$x \geq 0 \Longrightarrow x \in R^+$

<u>Ex(2)</u> let $f(x) = \sqrt{x-4}$, $g(x) = \frac{x+1}{3-x}$ then find $f_o g$, $g_o f$

<u>Sol :-</u>

$f(x) = \sqrt{x-4}$

$x-4 \geq 0 \Longrightarrow x \geq 4$, $\quad D_f = [4, \infty)$

$y = \sqrt{x-4}$, $y^2 = x-4 \Longrightarrow x = y^2+4 = R_f = R^+$

$g(x) = \frac{x+1}{3-x}$

$3-x \neq 0 \Longrightarrow x \neq 3$: $D_g = R \backslash \{3\}$

$y = \frac{x+1}{3-x} \Longrightarrow x+1 = y(3-x) \Longrightarrow x+1 = 3y = xy$

$x+xy = 3y-1 \Longrightarrow x = \frac{3y-1}{1+y}$

$1+y\neq 0 \Longrightarrow y\neq -1$, $R_g= R\backslash\{-1\}$

Now , to find f_og , is $R_g \subseteq D_f$?

$R\backslash\{-1\} \not\subseteq [4, \infty)$

$\therefore$fog is not exist

Now , to= find gof is $R_f \subseteq Dg$

$R^+ \not\subseteq R\backslash\{3\}$

$\therefore$gof is not exist

Homework:-

Find fog , gof to each of the following functions

(1) $F(x)=|x|$, $g(x)= \quad -x$
(2) If $f(t) = t^2$ find g ,h such that $(f_og)(x)=(f_oh)(x)=x^2-10x+25$

Limits :

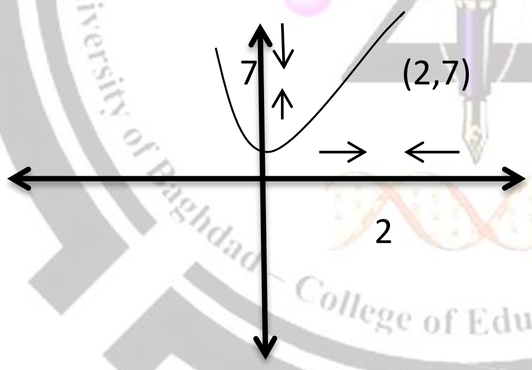
Definition : Let f(x) be defined on an open interval about x_0 , except possibly at x_0 itself If f(x) gets arbitrarily close to L for all x sufficiently close to x_0 , we say that f approaches the Limit L as x approaches x_0 , and we write .

$$\lim_{x \to x_0} f(x) = L$$

which is read "the Limit of f(x) as x approaches xo is L ", let f(x) be a function , we said that the limit of f(x) is L when x tend to x_0 , 0R $\underset{x \to x_0}{f(x)=L}$

Ex: Let $y=f(x) = x^2+ 3$, what happen when $x \to 2$

X	3	2.5	2.3	2.1	2.01	2.001	2.0001	At the right
F(x)	12	9.25	8.24	7.41	7.040	7.004	7.0004	
X	1	1.2	1.4	1.5	1.9	1.99	1.999	At the left
F(x)	4	4.44	4.96	5.25	5.98	6.96	6.999	



$L^+= \lim_{x \to^+ 2}(x^2+3)=7$ $\qquad D_f=R$

$L^- =\lim_{x \to^- 2}(x^2+3)=7$

$L^+ = L^-$

$L=\lim (x^2+3)=7$ limit exist

$x \to 2$

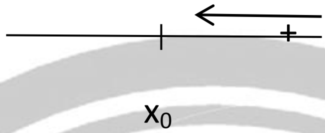
$F(2)=(2)^2+3=7$ function exist

<u>Note :-</u>

The limit of f(x) as x approaches xo from the right is :

$L^+=\lim f(x)$

$x \to x_0^+$

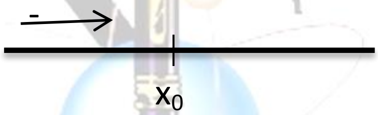


say that f(x) has right – hand limit L+

The limit of f(x) as x approaches x, from the left is :

$L=\lim f(x)$

$x \to x-$



say that f(x) has left – hand limit L-

function f(x) has a limit as approaches x_0 if and only f it has left – hand and right hand limits there and these one – sided limit are equat .

$\lim\limits_{x \to x0} f(x)=L \Leftrightarrow \lim\limits_{x \to x0+} f(x) = L$ and $\lim\limits_{x \to x0-} f(x) = L$

The approaches of finding limit :-   **حالات ايجاد الغاية**

To find limit we use some laws of limit

1) For any polynomial p(x) and any real number a , then $\lim\limits_{x \to a} p(x) = p(a)$, where polynomial is $p(x)=cnx^n+c_{n-1}x^{n-1}+\ldots+c_1x+co$

<u>Ex:</u> finding limits of $f(x)=x^2+x+5$ when x is approach n

$L=\lim\limits_{x \to 1} (x^2+x+s)=1+1+5=7$

$F(x) \to 7$ when $x \to 1$

2) Quotient

(a) The limit of a quotient of two functions is the quotient of the limits only when both limits exist and the limit in the denominator is not zero :

Ex: finding the limit of the function $\frac{x^3-5x+4}{x^2-2}$ when x is approach 3?

$$\lim_{x\to 3}\frac{x^2-5x+4}{x^2-2}=\frac{\lim(x^3-5x+4)}{\lim x^2-2}$$

$$=\frac{\lim x^3-5\lim x+\lim 4}{\lim x^2-\lim 2}=\frac{3^3-5x3+4}{3^2-2}=\frac{16}{7}$$

In this example we substitute in the limiting value of x

(b) Now in any case where the limits of both the numerator and denominator are o , we should try to algebraically simplify the expression to get cancellation .

Ex: Finding a limit $\lim_{x\to 1}\frac{x^2-1}{1-x}$

Sol : $\lim_{x\to 1}\frac{x^2-1}{1-x}\neq\frac{\lim x^2-1}{\lim 1-x}$

the limit in the denominator is zero can resolve this problem by factoring

$$lim_{x\to 1}\frac{x^2-1}{1-x}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{-\cancel{(x-1)}}=\lim_{x\to 1}\frac{x+1}{-1}=-2$$

Ex: finding a limit $\lim_{x\to o}\frac{\sqrt{x+2}-\sqrt{2}}{x}$

Sol: notice that both the numerator ($\sqrt{x+2}-\sqrt{2}$) and the denominator (x) approach o as x approaches o , we solve by rationalize the numerator $\frac{\sqrt{x+2}-\sqrt{2}}{x}=\frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})}=\frac{x+2-2}{x\,(\sqrt{x+2}+\sqrt{2})}$

$$=\frac{x}{x\,(\sqrt{x+2}+\sqrt{2})}=\frac{1}{(\sqrt{x+2}+\sqrt{2})}$$

Now

$$\lim_{x \to o} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x \to o} \frac{1}{(\sqrt{x+2}+\sqrt{2})} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(c) The limit of an nth root is the nth root of the limit

$$\lim_{x \to o} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to o} f(x)} = \sqrt[n]{\ell}$$

$$\begin{cases} \text{if } n \text{ is an odd positive integer} \\ \text{or} \\ \text{if } n \text{ is an even positive integer and } \ell > o \end{cases}$$

Ex: Evaluating the limit $\lim_{x \to 2} \sqrt[5]{3x2 - 2x}$

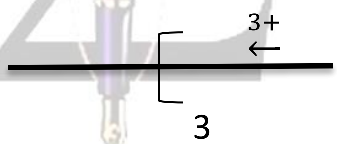
Sol : D_f = R

$$\lim_{x \to 2} \sqrt[5]{3x2 - 2x} = \sqrt[5]{\lim_{x \to 2}(3x2 - 2x)} = \sqrt[5]{8}$$

3) Find the limit of a function f(x)= $\sqrt{2x-6}$ when x if is the function contain (a) a square root :- in this case we must fine L+&L- and one of them not exist so the final limit not exist approach 3

Sol : 2x-6 ≥0 ⟹ 2x≥6⟹ x≥3

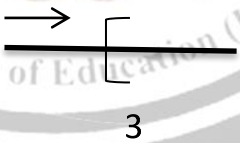
∴ Df = [3,∞)



Now , we find L+,L-

L+ = $\lim_{x \to 3+} \sqrt{2x-6} = \sqrt{6-6} = 0$

L- = $\lim_{x \to 3-} \sqrt{2x-6}$ not exist



∴ L= $\lim_{x \to 3} \sqrt{2x-6}$ not exist

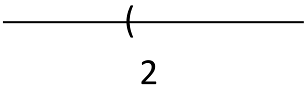
(b) odd root :- in this case the limit is exist since the domain is all real number

Ex:- Find the limit of function f(x) = $\frac{\sqrt{x-2}}{x-2}$ when x is approach 2

∴x-2≥0⟹ x≥ 2

x-2≠0⟹x≠2

∴ Df=(2, ∞)



L= $\lim\limits_{x\to 2} \frac{\sqrt{x-2}}{x-2}$ = $\lim\limits_{x\to 2} \frac{1}{\sqrt{x-2}}$

L+ = $\lim\limits_{x\to 2+} \frac{1}{\sqrt{x-2}}$ = + ∞

L- = $\lim\limits_{x\to 2-} \frac{1}{\sqrt{x-2}}$ not exist

∴ L = lim $\frac{1}{\sqrt{x-2}}$ not exist

Ex : find the limit of function f(x) =$\sqrt{2x-6}$ when x is approach 5

Sol : Df= [3, ∞)

3

5∈Df=[3, ∞)

L= $\lim\limits_{x\to 5} \sqrt{2x-6}$ = $\sqrt{10-6}$ = $\sqrt{4}$ = 2

Now , sometimes a function is described by using different for mulas on different parts of its domain and we illustrate such a function in the next example .

Ex: Evaluate $\lim\limits_{x\to 0}$ f(x) , where f is defined by

F(x) = $\begin{cases} x^2 + 2\cos x + 1 \, , for\ x < 0 \\ e^x - 4 \quad , for\ x \geq 0 \end{cases}$

Sol : since f is defined by different expressions for x $<$ 0 and for $x \geq 0$, we must consider one – sided limits we have

= $\lim\limits_{x\to 0-}$ f(x) = $\lim\limits_{x\to 0-}$ (x^2+2cos x +1) = 2cos 0+1 = 3

And

L+= $\lim\limits_{x\to 0+}$ f(x) = $\lim\limits_{x\to 0+}$ (e^x-4) = e^0 -4 = 1-4 = -3

Since the one – sided limits are different , we have that $\lim\limits_{x\to 0}$ f(x) does not exist .

Ex: Evaluate $\lim\limits_{y \to 0}$ f(y) , where f is defined by :

F(y) = $\begin{cases} y+2 & y \geq 0 \\ 2 & y < 0 \end{cases}$

Sol : since f is defined by different expressions for y <0 and y ≥0 , we must consider one – sided limits , we have

L+= $\lim\limits_{y \to 0+}$ f(y) = $\lim\limits_{y \to 0+}$ (y+2) = 0+2=2

L-= $\lim\limits_{y \to 0-}$ f(y) = $\lim\limits_{y \to 0-}$ 2=2

Since the one – sided limits are equal , we have that $\lim\limits_{y \to 0}$ f(y) exist .

Ex: Evaluate $\lim\limits_{y \to 1}$ f(y) , where f is defined in the above Example

L= $\lim\limits_{y \to 1}$ f(y) = $\lim\limits_{y \to 1}$ y+2 = 1+2=3

Note : The limit value does not depend on how the function is defined at x_0

<u>The limit Laws :</u>

The next theorem tells how to calculate limits of functions

Theorem (1) limit laws

If L, M, and k are real numbers and $\lim_{x\to xo}$ f(x)=L and $\lim_{x\to xo}$ g(x)=M , then

(1) Sum Rule : $\lim_{x\to xo}$ (f(x)+g(x))=L+M

The limit of the sum of two functions is the sum of their limits

(2) Difference Rule : $\lim_{x\to xo}$ (f(x)-g(x))=L-M

The limit of the difference of two functions is the difference of their limits

(3) Product Rule : $\lim_{x\to xo}$ (f(x)g(x))=L.M

The limit of a product of two functions is the product of their limits

(4) Constant multiple Rule : $\lim_{x\to xo}$ (k-f(x)) = k-L

The limit of a constant times a function is the constant times the limit of the function

(5) Quotient Rule : $\lim_{x\to xo} \frac{f(x)}{g(x)} = \frac{L}{M}$, M $\neq$ 0

The limit of a quotient of two functions is the quotient of their limits , provided the limit of the denominator is not zero

(6) Power Rule : If r and s are integers with no common factor and s≠0, then $\lim(f(x))^{rls}$= L^{rls} provided that L^{rls} is a real number (If s is even , we assume that L >0

Theorem (2) let p(x) is a polynomial function then $\lim_{x\to xo}$ p(x) = p(xo)

$\lim_{x\to xo}$ (c_o+c_ox+c_2x^2+n+c_nx^n) = c_o+ c_1x_o+c_2xo^2+n+c_nxo^n=p(xo)

Infinity limits :

$\lim_{x \to xo} f(x) = +\infty$ $\quad$ f(x) $\xrightarrow{\text{tends to}}$ $+\infty$

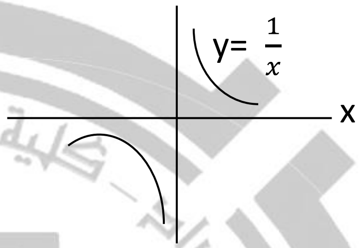
Or $\quad$ x $\xrightarrow{\text{approach to}}$ 0

$\lim_{x \to xo} f(x) = -\infty$

Ex(1) : find

$\lim_{x \to o} \frac{1}{x}$, Df = R\ {0}

L+ = $\lim_{x \to o+} \frac{1}{x} = \infty$, L- = $\lim_{x \to o-} \frac{1}{x} = -\infty$



$\therefore L^{+} \neq L^{-}$

L= $\lim_{x \to o} \frac{1}{x}$ does not exist

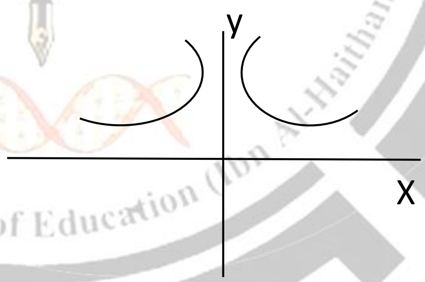
Ex(2): find

$\lim_{x \to o} \frac{1}{x^2}$, Df = R\ {0}

L+ = $\lim_{x \to o+} = \frac{1}{x^2} = \infty$

L- = $\lim_{x \to o-} = = \frac{1}{x^2} = \infty$ $\quad$ $\therefore L^{+} = L^{-}$

$\therefore$ L = $\lim_{x \to o} = \infty$



Ex: find $\lim_{x \to 1} \frac{-1}{(x-1)3}$

L+= $\lim_{x \to 1+} \frac{-1}{(x-1)3} = \frac{-1}{0+} = -\infty$

L-= $\lim_{x \to 1-} \frac{-1}{(x-1)3} = \frac{-1}{0-} = -\infty$

$\therefore \lim_{x \to 1} \frac{-1}{(x-1)3}$ not exist .

Finite limits as x → $\pm$ ∞ :-

$\lim\limits_{x\to+\infty} \frac{1}{x} = 0$, $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$

We say that f(x) has the limit L as x approaches infinity and write

$\lim\limits_{x\to\infty}$ f(x) = L

We say that f(x) has the limit L as x approaches minus infinity and write

$\lim\limits_{x\to-\infty}$ f(x) = L

Ex1) $\lim\limits_{x\to\infty} \frac{2x^2+1}{x+1}$

Sol : $\lim\limits_{x\to\infty} \frac{2\frac{x2}{x2}+\frac{1}{x2}}{\frac{x}{x2}+\frac{1}{x2}} = \lim\limits_{x\to\infty} \frac{2+\frac{1}{x2}}{\frac{1}{x}+\frac{1}{x2}} = \frac{lim2+lim\frac{1}{x2}}{lim\frac{1}{x}+lim\frac{1}{x2}}$

$\therefore \frac{2+0}{0+0} = \frac{2}{0} = \infty$

Ex2) $\boldsymbol{\lim\limits_{x\to\infty} \frac{-x}{7x+4}} = \lim\limits_{x\to\infty} \frac{-\frac{x}{x}}{7\frac{x}{x}+\frac{4}{x}} = \lim\limits_{x\to\infty} = \frac{-1}{7+\frac{4}{x}}$

$= \frac{-1}{7+0} = \frac{-1}{7}$

Ex3) $\lim\limits_{x\to\infty} \frac{5x+2}{2x3-1} = \lim\limits_{x\to\infty} \frac{\frac{5x}{x3}+\frac{2}{x3}}{\frac{2x3}{x3}-\frac{1}{x3}} = \lim\limits_{x\to\infty} \frac{\frac{5}{x2}+\frac{2}{x3}}{2+\frac{1}{x3}}$

$= \frac{0+0}{2+0} = 0$

Note Important : limits at infinity have properties similar to those of finite limits

Exercises : find the limit if it exist :

1) $\lim\limits_{n\to\infty} (\sqrt{n^2+1} - n)$

2) $\lim\limits_{x\to\infty} \frac{1}{\sqrt{x}}$

3) $\lim\limits_{x\to\infty} \sqrt{\frac{1}{x}}$

4) $\lim\limits_{x\to\infty} \frac{x^2+2x+3}{x^4+4x+4}$

5) $\lim\limits_{x\to o} \frac{1}{x3}$

6) $\lim\limits_{x\to\infty} \frac{x^3-2x-5}{3x^2+5x+1}$

7) $\lim\limits_{x\to 3} \frac{4}{(x-3)^4}$

8) $\lim\limits_{x\to\infty} \left(\frac{2x^2+3}{x^3+x-5} \text{ sgn } (x) \right)$

9) $\lim\limits_{x\to\infty} \frac{\sqrt{x}+2}{x+3}$

10) $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^2+10} \right)$

11) $\lim\limits_{x\to 3} \frac{[x]^2-9}{x^2-4}$

12) $\lim\limits_{x\to -4} |x+4|$

13) $\lim\limits_{x\to 5} \frac{1}{(x-5)^3}$

14) $\lim\limits_{x\to -1} \frac{2}{\sqrt{x+1}}$

15) $\lim\limits_{x\to\infty} \frac{1}{(x-2)^2}$

16) $\lim\limits_{x\to 0} \frac{1}{x^4+x^2+2}$

17) $\lim\limits_{x\to 3} \frac{x^2-16}{x^2-9}$

18) $\lim\limits_{x\to 0} f(x)$

$$f(x) = \begin{cases} x^2+7 & x \le 0 \\ x-4 & x > 0 \end{cases}$$

19) $\lim\limits_{x\to\infty} \frac{1+x}{1-x}$

Continuity

Let f be a function , then we say that f is continuous at x_0 if :-

(1) $f(x_0)$ is exist

(2) $\lim\limits_{x \to x0}$ f(x) is exist

(3) $\lim\limits_{x \to x0}$ f(x) = $f(_{x0})$

<u>Ex)</u> f(x)= x^2 if the function is continuous at x=2 ?

<u>Sol</u> : Df = R

(1) F(2) = 2^2 = 4

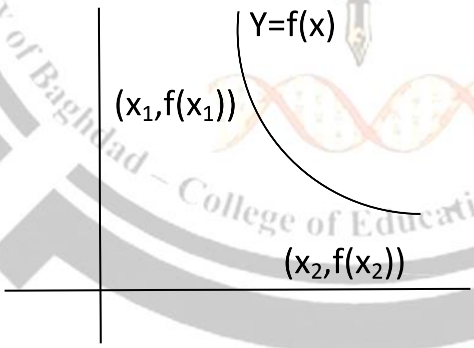
(2) $\lim\limits_{x \to 2}$ f(x) = $\lim\limits_{x \to 2} x^2 = 2^2 = 4$

(3) $\lim\limits_{x \to 2} x^2 = f(2) = 2^2 = 4$

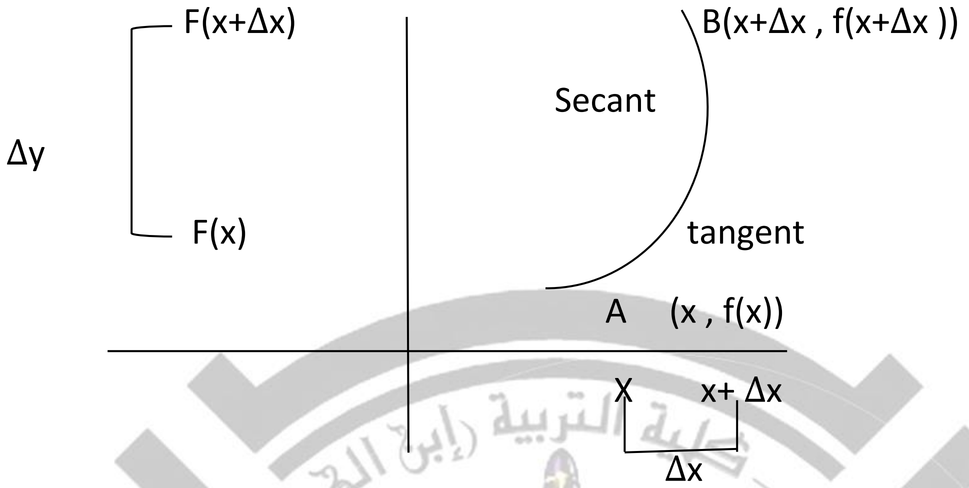
∴ f is cont at x = 2

Differentiation

Each point on the curve y = f(x) there is a single straight tangent at that point , the slope of straight tangent of the curve y=f(x) at the point (x,f(x)) it represents a derivative at that point .



Let A (x,f(x))be a fixed point on the curve , and B (x+Δx,f(x+Δx)) is a nother point therefore Δy=f(x+Δx)-f(x)



Note : that at Δx decreasing length (close to zero) the straight secant AB more and more a pliability begins on the straight tangent at the point (x,f(x)) , this means that slop straight secant AB be equal to slop straight tangent at the point (x,f(x)) , that's when (Δx→o) , knowing that slop straight tangent at the point (x,f(x)) represents a derived function at that point .

$$m_{tan} = \lim_{\Delta x \to o} m_{sec} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}$$

$$\boxed{\acute{f}(x) = \lim_{\Delta x \to o} \frac{f(x+\Delta x)-f(x)}{\Delta x}}$$

$\acute{y}, \acute{f}(x), \frac{dy}{dx}, \frac{df(x)}{dx}$

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}}$$

Note : when the value of the limit exist then the function called differentiable function , and f called the derivative of f at x

Ex(1) let f(x) = 4x -2 , find $\acute{f}$(x) by definition

Sol : $\acute{y}$= $\acute{f}$(x) = $\lim\limits_{\Delta x \to o} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

f(x)= 4x-2 , f(x+ Δx) = 4(x+Δx) -2

$\acute{f}$(x) = $\lim\limits_{\Delta x \to o} \frac{4(x+\Delta x)-2-[4x-2]}{\Delta x}$

= $\lim\limits_{\Delta x \to o} \frac{\cancel{4x}+4\Delta x-2-\cancel{4x}+2}{\Delta x}$

= $\lim\limits_{\Delta x \to o} \frac{4\cancel{\Delta x}}{\cancel{\Delta x}}$

= $\lim\limits_{\Delta x \to o}$ 4 = 4

Ex(2) let f(x) = $\sqrt{x}$, find the equation of the tangent line and normal line at the point (4,2) by definition .

