A Study of Some Generalizations of Fibrewise Bitopological Spaces

A Thesis
Submitted to the College of Education for Pure Sciences / Ibn Al-Haitham, University of Baghdad as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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2018 AC 1439 AH
بَشِّرِ اللَّهُ الرَّحْمَنِ الرَّحِيمِ

تَبَارَكَ الَّذِي بَيْنَ الْمَلَكَ وَهُوَ عَلَى كُلِّ شَيْءٍ قَدِيرٍ الَّذِي خَلَقَ الْمَوْتَ وَالْحَيَاةَ لِيُبَيِّنَ لَكُمْ أَيُّهُمُ أَحْسَنُ عَمَلُ وَهُوَ الْعَزِيزُ الْعَفُورُ الَّذِي خَلَقَ سَبْعَ سَمَوَاتٍ طَبَاقًا مَا تَرَى فِيهِ خَلَقُ الرَّحْمَانِ مِنْ تَفْنِيَّةٍ فَأَرَجِعُ الْبَصَرُ الْحَلَّ تَرَى مِن فُطُورٍ ثُمَّ أَرَجِعُ الْبَصَرُ كَرْتَنِينَ يَنْقِلُ إِلَيْكَ الْبَصَرُ حَاسِبًا وَهُوَ حِسَبُ وَلَقَدْ زَيَّنَا السَّمَاوَاتَ الْأَلَّهِيَّ بِمَصَبِيحٍ وَجَعَلْنَاهَا رَجُومًا لِّلشَّيَاطِينِ وَأَعْتَدْنَاهَا لَهُمْ عَذَابَ الْبَعْيُوتِ

صِدَاقِ اللَّهِ الْعَلِيِّ العَظِيمِ

الملك (1-5)
إهداء

إلى من يبلغ الرسالة ... وادي الأمانة ... ونصح الأمانة ... إلى نبي الرحمة ونور العالمين سيدنا محمد صلى الله عليه وسلم.

إلى سور الوطن الحصين وحماية ثغور المسلمين الجيش والجيش الشعبي سبيما الشهداء منهم في ميادين العز والكرامة.

إلى من أوصى الله سبحانه وتعالى بهم ووصينا الإنسان والذين إحسانا حملته أنكرها ووضعته كركها وحمله وقصاصه.

ثامن شهرا حكى إذا بلغ أشد وبلغ آربعين سنة فألمر وهو رجى أن استنكير عصماته التي أعمرت على والدي وأن أعمل صالحا ترضاه وأصحلي في دمريته إن بنت إليك وإنني من المسلمين (15).

الاحتفاء (15)

إلى من علمي وأرشدني في حياتي وشجعني على تحقيق حلمي أبي (رحمه الله).
إلى معني الخان من التضحية وسر نجاحي وسعاديامي الغالية.
إلى معلمي الأول وملهمي وقدوني الاستاذ الدكتور علي محسن عيسى (رحمه الله).
إلى من اعتمدهم في حياتي واكتسب منهم فوائدي اخوتى وأخواتي الأعزاء.
إلى SENI وعوني ومصدر ثقتي في مدلعيمات الصاع زوجتي الحبيبة.
إلى النور الذي يهد عني البصر والنسمة التي تنبع ما اضمر من كياني ابتنتاي كونر وريان.
إلى جميع أصدقائي واحبيتي أمنن كان لهم الآخر البلغ في نجاحي وتقدمي بالأخلاق اخي ورفيق دربي الاستاذ محمد عبد الحسين غافل.

اهدي لكم عملي هذا بتواضع...
Acknowledgment

Praise to Allah and peace and blessings be upon his prophets and messengers.
Blessing is also upon prophet Mohammed and his family and his extend.

I would like to thank who supervised on my thesis
Assist. Prof. Dr. YOUSIF YAQOUB YOUSIF

I would like to thank the
All staff of the department of mathematics-college of education for pure sciences / Ibn Al-Haitham- University of Baghdad.

Liwaa Ali


Abstract

In this research, we introduce and study the concept of fibrewise bitopological spaces. We generalize some fundamental results from fibrewise topology into fibrewise bitopological space. We also introduce the concepts of fibrewise closed bitopological spaces, (resp., open, locally sliceable and locally sectionable). We state and prove several propositions concerning with these concepts. On the other hand, we extend separation axioms of ordinary bitopology into fibrewise setting. The separation axioms we extend are called fibrewise pairwise $T_0$ spaces, fibrewise pairwise $T_1$ spaces, fibrewise pairwise $R_0$ spaces, fibrewise pairwise Hausdorff spaces, fibrewise pairwise functionally Hausdorff spaces, fibrewise pairwise regular spaces, fibrewise pairwise completely regular spaces, fibrewise pairwise normal spaces, and fibrewise pairwise functionally normal spaces. In addition, we offer some results concerning these extended axioms. Finally, we introduce some concepts in fibrewise bitopological spaces which are fibrewise $ij$-bitopological spaces, fibrewise $ij$-closed bitopological spaces, fibrewise $ij$-compact bitopological spaces, fibrewise $ij$-perfect bitopological spaces, fibrewise weakly $ij$-closed bitopological space, fibrewise almost $ij$-perfect bitopological space, fibrewise $ij^*$-bitopological spaces. We study several theorems and characterizations concerning these concepts.
Abbreviation

$$(M, \tau)$$

topological space

$$(M, \tau_1, \tau_2)$$

bitopological space

$${\cap}$$

intersection of set

$${\cup}$$

union of set

$$\epsilon$$

belong of set

$$\not\epsilon$$

not belong of set

$$p_M$$

projection function \( p: M \to B \)

$$M_b$$

$$p^{-1}(b) : b \in B$$

$$M_{B^*}$$

$$p^{-1}(B^*) : B^* \subseteq B$$

$$M \mid B^*$$

$$p^{-1}(B^*) : B^* \subseteq B$$

$$\emptyset$$

empty set

$$\varphi$$

fibrewise function

$$\Delta_{s \in S} \varphi_s$$

diagonal of function

$$id_M$$

identity function \( id_M : M \to M \)

$$\pi_2$$

projection function of product

$$\Delta: M \to M \times_B M$$

diagonal embedding

$$(M \times_B M, \tau_1 \times \tau_1, \tau_1 \times \tau_1)$$

product of two bitopology

$$\mathbb{R}$$

real numbers

$$\mathcal{A}$$

open cover

$$\Gamma$$

graph function

$$\lambda$$

continuous function \( \lambda : M_W \to [0,1] \)

$$\mathcal{F} < \mathcal{G}$$

\( \mathcal{G} \) is finer than \( \mathcal{F} \)

$$\prod_B M_r$$

product of function
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Introduction

Mathematics plays a vital and an important role in the development of civilization which mankind has witnessed since the dawn of the history up to nowadays. Mathematics has undoubtedly the big favor in accelerating the wheel of progress for producing ideas and laws helped to organize and coordinate the various natural sciences such as, Geometry, Physics, Chemistry, Biology, Astronomy, Economics and Computers, etc.

The middle of 19th century witnessed an important changes in mathematics structure, especially in Geometry. For the first time, the term of (Topology) has been used in 1847 in Germany by the German scientist (Johann Benedict Listing ). Topology is a Greek word, consisting of two syllables : “topo” means a place, “logos” means study. At the beginnings of 20th century as for 1925 up to 1975, this branch has clearly developed and formed an integrated competence. So, the topology is a science that deals with Geometry in a different way not as it used in Euclidean Geometry. This science distinguished by flexibility concerning the mathematical shapes. It could find the suitable solutions and remove the ambiguity of many problems that scientists couldn’t find the right solutions through the Euclidean Geometry.

Bitopological spaces are first introduced by Kelly [18] in (1963) followed by many researchers who developed and generalized bitopological space on different science.
The concept of fibrewise set over a given set was introduced by James in [9], [10], [11], [12], [13], [14] in 1989. In order to begin the work in the category of fibrewise (briefly F.W.) sets over a given set, called the base set, which is denoted by $B$. A F.W. set over $B$ consist of a set $M$ with a function $p: M \rightarrow B$, that is called the projection. The fibre over $b$ for every point $b$ of $B$ is the subset $M_b = p^{-1}(b)$ of $M$. Perhaps, fibre will be empty since we do not require $p$ is surjective, also, for every subset $B^*$ of $B$ we considered $M_{B^*} = p^{-1}(B^*)$ as a F.W. set over $B^*$ with the projection determined by $p$. The alternative notation $M \mid B^*$ is some time convenient. We considered the Cartesian product $B \times T$, for every set $T$, like a F.W. set $B$ by the first projection.

The functions are not only a fundamental but the most important concepts in Mathematics for having a wide applications. Thus, the mathematical scientists were interesting in inserting this vital concept within topology for finding new visions and opening a wide horizons. For this reason, the general topology idea for the continuous functions or the general fibrewise topology which deals with the topological spaces as a mapping from this space onto a one point space.

To put the foundation stone for fibrewise topological spaces, many attempts appeared during the last two decades, most of the results, obtained so far in this field can be found in the work of Dyckhoff [6] in (1972) and Niefield [28] in (1984). Some hope of this is provided by the link between fibrewise topology and topos theory, referred to by Lever [21] and [22] in (1983, 1984) and Johnstone [15] in (1981, 1984). Moreover, in Pasynkov [29] in (1984) and James [9], [10], [11], [12], [13] and [14] in (1986, 1989), we can find definitions of some fibrewise topological spaces. Also in Buhagiar [5] in (1997), we can find definitions of some topological mappings which are precisely the definitions of fibrewise topological spaces, where the codomain
is the base set. In (2003), Al-Zoubi and Hdeib [42] defined countably paracompact mappings, which are the fibrewise topological analogue of countably paracompact spaces finally Y.Y.Yousif and M. A. Hussain [35] and [36] in (2017) defined the concept of fibrewise soft topological spaces. Several characterizations of countably paracompact mappings are proved. As well as, we built on some of the result in [1], [2], [8], [17],[19], [20], [23], [31], [32], [33], [37], [38], [39], [40], [41].

The purpose of this thesis is to generalize fibrewise sets on the bitopological spaces, and to generalize some other mathematical concepts. The thesis will be entitled:

“*A Study of Some Generalizations of Fibrewise Bitopological Spaces*”

This thesis includes four chapters:

**Chapter one:** In this chapter we recall some of the fundamental definitions in the general topological spaces, bitopological spaces, and some basic concepts in the fibrewise spaces.

**Chapter two:** We introduce new definitions by mixing between the fibrewise sets and bitopological spaces and called it “fibrewise bitopological spaces”. We deal with many definitions and theorems which are generalized from general topology.

**Chapter three:** We study a basic concept and very important in topology which is called separation axioms in which we put new definitions of spaces, $T_0, T_1, T_2, T_3$, regular, normal in the light of the fibrewise bitopological space.

**Chapter four:** The aim of this chapter is to study compact fibrewise bitopological spaces, closed fibrewise bitopological spaces, rigid fibrewise bitopological spaces and the relationship among them and we give some basic definitions on the concept of filter and the point which is related with director filter and convergence of the filter.
Chapter 1

Preliminary Concepts
Chapter 1
Preliminary Concepts

This chapter consists of two sections. Section one contains fundamental concepts of topological spaces, Bitopological spaces, compact spaces, the concept of filters, and filter base and some examples about some of these concepts. Section two gives an explains fibrewise sets theories and some of their properties.

1.1. Fundamental Notions of Topological (bitopological) Spaces

Some basic concepts in topology which are useful for our study are given in this section.

Definition 1.1.1. [7] Let $X$ be a nonempty set and $\tau$ be a collection of subsets of $X$. The collection $\tau$ is said to be a topology on $X$ if $\tau$ satisfies the following three conditions:
(a) $\emptyset \in \tau$ and $X \in \tau$,
(b) $\tau$ is closed under finite intersection,
(c) $\tau$ is closed under arbitrary union.
If $\tau$ is a topology on $X$, then the pair $(X, \tau)$ is called a topological space or simply $X$ is a space. The subsets of $X$ belonging to $\tau$ are called open sets in the space and the complement of the subsets of $X$ which belongs to $\tau$ are called closed sets in the space.

Definition 1.1.2. [7] Let $(X, \tau)$ be a topological space and $A \subseteq X$. The closure (resp., interior) of $A$ is denoted by $Cl(A)$ (resp., $Int(A)$) and is defined as:

$$cl(A) = \bigcap \{ F \subseteq X; F \text{ is closed set and } A \subseteq F \}.$$

$$int(A) = \bigcup \{ O \subseteq X; O \text{ is open set and } O \subseteq A \}.$$
Evidently, \( cl(A) \) (resp., \( int(A) \)) is the smallest closed (resp., largest open) subset of \( X \) which contains (resp., contained in ) \( A \). Note that \( A \) is closed (resp., open) if and only if \( A = cl(A) \) (resp., \( A = int(A) \)).

**Definition 1.1.3.** [7] Let \((X, \tau)\) be a topological space and \( A \subseteq X \). The boundary of \( A \) is denoted by \( Bd(A) \) and is defined by:

\[
bd(A) = cl(A) - int(A).
\]

**Definition 1.1.4.** [4] Let \((X, \tau)\) be a topological space and \( A \subseteq X \). The subspace topology on \( A \) is denoted by \( \tau_A \) and is defined by:

\[
\tau_A = \{ A \cap O ; O \in \tau \}.
\]

The subspace topology is also called the relative topology or the induced topology or the trace topology.

**Definition 1.1.5.** [7] A function \( f : X \rightarrow Y \) is said to be continuous if the inverse image of each open set in \( Y \) is open in \( X \).

**Definition 1.1.6.** [7] A function \( f : X \rightarrow Y \) is said to be open if the image of each open set in \( X \) is open in \( Y \).

**Definition 1.1.7.** [7] A function \( f : X \rightarrow Y \) is said to be closed if the image of each closed set in \( X \) is closed in \( Y \).

The bitopological space was first created by Kelly [18] in 1963 and after that a large number of researches have been completed to generalize the topological ideas into bitopological setting.

Next Some basic concepts in bitopological spaces which are useful for our study are given.
Definition 1.1.8. [18] A triple \((M, \tau_1, \tau_2)\) where \(M\) is a non-empty set and \(\tau_1\) and \(\tau_2\) are two topologies on \(M\) is called bitopological space.

Example 1.1.9. Let \(M = \{1, 2, 3\}\), \(\tau_1 = \{M, \emptyset, \{1, 2\}, \{2\}\}\), \(\tau_2 = \{M, \emptyset, \{1, 3\}\}\). Then \((M, \tau_1, \tau_2)\) is bitopological space.

In this work \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) (briefly, \(M\) and \(N\)) always mean bitopological spaces on which no separation axioms are supposed unless clearly stated. By \(\tau_i\)-open (resp., \(\tau_i\)-closed), we shall mean the open (resp., closed) set with respect to \(\tau_i\) in \(M\), where \(i = 1, 2\). A set \(A\) is open (resp., closed) in \(M\) if it is both \(\tau_1\)-open (resp., \(\tau_1\)-closed) and \(\tau_2\)-open (resp., \(\tau_2\)-closed).

In what follows we consider \(i, j \in \{1, 2\}; i \neq j\).

Definition 1.1.10. [18] A function \(f : (M, \tau_1, \tau_2) \rightarrow (N, \sigma_1, \sigma_2)\) is said to be \(\tau_i\)-continuous (resp., \(\tau_i\)-open, \(\tau_i\)-closed), if the function \(f : (M, \tau_i) \rightarrow (N, \sigma_i)\) is continuous (resp., open, closed). \(f\) is called continuous (resp., open, closed) if it is \(\tau_i\)-continuous (resp., \(\tau_i\)-open, \(\tau_i\)-closed) for every \(i = 1, 2\).

Example 1.1.11. Let \(M = \{1, 2, 3\}\) and \(N = \{a, b, c\}\) be two sets. Let \(\tau_1\) and \(\tau_2\) (resp. \(\sigma_1\) and \(\sigma_2\)) be two topologies on \(M\) (res. \(N\)) such that \(\tau_1 = \{M, \emptyset, \{1, 2\}\}\) and \(\tau_2 = \{M, \emptyset, \{3\}, \{2, 3\}\}\), \(\sigma_1 = \{N, \emptyset, \{a, b\}\}\), \(\sigma_2 = \{N, \emptyset, \{a\}, \{a, c\}\}\). Define \(\varphi : M \rightarrow N\) such that \(\varphi(1) = b, \varphi(2) = a, \varphi(3) = c\). Then \(\varphi\) is continuous (open and closed).
Definition 1.1.12. [27] A bitopological space \((M, \tau_1, \tau_2)\) is said to be pairwise \(T_0\) space if for every pair of points \(x\) and \(y\) such that \(x \neq y\) there exists a \(\tau_i\)-open set containing \(x\) but not containing \(y\) or a \(\tau_j\)-open set containing \(y\) but not containing \(x\), where \(i, j = 1, 2, i \neq j\).

Definition 1.1.13. [16] A point \(x\) in \((M, \tau_1, \tau_2)\) is called an \(ij\)-contact point of a subset \(A \subseteq M\) iff for every \(\tau_i\)-open neighborhood (nbd) \(U\) of \(x\), \((\tau_j-cl(U)) \cap A \neq \emptyset\). The set of all \(ij\)-contact points of \(A\) is called the \(ij\)-closure of \(A\) and is denoted by \(ij-cl(A)\). \(A \subseteq M\) is called \(ij\)-closed iff \(A = ij-cl(A)\), where \(i, j = 1, 2\).

Definition 1.1.14. [4] A filter \(\mathcal{F}\) on a set \(M\) is a nonempty collection of nonempty subsets of \(M\) with the properties:
(a) If \(F_1, F_2 \in \mathcal{F}\), then \(F_1 \cap F_2 \in \mathcal{F}\).
(b) If \(F \in \mathcal{F}\) and \(F \subseteq F^* \subseteq M\), then \(F^* \in \mathcal{F}\).

Definition 1.1.15. [4] A filter base \(\mathcal{F}\) on a set \(M\) is a nonempty collection of nonempty subsets of \(M\) such that if \(F_1, F_2 \in \mathcal{F}\) then \(F_3 \subseteq F_1 \cap F_2\) for some \(F_3 \in \mathcal{F}\).

Definition 1.1.16. [4] If \(\mathcal{F}\) and \(\mathcal{G}\) are filter bases on \(M\), we say that \(\mathcal{G}\) is finer than \(\mathcal{F}\) (written as \(\mathcal{F} < \mathcal{G}\)) if for each \(F \in \mathcal{F}\), there is \(G \in \mathcal{G}\) such that \(G \subseteq F\) and that \(\mathcal{F}\) meets \(\mathcal{G}\) if \(F \cap G \neq \emptyset\) for every \(F \in \mathcal{F}\) and \(G \in \mathcal{G}\).

Definition 1.1.17. [4] A filter base \(\mathcal{F}\) on \(M\) is said to be \(ij\)-converges to a subset \(A\) of \(M\) (written as \(\mathcal{F} \xrightarrow{ij-con} A\)) iff for every \(\tau_i\)-open cover \(\mathcal{U}\) of \(A\), there is a finite subfamily \(\mathcal{U}_0\) of \(\mathcal{U}\) and a member \(F\) of \(\mathcal{F}\) such that \(F \subseteq \bigcup \mathcal{U}_0\).
\{\tau_j - cl(U) : U \in \mathcal{U}_0\}. Also if \( x \in M \), we say \( \mathcal{F} \xrightarrow{ij-con} x \) iff \( \mathcal{F} \xrightarrow{ij-con} \{x\} \) or equivalently, \( \tau_j \)-closure of every \( \tau_i \)-open nbd of \( x \) contains some members of \( \mathcal{F} \).

**Definition 1.1.18.** [3] A function \( f : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2) \) is called \( ij \)-continuous iff for any \( x \in M \), there exist \( \sigma_i \)-open nbd \( V \) of \( f(x) \), there exists a \( \tau_i \)-open nbd \( U \) of \( x \) such that \( f(\tau_j-cl(U)) \subseteq \sigma_j-cl(V) \), where \( i, j = 1, 2 \).

