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College of Education for Pure Science / Ibn Al-Haitham
Department of Mathematics

Numerical Solution of Fractional Order ODEs Using Linear Multistep Methods

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By
Hayder Hashem Khayoon

Supervised by
Asst. Prof. Dr. Fadhel Subhi Fadhel

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بسم الله الرحمن الرحيم

﴿ يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ

أُوتُوا الْعِلْمَ دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ ﴾

صَدَقَ اللَّهُ العَظِيم

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ABSTRACT

Fractional calculus is the subject for evaluating derivatives and integrals of non-integer orders of a given function, while fractional differential equations (considered in this work) is the subject of studying the solution of differential equations of fractional order, with contain initial or boundary condition. The solution of fractional differential equations has so many difficulties in their analytic solution, therefore, numerical methods may be in most cases are the suitable methods of finding the solution.

Therefore, the main objective of this work is to study ordinary differential equation of fractional order and solving the numerically equations using linear multistep method by utilizing the fractional Taylor series expansion. In addition, the numerical results have been improved using several approaches, such as variable step size method, predictor-corrector methods, Richardson extrapolation method and variable order method, the calculations are written using the mathematical software MATLAB 16a.

LIST OF SYMBOLS AND ABBREVIATIONS

${}^{C}D_{x}^{\alpha}$	The Caputo fractional derivative of order α.
${}^{R}D_{x}^{\alpha}$	The Riemann-Liouville fractional derivative of order α.
${}_{a}I_{x}^{\alpha}$	The Riemann-Liouville fractional integral of order α.
Γ	The gamma function.
B	The beta function.
ODEs	Ordinary differential equation.
FODEs	Fractional ordinary differential equation.
α	Order of fractional differentiation.
$E_{\alpha,1}$	One parameter Mittag-Leffler function.
$E_{\alpha,\beta}$	Two parameter Mittag-Leffler function.
${E^{(m)}}_{\propto,\beta}$	The m^{th} derivative of the Mittag-Leffler function.
LMMs	Linear Multistep Methods.
FLMMs	Fractional Linear Multistep Methods.
PCM	Predictor-Corrector Method.
IVPs	initial value problems.
$\mathbb{N}$	Set of natural number.
REM	Richardson Extrapolation Method.
Eq.	equation.
■	The end of the proof.

Contents

Introduction

Fractional calculus refers to the properties of the derivatives and the integrals of non-integer orders of given function. It is a branch of applied mathematics which generalizes the derivatives and the integrals of functions to non-integer orders; it is called fractional derivatives and integrals. Fractional calculus began in 1695 from several speculations of G.W. Leibniz (1646- 1716) and Leonhard Euler (1707- 1783) [33].

In the beginning, fractional calculus was studied only in pure mathematics without any real life applications. But in the last decades, it takes a large field in applications and paid attention to authors and researchers to seek in this object. However, this topic is an important of particular interest only the last thirty years [38] .

Perhaps, fractional calculus translates the reality of nature better. In other words, it talks with nature in this language, therefore, it is an efficient tool to formulate many natural phenomenas. The theory of fractional calculus comprises even complex orders for the derivatives and the integrals, briefly differ-integrals and left or right differ-integrals analogously to left or right derivatives.

Indeed, the differ-integrals are the operator which comprises both integer order of the derivatives and the integrals as a special case, the reason is that why in present-day fractional calculus be so popular and many applications arise.

Although fractional calculus subject with closely than 300 years' history, the progression of fractional calculus was a bit lazy at the early stage. The earliest systematic studies are made in the nineteenth century.

Fractional derivatives are an excellent tool for describing the memory and hereditary properties of various materials and processes while in integer-order models such effects are neglected. The first application of a semi-derivative which means derivative of difference orders equals 1/2 was made by Abel in 1823 [37] and this application of fractional calculus is in relationship with the solution of the integral equations for a tautochrones problem.

Fractional derivatives and the fractional integrals can be defined in different ways, like Riemann-Liouville, Caputo fractional derivatives, Grunwald-Letnikow [36].

There are two main ways to introduce fractional calculus, namely, the continuous and the discrete approaches. The continuous approach is based on the Riemann-Liouville fractional integral, which has Cauchy integral formula [38] as a starting point. The discrete approach is based on Grunwald-Letinikor fractional derivative.

As a generalization of the fact that ordinary derivatives are limits of difference quotients. The definition of Riemann-Liouville plays a paramount role mainly in the expansion of the theory of the fractional derivatives and fractional integrals and for its application in pure mathematics [39]. However, applied problems require proper definitions of fractional derivatives which can provide initial conditions with clear physical interpretation for the Fractional ordinary differential equation FODEs. This makes Caputo fractional derivative more suitable to be applied.

Fractional ordinary differential equation has been found to be efficient to describe some physical phenomena's, such as damping laws, rheology, fluid flow and so on [6]. With the expansion of fractional calculus theory, it is found only in latest years that the behavior of many systems can be described by using a system of FODEs. In the last few years, it is found that the FODEs described many of physical phenomena and various applications, so they are interested to look for solutions of this type of equations. There are only a few techniques for finding the solution of FODEs, since it is a new subject in mathematics.

The solution of FODEs are much involved. In general, there exists no method that yields an exact solution for FODEs and this has mandated the use of both analytical and numerical methods. Only approximate solution derived by using linearization or perturbation methods [6].

In recent years, many researchers have focused on the numerical solution of FODEs. Some numerical methods have been developed, such as: In 2010, Zurigat, Mamani and Alawneh [47], developed a framework to obtain approximate solutions to systems of FODEs by employing the homotopy analysis method. In 2011, Ibis and Bayram in 2011[19], presented fractional differential transform method for solving FODEs. In 2013, Kazem [23], applied the Laplace transform method for solving linear FODEs. In 2014, Jaradat, Zurigat and Safwan [21], proposed an algorithm which is a classy combination of Laplace transform method with the homotopy analysis which is called Laplace homotopy analysis method for the analytic solution of systems of FODEs. In 2014, Ding and jiang [14], presented wave form relaxation method and applied these methods on FODEs. In 2015, Damarla and Kundu [10], introduced a new applications of piecewise linear orthogonal triangular function to solve FODEs.

Some types of systems occur in the modeling of continuum and statistical mechanics [30], in chemistry [7], in fluid and seepage flow [17], in applied mathematics and computers [47].

The consideration of stability is one of the most important and essential issues for FODEs. It is an open object for researchers whom concern to study a system of FODEs.

The stability analysis is a central task in study of FODEs and stability analysis has been performed by many authors [3]. Matignon [33], in his Ph.D. thesis, was the first researcher who introduced some stability results related to a restrictive modeling of FODEs. Several authors [12] studied the stability and asymptotic stability of the linear fractional system with multiple order Caputo derivative.

Indeed, the stability results of FODEs has a lot of applications as in physical systems, dynamical systems and other application fields [27]. In fact, it is not easy to study the stability of nonlinear FODEs, some of researchers had been investigated some results of the stability

[28]. Mittag-Leffler stability is introduced to study the stability of FODEs by using Lyapunov direct method [28].

The first conference concerning fractional calculus, the merit is due to Betraim Ross who organized the first conference on the fractional calculus and it is applications at the University New Haven, June 1974 and edited the conferences proceedings. The first conference which is followed by the other conferences such as those conducted from Garry Roach and Adam Mc Bride (University of Strathclyde, Scotland) in 1984, by Katsuyuk Nishimoto (Nihon University, Japan) in 1989, and by Ivan Dimovski, Peter Rusev and Virginia Kiryakova (Varna, Bulgaria) in 1996, [17]. International conference on FODEs and its Applications, Novi Sad, Serbia, July 18 - 20, 2016, International Conference on FODEs and systems FSS 2017, Poland, 9-11 October, 2017.

For a first monograph, the deserve is ascribed to K.B Oldham and J. Spanier who, after a joint cooperation began in 1968, published a book devoted to fractional calculus in 1974. Nowadays, there are more other books of private issues and proceedings of journals published that referred to applications of the fractional calculus in various scientific areas included private functions, physics, chemical, control theory, stochastic processes, anomalous diffusion, rheology. Several private issues appeared in the last few decades, which contain some improved and selected papers presented at conferences and advanced schools, concerning several applications of the fractional calculus. Already since some years, there are two journals devoted about exclusively to the subject of the fractional calculus; namely Journal of the fractional calculus (Editor-in-chief: K. Japan) started in l992 and fractional calculus and applied analysis (Managing Editor: V. Kiryakova, Bulgaria) started in 1998. Recently the new journal of the fractional dynamic systems has been started to begin in 20l0, and Journal of the Fractional Calculus and has applications (JFCA) [35], (Managing Editor: A.M.A El-Sayed] started in 12/2010.

This thesis consists of three chapters. Chapter one is concerned with elementary concepts and basic definitions of necessary fractional calculus concepts, such as derivative and integration. which are related to the main subject of this work. In Chapter two, the derivation of certain of Fractional Linear Multistep Method (FLMMs) method will be presented that are used to solve initial value problems (IVPs) of FODEs. Also, the fractional tayler series expansion is presented which is necessary in the application of the (FLMMs).

In chapter three, the efficiency of the numerical results has been improved by introducing three methods in FODEs, namely the Predictor-Corrector Method (PCM), Richardson Extrapolation Method (REM) and Variable Order Methods. It notable that numerical examples are given for illustration and comparison purposes are given. It is an important to notice that, the calculations are written using the mathematical software MATLAB 16a, and the results are given in tabulated form.

Chapter

1

Basic Concepts

CHAPTER 1

BASIC CONCEPTS

Introduction

Fractional calculus has a long history whose infancy dates back to the beginning of classical calculus and it is a rich area having interesting applications in real life problems. This type of calculus has its origins in the generalizations of the differ-integrals calculus.

In this chapter, some basic fundamental computes concerning the fractional derivatives and the fractional integrals of functions will be given. Hence, this chapter consists of six sections; in section (1.1), the historical review of fractional calculus is given for completeness purpose; in section (1.2), the gamma and the beta functions are introduced, as well as, some of their important properties; in section (1.3) the fractional derivative, with some basic definitions for evaluating derivatives of an arbitrary order are given. Also, in section (1.4), basic definitions concerning fractional integrals are given; in section (1.5), some fundamental properties related to fractional derivatives and fractional integrals are given, fractional differential Finally, in section (1.6) the Fractional Order of ODEs are given as a model for the considered problem of this thesis.

1.1 Historical Background

In l730 Euler observed that the result of the valuation the derivative $\frac{d^q}{dx^q}$ of the power function x^n, $n > 0$ has a meaning for fractional order q. Pirre-Simon Laplace (l749- l827) in l812 suggested the notion of FODEs. In the research of Lacroix l8l9, the notion of Euler was iterated and the exact formula for the valuation the derivative:

$\frac{d^{1/2} x^n}{dx^{1/2}}$ and was already given: [43].

