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A survey on some analytical and geometrical properties of classes included univalent and multivalent functions

A Thesis
Submitted to the Council of College of Education for Pure Science Ibn Al-Haytham , University of Baghdad in Partial Fulfillments of the Requirements for the Degree of Doctor of Philosophy in Mathematics

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بسم الله الرحمن الرحيم

﴿اللَّهُ نُورُ السَّمَ اوَاتِ َوالأَرْضِ مَثَلُ نُورِهِ كَمِشْكَاةٍ فِيهَا مِصْبَاحٌ الْمِصْبَاحُ فِي
زُجَاجَةٍ الزُّجَاجَةُ كَأَنَّهَا كَوْكَبٌ دُرِّيٌّ يُوقَدُ مِنْ شَجَرَةٍ مُبَارَكَةٍ زَيْتُونِةٍ لا شَرْقِيَّةٍ وَلا غَرْبِيَّةٍ
يَكَادُ زَيْتُهَا يُضِيءُ وَلَوْ زلَمْ تَمْسَسْهُ نَارٌ نُورٌ عَلَى نُورٍ يَهْدِي اللَّهُ لِنُورِهِ مَنْ يَشَاءُ
وَيَضْرِبُ اللَّهُ الأَمْثَالَ لِلنَّاسِ وَاللَّهُ بِكُلِّ شَيْءٍ عَلِيمٌ ﴾

صدق الله العظيم

سورة النور ((الآية 35))

SUPERVISOR CERTIFICATION

I certify that this thesis entitled " A survey on some analytical and geometrical properties of classes included univalent and multivalent functions" was prepared by **Zainab hadi Mahmood** under my supervision at University of Baghdad ,College of Education For Pure Science - Ibn Al - Haytham as a partial fulfillment of the requirements for the **degree of Doctor of Philosophy in Mathematics**.

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DEDICATION

To who gave me everything in every world…
The Holy King

To the pure Souls that created my Soul…

my Mother…my Father…My husband

To the White birds that surrounded me with Love and Care…

my Brother…my Sisters…

my Young Family Flowers

To those who were absent from me and

Present in my Heart.....

They have all the gratitude…,

……,Praise and Thanks

CONTENTS

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AUTHORS PUBLICATIONS

1. Z. H. Mahmood, B. N. Shihab and K.A. Jassim, “On a class of analytic multivalent functions involving higher-order derivatives ”, Second International Conference for Applied and Pure Mathematics (SICAPM), College of Science, University of Baghdad, pp. 13-17, (2019).

2. Z. H. Mahmood, B. N. Shihab and K.A. Jassim, “Generalization of a subclass of multivalent function defined by dziok-srivastava linear operator”, accepted for publication in The Journal of The Indian Mathematical Society.

3. Z. H. Mahmood, B. N. Shihab and K.A. Jassim, “Certain subclasses of harmonic Multivalent functions of complex order”, accepted for publication in The Journal of The Indian Mathematical Society.
4. Z. H. Mahmood, B. N. Shihab and K.A. Jassim, “Certain family of multivalent functions associated with subordination”, accepted for publication in The Ibn Al-Haitham Journal for Pure and Applied Science.

LIST OF SYMBOLS

U	Open unit disk $\{z \in \mathbb{C}: \lvert z\rvert < 1\}$
U^*	The punctured unit disk $\{z \in \mathbb{C}: 0 < \lvert z\rvert < 1\}$
$\mathbb{C}$	The complex plane .
$\mathbb{N}$	The set of natural numbers .
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathcal{R}e\,(f(z))$	The real part of $f(z)$
$\mathcal{D}_p$	Class of multivalent analytic functions of the form $f(z) = z^p + \sum_{k=p+n}^{\infty} a_k\, z^k$, $(z \in U)$.
$\mathcal{D} = \mathcal{D}_1$	Class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k$, $(z \in U)$.
S	Class of normalized univalent functions of the form $f(z) = z + a_1 z^1 + \cdots$, $(z \in U)$
S_H	Class of harmonic univalent functions of the form $f = h + \bar{g}$,where $h(z) = z + \sum_{k=2}^{\infty} a_k\, z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k\, z^k$, $b_1 = 1$.
Σ_p	Class of meromorphic multivalent analytic functions of the form $f(z) = z^{-p} - \sum_{k=0}^{\infty} a_k\, z^k$, $(p \in N = \{1,2,\ldots\};\ z \in U^*)$.
Σ	Class of meromorphic univalent analytic functions of the form $f(z) = z^{-1} - \sum_{k=0}^{\infty} a_k\, z^k$, $(p \in N = \{1,2,\ldots\};\ z \in \bar{U})$.
$S_p^*(\alpha)$	Class of multivalent starlike of order α
$\mathcal{K}_p(\alpha)$	Class of multivalent convex of order α
$\mathcal{C}_p(\alpha)$	Class of multivalent close to convex of order α
$f * g$	Hadamard product of the function f and g .
$f \prec g$	f subordinate to g
$f = u + iv$	Complex- valued harmonic function .
$f = h + \bar{g}$	The harmonic function, sense-preserving and locally injective.
R_1	Radius of starlikeness.
R_2	Radius of convexity.
R_3	Radius of close to convex.
$(\lambda)_k$	The Pochammer symbol .
$k!$	The factorial function .
$\delta(k,q)$	The binomial coefficient equal $\frac{k!}{(k-q)!}$
$\mathcal{H}_p^{l,m}$	Dziok- Srivastava operator .
$N_\delta(h)$	The δ $-$neighborhood of a function h .
$I_{c,p}$	Integral operator
D_z^{μ}	Fractional calculus operator
$f^q(z)$	Derivative of function f q-times
$D_p^n(f^q(z))$	Differential operator for the Derivative of function f q-times
$S_{\boldsymbol{j+p+r}}$	Partial Sums of multivalent functions
$D_p^{n+m}(f^q(z))$	Differential operator of Higher order
$D_p^{n+1}(f^q(z))$	Differential operator of first order
$D_z^{-\mu}$	Inverse Differential operator
$I_{\gamma,\beta}^{m}$	Linear operator