Sol : $m_{tan}\Big]_{(4,2)}$ f(x) $\Big]_{(4,2)}$

$\acute{f}$(x) = $\lim\limits_{\Delta x \to o} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$

= $\lim\limits_{\Delta x \to o} (\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$

= $\lim\limits_{\Delta x \to o} \frac{\cancel{x}+\Delta x-\cancel{x}}{\Delta x\ (\sqrt{x+\Delta x}+\sqrt{x}}$

= $\lim\limits_{\substack{\Delta x \to o \\ \Delta x \neq o}} \frac{\cancel{\Delta x}}{\cancel{\Delta x}(\sqrt{x+\Delta x}+\sqrt{x}}$

= $\lim\limits_{\Delta x \to o} \frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$

= $\frac{1}{\sqrt{x}+\sqrt{x}}$ = $\frac{1}{2\sqrt{x}}$

$m_{tan}\Big]_{(4,2)}$= $\frac{1}{2\sqrt{4}}$ = $\frac{1}{4}$

(y-y_1) = m_{tan} (x-x_1)

y-2=$\frac{1}{4}$ (x-4)

$y = \frac{1}{4}x + 1$

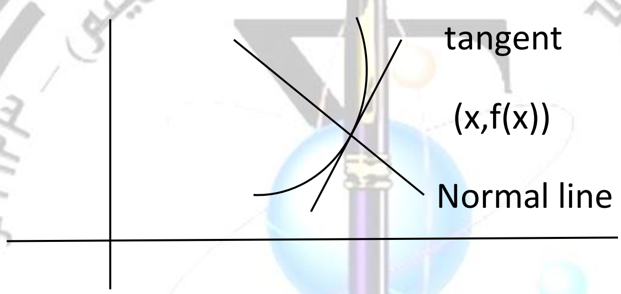
$m_{\perp} = \frac{-1}{m_{tan}} = \frac{-1}{\frac{1}{4}} = -4$

$(y - y_1) = m_{\perp}(x - x_1)$

$y-2=-4(x-4)$

$y=-4x+18$

Definition : the normal line to a curve is the line that is perpendicular to the tangent of the curve at a particular point



<u>Exc:</u> find $\acute{f}(x)$ by definition :

1- f(x) = x3
2- f(x) = x2 + $\frac{1}{x}$
3- Let f(x) = x2 , find the equation of the tangent line and normal line at the point (3,9) by definition
4- Using definition to prove that $\acute{f}(x) = m$ for $f(x) = y = mx + b$
5- Find the tangent line at (6,3) for $y = \sqrt{x+3}$

<u>Theorem :</u> Every function is differentiable at x, then _____ f is continuous at x_o .

<u>Proof :</u> to prove $\lim_{x \to xo} f(x) \to f(x_o)$

<u>i.e</u> $\lim_{x \to xo} [f(x) - f(x_o)] \to 0$

suppose that Δx = x-x_o ⇒ x= $x_o + \Delta x$

f(x) = f($x_o + \Delta x$)

when x ⟶ x_o , then $\Delta x \longrightarrow o$

$\lim\limits_{x \longrightarrow xo}$ [f(x) – f(x_o)] = $\lim\limits_{x \longrightarrow xo}$ [f(x_o + Δx) – f (x_o)]

= $\lim\limits_{\Delta x \longrightarrow o}$ [$\frac{f\ (\ x_o + \Delta x - f(x_o)}{\Delta x}$. Δx]

= $\lim\limits_{\Delta x \longrightarrow o} \frac{f(x_o+\Delta x)-f(x_o)}{\Delta x} \quad \lim\limits_{\Delta x \longrightarrow o} \Delta x$

= f(x_o) . 0

= 0

Note : The inverse of the above theorem is not true ____ if this function f continuous at the point , it is not necessary to be differentiable at that point as in the example :

Let f(x) = |x| , x_0 = 0

F(x)= $\lim\limits_{\Delta x \longrightarrow o} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ |x|= $\begin{cases} x & if \ \ x \geq 0 \\ -x & if \ \ x < 0 \end{cases}$

F(x)= $\lim\limits_{\Delta x \longrightarrow o} \frac{|x+\Delta x|-|x|}{\Delta x}$ |Δx |= $\begin{cases} \Delta x & if \ \ \Delta x \geq 0 \\ -\Delta x & if \ \ \Delta x < 0 \end{cases}$

F(0) = $\lim\limits_{\Delta x \longrightarrow o} \frac{|0+\Delta x|-|0|}{\Delta x}$ = $\lim\limits_{\Delta x \longrightarrow o} \frac{|\Delta x|}{\Delta x}$

L+ = $\lim\limits_{\Delta x \longrightarrow o+} \frac{\Delta x}{\Delta x}$ = 1

L- = $\lim\limits_{\Delta x \longrightarrow o-} \frac{-\Delta x}{\Delta x}$ = -1

L+ ≠ L- ⇒ limit is not exist

∴ f is not differentiable function at xo=o

(7) ($\frac{f}{g}$) (x) = $\frac{\acute{f}^{(x)}\ g(x)-f(x).\acute{g}^{(x)}}{(\acute{g}_{(x)})2}$, g(x) ≠ 0

Ex :- f(x) = $\frac{x+1}{x}$, find $\acute{f}$(x)

Sol : $\acute{f}$(x) = $\frac{x.1-(x+1).1}{x^2}$ = $\frac{x-x-1}{x^2}$ = $\frac{-1}{x^2}$

Corollary

Let y= f(x) = x^{-n} , for n$\in \mathbb{Z}_+, x \neq 0$, then y= f(x) = $\frac{dx^{-n}}{dx}$ = $-nx^{-n-1}$

Ex: let f(x) = -5 x^{-3} , find f(x)

F(x) = -5 (-3) x^{-3-1} = 15x^{-4}

$\frac{d}{dx}(x^r) = rx^{r-1} \forall r \in R$

(8) h(x) = f($g^{(x)}$) . g(x) , h= f_0g (x)

h=f_0g(x)=f($g^{(x)}$)

$\acute{h} = (f_0g)\acute{}^{(x)} =$

$\acute{f}\left(g^{(x)}\right).\acute{g}^{(x)}$

Corollary :-

Let y= $(f(x))^n$, n∈Z

$\frac{dy}{dx}$ = n $(f(x))^{n-1}$. f(x)

Derivation properties :-

(1) F(x) = c , c is constant , then $\acute{f}$(x) = 0

Ex : let f(x) = -5 , $\acute{f}$(x) = 0

(2) (c.f)́ (x) = c.$\acute{f}$(x) , c is constant

Ex: let f(x) = 3x ⇒ $\acute{f}$(x) = 3

(3) (f+g)́ (x) =$\acute{f}$ (x) + $\acute{g}$(x)

Ex: let f(x) = 2x , g(x) = 1

Sol: (f+g)́ (x) =$\acute{f}$ (x) + $\acute{g}$ (x) = (2x)́+(1)́=2(1)+0=2

Rem: $(f_1 \mp f_2 \mp f_3 \mp \ldots \mp f_n)\acute{}(x) = \acute{f}_1(x) \mp \acute{f}_2(x) \mp \ldots + \acute{f}_n(x)$

(4) $\acute{f}$ (x)= nx^{n-1} , n is positive integer

Ex: f(x) = x^6 ⇒ $\acute{f}$ (x)=$6x^{6-1}$=$6x^5$

(5) (f.g)́ (x): f(x). $\acute{g}$ (x)+g(x). $\acute{f}$ (x)

Ex: f(x) = (x^3+2) (1́ −x2) , find $\acute{f}$ (x)

Sol: $\acute{f}$ (x)=(x^3+2) $\frac{d}{dx}$ (1-x^2) + (1-x^2) $\frac{d}{dx}$ (x^3+2)

$= (x^3+2) . (-2x) + (1-x^3) . (3x^2)$

Rem : $(f.g.h)'(x) = f(x) . g(x) . h'(x) + f(x) . h(x) . g'(x) + g(x) . h(x) . f'(x)$

Ex(2) : let $y=t^2-1$, $x= 2t+3$, find $\frac{dy}{dx}$ at t=1

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{2t}{2}$

$= t$ $\qquad\qquad \left.\frac{dy}{dx}\right]_{t=1}$

$= 1$

Exc :

1- Let $y= x^3 - 3x^2 + 5x-4$, $x=t^2+t$, find $\frac{dy}{dx}$

Find $\frac{dy}{dx}$:

2- Let $y= u^3+1$, $u=x^2+3$
3- $Y=3t^2-1$, $x=6t-1$
4- $Y=\frac{t^2}{1+t}$, $x=\frac{t}{2+t}$
5- $Y=t^2$, $x=\frac{t}{1-t}$
6- $Y=Z^{2/3}$, $Z = x^2+1$
7- $Y= w^2-w^{-1}$, $w=3x$
8- $Y=2v^3+\frac{2}{v3}$, $v=(2x+2)^{2/3}$
9- $Y = \frac{u^2}{u^2+1}$, $u= \sqrt{2x + 1}$

Chain Rule

Let $y=f(x)$, $x=g(t)$, find $\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Let $y=f(t)$, $t=g(x)$, find $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Let y=f(t) , x=g(t) , find $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Ex(1): let y=3x-1 , x=2t , find $\frac{dy}{dt}$

Sol : $\frac{dy}{dt} = \frac{dy}{dx} - \frac{dx}{dt}$

= (3) . (2)

= 6

او y = 3x-1

= 3 (2t) – 1

Y= 6t – 1

$\frac{dy}{dt}$ = 6

Implicit differentiation الاشتقاق الضمنى

Ex : x2 + xy + y5 = 0 نشتق ضمنيا بالنسبة لـ x باعتبار y دالة ضمنية لــــx

$2x\frac{dx}{dx} + (x\frac{dy}{dx} + y\frac{dx}{dx}) + 5y^4\frac{dy}{dx} = 0$

2x+xy-+y+5y4y-=0

$Xy^{-}+5y^4y^{-}=-2x-y$

$(x+5y^4)\, y^{-} = -2x-y$

$y^{-}=\frac{-2x-y}{x+5y4}$

نشتق ضمنيا بالنسبة لـ y (بأعتبار x دالة ضمنية لــ y)

$2x\frac{dx}{dy} + (x\frac{dy}{dy} + y\frac{dx}{dy}) + 5y^4\frac{dy}{dy} = 0$

$2x\text{-}x^{-}\text{+x +y } x^{-}+5y^4=0$

$X+5y^4=-2x\, x^{-} \text{ -y } x^{-}$

$X+ 5y^4 = (-2x-y)\, x^{-}$

$x^{-} = \frac{x+5y^4}{-2x-y}$

$x^{-}= \frac{1}{y^{-}}$

<u>Ex :</u> find the equation of the tangent line and normal line of the curve x2+y2=2 at the point (1,1)

<u>Sol:</u> $2x\frac{dx}{dx} + 2y\,\frac{dy}{dx} = 0$

$2x+ 2y\, y^{-} = 0$

$2y\, y^{-} = -2x$

$y^{-} = \frac{-2x}{2y} = \frac{-x}{y}$

$y- = m]_{(1,1)} = \frac{-1}{1} = 1$

$(y-y_1) = m(x-x_1)$

y-1 = -1 (x-1)

y-1=-x+1

y= -x+2   the equation of the tangent line

$m\perp\ ]_{(1,1)} =\frac{-1}{m]_{(1,1)}}$

$m\perp\ ]_{(1,1)} =\frac{-1}{-1} =1$

(y-y1) = m (x- x1)

y-1 = 1 (x-1)

y=x the equation of the normal line

<u>Higher - order derivatives</u> المشتقات من مرتبة عليا

Let y= f(x)

$\bar{f}(x)= \lim\limits_{\Delta x \to o} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{dy}{dx} = \bar{y} = y^{(1)}$ المشتقة الاولى للدالة

$\bar{\bar{f}}(x) = \lim\limits_{\Delta x \to o} \frac{\bar{f}(x+\Delta x)-\bar{f}(x)}{\Delta x} = \frac{d^2y}{dx^2} = \bar{\bar{y}} = y^{(2)}$ المشتقة الثانية للدالة

$\bar{\bar{\bar{f}}}(x) = \lim\limits_{\Delta x \to o} \frac{\bar{\bar{f}}(x+\Delta x)-\bar{\bar{f}}(x)}{\Delta x} = \frac{d^3y}{dx^3} = \bar{\bar{\bar{y}}} = y^{(3)}$ المشتقة الثالثة للدالة

$f^{(n)}(x) = \lim\limits_{\Delta x \to o} \frac{f^{(n-1)}(x+\Delta x)-f^{(n-1)}(x)}{\Delta x} = \frac{d^n y}{dx^n} = y^{(n)} , n \in N$ المشتقة النونية للدالة

<u>ملاحظة :</u>

$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx}\right)$ second derivative

$\frac{d^3y}{dx^3} = \frac{d}{dx} \left(\frac{d^2y}{dx^2}\right)$ third derivative

<u>Ex :</u> let $y= 2x^3+x^2-1$

$y^{-}=6x^2+2x$

$y^{=}=12x+2$

$y^{\equiv}= 12$

$y^{(4)} = 0 , \ldots$

Chapter three

Applications of Differentiations Graphing function رسم المخطط البياني للدالة

Ex(1) : let f(x) = $\sqrt[3]{x} = x^{\frac{1}{3}}$

D_f=R   لا توجد محاذيات شاقوليه وافقية

x=0 ⇒ y=0

y=0 ⇒ $\sqrt[3]{x}$ = 0   بالتكعيب}⇐ نقطة التقاطع مع محورين (0,0)

⇒ x=0

f(-x) = $\sqrt[3]{-x}$ = -$\sqrt[3]{x}$ = -f(x) المنحني متناظر مع نقطة الاصل

f(-x) ≠ f(x)   المنحني غير متناظر مع المحور y

$\bar{f}(x) = \frac{1}{3}\, x^{\frac{-2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$ +++++ غير معرف +++++   اشارة $\bar{f}$

$\bar{f}$=0⇒ 1=0   وهذا غير ممكن   x< 0   x=0   x> 0

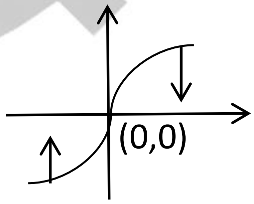
$\bar{\bar{f}}(x) = \frac{1}{3} \cdot \frac{-2}{3} \cdot x^{\frac{-5}{3}}$

$= \frac{-2}{9}\ \frac{1}{\sqrt[3]{x^5}}$

$= \frac{-2}{9x\ \sqrt[3]{x^2}}$ ++++ غير معروف - - - -   اشارة $\bar{\bar{f}}$

$\bar{\bar{f}}$=0⇒ -2= 0   x=0   (0,∞)

R_f=R



Ex(2): f(x) = $\frac{1}{x^2-1}$

$x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow \sqrt{x^2} = \sqrt{1} \Rightarrow |x| = 1 \Rightarrow x = \mp 1$

D_f=R|{1,-1}   المحاذيات الشاقولية

X=1,x=-1

$\frac{y}{1} = \frac{1}{x^2-1}$

$1 = yx^2 - y$

$1 + y = yx^2$

$x^2 = \frac{1+y}{y}$

$\sqrt{x^2} = \sqrt{\frac{1+y}{y}}$

$|x| = \sqrt{\frac{1+y}{y}}$

$x = \mp\sqrt{\frac{1+y}{y}}$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x^2-1}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}}$$

$$= \frac{\left(\lim\limits_{x\to\infty}\frac{1}{x}\right)2}{\lim\limits_{x\to\infty} 1-\left(\lim\limits_{x\to\infty}\frac{1}{x}\right)2}$$

$$= \frac{(0)^2}{1-(0)^2}$$

$$= \frac{o}{1}$$

$$= 0$$

y = 0 محاذي الافقي

If $x = 0 \Rightarrow y = \frac{1}{0^2-1} = -1$

If $y=0 \Rightarrow 0 = \frac{1}{x^2-1} \Rightarrow 0=1$

$f(-x) = \frac{1}{(-x)^2-1} = \frac{1}{x^2-1} = f(x)$

$f(-x) \neq -f(x)$

$\bar{f}(x) = \frac{0-2x}{(x^2-1)2}$

$\bar{f}(x) = \frac{-2x}{(x^2-1)2}$

$\bar{f}(x) = 0 \Rightarrow -2x = 0 \Rightarrow x=0$

++++ غير معرف +++ o - - - - غير معرف- - - - f^- اشارة

x=-1 x=0 x=1

$\bar{\bar{f}}(x)= \frac{(x^2-1)^2.-2+2x.2(x^2-1).2x}{(x^2-1)^4}$

$= \frac{-2\cancel{(x^2-1)}\{x^2-1-4x^2\}}{(x^2-1)^{\cancel{4}\,3}}$

$= \frac{-2\{-3x^2-1\}}{(x^2-1)^3}$

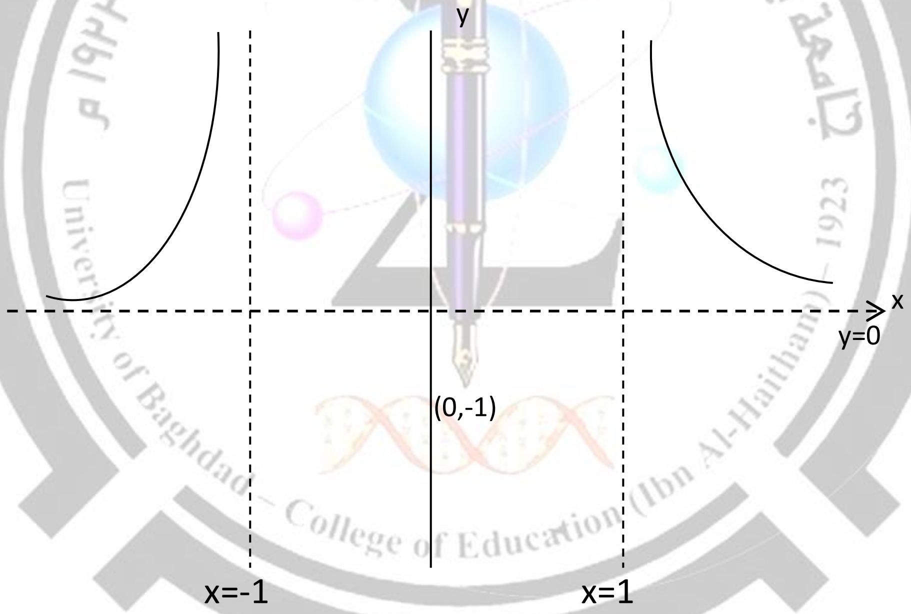
$f^{=}(x) = \frac{2(3x^2+1)}{(x^2-1)^3}$

$f^{=}(x) = 0 \Rightarrow 3x^2+1 = 0 \Rightarrow 3x^2 = -1 \Rightarrow x^2 = -\frac{1}{3}$

$\cup$ $\cap$ $\cup$

+++++ غير معرف - - - - - - غير معرف +++++ اشارة $f^{=}$

x=-1 0 x=1



$R_f=(-\infty, -1] \cup (0, \infty) - R \mid (1,0]$

Ex(3) : $y=|x^2-4|$

$$=\begin{cases} x^2-4 \quad if\ x^2-4>0 \Rightarrow x^2>4 \Rightarrow |x|>2 \Rightarrow x>2 \vee x<-2 \\ 0 \quad if\ x^2-4=0 \Rightarrow x^2=4 \Rightarrow |x|=2 \Longrightarrow x=2,-2 \\ -(x^2-4)\ if\ x^2-4<0 \Rightarrow x^2<4 \Rightarrow |x|<2 \Rightarrow 2<x<2 \end{cases}$$

$D_f = R$ لا يوجد محاذيات شاقولية وافقية

$x=0 \Longrightarrow y=|-4|=4$

y=0$\Longrightarrow$ $|x^2-4|$=0 $\Longrightarrow$ x=2 , -2

f(-x)= $|(-x)^2$-4$|$ = $|x^2-4|$ = f(x)

f(-x) $\neq$ -f(x)

$$y=\begin{cases} x^2-4 \;\; if \; x>2 \;\; \vee x<-2 \\ o \;\; if \;\; x=2,-2 \\ 4-x^2 \;\; if \;\; -2<x<2 \end{cases}$$

$$y^- = \begin{cases} 2x \quad if \quad x>2 \;\vee x<-2 \\ \text{غير معرف} \; if \;\; x=2,-2 \\ -2\,x \;\; if \;\; -2<x<2 \end{cases}$$

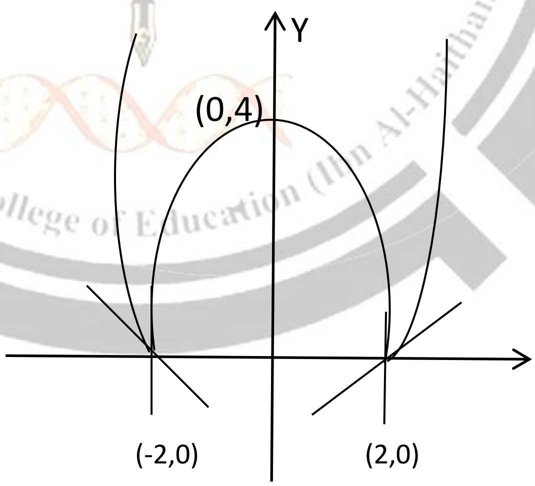
- - - غير معرف + + + o - - - غير معروف ++++ اشارة -y

-o o 2

$$y^{=} = \begin{cases} 2 \quad if \; x>2 \;\vee x<-2 \\ \text{غير معرف} \;\; if \; x=2,-2 \\ -2 \;\; if -2<x<2 \end{cases}$$

++++ غير معرف - - - - - غير معرف ++++ اشارة $y^{=}$

-2 o 2



R_f= R^+

<u>Ex(4) :</u> Graph of the following functions :-

1- y= $x^2 - 2x + 4$
2- y=$x^3 - 12x + 10$
3- y=x^5
4- y=$\frac{1}{x^2+3}$
5- y=$\frac{x}{x-1}$
6- y=$\frac{x^2}{x^2-1}$
7- y=$\frac{x^{-}+1}{x}$
8- y=$\frac{x+3}{x+2}$
9- y=$\sqrt{x^2 - 1}$
10- y=$x^4 - 1$
11- y=$\sqrt{x + 1}$
12- y=$\sqrt[3]{x^2 - 1}$

$R \setminus \{-1\} \nsubseteq [4, \infty)$
$\therefore f_0 g$ is not exist

Now , to find $g_0 f$ is $R_f \subseteq Dg$

$R^+ \nsubseteq R \setminus \{3\}$
$\therefore g_0 f$ is not exist

Homework :-

Find fog , gof to each the following functions

(1) f(x) = |x| , g(x) , -x
(2) If $f(t) = t^2$ find g, h such that $(f_0 g)(x)=(f_0 h)(x)=x^2-10x+25$

chapter four

The inverse trigonmetric functions معكوس الدوال المثلثية

(onto –&1-1) bijective f   اذا كانت الدالة متباينة

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
$\forall y \in y \quad \exists x \in x \quad \ni y=f(x)$
$\Leftrightarrow \therefore \exists \quad \begin{matrix} f: x \to y \ni \; y=f(x) \quad , f \; is \; 1-1 \; \& \; onto \\ f^{-1}: y \to x \ni \; x=f^{-1}(y) \end{matrix}$

معكوس دالةsin :

Let y= sin x

Sin : R → [-1,1]

Sin : $[-2\pi, 2\pi] \to [-1,1]$

سنعرف دالة جديدة هي معكوس الدالة sin ويرمز لها بالرمز sin-1 او arc sin

$\therefore \sin^{-1} y = \sin^{-1}(\sin x)$

$\sin^{-1} y = x$
$\therefore y = \sin x \Leftrightarrow x = \sin^{-1} y$

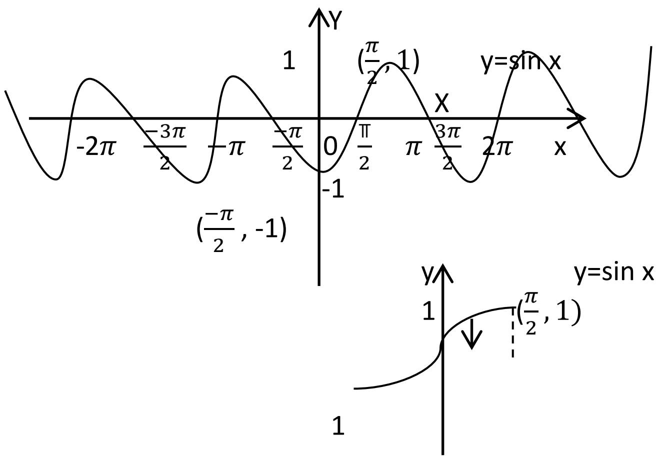
$\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1,1]$