Lacroix in (1820) [1] developed a formula for the n^{th} order derivative of $y = x^m$, n, m $\in \mathbb{N}$, which is:

$$D^n y = \frac{(m!)x^{m-n}}{(m-n)!}, \text{ where } D^n = \frac{d^n}{dx^n}$$

where $n \leq m$ is an integer, and with fractional order, he further obtained the following formula:

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \text{ where } D^\alpha = \frac{d^\alpha}{dx^\alpha}$$

where α , β are any two positive fractional numbers, such that $\alpha \leq \beta$ and Γ is the complete gamma function, which will be defined in section (1.2). In particular, he calculated the following derivative of fractional order [4]

$$D^{1/2}x = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} x^{1/2} = 2\sqrt{\frac{x}{\pi}}$$

The next stage was taken form Joseph Fourier (1768- 1830) in 1822 who suggested the notion of using his integral representations of f to define the derivative for fractional order. He obtained the following integral representations for a function f and its derivatives:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \cos s(x-t)\,ds$$

and

$$D^m f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(t)\,dt \int_{-\infty}^{\infty} s^m \cos\left[s(x-t) + \frac{m\pi}{2}\right] ds$$

and substituting the positive integer m by arbitrary real number α, yields officially to:

$$D^\alpha f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} s^\alpha \cos\left[s(x-t) + \frac{\alpha\pi}{2}\right] ds$$

In 1832, Joseph Liouville (1809- 1882) officially expanded the formula offered by Lacroix for the derivatives of the integral order m given by:

$$D^m e^{ax} = a^m e^{ax},$$

to the derivatives of the arbitrary order α as follows:

$$D^\alpha e^{ax} = a^\alpha e^{ax},\ a \in R.$$

using the series expansion of the function f, Liouville is derived the formula:

$$D^{\alpha} f(x) = \sum_{m=0}^{\infty} c_m a_m^{\alpha} e^{a_m x} , \tag{1.1}$$

where:

$$f(x) = \sum_{m=0}^{\infty} c_m \exp(a_m x), \ \mathrm{Re}(a_m) > 0. \tag{1.2}$$

Formula (1.1) is referred to as Liouville the first formula of fractional derivative. It can be used as that formula for the derivative of an the arbitrary order α, which may be rational or irrational or a complex. However, it used for functions of the form (1.2). In order to extend his first definition given by (1.1), Liouville formulated another definition of the fractional derivative based on gamma function, which is:

$$D^{\alpha} x^{-\beta} = (-1)^{\alpha} \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta}, \ \beta > 0.$$

This is called the Liouville second definition of fractional derivatives. He successfully applied his both definitions of problems in the potential theory. However, Liouville's first definition is restricted to a certain class of functions in the form of Eq. (1.2), and his second definition is useful only for rational functions. Neither of Liouville's first and second definitions was found to be suitable for the wide class of functions [11].

Bernhard Riemann (1826-1866) in 1847 proposed a different definition for fractional integration that involved a definite integral and was apply to power series with fractional exponents [39]:

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}},\ x>0$$

and since that at this time has be one of the fundamental formula of fractional integration jointly with Liouville construction of fractional integrations given by:

$$D^{-\alpha}f(x)=\frac{1}{(-1)^{\alpha}\Gamma(\alpha)}\int_0^{\infty}\phi(x+s)s^{\alpha-1}ds,\ -\infty<x<\infty,\ \operatorname{Re}(\alpha)>0$$

It is necessary to note here that both Riemann and Liouville dealt with the so called (complementary) function, which arise when one attempts to treat FODEs as fractional integration of the order $-\alpha$ [42].

In 1867 Anton Grunwald (1838- 1920) and Aleksey Letnikov (1837-1888) developed an approach to fractional differentiations based on the limit, as follows:

$$D^{\alpha}u(x)=\lim_{h\to 0}\frac{(\Delta^{\alpha}hu)(x)}{h^{\alpha}}$$

In 1949 Marcel Riesz (1886- 1969) has developed a theory of fractional order of integration for the fractional of more than one independent variable.

In 1965 Arthur Erdely (1908- 1977) has applied the fractional calculus to integral equations, while Higgin's in 1967 used the fractional integral operators to solved differentials Eq. [39].

1.2 Special Functions

In this section, the important special function related to this work are presented.,

1.2.1 The Gamma and Beta Functions, [10], [39]

It is important to recall that fractional calculus is so uneasy to understand and because that difficulty, we must be present in this section the definition and basic properties of the beta and gamma functions that are requisite for realizing this topic.

Certainly, one of the basic functions encountered in fractional calculus is the Euler's gamma function Γ, which generalizes the ordinary definition of factorial function of a positive integer number m which is allowed to take also any negative and even complex values or non-integer positive.

As it is known the gamma function Γ is defined by the next improper integral in which the variable appears as a parameter:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \; x > 0 . \qquad (1.3)$$

Following the important properties related to the gamma function:

1- By integrating (1.3) by parts, obtain the basic property of $\Gamma(x)$:

$$\Gamma(x+1) = x\,\Gamma(x), \text{ for } x > 0 .$$

2- In particular, when $x = n$ is, we may have:

$$\Gamma(n+1) = n!, \; n = 1, 2, \ldots; \text{ where } \Gamma(1) = 1$$

3-Also, substituting $t = u^2$ in Eq. (1.3) to obtain:

$$\Gamma(x) = 2\int_0^{\infty} \exp(-u^2)u^{2x-1}\,du\,,\ x > 0.$$

4- Gamma function also may be defined for negative values of x by:

$$\Gamma(-x) = \frac{-\pi\csc(\pi x)}{\Gamma(x+1)}\,,\ x \in \mathbb{R}\backslash\{0, -1, -2, \dots\}$$

5- $\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!\sqrt{\pi}}{4^n\, n!}\,,\ n \in \mathbb{N}.$

6- $\Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n\, n!\sqrt{\pi}}{(2n)!}\,,\ n \in \mathbb{N}.$

The following are some repeatedly faced examples of gamma functions for several values of x:

$$\lim_{x\to -1}\Gamma(x) = \infty,\ \lim_{x\to 0}\Gamma(x) = \infty,\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,,\ \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}\,,$$

$$\Gamma\left(\frac{-1}{2}\right) = -2\sqrt{\pi}\,,\ \Gamma\left(\frac{-3}{2}\right) = \frac{4}{3}\sqrt{\pi}\,,\ \Gamma(-1) = \infty,\ \Gamma(0) = \infty,$$

$$\Gamma(1) = 1,\ \Gamma(2) = 1,\ \Gamma(3) = 2,\ \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}.$$

Another type of the functions is called beta function. The beta function is an important function in fractional calculus. The beta function denoted by B(x, y) is defined by the following integral:

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}\,ds\,,\ y > 0, x > 0\ . \tag{1.4}$$

Similarly, as in gamma function, some properties concerning beta function may be given, which may be summarized as follows:

1- Beta function B(x, y) is symmetric with relation to its arguments x and y, that is; B(x, y) = B(y, x) which follows from equation (1.4) by change the variable $1-s=w$.

2- If the change of variable $s=\frac{w}{1+w}$ in Eq. (1.4) is made, then we will obtained :

$$B(x,y)=\int_0^\infty w^{x-1}(1+w)^{-(x+y)}dw, \ x, y \in \mathbb{Z}^+.$$

Finally, an important relationship between gamma and beta functions is given for all x, y > 0, by:

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad .$$

1.2.2 The Mittag-Leffler Function, [25]:

The function $E_\alpha(z)$ was defined by Mittag-Leffler in 1903. This function is a generalization of exponential function. this function plays a serious role in the solution of FODEs.

Definition (1.1), [22]:

The function of complex variable z is defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{Z^k}{\Gamma(\alpha k+\beta)}, \alpha > 0, \beta > 0. \tag{1.5}$$

is called the two parameters Mittag-Leffler function.

For β= 1 the following definition of the one parameter Mittag-Leffler function may be given.

Definition (1.2), [22]:

A function of the complex variable z defined by:

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\alpha>0 \tag{1.6}$$

sometimes it is denoted by $E_{\alpha,1}(Z)=E_{\alpha}(Z)$. For α= 1, we obtain

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z .$$

Also eq (1.5). satisfy the following relation

$$E_{\alpha,\beta}(z)=E_{\alpha,\alpha+\beta}(z)+\frac{1}{\Gamma(\beta)} .$$

Examples (1.1), [14]:

Some special cases of the Mittag-Leffler function are summarized as:

i. $E_0(z)=\frac{1}{1-z}\cdot|z|<1,\ E_{1,0}(z)=ze^Z\cdot$

ii. $E_{1,1}(z)=e^Z,\ E_{1,2}(z)=\frac{e^Z-1}{z},\ z\in C,$ and in general

$$E_{1,m}(z)=\frac{1}{z^{-1}}\left(e^z-\sum_{k=0}^{m-2}\frac{z^k}{k!}\right),\ z\in C\cdot$$

iii. $E_2(z^2)=E_{2,1}(z^2)=\cosh(z),\quad E_2(-z^2)=\cos(z),\ z\in C.$

iv. $E_{2,2}(z^2)=\frac{\sinh(z)}{z},\ z\in C\cdot$

v. $E_3(z)=\frac{1}{2}\left[e^{z^{1/3}}+2e^{-\frac{1}{2}z^{1/3}}\cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right],\ z\in C\cdot$

vi. $E_4(z)=\frac{1}{2}\left[e^{z^{1/34}}+\cosh(z^{1/4})\right],\ z\in C\cdot$

Definition (1.3), [44]:

An n^{th} derivative of the Mittage-Leffler function $E_{\alpha,\beta}(z)$ is defined by:

$$E^{(n)}_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{(j+n)!}{j!}\frac{z^j}{\Gamma(\alpha j+\alpha n+\beta)}. \quad (1.7)$$

where $\alpha,\beta\in R^+, z\in C$ and $n\in\mathbb{N}$.

1.3 The Fractional Derivative

In addition to the use of FODEs for the mathematical model of the real world physical problems, its popular in recent years, e.g., in the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of sticky material properties, etc., [26].

As it is known, FODEs are emerged as a new branch of the applied mathematics by which many physical and engineering approaches can be modeled, [32].

Fractional ordinary differential equation have acquired important and publicity through the past few decades, be the reason to mainly of its demonstrated applications in numerous seemingly various fields of the science. Fractional derivatives extend the perfect instrument for the adjective of the memory and genetically properties of several processes and materials [35].

Many definitions concerning fractional derivatives of the function of single variable are reported in literatures. Some of them are shown next:

1.3.1 Riemann-Liouville fractional derivatives, [39]:

The fractional derivatives of order α of a function f is defined to be:

$$^{RL}D^{\alpha}_{a,x} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(y)}{(x-y)^{\alpha+1-n}} dy, \quad x \geq a \qquad (1.8)$$

where α is a positive fractional number and n is a natural number, such that $n-1 < \alpha \leq n$.

Note that, this derivative is said to be left-handed fractional derivatives of order q of a function f at a point x since it depends on all function values to the left of the point x, that is, this derivative is a weighted average of such function values.

On the other hand, the right-handed fractional derivative of order q of a function f is defined to be:

$$^{RL}D^{\alpha}_{x,b} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b \frac{f(y)}{(y-x)^{\alpha+1-n}} dy, \quad x \leq b \qquad (1.9)$$

where α is a positive fractional number and n is a natural number, such that $n-1 < \alpha \leq n$.

Commonly used in most literatures, the function is defined by

$$ {}^{RL}D_x^{\alpha} f(x) = \frac{(1)}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(y)}{(x-y)^{\alpha-n+1}} dy, \quad (1.10) $$

is called the Riemann-Liouville fractional derivative, where $n-1 < \alpha \le n,\ n \in \mathbb{N}$.

Moreover, if α is a positive integer, then the above definitions give the standard integer derivatives, that is;

$$ (D_{a,x}^{\alpha} f)(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}} $$

and

$$ (D_{x,b}^{\alpha} f)(x) = (-1)^{\alpha} \frac{d^{\alpha} f(x)}{dx^{\alpha}} = \frac{d^{\alpha} f(x)}{d(-x)^{\alpha}}. $$

1.3.2 Caputo fractional derivatives, [5]:

The left-handed and the right handed fractional derivatives of order α of a function f are defined to be:

$$ {}_a^C D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{1}{(x-y)^{\alpha-n}} \left(\frac{d}{dy}\right)^n f(y) dy, \quad x \ge a \quad (1.11) $$

and

$$ {}_x^C D_b^{\alpha} f(x) = \frac{(-1)^{\alpha}}{\Gamma(n-\alpha)} \int_x^b \frac{1}{(y-x)^{\alpha-n}} \left(\frac{d}{dy}\right)^n f(y) dy, \quad x \le b \quad (1.12) $$

where α is positive fractional number and n is a natural number, such that $n-1 < \alpha \le n$.