**Definition 1.1.19.** [3] A point \( x \) in a bitopological space \( (M, \tau_1, \tau_2) \) is called an \( ij \)-adherent point of a filter base \( \mathcal{F} \) on \( M \) iff it is an \( ij \)-contact point of every number of \( \mathcal{F} \). The set of all \( ij \)-adherent points of \( \mathcal{F} \) is called the \( ij \)-adherence of \( \mathcal{F} \) and is denoted by \( ij-ad \mathcal{F} \), where \( i, j = 1, 2 \).

**Definition 1.1.20.** [24] A subset \( A \) in bitopological space \( (M, \tau_1, \tau_2) \) is called \( ij \)-H-set in \( M \) iff for each \( \tau_i \)-open cover \( \mathcal{A} \) of \( A \), there is a finite subcollection \( \mathcal{B} \) of \( \mathcal{A} \) such that \( A \subset \bigcup \{ \tau_j - cl(U) : U \in \mathcal{B} \} \), \( i, j = 1, 2 \). \( A \) is called a pairwise-H-set iff it is a 12- and 21-H-set. If \( A \) is an \( ij \)-H-set (pairwise-H-set) and \( A = M \), then the space is called an \( ij \)-QHC (resp., pairwise QHC) space, where \( i, j = 1, 2 \).

**Lemma 1.1.21.** [25] A subset \( A \) of a bitopological space \( (M, \tau_1, \tau_2) \) is an \( ij \)-H-set iff for each filter base \( \mathcal{F} \) on \( A \), \( (ij-ad \mathcal{F}) \cap A \neq \varnothing \), where \( i, j = 1, 2 \).

**Proof:** \((\Rightarrow)\) Clear.

\((\Leftarrow)\) Let \( \mathcal{A} \) be a \( \tau_i \)-open cover of \( A \) such that the union of \( \tau_j \)-closure of any finite sub collection of \( \mathcal{A} \) is not cover \( A \). Then \( \mathcal{F} = \{A \setminus \bigcup B \tau_j-cl(B) : B \) is finite sub collection of \( \mathcal{A} \} \) is a filter base on \( A \) and \((ij-ad \mathcal{F}) \cap A = \varnothing \). This is a contradiction. Thus, \( A \) is \( ij \)-H-set.
Definition 1.1.22 [25] A topological space \((M,\tau)\) is called Urysohn space iff for each \(x \neq y\) can be separated by closed nbd.

Definition 1.1.23. [25] A bitopological space \((M,\tau_1,\tau_2)\) is said to be pairwise Urysohn space if for \(x, y \in M\) with \(x \neq y\), there are \(\tau_i\)-open nbd \(U\) of \(x\) and \(\tau_j\)-open nbd \(V\) of \(y\) such that \(\tau_j - cl(U) \cap \tau_i - cl(V) = \emptyset\), where \(i, j = 1, 2\).

Lemma 1.1.24. [25] In a pairwise Urysohn bitopological space \((M,\tau_1,\tau_2)\) an \(ij\)-H-set is \(ij\)-closed, where \(i, j = 1, 2\).

Lemma 1.1.25. [16] In a bitopological space \((M,\tau_1,\tau_2)\). If \(U \in \tau_j\), then \(ij - cl(U) = \tau_j - cl(U)\), where \(i, j = 1, 2\).

Lemma 1.1.26. [30] The bitopological space \((M,\tau_1,\tau_2)\) is pairwise Hausdorff iff \(\{m\} = ij - cl\{m\}\), for each \(m \in M\).

1.2. Fundamental Notions of Fibrewise Topology

In order to begin the category in the classification of fibrewise (briefly, F.W.) sets over a given set, called the base set, which say \(B\). A F.W. set over \(B\) consists of a set \(M\) with a function \(p: M \to B\), that is called the projection. The fibre over \(b\) for every point \(b\) in \(B\) is the subset \(M_b = p^{-1}(b)\) of \(M\). Perhaps, fibre will be empty since we do not require \(p\) is surjective, also, for every subset \(B^*\) of \(B\), we consider \(M_{B^*} = p^{-1}(B^*)\) as a F.W. set over \(B^*\) with the projection determined by \(p\). The alternative notation of \(M_{B^*}\) is sometime referred to as \(M \mid B^*\). We consider the Cartesian product \(B \times T\), for every set \(T\), as a F.W. set over \(B\) by the first projection.
**Definition 1.2.1.** [9] If $M$ and $N$ are F.W. sets over $B$, with projections $p_M$ and $p_N$, respectively. A function $\varphi: M \to N$ is said to be F.W. function if $p_N \circ \varphi = p_M$, or $\varphi(M_b) \subseteq N_b$ for every point $b$ of $B$, where $p_N: N \to B$ and $p_M: M \to B$.

**Example 1.2.2.** Let $M = \{1, 2, 3\}$, $N = \{2, 4, 6\}$, $B = \{a, b, c\}$, let $p_M: M \to B$ where: $p_M(1) = a, p_M(2) = b, p_M(3) = c$. Let $p_N: N \to B$ where: $p_N(2) = a, p_N(4) = c, p_N(6) = b$. Let $\varphi: M \to N$ where: $\varphi(1) = 2, \varphi(2) = 6, \varphi(3) = 4$. Then $\varphi$ is a fibrewise function.

Note that a F.W. function $\varphi: M \to N$ over $B$ is determines, by a restriction, a F.W. function $\varphi_{B^*}: M_{B^*} \to N_{B^*}$ over $B^*$ for every subset $B^*$ of $B$.

**Definition 1.2.3.** [9] Let $(B, \Lambda)$ be a topological space. The F.W. topology on a F.W. set $M$ over $B$ means any topology on $M$ for which the projection $p$ is continuous.

**Definition 1.2.4.** [9] The F.W. topological space $(M, \tau)$ over $(B, \Lambda)$ is called F.W. closed (resp., F.W. open) if the projection $p$ is closed (resp., open).

**Example 1.2.5.** Let $B = \{1, 2, 3\}$, $\Lambda = \{B, \varphi, \{1, \{1, 2\}\}$. Let $M$ be fibrewise set over $B$ where $M = \{a, b\}$ and let $p: M \to B$ such that $p(a) = 1, p(b) = 2$. Let $\tau = \{M, \varphi, \{a\}\}$ be any topology on $M$. Then $p$ is continuous and $(M, \tau)$ is F.W. topology on $(B, \Lambda)$.

**Definition 1.2.6.** [9] The F.W. function $\varphi: M \to N$, where $M$ and $N$ are F.W. topological spaces over $B$ is called
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(a) continuous if for every point \( m \in M_b \); \( b \in B \), the inverse image of every open set of \( \varphi(m) \) is an open set of \( m \).

(b) open if for every point \( m \in M_b \); \( b \in B \), the image of every open set of \( m \) is an open set of \( \varphi(m) \).

**Example 1.2.7.** Let \( M = \{1, 2, 3\} \), \( \tau = \{M, \varphi, \{3\}, \{1, 3\}\} \), and let \( N = \{2, 3, 5\} \), \( \sigma = \{N, \emptyset, \{2, 5\}, \{2\}\} \). Let \( B = \{a, b, c\} \), and \( \Lambda = \{B, \emptyset, \{a\}, \{a, c\}\} \). Assume that \( \varphi : M \to N \) be function where \( \varphi(1) = 2, \varphi(3) = 5, \varphi(2) = 3 \). Let \( p_M : M \to B \) such that \( p_M(1) = c, p_M(2) = b, p_M(3) = a \). Let \( p_N : N \to B \) such that \( p_N(2) = a, p_N(3) = c, p_N(5) = a \). So \( p \) is continuous and open.

**Example 1.2.8.** Let \( M = \{1, 2, 3\} \), \( \tau = \{M, \varphi, \{1\}, \{2, 3\}\} \). Let \( B = \{a, b, c\} \), \( \Lambda = \{B, \emptyset, \{b\}, \{a, c\}\} \). Let \( p : M \to B \) where \( p(1) = b, p(2) = c, p(3) = a \). Let \( \tau^c = \{M, \varphi, \{1\}, \{2, 3\}\} \), \( \Lambda^c = \{B, \emptyset, \{b\}, \{a, c\}\} \). Then \( p \) is closed (resp., open).

**Definition 1.2.9.** [7] Assume that we are given a topological space \( M \), a family \( \{\varphi_s\}_{s \in S} \) of continuous functions, and a family \( \{N_s\}_{s \in S} \) of topological spaces where the function \( \varphi_s : M \to N_s \) that transfers \( x \in M \) to the point \( \{\varphi_s(x)\} \in \prod_{s \in S} N_s \) is continuous, it is called the diagonal of the functions \( \{\varphi_s\}_{s \in S} \) and is denoted by \( \Delta_{s \in S} \varphi_s \) or \( \varphi_1 \Delta \varphi_2 \Delta \ldots \Delta \varphi_k \) if \( S = \{1, 2, \ldots, k\} \).

**Definition 1.2.10.** [34] For every topological space \( M^* \) and any subspace \( M \) of \( M^* \), the function \( i_M : M \to M^* \) define by \( i_M(x) = x \) is called embedding of the subspace \( M \) in the space \( M^* \). Observe that \( i_M \) is continuous, since \( i_M^{-1}(U) = M \cap U \), where \( U \) is open set in \( M^* \). The embedding \( i_M \) is closed (resp., open) iff the subspace \( M \) is closed (resp., open).
**Definition 1.2.11.** [34] If $X$ is topological space and $x \in X$ a neighborhood of $x$ is a set $U$ which contain an open set $V$ containing $x$. If $A$ is open set and contains $x$ we called $A$ is open neighborhood for a point $x$.

**Definition 1.2.12.**[4] A topological space $(M, \tau)$ is called compact iff each open cover of $M$ has a finite subcover for $M$.

**Definition 1.2.13.** [26] Let $(M, \tau)$ and $(N, \sigma)$ be topological spaces. A function $f: M \rightarrow N$ is a local homeomorphism if for every point $x$ in $M$ there exists an open set $U$ containing $x$, such that the image is open in $N$ and the restriction is a homeomorphism.

**Definition 1.2.14.** [18] A bitopological space $(M, \tau_1, \tau_2)$ is said to be pairwise Hausdorff, if for each distinct points $x, y \in M$ there exist disjoint sets $\tau_i$-open set $U$ of $x$ and $\tau_j$-open set $V$ of $y$, for $i, j = 1, 2, i \neq j$. 
Chapter 2

Fibrewise Bitopological Spaces
Chapter 2

Fibrewise Bitopological Spaces

The aim of this chapter is to introduce a new bitopological structure which is called Fibrewise Bitopological space. We define the concept of bitopological space based on a fibrewise set. Some examples and theories related to this new structure are introduced. In section one, we defined the concept of fibrewise bitopological spaces and the notion of induced fibrewise bitopological spaces. In section two we studied the notions of fibrewise open and fibrewise closed bitopological spaces. The purpose of section three is to show the notions of Fibrewise locally sliceable and fibrewise locally sectionable bitopological spaces.

2.1. Fibrewise Bitopological Spaces

In this section we establish F.W. bitopological spaces. Several topological properties on this space are obtained and studied.

Definition 2.1.1. Let \((B, \Lambda_1, \Lambda_2)\) be a bitopological space. The F.W. bitopology on a F.W. set \(M\) over \(B\) means any bitopology on \(M\) for which the projection \(p\) is continuous.

Example 2.1.2. Let \(B = \{a, b, c\}, \Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{a, c\}\}\). Let \(M\) be a fibrewise set over \(B\) where \(M = \{1, 2, 3\}\). Let \(\tau_1 = \{M, \emptyset, \{1\}\}, \tau_2 = \{M, \emptyset\}\). Let \(p : M \to B\) where \(p(1) = a, p(2) = c = p(3)\). Then \((M, \tau_1, \tau_2)\) is a fibrewise bitopology on \((B, \Lambda_1, \Lambda_2)\).

For another example, we consider \((B, \Lambda_1, \Lambda_2)\) as a F.W. bitopological spaces over itself with the identity as a projection. Also, if we consider the bitopological product \(B \times T\), for every bitopological space \(T\), can be regarded as a
F.W. bitopological space over B, by the first projection. The latter situation can be applied for every subspace of $B \times T$.

**Remarks 2.1.3.**

(a) In F.W. bitopology, we work over bitopological base space $(B, \Lambda_1, \Lambda_2)$. If $B$ is a point–space, the theory changes to that of ordinary bitopology.

(b) A F.W. bitopological spaces over $B$ is just a bitopological space $(M, \tau_1, \tau_2)$ with a continuous projection $p : (M, \tau_1, \tau_2) \rightarrow (B, \Lambda_1, \Lambda_2)$.

(c) The coarsest such bitopology is obtained by $p$, in which the $\tau_i$–open set of $(M, \tau_1, \tau_2)$ is exactly the inverse image of the $\Lambda_i$–open set of $(B, \Lambda_1, \Lambda_2)$; called, the F.W. indiscrete bitopology, where $i = 1, 2$.

(d) The F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$ is defined to be a F.W. set over $B$ with F.W. bitopology.

(e) We consider the bitopological product $B \times T$, for every bitopological space $T$, as a F.W. bitopological spaces over B by the first projection.

**Definition 2.1.4.** The F.W. function $\varphi : M \rightarrow N$ where $(M, \tau_1, \tau_2)$ and $(N, \sigma_1, \sigma_2)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$ are said to be:

(a) $i$–continuous if for every point $m \in M_b, b \in B$, the inverse image of every $\sigma_i$–open set of $\varphi(m)$ is $\tau_i$–open set contain $m$. $\varphi$ is called continuous if it is $i$–continuous for every $i = 1, 2$.

(b) $i$–open if for every point $m \in M_b, b \in B$, the image of every $\tau_i$–open set of $m$ is $\sigma_i$–open set of $\varphi(m)$. $\varphi$ is called open if it is $i$–open for every $i = 1, 2$.

(c) $i$–closed if for every point $m \in M_b, b \in B$, the image of every $\tau_i$–closed set of $m$ is $\sigma_i$–closed set of $\varphi(m)$. $\varphi$ is called closed if it is $i$–closed for every $i = 1, 2$. 

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Example 2.1.5. Let $M = \{1, 2, 3\}$, $\tau_1 = \{M, \emptyset, \{1\}, \{2, 3\}\}$, and $\tau_2 = \{M, \emptyset, \{3\}\}$. Let $N = \{4, 5, 6\}$, $\sigma_1 = \{N, \emptyset, \{5\}, \{4, 6\}\}$, and $\sigma_2 = \{N, \emptyset, \{4\}\}$.

Let $B = \{a, b, c\}$, $\Lambda_1 = \{B, \emptyset, \{a\}, \{b, c\}, \{b\}, \{a, c\}, \{c\}, \{a, b\}\}$, $\Lambda_2 = \{B, \emptyset, \{b\}, \{a\}, \{a, b\}\}$. Define $p_M : M \to B$ such that $p_M(1) = a, p_M(2) = c, p_M(3) = b$. Define $p_N : N \to B$ such that $p_N(4) = a, p_N(5) = b, p_N(6) = c$. Let $\varphi : M \to N$ such that $\varphi(3) = 4, \varphi(1) = 5, \varphi(2) = 6$. Then $\varphi$ is continuous, open, closed.

If $\varphi : M \to N$ is a F.W. function where $M$ is a F.W. set and $(N, \sigma_1, \sigma_2)$ is a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. We can give $M$ the induced bitopology, in the ordinary sense and this is necessarily a F.W. bitopology. We may refer to it, therefore, like the induced F.W. bitopology and note the next characterizations.

**Proposition 2.1.6.** Let $\varphi : M \to N$ be a F.W. function, where $(N, \sigma_1, \sigma_2)$ is a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$ and $M$ has an induced F.W. bitopology. Then for every F.W. bitopological space $(Q, \delta_1, \delta_2)$ a F.W. function $\psi : (Q, \delta_1, \delta_2) \to (M, \tau_1, \tau_2)$ is continuous iff the composition $\varphi \circ \psi : Q \to N$ is continuous.

**Proof.** ($\Rightarrow$) Suppose that $\psi$ is continuous. Let $q \in Q_b ; b \in B$ and let $V$ be $\sigma_i$-open set of $\psi(q) = n \in N_b$ in $N$. Since $\varphi$ is continuous, then $\varphi^{-1}(V)$ is $\tau_i$-open set containing $\psi(q) = m \in M_b$ in $M$. Since $\psi$ is continuous, then $\psi^{-1}(\varphi^{-1}(V)) = (\varphi \circ \psi)^{-1}(V)$ is a $\delta_i$-open set containing $q \in Q_b$ in $Q$ and $\psi^{-1}(\varphi^{-1}(V)) = (\varphi \circ \psi)^{-1}(V)$ is a $\delta_i$-open set containing $q \in Q_b$ in $Q$, where $i = 1, 2$.

($\Leftarrow$) Suppose that $\varphi \circ \psi$ is continuous. Let $q \in Q_b ; b \in B$ and $U$ be a $\tau_i$-open set of $\psi(q) = m \in M_b$ in $M$. Since $\varphi$ is open then, $\varphi(U)$ is a $\sigma_i$-open set containing $\varphi(m) = \varphi(\psi(q)) = n \in N_b$ in $N$. Since $\varphi \circ \psi$ is...
continuous, then \((\varphi \circ \psi)^{-1}(\varphi(U)) = \psi^{-1}(U)\) is a \(\delta_i\)-open set containing \(q \in Q_b\) in \(Q\), where \(i = 1,2\).

**Proposition 2.1.7.** Let \(\varphi: M \to N\) be a F.W. function where, \((N, \sigma_1, \sigma_2)\) a F.W. bitopological space over \((B, \Lambda_1, \Lambda_2)\) and \(M\) has an induced F.W. bitopology. Then for every F.W. bitopological space \((Q, \delta_1, \delta_2)\), the surjective F.W. function \(\psi: (Q, \delta_1, \delta_2) \to (M, \tau_1, \tau_2)\) is open iff the composition \(\varphi \circ \psi: (Q, \delta_1, \delta_2) \to (N, \sigma_1, \sigma_2)\) is open.

**Proof.** \((\Rightarrow)\) Suppose that \(\psi\) is open. Let \(q \in Q_b; b \in B\) and let \(U\) be a \(\delta_i\)-open set of \(q\) in \(Q\). Since \(\psi\) is open, then \(\psi(U)\) is \(\tau_i\) open set containing \(\psi(q) = m \in M_b\) in \(M\) where \(i = 1,2\). Since \(\varphi\) is open, then \(\varphi(\psi(U))\) is \(\sigma_i\)-open set containing \(\varphi(m) = n \in N_b\) in \(N\) and \(\varphi(\psi(U)) = \varphi \circ \psi(U)\).

\((\Leftarrow)\) Suppose that \(\varphi \circ \psi\) is open. Let \(q \in Q_b; b \in B\). Let \(U\) be a \(\delta_i\) – open set of \(q\) in \(Q\). Since \(\varphi \circ \psi\) is open, then \(\varphi \circ \psi(U)\) is \(\sigma_i\)-open set containing \(\varphi \circ \psi(q) = n \in N_b\). Since \(\varphi\) is continuous, then \(\varphi^{-1}(\varphi \circ \psi(U))\) is \(\tau_i\)-open set of \(\psi(q) = m \in M_b\) in \(M\). But \(\varphi^{-1}(\varphi \circ \psi(U)) = \psi(U)\), where \(i = 1,2\).

Let us consider general cases of Propositions (2.1.6) and (2.1.7) as follows:

**Corollary 2.1.8.**

In the case of families \(\{\varphi_r\}\) of F.W. functions, where \(\varphi_r: M \to N_r\) with \((N_r, \sigma_{r1}, \sigma_{r2})\) F.W. bitopological space over \(B\) for every \(r\). Specially, given a family \(\{(M_r, \tau_{r1}, \tau_{r2})\}\) of F.W. bitopological space over \(B\), the F.W. bitopological product \(\prod_B M_r\) is defined to be the F.W. product with the F.W. bitopology generated by the family of projections \(\pi_r: \prod_B M_r \to M_r\). Then for every F.W. bitopological space \((Q, \delta_1, \delta_2)\) over \(B\), a F.W. function \(\theta: Q \to \prod_B M_r\) is continuous (resp., open). For example when \(M_r = M\) for every index \(r\) we
see that the diagonal $\Delta: M \rightarrow \prod_B M$ is continuous (resp., open) iff the composition $\pi_r \circ \Delta = id_M$ is continuous (resp., open).