D^m	Differential operator
$f_t(z)$	Functions of complex number with respect to time
$D^{\Omega}f(z)$	Differential operator of order Ω of a function f
$Q_{\beta}^{\alpha}f(z)$	Lin- Liu and Owa integral operator.

ABSTRACT

Our main objective in this thesis is to provide a survey on some analytical and geometrical properties of classes included univalent and multivalent functions. It is the study on a class $S_n(p\,, q, A\,, B\,, \lambda\,, \alpha\,, l, \beta)$ of analytic mulivalent functions involving higher – order derivatives . We obtain coefficient inequalities, distortion theorems, radii of convexity, closure theorems and modified Hadamard products for functions in this class. We have also discussed and study generalization of a subclass $HF_{\gamma}^{\lambda}(\alpha\,, \beta, A, B)$ of multivalent function defined by Dziok-Srivastava linear operator. We obtain some properties, such as , modified Hadamard product, Holder inequalities and closure properties under integral transforms are discussed. Also, we have given class $ST_w\,(k, \beta, c)$ of analytic functions convoluted with differential operator. We obtain some properties, such as, coefficient inequalities , distortion and covering theorem , radii of starlikeness and convexity and convex linear combination .Also, we have discussed and studied certain subclasses $A^*_{m,k}(ŋ, \theta, \delta)$, $A^*_{m0,k}(ŋ, \theta, \delta, z_0)$ and $A^*_{m1,k}(ŋ, \theta, \delta, z_0)$ of meromorphic univalent functions involving differential operator.We have obtained numerous sharp results including coefficient conditions, extreme points, distortion bounds and convex combinations for the above classes of meromorphic univalent functions. We have studied a certain subclasses $SC_H(b, \gamma, \lambda\,)$ of harmonic multivalent functions of complex order. Here , we obtain some results , like, Coefficient conditions, distortion bounds, extreme points, convolution, convex combinations, and neighborhoods for a new class of harmonic univalent functions in the open unit disc are investigated. Also , we study a certain subclass $\mathcal{W}_{\mathcal{H}}^{l,m}(b, \alpha, \beta, \gamma; t)$ of analytic functions of the form $f = h + \bar{g}$ defined by Dziok-Raina operator . The functions of these classes passes some result on coefficient conditions, extreme points and distortion bounds, convolution and convex combination properties and the closure property of the class under integral operator were determined. We study have studied a certain class $AJ_s(\alpha\,, \lambda, k\,, p\,)$ of multivalently harmonic meromorphic function of the form

$f = h + \bar{g}$. We obtain some properties, such as, coefficient conditions, extreme points and distortion bounds, convolution and convex combination properties and the closure property of the class under integral operator were determined. We discuss strong differential subordination properties for multivalent functions defined by integral operator. We some application of differential subordination and superordination result involving integral operator I_p^{α} for certain normalized analytic functions. We also deal with some applications of differential subordination Involving hadamard product. We obtain some subordination results for univalent functions in the open unit disk U. We studied of certain family of multivalent functions associated with subordination. We have introduced new classes by using subordination and we have obtained some geometric propertice , like , coefficient estimates and Distortion and Growth theorems , radius of starlikeness and radius of convexity, and other related results for subclasses $K\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ and $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$.

INTRODUCTION

The theory of univalent and multivalent functions is an age old branch of mathematics, particularly complex analysis attracting a large number of researchers owing to shear beauty, its geometrical aspects and a lot of avenues of research work. The present work is a part of the geometric function theory. The study of univalent and multivalent functions is one of the leading branch of the geometric function theory. An univalent function is an analytic or meromorphic function f in a domain of extended complex plane such that$f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$ where z_1 and z_2 are members of the domain.In other words f is a one-to-one mapping from a domain into the extendedcomplex plane. One of the fundamental problems in the study of univalentfunctions is whether there exists a univalent mapping from a simply connecteddomain onto a given simply connected domain. However, in view of RiemannMapping theorem above problem reduces to a problem of mapping a unitdisc onto a given simply connected domain such as starlike, convex, close-toconvex etc.

The class $\mathcal{D}$, of functions that are analytic and univalent on the unit disk

$U = \{z \in C: |z| < 1\}$, normalized by the two conditions $f(0) = 0$ and $f'(0) = 1$and having the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

playsan important role in the study of univalent function and for multivalent functions,

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

In context to Riemann mapping theorem, most of the geometric theorems which are concerned with family of functions $\mathcal{D}$ are translated in arbitrary simply connected domain with more than one boundary point. In1916, Bieberbach conjectured “The coefficients of each function $f \in \mathcal{D}$ satisfy $|a_n| \leq n$, for n = 2, 3,···. Strict inequality holds for all n unless f is a

Koebe function or one of its rotations. For many years this conjecture wasa challenge to all mathematicians and motivated for the development of thevarious new methods in the complex analysis. This conjecture was settled inthe summer of 1984 by Louis de Branges.