Example (1.2):

Consider f(x)=x, x > 0 then the fractional derivatives using above definitions are given by:

$$ {}^{R}D_{0,x}^{1.8}x = \frac{1}{\Gamma(2-1.8)}\frac{d^2}{dx^2}\int_0^x \frac{y}{(x-y)^{1.8+1-2}}dy = \frac{1}{\Gamma(0.2)x^{0.8}} $$

$$ {}^{C}D_{0,x}^{1.8}x = \frac{1}{\Gamma(2-1.8)}\int_0^x \frac{1}{(x-y)^{1.8-2}}\cdot\frac{d\ y^2}{dy}\ dy = \frac{1.667}{\Gamma(0.2)}x^{1.2} $$

$$ {}^{R}D_{x,1}^{1.8}x = \frac{1}{\Gamma(2-1.8)}\frac{d^2}{dx^2}\int_x^1 \frac{y}{(y-x)^{1.8+1-2}}dy = \frac{1}{\Gamma(0.2)}\left(\frac{-1.8}{(1-x)^{0.8}} - \frac{0.8x}{(1-x)^{1.8}}\right) $$

$$ {}^{C}D_{x,1}^{1.8}x^2 = \frac{1}{\Gamma\left(2-1.8\right)}\int_x^1 \frac{1}{\left(y-x\right)^{1.8-2}}\cdot\frac{d^2y^2}{dx^2}dy = \frac{1.667}{\Gamma(0.2)}(1-x)^{1.2} $$

$$ {}^{R}D_{0,x}^{0.5}x = \frac{1}{\Gamma(1-0.5)}\frac{d}{dx}\int_0^x \frac{y}{(x-y)^{0.5+1-1}}dy = \frac{2\sqrt{x}}{\Gamma(0.5)} $$

$$ {}^{C}D_{0,x}^{0.5}x = \frac{1}{\Gamma(1-0.5)}\int_0^x \frac{1}{\left(x-y\right)^{0.5-1}}\cdot\frac{dy}{dy}dy = \frac{2x^{1.5}}{3\Gamma\left(0.5\right)} $$

$$ {}^{R}D_{x,1}^{0.5}x = \frac{-1}{\Gamma(1-0.5)}\frac{d}{dx}\int_x^1 \frac{y}{(y-x)^{0.5+1-1}}dy = \frac{-1}{\Gamma(0.5)}\left(\sqrt{1-x} - \frac{x}{\sqrt{1-x}}\right) $$

$$ {}^{C}D_{x,1}^{0.5}x = \frac{-1}{\Gamma(1-0.5)}\int_x^1 \frac{1}{(y-x)^{0.5-1}}\frac{dy}{dx}dy = \frac{-2(1-x)^{1.5}}{3\Gamma(0.5)} $$

1.4 The Fractional Integral

The formulation of the fractional integrals and the fractional derivatives are natural development of integer order integrals and the derivatives in which the same approach that the fractional exponent is followed from the more classical integer of order exponent. For the latter, it is the notation that makes the leap seems obvious. While one cannot imagine a multiplication of the quantity from the fractional number of time, there are appear no procedural restriction to placing a non-integer into the exponential position.

Similarly, the common formulation for the fractional integral can be driven directly from a traditional expression of the repeated integration of a function. various definitions of a fractional integration may be given in: [29], [17].

1.4.1 Riemann-Liouville fractional integral [38]:

The right sided Riemann-Liouville fractional integral is defined by:

$$ {}_{a}^{RL}I_{x}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-s)^{\alpha-1}u(s)\,ds,\ \alpha > 0,\ a \in \mathbb{R}^{+} \qquad (1.14)$$

while the left hand sided fractional integral:

$$ {}_{x}^{RL}I_{b}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(s-x)^{\alpha-1}u(s)\,ds,\ \alpha > 0,\ b \in \mathbb{R}^{+} \qquad (1.15)$$

In most literatures, the function:

$$ {}^{RL}I_x^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (s-x)^{\alpha-1}u(s)\,ds, \qquad (1.16)$$

is called the Riemann-Liouville fractional integral, where $\alpha \in \mathbb{R}^+$ is the order of integral.

1.4.2 Weyl fractional integral [38]:

The Weyl definition of right and left hand sided fractional integrals are given respectively by:

$$ {}_{-\infty}I_x^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-s)^{\alpha-1}u(s)\,ds \qquad (1.17)$$

$$ {}_xI_{\infty}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (s-x)^{\alpha-1}u(s)\,ds \qquad (1.18)$$

Eq. (1.17) and (1.18) are used as the definition of the integration without any condition at the present time.

<u>Remark (1.1)</u> [6] [16] [8]:

The relation between the above types fractional order derivatives in definitions (1.10), (1.11), and (1.16) are:

1) The following is a well-known relation, for a finite interval for $x > 0$

$$ {}^{RL}D_x^{\alpha} f(x) = {}^{C}D_t^{\alpha} f(x) + \sum_{j=0}^{q-1} \frac{f^{j}(0)(x)^{j-\alpha}}{\Gamma(1+j-\alpha)}, \text{ for } q = [\alpha]+1,$$

$q - 1 < \alpha \le q$, $q \in \mathbb{N}$.

2) $ {}^{RL}D_x^{\alpha} f(x) = D^q\, {}_0^{RL}I_t^{q-\alpha} f(x) \neq {}_0^{RL}I_t^{q-\alpha} D^q f(x) = {}^{C}D_x^{\alpha} f(x), x>0$

$q - 1 < \alpha \le q$, $q \in \mathbb{N}$.

3) $^{RL}D_x^{\alpha}\left(f(x)-\sum_{j=0}^{q-1} f^{j}(0)\frac{x^{j}}{j!}\right)={}^{C}D_x^{\alpha}f(x),\ x>0,$

$q-1<\alpha\leq q,\ q\in\mathbb{N}$.

1.5 Properties of Fractional Derivatives and Integration [10], [39], [22].

In this section, some properties related to fractional derivatives and integration are given. These properties will provide our main means for realizing and utilizing FODEs.

1- <u>Linearity:</u> The linearity of the operator:

$$D^{\alpha}(a_1g_1 + a_2g_2) = a_1D^{\alpha}g_1 + a_2D^{\alpha}g_2 \qquad (1.19)$$

where g_1 and g_2 are any two function, $a_1, a_2 \in \mathbb{R}$, $\alpha \in \mathbb{R}^{+}$ and for any type of fractional derivatives[44].

3- ***<u>The Scale Change:</u>*** By a scale change of the function f with respect to a lower limit a, we mean its replacement by f(bx – ba + a), where b is a constant termed as the scaling factor, and hence the fractional derivative of order α with $Y = by - ba + a$, $X=x+\frac{(a-ba)}{b}$ for any type of derivatives, is given by:

$$\frac{d^{\alpha}g(cX)}{d(X-a)^{\alpha}}=c^{\alpha}\frac{d^{\alpha}g(cX)}{d(cX-a)^{\alpha}} \qquad (1.20)$$

3- <u>The Leibniz's Rule:</u> The rule for fractional order derivatives of a product of two functions g and f is a familiar rule in elementary calculus, which states that:

$$\frac{d^{\alpha}(fg)}{dx^{\alpha}} = \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-y-u+1)\Gamma(y+u+1)} \cdot \frac{d^{y+u}g}{dx^{y+u}} \cdot \frac{d^{\alpha-y-u}f}{dx^{\alpha-y-u}} du \quad (1.21)$$

where y is an arbitrary constant, α is an arbitrary order and for any type of derivatives [27].

4- ***<u>The Chain Rule:</u>*** As it is known, the chain rule for the fractional order α of derivatives is given by:

$$\frac{d^{\alpha}}{d(x-a)^{\alpha}} \phi(f(x)) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \phi(f(x)) + \sum_{n=1}^{\infty} \binom{\alpha}{j} \frac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} n!$$

$$\sum_{m=1}^{n} \varphi^{(m)} \sum \prod_{j=1}^{n} \frac{1}{(q_k)!} \left(\frac{f^{(j)}}{j!} \right)^{p_j}$$

where g and f are any functions, $q > -1$ and Σ is extended over all combinations of nonnegative integer values $q_1, q_2, \ldots, q_n$ for any type of derivative [29].

5- If $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$ and u is any function, then:

${}^{C}D^{\alpha\,RL}I^{\alpha}u(x) = u(x)$ and ${}^{C}D^{\alpha\,RL}I^{\alpha}u(x) = u(x)$.

$${}^{RL}I^{\alpha\,C}D^{\alpha}u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^{+}) \frac{x^{k}}{k!}, \; x > 0 \; .$$

6- ${}^{RL}I^{0}u(x) = D^{0}u(x) = u(x)$ [39].

7- ${}^{RL}I^{\alpha\,RL}I^{\beta}u(x) = {}^{RL}I^{\beta\,RL}I^{\alpha}u(x) = {}^{RL}I^{\alpha+\beta}u(x), \alpha, \beta \geq 0$ [39].

8- ${}^{RL}I^{\alpha}u(x) = {}^{C}D^{-\alpha}u(x), \alpha > 0$[39].

9- ${}^{C}D^{\alpha}u(x) = \frac{d^{m}}{dx^{m}} {}^{RL}I^{m-\alpha}u(x)$, $n - 1 < \alpha \leq n$, $n \in N$, $x > 0$[48].

10- ${}^{RL}D^{\alpha}\,{}^{RL}D^{\beta}u(x) = {}^{RL}D^{\beta}\,{}^{RL}D^{\alpha}u(x) = {}^{RL}D^{\alpha+\beta}u(x), \alpha\ , \beta > 0, x > 0.$

11- $${}^{C}D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \text{for}\, \beta > \alpha-1,\ \beta \in R^{+}\ [39].$$

12- ${}^{RL}D^{\alpha}\left({}^{RL}D^{-\beta}u(x)\right) = {}^{RL}D^{\alpha-\beta}u(x), -1 \le \alpha < m, m \in \mathbb{N},\ \beta > 0.$

13- $${}^{RL}D^{\beta}\left({}^{RL}D^{\alpha}f(x)\right) = {}^{RL}D^{\alpha-\beta}u(x) - \sum_{j=1}^{n}\left[{}^{RL}D^{\alpha-j}f(x)\right]_{x=0}\frac{x^{\beta-j}}{\Gamma(1+\beta-j)},$$

where $m-1 \le \alpha < m,\ m \in \mathbb{N},\ \beta > 0.$

14- ${}^{RL}D^{\alpha}e^{\alpha x} = x^{-\alpha}E_{1,1-\alpha}(ax), \alpha > 0, x > 0$ [48].

15- ${}^{RL}I^{\alpha}\,{}^{C}D^{\alpha}u(x) = u(x) - u(0), 0 < \alpha < 1$ [31].

16- $${}^{RL}I^{\alpha}\,{}^{C}D^{\alpha}(c) = c\frac{x^{\alpha}}{\Gamma(1+\alpha)}, \text{where } c \text{ is any constanant}$$

17- If $y^{(i)}(0) = 0, i = 0,1,\ldots,m-1,\ \alpha+\beta \le m, m \in N,$ then

a) ${}^{C}D^{\alpha}\,{}^{C}D^{\beta}u(x) = {}^{C}D^{\beta}\,{}^{C}D^{\alpha}u(x) = {}^{C}D^{\alpha+\beta}u(x)$ [40]. (1.22)

b) ${}^{C}D^{\alpha}\,{}^{RL}I^{\beta}u(x) = {}^{RL}I^{\beta}\,{}^{C}D^{\alpha}u(x) = {}^{C}D^{\alpha-\beta}u(x) = {}^{RL}I^{\beta-\alpha}u(x)$.

18- ${}^{RL}I^{\alpha}e^{\alpha x} = x^{\alpha}E_{1,\alpha+1}(ax), \alpha > 0, x > 0,$ [48].

1.6 Fractional Order of ODEs [12].

A relationship includes one or most derivatives of the unknown function y which is considered with respect to its independent variable x is known as an ODEs. Similar relationships involving at least one differ-integral of noninteger order may be termed as extraordinary differential equations.

As with ODEs, the situation of extraordinary differential equations often involves integrals and contains arbitrary constants. The differentials equation perhaps involve Riemann-Liouville differentials operators of fractional order $\alpha > 0$, which takes the form:

$$D_{x_0}^{\alpha}u(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_{x_o}^{x}\frac{u(y)}{(x-y)^{\alpha-m+1}}dy \tag{1.23}$$

The ODEs include such fractional derivatives has proved to become valuable tools in the modeling of many physical problems. Also, D^{α} has an m-dimensional kernel, then we need to describe m-initial conditions in order to get the unique solution to the FODEs:

$$D_{x_0}^{\alpha}\, y(x) = f\,(x, y(x)) \tag{1.24}$$

with several given function f. In the standard mathematical theory, the initial conditions corresponding to (1.24) should be of the form:

$$\frac{d^{\alpha-j}}{dt^{\alpha-j}}y(x)\Big|_{x=a} = b_j, \quad j = 1, 2, \ldots, m \tag{1.25}$$

with given values b_j. In other words, we must, specify some fractional derivatives of the function y.