2.2. Fibrewise Closed and Fibrewise Open Bitopological Spaces

In this section we introduce the F.W. closed and F.W. open bitopological spaces over $B$. Several topological properties on these concepts are studied.

**Definition 2.2.1.** The F.W. bitopological space $(M, \tau_1, \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is called F.W. closed if the projection $p$ is closed.

**Example 2.2.2.** Let $M = \{1, 2, 3\}$, $\tau_1 = \{M, \emptyset, \{1\}, \{2, 3\}\}$, $\tau_2 = \{M, \emptyset, \{2\}, \{1, 2\}\}$, $B = \{a, b, c\}$, $\Lambda_1 = \{B, \emptyset, \{b\}, \{a, c\}\}$, $\Lambda_2 = \{B, \emptyset, \{c\}, \{b, c\}\}$. Let $p: M \rightarrow B$ such that $p(1) = b$, $p(2) = c$, $p(3) = a$. Then $p$ is closed and $(M, \tau_1, \tau_2)$ is F.W. closed space.

For another example is to consider trivial F.W. bitopological space with compact fibre is F.W. closed.

**Proposition 2.2.3.** Let $\varphi: M \rightarrow N$ be a closed F.W. function where $(M, \tau_1, \tau_2)$ and $(N, \sigma_1, \sigma_2)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then $M$ is a F.W. closed if $N$ is a F.W. closed.

**Proof.** Assume that $\varphi: M \rightarrow N$ is a closed F.W. function and $N$ is F.W. closed i.e. the projection $p_N: N \rightarrow B$ is closed. To prove that $M$ is F.W. closed i.e. $p_M: M \rightarrow B$ is closed. Now, let $m \in M_b; b \in B$, and $F$ be $\tau_i$ – closed set of $m$ where $i = 1, 2$. Since $\varphi$ is closed, then $\varphi(F)$ is $\sigma_i$ – closed set of $\varphi(m)$ = $n \in N_b$ in $N$. Since $p_N$ is closed, hence $p_N(\varphi(F))$ is $\Lambda_i$ – closed set in $B$. 

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But, \( p_N \circ \varphi(F) = p_M(F) \) is \( \sigma_i \)–closed set of \( F \). Thus, \( p_M \) is closed and \( M \) is a F.W. closed where \( i = 1, 2 \).

**Proposition 2.2.4.** If \((M, \tau_1, \tau_2)\) is a F.W. bitopological space over \((B, \Lambda_1, \Lambda_2)\). Assume that \( M_j \) is a F.W. closed for every member \( M_j \) of a finite covering of \( M \). Then \( M \) is a F.W. closed.

**Proof.** Assume that \( M \) is a F.W. bitopological space over \( B \), then the projection \( p_M : M \to B \) exist. To prove that \( p \) is closed. Since \( M_j \) is F.W. closed, then the projection \( p_{M_j} : M_j \to B \) is closed for every member \( M_j \) of a finite covering of \( M \). Let \( F \) be \( \tau_i \)–closed subset of \( M \). Then \( p(F) = \bigcup p_j(M_j \cap F) \) which is a finite union of closed sets and so \( p \) is closed. Thus \( M \) is F.W. closed where \( i = 1, 2 \).

**Proposition 2.2.5.** Let \((M, \tau_1, \tau_2)\) be a F.W. bitopological space over \((B, \Lambda_1, \Lambda_2)\). Then \((M, \tau_1, \tau_2)\) is a F.W. closed iff for every fibre \( M_b, \ b \in B \) of \( M \) and every \( \tau_i \)–open set \( U \) of \( M_b \) in \( M \), there is a \( \Lambda_i \)–open set \( O \) of \( b \) where \( M_O \subset U, \ i = 1, 2 \).

**Proof.** (\( \Rightarrow \)) Assume that \( M \) is closed. i.e., \( p : M \to B \) is closed. Now, let \( b \in B \) and \( U \) be \( \tau_i \)–open set of \( M_b \) where \( i = 1, 2 \). Thus we have \( M - U \) is \( \tau_i \)–closed set and \( p(M - U) \) is \( \Lambda_i \)–closed set. Let \( O = B - p(M - U) \) is \( \Lambda_i \)–open set of \( b \). Hence, \( M_O = p^{-1}(B - p(M - U)) \) is a subset of \( U \).

(\( \Leftarrow \)) Suppose that the other direction is hold, to show that \( M \) is closed. Let \( F \) be \( \tau_i \)–closed set in \( M \) where \( i = 1, 2 \). Let \( b \in B - p(F) \) and every \( \tau_i \)–open set \( U \) of \( M_b \) in \( M \). By assumption there is \( \Lambda_i \)–open set \( O \) of \( b \) such that \( M_O \subset U \). It’s easy to show that \( O \subset B - p(F) \). Hence, \( B - p(F) \) is \( \Lambda_i \)–open set in \( B \). Hence, \( p(F) \) is a \( \Lambda_i \)–closed in \( B \), \( p \) is closed, and \( M \) is F.W. closed bitopological, where \( i = 1, 2 \).
**Definition 2.2.6.** The F.W. bitopoligical space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is called F.W. open if the projection \(p\) is open.

**Example 2.2.7.** Let \(M = \{x, y, z\}, \tau_1 = \{M, \emptyset, \{x\}, \{y, z\}\}, \tau_2 = \{M, \emptyset, \{y\}, \{x, y\}\}\). Let \(B = \{a, b, c\}, \Lambda_1 = \{B, \emptyset, \{b\}, \{a, c\}\}, \Lambda_2 = \{B, \emptyset, \{c\}, \{b, c\}\}\). Let \(p : M \to B\) such that \(p(x) = b, p(y) = c, p(z) = a\). Then \(p\) is open and \((M, \tau_1, \tau_2)\) is F.W. open space.

For another example, trivial F.W. bitopological spaces are always F.W. open.

**Proposition 2.2.8.** Let \(\varphi : M \to N\) be an open F.W. function where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \(N\) is F.W. open, then \(M\) is F.W. open.

**Proof.** Since \(N\) is F.W. open, we have \(p_N : N \to B\) is open. To prove that \(p_M\) is open. i.e., \(p_M : M \to B\) is open. Let \(m \in M_b ; b \in B\), and let \(U\) be \(\tau_i\) – open set of \(m\) where \(i = 1, 2\), since \(\varphi\) is open then \(\varphi(U)\) is \(\sigma_i\) – open set of \(\varphi(m) = n \in N_b\) in \(N\). Also, since \(N\) is F.W. open then \(p_N(\varphi(U))\) is \(\Lambda_i\) – open set in \(B\). Since \(p_N \circ \varphi(U) = p_M(U)\), then \(p_M\) is open and \(M\) is F.W. open, where \(i = 1, 2\).

**Proposition 2.2.9.** Let \(\varphi : M \to N\) be a F.W. function where \((M, \tau_1, \tau_2)\) and 
\((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). Assume that the product: \(id_M \times \varphi : (M \times_B M, \tau_1 \times \tau_1 \times \tau_1) \to (M \times_B N, \tau_1 \times \sigma_1 \times \tau_2 \times \sigma_1)\) is open and \(M\) is F.W. open. Then \(\varphi\) itself is open.

**Proof.** Consider the following figure:
The projection on the left is surjective while the projection on the right is open because $M$ is F.W. open bitopological space. Thus, $\pi_2 \circ (\text{id}_M \times \varphi) = \varphi \circ \pi_2$ is open and thus, $\varphi$ is open.

Our next three results apply equally to F.W. closed and F.W. open bitopological spaces, respectively.

**Proposition 2.2.10.** Let $\varphi: M \to N$ be a surjection F.W. continuous where $(M, \tau_1, \tau_2)$ and $(N, \sigma_1, \sigma_2)$ are F.W. bitopological spaces over $(B, A_1, A_2)$. Then $N$ is F.W. closed (resp., open) if $M$ is F.W. closed (resp., open).

**Proof.** Suppose that $M$ is a F.W. closed (resp., open). Then $p_M: M \to B$ is closed (resp., open). To prove that $N$ is a F.W. closed (resp., open) bitopological space over $B$. i.e., the projection $p_N: (N, \sigma_1, \sigma_2) \to (B, A_1, A_2)$ is closed (resp., open). Suppose that $\in N_b$; $b \in B$. Let $V$ be $\sigma_i$ - closed (resp., open) set of $n$ where $i = 1, 2$. Since $\varphi$ is continuous, then $\varphi^{-1}(V)$ is $\tau_i$ -closed (resp., $\tau_i$-open) set of $\varphi^{-1}(n) = m \in M_b$ in $M$ where $i = 1, 2$. Since $p_M$ is closed (resp., open) then $p_M(\varphi^{-1}(V))$ is closed (resp., open) set in $B$. But, $p_M(\varphi^{-1}(V)) = p_N(V)$. Thus $p_N$ is closed (resp., open), and $N$ is F.W. closed (resp., open).

**Proposition 2.2.11.** If $(M, \tau_1, \tau_2)$ is a F.W. bitopological space over $(B, A_1, A_2)$. Assume that $M$ is F.W. closed (resp., open) over $B$. Then $M_B^*$ is a F.W. closed (resp., open) over $B^*$ for every subspace $B^*$ of $B$. 

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Proof. Assume that $M$ is F.W. closed (resp., open) so that the projection $p:M \to B$ is closed (resp., open). To prove that $M_{B^*}$ is closed (resp., open), i.e., the projection $p_{B^*}:M_{B^*} \to B^*$ is closed (resp., open). Let $m \in M \cap B^*$, $G$ be $\tau_{i-}$ closed (resp., $\tau_{i-}$-open) set of $m$, where $i = 1, 2$. $G \cap M_{B^*}$ is $\tau_{iB^*}$-closed (resp., $\tau_{iB^*}$-open) set of $M_{B^*}$. $p_{B^*}(G \cap M_{B^*}) = p(G \cap M_{B^*}) = p(G) \cap p(M_{B^*}) = p(G) \cap B^*$ which is $\Lambda_{iB^*}$-closed (resp., $\Lambda_{iB^*}$-open) set in $B^*$. $p_{B^*}$ is closed (resp., open). Thus, $M_{B^*}$ is F.W. closed (resp., open), where $i = 1, 2$.

**Proposition 2.2.12.** Let $(M, \tau_1, \tau_2)$ be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that $(M_{B_j}, \tau_{1B_j}, \tau_{2B_j})$ is a F.W. closed (resp., open) bitopological spaces over $(B_j, \Lambda_{1B_j}, \Lambda_{2B_j})$ for every member of a $\Lambda_{iB_j}$-open covering of $B$. Then $M$ is a F.W. closed (resp., open) bitopological space over $B$, where $i = 1, 2$.

Proof. Assume that $M$ is F.W. bitopological space over $B$ then, the projection $p:M \to B$ exist. To prove that $p$ is closed (resp., open). Since $M_{B_j}$ is closed (resp., open) over $B_j$ for every member $\Lambda_i$-open covered of $B$ where $i = 1, 2$, then the projection $p_{B_j}:M_{B_j} \to B_j$ is closed (resp., open). Now, let $F$ be $\tau_{i}$-closed (resp., $\tau_{i}$-open) set of $M_b : b \in B$, $p(F) = \bigcup p_{B_j}(F \cap M_{B_j})$ which is a finite union of $\Lambda_i$-closed (resp., open) sets of $B$. Thus, $p$ is closed (resp., open) and $M$ is closed (resp., open) F.W. bitopological space over $B$, where $i = 1, 2$.

Actually, the proceeding proposition is true in locally finite closed covering see Theorem (1.1.11) and Corollary (1.1.12) in [7].

There are several subclasses of the class of F.W. open bitopological spaces which induced many important examples and have interesting properties.
2.3. Fibrewise Locally Sliceable and Fibrewise Locally Sectionable Bitopological Spaces

In this section, we generalize F.W. locally sliceable and F.W. locally sectionable bitopological spaces over \((B, A_1, A_2)\). Some topological properties related to these concepts are studied.

**Definition 2.3.1.** The F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, A_1, A_2)\) is called locally sliceable if for every point \(m \in M_b\), \(b \in B\), there exist an \(A_i\)–open set \(W\) of \(b\) and a section \(s: W \rightarrow M\) such that \(s(b) = m\), for \(i = 1\) or \(2\).

**Example 2.3.2.** Let \(M = \{1, 2\}\), \(\tau_1 = \{M, \emptyset, \{1\}\}\), \(\tau_2 = \{M, \emptyset, \{2\}\}\). Let \(B = \{a, b\}\), \(A_1 = \{B, \emptyset, \{a\}\}\), \(A_2 = \{B, \emptyset, \{b\}\}\). Let \(p : M \rightarrow B\) where \(p(1) = a, p(2) = b\). We have \(M_a = \{1\}, M_b = \{2\}\). Let \(s_1 : \{a\} \rightarrow \{1\}\) where \(s_1(a) = 1\), \(s_2 : \{b\} \rightarrow \{2\}\) where \(s_2(b) = 2\). Then \(M\) is a F.W. locally sliceable bitopological space.

The condition leads to \(p\) is open for if \(U\) is a \(\tau_i\)–open set of \(m\) in \(M\), then \(s^{-1}(M \cap U) \subset p(U)\) is a \(\Lambda_i\)–open set of \(b\) in \(W\), and hence, in \(B\), where \(i = 1, 2\). The class of locally sliceable bitopological space is finitely multiplicative.

**Proposition 2.3.3.** Let \(\{ (M_r, \tau_{r1}, \tau_{r2}) \}_{r=1}^k\) be a finite family of locally sliceable bitopological space over \((B, A_1, A_2)\). The F.W. bitopological product \(M = \prod_B M_r\) is locally sliceable.

**Proof.** Let \(m = (m_r)\) be a point of \(M_b\), \(b \in B\), so that \(m_r = \pi_r(m)\) for every index \(r\). Since \(M_r\) is a locally sliceable bitopological space, there is a \(\Lambda_i\)–open set \(W_r\) of \(b\) and a section \(s_r: W_r \rightarrow M_r | W_r\) where \(s_r(b) = m_r\).
Then the intersection \( W = W_1 \cap \ldots \cap W_n \) is a \( \Lambda_i \) -open set of \( b \) and a section \( s: W \to M_W \) is given by \( (\pi_r \circ s)(w) = s_r(w) \) for every index \( r \) and every point \( w \in W \), where \( i = 1, 2 \).

**Proposition 2.3.4.** Let \( \varphi: M \to N \) be a continuous F.W. surjection, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \( M \) is locally sliceable, then \( N \) is so.

**Proof.** Let \( n \in N_b : b \in B \). Then \( n = \varphi(m) \), for some \( m \in M_b \). If \( M \) is locally sliceable then, there is a \( \Lambda_i \) -open set \( W \) of \( b \) and a section \( s: W \to M_W \) where \( s(b) = m \). Then \( \varphi s: W \to N_W \) is a section such that \( s(b) = n \), where \( i = 1, 2 \), as required.

**Definition 2.3.5.** The F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is called F.W. discrete if the projection \( p \) is a local homeomorphism.

**Example 2.3.6.** Let \( M = \{1, 2\}, \tau_1 = \{M, \emptyset, \{1\}\}, \tau_2 = \{M, \emptyset, \{2\}\} \). Let \( B = \{a, b\}, \Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\} \). Let \( p : M \to B \) where: \( p(1) = a, p(2) = b \). We have \( M_a = \{1\}, M_b = \{2\} \). Let \( s_1 : \{a\} \to \{1\} \) such that \( s_1(a) = 1 \), \( s_2 : \{b\} \to \{2\} \) such that \( s_2(b) = 2 \). Then, \( p \) is local homeomorphism, and thus \( M \) is F.W. discrete.

**Remark 2.3.7.** It is not difficult to show examples of different F.W. discrete bitopologies on the same F.W. set which are equivalent, as F.W. bitopologies. For this reason, we must be careful not to say the F.W. discrete bitopology.
This means, we recall, that for every point \( b \) of \( B \) and every point \( m \) of \( M_b \) there is a \( \tau_i - \) open set \( V \) of \( m \) in \( M \) and a \( \Lambda_i - \) open set \( W \) of \( b \) in \( B \) where \( p \) maps \( V \) homeomorphically onto \( W \). In that case we say that \( W \) is evenly covered by \( V \), where \( i = 1, 2 \). It is clear that F.W. discrete bitopological spaces are locally sliceable therefor is F.W. open.

The class of F.W. discrete bitopological spaces are finitely multiplicative.

**Proposition 2.3.8.** Let \( \{(M_r, \tau_{r1}, \tau_{r2})\}_{r=1}^k \) be a finite family of F.W. discrete bitopological spaces over \( (B, \Lambda_1, \Lambda_2) \). Then the F.W. bitopological product \( (M = \prod B M_r, \tau_1, \tau_2) \) is F.W. discrete.

**Proof.** Given a point \( m \in M_b; \ b \in B \), then there is for every index \( r \) a \( \tau_i - \) open set \( U_r \) of \( \pi_r(m) \) in \( M_r \), where the projection \( p_r = p \circ \pi_r^{-1} \) maps \( U_r \) homeomorphically onto the \( \Lambda_i - \) open \( p_r(U_r) = W_r \) of \( b \). Then, the \( \tau_i - \) open \( \prod B U_r \) of \( m \) is mapped homeomorphically onto the intersection \( W = \cap W_r \) which is a \( \Lambda_i - \) open of \( b \), where \( i = 1, 2 \).

An attractive characterization of F.W. discrete bitopological spaces are given by the following proposition.

**Proposition 2.3.9.** If \( (M, \tau_1, \tau_2) \) is F.W. bitopological space over \( (B, \Lambda_1, \Lambda_2) \). Then, \( M \) is F.W. discrete iff:

(a) \( M \) is F.W. open

(b) The diagonal embedding \( \Delta: M \rightarrow M \times_B M \) is open

**Proof.** (\( \Leftarrow \)) Suppose that (a) and (b) are satisfied. Let \( m \in M_b; \ b \in B \), then \( \Delta(m) = (m, m) \) admits a \( \tau_i \times \tau_i - \) open set in \( M \times_B M \) which is entirely contained in \( \Delta(M) \). Without real lacking in general, we may suppose the
\[ \tau_i \times \tau_i \text{-open set is of the form } U \times_B U, \text{ where } U \text{ is a } \tau_i \text{-open set of } m \text{ in } M. \text{ Then } p\vert_U \text{ is a homeomorphism. Therefore, } M \text{ is F.W. discrete where } i = 1, 2. \]

(\Rightarrow) Assume that \( M \) is F.W. discrete. We have already seen that \( M \) is a F.W. open. To prove that \( \Delta \) is open, it is sufficient to show that \( \Delta(M) \) is \( \tau_i \times \tau_i \text{-open in } M \times_B M. \) So, let \( m \in M_b ; b \in B, \) and let \( U \) be a \( \tau_i \text{-open set of } m \text{ in } M \), where \( W = p(U) \) is a \( \Lambda_i \text{-open set of } b \text{ in } B \) and \( p \) maps \( U \) homeomorphically onto \( W \). Then, \( U \times_B U \) is contained in \( \Delta(M) \) since if not, then there exist distinct \( \xi, \xi^* \in M_W \), where \( w \in W \) and \( \xi, \xi^* \in U \), which is absurd.

Open subset of F.W. discrete bitopological spaces are also F.W. discrete. Actually, we have the following results.

**Proposition 2.3.10.** Assume that \( \varphi : M \to N \) is a continuous F.W. injection, where \((M, \tau_1, \tau_2) \) and \((N, \sigma_1, \sigma_2) \) are F.W. open bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \( N \) is F.W. discrete then \( M \) is so.

**Proof.** Consider the diagram shown below.

\[
\begin{array}{ccc}
M & \xrightarrow{\Delta} & M \times_B M \\
\varphi \downarrow & & \varphi \times \varphi \\
N & \xrightarrow{\Delta} & N \times_B N
\end{array}
\]

Figure 2.3.1. Diagram of Proposition 2.3.10.

Since \( \varphi \) is continuous so is \( \varphi \times \varphi \). Now \( \Delta(N) \) is \( \sigma_i \times \sigma_i \text{-open in } N \times_B N \), by Proposition (2.3.8.). Since \( N \) is a F.W. discrete, then \( \Delta(M) = \Delta\left( (\varphi^{-1}(N)) \right) = (\varphi \times \varphi)^{-1}(\Delta(N)) \) is a \( \tau_i \times \tau_i \text{-open in } M \times_B M. \) Thus, the conclusion follows from Proposition (2.3.9.) where \( i = 1, 2. \)
Proposition 2.3.11. Assume that \( \varphi: M \to N \) be an open F.W. surjection, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. open bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \( M \) is a F.W. discrete, then \( N \) is so.