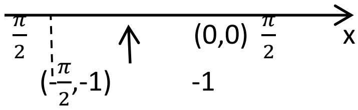
sin is 1-1 & onto $\therefore \exists \sin^{-1} \ni$

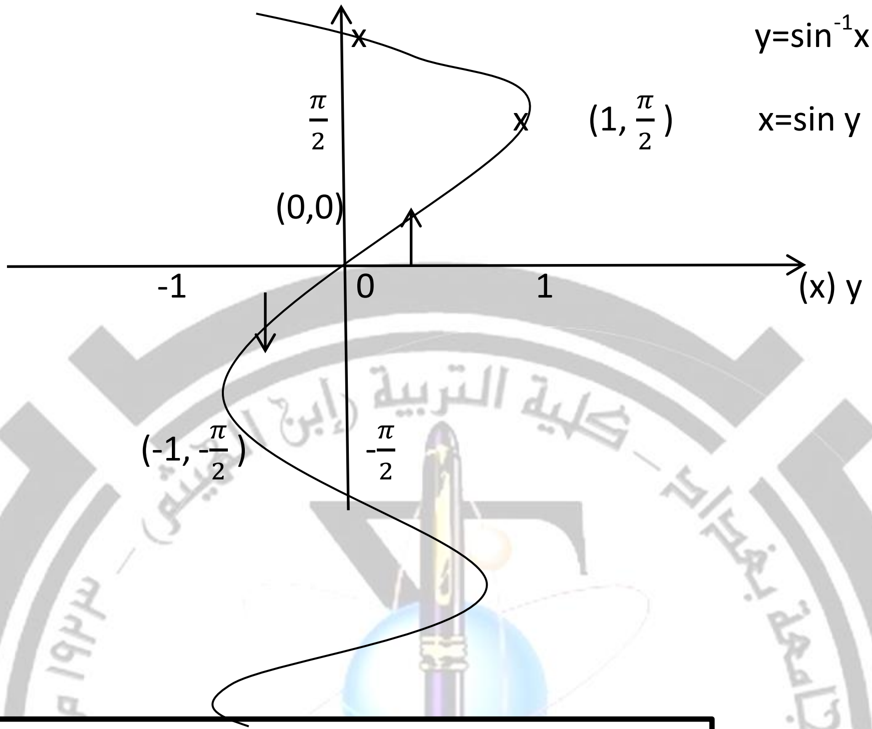
$\sin^{-1} : [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$D_{\sin} = [-\frac{\pi}{2}, \frac{\pi}{2}] = R_{\sin^{-1}}$

$R_{\sin} = [-1,1] = D_{\sin^{-1}}$







تعريف : $y=\sin x \Leftrightarrow x=\sin^{-1} y$ لكل $x\in[-\frac{\pi}{2} , \frac{\pi}{2}]$

ملاحظة : $\sin^{-1} y \neq \frac{1}{\sin y}$

دالة $\sin^{-1}$ هي دالة فردية حيث ان

$$\boxed{\sin^{-1}(-x)=-\sin^{-1}(x)}$$

البرهان :

Let $y=\sin^{-1}(-x) \Leftrightarrow \sin y = -x$ من تعريف المعكوس

$\Leftrightarrow x=-\sin y$

$\Leftrightarrow x=\sin (-y)$ لان دالة sin فردية

$\Leftrightarrow \sin^{-1} x = -y$

$\Leftrightarrow y= -\sin^{-1} (x)$

$\therefore \sin^{-1}(-x)= -\sin^{-1}(x)$

مثال :-

$\sin^{-1}(-1)=\sin^{-1}(1)$

$\sin^{-1}(-1)= -\frac{\pi}{2}$, $\sin^{-1}(1)= \frac{\pi}{2}$

<u>معكوس دالة (cos) :</u>

$Y= \cos x$

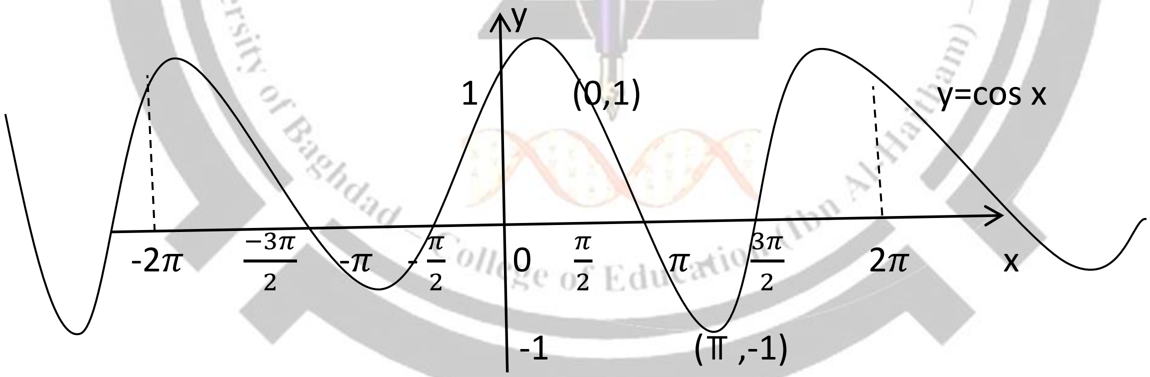
$\cos : R \longrightarrow [-1,1]$

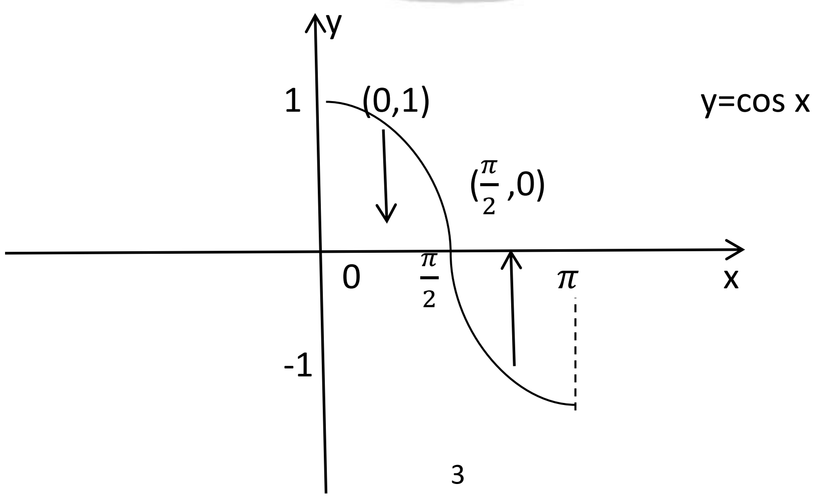
$\cos : [-2\pi , 2\pi] \longrightarrow [-1,1]$

$\cos : [0, \pi] \longrightarrow [-1,1]$

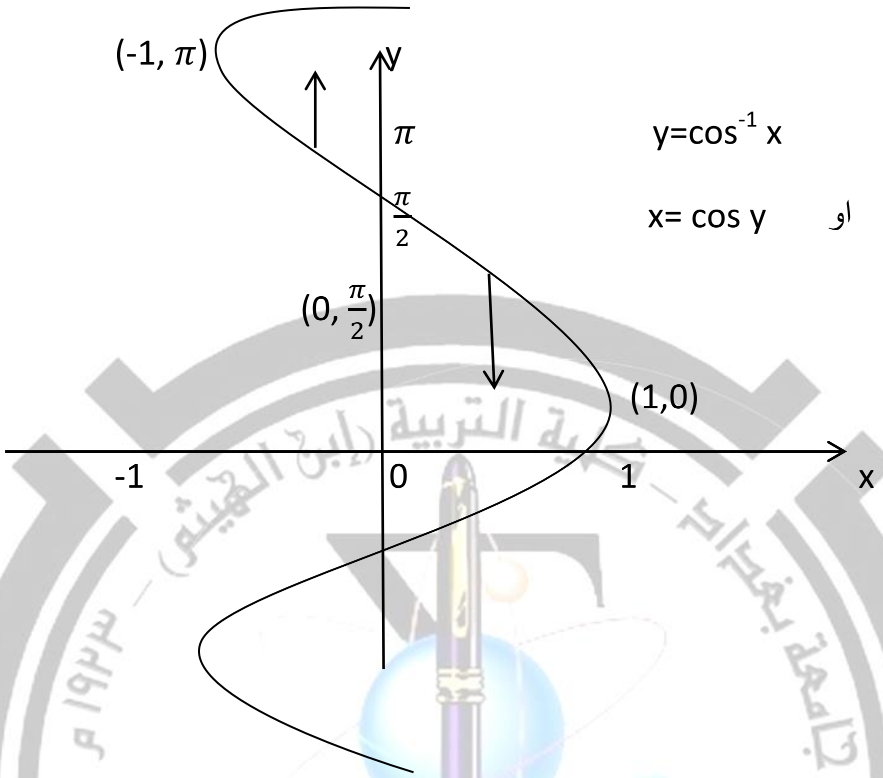
cos is 1-1 & onto $\therefore \exists \cos^{-1}$ أو Arc cos $\ni$

$\cos^{-1}: [-1,1] \longrightarrow [0, \pi]$





$(\pi\ ,-1)$



$x\in[0,\pi]$ لكل $\cos^{-1}(\cos x) = x$

$y\in[-1, 1]$ لكل $\cos(\cos^{-1} y) = y$

تعريف :- $y=\cos x \Leftrightarrow x= \cos^{-1} y$ لكل $x\in[0,\pi]$

ملاحظة : دالة $\cos^{-1}$ هي دالة ليست زوجية ولا فردية $\cos^{-1}(-x)= \pi - \cos^{-1} x$

البرهان : الطرف الايمن Let $y= \pi - \cos^{-1}(x)$

$\Rightarrow y-\pi = -\cos^{-1}(x)$

$\Rightarrow \cos^{-1}(x)= \pi - y$

$\Rightarrow x=\cos(\pi - y)$

$\Rightarrow x=-\cos y$

$\Rightarrow \cos y = -x$

$\Rightarrow y = \cos^{-1}(-x)$

$\Rightarrow y=$ الطرف الايسر

$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}(x)$

<u>طريقة اخرى</u> :-

Let $y= \cos^{-1}(-x)$

$\cos y=-x$ من تعريف المعكوس

$x=-\cos y$

$x= \cos(\pi -y)$

$\cos^{-1}x = \pi -y$

$\therefore y= \pi - \cos^{-1}(x)$

$\therefore \cos^{-1}(-x)= \pi -\cos^{-1}(x)$

<u>مثال</u> :- $\cos^{-1}(-1)= \pi \quad , \quad \cos^{-1}(1)=0$

نستنتج علاقة مهمة تجمع بين الدالتين $\cos^{-1}, \sin^{-1}$

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ لكل $x \in[-1,1]$

<u>برهن أن</u> :- $\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$ لكل $x \in[-1,1]$

<u>البرهان</u> :- لنفرض أن $w=\frac{\pi}{2} - \sin^{-1}(x)\ ,\ x \in[-1,1]$

$\sin^{-1}(x) = \frac{\pi}{2} -w$

$\sin(\sin^{-1}x)=\sin(\frac{\pi}{2} -w)$

$x=\sin(\frac{\pi}{2} -w)$

$x= \cos w$

$w \in D_{\cos}?\ \ w\in[0, \pi]\ ?$

وبما ان :- $-\frac{\pi}{2} \leq \sin^{-1}x\leq \frac{\pi}{2}$

$-\pi \le -\frac{\pi}{2} + \sin^{-1}x \le 0$ بأضافة $-\frac{\pi}{2}$ الى جميع الاطراف نضرب المتراجعة في -1

$\pi \ge \frac{\pi}{2} - \sin^{-1}x \ge 0$

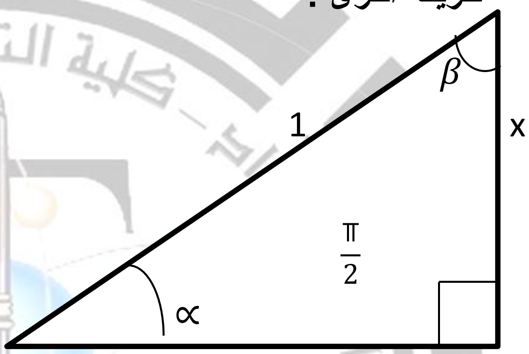
$0 \le \frac{\pi}{2} - \sin^{-1}x \le \pi$

$0 \le w \le \pi$

$\therefore\ w = \cos^{-1}(x)$ لكل $x \in [-1,1]$

$\therefore \cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$ لكل $x \in [-1,1]$

طريقة اخرى :-



$\sin \propto = x$, $\cos \beta = x$

$\therefore \propto = \sin^{-1}x$, $\beta = \cos^{-1} x$

$\propto + \beta = \frac{\pi}{2}$

$\beta = \frac{\pi}{2} = \propto$

$\therefore \cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$

لان مجموع زوايا المثلث 180

<u>معكوس دالة (tan)</u>

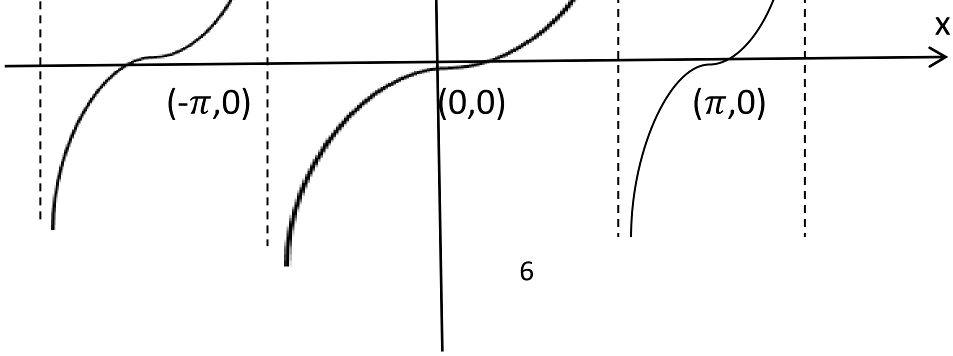
$y = \tan x$

$\tan : R \backslash \{x : x = \frac{\pi}{2} + n\pi, n \in I\} \longrightarrow R$

$\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow R$ محاذيات شاقولية $x = -\frac{\pi}{2}, x = \frac{\pi}{2}$

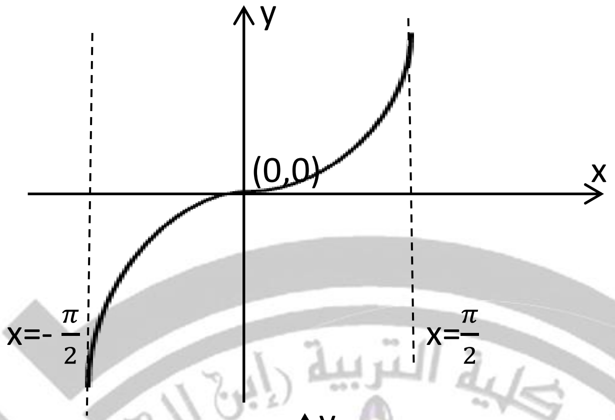
tan is 1-1 & onto $\therefore \exists \tan^{-1}$ او Arc tan $\exists$

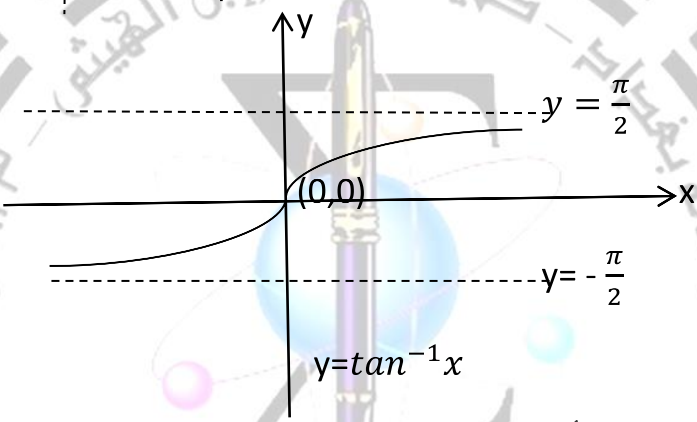
$\tan^{-1} : R \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ محاذيات افقية $y = -\frac{\pi}{2}, y = \frac{\pi}{2}$



x=- $\frac{3\pi}{2}$ x= - $\frac{\pi}{2}$ x = $\frac{\pi}{2}$ x=$\frac{3\pi}{2}$

y=tan x





$x = \tan y$ أو

تعريف : $y = \tan x \Leftrightarrow x = tan^{-1}y$ لكل $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

دالة tan-1 هي دالة فردية

$tan^{-1}(-x) = -tan^{-1}(x)$

البرهان : Let y=$tan^{-1}(-x)$

tan y=-x

X=-tan y

X=tan (-y)

-y = $tan^{-1}x$

Y= $-tan^{-1}x$

$\therefore tan^{-1}(-x) = -tan^{-1}(x)$

مثال : $tan^{-1}(1) = \frac{\pi}{4}$, $tan^{-1}(-1) = -\frac{\pi}{4}$

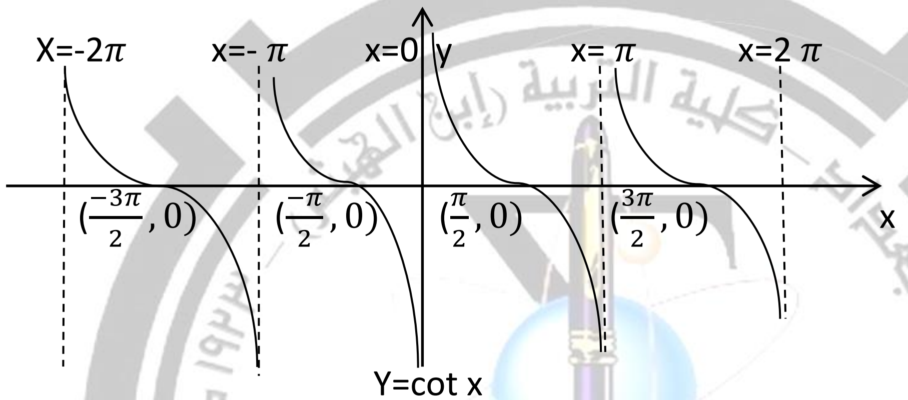
معكوس دالة (cot)

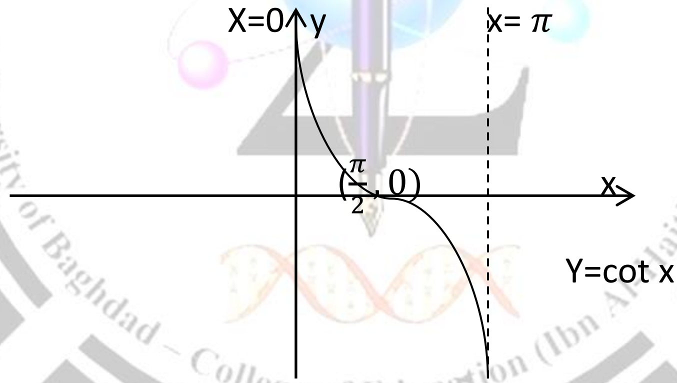
Y=cot x

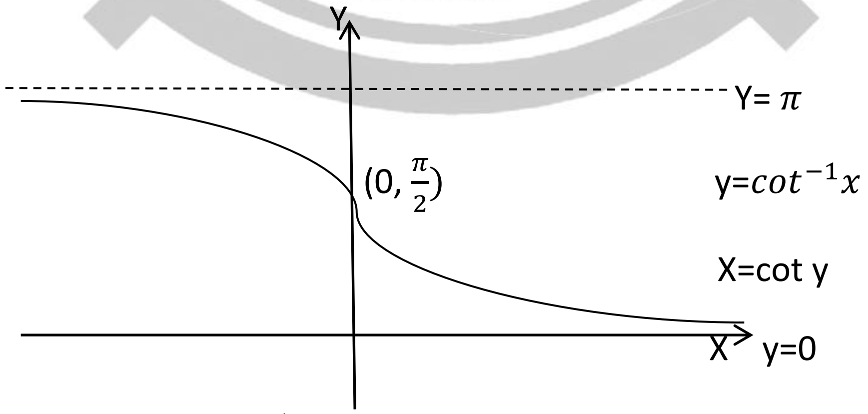
Cot : R\ {x: x= nπ , $n \in$ I} $\rightarrow R$

Cot: (0,π) $\rightarrow R$

Cot is 1-1 & onto $\therefore \exists cot^{-1}$ أو $Arc \cot \ni cot^{-1}: R \rightarrow (0,\pi)$







تعريف : $y = cot\ x \Leftrightarrow x = cot^{-1}y\ x \in (0,\pi)$لكل

$cot^{-1}x + tan^{-1}x = \frac{\pi}{2}$

برهن أن : $cot^{-1}(x) = \frac{\pi}{2} - tan^{-1}(x)$

البرهان : لتكن $w = \frac{\pi}{2} - tan^{-1}(x)$

$tan^{-1}(x) = \frac{\pi}{2} - w$

$x = \tan\left(\frac{\pi}{2} - w\right) = \frac{\sin(\pi/2 - w)}{\cos(\pi/2 - w)} = \frac{\cos w}{\sin w} = wtw$

X=cot w

$w \in D_{cot}$?

$w \in (0, \pi)$?

$0 < w < \pi$?