In practical applications, these values are frequently not available and so Caputo in l967 proposed that one must includes derivatives of integer-order of the function f as they are commonly used in initial value problem IVPs with integer-order equations, into the fractional-order equation, given as:

$$D^{\alpha}[f - T_{m-1}(u)](x) = F(x, u(x)) \quad (1.25)$$

where $T_{m-1}(u)$ is the Taylor polynomials of order (m-1) for f, centered at 0. Then, we can set the initial conditions in the classical form:

$$u^{(j)}(0) = u_0^{(j)}, \quad j = 0, 1, \ldots, m - 1 \quad (1.26)$$

As a classification, FODEs may be classified to be either linear or nonlinear, homogenous or non-humongous, etc. In which Eq. (1.22) is linear if it does not contain terms of independent variable alone, otherwise it is non-homogenous. Also FODEs are said to be linear if the dependent variable u(x) appears linearly in the FODEs, otherwise it is non-linear.

Chapter Two

Derivation of Numerical methods for solving FODEs Using LMMs

CHAPTER

2

DERIVATION OF NUMERICAL METHODS FOR SOLVING FODEs USING LMMs

INTRODUCTION

There is no general agreement on how the phrase "numerical analysis" should be interpreted. Some researchers see that “analysis” as the key word and wish to embedding the subject entirely in rigorous modern analysis, others suggest that "numerical" is the vital word and the algorithm or the approach is the only respectable yield. Numerical methods usually produce errors and we may refer that the numerical technique is said good if the error approach quickly or rapidly converges to zero and the method requires a minimum computer capacity and takes a less time as possible.

So, in this chapter, a study will be introduced for the derivation of some numerical methods for solving FODEs of the form

$$^{C}D^{\alpha}y(x) = f(x, y(x)),\ \ y(0) = y_0,\ \ 0 < \alpha \leq 1 \qquad (2,1)$$

This chapter consists of six sections; in section (2.1), the LMMs was introduced and studied. In section (2.2), the work of this section have been generalized for the fractional Linear Multistep Method (FLMMs). For derivation purpose. In section (2.3) the generalized Taylor's formula have been defined, in section (2.4) we presents the derivation of some FLMMs. In section (2.5) a modification of the result has been made using the variable step size method for solving FODEs. Finally in section (2.6), some numerical examples are given.

2.1 Linear Multistep Methods [2]

This section presents an introduction to the theory of LMMs. Consider the IVPs for a single first-order differential equation:

$$y'(x) = f(x, y(x)),\ \ y(x_0) = y_0. \qquad (2.2)$$

where f is a given continuous function and x_0, y_0 are fixed. We seek for a solution in the range $a \leq x \leq b$, where a and b are given and finite.

Consider the sequence of nod points $\{x_n\}$ over [a,b], defined by $x_n = a + nh$, n = 0, 1, …, N, $N \in \mathbb{N}$. The parameter h, will always be regarded as a constant. As fundamental property of the plurality of computation methods for the solution of equation (2.2), there is a step size of discretization, that is, we seek for the approximate solution, not on the continued interval $a \leq x \leq b$, but on the discrete set of point $\{x_n\}$, n = 0, 1, …, N. Let y_n be the approximat to the theoretical solution y at x_n, that is, to $y(x_n)$, and let $f_n = f(x_n, y_n)$, .

If a computational method for determining the sequence $\{y_n\}$ is taken form of LMMs of step number k, or a linear k-step method. Then the general form of LMMs may be written as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (2.3)$$

where α_j and β_j, are constants to be determined. We assume that $\alpha_k \neq 0$ and that not both of α_0 and β_0, equals zero. Since Eq. (2.3) can be multiplied both sides by the same fixed number without altering the relationship, then the coefficients α_j and β_j, are arbitrary to the extent of a fixed multiplier. We remove this arbitrariness by assuming throughout eq (2.3) that $\alpha_k = 1$. Thus the problem of determining the solution y, of the general non-linear IVPs we replace Eq. (2.2) by one that find the sequence of numerical solutions $\{y_n\}$, which satisfies the difference Eq. (2.3). Note that, since f_n is in general non-linear function of y_n, then Eq. (2.3) is a non-linear finite differential equation. Such equations are no easier to handle theoretically, as in linear differential equations, but they have the practical advantage of permitting us to compute the sequence $\{y_n\}$ numerically.

In order to do this, we must first supply the assistant of starting values y_0, y_1, …, y_{k-1} (in the case of a one-step method, only one of such value which is y_0 is needed and we normally choose y_0 to be constant).

As a classification to the LMMs we say that the LMMs is said to be explicit if $\beta_k = 0$ for all k and is said implicit if $\beta_k \neq 0$. For an explicit method, Eq. (2.3) yields the current value y_{n+k} directly in terms of previous y_{n+j}, f_{n+j}, j = 0, 1, …, k − 1, which is at this stage of the computations, have already been calculated while in implicit methods, however, will be called for the solution at each stage of the computations, of the equation:

$$y_{n+k} = h\beta_k f(x_{n+k}, y_{n+k}) + g \tag{2.4}$$

where g is a known function of the previously calculated values y_{n+j}, f_{n+j}, j = 0, 1, …, k – 1.

When the original differential Eq. (2.2) is linear, then Eq. (2.4) is also linear in y_{n+k}, and there is a unique solution for y_{n+k}, while when f is nonlinear, then there is a unique solution for y_{n+k}, which can be approached arbitrarily closely by the iteration:

$$y_{n+k}^{[s+1]} = h\beta_k f(x_{n+k}, y_{n+k}^{[s]}) + g(y_{n+k}^{[0]})$$

Thus, implicit methods in general entail a substantially greater computational effort than do explicit methods; on the other hand, for a given step number, implicit methods can be made more accurate than explicit ones and, moreover, enjoy more favorable stability properties. Then, the sufficient and necessary conditions for the stability of LMMs to have an order p that can be studied by using two associated polynomials, which are:

The first and the second characteristic polynomial which related to the LMMs (2.3), are given by:

$$\rho(r)=\sum_{j=0}^{k}\alpha_j r^j=\alpha_k r^k+\alpha_{k-1}r^{k-1}+\ldots+\alpha_0$$

$$\sigma(r)=\sum_{j=0}^{k}\beta_j r^j=\beta_k r^k+\beta_{k-1}r^{k-1}+\ldots+\beta_0$$

Also, it is important to notice that if $\sigma(r)$ is given, we can find a unique polynomial $\rho(r)$ of degree k, such that the method has an order $p \geq k$, therefore, we can consider the LMMs according to the roots of $\rho(r)$ and whether it is explicit or implicit.

(1) If the roots of $\rho(r)$ equal to 1 and 0, then the method is called of Adam's type, if the LMMs is explicit, then it is called of Adam Bashforth type, while if it is implicit then it is called of Adam-Moulton type, i.e., in Adam's methods, the following first characteristic polynomial is obtained:

$$\begin{aligned}\rho(r)&=r^k-r^{k-1}\\&=r^{k-1}(r-1)=0\end{aligned}$$

(2) If the roots of $\rho(r)$ equals to -1, 0 and 1, then the method is called of Nystrom type if it is explicit and if the method is implicit, then it is called of Milne-Simpson type, i.e., When:

$$\begin{aligned}\rho(r)&=r^k-r^{k-2}\\&=r^{k-2}(r^2-1)\\&=r^{k-2}(r-1)(r+1)\end{aligned}$$

Now, we explain the consistency, convergence and zero stability of LMMs, such that, a basic property which we shall demand of an acceptable LMMs is that the solution $\{y_n\}$ generated by the method converges, in some sense to the theoretical solution y(x) as the step length h tends to zero. The LMMs is said to be consistent with the IVPs

$$y' = f(x, y),\ y(x_0) = y_0$$

if it has an order at least p = 1, i.e., consistent method implies that

$$\sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$$

Finally the LMMs is said to zero-stable if all the roots r_j's, j = 1, 2, ..., k; of $\rho(r) = 0$ satisfy the condition $|r_j| \le 1$ and if r_j is a multiple zero of $\rho(r)$ then $|r_j| < 1$.

Definition (2.1):

The linear multistep method is said to be consistent if it has an order $p \ge 1$.

The following lemma gives an alternate definition to the consistency concept.

Lemma (2.2):

The linear multistep method is consistent if and only if:

$$\sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$$

Theorem (2.3):

If the linear multistep method is consistent, then the method is convergent if and only if it is zero stable.

2.2 Fractional Linear Multistep Methods [40]

The FLMMs is based upon using the involution square. In the leading work on discrete fractional calculus, Lubich suggests an elegance and effective strategy for classical LMMs which is originally devised for integer order ODEs. Although the possibility of FLMMs that have been known in various articles, in which their use for practical computation has not received interest the widespread so far, perhaps because of some difficulties in its explicitly wording FLMMs.

The key aspect in FLMMs is the approximation of the Riemann-Liouville fractional integral by means of the involution square

$$^{RL}I_h^{\beta} = h^{\beta}\sum_{j=0}^{n} w_{n\text{-}j}g(t_j) + h^{\beta}\sum_{j=0}^{s} w_{n,j}g(t_j) + O(h^p), \qquad (2.5)$$

on unified grids $t_n = t_0 + nh, h > 0,$ and where involution and starting square weights w_n and $w_{n,j}$ it is not depend on h. Starting weights $w_{n,j}$ have the important role, especially in the first section of the integral interval. Involution square weights w_n are the main component of the quadratic base and characterizes the certain FLMMs. They are obtained starting from any LMMs for ODEs.

$$\sum_{j=0}^{k} \rho_j y_{n-j} = \sum_{j=0}^{s} \sigma_j f(t_{n-j}, y_{n-j}) \qquad (2.6)$$

being $\rho(z) = \rho_0 z^k + \rho_1 z^{k-1} + ... + \rho_k$

and $\sigma(z) = \sigma_0 z^k + \sigma_1 z^{k-1} + ... + \sigma_k$

are the first and the second characteristic polynomials. As shown by Roberto Garrappa in [42], LMMs can be equivalently reformulated as (2.5) (but with β = 1 for ODEs with weights w_n obtained as the coefficients of the major power series

$$w(\xi) = \sum_{n=0}^{\infty} w_n \xi^n, \quad w(\xi) = \frac{\sigma(1/\xi)}{\rho(1/\xi)} \tag{2.7}$$

and the function $w(\xi)$ goes then under the generation name function of the LMMs. The idea proposed in [9] is to generate a square rule of the fractional problems (2.1) by evaluating the involution weights as a coefficient in the coefficients of the official power series of the fractional order power of the generate function

$$w(\xi) = \sum_{n=0}^{\infty} w_n \xi^n, \quad w(\xi) = \left(\frac{\sigma(1/\xi)}{\rho(1/\xi)} \right)^{\beta} \tag{2.9}$$

Methods of this kind, named as FLMMs, when applied to (2.1) read as

$$y_n = T_{m-1}(t) + h^{\alpha} \sum_{j=0}^{s} w_{n,j} f_j + h^{\alpha} \sum_{j=0}^{s} w_{n-j} f_j \tag{2.10}$$

where $T_{m-1}(t)$ is the Taylor expansion of $y(t)$ centered at t_0

$$T_{m-1}(t) = \sum_{k=0}^{m-1} \frac{(t - t_0)}{k!} y(t_0)^{(k)}$$

and the convergence property is given in the following result.

<u>Theorem (2.4)</u> [42] **(Convergence of FLMMs)**

Let (ρ,σ) be a stable and consistent form the order p implicit LMMs with zeros of $\sigma(\xi)$ haveing absolute value ≤ 1. The FLMMs (2.10) is convergent of order p.

One of the main difficulties in FLMMs (2.10) is in evaluating the weights w_n as coefficients in the coefficients of official power series (2.9).

Although some of the sophisticated algorithms are available for manipulating official power series.