**Proof.** From figure 2.3.1, with the assumption on \( \varphi \), if \( M \) is a F.W. discrete then \( \Delta(M) \) is an \( \tau_i \times \tau_i \) -open in \( M \times_B M \), by Proposition (2.3.9.). Hence \( \Delta(N) = \Delta((\varphi(M))) = (\varphi \times \varphi)(\Delta(M)) \) is an \( \sigma_i \times \sigma_i \) -open in \( N \times_B N \). Thus the conclusion follows again from Proposition (2.3.9.), where \( i = 1, 2 \).

Proposition 2.3.12. If \( \varphi, \psi: M \to N \) is a continuous F.W. functions, where \((M, \tau_1, \tau_2)\) is a F.W. bitopological and \((N, \sigma_1, \sigma_2)\) is a F.W. discrete bitopological space over \((B, \Lambda_1, \Lambda_2)\). Then the coincidence set \( K(\varphi, \psi) \) of \( \varphi \) and \( \psi \) is open in \( M \).

**Proof.** The coincidence set is precisely \( \Delta^{-1}(\varphi \times \psi)^{-1}(\Delta(N)) \), where:

![Figure 2.3.2. Diagram of Proposition 2.3.12.](image)

Hence the required result follows at once from Proposition (2.3.9.). In particular, take \( = N, \varphi = id_M \), and \( \psi = sop \) where \( s \) is a section. We conclude that \( s \) is an open embedding when \( M \) is a F.W. discrete.

Proposition 2.3.13. If \( \varphi: M \to N \) is a continuous F.W. functions, where \((M, \tau_1, \tau_2)\) is a F.W. open and \((N, \sigma_1, \sigma_2)\) is a F.W. discrete bitopological space over \((B, \Lambda_1, \Lambda_2)\). Then, the F.W. graph \( \Gamma: M \to M \times_B N \) of \( \varphi \) is an open embedding.
**Proof.** The F.W. graph is defined in the same way as the ordinary graph, but with values in the F.W. bitopological product. Therefore, the diagram shown below is commutative.

![Diagram](image)

Since $\Delta(N)$ is an $\sigma_i \times \sigma_i$-open in $N \times_B N$, by Proposition (2.3.9.), $\Gamma(M) = (\varphi \times id_N)^{-1}(\Delta(N))$ is an $\tau_i \times \sigma_i$-open in $M \times_B N$, where $i = 1, 2$, as asserted.

**Remark 2.3.14.** If $(M, \tau_1, \tau_2)$ is a F.W. discrete bitopological space over $(B, \Lambda_1, \Lambda_2)$ then for every point $m \in M_B : b \in B$, there is a $\Lambda_i$-open set $W$ of $b$ and a unique section $s: W \to M_W$ exist satisfying $s(b) = m$. We may refer to $s$ as the section through $m$.

**Definition 2.3.15.** The F.W. bitopological space $(M, \tau_1, \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is called locally sectionable if every point $b \in B$, admits an $\Lambda_i$-open set $W$ and a section $s: W \to M_W$, where $i = 1$ or 2.

**Example 2.3.16.** Let $M = \{1, 2\}$, $\tau_1 = \{M, \emptyset, \{1\}\}$, $\tau_2 = \{M, \emptyset, \{2 \} \}$. Let $B = \{a, b\}$, $\Lambda_1 = \{B, \emptyset, \{a\}\}$, $\Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p : M \to B$ where $p(1) = a$, $p(2) = b$. We have $M_a = \{1\}$, $M_b = \{2\}$. Let $s_1 : \{a\} \to \{1\}$ where $s_1(a) = 1$, $s_2 : \{b\} \to \{2\}$ where $s_2(b) = 2$. Thus $(M, \tau_1, \tau_2)$ is locally sectionable.
Remark 2.3.17. The F.W. non-empty locally sliceable bitopological spaces are locally sectionable, but the converse is false. In fact, locally sectionable bitopological spaces are not necessarily F.W. open. For example, take $M = (-1,1] \subset \mathbb{R}$ with $(M, \tau_1, \tau_2) : \tau_1 = \tau_2$, the natural projection onto $B = \mathbb{R} \mid \mathbb{Z}$; $(B, \Lambda_1, \Lambda_2) : \Lambda_1 = \Lambda_2$.

The class of locally sectionable bitopological spaces is finitely multiplicative as we show next.

Proposition 2.3.18. If $\{(M_r, \tau_{r_1}, \tau_{r_2})\}$ is a finite family of locally sectionable bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then the F.W. bitopological product $M = \prod_B M_r$ is locally sectionable.

Proof. Given a point $b$ of $B$, there exist an $\Lambda_i$-open set $W_r$ of $b$ and a section $s_r: W_r \to M_r \mid W_r$ for every index $r$. Since there are finite number of indices, the intersection $W$ of the $\Lambda_i$-open sets $W_r$ is also a $\Lambda_i$-open set of $b$, and a section $s: W \to (\prod_B M_r)_W$ is given by $\pi_r \circ s(w) = s_r(w)$, for $w \in W$, where $i = 1, 2$.

Our last two results apply equally well to every of the above three propositions.

Proposition 2.3.19. If $(M, \tau_1, \tau_2)$ is a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Suppose that $(M, \tau_1, \tau_2)$ is locally sliceable, F.W. discrete or locally sectionable over $(B, \Lambda_1, \Lambda_2)$. Then so is $M_B^*$ over $B^*$ for every $\Lambda_i$-open set $B^*$ of $B$, where $i = 1, 2$.

Proposition 2.3.20. Let $(M, \tau_1, \tau_2)$ be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that $M_{B_i}$ is a locally sliceable F.W. discrete or locally
sectionable over $B_j$ for every member $B_j$ of an $\Lambda_i$ – open covering of $B$. So is $M$ over $B$, such that, $i = 1, 2$. 
Chapter 3

Fibrewise Pairwise Separation Axioms
Chapter 3

Fibrewise Pairwise Separation Axioms

In this chapter, we define fibrewise bitopological space in the more important concept in topology which is the separation axioms. In Section one, we define and study the concepts of fibrewise pairwise $T_0$ spaces, fibrewise pairwise $T_1$ spaces, fibrewise pairwise $R_0$ spaces, fibrewise pairwise Hausdorff spaces, fibrewise pairwise functionally Hausdorff spaces. Some basic properties of these spaces are investigated. In Section two, we introduce the concepts of fibrewise pairwise regular spaces, fibrewise pairwise completely regular spaces, fibrewise pairwise normal spaces and fibrewise pairwise functionally normal spaces. Also, we give several results concerning them. Some of results in this chapter stated for the case of fibrewise topological space (see [1], [23]).

3.1. Fibrewise Pairwise $T_0$, Pairwise $T_1$, Pairwise $R_0$ and Pairwise Hausdorff Spaces.

The concepts of open sets have an important role in F.W. separation axioms. By using these concepts, we can construct many F.W. separation axioms. Now, we introduce the versions of F.W. pairwise $T_0$, F.W. pairwise $T_1$, F.W. pairwise $R_0$, and F.W. pairwise Hausdorff spaces as follows.

**Definition 3.1.1.** Let $(M, \tau_1, \tau_2)$ be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then $M$ is called a F.W. pairwise $T_0$ if whenever $x, y \in M_b$; $b \in B$ and $x \neq y$, either there exists a $\tau_i$-open set $U$ of $x$ which does not contains $y$ in $M$ or $\tau_j$-open set $V$ of $y$ which does not contains $x$ in $M$, where $i, j = 1, 2$ , $i \neq j$.
Example 3.1.2. Let $M = \{1, 2, 3\}$, $\tau_1 = \{M, \emptyset, \{1\}, \{3\}, \{1, 3\}\}$, $\tau_2 = \{M, \emptyset, \{2\}, \{1\}, \{2, 3\}\}$. Let $B = \{a, b\}$, $A_1 = \{B, \emptyset, \{a\}\}$, $A_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \to B$ where $p(1) = a$, $p(2) = b$, $p(3) = a$. Then $M$ is a F.W. pairwise space.

Remark 3.1.3.

(a) $(M, \tau_1, \tau_2)$ is a F.W. pairwise $T_0$ space iff each fiber $M_b$ is a pairwise $T_0$ space.

**Proof:** Let $x, y \in M_b, b \in B$ be a subset of F.W. pairwise $T_0$ spaces such that $x \neq y$, so $x, y \in M$. Since $M$ is $T_0$, there exist $\tau_i$-open set $U$ contain $x$ and $y \notin U$ or $\tau_j$-open set $V$ contain $y$ and $x \notin V$. Hence $U \cap M_b \in \tau_{ib}$ and $V \cap M_b \in \tau_{jb}$ and $(U \cap M_b) \cap (V \cap M_b) = (U \cap V) \cap M_b = \emptyset \cap M_b = \emptyset$. So $M_b$ is pairwise $T_0$ space.

(b) Subspaces of F.W. pairwise $T_0$ spaces are F.W. pairwise $T_0$ spaces.

**Proof:** Let $N$ be a subset of F.W. pairwise $T_0$ spaces. Let $x, y \in N_b, b \in B$ be a finite family of F.W. topological spaces such that $x \neq y$, then $x, y \in M_b, b \in B$ and since $M$ is $T_0$, then either there exist $\tau_i$-open set $U$ contain $x$, $y \notin U$ or $\tau_j$-open set $V$ contain $y$ and $x \notin V$. Since $U \cap N \in \tau_{ib}$, $V \cap N \in \tau_{ib}$ and $x \in U \cap N, y \notin U \cap N$ or $y \in V \cap N, x \notin V \cap N$, there for $N$ is F.W. pairwise $T_0$ spaces.

(c) The F.W. bitopological products of F.W. pairwise $T_0$ spaces with the family of F.W. pairwise projections are F.W. pairwise $T_0$ spaces.

**Proof:** Let $\{(M_r, \tau_{1r}, \tau_{2r})\}$ be a finite family of F.W. topological spaces, let $x, y \in M_b, b \in B$ be a finite family of F.W. topological spaces such that $x \neq y$, then $\pi_r(x) = x_r$ and $\pi_r(y) = y_r$ for some index $r$. Since $M_r$ is F.W. pairwise $T_0$ for all $r$, then either there exist $\tau_{ir}$-open set $U_r$ contain $x_r$, $y_r \notin U_r$ or $\tau_{jr}$-open set $V_r$ contain $y_r$ and $x_r \notin V_r$. Since $\pi_r$ is continuous, then the inverse images of $U_r$ and $V_r$ are open in $M$ and $x \in U, y \notin U$ or $y \in V, x \notin V$. Hence $M$ is F.W. pairwise $T_0$ space.
In a similar way, we can introduce the definition of F.W. pairwise $T_1$ space. Let $(M, \tau_1, \tau_2)$ be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then, $M$ is called a F.W. pairwise $T_1$ if whenever $x, y \in M_b; b \in B$ and $x \neq y$, there exist a $\tau_i$-open sets $U$, and a $\tau_j$-open set $V$ in $M$ such that $x \in U, y \notin U$ and $x \notin V, y \in V, i, j = 1, 2, i \neq j$. But it turns out that there is no real use for this in what we are going to do. In its place, we formulate some use of a new axiom. The axiom is that “every $\tau_i$-open set contains the $\tau_j$-closure of each of its points”, and use the word pairwise $R_0$ space. This is correct for pairwise $T_1$ spaces and for pairwise regular spaces. Thinking of it like a weak structure of pairwise regularity. For example, indiscrete spaces are pairwise $R_0$ spaces. The F.W. version of the pairwise $R_0$ axiom is defined as the following.

**Definition 3.1.4.** A F.W. bitopological space $(M, \tau_1, \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is called F.W. pairwise $R_0$ space if for every $x \in M_b; b \in B$, and every $\tau_i$-open set $V$ of $x$ in $M$, there exists a $\Lambda_i$-nbhd $W$ of $b$ in $B$ such that $V$ contains the $\tau_j$- closure of $\{x\}$ in $M_W$ (i.e., $M_W \cap \tau_j \cdot Cl\{x\} \subset V$) where $i, j = 1, 2, i \neq j$.

**Example 3.1.5.** Let $M = \{1, 2, 3\}, \tau_1 = \{M, \emptyset, \{1\}, \{3\}, \{1, 3\}\}, \tau_2 = \{M, \emptyset, \{2\}, \{2, 1\}, \{2, 3\}\}$. Let $B = \{a, b\}, \Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \rightarrow B$ where $p(1) = a, p(2) = b, p(3) = a$. Then, $M$ is F.W. pairwise $R_0$.

For another example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is a F.W. pairwise $R_0$ space for all pairwise $R_0$ spaces $T$.

**Remark 3.1.6.**

(a) The nbds of $x$ are given by a F.W. basis it is enough if the condition in Definition (3.1.4.) is satisfied for every F.W. basic nbds.
(b) If \((M, \tau_1, \tau_2)\) is a F.W. pairwise \(R_0\) space over \((B, \Lambda_1, \Lambda_2)\), then for each subspace \((B^*, \Lambda_1^*, \Lambda_2^*)\) of \((B, \Lambda_1, \Lambda_2)\), \((M_B^*, \tau_1^*, \tau_2^*)\) is a F.W. pairwise \(R_0\) space over \(B^*\).

**Proposition 3.1.7.** Let \(\varphi : M \to M^*\) be F.W. embedding function, where \((M, \tau_1, \tau_2)\) and \((M^*, \tau_1^*, \tau_2^*)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \(M^*\) is F.W. pairwise \(R_0\) then so is \(M\).

**Proof.** Let \(V\) be a \(\tau_i\)-open set of \(x\) in \(M\) where \(x \in M_b; b \in B\). Then \(V = \varphi^{-1}(V^*)\), where \(V^*\) is a \(\tau_i^*\)-open set of \(x^* = \varphi(x)\) in \(M^*\). Because \(M^*\) is a F.W. pairwise \(R_0\) then we have a nbd \(W\) of \(b\) in \(B\), where \(M^*_W \cap \tau_i^*\text{-Cl}\{x^*\} \subset V^*\). Hence, \(M^*_W \cap \tau_j\text{-Cl}\{x\} \subset \varphi^{-1}(M^*_W \cap \tau_j^*\text{-Cl}\{x^*\}) \subset \varphi^{-1}(V^*) = V\), and hence \(M\) is a F.W. pairwise \(R_0\) where \(i, j = 1, 2, \ldots, i \neq j\).

The class of F.W. pairwise \(R_0\) spaces is finitely multiplicative as we show in the following.

**Proposition 3.1.8.** If \(\{(M_r, \tau_{1r}, \tau_{2r})\}\) is a finite family of F.W. pairwise \(R_0\) spaces over \(B\). Then the F.W. bitopological product \(M = \prod_B M_r\) is a F.W. pairwise \(R_0\).

**Proof.** Let \(x \in M_b; b \in B\). Consider a \(\tau_i\)-open set \(V = \prod_B V_r\) of \(x\) in \(M\), where \(V_r\) is a \(\tau_{ir}\)-open set of \(\pi_r(x) = x_r\) in \(M_r\) for each index \(r\). Since \(M_r\) is a F.W. pairwise \(R_0\), then we have a nbd \(W_r\) of \(b\) in \(B\) where \((M_r \mid W_r) \cap \tau_{jr}\text{-Cl}\{x_r\} \subset V_r\). Then, we regard \(W\) as a nbd of \(b\) where \(W\) is an intersection of \(W_r\) and \(M_W \cap \tau_j\text{-Cl}\{x\} \subset V\) and hence \(M = \prod_B M_r\) is F.W. pairwise \(R_0\) where \(i, j = 1, 2, \ldots, i \neq j\).

Similar conclusion holds for infinite F.W. products provided all that of the factors is F.W. nonempty.
**Proposition 3.1.9.** Let \( \varphi: M \to N \) is closed, continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \(B\). If \(M\) is F.W. pairwise \(R_0\) then so is \(N\).

**Proof.** Assume that \(V\) is an \(\sigma_i\)-open set of \(y\) in \(N\), where \(y \in N_b; \ b \in B\), choose \(x \in \varphi^{-1}(y)\). Then \(U = \varphi^{-1}(V)\) is a \(\tau_i\)-open set of \(x\) in \(M\). Since \(M\) is F.W. pairwise \(R_0\), then we have a nbd \(W\) of \(b\) in \(B\), where \(M_W \cap \tau_j - cl\{x\} \subset U\). Therefore \(N_W \cap \varphi(\tau_j - cl\{x\}) \subset \varphi(U) = V\). Because \(\varphi\) is closed, \(\varphi(\tau_j - cl\{x\}) = \sigma_j - cl(\varphi\{x\})\). Hence, \(N_W \cap \sigma_j - cl(\varphi\{x\}) \subset V\) and \(N\) is F.W. pairwise \(R_0\) where \(i, j = 1, 2, \ i \neq j\).

Now we introduce the concept of F.W. pairwise Hausdorff spaces.

**Definition 3.1.10.** A F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is called F.W. pairwise Hausdorff if whenever \(x, y \in M_b; \ b \in B\) and \(x \neq y\), there exist a disjoint pair of \(\tau_i\)-open set \(U\) of \(x\) and \(\tau_j\)-open set \(V\) of \(y\) in \(M\), where \(i, j = 1, 2, \ i \neq j\).

**Example 3.1.11.** Let \(M = \{1, 2, 3\}, \tau_1 = \{M, \emptyset, \{1\}, \{3\}, \{1, 3\}\}, \tau_2 = \{M, \emptyset, \{2\}, \{2, 1\}, \{2, 3\}\}\). Let \(B = \{a, b\}, \Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}\). Let \(p: M \to B\) where \(p(1) = a, p(2) = b, p(3) = a\). Then \(M\) is a F.W. pairwise \(T_2\).

Another example, \((B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)\) is F.W. pairwise Hausdorff space for any pairwise Hausdorff spaces \(T\).

**Remark 3.1.12.** If \((M, \tau_1, \tau_2)\) is F.W. pairwise Hausdorff space over \((B, \Lambda_1, \Lambda_2)\) then \(M^*_B\) is F.W. pairwise Hausdorff over \(B^*\) for every subspace \(B^*\) of \(B\). Especially, the fibers of \((M, \tau_1, \tau_2)\) are pairwise Hausdorff spaces.
On the other hand, a F.W. bitopological space with pairwise Hausdorff fibres is not necessarily pairwise Hausdorff.

Example 3.1.13. Let, \( \tau_1 = \{ M, \emptyset, \{ 1 \}, \{ 1, 2 \} \} \), \( \tau_2 = \{ M, \emptyset, \{ 1 \}, \{ 1, 3 \} \} \), where \( M = \{ 1, 2, 3 \} \). Let \( B = \{ a, b \} \), \( \Lambda_1 = \{ B, \emptyset, \{ a \} \} \), \( \Lambda_2 = \{ B, \emptyset \} \). Let \( p: M \to B \) where \( p(1) = a, p(2) = b = p(3) \). Then, we have \( M_b = \{ 2, 3 \} \), \( \tau_{1M_b} = \{ M_b, \emptyset, \{ 2 \} \} \), \( \tau_{2M_b} = \{ M_b, \emptyset, \{ 3 \} \} \). Then, there exist \( \tau_{1M_b} \)-open set \( U = \{ 2 \} \) where \( 2 \in U \), and there exist \( \tau_{2M_b} \) open set \( V = \{ 3 \} \) where \( 3 \in V \) where \( U \cap V = \emptyset \). But \( M \) is not pairwise Hausdorff since \( 2 \) and \( 3 \in M \) and \( 2 \neq 3 \), and there is no disjoint pair of open sets of \( 2 \) and \( 3 \).

Proposition 3.1.14. The F.W. bitopological space \(( M, \tau_1, \tau_2 \) \) over \(( B, \Lambda_1, \Lambda_2 \) \) is F.W. pairwise Hausdorff iff the diagonal embedding \( \Delta: M \to M \times_B M \) is \( \tau_i \times_B \tau_i \) -closed.