Many distinguished researchers in mathematicians like St.Ruscheweyh, H. M. Srivastava, H. Silverman, S. S. Miller, P. T. Mocanu,S. Owa, P. L. Duren, J. M. Jahangiri et. al., have opened new avenues in the field of complex analysis, particularly in geometric function theory. Thepresent work enveils beautiful applications of generalized Ruscheweyh derivatives, various integral operators, convolution theorems, neighbourhood andpartial sums, differential subordination, hypergeometric functions, fractionalcalculus. The references which have been cited are enclosed at the end of this thesis as well as with list of publications .

This thesis is divided to five chapters . These chapters arranged as follows;

Chapter One , We have given an exhaustive list of essential definitions ofthe family of univalent functions such as, starlike, convex, close-to-convex, thegeometrical behaviour in the form of bounds like growth theorem, region ofunivalency, i.e. the radius of starlikeness, convexity, close-to-convexity, etc.has been defined. The terms like hypergeometric functions, fractional derivative, fractional integration, subordination principle, etc. have been defined in detail.

Chapter Two, is devoted to study some properties of certain subclasses of univalent , meromorphic univalent and multivalent functions defined by some operators.

This chapter consists of three sections. In section one , we introduce new subclass $S_n(p\,,q;,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ of multivalent functionswith higher order derivatives defined in the unit disk. we obtain coefficient inequalities, distortion theorems, radii of convexity, closure theoremsand modified Hadamard products for functions in this class.

Insection two ,we introduced a new subclass $HF_{\gamma}^{\lambda}(\alpha\,,\beta,A,B)$ of multivalent functions involving DziokSrivastava Operator .The results

on modified Hadamard product ,Holder inequalities and closure properties underintegral transforms are discussed.

Section three, is study on new subclasses $S_w\,(k,\beta) and\, ST_w\,(k,\beta,\alpha,p,q)$ of functions with fixed second position coefficients .We obtain some interesting properties, such as , coefficient inequalities, distortion and growth property , closure property, radii of stalikness and convexity and Hadamard product .

Chapter Three ,is divided into two sections ,we have introduced and studied some new subclasses $A_{m,k}^{*}(\eta,\theta,\delta)$, $A_{m0,k}^{*}(\eta,\theta,\delta,z_0)$ and $A_{m1,k}^{*}(\eta,\theta,\delta,z_0)$ of meromorphic univalent functions which are defined by means of a differential operator .We have obtained numerous sharp results including coefficient conditions, extreme points, distortion bounds and convex combinations for the above classes of meromorphic univalent functions.

Chapter Four , is divided into four sections, section one and two ,we have introduced and studied certain subclasses $SC_H(b,\gamma,\lambda)$ and $\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ of harmonicMultivalent functions of complex order.We investigate coefficient conditions, extreme pointsand distortion bounds .We also examinetheir convolution and convex combination properties and the closure propertyof this class under integral operator.

An attempt isalso made in undertaking study of multivalent harmonic meromorphic functions in Section three, we have introduced a subclass$AJ_s(\alpha,\lambda, k, p)$ of multivalent harmonicmeromorphic functions defined in the exterior of the unit disk .We obtain theseveral geometric results.InSection four, we study univalent harmonic function defined by Ruscheweyh derivative in the unit disk .We also obtain several interesting properties of class $AS_{\mathcal{H}}(\lambda,\alpha,k,\gamma)$such as coefficientestimates, distortion bound, extreme points, Hadamard product and otherseveral results. We have also attempted for deriving applications of fractional calculus operator in establishing distortion theorem.

Chapter Five is fully devoted to the study of differential subordination properties of classes of univalent and multivalent functions defined .This is study strong differential subordination properties for multivalent functions defined by Ruscheweyh derivative operator. The secondsection deals with some applications of differential subordination

involving hadamard product. We obtain some subordination results for univalent functions in the open unit disk U. Section three has been fully dealt with study of certain family of multivalent functions associated with subordinationand in section four we have introduced new classes by using subordination and we have obtained coefficient estimates and properties which contains distortion and growth theorems , radius of starlikeness and radius of convexity, and other related results for subclasses $K\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)and\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$.

CHAPTER 1

BASIC DEFINITIONS AND FUNDAMENTAL RESULTS

CHAPTER ONE

BASIC DEFINITIONS AND FUNDAMENTAL RESULTS

INTRODUCTION

This chapter, included two sections. Section one presents some important definitions of analytic ,holomorphic, meromorphic, univalent and multivalent functions. We also presents the concept of neighbourhood of these functions. Some operators such as Differential, integral , Dziok- Srivastava operator and other operators which are the basis for obtaining new classes of functions, some of which were generalizations of the preceding functions. These classes were known by using these operators as well as this section contains some examples for those concepts.

Section two included some fundemential results and theorems about the functions which mentioned in the first section of the class of functions that we studies in the subsequent chapters, which defined by using these operators. This chapter represents the basis for the expansion and access to the results of subsequent chapters .