2.3 Euler's Method for Solving FODEs

To transform the explicit Euler's method for solving FODEs, the following approached is followed:

Consider the FODEs:

$$^{C}D^{\alpha}y(x) = f(x, y(x)),\ \ y(x_0) = y_0, x \in [a,b],\ 0 < \alpha \leq 1 \qquad (2.11)$$

Since Euler's method reads as follows:

$$y_{n+1} = y_n + hy'_n + O(h^2)$$

Hence, by using property (1.22), one may get:

$$y_{n+1} = y_n + h\,{}^{C}D^{1-\alpha}\,{}^{C}D^{\alpha}y_n + O(h^2)$$

$$y_{n+1} = y_n + h\,{}^{C}D^{1-\alpha}f(x_n, y_n) + O(h^2) \qquad (2.12)$$

where ${}^{C}D^{1-\alpha}f(x,y)$ could be evaluated easily using fractional calculus.

Similarity to transform the implicit Euler's method to solve FODEs, the following approach is followed:

Consider the FODEs. (2.11) and the implicit Euler's method which reads as follows:

$$y_{n+1} = y_n + hy'_{n+1} + O(h^3)$$

Then, by using property (1.22), we get:

$$y_{n+1} = y_n + h\,{}^{C}D^{1-a}\,{}^{C}D^{a}y_{n+1} + O(h^3)$$

$$y_{n+1} = y_n + h\,{}^{C}D^{1-a}f(x_{n+1}, y_{n+1}) + O(h^3)$$

<u>Example (2.1):</u> [46]

Consider the FODEs:

$$D^{1/2}y(x) = x^2, y(1) = 4, x \in [1,2].$$

Taking $h = 0.1$ and to solve this problem using Euler's method, recall that:

$$y_{n+1} = y_n + h\,{}^{C}D^{1-1/2}\,{}^{C}D^{1/2}y(x)$$

and since $D^{1/2}y(x) = x^2$, then:

$$y_{n+1} = y_n + h\,{}^{C}D^{1/2}x^2$$

and since ${}^{C}D^{1/2}x^2 = \dfrac{8x^{3/2}}{3\sqrt{\pi}}$

Hence, Euler's method leads to:

$$y_{n+1} = y_n + h\frac{8x_n^{3/2}}{3\sqrt{\pi}}$$

Hence, upon carrying out the above explicit Euler's method and implicit Euler's method for $x \in [1,2]$, with its comparison with the exact solution, we get the results presented in table (2.1).

Table (2.1). Numerical values for Example (2.1) when $\alpha = 0.5$ and h=0.01.

n	x_n	***explicit Euler's Method***	**Absolute Errors**	***implicit Euler's Method***	**Absolute Errors**	***Exact Solution***
0	1	4	0	4	0	4
1	1.1	4.1504	0.0113	4.1537	0.0080	4.1617
2	1.2	4.3239	0.0231	4.3305	0.0165	4.3470
3	1.3	4.5754	0.0108	4.5787	0.0075	4.5862
4	1.4	4.7823	0.0131	4.7853	0.0101	4.7954
5	1.5	4.9212	0.0122	4.9254	0.0080	4.9334
6	1.6	5.1834	0.0130	5.1873	0.0910	5.1964
7	1.7	5.3976	0.0106	5.3992	0.0090	5.4082
8	1.8	5.5669	0.0074	5.5693	0.0050	5.5743
9	1.9	5.7348	0.0091	5.7379	0.0060	5.7439

Example (2.2): [46]

Consider the FODEs:

$$D^{3/2}y(x)=\cos(x+\frac{\pi}{4}),\ y(0)=1,\ x\in[1,2].$$

Then using implicit Euler's method, we have:

$$y_{n+1}=y_n+h\,{}^{C}D^{1/2}\,{}^{C}D^{\frac{1}{2}}y+\frac{h^2}{2}\,{}^{C}D^{1/2}\,{}^{C}D^{3/2}y+O(h^3)$$

Since, $D^{1/2}y(x)=\sin(x+\frac{\pi}{4})$ Then:

$$y_{n+1}=y_n+h\,{}^{C}D^{1/2}\sin(x+\frac{\pi}{4})+\frac{h^2}{2}\,{}^{C}D^{1/2}\cos(x+\frac{\pi}{4})+O(h^3)$$

Also:

$$D^{1/2}\sin(x+\frac{\pi}{4})=\sin(x+\frac{\pi}{4})$$

and:

$$D^{1/2}\cos(x+\frac{\pi}{4})=\frac{1}{\sqrt{\pi(x+\frac{\pi}{4})}}+\cos(x+\frac{\pi}{4})$$

Therefore:

$$y_{n+1}=y_n+h\sin(x+\frac{\pi}{4})+\frac{h^2}{2}\left[\frac{1}{\sqrt{\pi(x+\frac{\pi}{4})}}+\cos(x+\frac{\pi}{4})\right]+O(h^3)$$

For comparison purpose, the exact solution is given by:

$$y(x) = \sin(x) + 1$$

The next results presented in table (2.2) are obtained upon carrying explicit Euler's Method and implicit Euler's Method, as well as, their comparison with the exact solution.

Table (2.2). Numerical values for Example (2.2) when $\alpha = 1.5$ and h=0.01.

n	x_n	***explicit Euler's Method***	Absolute Errors of explicit Euler's	***implicit Euler's Method***	Absolute Errors of implicit Euler's	***Exact Solution***
0	1	1	0	1	0	1
1	1.1	1.10314	0.00331	1.10212	0.00229	1.09983
2	1.2	1.03645	0.16221	1.09891	0.09975	1.19866
3	1.3	1.13468	0.15297	1.18423	0.10333	1.28765
4	1.4	1.26653	0.12002	1.29754	0.08901	1.38655
5	1.5	1.35432	0.14216	1.47653	0.01995	1.49648
6	1.6	1.44312	0.13086	1.48254	0.09144	1.57398
7	1.7	1.55322	0.13018	1.59763	0.08577	1.68340
8	1.8	1.68761	0.10537	1.70121	0.09177	1.79298
9	1.9	1.76542	0.10802	1.79875	0.07467	1.87344

2.4 Generalized Taylor's Formula

In this section, we will insert a generalization of Taylor's formula which includes Caputo fractional derivatives. We begin with introducing the generalized mean value theorem.

Definition (2.5) [46]

A real function f(x), x > 0, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number p(> μ), such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and is said in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Lemma (2.6) [46]

If $m - 1 < \alpha \le m$, $m \in \mathbb{N}$ and $f \in C_\mu^m$, $\mu \ge -1$, then

$$D^\alpha J^\alpha f(x) = f(x)$$

$$D^\alpha J^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0+)\frac{x^k}{k!},\ x > 0\ . \tag{2.13}$$

Theorem (2.7) (Generalized Mean Value Theorem) [46]

Assume that $f(x) \in C[0, a]$ and $\left(D^\alpha f\right)(\zeta) \in C[0, a]$ for $0 < \alpha \le 1$.

then we have

$$f(x)=f\left(0+\right) - \frac{1}{\Gamma(\alpha)}\left(D^\alpha f\right)(\xi)\, x^\alpha \tag{2.14}$$

with $0 \le \xi \le x$, $\forall x \in (0, a]$.

In case α = 1, then (2.14) reduced to the classical mean value theorem.

Theorem (2.8).[46]

Suppose that $D^{n\alpha}f(x)$, $D^{(n+1)\alpha} f(x) \in C[0, a]$, for $0 < \alpha \leq 1$ then

$$(J^{n\alpha}D^{n\alpha}f)(x) - (J^{(n+1)\alpha}D^{(n+1)\alpha}f)(x) = \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}(D^{n\alpha}f)(0+) \qquad (2.18)$$

where:

$$D^{n\alpha} = D^{\alpha}D^{\alpha}\ldots D^{\alpha} \qquad (n - \text{times}).$$

Theorem (2.9) **(Generalized Taylor's formula)** [46]

Assume that $D^{k\alpha}f(x) \in C(0, a]$ for k = 0, 1,...,n + 1, where $0 < \alpha \leq 1$;

then:

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)}\left(D^{i\alpha}\right)(0+) + \frac{\left(D^{(n+1)\alpha}f\right)(\xi)}{\Gamma\left((n+1)\alpha+1\right)} x^{(n+1)\alpha} \qquad (2.19)$$

with $0 \leq \xi \leq x$, $\forall x \in (0, a]$.

Also, the case of α = 1, the generalized Taylor's formula (2.19) reduces to the classical Taylor's formula.

2.5 Derivation of Some FLMMs

For convenience we subdivide the interval [0, a] into j subintervals $[t_n, t_{n+1}]$ of equal step size h = a/j by using the nodes points $t_n = nh$, for n = 0,1,..., N Assume that y(t), $D^{\alpha}y(t)$ and $D^{2\alpha}y(t)$ are continuous on [0, a] and use the formula (2.1) to expand y(t) about $t = t_0 = 0$. For every value t, there is a value c_1 so that:

$$y(t)=y(t_0)+\left(D^{\alpha}y(t)\right)(t_0)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\left(D^{2\alpha}y(t)\right)(c_1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (2.23)$$

when $\left(D^{\alpha}y(t)\right)(t_0)=f\left(t_0,y(t_0)\right)$ and $h = t_1$ are substituted into Eq. (2,23), the result is an expression for $y(t_1)$:

$$y(t_1)=y(t_0)+f\left(t_0,y(t_0)\right)\frac{h^{\alpha}}{\Gamma(\alpha+1)}+\left(D^{2\alpha}y(t)\right)(c_1)\frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \quad (2.24)$$

if the step size h is chosen sufficiently small, then we may omit the second-order term (involving $h^{2\alpha}$) and get:

$$y(t_1)=y(t_0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f\left(t_0,y(t_0)\right) \quad (2.25)$$

The process is repeated and will generates a sequence of points an approximation to the solution y(t) at a special node point. the general Fractional Euler's Method at $t_{n+1} = t_n + h$, is:

$$y(t_{n+1})=y(t_n)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f\left(t_n,y(t_n)\right) \quad (2.26)$$

for n = 0, 1, ..., j − 1. It is clear that if $\alpha = 1$, then the explicit Fractional Euler Method (2.26) reduces to the classical Euler's method, [44].

The next method will be address as the modified explicit Fractional Euler Method, from Eq. (2.24),

$$y(t_{n+1})=y(t_n)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}y'(t_n)+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y''(c_1) \tag{2.27}$$

when the flowing abbreviation is used, $f\left(t_j, y\left(t_j\right)\right)=y'\left(t_n\right)$ and

$\left(D^{2\alpha}y(t)\right)(c_1)=y''(c_1)$

Derive the Eq. (2.27) with integer order, yields:

$$y'(t_{n+1})=y'(t_n)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}y''(t_n)+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y'''(c_1)$$

$$=y'(t_{n+1})-\frac{h^{\alpha}}{\Gamma(\alpha+1)}y''(t_n)-\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y'''(c_1) \tag{2.28}$$

By reparation Eq. (2.28) in Eq. (2.27)

$$y(t_{n+1})=y(t_n)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}[y'(t_{n+1})-\frac{h^{\alpha}}{\Gamma(\alpha+1)}y''(t_n)$$

$$-\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y'''(c_1)]+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y''(c_1)$$

$$=y(t_n)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}y'(t_{n+1})-\frac{h^{2\alpha}}{\left(\Gamma(\alpha+1)\right)^2}y''(t_n)$$

$$-\frac{h^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}y'''(c_1)+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}y''(c_1)$$

If the step size h is chosen small enough, then we may neglect the second-order term (involving $h^{2\alpha}$ and $h^{3\alpha}$) and get the implicit Fractional Euler Method.

$$y(t_{n+1})=y(t_n)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}y'(t_{n+1})$$

$$=y(t_n)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f\left(t_{n+1},y(t_{n+1})\right) \tag{2.29}$$

when $f\left(t_{n+1},y(t_{n+1})\right)=y'(t_{n+1})$ and hence, by adding Eq. (2.26) and (2.29) we get the implicit fractional Trapezoidal rule:

$$y(t_{n+1})=y(t_n)+\frac{h^{\alpha}}{2\Gamma(\alpha+1)}\left[f\left(t_n,y(t_n)\right)+f\left(t_{n+1},y(t_{n+1})\right)\right] \tag{2.30}$$

By subtraction Eq. (2.26) from Eq. (2.29) we get

$$f\left(t_n,y(t_n)\right)=f\left(t_{n+1},y(t_{n+1})\right) \tag{2.31}$$

By compensation Eq. (2.30) and (2.31), the explicit fractional Trapezoidal rule is obtained:

$$y(t_{n+1})=y(t_n)+\frac{h^{\alpha}}{2\Gamma(\alpha+1)}\left[f\left(t_{n-1},y(t_{n-1})\right)+f\left(t_n,y(t_n)\right)\right] \tag{2.32}$$

It is clear that this methods (2.26), (2.29), (2.31) and (2.32), are consists, since it has order $p \geq 1$. Also, it is zero stable, since the poly roots of the first characteristic poly. $\rho(r) = r - 1 = 0$ is r =1. Therefore, using theorem (2.3), there are convergent.