Proof. \(( \Rightarrow )\) Let \( x, y \in M_b \), \( b \in B \) and \( x \neq y \). Since \( \Delta(M) \) is \( \tau_i \times_B \tau_i \)-closed in \( M \times_B M \), then \( (x, y) \) a point of the complement admits a F.W. product \( \tau_i \times_B \tau_j \)-open set \( U \times_B V \) which does not meet \( \Delta(M) \). Then \( U, V \) are disjoint pair of \( x, y \) where \( U \) is \( \tau_i \)-open set of \( x \), and \( V \) is \( \tau_j \)-open set of \( y \) such that \( i, j = 1, 2 \), \( i \neq j \). 

\(( \Leftarrow )\) Let \( (x, y) \in M \times_B M - \Delta(M) \), so \( (x, y) \notin \Delta(M) \), and \( x \neq y \) since \( M \) is F.W. pairwise \( T_2 \) space then there exist disjoint pair \( \tau_i \)-open set \( U \) of \( x \) and \( \tau_j \)-open set \( V \) of \( y \), so \( U \times_B V \) is \( \tau_i \times_B \tau_j \)-open set in \( M \times_B M \). Hence \( M \times M - \Delta(M) \) is \( \tau_i \times_B \tau_i \) is open and \( \Delta(M) \) is \( \tau_i \times_B \tau_i \) closed.

Subspaces of F.W. pairwise Hausdorff spaces are F.W. pairwise Hausdorff spaces. Actually, we have the following proposition.
**Proposition 3.1.15.** Assume that \( \varphi : M \rightarrow M^* \) is embedding F.W. function, where \((M, \tau_1, \tau_2)\) and \((M^*, \tau^*_1, \tau^*_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \(M^*\) is F.W. pairwise Hausdorff then so is \(M\).

**Proof.** Let \(x, y \in M_B\); \(b \in B\) and \(x \neq y\). Then \(\varphi(x), \varphi(y) \in M^*_B\) are distinct. Since \(M^*\) is a F.W. pairwise Hausdorff, then we have a \(\tau^*_i\)-open sets \(U^*\) of \(\varphi(x)\) and a \(\tau^*_j\)-open set \(V^*\) of \(\varphi(y)\) in \(M^*\) which are disjoint. Because \(\varphi\) is continuous, the inverse images \(\varphi^{-1}(U^*) = U\) and \(\varphi^{-1}(V^*) = V\) such that \(U\) is a \(\tau_i\)-open set of \(x\) and \(V\) is a \(\tau_j\)-open set of \(y\) in \(M\) such that \(V\) and \(U\) are disjoint. Hence, \(M\) is a F.W. pairwise Hausdorff where \(i, j = 1, 2, i \neq j\).

**Proposition 3.1.16.** Let \(\varphi : M \rightarrow N\) be a continuous F.W. function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \(N\) is F.W. pairwise Hausdorff, then the F.W. graph \(\Gamma : M \rightarrow M \times_B N\) of \(\varphi\) is a \(\tau_i \times_B \sigma_j\)-closed embedding.

**Proof.** The F.W. graph is defined in a similar way to the ordinary graph, but with values in the F.W. product. Hence, the figure shown below is commutative.

![Diagram of Proposition 3.1.16.](image)

Since \(\Delta(N)\) is a \(\sigma_i \times_B \sigma_i\)-closed in \(N \times_B N\), by Proposition (3.1.14.), then \(\Gamma(M) = (\varphi \times id_N)^{-1}(\Delta(N))\) is a \(\tau_i \times_B \sigma_j\)-closed in \(M \times_B N\), as asserted, where \(i, j = 1, 2, i \neq j\).
The category of F.W. pairwise Hausdorff spaces is multiplicative, in the following sense.

**Proposition 3.1.17.** Assume that \([\{M_r, \tau_{1r}, \tau_{2r}\}\] is a family of F.W. pairwise Hausdorff spaces over \((B, A_1, A_2)\). The F.W. bitopological product \(M = \prod_B M_r\) is a F.W. pairwise Hausdorff.

**Proof.** Let \(x, y \in M_b; b \in B\) and \(x \neq y\). Then \(\pi_r(x) = x_r \neq \pi_r(y) = y_r\) for some index \(r\). Because \(M_r\) is F.W. pairwise Hausdorff, then we have a \(\tau_{ir}\)-open set \(U_r\) of \(x_r\), and a \(\tau_{jr}\)-open set \(V_r\) of \(y_r\) in \(M_r\) where \(U_r\) and \(V_r\) are disjoint. Because \(\pi_r\) is continuous, the inverse images \(U\) and \(V\) are disjoint \(\tau_i\)-open and \(\tau_j\)-open sets, respectively, of \(x, y\) in \(M\), where \(i, j = 1, 2, i \neq j\).

The pairwise functionally version of the F.W. pairwise Hausdorff axiom is stronger than the non-pairwise functional version but their properties are similar. From now on, we denote by \(I\) the closed unit interval \([0, I]\) in the real line \(\mathbb{R}\).

**Definition 3.1.18.** A F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, A_1, A_2)\) is F.W. pairwise functionally Hausdorff if for every \(x, y \in M_b; b \in B\) and \(x \neq y\), there exists a nbhd \(W\) of \(b\) in \(B\) and disjoint pair \(\tau_i\)-open sets \(U\) of \(x\) and \(\tau_j\)-open set \(V\) of \(y\) in \(M\) and a continuous function \(\lambda: M_W \to I\) such that \(M_b \cap U \subset \lambda^{-1}(0)\) and \(M_b \cap V \subset \lambda^{-1}(1)\) where \(i, j = 1, 2, i \neq j\).

**Example 3.1.19.** Let \(M = \{2, 4, 6\}, \tau_1 = \{M, \emptyset, \{2\}, \{6\}, \{2, 6\}\}, \) let \(\tau_2 = \{M, \emptyset, \{4\}, \{2, 4\}, \{4, 6\}\}. \) Let \(B = \{a, b\}, A_1 = \{B, \emptyset, \{a\}\}, A_2 = \{B, \emptyset, \{b\}\}. \) Let \(p: M \to B\) where \(p(2) = a = p(6), p(4) = b.\) Hence, \(M\) is F.W. pairwise Hausdorff and \(M_a = \{2, 6\}\) and \(2 \neq 6,\) so \(W = \{a\}. \) Let, \(\lambda: M_W \to I\) where \(\lambda(2) = 0, \lambda(6) = 1, \tau_{1M_W} = \{M_W, \emptyset, \{2\}, \{6\}\}, \tau_{2M_W} = \{M_W, \emptyset, \{2\}, \{6\}\}.\)
Thus $\lambda$ is continuous and $M_b \cap U \subset \lambda^{-1}(0)$ and $M_b \cap V \subset \lambda^{-1}(1)$. Therefor, $M$ is F.W. pairwise functionally Hausdorff.

For another example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise functionally Hausdorff space for each pairwise functionally Hausdorff spaces $T$.

**Remark 3.1.20.** If $(M, \tau_1, \tau_2)$ is F.W. pairwise functionally Hausdorff space over $(B, \Lambda_1, \Lambda_2)$ then $M_B^*$ is F.W. pairwise functionally Hausdorff over $B^*$ for every subspace $B^*$ of $B$. In particular, the fibers of $M$ are pairwise functionally Hausdorff spaces.

Subspaces of F.W. pairwise functionally Hausdorff spaces are F.W. pairwise functionally Hausdorff spaces. Actually, we have the following result.

**Proposition 3.1.21.** Assume that $\varphi: M \to M^*$ is a embedding F.W. function, where $(M, \tau_1, \tau_2)$ and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If $M^*$ is F.W. pairwise functionally Hausdorff then so is $M$.

**Proof.** Let $x, y \in M_b$ and $x \neq y$; $b \in B$. Then $\varphi(x) = x^*$, $\varphi(y) = y^* \in M_b^*$, $x^* \neq y^*$. Since $M^*$ is F.W. pairwise functionally Hausdorff, then we have a nbd $W$ of $b$ in $B$ and disjoint pair of $\tau_1^*$-open set $U^*$ of $x^*$ and $\tau_2^*$-open set $V^*$ of $y^*$ and a continuous function $\lambda^*: M^* \mid W \to I$ such that $M_b^* \cap U^* \subset (\lambda^*)^{-1}(0)$ and $M_b^* \cap V^* \subset (\lambda^*)^{-1}(1)$. Now, since $\varphi$ is continuous, then $\varphi^{-1}(U^*) = U$ and $\varphi^{-1}(V^*) = V$ are disjoint pair of $\tau_i$-open set of $x$ and $\tau_j$-open set of $y$, respectively and the continuous function $\lambda$ where $\lambda = \lambda^* \circ \varphi: M_W \to I$ such that $M_b \cap U \subset \lambda^{-1}(0)$ and $M_b \cap V \subset \lambda^{-1}(1)$, where $i, j = 1, 2, i \neq j$. 

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Furthermore the category of F.W. pairwise functionally Hausdorff spaces is multiplicative, as in the following proposition.

**Proposition 3.1.22.** Assume that \(\{(M_r, \tau_{1r}, \tau_{2r})\}\) is a family of F.W. pairwise functionally Hausdorff spaces over \((B, A_1, A_2)\). The F.W. bitopological product \(M = \prod_B M_r\) is F.W. pairwise functionally Hausdorff.

**Proof.** Let \(x, y \in M_b; b \in B\), and \(x \neq y\). Then, \(\pi_r(x) = x_r, \pi_r(y) = y_r \in (M_r)_b\) for some index \(r\) where \(x_r \neq y_r\). Since \(M_r\) is F.W. pairwise functionally Hausdorff, then we have a nbhd \(W_r\) of \(b\) in \(B\) and disjoint pair of \(\tau_{ir}\)-open set \(U_r\) of \(x_r\), and \(\tau_{jr}\)-open set \(V_r\) of \(y_r\) and a continuous function \(\lambda: M_r \mid W_r \to I\) such that \((M_r)_b \cap U_r \subset \lambda^{-1}(0)\) and \((M_r)_b \cap V_r \subset \lambda^{-1}(1)\). Now, the intersection of \(W_r\) is a nbhd \(W\) of \(b\) in \(B\), and since \(\pi_r\) is continuous, then \(\pi_r^{-1}(U_r) = U\) and \(\pi_r^{-1}(V_r) = V\) are disjoint pair of \(\tau_i\)-open set of \(x\) and \(\tau_j\)-open set of \(y\), respectively, and the continuous function \(\Omega\) where \(\Omega = \lambda \circ \pi_r: M_b \to I\) where \(M_b \cap U \subset \Omega^{-1}(0)\) and \(M_b \cap V \subset \Omega^{-1}(1)\) where \(i, j = 1, 2, i \neq j\).

### 3.2. Fibrewise Pairwise Regular and Pairwise Normal Spaces

In this section we consider the F.W. Concept advanced pairwise separation axioms. Namely, F.W. pairwise regularity and F.W. pairwise completely regularity.

**Definition 3.2.1.** The F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, A_1, A_2)\) is called F.W. pairwise regular if for every \(x \in M_b; b \in B\), and for every \(\tau_i\)-open set \(V\) of \(x\) in \(M\), there exists a nbhd \(W\) of \(b\) in \(B\), and a \(\tau_i\)-open set \(U\) of \(x\) in \(M_W\) such that \(V\) is containing the \(\tau_j\)-closure of \(U\) in \(M_W\) (i.e., \(M_W \cap \tau_j - cl(U) \subset V\)), where \(i, j = 1, 2, i \neq j\).
Example 3.2.2. Let $M = \{1, 2, 3\}, \tau_1 = \{M, \emptyset, \{3\}\}, \tau_2 = \{M, \emptyset, \{1, 2\}\}$. Let $B = \{a, b\}, \Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p : M \to B$ such that $p(1) = b = p(2), p(3) = a$. Then $M$ is F.W. pairwise regular.

We can consider another example, trivial F.W. spaces with pairwise regular fibre are F.W. pairwise regular.

Remark 3.2.3.

(a) The nbds of $x$ are given by a F.W. basis it is enough if the condition in Definition (3.2.1) is satisfied for every F.W. basic nbds.

(b) If $(M, \tau_1, \tau_2)$ is F.W. pairwise regular space over $(B, \Lambda_1, \Lambda_2)$ then $(M^*_B, \tau_1^*, \tau_2^*)$ is F.W. pairwise regular space over $(B^*, \Lambda_1^*, \Lambda_2^*)$ for every subspace $B^*$ of $B$.

Subspaces of F.W. pairwise regular spaces are F.W. pairwise regular spaces. Actually we have the following proposition.

Proposition 3.2.4. Assume that $\varphi : M \to M^*$ is embedding F.W. function, where $(M, \tau_1, \tau_2)$ and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If $M^*$ is F.W. pairwise regular then so is $M$.

Proof. Let $V$ be a $\tau_i$-open set of $x$ in $M$ where $x \in M_b; b \in B$. Then $V = \varphi^{-1}(V^*)$, where $V^*$ is a $\tau_i^*$-open set of $x^* = \varphi(x)$ in $M^*_b$. Because $M^*$ is F.W. pairwise regular, then we have a nbd $W$ of $b$ in $B$ and a $\tau_i^*$-open set $U^*$ of $x^*$ in $M_{W^*}^*$ where $M_{W}^* \cap \tau_j^* - cl(U^*) \subset V^*$. Then $U = \varphi^{-1}(U^*)$ is a $\tau_i$-open set of $x$ in $M_W$ such that $M_{W} \cap \tau_j - cl(U) \subset V$. Hence, $M$ is F.W. pairwise regular, where $i, j = 1, 2, i \neq j$ as required.
The class of F.W. pairwise regular spaces is F.W. multiplicative as in the following proposition.

**Proposition 3.2.5.** Assume that \(\{(M_r, \tau_{1r}, \tau_{2r})\}\) is a finite family of F.W. pairwise regular spaces over \(B\). The F.W. bitopological product \(M = \prod_B M_r\) is F.W. pairwise regular.

**Proof.** Consider a \(\tau_i\)-open set \(V = \prod_B V_r\) of \(x\) in \(M\), where \(x \in M_b; b \in B\) and \(V_r\) is a \(\tau_{ir}\)-open set of \(\pi_r(x) = x_r\) in \(M_r\) for each index \(r\). Since \(M_r\) is F.W. pairwise regular we have a nbd \(W_r\) of \(b\) in \(B\), and a \(\tau_{ir}\)-open set \(U_r\) of \(x_r\) in \(M_r \mid W_r\) such that the \(\tau_{jr}\)-closure of \(U_r\) in \(M_r \mid W_r\) is contained in \(V_r\). (i.e. \((M_r \mid W_r) \cap \tau_{jr} - cl(U_r) \subset V_r\)). Then we regard \(W\) as a nbd of \(b\) in \(B\), where \(W\) is the intersection of \(W_r\), and \(U = \prod_B U_r\) is a \(\tau_i\)-open set of \(x\) in \(M_W\) where the \(\tau_j\)-closure of \(U\) in \(M_W\) is contained in \(V\). (i.e. \(M_W \cap \tau_j - cl(U) \subset V\)). Hence, \(M = \prod_B M_r\) is F.W. pairwise regular, where \(j = 1, 2, i \neq j\).

Similar conclusion holds for infinite F.W. products provided that every of the factors is F.W. non-empty.

**Proposition 3.2.6.** Assume that \(\varphi : M \to N\) is a closed, open and continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \(B\). Then \(M\) is F.W. pairwise regular iff \(N\) is F.W. pairwise regular.

**Proof.** \((\Rightarrow)\) Let \(V\) be a \(\sigma_i\)-open set of \(y\) in \(N\) where \(y \in N_b; b \in B\), choose \(x \in \varphi^{-1}(y)\). Then \(U = \varphi^{-1}(V)\) is a \(\tau_i\)-open set of \(x\) in \(M\). Because \(M\) is F.W. pairwise regular , we have a nbd \(W\) of \(b\) in \(B\), and a \(\tau_i\)-open set \(U^*\) of \(x\) such that \(M_W \cap \tau_j - cl(U^*) \subset U\). Then \(N_W \cap \varphi(\tau_j - cl(U^*)) \subset V\). Because \(\varphi\) is closed, \(\varphi(\tau_j - cl(U^*)) = \sigma_j - cl(\varphi(U^*))\), and because \(\varphi\) is open, then
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$\varphi(U^*)$ is a $\sigma_i$-open set of $y$. Hence, $N$ is F.W. pairwise regular, where $i, j = 1, 2, i \neq j$, as asserted.

$(\Leftarrow)$ By a similar way of the first direction.

The pairwise functionally version of the F.W. pairwise regularity axiom is stronger than the non-pairwise functionally version. However, their properties are similar. In the ordinary theory, the word completely regular is used instead of functionally regular. We widen this usage to the F.W. theory.

**Definition 3.2.7.** A F.W. bitopological space $(M, \tau_1, \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is called F.W. pairwise completely regular if for every $x \in M_b; b \in B$, and for every $\tau_i$-open set $V$ of $x$ there exists a nbd $W$ of $b$ in $B$ and a $\tau_j$-open set $U$ of $x$ in $M_W$ and a continuous function $\lambda: (M_W, \tau_{1W}, \tau_{2W}) \to I$ such that $M_b \cap U \subset \lambda^{-1}(0)$ and $M_W \cap (M_W - V) \subset \lambda^{-1}(1)$, where $i, j = 1, 2, i \neq j$.

**Example 3.2.8.** Let $M = \{1, 2, 3, 4\}, \tau_1 = \{M, \emptyset, \{1, 3\}\}, \tau_2 = \{M, \emptyset, \{2, 4\}\}$. Let $B = \{a, b\}, \Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \to B$ such that $p(1) = a = p(3), p(2) = b = p(4), M_a = \{1, 3\}$, let $x = 1, V = \{1, 3\}, W = \{a\}, M_W = \{1, 3\}, \tau_{1M_W} = \{M_W, \emptyset\} = \tau_{2M_W}$, let $U = M_W$. Let $\lambda: M_W \to I$ such that $\lambda(1) = 0 = \lambda(3)$. $\lambda$ is continuous and $M_b \cap U \subset \lambda^{-1}(0), M_W \cap (M_W - V) \subset \lambda^{-1}(1)$ Similar if $x = 3$. $M_b = \{2, 4\}$, let $x = 2, V = \{2, 4\}, W = \{b\}, M_W = \{2, 4\}, \tau_{1M_W} = \{M_W, \emptyset\} = \tau_{2M_W}$, let $U = M_W$. Let $\lambda: M_W \to I$ such that $\lambda(2) = 0 = \lambda(4)$. $\lambda$ is continuous and $M_b \cap U \subset \lambda^{-1}(0), M_W \cap (M_W - V) \subset \lambda^{-1}(1)$. Similarly if $x = 4$.

For another example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise completely regular space for every pairwise completely regular spaces $T$. 
Remark 3.2.9.

(a) The nbds of \( x \) are given by a F.W. basis it is enough if the condition in Definition (3.2.7.) is satisfied for every F.W. basic nbds.

(b) If \((M, \tau_1, \tau_2)\) is F.W. pairwise completely regular space over \((B, \Lambda_1, \Lambda_2)\) then \((M_B^*, \tau_1^*, \tau_2^*)\) is F.W. pairwise completely regular space over \((B^*, \Lambda_1^*, \Lambda_2^*)\) for every subspace \(B^*\) of \(B\).

Subspaces of F.W. pairwise completely regular spaces are F.W. pairwise completely regular spaces. In fact, we have the following result.

**Proposition 3.2.10.** Assume that \( \varphi: M \to M^* \) is embedding F.W. function, where \((M, \tau_1, \tau_2)\) and \((M^*, \tau_1^*, \tau_2^*)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \(M^*\) is F.W. pairwise completely regular then so is \(M\).

**Proof.** Let \( V \) be a \( \tau_i^* \)-open set of \( x \) in \( M \) where \( x \in M_b; b \in B \), then \( \varphi(x) = x^* \in M_b^* \) and \( V = \varphi^{-1}(V^*) \) is a \( \tau_i^* \)-open set of \( x^* \). Because \( M^* \) is F.W. pairwise completely regular, then we have a nbd \( W \) of \( b \) in \( B \) and \( \tau_j^* \)-open set \( U^* \) of \( x^* \) and a continuous function \( \lambda: M^*_{W} \to I \) such that \( M_b^* \cap U^* \subset \lambda^{-1}(0) \) and \( M_W^* \cap (M_W^* - V^*) \subset \lambda^{-1}(1) \). Now, because \( \varphi \) is continuous, then \( \varphi^{-1}(U^*) = U \) is \( \tau_i^* \)-open set of \( x \) in \( M_W \) and the continuous function \( \Omega = \lambda \circ \varphi \) such that \( \Omega: M_W \to I \) and \( M_b \cap U \subset \Omega^{-1}(0) \) and \( M_W \cap (M_W - V) \subset \Omega^{-1}(1) \) where \( i, j = 1, 2, i \neq j \).