1.1 BASIC DEFINITIONS

DEFINITION (1.1.1)[1]**:**A function f of the complex variable is analytic at a point z_0 if its derivative exists not only at z_0 but at each point z in some neighborhood of z_0. It is analytic in region $\mathbb{U}$if it is analytic at every point in $\mathbb{U}$. We say that f is entire function if it is analytic at every point in complex plane $\mathbb{C}$.

DEFINITION (1.1.2)[2] **.** A function f is said to be holomorphic on U if f is complex differentiable at every point z_0 in an open set U , and f is called holomorphic at the point z_0 if it is holomorphic on some neighborhood of z_0 .

EXAMPLE (1.1.1)[2]**.** The principle branch of the complex logarithm function is holomorphic on the set $C\backslash\{z \in R: z \leq 0\}$.The root function can be defined as $\sqrt{z} = e^{\frac{1}{2}logz}$, therefore it is holomorphic .

DEFINITION (1.1.3)[2] **.** A meromorphic function is single- valued function that is analytic in all the points of its domain ,and at those singularities it most go to infinity like a polynomial. That is these exceptional points most be poles and not essential singularities .

EXAMPLE (1.1.2)[2] **.** Riemann zeta function

$\xi(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{u^{z-1}}{e^u - 1} du$,where $\xi(z) = \sum_{n=1}^\infty \frac{1}{n^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$, and $z = x + iy$, x,y$\in R$.

is meromorphic function on whole complex plane .It is reduces to the Harmonic series(which is diverges) and therefore has a singularity. In the complex plane, trivial zeros occur at -2,-4,-6,... and non-trivial zeros at $s = \sigma + it, \sigma, t \in R$.

DEFINITION (1.1.4)[2]**.**A function f analytic in the open unit disk U={z∈ ℂ: $|z| < 1\}$ is said to be univalent there, if it does not take the same value twice, that is $f(z_1) \neq f(z_2)$if $z_1 \neq z_2 \in D$. In other words, f is one-to-one (or injective) mapping of onto another domain.

EXAMPLE (1.1.3)[2] **.** Consider the application $\emptyset_a(z) = \frac{z-a}{1-\bar{a}z}$ is univalent when $|a| < 1$.

DEFINITION (1.1.5)[4]**.**Let $D \subset \mathbb{C}$ be a non-empty subset . An analytic function $f: D \to \mathbb{C}$ is said to be multivalent of order p (or p -valent) if the equation $f(z) = w$ has at most p roots in D and some w exist, in which the equation $f(z) = w$ has exactly p roots in D .that is the concepts of multivalent (or $p-$valent) functions,which are generalized univalent functions , over the open unit disk U .

DEFINITION (1.1.6)[2]**.**Let $D \subset \mathbb{C}$ be a non-empty subset . A function $f: D \to \mathbb{C}$ is called locally univalent at a point $z_0 \in D$ if it is univalent in some neighborhood of z_0 . For analytic function f, the condition $f(z_0) \neq 0$ is equivalent to local univalent at z_0 .

DEFINITION (1.1.7)[1]**.**Let $D \subset \mathbb{C}$ be a non-empty subset . A function $f: D \to \mathbb{C}$ is called conformal at a point z_0 on D if it preserves the angle between oriented curves passing through z_0 in magnitude and in sense . A function f is called conformal in the domain D , if it is conformal at each point of the domain .

EXAMPLE (1.1.4)[1] **.** The map $f(z) = e^{i\theta}z$ for some angle θ. This takes a number z and increases its argument by θ ,while keeping its magnitude fixed. In other words , f is anti-clockwise rotation around the origin by θ and dilates by $r = 1$.

DEFINITION (1.1.8)[1]**.**A domain Din the complex plane $\mathbb{C}$ is said to be convex if for every pair of points, the line segment joining them lies completely in the interior of D . In other word , $w_1, w_2 \in D$ implies $tw_1 + (1-t)w_2 \in D$ for $0 \leq t \leq 1$.

DEFINITION (1.1.9)[5]**.** Let$X$be a topological vector space over the field $\mathbb{C}$and let E be a subset of X . A point $x \in E$ is called an extreme point of E if it has no representation of the form $x = ty + (1-t)z , 0 \leq t \leq 1$ as a proper convex combination of two distinct points y and z in E .

DEFINITION (1.1.10)[5]**.** Let$X$ and E be as mentioned in Definition (1.1.10) . Then the convex hull of E is the smallest convex set containing E and the closed convex hull of E is the smallest closed convex set containing E , it is the closure of the convex hull of E , we denote the closed convex hull of E by $\overline{co}E$.

EXAMPLE (1.1.5) [1]**.** Any circular disk is a convex set .

DEFINITION (1.1.11) [3]**.**Class $\mathcal{D}$ is defined

$$\mathcal{D} = \{f \in \mathcal{H}(U): f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , z \in U\}, \qquad (1.1)$$

and is called the class of analytic functions with positive coefficients .

DEFINITION (1.1.12)[2].A function $f \in \mathcal{D}$ is said to be normalized if it satisfies the condition $f(0) = 0 = f'(0) - 1$.

DEFINITION (1.1.13)[3]**.** A function $f \in \mathcal{D}$ is said to be convex function if the image $f(D)$ is convex .

EXAMPLE (1.1.6)[3]**.** The function $f(z) = \frac{z}{1-z}$ is a covex function , since $f(D)$ is convex set . where D is the open unit disk , that is , it maps D onto a half plane .