بما ان $\frac{-\pi}{2} < tan^{-1}x < \frac{\pi}{2}$

$\frac{\pi}{2} > -tan^{-1}x > -\frac{\pi}{2}$ $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \underset{-tan}{\overset{tan}{\rightleftarrows}} R$

$-\frac{\pi}{2} < -tan^{-1}x < \frac{\pi}{2}$ $(0,\pi) \underset{cot^{-1}}{\overset{cot}{\rightleftarrows}} R$

$0 < \frac{\pi}{2} - tan^{-1}x < \pi$ بأضافة $\frac{\pi}{2}$ لجميع اطراف المتراجحة

$0 < w < \pi$

$\therefore w \in (0, \pi)$

$w \in D_{cot}$

$\therefore w = cot^{-1}x$

$\therefore cot^{-1}x = \frac{\pi}{2} - tan^{-1}x$

طريقة اخرى : $cot^{-1}(x) = \frac{\pi}{2} - tan^{-1}(x)$

Let $tan^{-1}(x) = y$

$\therefore x = \tan y$

$\left(\frac{\pi}{2} - y\right) = Z = cot^{-1}(x)$

$\cot\left(\frac{\pi}{2} - y\right) = \cot Z$

$\tan y = \cot Z$

$\tan y = x$ لكن

$\therefore x = \cot Z$

$\therefore cot^{-1}(x) = \frac{\pi}{2} - tan^{-1}(x)$

ملاحظة : $tan^{-1}(x) \neq \frac{sin^{-1}(x)}{cos^{-1}(x)}$

$cot^{-1}(x) \neq \frac{cos^{-1}(x)}{sin^{-1}(x)}$

اثبت ان : $tan^{-1}\left(\frac{1}{x}\right) = cot^{-1}(x)$

البرهان : لتكن $\propto = tan^{-1}\left(\frac{1}{x}\right)$

$\tan \propto = \frac{1}{x}$ من تعريف المعكوس

$x = \frac{1}{\tan\propto}$

$x = \cot \propto$

$\propto = cot^{-1}(x)$

$\therefore tan^{-1}\left(\frac{1}{x}\right) = cot^{-1}(x)$

معكوس دالة (sec) :

$y = sec\, x$

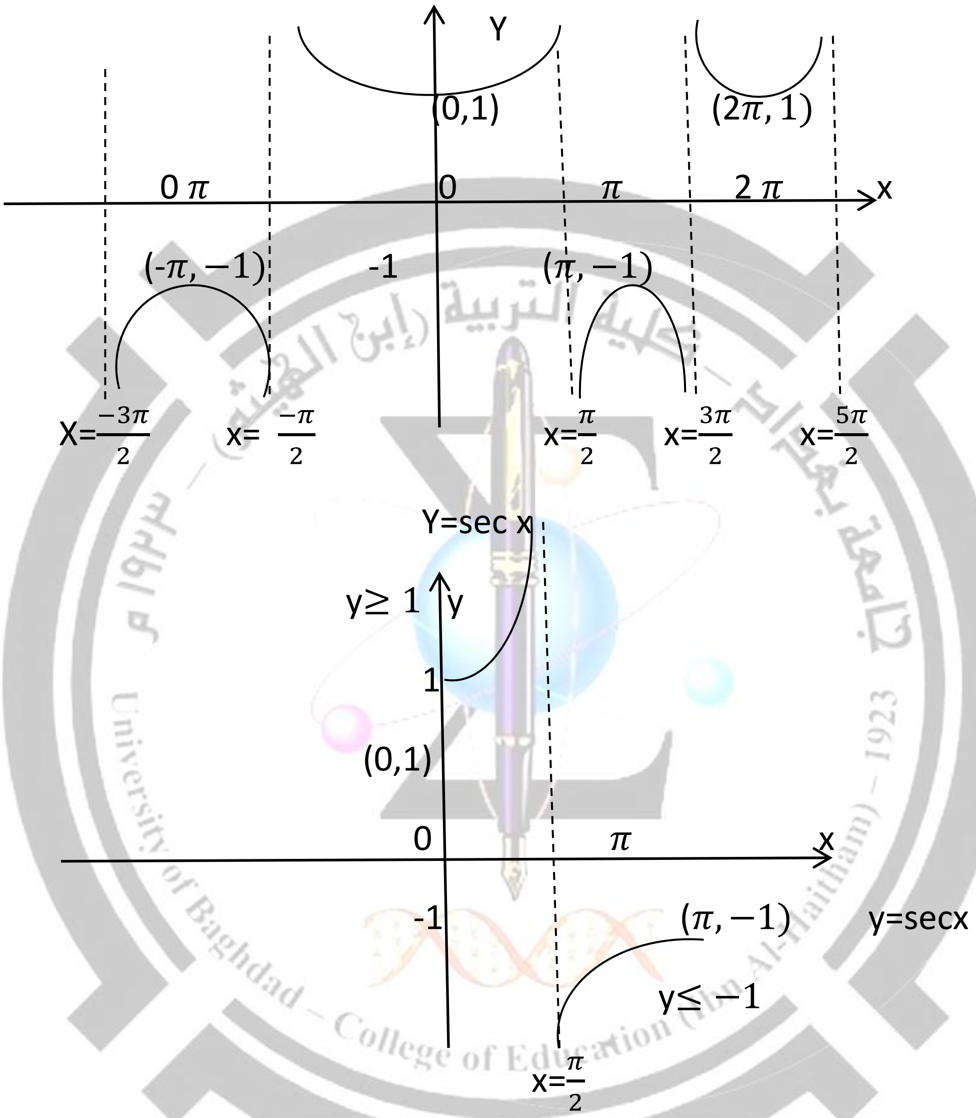
$sec : R \left\{x : x = \frac{\pi}{2} + n\,\pi, n \in \mathrm{I}\right\} \rightarrow |y| \geq 1$

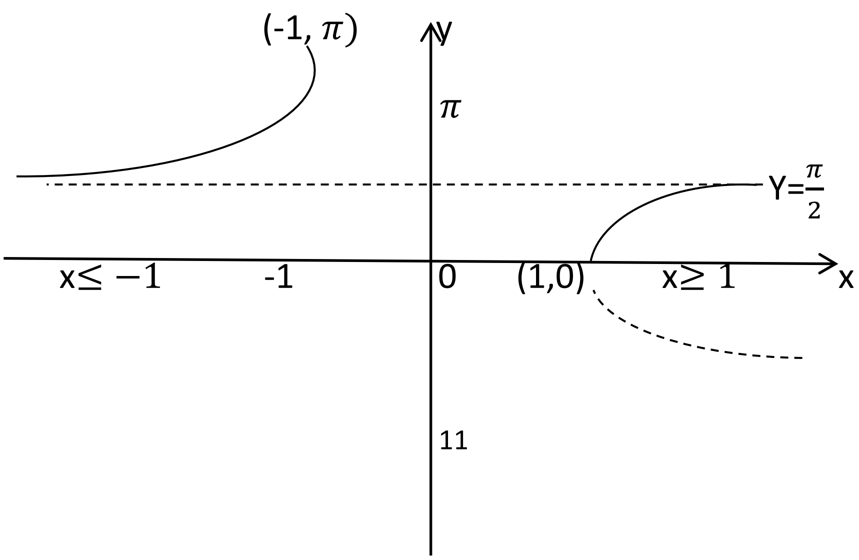
$sec : [0, \pi] \backslash \left\{\frac{\pi}{2}\right\} \rightarrow |y| \geq 1$

أو $y \geq 1 \vee y \leq -1$

أو R\ (-1,1)

Sec is 1-1 & onto $\therefore \exists sec^{-1}$ او $Arc \sec \ni \ sec^{-1}: |x| \geq 1 \rightarrow [0,\pi] \backslash \{\frac{\pi}{2}\}$





y=$sec^{-1}(x)$

أو x=sec y

تعريف: $sec^{-1}(x) = cos^{-1}(\frac{1}{x})$ لكل $|x| \geq 1$

Let y= $sec^{-1}(x)$

X=sec y

X=$\frac{1}{\cos y}$

Cos y = $\frac{1}{x}$

$\therefore y = cos^{-}(\frac{1}{x})$

$\therefore sec^{-1}(x) = cos^{-1}(\frac{1}{x})$

اثبت ان : $sec^{-1}(-x) = \pi - sec^{-1}(x)$

البرهان : Let $sec^{-1}(-x) = y$

$\therefore \sec y = -x$ من تعريف المعكوس

X= sec y

X=-cos y $\quad \cos(\pi - y) = -\cos y$

X= $\frac{1}{\cos(\pi - y)}$

X= sec $(\pi - y)$

$\pi - y = sec^{-1}(x)$

$\therefore y = \pi - sec^{-1}(x)$

$\therefore sec^{-1}(-x) = \pi - sec^{-1}(x)$

معكوس دالة (csc) :

Y=csc(x)

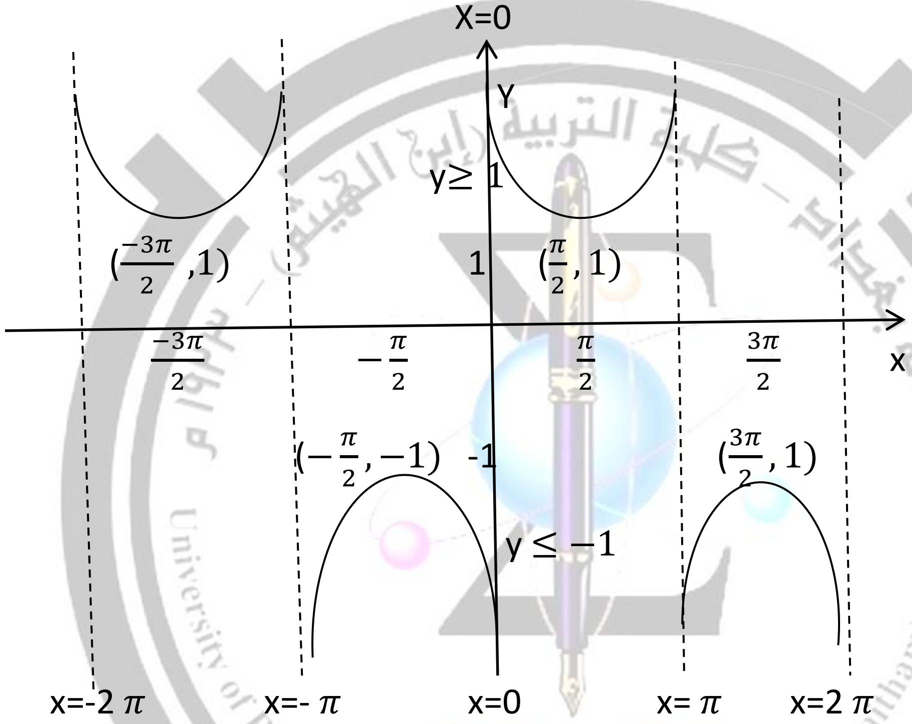
Csc:R\ { x: x= n $\pi, n \in \mathrm{I}\} \rightarrow |y| \geq 1$

Csc: [-$\frac{\pi}{2}$,$\frac{\pi}{2}$] \ {0} $\rightarrow |y| \geq 1$

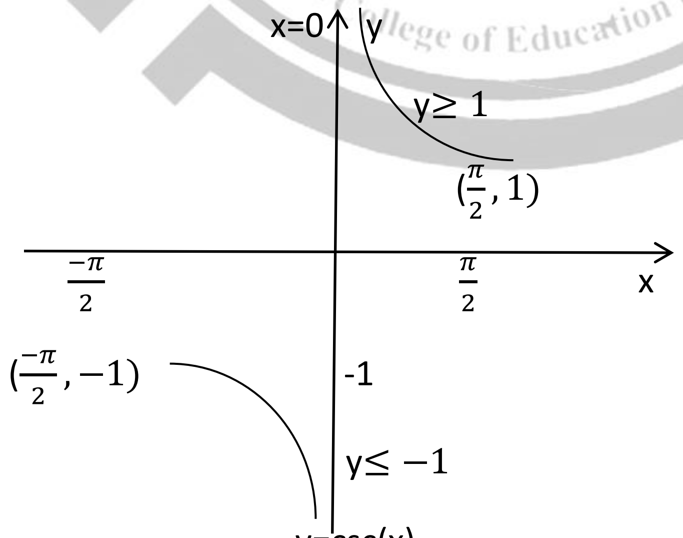
او y$\geq 1 \ \vee y \leq -1$

أو R\(-1,1)

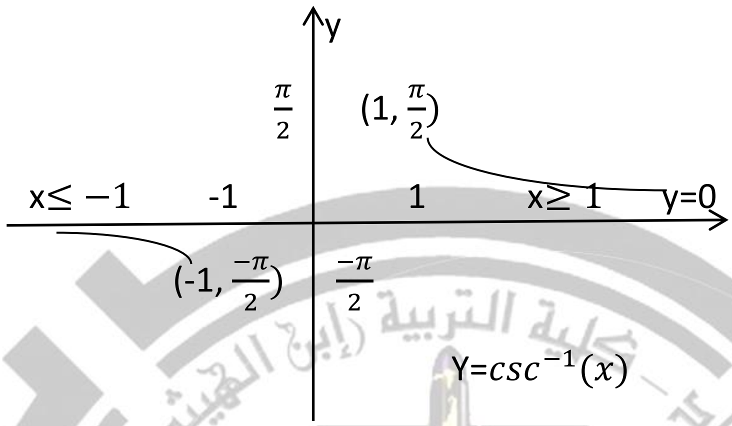
Csc is 1-1 & onto $\therefore \exists csc^{-1} \ni csc^{-1}: |x| \geq 1 \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ \ {0}



y=csc(x)



y=csc(x)



برهن على ان : لكل $|x| \geq 1$ $csc^{-1}(x) = sin^{-1}(\frac{1}{x})$

البرهان :

Let y= $csc^{-1}(x)$

Csc y =x

$\therefore x = \frac{1}{\sin y}$

Sin y = $\frac{1}{x}$

$\therefore y = sin^{-1}(\frac{1}{x})$

$\therefore csc^{-1}(x) = sin^{-1}(\frac{1}{x})$

بسط التعبير الاتي :-

$\sec \sin^{-1}(\frac{-2}{3})$

لتكن $\sigma = sin^{-1}\left(\frac{-2}{3}\right)$

$\sin \sigma = \frac{-2}{3}$

$\sin \sigma < 0$ *سالب في الربعين* (3) و (4)

$\sigma \in D_{sin}$

$\sigma \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\qquad$ $\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$

σ في الربعين (1) و (4)

$\therefore$ الزاوية σ تقع في الربع الرابع

$\sigma\ in\ the\ fourth\ quarter$

حسب مبرهنة فيثاغورس

$(3)^2 = x^2 + 4$

$9 = x^2 + 4$

$x^2 = 9 - 4 = 5$

$x = \mp\sqrt{5}$

$x = +\sqrt{5}$

$\sec\left(sin^{-1}\left(\frac{-2}{3}\right)\right) = \sec\sigma = \frac{1}{\cos\sigma} = \frac{\text{الوتر}}{\text{المجاور}} = \frac{3}{\sqrt{5}} > 1$

Sec:R\ {x: x= $\frac{\pi}{2} + n\pi\ , n \in I\}\ \rightarrow |y|\ \geq 1$

Sec x $\geq 1\ \vee \sec x\ \leq -1$

Ex) Find the value $sin^{-1}(-\frac{1}{2})$

Sol:- $\because\ sin^{-1}: [-1,1] \rightarrow [\frac{-\pi}{2}, \frac{\pi}{2}]$

$\therefore\ \sigma \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

$sin^{-1}\left(\frac{-1}{2}\right) = -\ sin^{-1}\left(\frac{1}{2}\right)\ \ [\ since\ sin^{-1}\ odd\ func.]$

Let $y = sin^{-1}(\frac{-1}{2})$

$\sin y = \frac{-1}{2} \Longrightarrow y = \frac{-\pi}{6}$

$\therefore\ \ \sigma\ in\ the\ four\ th\ quarter$

Ex) Find the value $sec^{-1}(2)$

Sol: $\because\ \ sec^{-1}: |x| \geq 1\ \ \rightarrow \{0, \pi\} \backslash \{\frac{\pi}{2}\}$

$\therefore \sigma \in [0,\pi] \setminus \{\frac{\pi}{2}\}$

Let $y = sec^{-1}(2) \Rightarrow \sec y = 2$

$\frac{1}{\cos y} = 2 \Rightarrow \cos y = \frac{1}{2} > 0$

$\Rightarrow y = \frac{\pi}{3}$

$\therefore \sigma\ in\ the\ first\ quarter$

مشتقات الدوال المثلثية العكسية :-

تعطى بعد البرهان

1- $\frac{d(sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}} . \frac{du}{dx}$

2- $\frac{d(cos^{-1}u)}{dx} = -\frac{1}{\sqrt{1-u^2}} . \frac{du}{dx}$

3- $\frac{d(tan^{-1}u)}{dx} = \frac{1}{1+u^2} . \frac{du}{dx}$

4- $\frac{d(cot^{-1}u)}{dx} = -\frac{1}{1+u^1} . \frac{du}{dx}$

5- $\frac{d(sec^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} . \frac{du}{dx}$

6- $\frac{d(csc^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}} . \frac{du}{dx}$

امثلة : جد المشتقة

1- $f(x) = sin^{-1}(x^2)$

$\acute{f}(x) = \frac{1}{\sqrt{1-(x^2)^2}} . 2x = \frac{2x}{\sqrt{1-x^4}}$

2- $y = cos^{-1}\sqrt{x}$

$\acute{y} = -\frac{1}{\sqrt{1-(\sqrt{x})^2}} . \frac{1}{2} x^{\frac{1}{2}} = \frac{-1}{2\sqrt{x}\sqrt{1-x}}$

3- $y = tan^{-1}(e^x)$

$\acute{y} = \frac{1}{1+(e^x)^2} . e^x . Lne . 1 = \frac{e^x}{1+e^{2x}}$

4- $y = sin^{-1}\sqrt{1-\sqrt{x}}$

$\acute{y} = \frac{1}{\sqrt{1-(\sqrt{1-\sqrt{x}})^2}} . \frac{1}{2} (1-\sqrt{x})^{-\frac{1}{2}} . -\frac{1}{2} x^{-\frac{1}{2}} =$

$-\frac{1}{4} \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1-1+\sqrt{x}}} \frac{1}{\sqrt{(1-\sqrt{x})}}$

$$= \frac{-1}{4\sqrt{x}\,\sqrt{1-\sqrt{x}}\,\sqrt{1-(1-\sqrt{x})}} = \frac{-1}{4\sqrt{x}\,\sqrt{1-\sqrt{x}}\,\sqrt{x-x+\sqrt{x}}}$$

$$= \frac{-1}{4\sqrt{x}\sqrt{1-\sqrt{x}}\sqrt[4]{x}}$$

5- $f(x)cot^{-1}(\frac{1-x}{1+x})$

$$\acute{f}(x) = \frac{1}{1+(\frac{1-x}{1+x})^2} \cdot \frac{(1+x).-1-(1-x)1}{(1+x)^2}$$

$$= -\frac{1}{\frac{(1+x)^2+(1-x)^2}{(1+x)^2}} \cdot \frac{-1-x-1+x}{(1+x)^2}$$

$$= -\frac{(1+x)^2}{1+2x+x^2+1-2x+x^2} \cdot \frac{-2}{(1+x)^2}$$

$$= -\frac{1}{2+2x^2} - 2 = \frac{2}{2(1+x^2)} = \tan x$$

$$\acute{f}(x) = \frac{1}{1+x^2}$$

6- $f(x) = sec^{-1}(\frac{\sqrt{1+x^2}}{x})$

$$\acute{f}(x) = \frac{1}{\left|\frac{\sqrt{1+x^2}}{x}\right|\sqrt{(\frac{\sqrt{1+x^2}}{x})^2-1}} \cdot \frac{x.\frac{1}{2}(1+x^2)^{-\frac{1}{2}}.2x-\sqrt{1+x^2}-1}{x^2}$$

$$= \frac{1}{\frac{\sqrt{1+x^2}}{|x|}\sqrt{\frac{1+x^2}{x^2}-1}} \cdot \frac{\frac{x}{2\sqrt{1+x^2}}.2x-\sqrt{1+x^2}}{x^2}$$

$$= \frac{1}{\frac{\sqrt{1+x^2}}{|x|}\sqrt{\frac{1+x^2-x^2}{x^2}}} \cdot \frac{\frac{x^2-(1+x^2)}{\sqrt{1+x^2}}}{\frac{x^2}{1}} = \frac{1}{\frac{\sqrt{1+x^2}}{\sqrt{x^2}}.\frac{1}{\sqrt{x^2}}} \cdot \frac{x-1-x^2}{\sqrt{1+x^2}} \cdot \frac{1}{x^2}$$

$$= \frac{1}{\frac{\sqrt{1+x^2}}{x^2}} \cdot \frac{-1}{\sqrt{1+x^2}} \cdot \frac{1}{x^2} = \frac{x^2}{\sqrt{1+x^2}} \cdot \frac{-1}{\sqrt{1+x^2}} \cdot \frac{1}{x^2}$$

$$\acute{f} = \frac{-1}{1+x^2}$$

تمارين : احسب قيمة :-

1- $sin^{-1}(1) - sin^{-1}(-1)$
2- $tan^{-1}(1) - tan^{-1}(-1)$
3- $sec^{-1}(2) - sec^{-1}(-2)$

بسط التعبيرات التالية :-

1- $\cos(sin^{-1} 0.8)$
2- $\sin(2\, sin^{-1} 0.8)$
3- $cos^{-1}(-\sin\frac{\pi}{6})$

4- $sec^{-1}(\sec(-30°))$

جد قيمة كل مما يأتي :-

1- $\sin(cos^{-1}\frac{\sqrt{2}}{2})$

2- $\sec(cos^{-1}\frac{1}{2})$

3- $\cos(cos^{-1}\frac{1}{2})$

4- $\csc(sec^{-1}2)$

5- $\cos(cot^{-1}1)$

6- $\tan(sin^{-1}\left(-\frac{1}{2}\right))$

7- $\cot(sin^{-1}\left(-\frac{1}{2}\right))$

8- $\cot(tan^{-1}(-\sqrt{3}))$

9- $\csc(sin^{-1}\left(\frac{-\sqrt{2}}{2}\right))$

10- $\tan(sec^{-1}(1))$

11- $\cot(cos^{-1}0)$

دالة اللوغارتم الطبيعي (Natural Logarithm function) :

$y = Lnx \quad ,x > o$

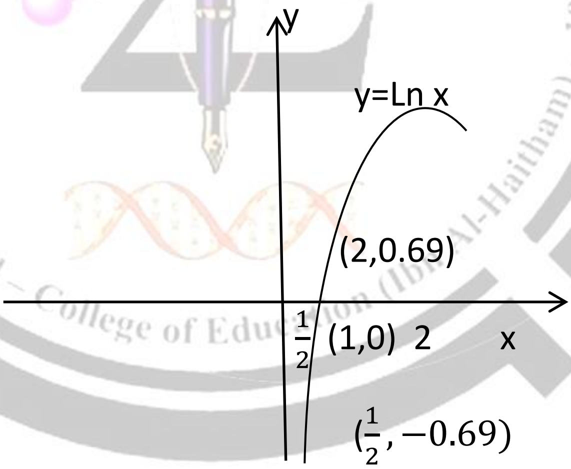
$Ln : (0, \infty) \to R$

$Lne = 1$

$Ln\ 1 = 0$

$Ln\ 2 = 0.69$

$Ln\ 10 = x = 1$



$y = Ln\left(u(x)\right)$ تعميم

حيث u(x) دالة موجبة لـX

مثال. y=Ln(2x)

Y= Ln (cos x)

y= Ln $(e^{x}) = x$

خواص الدالة اللوغارتمية :-

1- $\text{Ln}\,(x.y) = \text{Ln}\,x + \text{Ln}\,y \quad , x > 0 , y > 0$
2- $\text{Ln}\,(\frac{x}{y}) = Ln\,x - Ln\,y$
3- $\text{Ln}(\frac{1}{x}) = -Ln\,x \Longrightarrow Ln\,\left(\frac{1}{x}\right) = Ln\,1 - Ln\,x = -Ln\,x$
4- $\text{Ln}\,x^a = a\,Ln\,x$

امثلة : جد الاتي : (اذا علمت ان Ln2 = 0.69)

$\text{Ln}\,16 = \text{Ln}\,2^4 = 4\,Ln\,2 = 4\,(0.69) = 2.76$

$\text{Ln}\,\sqrt{2} = Ln\,2^{\frac{1}{2}} = \frac{1}{2}\,Ln\,2 = (0.5)(0.69) = 0.345$

$\text{Ln}\,8 = \text{Ln}\,2^3 = 3\,Ln\,2 = 3\,(0.69) = 2.07$

$\text{Ln}\,\frac{1}{2} = -Ln\,2 = -0.69$

امثلة : برهن كلا مما يأتي :-

1- $2\,\text{Ln}\,(\cos\frac{\sigma}{2}) = Ln\,\frac{1+\cos\sigma}{2}$

الطرف الايمن= $\text{Ln}\,\frac{1+\cos\sigma}{2}$

$= \text{Ln}\,\frac{2\,cos^2(\frac{\sigma}{2})}{2}$

$= \text{Ln}\,cos^2(\frac{\sigma}{2})$

$= 2\,\text{Ln}\,\cos(\frac{\sigma}{2})$

= الطرف الايسر

2- $Ln\,(x + \sqrt{x^2 - 1}) = -\text{Ln}\,(x - \sqrt{x^2 - 1})$

الطرف الايسر= $Ln\,(x + \sqrt{x^2 - 1})$

$= Ln\,\frac{(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})}{(x-\sqrt{x^2-1})}$

$= Ln\,\frac{x^2-(x^2-1)}{x-\sqrt{x^2-1}}$

$= Ln\,\frac{x^2-x^2+1}{x-\sqrt{x^2-1}}$

$= Ln\,\frac{1}{x-\sqrt{x^2-1}} \qquad = Ln 1 - Ln\,(x - \sqrt{x^2 - 1})$

$= -Ln\left(x - \sqrt{x^2 - 1}\right) =$ الطرف الايمن

3- $2\,Ln\sin\sigma = Ln\,(1 - \cos\sigma) + Ln\,(1 + \cos\sigma)$

الطرف الايمن= $Ln\,(1 - \cos\sigma) + Ln\,(1 + \cos\sigma)$

$= Ln\ ((1 - \cos\sigma)\ (1 + \cos\sigma))$
$= Ln\ (1 - cos^2\sigma)$
$= Ln\ sin^2\sigma$
$= 2\ Ln \sin\sigma =$ الطرف الايسر