The class of F.W. pairwise completely regular spaces is finitely multiplicative, as we show next.

**Proposition 3.2.11.** Assume that \( \{(M_r, \tau_{1r}, \tau_{2r})\} \) is a finite family of F.W. pairwise completely regular spaces over \((B, \Lambda_1, \Lambda_2)\). The F.W. bitopological product \( M = \prod_B M_r \) is F.W. pairwise completely regular.
Proof. Let \( x \in M_b \); \( b \in B \). Consider a F.W. \( \tau_i \)-open set \( \prod_B V_r \) of \( x \) in \( M \), where \( V_r \) is a \( \tau_{ir} \)-open set of \( \pi_r(x) = x_r \) in \( M_r \) for all index \( r \). Because \( M_r \) is F.W. pairwise completely regular, we have a nbd \( W_r \) of \( b \) in \( B \), and a \( \tau_{jr} \)-open set \( U \) of \( x_r \) in \( M_r \) and a continuous function \( \lambda_r : (M_r)_W \rightarrow I \) where \((M_r)_b \cap U \subset \lambda_r^{-1}(0) \) and \((M_r)_W \cap ((M_r)_W - V_r) \subset \lambda_r^{-1}(1) \). Then we regard \( W \) as a nbd of \( b \) in \( B \) where \( W \) is the intersection of \( W_r \) and \( \lambda : M_W \rightarrow I \) is a continuous function where

\[
\lambda(\xi) = \inf_{r=1,2,\ldots,n} \{\lambda_r \xi_r\} \text{ for } \xi = (\xi_r) \in M_W.
\]

Since \((M_r)_b \cap \pi_r^{-1}(U) \subset \pi_r^{-1}[(M_r)_b \cap U] \subset \pi_r^{-1}(\lambda_r^{-1}(0))(\lambda_r \circ \pi_r)^{-1}(0)\) and \((M_r)_W \cap \pi_r^{-1}((M_r)_W - V_r) \subset \pi_r^{-1}[(M_r)_W \cap ((M_r)_W - V_r)] \subset \pi_r^{-1}(\lambda_r^{-1}(1)) = (\lambda_r \circ \pi_r)^{-1}(1) \) where \( i, j = 1, 2, i \neq j \).

A similar conclusion holds for infinite F.W. products if all of the factors is F.W. non-empty.

Lemma 3.2.12. Assume that \( \varphi : M \rightarrow N \) is a closed, open F.W. surjection function, where \( M \) and \( N \) are F.W. bitopological spaces over \( B \). Let \( \alpha : M \rightarrow \mathbb{R} \) be a continuous real-valued function which is F.W. bounded above, in the sense that \( \alpha \) is bounded above on each fibre of \( M \). Then, \( \beta : N \rightarrow \mathbb{R} \) is continuous, where:

\[
\beta(\eta) = \sup_{\xi \in \varphi^{-1}(\eta)} \alpha(\xi)
\]

Proposition 3.2.13. Assume that \( \varphi : M \rightarrow N \) is a closed, open and continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopo-
logical spaces over \((B, \Lambda_1, \Lambda_2)\). If \(M\) is F.W. pairwise completely regular then so is \(N\).

**Proof.** Let \(V_y\) be a \(\sigma_i\)-open set of \(y\) in \(N\) where \(y \in N_b; b \in B\). Choose \(x \in \varphi^{-1}(y)\) such that \(V_x = \varphi^{-1}(V_y)\) is a \(\tau_i\)-open set of \(x\). Because \(M\) is F.W. pairwise completely regular, we have a nbd \(W\) of \(b\) in \(B\), and a \(\tau_j\)-open set \(U_x\) of \(x\) in \(M_W\) and a continuous function \(\lambda : M_W \to I\) such that \(M_b \cap U_x \subset \lambda^{-1}(0)\) and \(M_W \cap (M_W - V_y) \subset \lambda^{-1}(1)\). Using Lemma (3.2.12.), we get a continuous function \(\Omega : N_W \to I\) such that \(N_b \cap U_y \subset \Omega^{-1}(0)\) and \(N_W \cap (N_W - V_y) \subset \Omega^{-1}(1)\), where \(i, j = 1, 2, i \neq j\).

Next, we define the version of F.W. pairwise normal space.

**Definition 3.2.14.** A F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is called F.W. pairwise normal if for every \(b \in B\) and every disjoint pair of \(\tau_i\)-closed set \(H\), and \(\tau_j\)-closed set \(K\) of \(M\), there exists a nbd \(W\) of \(b\) in \(B\) and a disjoint pair of \(\tau_j\)-open set \(U\), and \(\tau_i\)-open set \(V\) of \(M_W \cap H, M_W \cap K\) in \(M_W\), where \(i, j = 1, 2, i \neq j\).

**Example 3.2.15.** Let \(M = \{1, 2\}, \quad \tau_1 = \{M, \varphi, \{1\}\}, \tau_2 = \{M, \varphi, \{2\}\}\). \(H=\{1\}, K = \{2\}\). Let \(B = \{a, b\}, \quad \Lambda_1 = \{B, \varphi, \{a\}\}, \Lambda_2 = \{B, \varphi, \{b\}\}\). Let \(p : M \to B\) where \(p(1) = a, p(2) = b\). We have \(M_a = \{1\}, M_b = \{2\}\), where the nbd of \(a\) is \(\{a\}\), and the nbd of \(b\) is \(\{b\}\), \(M_a \cap H = \{1\}, M_a \cap K = \varphi\), \(M_b \cap H = \varphi, M_b \cap K = \{2\}\). Let \(V = \{2\}, U = \{1\}\). So, \(M\) is F.W. pairwise normal.

**Remark 3.2.16.** If \((M, \tau_1, \tau_2)\) is a F.W. pairwise normal space over \((B, \Lambda_1, \Lambda_2)\), then for each subspace \(B^*\) of \(B\) and \((M^*_B, \tau^*_1, \tau^*_2)\) is F.W. pairwise normal space over \((B^*, \Lambda^*_1, \Lambda^*_2)\).
Closed subspaces of F.W. pairwise normal spaces are F.W. pairwise normal. Actually, we have.

**Proposition 3.2.17.** Assume that \(\varphi: M \rightarrow M^*\) is a closed, embedding F.W. function where \((M, \tau_1, \tau_2)\) and \((M^*, \tau_1^*, \tau_2^*)\) are F.W. bitopological spaces over \(B\). If \((M^*, \tau_1^*, \tau_2^*)\) is F.W. pairwise normal then so is \((M, \tau_1, \tau_2)\).

**Proof.** Let \(H\) and \(K\) be disjoint pair of \(\tau_i\)-closed and \(\tau_j\)-closed sets of \(M\) and let \(b \in B\). Then \(\varphi(H)\) and \(\varphi(K)\) are disjoint pair of \(\tau_i^*\)-closed set and \(\tau_j^*\)-closed set of \(M^*\). Since \(M^*\) is F.W. pairwise normal then, we have a nbd \(W\) of \(b\) in \(B\) and a \(\tau_j^*\)-open set \(U^*\) and \(\tau_i^*\)-open set \(V^*\) of \(M^*_W \cap \varphi(H), M^*_W \cap \varphi(K)\), in \(M^*_W\). Since \(\varphi\) is continuous, then \(\varphi^{-1}(U^*) = U\) and \(\varphi^{-1}(V^*) = V\) are disjoint pair of \(\tau_j\)-open and \(\tau_i\)-open sets of \(M_W \cap H, M_W \cap K\) in \(M_W\), where \(i, j = 1, 2, i \neq j\).

**Proposition 3.2.18.** Let \(\varphi: M \rightarrow N\) be a closed continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). Then \((M, \tau_1, \tau_2)\) is F.W. pairwise normal iff \((N, \sigma_1, \sigma_2)\) is F.W. pairwise normal.

**Proof.** \((\Rightarrow)\) Let \(H\) and \(K\) be disjoint pair of \(\sigma_i\)-closed and \(\sigma_j\)-closed sets of \(N\) and let \(b \in B\). Then, \(\varphi^{-1}(H)\) and \(\varphi^{-1}(K)\) are disjoint pair of \(\tau_i\)-closed and \(\tau_j\)-closed sets of \(M\). Because \(M\) is F.W. pairwise normal, then we have a nbd \(W\) of \(b\) in \(B\) and a disjoint pair of \(\tau_j\)-open set and \(\tau_i\)-open set \(U, V\) of \(M_W \cap \varphi^{-1}(H)\) and \(M_W \cap \varphi^{-1}(K)\). Since \(\varphi\) is closed then, the sets \(N_W - \varphi(M_W - U)\) and \(N_W - \varphi(M_W - V)\) are open in \(N_W\), and structure a disjoint pair of \(\sigma_j\)-open, \(\sigma_i\)-open sets of \(N_W \cap H, N_W \cap K\) in \(N_W\), as required, where \(i, j = 1, 2, i \neq j\).

\((\Leftarrow)\) By similar way of first direction.
Lastly, we define the version of F.W. pairwise functionally normal space.

**Definition 3.2.19.** A F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is called F.W. pairwise functionally normal if for every \(b \in B\) and every disjoint pair of \(\tau_i\)-closed set \(H\), and \(\tau_j\)-closed set \(K\) of \(M\), there exists a nbd \(W\) of \(b\) in \(B\) and a disjoint pair of \(\tau_j\)-open set \(U\), and \(\tau_i\)-open set \(V\) and a continuous function \(\lambda : M_W \to I\) such that \(M_W \cap H \cap U \subset \lambda^{-1}(0)\) and \(M_W \cap K \cap V \subset \lambda^{-1}(1)\) in \(M_W\), where \(i, j = 1, 2, i \neq j\).

**Example 3.2.20.** Let \(M = \{1, 2, 3, 4\}\), \(\tau_1 = \{M, \emptyset, \{1, 2\}\}\), \(\tau_2 = \{M, \emptyset, \{3, 4\}\}\). Let \(B = \{a, b\}\), \(\Lambda_1 = \{B, \emptyset, \{a\}\}\), \(\Lambda_2 = \{B, \emptyset, \{b\}\}\). Let \(p : M \to B\) such that \(p(1) = a = p(2), p(3) = b = p(4)\). Let \(H = \{3, 4\}, K = \{1, 2\}\). Let \(b = a\), nbd of \(a\) is \(W = \{a\}\), \(M_W = \{1, 2\}\), \(\tau_{1M_W} = \{M_W, \emptyset\} = \tau_{2M_W}\) let \(U = \{3, 4\}, V = \{1, 2\}\). Let \(\lambda : M_W \to I\) such that \(\lambda(1) = 1 = \lambda(2)\), \(\lambda\) is continuous and \(M_W \cap H \cap U = \emptyset \subset \lambda^{-1}(0), M_W \cap K \cap V = \{1, 2\} \subset \lambda^{-1}(1)\). Let \(b = b\), nbd of \(b\) is \(W = \{b\}\), \(M_W = \{3, 4\}\) \(\tau_{1M_W} = \{M_W, \emptyset\} = \tau_{2M_W}\). Let \(\lambda : M_W \to I\) such that \(\lambda(3) = 0 = \lambda(4)\), \(\lambda\) is continuous and \(M_W \cap H \cap U = \{3, 4\} \subset \lambda^{-1}(0), M_W \cap K \cap V = \subset \lambda^{-1}(1)\). So \(M\) is F.W. pairwise functionally normal.

For another example, \((B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)\) is F.W. pairwise functionally normal space when \(T\) is pairwise functionally normal space.

**Remark 3.2.21.** If \((M, \tau_1, \tau_2)\) is F.W. pairwise functionally normal space over \((B, \Lambda_1, \Lambda_2)\) then for every subspace \(B^*\) of \(B\) we have \((M_B^*, \tau_1^*, \tau_2^*)\) is F.W. pairwise functionally normal space over \((B^*, \Lambda_1^*, \Lambda_2^*)\).
Closed subspaces of F.W. pairwise functionally normal spaces are F.W. pairwise functionally normal. Actually we have.

**Proposition 3.2.22.** Assume that \( \varphi: M \to M^* \) is a closed, embedding F.W. function where \((M, \tau_1, \tau_2)\) and \((M^*, \tau_1^*, \tau_2^*)\) are F.W. bitopological spaces over \(B\). If \(M^*\) is F.W. pairwise functionally normal then so is \(M\).

**Proof.** Let \(H\) and \(K\) be disjoint pair of \(\tau_i\)-closed and \(\tau_j\)-closed sets of \(M\) and let \(b \in B\). Then \(\varphi(H), \varphi(K)\) are disjoint pair of \(\tau_i^*\)-closed set and \(\tau_j^*\)-closed set of \(M^*\). Since \(M^*\) is F.W. pairwise functionally normal, we have a nbd \(W\) of \(b\) in \(B\) and a disjoint pair of \(\tau_j^*\)-open set \(U\) and \(\tau_i^*\)-open set \(V\) and a continuous function \(\lambda: M^*_W \to I\) such that \(M^*_W \cap \varphi(H) \cap U \subset \lambda^{-1}(0)\) and \(M^*_W \cap \varphi(K) \cap V \subset \lambda^{-1}(1)\) in \(M^*_W\). Since \(\varphi\) is continuous, then \(\varphi^{-1}(U), \varphi^{-1}(V)\) are \(\tau_j\)-open set, \(\tau_i\)-open set and the function, \(\Omega = \lambda \circ \varphi\) is a continuous, \(\Omega: M_W \to I\) such that \(M_W \cap H \cap \varphi^{-1}(U) \subset \Omega^{-1}(0)\) and \(M_W \cap K \cap \varphi^{-1}(V) \subset \Omega^{-1}(1)\) in \(M_W\) as required where \(i, j = 1, 2, \ i \neq j\).

**Proposition 3.2.23.** Assume that \( \varphi: M \to N \) is a closed, open and continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \((M, \tau_1, \tau_2)\) is F.W. pairwise functionally normal then so is \((N, \sigma_1, \sigma_2)\).

**Proof.** Let \(H, K\) be disjoint pair of \(\sigma_i\)-closed and \(\sigma_j\)-closed sets of \(N\) and let \(b \in B\). Then \(\varphi^{-1}(H), \varphi^{-1}(K)\) are disjoint pair of \(\tau_i\)-closed and \(\tau_j\)-closed sets of \(M\). Because \(M\) is F.W. pairwise functionally normal, then we have a nbd \(W\) of \(b\) in \(B\) and a disjoint pair of \(\tau_j\)-open set and \(\tau_i\)-open set \(U, V\) and a continuous function \(\lambda: M_W \to I\) such that \(M_W \cap \varphi^{-1}(H) \cap U \subset \lambda^{-1}(0)\) and \(M_W \cap \varphi^{-1}(K) \cap V \subset \lambda^{-1}(1)\) in \(M_W\). Hence, a function \(\Omega: N_W \to I\) is given by \(\Omega(y) = \sup_{x \in \varphi^{-1}(y)} \lambda(x); y \in N_W\). Because \(\varphi\) is open and closed, in addi-
tion to continuous, it leads to that $\Omega$ is continuous. Hence, $N_W \cap H \cap \varphi(U) \subset \Omega^{-1}(0)$ and $N_W \cap K \cap \varphi(V) \subset \Omega^{-1}(1)$ in $M_W$ where $i, j = 1, 2, i \neq j$. 
Chapter 4

Fibrewise IJ-Perfect Bitopological Spaces
In many times, it has been mixed between topological spaces and some of the basic concepts to get a new topological structure. In this chapter, we shall give a new definition for fibrewise bitopological space in the light of compactness to get a space which has big importance characteristics in topology. Filter concept is considered as one of the rich concepts in topology for having a notable role in the modern directions for topology.

4.1. Fibrewise IJ-Perfect Bitopological Spaces

**Definition 4.1.1.** Let \((B, \Lambda_1, \Lambda_2)\) be a bitopological space. A F.W. ij-bitopology on a F.W. set \(M\) over \(B\) means any bitopology on \(M\) for which the projection \(p\) is ij-continuous, where \(i, j = 1, 2\).

**Definition 4.1.2.** A function \(f: (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2)\) is called ij-closed if the image of each ij-closed set in \(M\) is ij-closed set in \(N\), where \(i, j = 1, 2\).

**Theorem 4.1.3.** A function \(f: (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2)\) is ij-closed iff \(ij - \text{cl}(f(A)) \subseteq f(ij - \text{cl}(A))\) for each \(A \subseteq M\), where \(i, j = 1, 2\).

**Proof:** (\(\Rightarrow\)) Suppose that \(f\) is ij-closed. Let \(A \subseteq M\), since \(f\) is ij-closed then \(f(ij - \text{cl}(A))\) is ij-closed set in \(N\), since \(ij - \text{cl}(A)\) is closed set in \(M\). so, \(ij - \text{cl}(f(A)) \subseteq f(ij - \text{cl}(A))\).

(\(\Leftarrow\)) Suppose that \(A\) is ij-closed set in \(M\), so \(A = ij - \text{cl}(A)\), but we have \(ij - \text{cl}(f(A)) \subseteq f(ij - \text{cl}(A))\), thus \(ij - \text{cl}(f(A)) \subseteq f(A)\). Thus, \(f(A)\) is ij-closed in \(M\). Therefore \(f\) is ij-closed.
Definition 4.1.4. A filter base $\mathcal{F}$ on bitopological space $(M, \tau_1, \tau_2)$ is said to be $ij$-directed toward a set $A \subseteq M$, written as $\mathcal{F} \xrightarrow{ij-d} A$, iff every filter base $\mathcal{G}$ finer than $\mathcal{F}$ has an $ij$-adherent point in $A$, i.e. $(ij-\text{ad} \mathcal{G}) \cap A \neq \emptyset$. We write $\mathcal{F} \xrightarrow{ij-d} x$ to mean $\mathcal{F} \xrightarrow{ij-d} \{x\}$, where $x \in M$, where $i,j = 1, 2$.

Now, we introduce a characterization of $ij$-adherent point $x$ of a filter base $\mathcal{F}$.

Theorem 4.1.5. A point $x$ in bitopological space $(M, \tau_1, \tau_2)$ is an $ij$-adherent point of a filter base $\mathcal{F}$ on $M$ iff there exists a filter base $\mathcal{F}^*$ finer than $\mathcal{F}$ such that $\mathcal{F}^* \xrightarrow{ij-con} x$, where $i,j = 1, 2$.

Proof: ($\Rightarrow$) Let $x$ be an $ij$-adherent point of a filter base $\mathcal{F}$ on $M$, so it is an $ij$-contact point of every number of $\mathcal{F}$. This yields, for every $\tau_i$-open nbd $U$ of $x$, we have $\tau_j - \text{cl}(U) \cap F \neq \emptyset$ for every number $F$ in $\mathcal{F}$. Consequently, $\tau_j - \text{cl}(U)$ contains a some member of any filter base $\mathcal{F}^*$ finer than $\mathcal{F}$, such that $\mathcal{F}^* \xrightarrow{ij-con} x$.

($\Leftarrow$) Suppose that $x$ is not an $ij$-adherent point of a filter base $\mathcal{F}$ on $M$, then there exists $F \in \mathcal{F}$ such that $x$ is not an $ij$-contact of $F$. Hence, there exists an $\tau_i$-open nbd $U$ of $x$ such that $\tau_j - \text{cl}(U) \cap F = \emptyset$. Denote by $\mathcal{F}^*$ the family of sets $\mathcal{F}^* = F \cap (M - \tau_j-\text{cl}(U))$ for $F \in \mathcal{F}$, then the sets $\mathcal{F}^*$ are nonempty. Also $\mathcal{F}^*$ is a filter base and indeed it is finer than $\mathcal{F}$. This is, given $F_1^* = F_1 \cap (M \setminus \tau_j-\text{cl}(U))$ and $F_2^* = F_2 \cap (M \setminus \tau_j-\text{cl}(U))$, there is an $F_3 \subseteq F_1 \cap F_2$ and this gives $F_3^* = F_3 \cap (M \setminus \tau_j-\text{cl}(U)) \subseteq F_1 \cap F_2 \cap (M \setminus \tau_j-\text{cl}(U)) = F_1 \cap (M \setminus \tau_j-\text{cl}(U)) \cap F_2$. By construction $\mathcal{F}^*$ is not $ij$-convergent to $x$. This is a contradiction, and thus, $x$ is an $ij$-adherent point of a filter base $\mathcal{F}$ on $M$. 