DEFINITION (1.1.14)[7].A domain D in the complex plane $\mathbb{C}$ is said to be starlike with respect to point $w_0 \in D$ if the line segment joining w_0 to every other point $w \in D$ lies in the interior of D .That is , for any $w \in D$ implies $tw_1 + (1-t)w_2 \in D$,where $0 \leq t \leq 1$.

DEFINITION (1.1.15)[7]**.** A function $f \in \mathcal{D}$ is called starlike if the image $f(D)$ is starlike with respect to the origin .

EXAMPLE (1.1.7)[8]. The Koebe function $f(z) = \frac{z}{(1-z)^2}$ is a starlike function and the domain $k(D)$ is starlike with respect to each $w_0 > \frac{-1}{4}$.

DEFINITION (1.1.16)[3]**. .** The class $\mathcal{D}\rho$ is called the class of multivalent functions with positive coefficients in the open unit disk U and is defined as

$$\mathcal{D}\rho = \{f \in \mathcal{H}(D): f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ , p \in \mathbb{N} = \{1,2,\dots\} \ , z \in D\} \qquad (1.2)$$

DEFINITION (1.1.17)[9]**.**A function $f \in \mathcal{D}\rho$ is said to be multivalent starlike if f satisfies the following condition f

$$\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \ , z \in U \ , p \in \mathbb{N} = \{1,2 \dots\} \ , f(z) \neq 0$$

and the subclass of all multivalent starlike functions is denoted by S_p^*.

DEFINITION (1.1.18)[9]**.** A function $f \in \mathcal{D}\rho$ is said to be multivalent convex if f satisfies the following condition f

$$\mathcal{R}e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \ , z \in U \ , p \in \mathbb{N} = \{1,2 \dots\} \ , f'(z) \neq 0 \ ,$$

and the subclass of all multivalent starlike functions is denoted by $\mathcal{K}_p$.

DEFINITION (1.1.19)[9]. A function $f \in \mathcal{D}\rho$ is said to be multivalent close to convex if f satisfies the following conditionif there is a convex function g such that

$\mathcal{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0$, $z \in U$, $p \in \mathbb{N} = \{1,2 \dots\}$, $g'(z) \neq 0$,

and the subclass of all multivalent close to convex functions is denoted by $\mathcal{C}_p$.

Note that $\mathcal{K}_p \subset S_p^* \subset \mathcal{C}_p$.

DEFINITION (1.1.20)[10]**.** A function $f \in \mathcal{D}\rho$is said to be multivalent starlike of order αif f satisfies the following condition$f$

$\mathcal{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$,,$z \in U; 0 \leq \alpha < p$,$p \in \mathbb{N} = \{1,2 \dots\}$(1.3)

and the subclass of all multivalent starlike of order α is denoted by $S_p^*(\alpha)$.

EXAMPLE (1.1.8)[11]**.** If $p = 1$ The function $k(z,p) = \frac{z}{(1-z)^{2(1-\alpha)}}$ is starlike function of order α ,since it satisfies condition (1.3) , $z \in U; 0 \leq \alpha < 1$.

DEFINITION (1.1.21)[9]**.** A function $f \in \mathcal{D}\rho$is said to be multivalent convex of order α if $f$satisfies the following condition$f$

$\mathcal{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$, $z \in U; 0 \leq \alpha < p$,$p \in \mathbb{N} = \{1,2,\dots\}$,$f'(z) \neq 0$,(1.4)

and the subclass of all multivalent convex of order α is denoted by $\mathcal{K}_p(\alpha)$.

DEFINITION (1.1.22)[2]**.**A function $f \in \mathcal{D}\rho$is said to be multivalent close to convex of order α if $f$satisfies the following condition if there is a convex function g such that

$\mathcal{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha$,$z \in U; 0 \leq \alpha < p$,$p \in \mathbb{N} = \{1,2 \dots\}$, $z^{p-1} \neq 0$**,**(1.5)

and the subclass of all multivalent close to convex functions is denoted by $\mathcal{C}_p(\alpha)$ .We note that $\mathcal{K}_p(\alpha) \subset S_p^*(\alpha) \subset \mathcal{C}_p(\alpha)$Owa(1992) [2] and Jeyaraman et al.(2013) [62] .The classes $\mathcal{K}_p(\alpha)$ and $S_p^*(\alpha)$ are studied also by Owa (1992)[97], Goodman and Robertson (1950)[48] and Learch(1978) [72].we note that $S_p^*(\alpha) \subseteq S_p^*(0) \equiv S_p^*$ and $\mathcal{K}_p(\alpha) \subseteq \mathcal{K}_p(0) \equiv \mathcal{K}_p$,where S_p^* and $\mathcal{K}_p$ are denote the subclass of S_p ,consisting of functions which are multivalent starlike and convex in U ,respectively Aouf and Hossen (2000)[17] and Silverman and Owa (1992) [120]

DEFINITION (1.1.23)[2]. Radius of starlikeness of a function f is the largestr_1, $0 < r_1 < 1$, for which it is starlike in $|z| < R_1$. (1.6)**DEFINITION (1.1.24)**[2]. Radius of convexity of a function f is the largestR_2, $0 < R_2 < 1$, for which it is convex in $|z| < R_2$ (1.7)