Y=$r \sin\theta\ , x = r\cos\theta$

$r^2 \cos\theta^2 +\ r^2 \sin\theta^2 -\ r^2$

$x^2 + y^2 = 1$

$\frac{(x)^2}{r} + \frac{(y)^2}{r} = \frac{(r)^2}{r}, r = 1$

U=u(x)

Y=a^u

Ln y=$Ln\ a^u = u\ Ln\ a$

$\frac{1}{y}\ \frac{dy}{dx} = \frac{du}{dx}\ .Ln\ a$

$\acute{y} = \frac{dy}{dx} = y.Ln\ a\ \frac{du}{dx}$

$\acute{y} = Ln\ a\ a^u \frac{du}{dx}$

$\log_b^a = \frac{Ln\ a}{Ln\ b}$

$\log_{10}^{a} = \frac{Ln\ a}{Ln\ 10} = \frac{Ln\ a}{1} = Ln\ a$

مشتقة الدالة اللوغارتمية : حيث u دالة موجبة بالنسبة الى x

$\frac{d\ Ln\ u}{dx} = \frac{d\ Ln\ u}{du}.\frac{du}{dx}$ حسب قانون السلسلة

$= \frac{1}{u}\ .\frac{du}{dx}$

$\frac{d\ Ln\ u}{dx} = \frac{\frac{du}{dx}}{u}$

دخول الدالة اللوغارتمية في الغاية

$Lim_{x\to\infty}\ Lnx =\ +\infty$

$Lim_{x\to 0^+} Ln\, x = -\infty$

امثلة : جد الغاية ان وجدت :

1- $\underset{x\to 1}{lim}\,(x - Lnx) = 1 - Ln1 = 1 - 0 = 1$
2- $\lim_{x\to 1} : \cos(Lnx) = \cos(Ln1) = \cos(0) = 1$
3- $\lim_{x\to 1} Ln\, x^{(x+1)} = \underset{x\to 1}{lim}\,\{(x+1).Lnx\}$

$= \underset{x\to 1}{lim}\,(x+1)\ \underset{x\to 1}{lim}\, Lnx$

$= (1+1).Ln\,1 = 2(0) = 0$

4- $\lim_{x\to 0} \frac{Ln(x+1)}{x} \qquad \frac{Ln(1)}{\sigma} = \frac{0}{0}$

$= \lim_{x\to 0} \frac{\frac{1}{x+1}}{1} = \underset{x\to 0}{lim}\ \frac{1}{x+1} = \frac{1}{0+1} = \frac{1}{1} = 1$

امثلة : جد $\acute{y}$

1- $y = Ln\,(x^2 + 2x)$

$\acute{y} = \frac{2x+2}{x^2+2x}$

2- $y = Ln\,(\tan x + \sec x)$

$\acute{y} = \frac{sec^2x + secx.tanx}{tanx+secx} = \frac{secx\cancel{(secx+tanx)}}{\cancel{(tanx+secx)}}$

$\acute{y} = \sec x$

3- $y = (Lnx)^3$

$\acute{y} = 3(Lnx)^2.\frac{1}{x}$

واجب (.H.W) : جد $\acute{y}$

1- $y = Ln\,(x\sqrt{x^2+1})$
2- $y = Ln\,(3x\,\sqrt{x+2})$
3- $y = xLnx - x$
4- $y = x^3\, Ln\,(2x)$
5- $y = \frac{1}{2}\, Ln\, \frac{1+x}{1-x}$
6- $y = \frac{1}{3}\, Ln\, \frac{x^3}{1+x^3}$
7- $y = Ln\, \frac{x}{2+3x}$

8- $y = Ln\,(x^2+4) - x\,tan^{-1}\frac{x}{2}$
9- $y = x(Lnx)^3$

الدالة الاسية (The exponential function)

$y = e^x$

$e^x : R \to (0, \infty)$

$e^x = Ln^{-1}(x)$

$e = 2.718$

$e^\circ = (2.7)^\circ = 1 > 0$

$e^2 = (2.7)^2 = 7.29 > 0$

$e^{-1} = (2.7)^{-1} = \frac{1}{2.7} > 0$

رمزها : $e^x = \exp(x) = Exp(x)$

تعميم : $y = e^{u(x)}$

امثلة : $y = e^{\sin x}$

$y = e^{\tan^{-1} x}$

$y = e^{\sqrt{x}}$

$y = e^{3x}$

$y = e^{x^2}$

$y = \frac{1}{e^x}$

العلاقة بين الدالة الاسية والدالة اللوغارتمية :

$y = e^x \Leftrightarrow x = Lny$

خواص الدالة الاسية :-

1- $e^\circ = 1$
2- $e^{x_1}.e^{x_2} = e^{x_1+x_2}$
3- $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$

4- $(e^x)^r = e^{rx} \qquad \forall r \in R$

5- $e^{-x} = \frac{1}{e^x}$

6- $e^{Lnx} = x = Ln\, e^x$

امثلة : احسب قيمة كل مما يأتي :-

1- $Ln\,(e^{-x2}) = -x^2$

2- $Ln\left(e^{\frac{1}{x}}\right) = \frac{1}{x}$

3- $e^{Ln\frac{1}{x}} = \frac{1}{x}$

4- $e^{2Lnx} = e^{Lnx^2} = x^2$

5- $\exp(Lnx - 2Lny)$

$= \exp(Lnx - Lny^2) = \exp(Ln\,\frac{x}{y^2}) = \frac{x}{y^2}$

6- $e^{x+Lnx} = e^x . e^{Lnx}$

$= e^x . x$

$= x\, e^x$

دخول الدالة الاسسية في الغاية :

$\lim_{x\to+\infty} e^x = +\infty$

$\lim_{x\to-\infty} e^x = 0$

مشتقة الدالة الاسسية :

$y = e^u$ حيث $u=f(x)$ دالة قابلة الاشتقاق

$\frac{d\,e^u}{dx} = e^u . \frac{du}{dx} \qquad e=2.718$

امثلة : جد المشتقة

1- $y = e^{tan^{-1}x}$

$\acute{y} = e^{tan^{-1}x} . \frac{1}{1+x^2}$

2- $y = Ln\,\frac{e^x}{1+e^x}$

$\acute{y} = \frac{1}{\frac{e^x}{1+e^x}} . \frac{(1+e^x).e^x.1 - e^x.e^x.1}{(1+e^x)^2}$

$= \frac{1}{e^x} . \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)}$

$= \frac{1}{e^x} . \frac{e^x}{(1+e^x)}$

$$= \frac{1}{(1+e^x)}$$

$$\tan y = e^x + Lnx$$

$$sec^2 y . \acute{y} = e^x . 1 + \frac{1}{x}$$

$$\acute{y} = \frac{e^x + \frac{1}{x}}{sec^2 y}$$

واجب بيتي (H.W.) جد $\acute{y}$

1- $y = x^2 . e^x$
2- $y = \frac{1}{2}(e^x + e^{-x})$
3- $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4- $y = e^{\sin^{-1}\sqrt{x}}$
5- $y = (1+2x). e^{-2x}$
6- $y = (9x^2 - 6x + 2)e^{3x}$
7- $y = \frac{ax-1}{a^2} . e^{ax}$ a= constant
8- $y = e^{-x^2}$
9- $y = x^2 - e^{-x^2}$
10- $y = e^x . Lnx$

اللوغارتم الاعتيادي The function y= Lo gu

$$\log_a x = \frac{Ln\, x}{Ln\, a}, x > 0, a > 0, a \neq 1$$

تعميم :

$$\log_a u = \frac{Ln\, u}{Ln\, a}$$

$$u > 0, a > 0, a \neq 1$$

الخواص :

1- $\log_a (x.y) = \log_a x + \log_a y$
2- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
3- $\log_a x^y = y . \log_a x$
4- $\log_a a = 1 \leftrightarrow Ln\, e = 1$
5- $\log_a 1 = 0 \leftrightarrow Ln\, 1 = 0$

امثلة :

$$\log_2 16 = \frac{Ln\ 16}{Ln\ 2} = \frac{Ln\ 2^4}{Ln 2} = \frac{4\ Ln\ 2}{Ln\ 2} = 4$$

6- $\log_{\frac{1}{7}} 49 = \frac{Ln\ 49}{Ln\ \frac{1}{7}} = \frac{Ln\ 7^2}{-\ Ln\ 7} = \frac{2\ Ln 7}{-\ Ln\ 7} = -2$

$$\log_{10} 10 = 1$$

$$\log_{10} 100 = \log_{10} 10^2 = 2\ \log_{10} 10 = 2(1) = 2$$

$$\log_{10} \frac{1}{1000} = \log_{10} 10^{-3} = -3\ \log_{10=1} 10 = 1 = -3(1) = -3$$

مشتقة الدالة اللوغارتمية العامة :-

$\log_a u = \frac{Ln\ u}{Ln\ a}$ حسب التعريف

$$\frac{d}{dx} \log_a u = \frac{d}{dx} (\frac{Ln\ u}{Ln\ a})$$

$$= \frac{1}{Ln\ a} . \frac{d}{dx}\ Ln\ u$$

$$= \frac{1}{Ln\ a} . \frac{\frac{du}{dx}}{u}$$

$$\frac{d\ \log_a u}{dx} = \frac{du \mid dx}{u.Lna}$$

$$u > 0\ , a > 0\ , a \neq 1$$

امثلة : جد $\acute{y}$

1- $y = \log_2(x^2 + 3x)$

$\acute{y} = \frac{2x+3}{(x^2+3x).Ln2}$

2- $y = \log_7(\tan x + sinx)$

$\acute{y} = \frac{sec^2 x + \cos x}{(\tan x + \sin x)\ .Ln\ 7}$

3- $y = Lnx. \log_{10} x$

$\acute{y} = Lnx. \frac{1}{x.Ln10} + \log_{10} x . \frac{1}{x}$

4- $y = \log_a sin^{-1}x + \frac{x}{e^x}$ a= constant

$\acute{y} = \frac{\frac{1}{\sqrt{1-x^2}}}{sin^{-1}x.Lna} + \frac{e^x.1 - x.e^x.1}{e^{2x}}$

واجب : جد $\acute{y}$

1- $y = \log_4 \sin x$

2- $y = \log_a(e^x + \sin x)$

الدالة الاسسية العامة :

$\because x = e^{Lnx}$

$a = e^{Lna}$

$a^u = (e^{Lna})^u$

$a^u = e^{u\, Lna}$

حيث u عدد حقيقي $-\infty < u < \infty$ والاساس $a > 0$

خواص الدالة الاسسية العامة :-

1- $a^1 = a \qquad (a > 0)$
2- $a^\circ = 1$
3- $a^u . a^\vartheta = a^{u+\vartheta}$
4- $(a^{m/n})^n = a^m$
5- $(a.b)^u = a^u . b^u \qquad$ ($a > 0, b > 0$ حيث)

مشتقة الدالة الاسسية العامة :-

$a^u = e^{uLna}$ حسب التعريف

$\frac{da^u}{dx} = \frac{d}{dx}(e^{uLna})$

$\frac{da^u}{dx} = e^{uLna} . Lna . \frac{du}{dx}$

$\frac{da^u}{dx} = a^u . Lna . \frac{du}{dx}$

امثلة : جد المشتقة :-

1- $y = 2^{(x^2+secx)}$
$\acute{y} = 2^{(x^2+secx)} . Ln2(2x + secx.tanx)$
2- $y = 4^{sin^{-1}x}$
$\acute{y} = 4^{sin^{-1}x} . Ln4 . \frac{1}{\sqrt{1-x^2}}$
3- $y = x^\pi . \pi^x = a^u$
$\acute{y} = x^\pi . \pi^x . Ln\pi . 1 + \pi^x . \pi x^{\pi-1} . 1$

واجب : جد المشتقة :-

1- $y = 5^{x^2+x-1}$

2- $y = 6^{\sin x + Lnx + 3}$

3- $y = 2^{\sec x}$

4- $y = 3^{\tan x}$

5- $y = Ln\left(\frac{x^4}{1+x^3}\right) + 7^{x^{2/3}}$

6- $S = 2^{-t2}$

امثلة :-

1- Find x if $3^x = 2^{x+1}$

$Ln\, 3^x = Ln\, 2^{x+1}$

$x\, Ln3 = (x+1)Ln2$

$xLn3 = xLn2 + Ln2$

$xLn3 - xLn2 = Ln2$

$(Ln3 - Ln2)x = Ln2$

$x = \frac{Ln\, 2}{Ln\, 3 - Ln2} = \frac{Ln2}{Ln\, \frac{3}{2}} = \frac{Ln2}{Ln\, 1{:}5}$

2- $3^{\log_3 7} + 2^{\log_2 5} = 5^{\log_5 x}$ $\quad Find\ x$

$e^{log_3 7.Ln3} + e^{log_2 5.Ln2} = e^{log_5 x\, .Ln5}$

$e^{\frac{Ln7}{Ln3}.Ln3} + e^{\frac{Ln5}{Ln2}.Ln2} = e^{\frac{Lnx}{Ln5}.Ln5}$

$e^{Ln7} + e^{Ln5} = e^{Lnx}$

$7 + 5 = x$

X=12

العلاقة بين الدالة الاسسية العامة والدالة اللوغارتمية العامة :-

$\log_a y = x \Leftrightarrow y = a^x, a > 0\,, a \neq 1$

$\log_{10} y = x \Leftrightarrow y = 10^x$

$\log_e y = Lny = x \Leftrightarrow y = e^x$

$\log_2 32 = 5 \Leftrightarrow 32 = 2^5$

الدالة الاسسية واللوغارتمية هي دوال عكسية لبعضها البعض ولكن بنفس الاساس

طريقة الاشتقاق اللوغارتمي (Logarithmic differentiation)

اذا كان الاسس والاساس متغيرات

امثلة : جد المشتقة :-

1- $y = x^{x^2 .v} = u \quad , x > 0$

$Ln\, y = Ln\, x^{x^2}$

$Lny = \ x^2 . Lnx$

$\frac{\acute{y}}{y} = x^2 . \frac{1}{x} + Lnx . 2x$

$\acute{y} = y(x + 2xLnx)$

$\acute{y} = x^{x2}(x + 2x\, Lnx)$

2- $y = (x^2 + 1)$

$Ln\, y = Ln\, (x^2 + 1)^{Lnx}$

$Lny = Lnx\, . Ln(x^2 + 1)$

$\frac{\acute{y}}{y} = Lnx\, . \frac{2x}{x^2+1} + Ln\, (x^2 + 1). \frac{1}{x}$

$\acute{y} = y\, \left(\frac{2xLnx}{x^2+1} + \frac{Ln(x^2+1)}{x} \right)$

$\acute{y} = \ (x^2 + 1)^{Lnx}\, \left(\frac{2x\, Lnx}{x^2+1} + \frac{Ln(x^2+1)}{x} \right)$

3- $y = (sinx)^{\tan x} \quad , \sin x \ > 0$

$Ln\, y = Ln\, (\sin x)^{\tan x}$

$Ln\, y = \tan x\, . Ln\, (\sin x)$

$\frac{\acute{y}}{y} = \tan x\, . \frac{\cos x}{\sin x} + Ln\, (\sin x\,)\, . sec^2 x$

$\acute{y} = y\, (\frac{\sin x}{\cos x}\, . \frac{\cos x}{\sin x} + \ sec^2 x\, . Ln\, (\sin x))$

$\acute{y} = \ (\sin x)^{\tan x}\ (1 + \ sec^2 x\, . Ln\, (\sin x))$

H.W.

1) $y = \ x^x, x^{\circ} > 0$

2) $y = \ \sqrt{e^{x^2} + e^x}$

3) $x^{(\sqrt{5}+Lnx)}$

4) $y = x^{Lnx}$

Chapter Five

Hyperbolic functions الدوال الزائدية

1- Sinh (x) = $\frac{e^{x}-e^{-x}}{2}$ دالة الجيب الزائدي

2- Cosh = $\frac{e^{x}-e^{-x}}{2}$ دالة الجيب تمام الزائدي

3- Tanh(x)= $\frac{e^{x}-e^{-x}}{e^{x}-e^{-x}}$ tanh(x)=$\frac{\sinh(x)}{\cosh(x)}$

4- Coth(x)= $\frac{e^{x}-e^{-x}}{e^{x}-e^{-x}}$ coth(x)=$\frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)}$

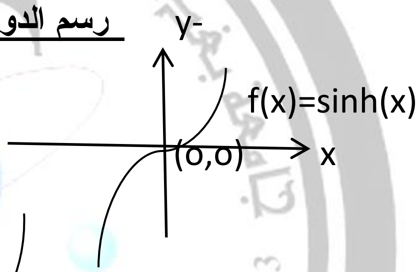
5- Sech(x)= $\frac{2}{e^{x}-e^{-x}}$ sech(x)= $\frac{1}{\cosh(x)}$

6- Csch(x)= $\frac{2}{e^{x}-e^{-x}}$ csch(x)= $\frac{1}{\sinh(x)}$

Graph hyperbolic functions :- رسم الدوال الزائدية

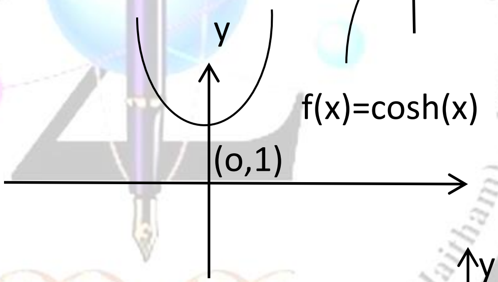
1- f(x)=sinh(x)

D_f=R , R_f=R



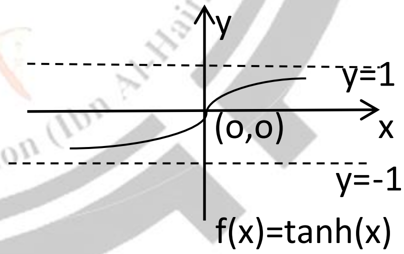
2- f(x)= cosh(x)

D_f=R , R_f=[1,∞)



3- f(x)=tanh(x)

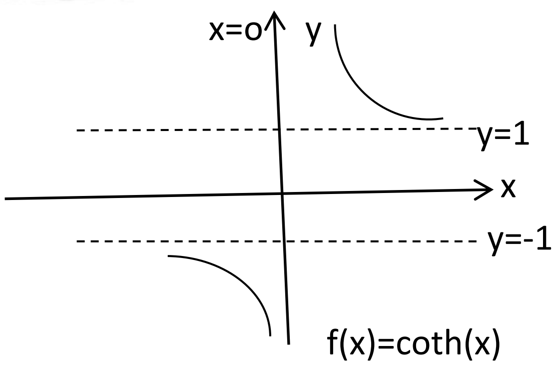
D_f=R , R_f=(-1,1)



4- f(x)=coth(x)

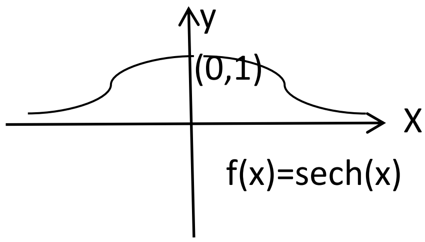
D_f=(-∞ ,o) ∪ (o, ∞)

R_f=(-∞,-1) ∪(1, ∞)



5- $f(x)= \text{sech}(x)$

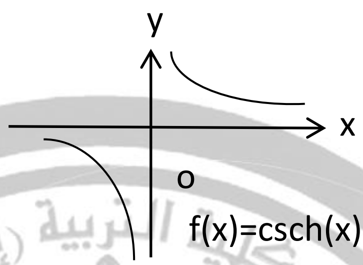
$D_f=R \quad , R_f=(o,1]$



6- $f(x)=\text{csch}(x)$

$D_f=(-\infty,o) \cup (o, \infty)$

$R_f=(-\infty,o) \cup (o, \infty)$



<u>Identities for hyperbolic functions</u> العلاقات بين الدوال المثلثية الزائدية

$\cosh^2(x)-\sinh^2(x)=1$

$1-\tanh^2(x)=\text{sech}^2(x)$ $\quad$ $\text{tonh}^2(x)+\text{sech}^2(x) =1$

$\coth^2(x)-1=\text{csch}^2(x)$ $\quad$ $\coth^2(x)-\text{csch}^2(x) =1$

$\cosh(-x) = \cosh(x)$ زوجية

$\sinh(-x) = -\sinh(x)$ فردية

$\tanh(-x) = -\tanh(x)$ فردية

$\cosh(x) + \sinh(x) = e^{x}$

$\cosh(x) - \sinh(x) = e^{-x}$

$\cosh(x+y)=\cosh(x)\cosh(y)+\sinh(y)$

$\sinh(x+y)=\sinh(x)\cosh(y)+\cosh(x)\sinh(y)$

$\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1-\tanh(x)\tanh(y)}$

$\cosh(2x)=2\cosh^2(x)-1$

$\cosh^2(x)= \frac{1}{2}(1+\cosh(2x))$

$\sinh^2(x)= \frac{1}{2}(\cosh(2x)-1)$

$\left.\begin{array}{l} \cosh(2x)+1=2\cosh^2(x)-1 \\ \cosh(2x)-1=2\sinh^2(x) \end{array}\right\}$

$\text{Cosh}(2x)=2\sinh^2(x)+1$

$\left.\begin{array}{l} \text{Cosh}(2x)=\cosh^2(x)+\sinh^2(x) \\ \text{Sinh}(2x)=2\sinh(x)\cosh(x) \end{array}\right\}$

Derivatives of hyperbolic functions :- مشتقات الدوال الزائدية

1- $\frac{d\, snh(u)}{dx} = \cosh(u)\,\frac{du^{-1}}{dx}$
2- $\frac{d\cosh(u)}{dx} = \sinh(u)\,\frac{du}{dx}$
3- $\frac{d\,\tanh\,(u)}{dx} = \text{sech}^2(u)\,\frac{du}{dx}$
4- $\frac{d\,\coth\,(u)}{dx} = -\text{csch}^2(u)\frac{du}{dx}$
5- $\frac{d\,\text{sech}(u)}{dx} = -\text{sech}(u)\,.\,\tanh(u)\,\frac{du}{dx}$
6- $\frac{d\,\text{csch}(u)}{dx} = -\text{csch}(u)\,.\,\coth(u)\,\frac{du}{dx}$

Examples :- finding Derivatives

1- Y=sinh(3x)
   y-=3cosh(3x)

2- Y=cosh2(5x)
   y=2cosh(5x) . sinh(5x) . 5=5sinh(lox)
3- y=tanh(2x)
   y-=2sech^2(2x)

4- y=coth(tan x)

y^{-}=-csch^2(tan x) . see^{2}x.1

5- y=sech^{3}x

y^{-}=3sech^2(x)-sech(x).tanh(x).1

y^{-}=-3sech^3(x).tanh(x)

6- y=4csch($\frac{x}{4}$)

y^{-}=$\cancel{4}$-csch($\frac{x}{4}$) . coth($\frac{x}{4}$) $\frac{1}{\cancel{4}}$

y^{-}=-csch($\frac{x}{4}$) . coth($\frac{x}{4}$)

<u>Exercises:-</u> finding derivatives :-

1- f(x)$\frac{\cosh(x)}{x}$
2- f(x)=e^x.cosh(x)
3- f(x)=Ln(sinh(x^2))
4- f(x)=tanh($\frac{4x+1}{5}$)
5- f(x)=Ln(tanh(x))
6- f(x)=e^x.tanh(2x)
7- f(x)=$tan^{-1}(sinh^2 x)$
8- f(x)=coth($\frac{1}{x}$)
9- y=$cosh^2(5x) - sinh^2(5x)$
10- sinh y=tan x
11- y=sinh^2(3x)
12- sin $^{-1}$ x= sech y
13- x=cosh(Ln y)
14- tan x=tanh^2 y
15- sinh y=see x
16- y= tanh (Ln x)
17- y=sinh(tan^{-1}e^{3x})
18- y^2+x cosh y +sinh ^{2} x = 50
19- y= cot (csch (e^x))
20- f(x)= csch^3 ($\sqrt{2x}$)