2.6 Variable Step Size Method for Solving FODEs

In this section, the variable step size methods for solving FODEs will be derived that may be considered as a new approach for solving FODEs, in all fixed step-size methods, the local truncation error will depend on step size h and on the numerical method used. But, in variable step-size methods, we shall find the numerical solution y_t for the FODEs given in Eq. (2.26), (2.29), (2.31) and (2.32), with $y_{t_0} = y_0$ that is accurate to within a predefined tolerance **ε**. Therefore, it turns out for acceptable effective estimates of the step-size, it is required to attain a custom local truncated error (tolerance) ε. The variable step-size method which will be consider here, is based upon comparing to between the estimates of the one and two steps of the numerical value of y_t at some time obtained by the numerical method with local truncation error term that is of the form Ch^p, where C is unknown constant and p is the order of the method. Assume that we started with the initial condition y_0 with step-size h^α using certain Fractional Euler Method to find the solution $y^{(1)}_{t_0+h^\alpha}$ and $y^{(2)}_{t_0+h^\alpha}$ using the step-size h^α and $\frac{h^\alpha}{2}$, respectively.

Let: $E_{est} = \left\| y^{(1)}_{t_0+h^\alpha} - y^{(2)}_{t_0+h^\alpha} \right\|$

Hence if $E_{est.} \le \varepsilon$, then there is no problem and one may consider $y^{(2)}_{t_0+h^\alpha}$ as the solution at $t_0 + h$. Otherwise if $E_{est.} > \varepsilon$, then one can find another estimation of the step- size say h_{new} . If this approximation was accepted then this value of h_{new} will be used as the new value of h in the next step; if not, then it will be used as an old h and repeat similarly as above [20].

Theorem (2.10)

Suppose the $y^{(1)}_{t_0+h^\alpha}$ and $y^{(2)}_{t_0+h^\alpha}$ are the numerical solution obtained by the Fractional Euler Method given in Eq. (2.1) with step sizes h^α and $\frac{h^\alpha}{2}$, respectively. If ε is the tolerance and $E_{est} = \left\| y^{(1)}_{t_0+h^\alpha} - y^{(2)}_{t_0+h^\alpha} \right\|$, then the new value of the step size is giving:

$$h_{new} = \left[\frac{\frac{\varepsilon}{2}}{E_{est}} \right]^{\frac{1}{\alpha}} h_{old} \tag{2.33}$$

Proof:

Suppose y is the actual solution at t_0+h, then

$$E_{est} = \left| y^{(1)}_{t_{0+h}} - y^{(2)}_{t_{0+h}} \right| = ch^\alpha - c\left(\frac{h^\alpha}{2}\right) = c\left(\frac{h^\alpha}{2}\right)$$

this gives the estimate $\varepsilon = ch^\alpha_{new} = \left[\frac{E_{est}}{(\frac{h^\alpha_{old}}{2})} \right] h^\alpha_{new}$

since $c = \left[\frac{E_{est}}{(\frac{h^\alpha_{old}}{2})} \right]$

and so:

$$h_{new} = \left[\frac{\frac{\varepsilon}{2}}{E_{est}} \right]^{\frac{1}{\alpha}} h_{old}$$

where h_{old} refers to the old value of the step size. ■

Similarly, the variable step size (2.33) may be obtained for methods (2.26), (2.29), (2.30) and (2.32).

2.7 Numerical Examples

In this section, two example will be given for comparing purposes between the different used methods.

Example (2.3). [46] Our first example deals with the homogeneous linear FODEs

$$D^{\alpha} y(t) = -y(t), \quad y(0) = 1, \; t > 0, \; 0 < \alpha \leq 1. \tag{2.13}$$

The exact solution of Eq. (2.13) is given by

$$y(t) = E_{\alpha}\left(-t^{\alpha}\right)$$

Where E_{α} is the mittag-leffler .

Table (2.3). Numerical values for Example (2.3) when $\alpha = 0.5$ and h=0.01.

t	Exact solution	Absolute Errors explicite fractional Euler's	Absolute Errors implicit fractional Euler's	Absolute Errors implicit fractional Trapezoidal	Absolute Errors explicite fractional Trapezoidal
0	1.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.723578	0.167213	0.155995	0.130745	0.137244
0.2	0.643788	0.147135	0.134112	0.117126	0.119564
0.3	0.592018	0.102264	0.100434	0.091129	0.092389
0.4	0.553606	0.089545	0.082064	0.074409	0.077334
0.5	0.523157	0.079655	0.070409	0.064279	0.656235
0.6	0.498025	0.067576	0.063498	0.051468	0.052886
0.7	0.476703	0.047554	0.046608	0.033885	0.032418
0.8	0.458246	0.031198	0.027358	0.013057	0.018186
0.9	0.442021	0.020405	0.016427	0.008685	0.009423
1.0	0.427584	0.016023	0.010825	0.001385	0.005612

While upon using the variable step size method in connection with the methods the explicite fractional Euler's, the implicit fractional Euler's, the implicit fractional Trapezoidal and the explicite fractional Trapezoidal which are presented in table (2.3) and table (2.4).

Table (2.4). Numerical values for Example (2.4) when $\alpha = 0.5$ with h=0.1 and ε =0.5 .

T	Exact solution	Absolute Errors explicite fractional Euler's	Absolute Errors implicit fractional Euler's	Absolute Errors implicit fractional Trapezoidal	Absolute Errors explicite fractional Trapezoidal
0	1.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.723578	0.038724	0.033786	0.029024	0.029987
0.2	0.643788	0.025795	0.021898	0.019893	0.019979
0.3	0.592018	0.013606	0.012463	0.009667	0.009857
0.4	0.553606	0.010369	0.00969	0.007996	0.008215
0.5	0.523157	0.009424	0.008609	0.006298	0.006758
0.6	0.498025	0.008798	0.008184	0.005876	0.005956
0.7	0.476703	0.007209	0.006546	0.005054	0.005286
0.8	0.458246	0.00678	0.005323	0.004984	0.005054
0.9	0.442021	0.00596	0.005046	0.003243	0.003554
1.0	0.427584	0.00408	0.003928	0.002847	0.002998

Example (2.4) . [44] The second example deal with the nonlinear equation

$$D^{\alpha} y(t) = y(t)^2 - \frac{2}{(t-1)^2}, \quad y(0) = -2, \text{ where } 0 < \alpha \le 1. \quad (2.14)$$

Table (2.5). Numerical values for Example (2.4) when $\alpha = 0.5$ and h=0.01.

t	Exact solution	Absolute Errors explicite fractional Euler's	Absolute Errors implicit fractional Euler's	Absolute Errors implicit fractional Trapezoidal	Absolute Errors explicite fractional Trapezoidal
0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	-0.090000	0.112765	0.105022	0.091132	0.09314
0.2	-0.160000	0.110098	0.100212	0.090122	0.092176
0.3	-0.210000	0.102565	0.100115	0.087662	0.088565
0.4	-0.240000	0.095425	0.092214	0.072029	0.073245
0.5	-0.250000	0. 096522	0.090423	0.070279	0.071001
0.6	-0.240000	0.095769	0.088363	0.061433	0.062234
0.7	-0.210000	0.08654	0.080811	0.058542	0.059461
0.8	-0.160000	0.08023	0.076352	0.043644	0.044766
0.9	-0.090000	0.07470	0.067672	0.033256	0.034121
1.0	0.000000	0.05022	0.04812	0.012520	0.013224

While upon using the Variable Step Size Method in connection with the methods the explicite fractional Euler's, the implicit fractional Euler's, the implicit fractional Trapezoidal and the explicite fractional Trapezoidal which are presented in table (2.5) and table (2.6).

Table (2.6). Numerical values for Example (2.4) when $\alpha = 0.5$ with h=0.1 and ε =0.5.

t	Exact solution	Absolute Errors explicite fractional Euler's	Absolute Errors implicit fractional Euler's	Absolute Errors implicit fractional Trapezoidal	Absolute Errors explicite fractional Trapezoidal
0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	-0.090000	0.030651	0.029077	0.011022	0.012873
0.2	-0.160000	0.023218	0.022120	0.010542	0.011644
0.3	-0.210000	0.011678	0.010981	0.008732	0.008843
0.4	-0.240000	0.010505	0.009643	0.007632	0.077454
0.5	-0.250000	0. 009843	0.009122	0.007071	0.007112
0.6	-0.240000	0.009277	0.008921	0.006621	0.006822
0.7	-0.210000	0.008905	0.008010	0.006011	0.06243
0.8	-0.160000	0.008021	0.007263	0.005121	0.005322
0.9	-0.090000	0.007704	0.006971	0.004290	0.044322
1.0	0.000000	0.006023	0.005511	0.003522	0.003722

Chapter Three

Predictor-Corrector, Richardson Extrapolation and Variable Order Methods for Solving FODEs

CHAPTER

3

PREDICTOR-CORRECTOR, RICHARDSON EXTRAPOLATION AND VARIABIE ORDER METHODS FOR SOLVING FODES

Introduction

Sometimes, numerical methods for solving ODEs are more reliable than analytic methods, especially in solving FODEs, since such type of equations has some difficulties in their methods of solution, which could not be handled easily.

This chapter consists of four sections. In section (3.1) the predictor-corrector method (PCM) for solving FODEs is presented. In section (3.2) the Richardson extrapolation method (REM) for solving FODEs is derived to improve the accuracy of the numerical results for solving FODEs by means of the second and third order methods. In sections (3.3) the variable order methods have been modified and introduced to solve numerically the FODEs. Finally, in section (3.4) an illustrative examples are considered in order to compare between the presented methods in this chapter.

3.1 Predictor-Corrector Methods for Solving FODEs [16]

In this section, we will derive the fundamental algorithm that we have developed for the solution of IVPs with Caputo derivatives of the form

$$^{C}D^{\alpha}y(x)=f(x,y(x)),\ \ y(0)=y_0,\ 0<\alpha\le 1. \qquad (3.1)$$

where $^{C}D^{\alpha}y(x)$ denotes the Caputo derivatives. The algorithm is generalization of classical Fractional Euler Method that is well known for the numerical solution of problems [25]. Our approach is based on the analytical properties that the IVPs will depends on.

As it is known, the Fractional Euler Method of PCM are explicit, linear, multistep techniques. Each successive member of the family has the higher order of convergence, and the family can be extended indefinitely. The Fractional Euler Method of PCM, can be similarly extended to an arbitrarily high order of convergence. This PCM combined method will be termed as Fractional Euler Method. For clarity, we will refer to the order of convergence of both the implicit Euler method predictor phase “explicit Fractional Euler Method”. [36]

Now, the Fractional Euler Method of PCM can be constructed from the Fractional Euler Method (an explicit method) and the (an implicit method).