DEFINITION (1.1.25)[13].Radius of close to convexity of a function f is the largestR_3, $0 < R_3 < 1$,for which it is close to convex in $|z| < R_3$. (1.8)

DEFINITION (1.1.26)[14].The convolution (Hadamard product) of two functionsf_i , $i = 1,2$,of the form

$$f_i(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,i} z^k \ , j = 1,2, p \in \mathbb{N} = \{1,2, \dots\} \ , z \in D$$

belonging to the class $T(j,p)$ is denoted by $f_1 * f_2$ and defined as follows :

$(f_1 * f_2)(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,1} a_{k,2} z^k$,$a_{k,i} \geq 0$,$p \in \mathbb{N} = \{1,2, \dots\}$.(1.9)

If $p = 1$, then the convolution (or Hadamard pdoduct)for f_i in $T(j,1)$

DEFINITION (1.1.27)[15].The convolution (Hadamard product) of two functionsf_i, $i = 1,2$, of the form

$f_i(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,i} z^k$, $p \in \mathbb{N} = \{1,2, \dots\}$, $z \in D$,

Belonging to the class Dp is denoted by $f_1 * f_2$and defined as

$(f_1 * f_2)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k$. (1.10)

If $p = 1$, then the convolution (or Hadamard pdoduct) for f_i .

EXAMPLE (1.1.9)[15].. If $f(z) = \sum_{k=1}^{\infty} \frac{1}{2} z^k$, $z \in D$ and $h(z) = \sum_{k=1}^{\infty} z^k = \frac{z}{1-z}$, $z \in D$ are functions in$\mathcal{D}$, the convolution of these function is $(f * h)(z) = \sum_{k=1}^{\infty} \frac{1}{2} z^k = f(z)$, $z \in D$, where D is the open unit disk .

DEFINITION (1.1.28)[15].For $\delta > 0$,a δ −neighborhood of a function $f \in S_p(\alpha)$ of the form

$$f\ (z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0 \quad (z \in U, \qquad p \in N = \{1,2, \dots.\})$$

is define by

$$N_\delta(f) = \{g: g \in S_p(\alpha): g(z) = z^p - \sum_{k=p+1}^{\infty} c_k z^k$$

and $\sum_{k=p+1}^{\infty} k|a_k - c_k| \leq \delta\}$(1.11)

It follows from (1.11) that if

$h(z) = z^p \quad (p \in N)$(1.12)

Then

$$N_\delta(h) = \{g: g \in S_p(\alpha), g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k \ and \quad \sum_{k=p+1}^{\infty} k|b_k| \leqq \delta\} \qquad (1.13)$$

DEFINITION (1.1.29)[15]**.**For $\delta > 0$, where δ – neighborhood of $f(z) \in T(j,p)$of the form

$f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k$, ($a_k \geq 0; j,p \in N = \{1,2, \ldots .\}$;

is defined by$N_\delta(f) = \{g: g \in T(j,p), g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k$

$and \quad \sum_{k=j+p}^{\infty} k|a_k - b_k| \leq \delta$ }. (1.14)

It follows from (1.15) that if

$h(z) = z^p \quad (k \geq j+p; n,p \in N; q \in N_0 = N \cup \{0\})$. (1.15)

So that ,obviously ,

$N_\delta(h) = \{g: g \in T(j,p), g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k , \sum_{k=j+p}^{\infty} |b_k| \leq \delta\}$.(1.16)

DEFINITION (1.1.30)[16]**.**A function $f \in \mathcal{D}\rho$ is said to be in the class J for which there exist another function$g \in L$ such that

$\left|\frac{f(z)}{g(z)} - 1\right| < p - \rho \quad (z \in U: 0 \leq \rho < p$).

where $J \subset D$ .This for :

(i) $D = S_p(\alpha)$; $J = \mathcal{HF}_\gamma^\lambda(\mu,p,\alpha,\beta,A,B)$; $L = \mathcal{HF}_\gamma^\lambda(p,\alpha,\beta,A,B)$

(ii) $D = S_p(\alpha)$;$J = \mathcal{HF}(\rho,\lambda,\alpha,\gamma,\beta,p,A,B)$; $L = \mathcal{HF}(\lambda,\alpha,\gamma,\beta,p,A,B)$

(iii) $D = T(j,p)$; $J = T_j(n,p,q,\alpha,\lambda,\gamma)$; $L = T_j(n,p,q,\alpha,\lambda,\gamma)$

DEFINITION (1.1.31)[15]**.**The pPochhammer symbol is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2)\ldots\ldots(a+n-1) & for\ n \in \mathbb{N} \\ 1 & for\ n=0 \end{cases} , \qquad (1.17)$$

where Γ denotes the gamma function .

DEFINITION (1.1.32)[16]**.**Let $\alpha_1, \alpha_2, \ldots\ldots, \alpha_l, \beta_1, \beta_2, \ldots\ldots, \beta_m$real or complex numbers with$\beta_j \neq 0,-1,-2, \ldots$, $j = 1,2, \ldots, m$, the generalized hypergeometric function is denoted by ${}_lF_m(z)$ and is defined by

$${}_lF_m(z) = {}_lF_m(z)(\alpha_1, \alpha_2, \ldots, \alpha_l, \beta_1, \beta_2, \ldots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots .(\alpha_l)_n z^n}{(\beta_1)_n \ldots .(\beta_m)_n n!}$$

$$= 1 + \frac{(\alpha_1)_1 \ldots (\alpha_l)_1}{(\beta_1)_1 \ldots (\beta_m)_1} z + \frac{\alpha_1(\alpha_1+1) \ldots \alpha_l(\alpha_l+1)}{\beta_1(\beta_1+1) \ldots \beta_m(\beta_m+1)} \frac{z^2}{2!} + \cdots \quad ,|z| < 1$$

; $(l \leq m+1, l,m \in N_0 = N \cup \{0\}; z \in U)$, (1.18)

where $(x)_n$ is the pochhammer symbol defined by (1.17) .