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Theorem 4.1.6. Let $F$ be a filter base on bitopological space $(M, \tau_1, \tau_2)$. Let $x \in M$, then $F \xrightarrow{ij\text{-}con} x$ iff $F \xrightarrow{ij\text{-}d} x$, where $i, j = 1, 2$.

Proof: ($\Leftarrow$) If $F$ does not ij-converge to $x$, then there exists a $\tau_i$-open nbd $U$ of $x$ such that $F \notin \tau_j$-$\text{cl}(U)$, for all $F \in F$. Then $G = \{(M-\tau_j-\text{cl}(U) \cap F : F \in F \}$ is a filter base on $M$ finer than $F$, and clearly $x \notin ij$-adherence of $G$. Thus, $F$ cannot be ij-directed towards $x$ which is contradiction. Hence, $F$ is ij-converge to $x$.

($\Rightarrow$) Clear.

Definition 4.1.7. A function $f : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2)$ is said to be ij-perfect iff for each filter base $F$ on $f(M)$, such that $F$ ij-directed towards some subset $A$ of $f(M)$, the filter base $f^{-1}(F)$ is ij-directed towards $f^{-1}(A)$ in $M$. $f$ is called pairwise ij-perfect iff $f$ is $12$ and $21$-perfect, where $i, j = 1, 2$.

Definition 4.1.8. The F.W. bitopological space $(M, \tau_1, \tau_2)$ over bitopological space $(B, \Lambda_1, \Lambda_2)$ is called F.W. ij-perfect iff the projection $p$ is ij-perfect, where $i, j = 1, 2$.

In the following theorem we show that only points of $N$ could be sufficient for the subset $A$ in Definition (4.1.7.) and hence ij-direction can be replaced in view of Theorem (4.1.5.) by ij-convergence.

Theorem 4.1.9. Let $(M, \tau_1, \tau_2)$ be a F.W. bitopological space over bitopological space $(B, \Lambda_1, \Lambda_2)$. Then the following are equivalent:

(a) $(M, \tau_1, \tau_2)$ is F.W. ij-perfect bitopological space.

(b) For each filter base $F$ on $p(M)$, which is ij-convergent to a point $b$ in $B$, $M_F \xrightarrow{ij\text{-}d} M_b$. 

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(c) For any filter base $\mathcal{F}$ on $M$, $\text{ij-ad } p(\mathcal{F}) \subset p(\text{ij-ad } \mathcal{F})$.

**Proof:** (a)$\Rightarrow$(b) Follows from Theorem (4.1.6).

(b)$\Rightarrow$(c) Let $b \in \text{ij-ad } p(\mathcal{F})$. Then by Theorem (4.1.5), there is a filter base $\mathcal{G}$ on $p(M)$ finer than $p(\mathcal{F})$ such that $\mathcal{G} \xrightarrow{\text{ij-con}} b$. Let $\mathcal{U} = \{M_g \cap F : G \in \mathcal{G} 	ext{ and } F \in \mathcal{F}\}$. Then $\mathcal{U}$ is a filter base on $M$ finer than $M_G$. Since $\mathcal{G} \xrightarrow{\text{ij-d}} b$, by Theorem (4.1.6) and $p$ is $\text{ij}$-perfect, $M_g \xrightarrow{\text{ij-d}} M_b$. $\mathcal{U}$ being finer than $M_G$, we have $M_b \cap (\text{ij-ad } \mathcal{U}) \neq \emptyset$. It is then clear that $M_b \cap (\text{ij-ad } \mathcal{F}) \neq \emptyset$. Thus $b \in p(\text{ij-ad } \mathcal{F})$.

(c)$\Rightarrow$(a) Let $\mathcal{F}$ be a filter base on $p(M)$ such that it is $\text{ij}$-directed towards some subset $A$ of $p(M)$. Let $\mathcal{G}$ be a filter base on $M$ finer than $M_F$. Then $p(\mathcal{G})$ is a filter base on $p(M)$ finer than $\mathcal{F}$ and hence $A \cap (\text{ij-ad } p(\mathcal{G})) \neq \emptyset$. Thus, by (c), $A \cap p(\text{ij-ad } \mathcal{G}) \neq \emptyset$ such that $M_A \cap (\text{ij-ad } \mathcal{G}) \neq \emptyset$. This shows that $M_F$ is $\text{ij}$-directed towards $M_A$. Hence, $p$ is $\text{ij}$-perfect.

**Definition 4.1.10.** The function $f : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2)$ is called $\text{ij}$-compact function if it is $ij$-continuous, $ij$-closed and for each filter base $\mathcal{F}$ in $N$ then $f^{-1}(\mathcal{F})$ is filter base in $M$, where $i, j = 1, 2$.

**Definition 4.1.11.** The F.W. $ij$-bitopological space $(M, \tau_1, \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is called F.W. $ij$-compact iff the projection $p$ is $ij$-compact, where $i, j = 1, 2$.

For example the bitopological product $B \times_B T$ is F.W. $ij$-compact over $B$, for all $ij$-compact space $T$, where $i, j = 1, 2$. 
Definition 4.1.12. The F.W. ij-bitopological space $(M, \tau_1, \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is called F.W. ij-closed if and only if the projection $p$ is ij-closed, where $i, j = 1, 2$.

Theorem 4.1.13. If the F.W. bitopological space $(M, \tau_1, \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is ij-perfect, then it is ij-closed, where $i, j = 1, 2$.

Proof: Assume that $M$ is a F.W. ij-perfect bitopological space over $B$, then the projection $p_M : M \to B$ is ij-perfect, to prove that it is ij-closed, by [(4.1.9.) (a)$\Rightarrow$(c)] for any filter base $F$ on $M$ ij-ad $p (F) \subset p (ij-ad (F))$, by Theorem (4.1.3.) $f$ is ij-closed if $ij - cl f(A) \subset f (ij - cl(A))$ for all $A \subset M$, therefore $p$ is ij-closed where $F = \{A\}$.

4.2. Fibrewise IJ-perfect Bitopological Spaces and IJ-Rigidity

In this section, we introduce the notion of ij-perfect bitopological, ij-rigidity spaces and investigate some of their basic properties.

Definition 4.2.1. A subset $A$ of bitopological space $(M, \tau_1, \tau_2)$ is said to be ij-rigid in $M$ iff for each filter base $F$ on $M$ with $(ij \text{-} adF) \cap A = \varnothing$, there is a $\tau_i$-open set $U$ and $F \in F$ such that $A \subset U$ and $\tau_j \text{-} cl(U) \cap F = \varnothing$, or equivalently, iff for each filter base $F$ on $M$ and whenever $A \cap (ij \text{-} adF) = \varnothing$, then for some $F \in F$, $A \cap (ij - \text{cl}(F)) = \varnothing$, where $i, j = 1, 2$.

Theorem 4.2.2. If $(M, \tau_1, \tau_2)$ is a F.W. ij-closed bitopological space over $(B, \Lambda_1, \Lambda_2)$ such that each $M_b$ where $b \in B$ is ij-rigid in $M$, then $(M, \tau_1, \tau_2)$ is a F.W. ij-perfect, where $i, j = 1, 2$.

Proof: Assume that $M$ is a F.W. ij-closed bitopological space over $B$, then the projection $p_M : M \to B$ exist. To prove that it is ij-perfect, let $F$ be a filter
base on $p(M)$ such that $\mathcal{F} \xrightarrow{ij-con} b$ in $B$, for some $b \in B$. If $\mathcal{G}$ is a filter base on $M$ finer than the filter base $M_\mathcal{F}$, then $p(\mathcal{G})$ is a filter base on $B$, finer than $\mathcal{F}$. Since $\mathcal{F} \xrightarrow{ij-d} b$ by Theorem (4.1.5.), $b \in ij-ad p(\mathcal{G})$, i.e. $b \in \cap \{ij-ad p(G) : G \in \mathcal{G}\}$ and hence $b \in \cap \{p(ij-ad G) : G \in \mathcal{G}\}$ by Theorem (4.1.3.). Since $p$ is ij-closed, then $M_b \cap ij-ad (G) \neq \varnothing$, for all $G \in \mathcal{G}$. Hence, for all $U \in \tau_i$ with $M_b \subseteq U$, $\tau_j-cl(U) \cap G \neq \varnothing$, for all $G \in \mathcal{G}$. Since, $M_b$ is ij-rigid, it then follows that $M_b \cap (ij-ad \mathcal{G}) \neq \varnothing$. Thus $M_\mathcal{F} \xrightarrow{ij-d} M_b$.

Hence by Theorem [(4.9)(b)⇒(a)], $p$ is ij-perfect.

**Theorem 4.2.3.** If the F.W. ij-bitopological space $(M, \tau_1, \tau_2)$ over $(B, A_1, A_2)$ is ij-perfect, then it is ij-closed and for each $b \in B$, $M_b$ is ij-rigid in $M$, where $i, j = 1, 2$.

**Proof:** Assume that $M$ is a F.W. ij-bitopological space over $B$, then the projection $p_M : M \rightarrow B$ exist and it is ij-continuous. Since $p$ is an ij-perfect so it is ij-closed. To prove the other part, let $b \in B$, and suppose $\mathcal{F}$ is a filter base on $M$ such that $(ij-ad \mathcal{F}) \cap M_b = \varnothing$. Then $b \notin p(ij-ad \mathcal{F})$. Since $p$ is ij-perfect, by Theorem [(4.1.9)(a)⇒(c)], $b \notin ij-ad p(\mathcal{F})$. Thus there exists an $F \in \mathcal{F}$ such that $b \notin ij-ad p(F)$. There exists an $A_i$-open nbd $V$ of $b$ such that $A_j - cl(V) \cap p(F) = \varnothing$. Since $p$ is ij-continuous, for each $x \in M_b$ we shall get a $\tau_i$-open nbd $U_x$ of $x$ such that $p(\tau_j - cl(U_x)) \subseteq A_j - cl(V) \subseteq B - p(F)$. Then $p(\tau_j - cl(U_x)) \cap p(F) = \varnothing$, so that $\tau_j-cl(U_x) \cap F = \varnothing$. Then $x \notin ij-cl(F)$, for all $x \in M_b$, so that $M_b \cap (ij-cl(F)) = \varnothing$, Hence $M_b$ is ij-rigid in $M$.

**Corollary 4.2.4.** A F.W. ij-bitopological space $(M, \tau_1, \tau_2)$ over $(B, A_1, A_2)$ is ij-perfect iff it is ij-closed and each $M_b$, where $b \in B$ is ij-rigid in $M$, where $i, j = 1, 2$. 

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Next we show that the above theorem remains valid if F.W. ij-closedness bitopological space replaced by a strictly weaken condition which we shall called F.W. weak ij-closedness bitopological space. Thus we define as follows.

**Definition 4.2.5.** A function \( f : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2) \) is said to be weakly ij-closed if for every \( y \in f(M) \) and every \( \tau_i \)-open set \( U \) containing \( f^{-1}(y) \) in \( M \), there exists a \( \sigma_i \)-open nbd \( V \) of \( y \) such that \( f^{-1}(\sigma_j\text{-cl}(V)) \subset \tau_j\text{-cl}(U) \), where \( i, j = 1, 2 \).

**Definition 4.2.6.** The F.W. ij-bitopological space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is called F.W. weakly ij-closed iff the projection \( p \) is weakly ij-closed, where \( i, j = 1, 2 \).

**Theorem 4.2.7.** The F.W. ij-closed bitopological space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is weakly ij-closed, where \( i, j = 1, 2 \).

**Proof:** Assume that \( M \) is a F.W. ij-closed bitopological space over \( B \), then the projection \( p_M : M \to B \) exist and to prove its weakly ij-closed. Let \( b \in p(M) \) and let \( U \) be a \( \tau_i \)-open set containing \( M_b \) in \( M \). Now, by Lemma (1.1.25.) \( \tau_j - \text{cl} \left( M - \tau_j - \text{cl}(U) \right) = i j - \text{cl} \left( M - \tau_j - \text{cl}(U) \right) \) and hence by theorem (4.1.3.) and since \( p \) is ij-closed, we have ij-cl \( p(M - \tau_j - \text{cl}(U)) \subset p[ij - \text{cl}(M - \tau_j - \text{cl}(U))] \). Now since \( b \notin p[ij - \text{cl}(M - \tau_j - \text{cl}(U))] \), \( b \notin ij - \text{cl} p(M - \tau_j - \text{cl}(U)) \) and thus there exists an \( \sigma_i \)-open nbd \( V \) of \( b \) in \( B \) such that \( \sigma_j\text{-cl}(V) \cap p(M - \tau_j - \text{cl}(U)) = \varnothing \) which implies that \( M_{\sigma_j\text{-cl}(V)} \cap (M - \tau_j - \text{cl}(U)) = \varnothing \) i.e., \( M_{\sigma_j\text{-cl}(V)} \subset \tau_j\text{-cl}(U) \), and thus \( p \) is weakly ij-closed.
A F.W. weakly ij-closed is not necessarily to be F.W. ij-closed and the following example show this.

**Example 4.2.8.** Let $\tau_1$, $\tau_2$, $\Lambda_1$ and $\Lambda_2$ be any topologies and $p : (M, \tau_1, \tau_2) \to (B, \Lambda_1, \Lambda_2)$ be a constant function, then $p$ is weakly ij-closed for $i, j = 1, 2$ and $(i \neq j)$. Now, let $M = B = \mathbb{R}$. If $\Lambda_1$ or $\Lambda_2$ is the discrete topology on $B$, then $p : (M, \tau_1, \tau_2) \to (B, \Lambda_1, \Lambda_2)$ given by $p(x) = 0$, for all $x \in M$, is neither 12-closed nor 21-closed, irrespectively of the topologies $\tau_1$, $\tau_2$ and $\Lambda_2$ (or $\Lambda_1$).

**Theorem 4.2.9.** Let $(M, \tau_1, \tau_2)$ be F.W. ij-bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then $(M, \tau_1, \tau_2)$ is F.W. ij-perfect if :

(a) $(M, \tau_1, \tau_2)$ is F.W. weakly ij-closed bitopological space, and

(b) $M_b$ is ij-rigid, for each $b \in B$.

**Proof:** Assume that $M$ is a F.W. ij-bitopological space over $B$ satisfying the conditions (a) and (b), then the projection $p_M : M \to B$ exist. To prove that $p$ is ij-perfect we have to show in view of Theorem (4.2.2.) that $p$ is ij-closed. Let $b \in ij - cl p(A)$, for some non-null subset $A$ of $M$, but $b \notin p (ij - cl(A))$. Then $\mathcal{H} = \{A\}$ is a filter base on $M$ and $(ij-ad \mathcal{H}) \cap M_b = \varnothing$. By ij-rigidity of $M_b$, there is a $\tau_i$-open set $U$ containing $M_b$ such that $\tau_j - cl(U) \cap A = \varnothing$. By weak ij-closedness of $p$, there exists an $\Lambda_i$-open nbd $V$ of $b$ such that $M_{(\Lambda_j - cl(V))} \subset \tau_j - cl(U)$, which implies that $M_{(\Lambda_j - cl(V))} \cap A = \varnothing$, i.e., $(\Lambda_j - cl(V)) \cap p(A) = \varnothing$, which is impossible since $b \in ij - cl p(A)$. Hence $b \notin p (ij - cl(A))$. So $f$ is ij-closed.

**Theorem 4.2.10.** If $(M, \tau_1, \tau_2)$ is F.W. ij-perfect bitopological space over $(B, \Lambda_1, \Lambda_2)$ and $B^* \subset B$ is an ij-H-set in $B$, then $M_{B^*}$ is an ij-H-set in $M$, where $i, j = 1, 2$. 
**Proof:** Assume that \( M \) is a F.W. \( ij \)-perfect bitopological space over \( B \), then the projection \( p_M : M \to B \) exist. Let \( \mathcal{F} \) be a filter base on \( M_{B^*} \), then \( p(\mathcal{F}) \) is a filter base on \( B^* \). Since \( B^* \) is an \( ij \)-H-set in \( B, B^* \cap ij - ad \ p(\mathcal{F}) \neq \emptyset \) by Lemma (1.1.21.). By Theorem [(4.1.9) (a)⇒(c)], \( B^* \cap p(ij - ad \ (F)) \neq \emptyset \), so that \( M_{B^*} \cap ij-ad \ (\mathcal{F})\neq \emptyset \). Hence by Lemma (1.1.21.), \( M_{B^*} \) is an \( ij \)-H-set in \( M \).

The converse of the above theorem is not true, is shown in the next example.

**Example 4.2.11.** Let \( M = B = \mathbb{R}, \tau_1 \) and \( \tau_2 \) be the cofinite and discrete topologies on \( M \) and \( \Lambda_1, \Lambda_2 \) respectively denote the indiscrete and usual topologies on \( B \). Suppose \( p : (M, \tau_1, \tau_2) \to (B, \Lambda_1, \Lambda_2) \) is the identity function. Each subset of either of \( (M, \tau_1, \tau_2) \) and \( (B, \Lambda_1, \Lambda_2) \) is a 12-set. Now, any non-void finite set \( A \subset M \) is 12-closed in \( M \), but \( p(A) \) (i.e., \( A \)) is not 12-closed in \( B \) (in fact, the only 12-closed subsets of \( B \) are \( B \) and \( \emptyset \)).

**Definition 4.2.12.** A function \( f : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2) \) is said to be almost \( ij \)-perfect if for each \( ij \)-H-set \( K \) in \( N \), \( f^{-1}(K) \) is an \( ij \)-H-set in \( M \), where \( i, j = 1, 2 \).

**Definition 4.2.13.** The F.W. \( ij \)-bitopological space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is called F.W. almost \( ij \)-perfect iff the projection \( p \) is almost \( ij \)-perfect, where \( i, j = 1, 2 \).

By analogy to Theorem (4.2.2.), a sufficient condition for a function to be almost \( ij \)-perfect, is proved as follows.
**Theorem 4.2.14.** Let \((M, \tau_1, \tau_2)\) be F.W. \(ij\)-bitopological space over \((B, \Lambda_1, \Lambda_2)\) such that:

(a) \(M_b\) is \(ij\)-rigid, for each \(b \in B\), and
(b) \((M, \tau_1, \tau_2)\) is F.W. weakly \(ij\)-closed bitopological space.

Then \((M, \tau_1, \tau_2)\) is F.W. almost \(ij\)-perfect bitopological space.

**Proof:** Assume that \(M\) is a F.W. \(ij\)-bitopological space over \(B\), then the projection \(p_M : M \to B\) exist and it is \(ij\)-continuous. Let \(B^*\) be an \(ij\)-H-set in \(B\) and let \(\mathcal{F}\) be a filter base on \(M_{B^*}\). Now \(p(\mathcal{F})\) is a filter base on \(B^*\) and so by Lemma (1.1.21.), \((ij - \text{ad } p(\mathcal{F})) \cap B^* \neq \emptyset\). Let \(b \in [ij - \text{ad } p(\mathcal{F})] \cap B^*\). Suppose that \(\mathcal{F}\) has no \(ij\)-ad point in \(M_{B^*}\) so that \((ij\text{-ad } (\mathcal{F})) \cap M_b = \emptyset\).

Since \(M_b\) is \(ij\)-rigid, there exists an \(F \in \mathcal{F}\) and a \(\tau_i\)-open set \(U\) containing \(M_b\) such that \(F \cap \tau_j - \text{cl}(U) = \emptyset\). By weak \(ij\)-closedness of \(p\), there is a \(\Lambda_i\)-open nbd \(V\) of \(b\) such that \(M_{(\Lambda_j - \text{cl}(V))} \subset \tau_j - \text{cl}(U)\) which implies that \(M_{(\Lambda_j - \text{cl}(V))} \cap F = \emptyset\), i.e., \(\Lambda_j\text{-cl}(V) \cap p(F) = \emptyset\), which is a contradiction. Thus by Lemma (1.1.21.), \(M_{B^*}\) is an \(ij\)-H-set in \(M\) and hence \(p\) is almost \(ij\)-perfect.