First, for the predictor step; starting from the correct value y_0, calculate an initial value y_n via the explicit Fractional Euler Method:

$$y^{p}(t_{n+1})=y^{c}(t_n)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_n,y^{c}(t_n)) \qquad (3.2)$$

Next, for the corrector steps; improve the initial guess through iteration of implicit Fractional Euler Method:

$$y^{c}(t_{n+1}) = y^{c}(t_{n}) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_{n}, y^{p}(t_{n+1})) \tag{3.3}$$

Hence, the pair of Euler's scheme for PCM is as follows

$$\text{P: } y^{p}(t_{n+1}) = y^{c}(t_{n}) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_{n}, y^{c}(t_{n})) \tag{3.4}$$

$$\text{C: } y^{c}(t_{n+1}) = y^{c}(t_{n}) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_{n}, y^{p}(t_{n+1})) \tag{3.5}$$

This iteration may repeated for some fixed n-times or until the guesses converge to within some error tolerance ε:

$$|y^{p}{}_{n} - y^{p}{}_{n-1}| \le \varepsilon, \; n = 1, 2, 3, \dots$$

Similarly, the PCM for the pair fractional trapezoidal rule (an explicit method) and certain corrector method (an implicit method) is as follows.

$$\text{P: } y^{p}(t_{n+2}) = y^{c}(t_{n+1}) + \frac{h^{\alpha}}{2\Gamma(\alpha+1)} \left[f(t_{n}, y^{c}(t_{n})) + f(t_{n+1}, y^{c}(t_{n+1})) \right] \tag{3.6}$$

$$\text{C: } y^{c}(t_{n+2}) = y^{c}(t_{n+1}) + \frac{h^{\alpha}}{2\Gamma(\alpha+1)} \left[f(t_{n+1}, y^{p}(t_{n+1})) + f(t_{n+2}, y^{p}(t_{n+2})) \right] \tag{3.7}$$

3.2 Richardson Extrapolation Method for Solving FODEs [21]

Let $y(x_n)$ be the numerical solution at x_n obtained using certain numerical scheme say fractional Euler's method for simplicity, which has an order one and let $y(x_n)$ be the exact solution at x_n. The numerical solution will depends on the step size h and therefore the error approximation may be extended using power series expansion in certain time step h_i of the form

$$y(x_n) - y_n(h^\alpha) = \sum_{i=1}^{\infty} a_i h_i^\alpha$$

or equivalently

$$y(x_n) = y_n(h_1^\alpha) + \sum_{i=1}^{\infty} a_i h_i^\alpha \tag{3.8}$$

Also, for another step size h_2 we have similarly

$$y(x_n) = y_n(h_2^\alpha) + \sum_{i=1}^{\infty} a_i h_i^\alpha \tag{3.9}$$

multiply Eq. (3.8) by h_2^α and Eq. (3.9) by h_1^α and subtract them, to get:

$$h_1^\alpha y(x_n) - h_2^\alpha y(x_n) = h_1^\alpha y_n(h_2^\alpha) - h_2^\alpha y_n(h_1^\alpha) + a_1(h_1^\alpha h_2^\alpha - h_1^\alpha h_2^\alpha)$$
$$+ a_2(h_1^\alpha h_2^{2\alpha} - h_2^\alpha h_1^{2\alpha}) + \ldots$$

and so

$$y(x_n) = \frac{h_1^\alpha y_n(h_2^\alpha) - h_2^\alpha y_n(h_1^\alpha)}{h_1^\alpha - h_2^\alpha} + \frac{a_2 h_1^\alpha h_2^{2\alpha} - a_2 h_2^\alpha h_1^{2\alpha}}{h_1^\alpha - h_2^\alpha},\ h_1^\alpha \neq h_2^\alpha$$

and to make the $h^{2\alpha}$ terms cancel, will implies to:

$$y(x_n) = \frac{h_1^{\alpha} y_n(h_2^{\alpha}) - h_2^{\alpha} y_n(h_1^{\alpha})}{h_1^{\alpha} - h_2^{\alpha}} + O(h_1^{\alpha} h_2^{\alpha}) \tag{3.10}$$

Eq. (3.10) $O(h_1^{\alpha} h_2^{\alpha})$ is a higher order terms in h_1^{α} and h_2^{α}. Clear that Eq. (3.10) is a better approximation for $y(x_n)$.

As an illustration and for simplicity, let $h_1 = h$, $h_2 = \frac{h_1}{2} = \frac{h}{2}$, and therefore Eq. (3.10) will take the form:

$$y(x_n) = 2y_n(\frac{h^{\alpha}}{2}) - y_n(h^{\alpha}) + O(h^{2\alpha}) \tag{3.11}$$

It is remarkable that Similarity to the above approach may be followed to obtain solution of higher order $O(h^{3\alpha}), O(h^{4\alpha}),...$ as we will show next . This approach for improving the accuracy is called REM.

Similarly, for the step size h_0, h_1 and h_2, we have

$$y(x_n) = y_n(h_0^{\alpha}) + \sum_{i=1}^{\infty} a_i h_i^{\alpha} \tag{3.12}$$

$$y(x_n) = y_n(h_1^{\alpha}) + \sum_{i=1}^{\infty} a_i h_i^{\alpha} \tag{3.13}$$

$$y(x_n) = y_n(h_2^{\alpha}) + \sum_{i=1}^{\infty} a_i h_i^{\alpha} \tag{3.14}$$

Multiplying Eq. (3.12) by $h_1^{\alpha} h_2^{\alpha}$ and Eq. (3.13) by $\left(-3h_0^{\alpha} h_2^{\alpha}\right)$ and Eq. (3.14) by $\left(2h_0^{\alpha} h_1^{\alpha}\right)$. By combining the three equations, we get.

$$h_2^{\alpha}h_3^{\alpha}y(x_n)-3h_1^{\alpha}h_3^{\alpha}y(x_n)+2h_1^{\alpha}h_2^{\alpha}y(x_n)=h_2^{\alpha}h_3^{\alpha}y_n(h_1^{\alpha})-3h_1^{\alpha}h_3^{\alpha}y_n(h_2^{\alpha})$$
$$+2h_1^{\alpha}h_2^{\alpha}y_n(h_3^{\alpha})+a_1h_1^{\alpha}h_2^{\alpha}h_3^{\alpha}-3a_1h_1^{\alpha}h_2^{\alpha}h_3^{\alpha}+2a_1h_1^{\alpha}h_2^{\alpha}h_3^{\alpha}+a_2h_1^{2\alpha}h_2^{\alpha}h_3^{\alpha}$$
$$-3a_2h_1^{\alpha}h_2^{2\alpha}h_3^{\alpha}+2a_2h_1^{\alpha}h_2^{\alpha}h_3^{2\alpha}+a_3h_1^{3\alpha}h_2^{\alpha}h_3^{\alpha}-3a_3h_1^{\alpha}h_2^{3\alpha}h_3^{\alpha}+2a_3h_1^{\alpha}h_2^{\alpha}h_3^{3\alpha}$$

$$y(x_n)=\frac{h_2^{\alpha}h_3^{\alpha}y_n(h_1^{\alpha})-3h_1^{\alpha}h_3^{\alpha}y_n(h_2^{\alpha})+2h_1^{\alpha}h_2^{\alpha}y_n(h_3^{\alpha})}{\left(h_2^{\alpha}h_3^{\alpha}-3h_1^{\alpha}h_3^{\alpha}+2h_1^{\alpha}h_2^{\alpha}\right)}$$
$$+\frac{a_2h_1^{2\alpha}h_2^{\alpha}h_3^{\alpha}-3a_2h_1^{\alpha}h_2^{2\alpha}h_3^{\alpha}+2a_2h_1^{\alpha}h_2^{\alpha}h_3^{2\alpha}}{\left(h_2^{\alpha}h_3^{\alpha}-3h_1^{\alpha}h_3^{\alpha}+2h_1^{\alpha}h_2^{\alpha}\right)}$$
$$+\frac{a_3h_1^{3\alpha}h_2^{\alpha}h_3^{\alpha}-3a_3h_1^{\alpha}h_2^{3\alpha}h_3^{\alpha}+2a_3h_1^{\alpha}h_2^{\alpha}h_3^{3\alpha}}{\left(h_2^{\alpha}h_3^{\alpha}-3h_1^{\alpha}h_3^{\alpha}+2h_1^{\alpha}h_2^{\alpha}\right)}$$

take $h_0=h$, $h_1=\frac{h}{2}$ and $h_2=\frac{h}{4}$; then

$$y(x_n)=\frac{\frac{h^{2\alpha}}{8}y_n(h^{\alpha})-\frac{3h^{\alpha}}{4}y_n\left(\frac{h^{2\alpha}}{2}\right)+h^{2\alpha}y_n\left(\frac{h^{\alpha}}{4}\right)}{\left(\frac{h^{2\alpha}}{8}-\frac{3h^{2\alpha}}{4}+h^{2\alpha}\right)}+$$
$$\frac{a_2\left(\frac{h^{4\alpha}}{8}\right)-\left(\frac{3}{16}\right)h^{4\alpha}+\left(\frac{h^{4\alpha}}{16}\right)}{\left(\frac{h^{2\alpha}}{8}-\frac{3h^{2\alpha}}{4}+h^{2\alpha}\right)}+\frac{a_3\left(\frac{h^{5\alpha}}{8}\right)-3a_3\left(\frac{h^{5\alpha}}{32}\right)+2\left(\frac{h^{5\alpha}}{128}\right)}{\left(\frac{h^{2\alpha}}{8}-\frac{3h^{2\alpha}}{4}+h^{2\alpha}\right)}$$

$$y(x_n)=\frac{\frac{3h^{2\alpha}}{8}\left[\frac{1}{3}y_n(h^{\alpha})-2y_n\left(\frac{h^{\alpha}}{2}\right)+\frac{8}{3}y_n\left(\frac{h^{\alpha}}{4}\right)\right]}{\left(\frac{3h^{2\alpha}}{8}\right)}+\frac{\left(\frac{3h^{5\alpha}}{64}\right)}{\left(\frac{3h^{2\alpha}}{8}\right)}$$

$$y(x_n)=\frac{\frac{3h^{2\alpha}}{8}\left[\frac{1}{3}y_n(h^\alpha)-2y_n\left(\frac{h^\alpha}{2}\right)+\frac{8}{3}y_n\left(\frac{h^\alpha}{4}\right)\right]}{\left(\frac{3h^{2\alpha}}{8}\right)}+\frac{3h^{3\alpha}}{8}$$

and to make the $h^{3\alpha}$ terms cancel, we get:

$$y(x_n)=\frac{1}{3}y_n(h^\alpha)-2y_n\left(\frac{h^\alpha}{2}\right)+\frac{8}{3}y_n\left(\frac{h^\alpha}{4}\right)+O(h^{3\alpha})$$

3.3 Variable Order Method for Solving FODEs [20]

This method is considered as a generalization of REM using the FLMMs in connection with variable order methods used for solving ODEs to derive a new approach for solving FODEs with more accurate results. This method will be referred to as the variable order method for solving FODEs.

Consider the FODEs:

$$^{C}D^{\alpha}y(x)=f(x,y(x)),\ y(0)=y_0,\ 0<\alpha\leq 1. \quad (3.15)$$

In this investigation, approximation is studied for expectations of functions of the solution, i.e., $y(x_n)$ that is, weak approximation. The weak error is defined as:

$$y(x_n)-y\ (h^\alpha) \quad (3.16)$$

The primary goal of this investigation is to prove that the variable order method has a weak error power series expansion of the form:

$$y(x_n)-y\ (h^\alpha)=a_1h^\alpha+a_2h^{2\alpha}+a_3h^{3\alpha}+\ldots \quad (3.17)$$

where a_1, a_2, … are some constants independent of h^α and by using several approximations $y(h_0^\alpha), y(h_1^\alpha), y(h_2^\alpha)$ …; with $h_0^\alpha > h_1^\alpha > h_2^\alpha > \ldots$; where h_0^α, h_1^α, h_2^α, … are the step sizes.