EXAMPLE (1.1.10)[16] **.** ${}_0F_0(z)(-1;-1;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.

DEFINITION (1.1.33)[17].A single-valued function of complex variable is a complex function $f\colon \mathbb{C} \to \mathbb{C}$ that has the same value at every point z_0.

DEFINITION (1.1.34)[18].A function f of a complex variable z which is single-valued in this domain and has afinite derivative at every point (analytic) is called regular .

DEFINITION (1.1.35)[19].A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain $D \subseteq C$ if both u and $v$are real harmonic in D . If $f = u + iv$ be harmonic ,then we can find the analytic functions G, H such that $u = Re\ G$ and $v = Im\ H$,thus

$$h + \bar{g} = \frac{G+H}{2} + \frac{\bar{G}-\bar{H}}{2},$$

where h and g are analytic in D,call h the analytic part and g the co-analytic part of f .

DEFINITION (1.1.36)[19].The harmonic function $f = h + \bar{g}$ is sense- preserving and locally univalent if $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$, $\forall z \in U$ .Where J_f denotes the Jacobian of f . If $f = h + \bar{g}$ is Harmonic and sense- preserving and injective , then we say that f is harmonic univalent .Let $\tau = \tau(U)$ be denote the class of analytic functions in U. For positive integer n and $a \in \mathbb{C}$, let

$\tau[a, n] = \{f \in \tau\colon f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$,with $\tau_0 = \tau[0,1]$, $\tau_1 = \tau[1,1]$.

EXAMPLE (1.1.11)[20].To show the image of U under the harmonic function $f(z) = z + \frac{\bar{z}^2}{2}$enter this function in complex tool in the form : $z + \frac{conj(z^2)}{2}$ (see the following fig.)

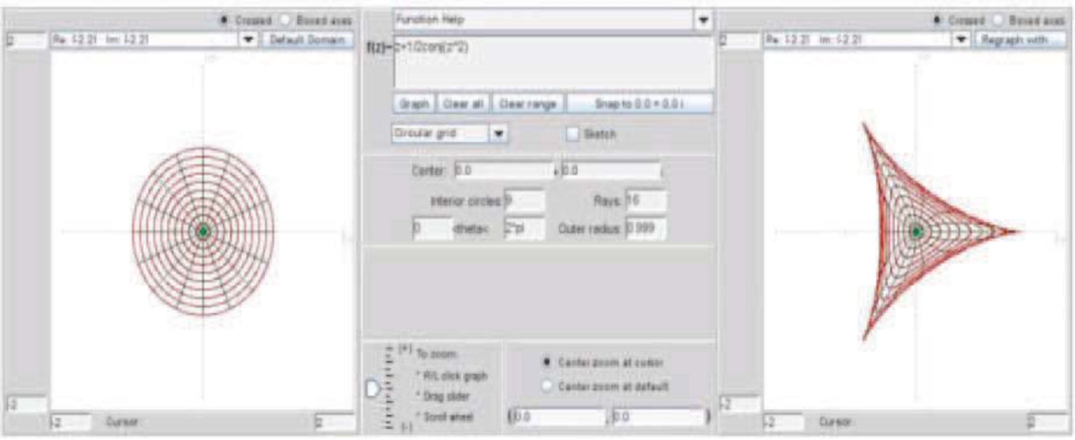

Figure (1.1.): Image of U under the harmonic function.

Note that the harmonic function $f(z) = h(z) + \overline{g(z)}$ can be written in the form

$f(z) = Re\big(h(z) + g(z)\big) + iIm\big(h(z) - g(z)\big).$

Thus for the function above it can be written as

$f(z) = Re\left(z + \frac{(z^2)}{2}\right) + iIm\left(z - \frac{(z^2)}{2}\right)$,

In complex Tool you can also enter the harmonic function in the form. where this function must be in the form $Re\left(z + \frac{(z^2)}{2}\right) + iIm\left(z - \frac{(z^2)}{2}\right)$.

DEFINITION (1.1.37)[3].Let $f$and g are two analytic functions in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$,we say that f is subordinate to g ,written by $f \prec g$ or g is superordinate to f in U if there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$, and $|w(z)| < 1\ (z \in U)$, such that $f(z) = g(w(z))$.Furthermore , if the function g is univalent in U , then we have the following equivalence $f \prec g \leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

DEFINITION (1.1.38)[3].Let $f: R \to \mathbb{C}$ be a function . We call f a Schwarz function , if for all $c \in R, n \in N_0 = N \cup \{0\}$,

$|f^n(x)| = O(|z|^c)$,

where " capital O " is defined as follows :

Let a_n and b_n be any two sequences and $b_n \geq 0$ for all n . If there exists a fixed number $m > 0$ such that $a_n \leq mb_n$(for all n) , then we write $a_n = O(b_n)$.