Inverse Hyperbolic Functions :- الدوال الزائدية العكسية

$x=\sinh y \Leftrightarrow y=\sinh^{-1}(x)$

$-\infty < x < \infty \quad , -\infty < y < \infty$

$x=\cosh y \Leftrightarrow y=\cosh^{-1}(x)$

$x \geq 1 \quad , y \geq 0$

$x=\tanh y \Leftrightarrow y=\tanh^{-1}(x)$

$x=\coth y \Leftrightarrow y=\coth^{-1}(x)$

$x=\operatorname{sech} y \Leftrightarrow y=\operatorname{sech}^{-1}(x)$

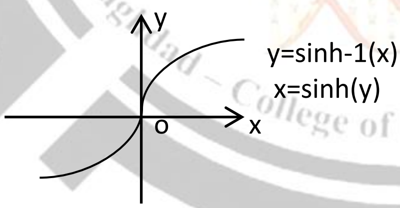
$0 < x \leq 1 \quad , \quad y > 0$

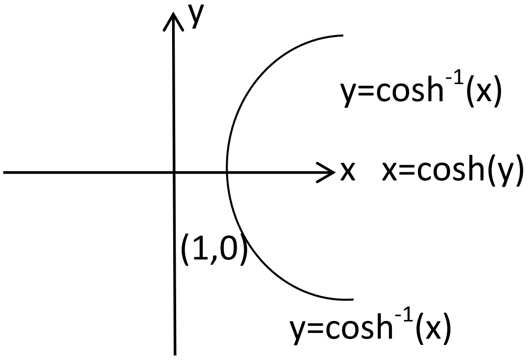
$x=\operatorname{csch} y \Leftrightarrow y=\operatorname{csch}^{-1}(x)$

$\operatorname{sech}^{-1}(x)=\cosh^{-1}(\frac{1}{x})$

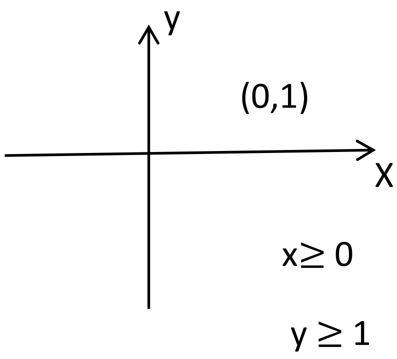
$\operatorname{csch}^{-1}(x) = \sinh^{-1}(\frac{1}{x})$

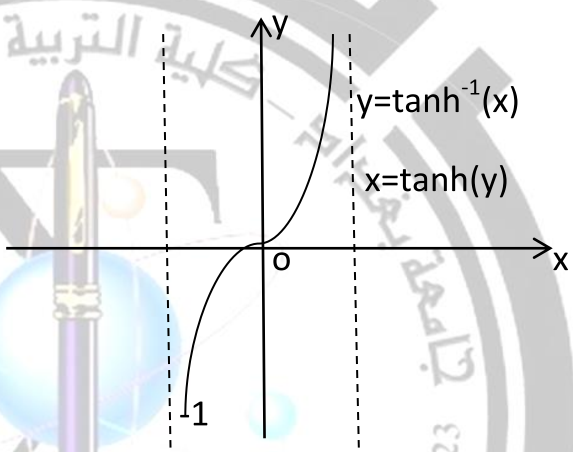
Graph Inverse hyperbolic functions -: رسم الدوال الزائدية العكسية

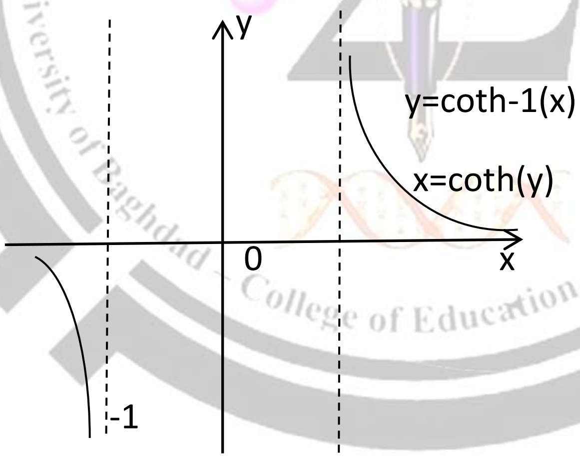


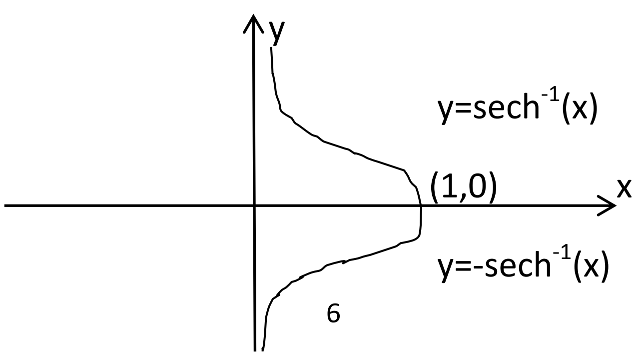


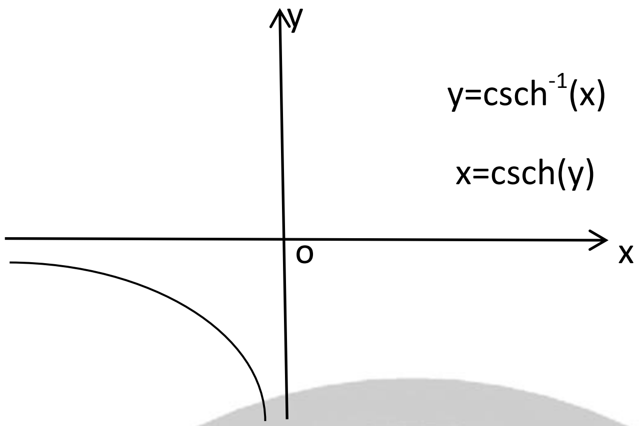
1-1&onto


y
(0,1)
X
x≥ 0
y ≥ 1


y
y=tanh-1(x)
x=tanh(y)
o
x
-1


y
y=coth-1(x)
x=coth(y)
0
x
-1


y
y=sech-1(x)
(1,0)
x
y=-sech-1(x)



$\sinh^{-1}(x)=\text{Ln}(x+\sqrt{x^2+1})$, $-\infty < x < \infty$, x∈R لكل

$\cosh^{-1}(x)=\text{Ln}(x+\sqrt{x^2-1})$, x≥1 لكل

$\tanh^{-1}(x)=\frac{1}{2}\,\text{Ln}\,\frac{1+x}{1-x}$ $|x| < 1$ لكل

$\coth^{-1}(x) = \frac{1}{2}\,\text{Ln}\,\frac{x+1}{x-1} = \tanh^{-1}(\frac{1}{x})$, $|x| > 1$ لكل

$\text{sech}^{-1}(x)= \text{Ln}\,(\frac{1+\sqrt{1-x^2}}{x})= \cosh^{-1}(\frac{1}{x})$, $0< x \le 1$

$\text{csch}^{-1}(x) = \text{Ln}\,(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}) = \sinh^{-1}(\frac{1}{x})$, $x\neq 0$

$$\text{csch}^{-1}(x)=\begin{cases} Ln\,\frac{1+\sqrt{1+x^2}}{x} & ,\ x>0 \\ -Ln\,\frac{1+\sqrt{1+x^2}}{-x} & ,x<0 \end{cases}$$

Chapter Six

Integration التكامل

The indefinite integral التكامل الغير محدد

تعريف يعرف التكامل الغير محدد للدالة f بـ :

$$\int f(x)dx = F(x) + c$$

حيث c هو ثابت التكامل و F(x) هو عكس مشتقة الدالة f(x) حيث :

F^{-}(x)=f(x)

ويمكن تعريف التكامل على انه عكس المشتقة

Theorems :-

(1) For any rational power r≠-1

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c$$

(2) For any constant a

$$\int a\, dx = ax + c$$

(3) For any constant a

$$\int a\, f(x)dx = a \int f(x)dx$$

(4) $\int [f(x) \mp g(x)]dx = \int f(x)dx \mp \int g(x)dx$

∴ from (3) and (4) we have :

$$\int [a\, f(x) \mp bg(x)]dx = a \int f(x)dx \mp b \int g(x)dx$$

s.t a,b constants

Examples :-

(1) $\int x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$

(2) $\int \frac{1}{x^3}\, dx = \int x^{-3}\, dx = \frac{x^{-3+1}}{-3+1} + c = \frac{x^{-2}}{-2} + c = \frac{-1}{2x^2} + c$

(3) $\int 3x^5\, dx = 3\int x^5 dx = 3.\frac{x^{5+1}}{5+1} + c = \cancel{3}^{1}.\frac{x^6}{\cancel{6}_{2}} + c = \frac{x^6}{2} + c$

(4) $\int \sqrt{x}dx = \int x^{1/2}\, dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} + c = \frac{x^{3/2}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x^3} + c$

(5) $\int (3 + 6x^2)dx = \int 3dx + \int 6x^2 dx = 3\int dx + 6\int x^2 dx$

$$= 3x + \cancel{6}^{2} . \frac{x^3}{\cancel{3}_1} + c = 3x + 2x^2 + c$$

(6) $\int (5x^4 - \frac{10}{x^2})\, dx = \int 5x^4 dx - \int \frac{10}{x^2} dx$

$$= 5\int x^4 dx - 10 \int x^{-2} dx$$

$$= \cancel{5}^{1} . \frac{x^5}{\cancel{5}_1} - 10 . \frac{x^{-1}}{-1} + c$$

$$= x^5 + \frac{10}{x} + c$$

Integration of the power functions [f(x)]^n تكامل الدالة

$$\int [f(n)]^n . f^-(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

s.t n≠ -1 and c any constant

Examples :-

- $\int \underbrace{[5+6x]^2}_{[f(x)]^n} \underbrace{. 6dx}_{f^-(x)} = \frac{[5+6x]^3}{3} + c$
- $\int [1 + 5x^2]^9 . xdx$

$$f(x) = 1 + 5x^2$$
$$f^-(x) = 10x$$

$$\Longrightarrow \frac{10}{10} \int [1 + 5x^2]^9 . xdx = \frac{1}{10} \int \underbrace{[1 + 5x^2]^9}_{[f(x)]^4} . \underbrace{10x\, dx}_{f^-(x)}$$

$$= \frac{1}{10} . \frac{[1+5x^2]^{10}}{10} + c = \frac{[1+5x^2]^{10}}{100} + c$$

تكاملات الدوال المثلثية :-

1) $\int \sin(x) dx = -\cos(x) + c$

$\int \sin(u) \frac{du}{dx} = -\cos(u) + c$

2) $\int \cos(x)\, dx = \sin(x) + c$

$\int \cos(u) \frac{du}{dx} = \sin(u) + c$

3) $\int sec^2(x) dx = \tan(x) + c$

$\int sec^2(u) \frac{du}{dx} = \tan(u) + c$

4) $\int csc^2(x)dx = -\cot(x) + c$

$\int csc^2(u)\frac{du}{dx} = -\cot(u) + c$

5) $\int[\sec(x)\,.\tan(x)]dx = \sec(x) + c$

$\int[\sec(u)\,.\tan(u)]\frac{du}{dx} = \sec(u) + c$

6) $\int[\csc(x)\,.\cot(x)]dx = -\csc(x) + c$

$\int[\csc(u)\,.\cot(u)]\frac{du}{dx}$ = - csc(u)+c

Examples :-

(1) $\int \cos(2x)\,dx = \frac{2}{2} \quad \int \cos(2x)dx$

$= \frac{1}{2}\int \cos(2x).\,2dx$

$= \frac{1}{2}$ sin (2x) +c

(2) $\int x\sin(2x^2)dx = \frac{4}{4}\int x.\sin(2x^2)\,dx$

$= \frac{1}{4}\int \sin(2x^2).\,4x\,dx$

$= \frac{1}{4}.-\cos(2x^2) + c$

$= \frac{-1}{4}\cos(2x^2) + c$

(3) $\int(sec^2\,(x) + 4x^8)dx = \int sec^2(x)dx + 4\int x^8dx$

= tan(x) +4. $\frac{x^9}{9}$ +c

(4) $\int 3\csc(x)\,.\cot(x)\,dx = 3\int[\csc(x)\,.\cot(x)]dx$

= -3 csc(x) +c

(5) $\int csc^2(\sqrt{x}).\frac{dx}{\sqrt{x}}$ $\quad (\sqrt{x})^- = (x^{\frac{1}{2}})^- = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$

$\frac{2}{2}\int csc^2(\sqrt{x}).\frac{dx}{\sqrt{x}} = 2\int csc^2(\sqrt{x}).\frac{1}{2\sqrt{x}}\,dx$

= -2 cot($\sqrt{x}$)+c

التكامل المحدد :-

اذا كانتb,aواقعة ضمن الفترة $[\propto,\beta]$ وكانتFعكس تفاضل للدالةf عليها فأن$\int_a^b f$ يعرف

كما يلي :

$\int_a^b f = \int_a^b f(x)dx$

$= F(x)+c\Big|_{a}^{b}$

= التعويض بالحد الاعلى – التعويض بالحد الادنى

$= F(b)+c-(F(a)+c)$

$= F(b)-F(a)$

يسمى العدد $\int_a^b f$ بالتكامل المحدد للدالة f من a الى b

وتسمى a بالحد الادنى للتكامل و b بالحد الاعلى للتكامل كما يقال

ان للدالة f تكون قابلة للتكامل على الفترة [a,b] اذ وجد $\int_a^b f$

خواص التكامل المحدد :-

(1) $\int_a^b (k_1 f \mp k_2 g) = k_1 \int_a^b f \mp k_2 \int_a^b g$

(2) $\int_a^b f = \int_a^c f + \int_c^b f \ni c \in [a,b]$

(3) $\int_a^b f = F(b) - F(a) = -\big(F(a) - F(b)\big) = -\int_a^b f$

(4) $\int_a^b f = F(a) - F(a) = 0$

(5) $\int_{-a}^{a} f = 2\int_0^a f$

او $= 2\int_{-a}^{0} f$

$\int_{-a}^{a} f = 0$

Ex: $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{\cos x}{\sin^2 x} + \left(4x + 1^{1/2}\right)\right] dx$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \underset{u}{sin^{-2}x} \underset{du}{\cos x\ dx} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \underset{u^n}{(4x+1)^{1/2}} \underset{du}{dx} \quad \frac{4}{4}\ du = 4dx$

$\frac{sin^{-1}x}{-1}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{1}{4_2}\ \frac{(4x+1)^{3/2}}{3/2}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$-\frac{1}{\sin x}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{1}{6}\sqrt{(4x+1)^3}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \qquad \int_a^b f = F(b) - F(a)$

$- \frac{1}{\sin\frac{\pi}{2}} - (-\frac{1}{\sin\frac{\pi}{4}}) + \frac{1}{6}\sqrt{(4\frac{\pi}{2}+1)^3} - \frac{1}{6}\sqrt{(4\frac{\pi}{4}+1)^3}$

$- \frac{1}{1} + \frac{1}{\frac{1}{\sqrt{2}}} + \frac{1}{6}\sqrt{(2\pi\ +1)^3} - \frac{1}{6}\sqrt{(\pi\ +1)^3}$

تطبيقات على التكامل المحدد :-

المساحة تحت المنحني : (Area under a carve)

لتكن A المساحة المحددة بالمنحني y=f(x) ومحور x (y=0) والمستقيمان x=a و x =b فأن :-

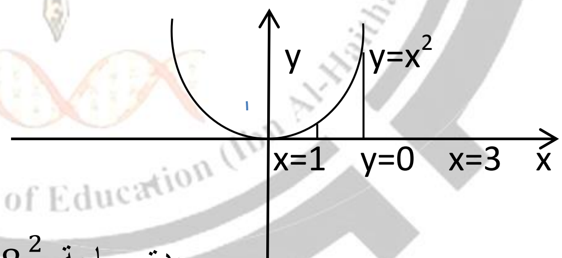
$$A_a^b = \left|\int_{x=a}^{x=b} f(x)dx\right|$$

واذا كانت المساحةA محددة بالمنحني x=g(y) ومحورy (x=0) والمستقيمات y=c و y=d فأن:-

$$A_c^d = \left|\int_{y=c}^{y=d} g(y)dy\right|$$

Ex1) Find the area bounded by the x- axis and the cecrie $y=x^2$ & x= 1 & x =3

احسب مساحة المنطقة المحددة بالمنحني والمستقيمان x=1 وx=3



$A_1^3 = \left|\int_{x=1}^{x=3} f(x)dx\right|$

$= \left|\int_1^3 x^2 dx\right| = \left|\frac{x^3}{3}\right]_1^3\Big|$

$= \left|\frac{(3)^3}{3} - \frac{(1)^3}{3}\right| = \left|9 - \frac{1}{3}\right| = \left|8\frac{2}{3}\right| = 8\frac{2}{3}$ وحدة مساحة

Ex2) Find the area bounded by the x-axis and the cuwe $y=6-x-x^2$

المحور x ⇐ y=0

$y_1=y_2$

$(6-x-x^2=0)$ x-1

$x^2+x-6=0 \Rightarrow (x+3)(x-2)=0 \Rightarrow x=-3, x=2$

نقاط التقاطع مع المحورx

$A^{2}_{-3} = \left|\int_{-3}^{2} f(x)dx\right| = \left|\int_{-3}^{2}(6-x-x^2)dx\right|$

= $\left|\int_{-3}^{2} 6dx - \int_{-3}^{2} xdx - \int_{-3}^{2} x^2dx\right|$

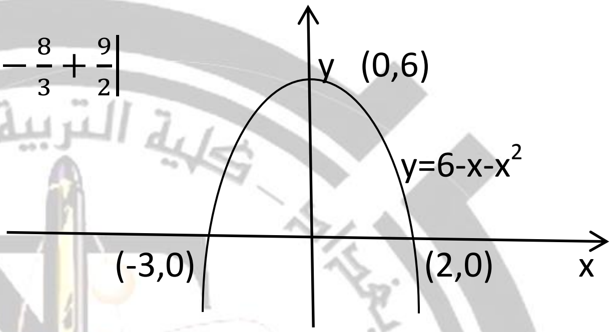
= $\left|6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{3}^{2}\Big|$

= $\left|\left(12 - 2 - \frac{8}{3}\right) - (-18 - \frac{9}{2} + \frac{27}{3})\right|$

= $\left|10 - \frac{8}{3} + 18 + \frac{9}{2} - 9\right| = \left|19 - \frac{8}{3} + \frac{9}{2}\right|$

= $\left|\frac{114-16+27}{6}\right|$

= $\left|\frac{114+11}{6}\right| = \left|\frac{125}{6}\right|$

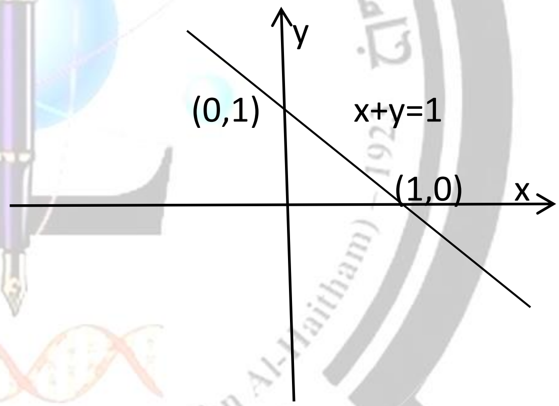


Ex3) Find the area bounded by the line x+y=1 and the coordinate axes

f(x)=y=1-x

x=0⇒y=1 (0,1)

y=0 ⇒ x=1 (1,0)



y_1=1-x , y_2=0

y_1=y_2

1-x=0 ⇒ x=1

A_{0}^{1}= $\left|\int_{x=0}^{x=1} f(x)dx\right| = \left|\int_{0}^{1}(1-x)dx\right| = \left|\int_{0}^{1} dx - \int_{0}^{1} xdx\right| = \left|x - \frac{x^2}{2}\right]_{0}^{1}\Big|$

= $\left|\left(1 - \frac{1}{2}\right) - 0\right| = \left|\frac{1}{2}\right| = \frac{1}{2}$ وحدة مساحة

X1=1-y,x2=0

X1=x2

1-y=0⇒ y=1

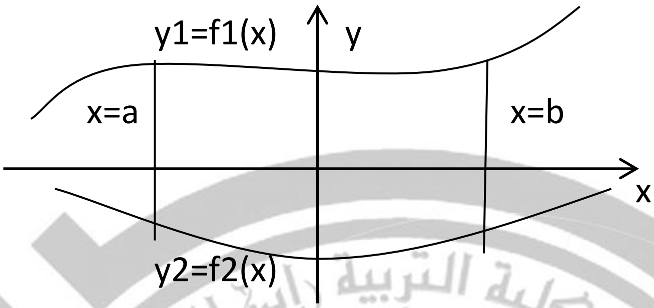
X=g(y) = 1-y

A_{0}^{1}= $\left|\int_{0}^{1} g(y)dy\right| = \left|\int_{0}^{1}(1-y)dy\right| = \left|\int_{0}^{1} dy - \int_{0}^{1} y\, dy\right|$

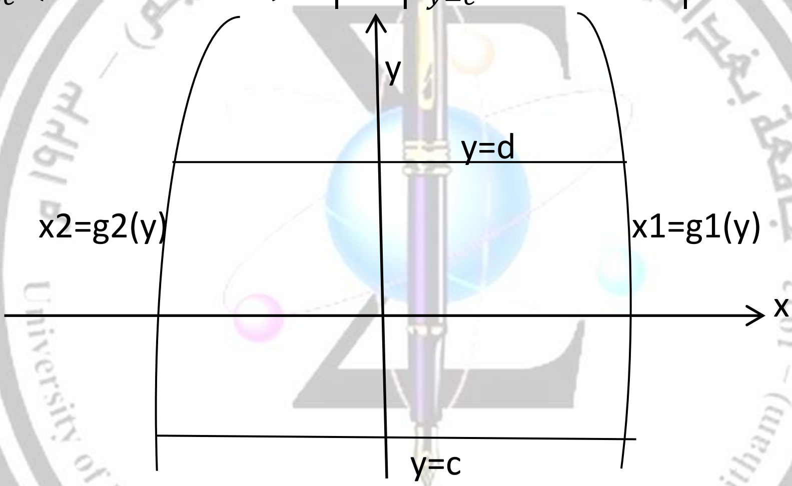
$= \left|y - \frac{y^2}{2}\right]_0^1\Big| = \left|\left(1 - \frac{1}{2}\right) - 0\right| = \left|\frac{1}{2}\right| = \frac{1}{2}$ وحدة مساحة

المساحة المحصورة بين منحنيين (Area between two curves)

$A_a^b = \left|\int_{x=a}^{x=b} (f_1(x) - f_2(x))dx\right| = \left|\int_{x=a}^{x=b} (y_1 - y_2)dx\right|$



$A_c^d = \left|\int_{y=c}^{y=d} (g_1(y) - g_2(y))dy\right| = \left|\int_{y=c}^{y=d} (x_1 - x_2)dy\right|$



Ex4) find the area bounded by the curve y=2-x2 and the line y=-x

Y1=y2

2-x2=-x

X2-x-2=0 ⇒ (x-2) (x+1)=0

X=2 , x=-1

Y=-x او y=2-x2

X=-1 ⇒ y=1 x=-1 ⇒ y= 2-1 =1

X=2 ⇒ $y = -2$ x=2 ⇒ y=2-4=-2

A_{-1}^{2}= $\left|\int_{x=-1}^{x=2}\left(f_1(x)-f_2(x)\right)dx\right| = \left|\int_{-1}^{2}(y_1-y_2)dx\right|$

= $\left|\int_{-1}^{2}(2-x^2-(-x)dx\right|$ = $\left|\int_{-1}^{2}(2-x^2+x)dx\right|$

= $\left|\int_{-1}^{2} 2dx-\int_{-1}^{2} x^2dx+\int_{-1}^{2} xdx\right|$ = $\left|2x-\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{2}\Big|$