Now, to successively eliminate the terms in the error expansion, thereby producing approximations using methods of higher order. The sequence of step sizes used was $h^\alpha = \frac{h^\alpha}{2^j}$; j = 0, 1, 2, …; where h is some starting step size. If a_1 in Eq. (3.17) is not zero, then the approximation scheme $y(x_n)$ is only of order h. To obtain approximations of order $h^{2\alpha}$, and we proceed as follows:

Find the weak error expansion using two different step sizes h_0^α and h_1^α, such that $h_1^\alpha < h_0^\alpha$, as follows:

$$y(x_n) = y(h_0^\alpha) + a_1 h_0^\alpha + a_2 h_0^{2\alpha} + a_3 h_0^{3\alpha} + \ldots \tag{3.18}$$

$$y(x_n) = y(h_1^\alpha) + a_1 h_1^\alpha + a_2 h_1^{2\alpha} + a_3 h_1^{3\alpha} + \ldots \tag{3.19}$$

and upon subtracting h_0^α times the second equation from h_1^α times the first equation and solving for $y(x_n)$, one may get:

$$h_1^\alpha y(x_n) - h_0^\alpha y(x_n) = h_1^\alpha y(h_0^\alpha) - h_0^\alpha y(h_1^\alpha) + a_1 h_0^\alpha h_1^\alpha - a_1 h_0^\alpha h_1^\alpha + a_2 h_1^\alpha h_0^{2\alpha} \text{-} a_2 h_0^\alpha h_1^{2\alpha} + \ldots$$

and upon eliminating the terms involving a_1, we obtain:

$$y(x_n) = \frac{h_1^\alpha y(h_0^\alpha) - h_0^\alpha y(h_1^\alpha)}{h_1^\alpha - h_0^\alpha} + \frac{a_2 h_1^\alpha h_0^{2\alpha} - a_2 h_0^\alpha h_1^{2\alpha}}{h_1^\alpha - h_0^\alpha}, \; h_1^\alpha \neq h_0^\alpha$$

Thus, letting:

$$y(x_n) = \frac{h_1^\alpha y(h_0^\alpha) - h_0^\alpha y(h_1^\alpha)}{h_1^\alpha - h_0^\alpha} + \frac{(-1)a_2 h_1^\alpha h_0^\alpha \left(h_1^\alpha - h_0^\alpha\right)}{h_1^\alpha - h_0^\alpha}$$

and also

$$y(x_n) = \frac{h_1^\alpha y\ (h_0^\alpha) - h_0^\alpha y\ (h_1^\alpha)}{h_1 - h_0} - a_2 h_1^\alpha h_0^\alpha$$

Therefore, and to make the $h^{2\alpha}$ terms cancel, we get:

$$y(x_n) = \frac{h_1^\alpha y(h_0^\alpha) - h_0^\alpha y(h_1^\alpha)}{h_1^\alpha - h_0^\alpha} + O(h_0^\alpha h_1^\alpha)$$

$$= y(h_1^\alpha) + \frac{y(h_1^\alpha) - y(h_0^\alpha)}{\frac{h_0^\alpha}{h_1^\alpha} - 1} + O(h_0^\alpha h_1^\alpha)$$

where $O(h_0^\alpha h_1^\alpha)$ is higher order terms in h_0^α and h_1^α

Thus, letting

$$y_1(h_0^\alpha) = y(h_1^\alpha) + \frac{y(h_1^\alpha) - y(h_0^\alpha)}{\frac{h_0^\alpha}{h_1^\alpha} - 1} + O(h_1^\alpha h_0^\alpha)$$

that is an O($h_0^{2\alpha}$) approximation to $y(x_n)$. Since $h_1^\alpha < h_0^\alpha$ and any two pair h_j^α, h_{j+1}^α may be used in the above elimination process, one may see that in general:

$$y_1(h_j^\alpha) = y(h_{j+1}^\alpha) + \frac{y(h_{j+1}^\alpha) - y(h_j^\alpha)}{\frac{h_j^\alpha}{h_{j+1}^\alpha} - 1} + O(h^{2\alpha}) \qquad (3.20)$$

which is also an O($h_j^{2\alpha}$) approximation to $y(x_n)$. Now, we have:

$$y(x_n) = y_1(h_0^\alpha) - a_2 h_0^\alpha h_1^\alpha - a_3 h_0^\alpha h_1^\alpha (h_0^\alpha + h_1^\alpha) - a_4 h_0^\alpha h_1^\alpha (h_0^{2\alpha} + h_1^\alpha h_2^\alpha + h_2^{2\alpha}) - \dots$$

and

$$y(x_n) = y_1(h_1^\alpha) - a_2 h_1^\alpha h_2^\alpha - a_3 h_1^\alpha h_2^\alpha (h_1^\alpha + h_2^\alpha) - a_4 h_1^\alpha h_2^\alpha (h_1^{2\alpha} + h_1^\alpha h_2^\alpha + h_2^{2\alpha}) - \dots$$

and upon eliminating the terms involving a_2, we obtain:

$$y(x_n) = y_2(h_1^\alpha) + a_3 h_0^\alpha h_1^\alpha h_2^\alpha + a_4 h_0^\alpha h_1^\alpha h_2^\alpha (h_0^\alpha + h_1^\alpha + h_2^\alpha) + \dots$$

where

$$y_2(h_0^\alpha) = y_1(h_0^\alpha) + \frac{y_1(h_1^\alpha) - y_1(h_0^\alpha)}{\frac{h_0^\alpha}{h_1^\alpha} - 1} + O(h^{3\alpha})$$

which is an $O(h^{3\alpha})$ approximation to $y(x_n)$. More generally:

$$y_2(h_j^\alpha) = y_1(h_j^\alpha) + \frac{y_1(h_{j+1}^\alpha) - y_1(h_j^\alpha)}{\frac{h_j^\alpha}{h_{j+1}^\alpha} - 1} + O(h^{3\alpha}) \qquad (3.21)$$

Similarly, continuing in this manner, the following recursively sequence may be derived:

$$y_0(h_j^\alpha) = y(h_j^\alpha),$$

$$y_n(h_j^\alpha) = y_{n-1}(h_{j+1}^\alpha) + \frac{y_{n-1}(h_{j+1}^\alpha) - y_{n-1}(h_j^\alpha)}{\frac{h_j^\alpha}{h_{j+1}^\alpha} - 1}. \qquad (3.22)$$

for all n = 1, 2, …; j = 0, 1, …

On the basis of the results for $y(h_j^{\alpha})$ and $y_2(h_j^{\alpha})$, it seems that $y_n(h_j^{\alpha})$ provides an $O(h_j^{(n+1)\alpha})$ approximation to $y(x_n)$. This may be verified directly by following the evolution of the general term $a_n\, h^{n\alpha}$ in the error expansion, but is perhaps obtained more easily by the following alternative approach obtained from Eq. (3.22) and (2.26), which is given in the following table:

Level	***O(*** h_j^{α} ***)***	***O(*** $h_j^{2\alpha}$ ***)***	***O(*** $h_j^{3\alpha}$ ***)***	***O(*** $h_j^{4\alpha}$ ***)***	
0	$y_0(h_0^{\alpha})$				
1	$y_0(h_1^{\alpha})$	$y_1(h_0^{\alpha})$			
2	$y_0(h_2^{\alpha})$	$y_1(h_1^{\alpha})$	$y_2(h_0^{\alpha})$		
3	$y_0(h_3^{\alpha})$	$y_1(h_2^{\alpha})$	$y_2(h_1^{\alpha})$	$y_3(h_0^{\alpha})$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

3.4 Numerical Examples

In this section, two example will be given for comparing purposes between the different proposed methods.

Example (3.1) [46] Our first example deals with the homogeneous linear FODEs.

$$D^{\alpha}y(t) = -y(t),\quad y(0)=1,\ t\geq 0,\ \ 0<\alpha\ \leq 1$$

The exact solution of Eq. (2.13) is given by

$$y(t) = E_{\alpha}\left(-t^{\alpha}\right)$$

Where E_α is the mittag-leffler function.

Table (3.1). Numerical values for Example (3.1) when $\alpha = 0.5$ and h=0.01, n = 10, $\varepsilon = 0.5$.

t	*Absolute Errors PCM of Euler's*	*Absolute Errors PCM of Trapezoidal*	*Absolute Errors REM O($h^{2\alpha}$)*	*Absolute Errors REM O($h^{3\alpha}$)*	*Absolute Errors Variable order*
0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.039874	0.027654	0.146547	0.132871	0.139775
0.2	0.026578	0.015689	0.131756	0.119321	0.128564
0.3	0.017699	0.003311	0.094786	0.092876	0.093765
0.4	0.008754	0.004234	0.072163	0.071002	0.071876
0.5	0.007592	0.003966	0.062386	0.061988	0.620125
0.6	0.005538	0.003678	0.053297	0.051787	0.052765
0.7	0.004679	0.003076	0.032265	0.030889	0.031668
0.8	0.003434	0.002867	0.027954	0.025129	0.026118
0.9	0.002387	0.001765	0.019817	0.017776	0.018556
1.0	0.001765	0.001243	0.009781	0.008385	0.009088

Example (3.2) [46] The second example deal with the nonlinear equation

$$D^\alpha y(t) = y(t)^2 - \frac{2}{(t-1)^2}\ ,\ \ y(0) = -2,\ \text{where}\ 0 < \alpha \le 1 \qquad (3.23)$$

Table (3.2). Numerical values for Example (3.2) when $\alpha = 0.5$ and h=0.01, n = 10, $\varepsilon = 0.5$.

t	***Absolute Errors PCM of Euler's***	***Absolute Errors PCM of Trapezoidal***	***Absolute Errors REM O($h^{2\alpha}$)***	***Absolute Errors REM O($h^{3\alpha}$)***	***Absolute Errors Variable order***
0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.026543	0.015777	0.107112	0.102434	0.106542
0.2	0.017432	0.010229	0.102356	0.098775	0.101987
0.3	0.015743	0.009466	0.099212	0.092245	0.098321
0.4	0.012547	0.007443	0.091001	0.087321	0.090122
0.5	0. 010211	0. 007060	0.090988	0.086752	0.090002
0.6	0.009653	0.005221	0.089962	0.080211	0.087865
0.7	0.007223	0.004552	0.082765	0.079443	0.080234
0.8	0.005887	0.002234	0.073987	0.067882	0.072987
0.9	0.003876	0.001088	0.072335	0.053217	0.071876
1.0	0.001665	0.000989	0.048342	0.044876	0.047844

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CONCLUSIONS AND RECOMMENDATION

The fundamental objective of this work has been to construct a numerical scheme to the numerical solution of the linear and nonlinear (FODE,s). Those objective has been obtained by using the submitted Modified Fractional Euler's method. From the results or the table (2.1) to (2.5) we can see the accuracy of the optioned used approaches and there are step size methods, in which the efficiency of the results is increased, and more precisely. the Variable Step Size Method approximate solution in this case is in high agreement with the exact solution.

As a results of the table (3.1) and (3.2) we can see that the PCM is very high in accuracy, which is the most accuracy of all methods. The REM gives a good result, but on other hand the Variable order method gives more accuracy results than REM.

For future work the following problems could be recommended:

1- Solve the initial value problem given by equations (2.1) by using Runge-Kutta method.
2- Derive methods that highest order.
3- Solve FODEs with multiple order.
4- Solve system of FODEs.

الخلاصة

حساب التفاضل الكسري هو موضوع لحساب المشتقات والتكاملات لرتب كسرية من دالة معينة، للمعادلات التفاضلية الكسرية، موضوع الدراسة هو حل المعادلات التفاضلية للرتب الكسري، التي تحتوي على شرط أولي أو حدودي. حل المعادلات التفاضلية ذات الرتب الكسرية لديها الكثير من الصعوبات في الحل التحليلي، وبالتالي قد تكون الطرق العددية في معظم الحالات هي الطريقة المناسبة للحل. ولذلك، فإن الهدف الرئيسي من هذا العمل هو دراسة الحل العددي للمعادلات التفاضلية الاعتيادية ذات الرتب الكسرية باستخدام طريقة متعددة الخطوات الخطية عن طريق الاستفادة من نشر متسلسلة تايلر الكسرية. وبالإضافة إلى ذلك، تم تحسين النتائج العددية باستخدام العديد من النهج، مثل طريقة طريقة حجم الخطوة المتغيرة ، طريقة التخمين والتصحيح، طريقة ريكاردسون للاستكمال وطريقة متغير الرتبة.الحسابات كتبت باستخدام البرمجيات الرياضية ماتلاب 16أ.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بغداد
كلية التربية للعلوم الصرفة / ابن الهيثم
قسم الرياضيات

الحل العددي للمعادلات التفاضلية الاعتيادية ذات الرتب الكسرية باستخدام طرائق متعددة الخطوات الخطية

رسالة

مقدمة إلى كلية التربية للعلوم الصرفة / ابن الهيثم – جامعة بغداد

وهي جزء من متطلبات نيل درجة ماجستير علوم

في الرياضيات

من قبل

حيدر هاشم خيون

إشراف

أ.م.د. فاضل صبحي فاضل

1439هـ 2018م