### 4.3. Application of Fibrewise IJ-Perfect Bitopological Spaces

We now give some applications of fibrewise \(ij\)-perfect bitopological spaces. The following characterization theorem for an \(ij\)-continuous function is recalled to this end.

**Theorem 4.3.1.** A bitopological space \((M, \tau_1, \tau_2)\) is F.W. \(ij\)-bitopological space over \((B, \Lambda_1, \Lambda_2)\) iff \(p(ij - \text{cl}(A)) \subset ij - \text{cl}(p(A))\), for each \(A \subset M\), where \(i, j = 1, 2\).

**Proof:** \((\Rightarrow)\): Assume that \(M\) is a F.W. \(ij\)-bitopological space over \(B\), then the projection \(p_M : M \to B\) exist and it is \(ij\)-continuous. Suppose that \(x \in ij - \text{cl}(A)\) and \(V\) is \(\Lambda_i\)-open nbd of \(f(x)\). Since \(p\) is \(ij\)-continuous, there exists an \(\tau_i\)-open nbd \(U\) of \(x\) such that \(p(\tau_j - \text{cl}(U)) \subset \Lambda_j - \text{cl}(V)\). Since \(\tau_j\text{-cl } (U) \cap \)
A \neq \varnothing$, then $A_j \text{-cl}(V) \cap p(A) \neq \varnothing$. So, $p(x) \in ij - \text{cl}(p(A))$. This shows that $p(ij - \text{cl}(A)) \subset ij - \text{cl}(p(A))$.

($\Leftarrow$) Clear.

**Theorem 4.3.2.** Let $(M, \tau_1, \tau_2)$ be a F.W. ij-perfect bitopological space over $(B, A_1, A_2)$. Then $M_A$ preserves ij-rigidity, where $i, j = 1, 2$.

**Proof:** Assume that $M$ is a F.W. ij-bitopological space over $B$, then the projection $p_M : M \to B$ exist and it is ij-continuous. Let $A$ be an ij-rigid set in $B$ and let $\mathcal{F}$ be a filter base on $M$ such that $M_A \cap (ij - \text{ad}(\mathcal{F})) = \varnothing$. Since $p$ is ij-perfect and $A \cap p(ij - \text{ad}(\mathcal{F})) = \varnothing$ by Theorem [(4.1.9.) (a)$\Rightarrow$(c)] we get $A \cap (ij - \text{ad} p(\mathcal{F})) = \varnothing$. Now $A$ being an ij-rigid set in $B$, there exists an $F \in \mathcal{F}$ such that $A \cap ij - \text{clp}(F) = \varnothing$. Since $p$ is ij-continuous, by Theorem (4.3.1.) it follows that $A \cap p(ij - \text{cl}(F)) = \varnothing$. Thus $M_A \cap (ij - \text{cl}(F)) = \varnothing$. This proves that $M_A$ is ij-rigid.

In order to investigate the conditions under which a F.W. almost ij-perfect bitopological space may be F.W. ij-perfect bitopological space, we introduce the following definition.

**Definition 4.3.3.** A function $f : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2)$ is said to be $ij^*$-continuous iff for any $\sigma_j$-open nbd $V$ of $f(x)$, there exists a $\tau_i$-open nbd $U$ of $x$ such that $f(\tau_j \text{-cl}(U)) \subset \sigma_i \text{-cl}(V)$, where $i, j = 1, 2$.

**Definition 4.3.4.** The F.W. ij-bitopological space $(M, \tau_1, \tau_2)$ over $(B, A_1, A_2)$ is called F.W. $ij^*$-bitopological space iff the projection $p$ is $ij^*$-continuous, where $i, j = 1, 2$.

The relevance of the above definition to the characterization of F.W. ij-perfect bitopological space is quite apparent from the following result.
**Theorem 4.3.5.** If \((M, \tau_1, \tau_2)\) is F.W. \(ij^*\)-bitopological space on a pairwise Urysohn space \((B, \Lambda_1, \Lambda_2)\), then it is F.W. \(ij\)-perfect bitopological space iff for every filter base \(\mathcal{F}\) on \(M\), if \(p(\mathcal{F}) \xrightarrow{ij-\text{con}} b\) where \(b \in B\), then \(ij - ad \mathcal{F} \neq \varnothing\), where \(i, j = 1, 2\).

**Proof:** \((\Rightarrow)\) Let \((M, \tau_1, \tau_2)\) be a F.W. \(ij^*\)-bitopological space on a pairwise Urysohn space \((B, \Lambda_1, \Lambda_2)\), then there is a \(ij^*\)-continuous projection function \(p: (M, \tau_1, \tau_2) \to (B, \Lambda_1, \Lambda_2)\) and \(p(\mathcal{F}) \xrightarrow{ij-\text{con}} b\) where \(b \in B\), for a filter base \(\mathcal{F}\) on \(M\). Then \(M_{p(\mathcal{F})} \xrightarrow{ij-\text{dir.}} M_b\). Since \(\mathcal{F}\) is finer than \(M_{p(\mathcal{F})}\), \(M_b \cap ij - ad \mathcal{F} \neq \varnothing\), so that \(ij - ad \mathcal{F} \neq \varnothing\).

\((\Leftarrow)\) Suppose that for every filter base \(\mathcal{F}\) on \(M\), \(p(\mathcal{F}) \xrightarrow{ij-\text{con}} b\) where \(b \in B\) implies \(ij - ad \mathcal{F} \neq \varnothing\). Let \(\mathcal{G}\) be a filter base on \(B\) such that \(\mathcal{G} \xrightarrow{ij-\text{con}} b\), and suppose that \(\mathcal{G}^*\) is a filter base on \(M\) such that \(\mathcal{G}^*\) is finer than \(M_{\mathcal{G}}\). Then \(p(\mathcal{G}^*)\) is finer than \(\mathcal{G}\). So \(p(\mathcal{G}^*) \xrightarrow{ij-\text{con}} b\). Hence \(ij - ad \mathcal{G}^* \neq \varnothing\). Let \(z \in B\) such that \(z \neq b\). Then since \(B\) is pairwise Urysohn, there exist a \(\Lambda_i\)-open nbd \(U\) of \(b\) and \(\Lambda_j\)-open nbd \(V\) of \(z\) such that \((\Lambda_j - cl(U)) \cap (\Lambda_i - cl(V)) = \varnothing\). Since \(p(\mathcal{G}^*) \xrightarrow{ij-\text{con}} b\), there exist a \(G \in \mathcal{G}^*\) such that \(p(G) \subset \Lambda_j - cl(U)\). Now, since \(p\) is \(ij^*\)-continuous, corresponding to each \(x \in M_z\) there is a \(\tau_i\)-open nbd \(W\) of \(x\) such that \(p(\tau_j - cl(W)) \subset \Lambda_i - cl(V)\). Thus \(\Lambda_j - cl(W) \cap G = \varnothing\).

It follows that \(M_z \cap ij - \mathcal{G}^* = \varnothing\), for each \(z \in B - \{b\}\). Consequently \(M_b \cap ij - ad \mathcal{G}^* \neq \varnothing\), and \(p\) is \(ij\)-perfect and hence \((M, \tau_1, \tau_2)\) is F.W. \(ij^*\)-bitopology.

**Definition 4.3.6.** A bitopological space \((M, \tau_1, \tau_2)\) is said to be locally \(ij\)-QHC iff for every \(x \in M\), there is a \(\tau_i\)-open nbd of \(x\), which is an \(ij\)-H-set, where \(i, j = 1, 2\).
**Corollary 4.3.7.** Let \((M, \tau_1, \tau_2)\) be a F.W. \(ij^*\)-bitopological space over \(ij\)-QHC on a pairwise Urysohn bitopological space \((B, A_1, A_2)\), then \((M, \tau_1, \tau_2)\) is F.W. \(ij\)-perfect bitopological space, where \(i, j = 1, 2\).

**Theorem 4.3.8.** Let \((M, \tau_1, \tau_2)\) be a F.W. \(ij^*\)-bitopological space over locally \(ij\)-QHC on a Urysohn space \((B, A_1, A_2)\), then \((M, \tau_1, \tau_2)\) is F.W. \(ij^*\)-bitopological space iff it is F.W. almost \(ij\)-perfect, where \(i, j = 1, 2\).

**Proof:** (\(\Rightarrow\)) If \((M, \tau_1, \tau_2)\) is F.W. \(ij^*\)-bitopological space, then by corollary (4.3.7.), it is F.W. almost \(ij\)-perfect.

(\(\Leftarrow\)) Let \((M, \tau_1, \tau_2)\) is F.W. almost \(ij\)-perfect, then there exist almost \(ij\)-perfect projection function \(p: (M, \tau_1, \tau_2) \to (B, A_1, A_2)\), and let \(\mathcal{F}\) be any filter base on \(M\) and let \(p(\mathcal{F}) \xrightarrow{ij\text{-con.}} b\) where \(b \in B\). There are an \(ij\)-H-set \(B^*\) in \(B\) and \(A_i\)-open nbd \(V\) of \(b\) such that \(b \in V \subseteq B^*\). Let \(\mathcal{H} = \{A_j - cl(U) \cap p(F) \cap B^*; F \in \mathcal{F}\text{ and } U\text{ is a }A_i\text{-open nbd of } b\}\). By Lemma (1.1.24.), \(B^*\) is \(ij\)-closed and hence no member of \(\mathcal{H}\) is void. In fact, if not, let for some \(A_i\)-open nbd \(U\) of \(b\) and some \(F \in \mathcal{F}\), \(A_j - cl(U) \cap p(F) \cap B^* = \emptyset\). Then \(W = U \cap V\) since \(y \in U \cap V \subseteq A_i\) and \(A_j - cl(W) = ij - cl(W) \subseteq ij - cl(B^*) = B^*\) by Lemma (1.1.25.). Now \(\emptyset = A_j - cl(W) \cap p(F) \cap B^* = A_j - cl(W) \cap p(F)\), which is not possible, since \(p(\mathcal{F}) \xrightarrow{ij\text{-con.}} b\). Thus \(\mathcal{H}\) is filter base on \(B\), and is clearly finer than \(p(\mathcal{F})\), so that \(\mathcal{H} \xrightarrow{ij\text{-con.}} b\). Also \(\mathcal{G} = \{M_H \cap F; H \in \mathcal{H}\text{ and } F \in \mathcal{F}\}\) is clearly a filter on \(M_{B^*}\). Since \(p\) is almost \(ij\)-perfect, \(M_{B^*}\) is an \(ij\)-H-set and hence \(ij - ad\mathcal{G} \cap M_{B^*} \neq \emptyset\). Thus \(ij - ad\mathcal{F} \neq \emptyset\). Thus \(p\) is \(ij\)-perfect and by Theorem (4.3.5.) \((M, \tau_1, \tau_2)\) is F.W. \(ij^*\)-bitopological space.

The following characterization theorem for a F.W. \(ij\)-bitopological space is recalled to this end.
Chapter 4
Fibrewise IJ-Perfect Bitopological Spaces

Theorem 4.3.9. A F.W. set \( M \) over \((B, \Lambda_1, \Lambda_2)\) is F.W. ij-bitopological space iff \( p(ij-cl(A)) \subset ij-clp(A) \) for each \( A \subset M \), where \( i, j = 1, 2 \).

Proof: Since \( M \) is a F.W. set over \( B \), then there is projection \( p \) where \( p: M \to B \). Now we have to prove that \( p \) is ij-continuous. But it directly by Theorem (4.3.1.).

Theorem 4.3.10. If \((M, \tau_1, \tau_2)\) is a F.W. ij-perfect bijective bitopological space with \( M \) is a pairwise Hausdorff space on \((B, \Lambda_1, \Lambda_2)\), Then \( B \) is also pairwise Hausdorff.

Proof: Let \( b_1, b_2 \in B \) such that \( b_1 \neq b_2 \). Since \( p \) is onto, then \( M_{b_1}, M_{b_2} \in M \) and since \( p \) is one to one, then \( M_{b_1} \neq M_{b_2} \). Since \( p \) is ij-perfect, so by Theorem (4.1.13) it is ij-closed. By Lemma (1.1.26) we have \( \{ M_{b_1} \} = ij - cl \{ M_{b_1} \} \) and \( \{ M_{b_2} \} = ij - cl \{ M_{b_2} \} \). Since \( p \) is pairwise Hausdorff. Now \( p(ij - cl \{ M_{b_1} \}) = ij - cl \{ b_1 \} \) and \( p(ij - cl \{ M_{b_2} \}) = ij - cl \{ b_2 \} \) since \( p \) is ij-closed. This mean \( \{ b_1 \} = ij - cl \{ b_1 \} \) and \( \{ b_2 \} = ij - cl \{ b_2 \} \). Hence \( B \) is pairwise Hausdorff.

Our next theorem give a characterization of an important class of F.W. bitopological space viz. the ij-QHC spaces in terms of F.W. ij-perfect bitopological space.

Theorem 4.3.11. For a bitopological space \((M, \tau_1, \tau_2)\), the following statement are equivalent:

a) \( M \) is ij-QHC

b) The F.W. \((M, \tau_1, \tau_2)\) is ij-perfect bitopological space with constant projection over \( B^* \) where \( B^* \) is a singleton with two equal bitopologies viz. the unique bitopology on \( B^* \).
c) The F.W. $(B \times M, Q_1, Q_2)$ is ij-perfect bitopological space over $(B, \Lambda_1, \Lambda_2)$, where $Q_i = \Lambda_i \times \tau_j, i, j = 1, 2$ and $i \neq j$.

**Proof:** (a) $\Rightarrow$ (b) Let $p : (M, \tau_1, \tau_2) \to (B^*, \Lambda_1, \Lambda_2)$ is a constant projection over $B^*$ where $B^*$ is a singleton with two equal bitopologies viz the unique bitopology on $B^*$. $P$ is clearly ij-closed. Also, $M_{B^*}$, i.e. $M$ is obviously ij-rigid since $B^*$ is ij-QHC. Then by Theorem (4.2.2.) $p$ is ij-perfect.

(b) $\Rightarrow$ (a) Follows from Theorem (4.3.2.).

(a) $\Rightarrow$ (c) Suppose that $(B \times M, Q_1, Q_2)$ is F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$ where $Q_i = \Lambda_i \times \tau_j, i, j = 1, 2$ and $i \neq j$, then there is a projection $p = \pi_i : (B \times M, Q_1, Q_2) \to (B, \Lambda_1, \Lambda_2)$. We show that $\pi_i$ is ij-closed and for each $b \in B, M_B$ is ij-rigid in $B \times M$. Then the result will follow from Theorem (4.2.2). Let $A \subset B \times M$ and $a \not\in \pi_i(ij - cl(A))$. For each $m \in M, (a, m) \not\in ij - cl(A)$, so that there exist a $\Lambda_j$-open nbd $G_m$ of $a$ and a $\tau_i$-open nbd $H_m$ of $m$ such that $[Q_i - cl(G_m \times H_m)] \cap A = \varnothing$. Since $M$ is ij-QHC, $\{a\} \times M$ is a ij–H-set in $B \times M$. Thus there exist finitely many elements $m_1, m_2, m_3, ..., m_n$ with $\{a\} \times M \subset \bigcup_{k=1}^{n} Q_i - cl(G_{m_k} \times H_{m_k})$. Now, $a \in \bigcap_{k=1}^{n} G_{m_k} = G$ which is a $\Lambda_i$-open nbd of $a$ such that $(\Lambda_i - cl(G) \cap \pi_i(A) = \varnothing$. Hence $a \not\in ij - cl \pi_i(A)$ and thus $ij - cl \pi_i(A) \subset \pi(ij - cl(A))$.

So $\pi$ is ij-closed, by Theorem (4.1.3.). Next, let $b \in B$. To show that $(B \times M)_b = \pi_i^{-1}(b)$ to be ij-rigid in $B \times M$. Let $\mathcal{F}$ be a filter base on $B \times M$ such that $\pi_i^{-1}(b) \cap ij - ad \mathcal{F} = \varnothing$. For each $m \in M, (b, m) \not\in ij - ad \mathcal{F}$.

Thus there exist $\Lambda_j$-open nbd $U_m$ of $b$ in $B$, a $\tau_i$–open nbd $V_m$ of $m$ in $M$ and and $F_m \in \mathcal{F}$ such that $Q_i - cl(U_m \times V_m) \cap F_m = \varnothing$. As show above, there exist finitely many elements $m_1, m_2, m_3, ..., m_n$ of $M$ such that $\{b\} \times M \subset \bigcup_{k=1}^{n} Q_i - cl(U_{m_k} \times V_{m_k})$. Putting $U = \bigcap_{k=1}^{n} U_{m_k}$ and choosing $F \in \mathcal{F}$ with $F \subset \bigcap_{k=1}^{n} F_{m_k}$, we get $\{b\} \times M \subset U \times M \subset Q_j$ such that $Q_i - cl(U \times M) \cap
Thus \((\langle ij - cl(F) \rangle \cap [\pi_i^{-1}(b)] = \emptyset)\). Hence \(\pi_i^{-1}(b)\) is ij-rigid in \(B \times M\).

\((c) \Rightarrow (a)\) Taking \(B^* = B\), we have that \(p = \pi_i: B^* \times B \rightarrow B^*\) is ij-perfect. Therefore by Theorem. (4.2.10.) \(B^* \times M\) is an ij-H-set and Hence \(M\) is ij-QHC.
Conclusions

The main purpose of the present work is the starting point for the applications of abstract topological structures in fibrewise theory by using bitopological systems. We believe that fibrewise bitopological structure will be an important base for modification of knowledge extraction and processing.

We used separation axioms concept in fibrewise bitopological space to introduce a new notion namely fibrewise pairwise separation axioms. The suggested methods of fibrewise pairwise separation axioms open way for constructing new types of fibrewise topologies.

Finally, the generalization of fibrewise bitopology in the ij-perfect space are introduced, we believe such generalization will be useful in compact bitopology, as well as soft bitopology.
The following are some open problems for the future works:

In the future we can use the concepts fibrewise bitopological spaces in define fibrewise soft bitopological spaces, also we can define fibrewise soft bitopological-$T_i$ where $i=1,2,3,4$. On the other hand we can discuss the relation between fibrewise soft bitopological spaces and fibrewise soft $j$-bitopological spaces, where $j \in \{\alpha, S, P, b, \beta\}$. Furthermore, we will study fibrewise bitopological digital (resp., di, tri, nano, filte, girll, fuzzy) topological spaces.
References


المستخلص

قدمنا في هذه الرسالة دراسة حول بعض الفضاءات التوبولوجية الثنائية في نظرية المجموعات الليفية وبعض النتائج المتعلقة بها والمفاهيم الأساسية مثل الفضاءات التوبولوجية الثنائية الليف القابلة للتقطيع و الفضاءات التوبولوجية الثنائية الليف القابلة للتجزئة.

كما وتطرقا لبديهيات الفصل في الفضاءات التوبولوجية الثنائية الليفية مثل والهاوسدروف والفضاءات التوبولوجية الثنائية الليف هاوسدروف دانيا والفضاءات التوبولوجية الثنائية الليف المنتظمة والفضاءات التوبولوجية الثنائية الليف المنتظمة بالكامل و الفضاءات التوبولوجية الثنائية الليف الطبيعية و الفضاءات التوبولوجية الثنائية الليف الطبيعية دانيا.

ثم تناولنا مفهوم التراص في الفضاءات التوبولوجية الثنائية الليف والعلاقة بينه وبين مفهوم الفضاءات التوبولوجية الثنائية الليف الصلبة و الفضاءات التوبولوجية الثنائية الليف المغلقة والفضاءات التوبولوجية الثنائية الليف ضعيفة الإغلاق.

كما وتمت دراسة مفهوم المرشحات والمرشحات الأساسية والمرشح الموجه واقتراب المرشح إلى نقطة معينة في الفضاءات التوبولوجية الثنائية الليف.
دراسة بعض تعميمات الفضاءات التوبولوجية
الليفية الثنائية

رسالة

مقدمة إلى كلية التربية للعلوم الصرفة / ابن الهيثم، جامعة بغداد
كجزء من متطلبات نيل درجة ماجستير علوم في الرياضيات
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