= $\left|\left(4-\frac{8}{3}+\frac{\cancel{4}^{2}}{\cancel{2}_{1}}\right)-(-2+\frac{1}{3}+\frac{1}{2})\right|$

= $\left|\left(6-\frac{8}{3}\right)-(\frac{-12+2+3}{6})\right|$

= $\left|\frac{18-8}{3}-\frac{-7}{6}\right| = \left|\frac{10}{3}+\frac{7}{6}\right| = \left|\frac{20+7}{6}\right| = \left|\frac{27}{6}\right| = 4\frac{\cancel{3}^{1}}{\cancel{6}_{2}}$

= $4\frac{1}{20}$

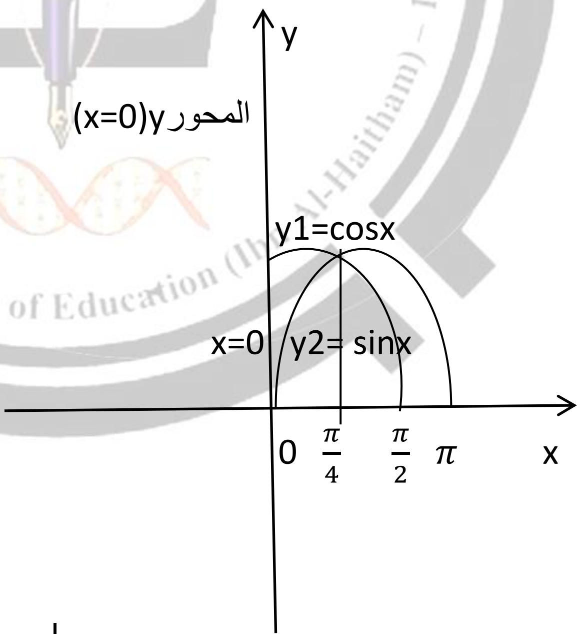
Ex(2)

أوجد مساحة المنطقة المثلثية الشكل في الربع الاول والمحددة بالمحور y والمنحنيين y=sin x و y=cos x ؟

y1=y2

cos x = sin x

x= $\frac{\pi}{4}$



$A_{0}^{\frac{\pi}{4}}$= $\left|\int_{0}^{\frac{\pi}{4}}\left(f_1(x)-f_2(x)\right)dx\right|$

= $\left|\int_{0}^{\frac{\pi}{4}}(\cos x-\sin x)dx\right|$

=$\left|\int_{0}^{\pi/4}\cos x\, dx-\int_{0}^{\pi/4}\sin x\, dx\right|$

= $\left|\sin x\,\right]_{0}^{\pi/4}+\cos x\,\Big]_{0}^{\pi/4}\Big|$

= $\left|(\sin\frac{\pi}{4}-\sin 0)+(\cos\frac{\pi}{4}-\cos 0\,)\right|$

= $\left|\left(\frac{1}{\sqrt{2}}-0\right)+(\frac{1}{\sqrt{2}}-1)\right|$ = $\left|\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right|$ = $\left|\frac{\cancel{2}}{\cancel{\sqrt{2}}}-1\right|$

$= \sqrt{2} - 1 = |1.4 - 1| = |0.4| = 0.4$

(1) التكامل بطريقة التجزئة :

d(u.v)=u.dv+v.du

$\int u.dv = \int [d(u.v) - v.du]$ نوزع التكامل

$\int u.dv = u.v - \int v.du$

Ex:- $\int \frac{x^3}{(x^2+1)^{3/2}}$ dx

$\int \underbrace{x^2}_{u} \frac{x}{(x^2+1)^{3/2}}$ dx

Let u=$x^2 \Longrightarrow du = 2x\, dx$

dv=$\frac{x}{(x^2+1)^{3/2}}\, dx \Longrightarrow v = \frac{1}{2} \int (x^2+1)^{-3/2} .2x\, dx$

= $\frac{1}{2} \quad \frac{(x^2+1)^{-1/2}}{-\frac{1}{2}}$

= $\frac{-1}{(x^2+1)^{1/2}}$

$\int u.dv = u.v - \int v.du$

$\int \frac{x^3 dx}{(x^2+1)^{1/2}} = x^2 . \frac{-1}{(x^2+1)^{1/2}} - \int \frac{-1}{(x^2+1)^{1/2}} .2x\, dx$

= $\frac{-x^2}{(x^2+1)^{1/2}} + \int (x^2+1)^{-1/2} .2x\, dx$

= $\frac{-x^2}{(x^2+1)^{1/2}} + \frac{(x^2+1)^{1/2}}{\frac{1}{2}} + c$

= $\frac{-x^2}{\sqrt{x^2+1}} + 2\sqrt{x^2+1}$ +c

(2) التكامل بطريقة تجزئة الكسور :-

اذا كان البسط اكبر او يساوي المقام فأننا نقوم بعملية القسمة الطويلة ثم نكامل كما موضح في المثال التالي :

Ex: $\int \frac{4+x^2}{8+x}$

نرتب حدود x من الاعلى الى الادنى

$$\begin{array}{r|l} & x-8 \\ \hline x+8 & x^2+4 \\ & \mp x^2 \mp 8x \\ \hline & -8x+4 \\ & \pm 8x \pm 64 \\ \hline & 68 \text{ الباقي} \end{array}$$

$\int \frac{x^2+4}{x+8}$ dx

$\int \left[(x-8)+\frac{68}{x+8}\right] dx$

$\int \underset{u}{(x-8)}\underset{du}{dx} + 68\int \frac{1}{\underset{u}{x+8}} \underset{du}{dx}$

$\frac{(x-8)^2}{2} + 68\ln|x+8| + c$

اما اذا كان المقام اكبر من البسط فأننا نقوم بتجزئة الكسور الى مجموعة من الكسور يسهل علينا ايجاد تكاملة .

ملاحظة (1) اذا كان العامل بالصورة $\frac{1}{(x-a)^n}$ حيث $n\in \mathrm{I}^+$ فأن تجزئته تكون بالشكل :

$$\frac{1}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

(2) اما اذا كان العامل بالصورة ax^2+bx+c ولا يمكن تحليله فأن بسطه يكون بالشكل Ax+B اي تقليل درجة واحدة .

كما موضح في المثال التالي :

Ex : $\int \frac{4-2x}{(x^2+1)(x-1)^2} dx$

$\frac{4-2x}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{c}{x-1} + \frac{D}{(x-1)^2}$

$= \frac{(Ax+B)(x-1)^2+c(x^2+1)(x-1)+D(x^2+1)}{(x^2+1)(x-1)^2}$

$= \frac{(Ax+B)\ (x^2-2x+1)+(cx^2+c)(x-1)+Dx^2+D}{(x^2+1)(x-1)^2}$

$= \frac{Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+cx^3-cx^2+cx-c+Dx^2+D}{(x^2+1)(x-1)^2}$

$= \frac{(A+c)x^3+(-2A+B-c+D)x^2+(A-2B+c)x+(B-c+D)}{(x^2+1)(x-1)^2}$

A+c=0 ..... (1) -

2A+B-c+D=0 ..... (2)]

$$\begin{array}{r} -2A+B-c+D=0 \\ \mp B \pm c \mp D = \mp 4 \\ \hline -2A = -4 \\ \therefore A = 2 \end{array}$$

A-2B+c=-2....(3)

B-c+D=4.....(4)

C=-2 $\Leftarrow c = -A \ldots (1)$ من معادلة

B=1 $\Leftarrow$ 2-2B+(-2) = -2 ...(3) من معادلة

D=1 $\Leftarrow$ 1-(-2)+D=4 ...(4) من معادلة

$\therefore \ \frac{4-2x}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}$

$\int \frac{4-2x}{(x^2+1)(x-1)^2}\, dx = \int \left[\frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}\right] dx$

$= \int \frac{2x+1}{x^2+1}\, dx - 2\int \frac{dx}{x-1} + \int \frac{1}{(x-1)^2}\, dx$

$= \int \frac{2x\ \ dx}{\underset{u}{x^2+1}} + \int \frac{dx}{x^2+1} - 2\int \frac{dx}{\underset{u}{x-1}} + \int \underset{u}{(x-1)}^{-2} \underset{du}{dx}$

$= \text{Ln}\ |x^2+1| + \ tan^{-1}(x) - 2\ Ln\ |x-1| + \frac{(x-1)^{-1}}{-1} + c$

أو $\text{Ln}|x^2+1| - \ cot^{-1}(x) - 2Ln|x-1| + \frac{(x-1)^{-1}}{-1} + c$

(3)<u>**التكامل بأستخدام التعويضات المثلثية :**</u> $a^2 + u^2, \sqrt{u^2-a^2}, \sqrt{a^2-u^2}$

- (1) $\int \frac{du}{\sqrt{a^2-u^2}} = \ sin^{-1}\frac{u}{a} + c$ او $- \ cos^{-1}\frac{u}{a} + \acute{c}$
- (2) $\int \frac{du}{u\sqrt{u^2-a^2}} = \ \frac{1}{a}\ sec^{-1}\frac{u}{a} + c$ او $\frac{-1}{a} csc^{-1}\frac{u}{a} + \acute{c}$
- (3) $\int \frac{du}{a^2+u^2} = \ \frac{1}{a}\ tan^{-1}\frac{u}{a} + c$ أو $- \ \frac{1}{a} cot^{-1}\frac{u}{a} + \acute{c}$

يتم تعويض عن u بالاعتماد على شكل الحد :-

- (1) اذا كان $\sqrt{a^2-u^2}$ نستخدم التعويض u=a sinθ

<u>Ex:</u> $\int \frac{dx}{\sqrt{9-x^2}}$ $\qquad \sqrt{9-x^2} = \sqrt{a^2-u^2} \ \Longrightarrow a = 3\ , u = x$

Let u=a sin θ

x=3sin $\theta \Longrightarrow dx = 3\cos\theta \ \ d\theta$

$\int \frac{3\cos\theta\ d\theta}{3\cos\theta}$ $\qquad \sqrt{9-x^2} = \sqrt{9-9sin^2\theta}$

$= \sqrt{9\ cos^2\theta}$

$\int d\theta$ $\qquad = 3\cos\theta$

$\theta + c$ $\qquad$ x= 3 sin $\theta \Longrightarrow \sin\theta = \frac{x}{3} \Longrightarrow \theta = sin^{-1}\frac{x}{3}$

$sin^{-1}\frac{x}{3}$ + c

- (2) اذا كان $a^2 + u^2$ نستخدم التعويض u=a tan θ

Ex: $\int \frac{dx}{\sqrt{25+x^2}}$ $\qquad$ $\sqrt{25+x^2} = \sqrt{a^2+u^2} \Longrightarrow a = 5, u = x$

Let u=a tan θ

x=3sin $\theta \Longrightarrow dx = 3\cos\theta\, d\theta$

$\int \frac{5\, sec^2\theta\ d\theta}{5\, sec\theta}$ $\qquad$ $\sqrt{9-x^2} = \sqrt{9-9sin^2\theta} = \sqrt{9(1-sin^2\theta)}$

$= \sqrt{9cos^2\theta}$

$\int d\,\theta$ $\qquad$ = 3 cos θ

$\theta + c$ $\qquad$ x= 3sin $\theta \Longrightarrow \sin\theta = \frac{x}{3} \Longrightarrow \theta = sin^{-1}\frac{x}{3}$

$sin^{-1}\frac{x}{3} + c$

- (2) اذا كان $a^2 + u^2$ نستخدم التعويض u= a tan θ

Ex : $\int \frac{dx}{\sqrt{25+x^2}}$ $\qquad$ $\sqrt{25+x^2} = \sqrt{a^2+u^2} \Longrightarrow a = 5, u = x$

Let u=a tan θ

x= 5 tan $\theta \Longrightarrow dx = 5\, sec^2\theta\, d\theta$

$\int \frac{5\, sec^2\theta\, d\theta}{5\, sec\theta}$ $\qquad$ $\sqrt{25+x^2} = \sqrt{25+25\, tan^2\theta}$

$\int \sec\theta\, d\theta$ $\qquad$ $= \sqrt{25\,(1+tan^2\theta)} = \sqrt{25\, sec^2\theta}$

In $|\sec\theta + \tan\theta| + c$ $\qquad$ = 5 sec θ

In $\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + c$ $\qquad$ sec $\theta = \frac{\sqrt{25+x^2}}{5}$,tan $\theta = \frac{x}{5}$

- (3) اذا كان $\sqrt{a^2+u^2}$ نستخدم التعويض u=a sec θ

Ex: $\int \frac{dx}{x\sqrt{x^2-16}}$ $\qquad$ $\sqrt{x^2-16} = \sqrt{u^2-a^2}, u = x,\ a = 4$

Let u = a sec θ

x=4 sec $\theta \Longrightarrow dx = 4\sec\theta\tan\theta\, d\theta$

$= \int \frac{4\sec\theta\tan\theta\, d\theta}{4\,sec\theta\,.4tan\theta}$ $\sqrt{x^2-16} = \sqrt{16\,sec^2\theta - 16}$

$= \frac{1}{4}\int d\theta$ $= \sqrt{16\,(sec^2\theta - 1)}$

$= \frac{1}{4}\theta + c$ $= \sqrt{16\,tan^2\theta} = 4\tan\theta$

$= \frac{1}{4}\,sec^{-1}\frac{x}{4} + c$ $x = 4\sec\theta \Rightarrow sec\theta = \frac{x}{4} \Rightarrow \theta = sec^{-1}\frac{x}{4}$

(4) التكامل بطريقة اكمال المربع :-

اذا كان من الممكن تحويل الدالة $f(x) = ax^2 + bx + c$ حيث $a \neq 0$ الى الشكل $au^2 + B$ بأستخدام طريقة اكمال المربع وكالاتي :-

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

بأضافة وطرح ^{2}(نصف معامل x) للقوس اعلاه

$$= a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + c - \frac{b^2}{4a}$$

$$= a\,(x+\frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

$$= a\,u^2 + B$$

حيث ان : $B = c - \frac{b^2}{4a}$ & $u = x + \frac{b}{2a}$

ملاحظة :- ان استخدام هذا الاسلوب يكون عندما :

1) لا يمكن تحليل الحد $ax^2 + bx + c$
2) عند اختفاء الثابت c
3) عندما يكون الحد $ax^2 + bx + c$ واقعا تحت جذر اي (مرفوع لقوى كسرية) .

والسبب في تحويل الحد $ax^2 + bx + c$ الى الصيغة $au^2 + B$ وذلك لغرض استخدام التكامل بالتعويضات المثلثية او التكامل المباشر ان امكن .

Ex: $\int \frac{dx}{\sqrt{2x - x^2}}$ جد التكامل بأستخدام طريقة اكمال المربع

$$\sqrt{2x - x^2} = \sqrt{-(x^2-2x)} = \sqrt{-(x^2-2x+1-1)}$$

$$= \sqrt{-(x^2-2x-1)-1}$$

$$= \sqrt{-(x-1)^2-1}$$

$$= \sqrt{1-(x-1)^2}$$

$$\sqrt{a^2-u^2}$$

a=1 , u= x-1 $\Longrightarrow$ $du = dx$

$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$ $\int \frac{du}{\sqrt{1-u^2}}$

$= sin^{-1}(x-1) + c$ $= sin^{-1} u + c$

أو $-cos^{-1}(x-1) + \acute{c}$ $= -cos^{-1}u + \acute{c}$

Double Integral 1-A

(1) $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\, dx\, dy$

$= \int_0^{\sqrt{2}} yx \,\Big]_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}}\, dy$

$= \int_0^{\sqrt{2}} [\,(y * \sqrt{4-2y^2}\,) - (\,y* -\sqrt{4-2y^2})\,]\, dy$

$= \int_0^{\sqrt{2}} 2y\, \sqrt{4-2y^2}\, dy$ $= \frac{2}{-2} \int_0^{\sqrt{2}} -2y \,.\sqrt{4-2y^2}\, dy$

$= 2.\, \frac{1}{-4} \quad \frac{(4-2y^2)^{3/2}}{\frac{3}{2}} \,\Big]_0^{\sqrt{2}}$ $= \frac{-1}{2} \int \sqrt{4-2y^2}\,.\; 4y\, dy$

$= -\frac{1}{3}\, (4-2y^2)^{3/2}\Big]_0^{\sqrt{2}}$

$= -\frac{1}{3}\, [\,0-(4)^{3/2}\,] = \frac{8}{3}$

(2) $\int_0^1 \int_0^{e^x} dy\, dx$

$= \int_0^1 y\,\Big]_0^{e^x} dx = \int_0^1 e^x dx = e^x\Big|_0^1 = e^1 - e^0 = e - 1$

(3) $\int_{-1}^{1} \int_0^2 (1-6x^2y) dx\, dy$

$\int_{-1}^{1} [\int_0^2 (1-6x^{-}y) dx] - dy = \int_{-1}^{1} (x-2x^3y)\Big|_0^2 dy = \int_{-1}^{1} (2 - 16y) dy$

=-2 y -8$y^2\Big|_{-1}^{1} = 2^{-6} - 8 - (-2^{-10} - 8) = -2^{-6+10} - 8 + 2 + 8 = 4$

Triple Integral – Volume

1- $\int_0^2 \int_0^{4-y^2} \int_0^{y} dzdxdy = \int_0^2 \int_0^{4-y^2} z \Big|_0^{y} dxdy = \int_0^2 \int_0^{4-y^2} (y - 0)dxdy$

= $\int_0^2 yx \Big|_0^{4-y^2} - dy = \int_0^2 (y * (4 - y^2) - 0)dy = \int_0^2 (4y - y^3)dy$

= $2y^2 - \frac{y^4}{4}\Big|_0^2 = 8 - 4 = 4$

2- $\int_{-2}^{1} \int_0^{2x} \int_0^{2-x} dz\, dy\, dx = \int_{-2}^{1} \int_0^{2x} z \Big|_0^{2-x} dy\, dx = \int_{-2}^{1} \int_0^{2x} (2 - x)dy\, dx$

= $\int_{-2}^{1} (2y - xy) \Big|_0^{2x} dx = \int_{-2}^{1} (4x - 2x^2)dx$

= $2x^2 - \frac{2}{3}x^3 \Big|_{-2}^{1} = (2 - \frac{2}{3})$- (8- $\frac{-8*2-}{3}$)

= 2 - $\frac{2}{3}$ – (8 - $\frac{16}{3}$) -(8+$\frac{16}{3}$)

= -6 - $\frac{18}{3}$= -6 -6 = 1 - |2| = 12

= $\frac{-18-18}{3}$ = $\frac{\cancel{-36}^{-12}}{\cancel{3}_1}$ = -12

3- $\int_{-1}^{1} \int_0^{1-x} \int_{4x^2}^{5-x^2} dz\, dy\, dx = \int_{-1}^{1} \int_0^{1-x} z^{5-x^2}\, dy\, dx$

= $\int_{-1}^{1} \int_0^{1-x} (5 - x^2 - 4x^2)dy\, dx = \int_{-1}^{1} \int_0^{1-x} (5 - 5x^2)dy\, dx$

= $\int_{-1}^{1} (5y - 5x^2 y)\Big|_0^{1-x} dx$

= $\int_{-1}^{1} (5(1 - x) - 5x^2(1 - x)dx = \int_{-1}^{1} (5 - 5x - 5x^2 + 5x^3)dx$

= 5x - $\frac{5x^2}{2} - \frac{5}{3}x^3 + \frac{5}{4}x^4\Big|_{-1}^{1}$

4- $\int_0^1 \int_{\frac{-1}{2}}^0 \int_{x-y}^{x+y+1} dz\, dy\, dx$

$\int_0^1 \int_{\frac{-1}{2}}^0 z \Big|_{x-y}^{x+y+1} dy\, dx = \int_0^1 \int_{\frac{-1}{2}}^0 [(x+y+1)-(x-y)] dy\, dx$

$= \int_0^1 \int_{\frac{-1}{2}}^0 (1+2y) dy\, dx = \int_0^1 (y+y^2) \Big|_{\frac{-1}{2}}^0 dx$

$= \int_0^1 0 - \left(\frac{-1}{2} + \frac{1}{4}\right) dx = \int_0^1 \frac{1}{4}\, dx = \frac{1}{4}\, x \Big|_0^1$

$= \frac{1}{4}$

H.W

(1) $\int_0^A \int_0^x x \sin y\, dy\, dy$

(2) $\int_{-2}^1 \int_{-y^2}^{y-2} dx\, dy$

(3) $\int_{-2}^0 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} y\, dy\, dx$

(4) $\int_{-1}^1 \int_{2y\sqrt{1-y^2}}^{3y\sqrt{1-y^2}} \int_{8+x^2+9y^2}^{x^2+9y^2} dZ\, dx\, dy$

(5) $\int_{-3}^3 \int_{-\frac{9-x^2}{3}}^{\frac{9-x^2}{3}} \int_{18-x^2-9y^2}^{8+x^2+9y^2} dZ\, dx\, dy$

Chapter Eight

Conic Sections

The distance between two points $p_1\ (x_1\ , y_1)$ and $p_2\ (x_2\ , y_2)$ is :

d=$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The distance between point p(x,y) and line Ax+By+c=0 is :

d=$\frac{|Ax+By+c|}{\sqrt{A^2+B^2}}$

1- The circle

The set of points in a plane whose distance from some fixed center point is constant radius value .

The general equation for circle of second degree is :

(x-h)$^2 + (y-k)^2 = r^2$

Let c(h,k) and r are the center and the radius of circle respectively . if p(x,y) be a point on this circle then the standard equation for circle is :

cp=r where r> 0

and $\sqrt{(x-h)^2 + (y-k)^2 = r}$

Note : if the center of circle is (0,0) then $x^2 +\ y^2 =\ r^2$

Example 1 : Find the center and radius of the circle has the following equation : $x^2 +\ y^2 - 4x + 6y - 3 = 0$

Solution : ($x^2 - 4x + 4$) + ($y^2 + 6y + 9) = 4 + 9 + 3$

(x- 2)$^2 + (y+3)^2 = 16$

The center is (2,-3) and the radius equal 4 .

Example 2 : Find the equation of circle if the center (-1,1) and through tangent the line x+ 2y = 4

Solution : d= $\frac{|Ax+By+c|}{\sqrt{A^2+B^2}} = \frac{|1(-1)+2(1)-4|}{\sqrt{1^2+2^2}} = \frac{3}{\sqrt{5}}$

$(x+1)^2 + (y-1)^2 = \frac{9}{5}$

Exercises :

1- Prove that the loci of the point p(x,y) is circle . if the distance between p and A(6,0) equal twice the distance between p and B(0,3)?

2- Find in equation for the circle through the points (1,0) , (0,1) and (2,2)?

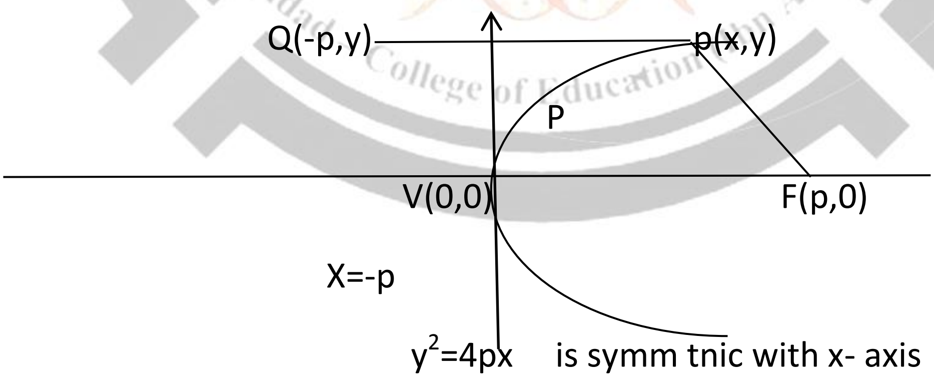
3- Find an equation for the circle through the points (2,3) , (3,2) and (-4,3) ?

4- Find an equation for the circle centered at (-2,1) that passes through the point (1,3) Is the point (1.1 , 2.8) inside , outside , or on the circle ?

5- Find equations for the tangents to the circle $(x-2)^2 + (y-1)^2 = 5$, at the points where the circle crosses the coordinate axis?

2- Parabola

Parabola is a set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane the fixed point is called focus (F) and the fixed line is called directory (L)



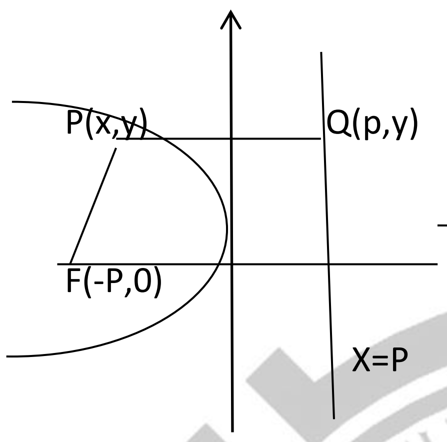
is symm tnic with x- axis

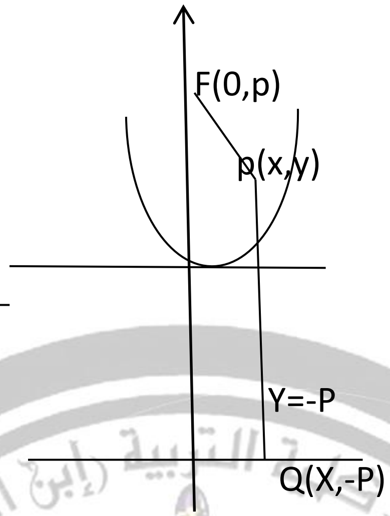
pF=pQ

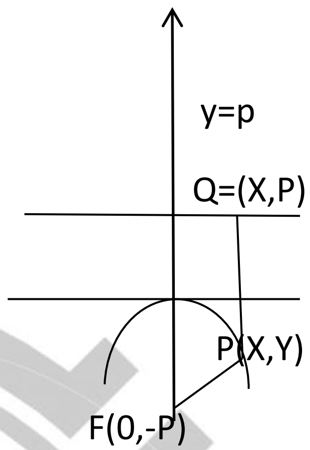
$\sqrt{(x-p)^2 + (y-0)^2} = \sqrt{(x+p)^2 + (y-y)^2}$

$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$

$y^2 = 4px \quad , x \geq 0 \quad is\ symmtric\ with\ x-axis$







$y^2 = -4px$ — Symmetric with x-axis & open to left — V(h,k) : (y,0)

$x^2 = 4py$ — sym – with – axis & open to up — v(h,k)= (0,0)

$x^2 = -4py$ — sym – with y – axis & open to down — v(u,k)=(0,0)

	Equation	Focus(F)	Directrix	Axis
1	$y^2 = 4px$ (y-k)$^2 = 4p(x-h)$	(p,0) (h+p,k)	X=-p X=h-p	x-axis
2	$y^2 = -4px$ (y-k)$^2 = -4p(x-h)$	(-p,0) (h-p,k)	X=p X=h+p	x-axis
3	$x^2 = 4py$ (x-h)$^2 = -4p(y-k)$	(0,p) (h,k+p)	Y=-p Y=k-p	y-axis
4	$x^2 = -4py$ (x-h)$^2 = -4p(y-k)$	(0,-p) (h,k-p)	Y=p Y=k+p	y-axis

<u>Example1 :</u> Discuss and sketch the following equation :

$$y^2 + 6y + 2x + 5 = 0$$

<u>Solution :</u>

(y+3)$^2 + 2x - 4 = 0$

$(y+3)^2 = -2x - 4$

$(y+3)^2 = -2(x-2)$

2=4p

P= $\frac{1}{2}$

V(h,k) = (2,-3) , F(h-p,k) = $(2-\frac{1}{2}, -3) = (\frac{3}{2}, -3)$

X=h+p=$2+\frac{1}{2} = \frac{5}{2}$

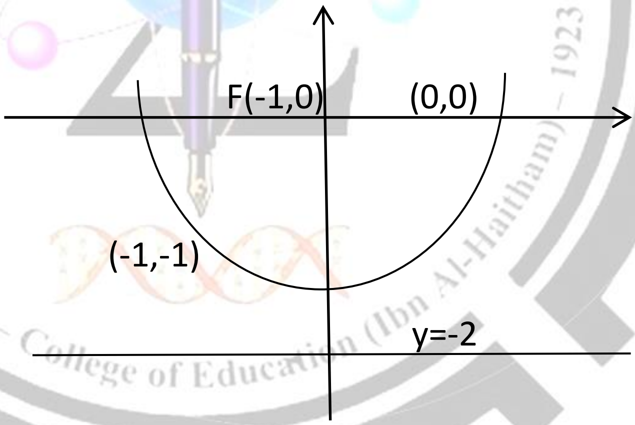
Example2 : find the focus and vertex and of the following parabola :

$$x^2 + 2x + 1 - 4y - 4 = 0$$

Solution : $(x+1)^2 = 4(y+1)$

V(h,k)= (-1,-1) , p = 1 , F(h,k+p)=(-1-1+1)=(-1,0)

y=k-p=-1-1=-2



Exercises :

1) Find an equation , focus and directory of each of the following parabola , then sketch each one :
   $y^2 = 12x$, $y^2 = -2x$, $x^2 = 6y$, $x^2 = -8y$, $y = 4x^2$

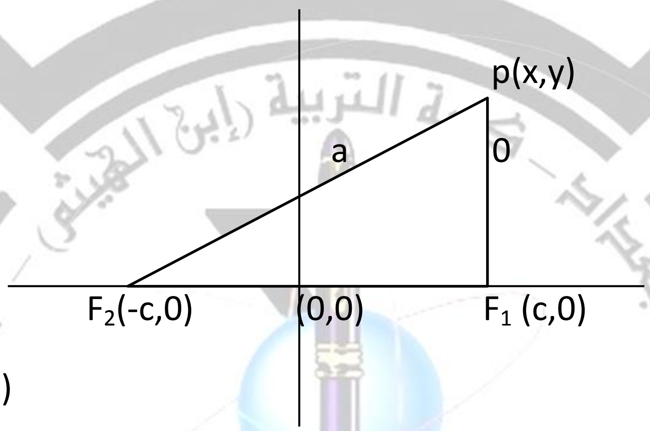
   $$y = -8x^2 , x = 2y^2 , x = -3y^2$$

2) Find focus , vertex and directory , and sketch of each of the following parabola :

a.$(x+1)^2 = -4(y-3), b.(y+2)^2 = 8(x-1)$

3- Ellipse

Ellipse is the set of points in a plane whose distances from two fixed points in the plane have a constant sum . The two fixed points are called foci (F_1 , F_2) of the Ellipse



$F_1(c,0)$, $F_2(-c,0)$

pF_1+pF_2=constant

$\sqrt{(x-c)^2+(y-0)^2} + \sqrt{(x+c)^2+(y-0)^2} = 2a \; where \; a > 0$

$\sqrt{x^2-2cx+c^2+y^2} = 2a - \sqrt{x^2+2cx+c^2+y^2}$

$x^2-2cx+c^2+y^2 = 4a^2 - 4a\sqrt{x^2+2cx+c^2+y^2} + x^2+2cx+c^2+y^2$

$4cx+4a^2 = 4a\sqrt{x^2+2cx+c^2+y^2}$

$cx+a^2 = a\sqrt{x^2+2cx+c^2+y^2}$

$(cx+a^2)^2 = a^2(x^2+2cx+c^2+y^2)$

$c^2x^2+2ca^2x+a^4 = a^2x^2+2ca^2x+a^2c^2+a^2y^2 \; where \; a > c$

$(a^2-c^2)x^2+a^2y^2 = (a^2-c^2)a^2$

$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1$ where $a > c$

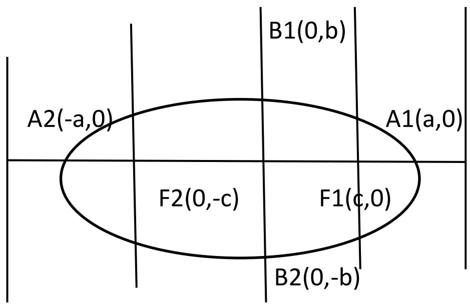
$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \quad \rightarrow c^2 = a^2-b^2 \; where \; a > b, b^2 = a^2-c^2$

The Ellipse equation of x-axis with center (0,0) and foci $(\pm c, 0)$

$e=\frac{c}{a}$, $c < a \rightarrow e < 1$ (Eccentricity)

The Directory L_1,L_2 are :

x=$\pm\ \frac{a}{e}$ $\quad or\ x = \pm\frac{a^2}{c}$



if x=0

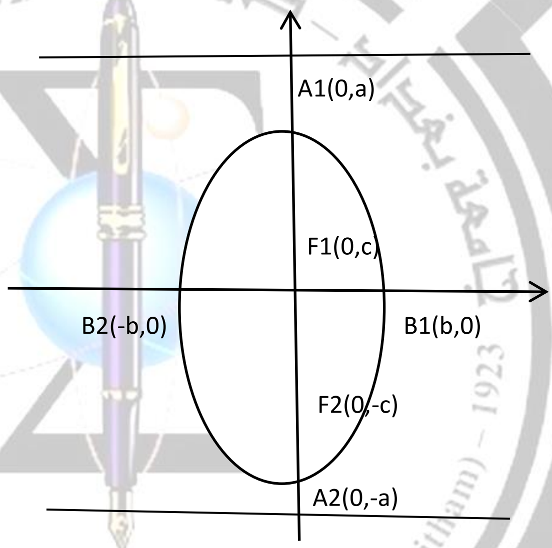
(0,-b) , (0,b)

If y=0

(-a,0) , (a,0)

But the Ellipse equation of y-axis with center (0,0) and foci (0, $\pm c$)

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$



If x=0

(0,-a) , (0,a)

If y=0

(-b,0) , (b,0)

The Directory L_1 , L_2 are :

Y= $\pm\ \frac{a}{e}$ $\quad or\ y = \ \pm\frac{a^2}{c}$

x-axis	y-axis
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ F_1(c,0) , F_2(-c,0) Centre C(0,0) A_1(a,0) , A_1(-a,0) B_1(0,b) , B_2(0,-b) x= $\pm\ \frac{a}{e}$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ F_1(0,c) , F_2(0,-c) Centre C(0,0) A_1(0,a) , A_2(0,-a) B_1(b,0) , B_2(-b,0) y= $\pm\ \frac{a}{e}$

$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ $F_1(h+c,k)$, $F_2(h-c,k)$ Centre C(h,k) $A_1(h+a,k)$, $A_1(h-a,k)$ $B_1(h,k+b)$, $B_2(h,k-b)$ $x=h\pm\frac{a}{e}$	$\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$ $F_1(h,k+c)$, $F_2(h,k-c)$ Centre C(h,k) $A_1(h,k+a)$, $A_2(h,k-a)$ $B_1(h+b,k)$, $B_2(h-b,k)$ $y=k\pm\frac{a}{e}$

Example 1 : Find and sketch the equation of Ellipse that the focus F(0,0) and a=3 , c(0,2) ?

Solution :

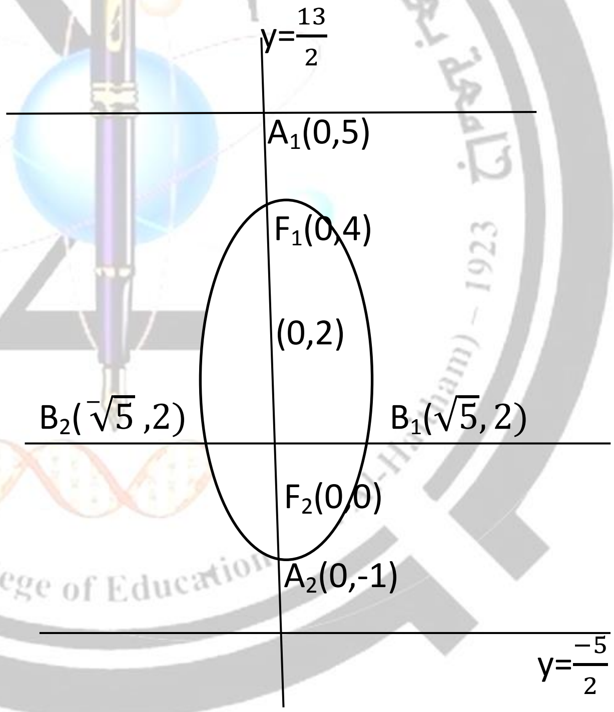
$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$

$b^2=a^2-c^2 \rightarrow b^2=9-4 \rightarrow b=\pm\sqrt{5}$

$\frac{(y-2)^2}{9}+\frac{(x-0)^2}{5}=1$

$e=\frac{c}{a}=\frac{2}{3}$

$y=k\mp\frac{a}{e}=2\mp\frac{3}{\frac{2}{3}}=2\mp\frac{9}{2}$



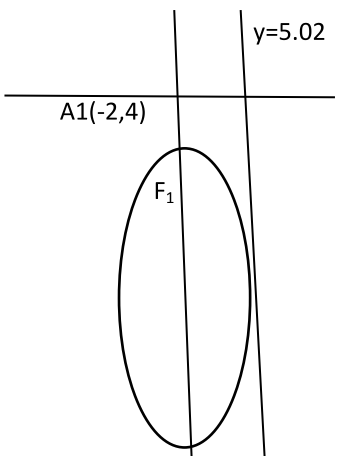
Example 2: Discuss and sketch the following equation :

$$9x^2+4y^2+36x-8y+4=0$$

Solution :

$(9x^2+36x+36)+(4y^2-8y+4)=36$

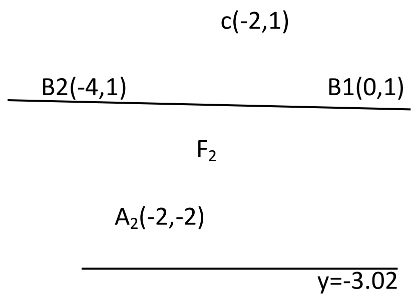
$9(x^2+4x+4)+4(y^2-2y+1)=36$



$$9(x+2)^2 + 4(y-1)^2 = 36$$

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{9} = 1$$

$$a^2 = 9 \rightarrow a = \mp 3, b^2 = 4 \rightarrow b = \mp 2$$



$$c^2 = a^2 - b^2 \rightarrow c^2 = 9 - 4 = 5 \rightarrow c = \pm\sqrt{5}$$

h=-2,k=1→ $c(-2,1) centre\ of\ Ellipse$

F_1(h,k+c),F2(h,k-c) → $F_1(-2,1+\sqrt{5}), F_2(-2,1-\sqrt{5})$

A_1(h,k+a),A_2(h,k – a)→ $A_1(-2,1+3), A_2(-2,1-3) \rightarrow A_1(-2,4), A_2(-2,-2)$

B_1(h+b,k),B(h-b,k) → $B_1(-2+2,1), B_2(-2-2,1) \rightarrow B_1(0,1), B_2(-4,1)$

y=k$\pm\frac{a^2}{c} \rightarrow y = 1 \pm \frac{9}{\sqrt{5}}$

Exercises :

1- Find an equation of Ellipse if the vertices A1(1,1) , A2(1,7) , B1(3,4) and B2(-1,4) ?

2- Prove that the tangent of Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point p(x_1,y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$?

3- Find the equation of an Ellipse of eccentricity $\frac{2}{3}$ if the line x=9 is one directrix and the focus is at (4,0) ?

4- Find the equation of Ellipse having the center C, focus F and sketch graph :

1) C(0,0) , F(0,2) , a=4
2) C(0,0) , F(-3,0) , a=5
3) C(0,2) , F(0,0) , a=4
4) C(-3,0) , F(-3,-2) , a=4

5- Find and sketch the center , vertices and foci of the Ellipse :

1) $25x^2 + 9y^2 - 100x + 54y - 44 = 0$
2) $9x^2 + 4y^2 = 36$

3) $4x^2 + 9y^2 = 144$

4- Hyperbole

Hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference , The two fixed points are called foci (F_1,F_2) of the Hyperbola .

F1(-c,0) , F2(c,0)

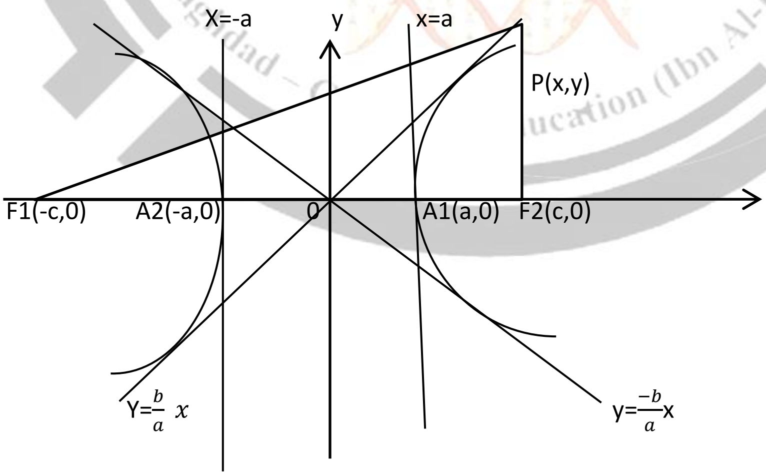
pF_1-pF_2=2a

$$\sqrt{(x+c)^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2}=a2 \; where \; a>0$$

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1 \qquad where \quad c>a\,,b^2=c^2-a^2$$

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1 \quad \rightarrow \quad c^2=a^2+b^2$$

The Hyperbola equation the foci on x-axis with center (0,0) , intersection with x-axis at $(\pm\, a, 0)$ and its asymptotes y=$\pm\,\frac{bx}{a}$



Eccentricity : e=$\frac{c}{a}$ $, c>a$

Directrix: x=$\pm\ \frac{a}{e}$

x-axis	y-axis
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $F_1(c,0)$, $F_2(-c,0)$ Centre C(0,0) $A_1(a,0)$, $A_1(-a,0)$ $B_1(0,b)$, $B_2(0,-b)$ y=$\pm\ \frac{b}{a}\ x$ x= $\pm\frac{a}{e}=\pm\frac{a^2}{c}$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ $F_1(0,c)$, $F_2(0,-c)$ Centre C(0,0) $A_1(0,a)$, $A_2(0,-a)$ $B_1(b,0)$, $B_2(-b,0)$ y=$\pm\ \frac{a}{b}\ x$ y= $\pm\frac{a}{e}=\pm\frac{a^2}{c}$
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ $F_1(h+c,k)$, $F_2(h-c,k)$ Centre C(h,k) $A_1(h+a,k)$, $A_1(h-a,k)$ $B_1(h,k+b)$, $B_2(h,k-b)$ (y-k)=$\pm\ \frac{b}{a}\ (x-h)$ x= $h\pm\frac{a}{e}$ x= $h\pm\frac{a^2}{c}$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ $F_1(h,k+c)$, $F_2(h,k-c)$ Centre C(h,k) $A_1(h,k+a)$, $A_1(h,k-a)$ $B_1(h+b,k)$, $B_2(b-b,k)$ (y-k)=$\pm\ \frac{b}{a}\ (x-h)$ y= $k\pm\frac{a}{e}$ y= $k\pm\frac{a^2}{c}$

Example 1: sketch the following hyperbolas :

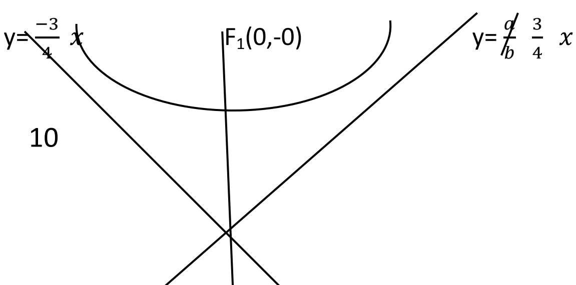
1. $\frac{y^2}{9}-\frac{x^2}{16}=1$ 2. $\frac{x^2}{16}-\frac{y^2}{9}=1$

Solution : 1. C(0,0)

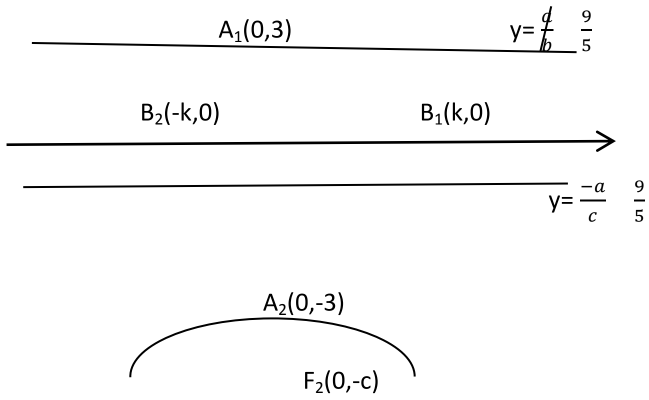
$a^2=9\ \rightarrow a=3\ , b^2=16\ \rightarrow b=4$

$c^2=\ a^2+\ b^2\ \rightarrow c\ \sqrt{a^2+b^2}=\ \pm5$

$F_1(0,5)$, $F_2(0,-5)$



$A_1(0,3)$, $A_1(0,-3)$



$B_1(4,0)$, $B_2(-4,0)$

y=$\pm \frac{a}{b} x \rightarrow y = \pm \frac{3}{4} x$

y=$\pm \ \frac{a^2}{c} \ \rightarrow y = \pm \frac{9}{5}$

<u>Example 2:</u> Sketch the following hyperbolas :

1. $9(x-2)^2 - 4(y+3)^2 = 36$
2. $4(x-2)^2 - 9(y+3)^2 = 36$
3. $4x^2 - 5y^2 - 16x + 10y + 31 = 0$

<u>Solution :</u> 1. $\frac{(x-2)^2}{4} - \ \frac{(y+3)^2}{9} = 1$

C(2,-3)

$a^2 = 4 \ \rightarrow a = 2, b^2 = 9 \rightarrow b = \ \pm 3$

$c^2 = \ a^2 + b^2 \rightarrow c = \ \sqrt{a^2 + b^2} = \ \pm \sqrt{13}$

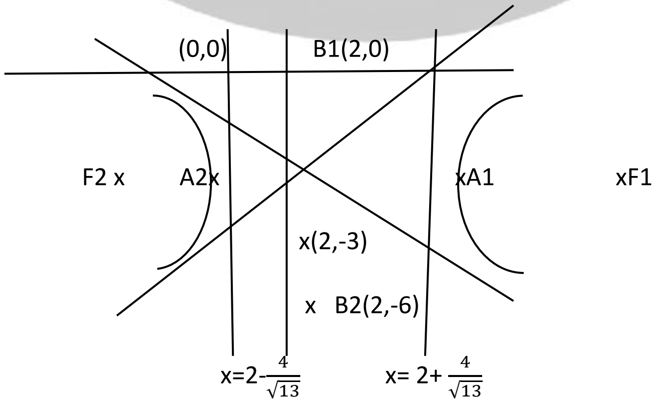
$F_1(2+\sqrt{13}$, -3) F_2 (2- $\sqrt{13}$, -3)

A_1(2+2,-3) , A_1(2-2,-3) → A_1 (4,-3) , A_1 (0,-3)

B_1(2,-3+3) , B_2(2,-3-3) → B_1(2,0) , B_1(2,-6)

x=h+ $\pm \ \frac{a^2}{c} \ \ \rightarrow x = 2 \ \pm \ \frac{4}{\sqrt{13}}$

(y-k) = $\pm \ \frac{b}{a} \ (x-h) \rightarrow (y+3) = \ \pm \ \frac{3}{2} \ (x-2)$



Exercises

1- Prove that tangent of Hyperbole $b^2x^2 - a^2y^2 = a^2b^2$ at the points p1 (x1,y1) is b2xx1 – a2yy1 = a2 b2 ?

2- Find an equation of Hyperbola from the given in formation , F1(0,0) , F2(0,4) and pass through the point (12,9) ?

3- Find an equation of Hyperbola that the eccentricity $\sqrt{2}=\frac{c}{a}$ and the vertices $(0,\pm 2)$?