EXAMPLE (1.1.12)[21].Under the condition on $b(z)$in Schwarzian Lemma

$$b(z) \prec g(z) = e^{i\alpha}z$$

Suppose that $f \prec g$ and $g(U) = D$. Then the inverse g^{-1} is analytic in D and maps onto U with $g^{-1}(a_0) = 0$.Hence the composite function $b(z) = g^{-1}(f(z)) = g^{-1}(a_0) = 0$.

Thus $b(z)$ is a Schwarzian function ,and $f(z) = g(b(z))$.

DEFINITION (1.1.39)[22].Let $\Omega$and Λ be any sets in$\mathbb{C}$**,** let p be an analytic function in the open unit disk U with $p(0) = a$ and let $\psi(r, s, t; z): \mathbb{C}^3 \times U \to \mathbb{C}$.

The heart of this monograph deals with the generalizations of the following implication :satisfy the admissibility condition :

$\left\{\psi(p(z), zp'(z), z^2\, p''(z); z: z \in U\right\} \subset \Omega \Longrightarrow p(U) \subset \Lambda$(1.19)

If Λ is simple connected domain containing the point a and $\Lambda \neq \mathbb{C}$,then there is a conformal mapping p of U on to Λ such that $p(0) = a$ in this case ,(1.19) can be written

as **:**

$\left\{\psi(p(z), zp'(z), z^2\, p''(z); z: z \in U\right\} \subset \Omega \Longrightarrow p(U) \subset q(U)$.

If $\Omega$is alsosimple connected domain and $\Omega \neq \mathbb{C}$,then there is a conformal mapping h of U onto Ω such that $h(0) = \psi(a, 0,0; 0)$. If in addition, the function $\psi(p(z), zp'(z), z^2\, p''(z); z)$ is analytic in U ,then (1.19) can be written as :

$\psi(p(z), zp'(z), z^2\, p''(z); z) \prec h(z) \Longrightarrow p(z) \subset q(z)$(1.20)

<u>DEFINITION (1.1.40)</u>[22].Let$\psi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$\psi(p(z), zp'(z), z^2\, p''(z); z) \prec h(z)$(1.21)

Then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply dominant if $p \prec q$ for all p satisfying (1.21) .

A dominant $\check{q}$ that satisfies $\check{q} \prec q$ for all dominants q of (1.21) is said to be the best dominant of (1.21) .

Note that the best dominant is unique up to a relation of U .

<u>DEFINITION (1.1.41)</u>[22].Denoted by Q the set of all functions $q(z)$ that are analytic and injective on $\overline{U} \setminus E(q)$ where

$E(q) = \{\zeta \in \partial U\colon \lim_{z\rightarrow\zeta} q(z) = \infty\}$, (1.22)

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$ .Further , let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$ with $Q(1) \equiv Q_1$.

<u>DEFINITION (1.1.43)</u>[22].Let Ωbe a set in$\mathbb{C}$ **,** $q \in Q$ and let n be positive integer **.**The classofadmissible function $\Psi_n[\Omega, q]$ consist of those functions $\psi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$that satisfy the admissibility condition :

$\psi(r, s, t; z) \notin \Omega$,

whenever $u = q(\xi)$, $v = k\xi q'(\xi)$,

and$\mathcal{R}e\left\{\frac{t}{s} + 1\right\} \geq k$ $\mathcal{R}e\left\{1 + \frac{\xi q''(\xi)}{q'(\xi)}\right\}$, (1.23)

$z \in U$, $\xi \in \partial U \setminus E(q)$, and $k \geq n$. $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

In particular ,if

$q(z) = M\frac{Mz+a}{M+\bar{a}z}$,

where $M > 0$, $|a| < M$, then $q(U) = U_M = \{w\colon |w| < M\}$, $q(0) = a$, $E(q) = \phi$ and $q \in Q(a)$ .In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = U_M$,the class is simple denoted by $\Psi_n[\mathrm{M}, a]$.

DEFINITION (1.1.44)[23]**.**Let Ωbe a set in$\mathbb{C}$ **,** $q \in Q$ and$n$ be positive integer **.**The classofadmissible function $\Psi_n[\Omega, q]$ consists of those function $\psi: \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$that satisfies the admissibility condition :

$\psi(r, s, t; z, \xi) \notin \Omega$**,**

whenever

$r = q(\xi) \quad , s = k\xi q'(\xi)$,

and$\mathcal{R}e\left\{\frac{t}{s} + 1\right\} \geq k \ \mathcal{R}e\left\{1 + \frac{\xi q''(\xi)}{q'(\xi)}\right\}$, (1.24)

$z \in U$, $\xi \in \partial U \setminus E(q), \xi \in \bar{U}$,and $k \geq n$. $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

DEFINITION (1.1.45)[23]**.**Let Ωbe a set in$\mathbb{C}$ **,** $q \in \tau[a, n]$ and$n$ be positive integer **.**The classofadmissible function $\Psi'_n[\Omega, q]$ consists of those functions $\psi: \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$that satisfies the admissibility condition :

$\psi(r, s, t; \xi, \zeta) \notin \Omega$**,**

whenever

$r = q(z) \quad , s = \frac{1}{m} z q'(z)$,

and $\mathcal{R}e\left\{\frac{t}{s} + 1\right\} \geq \frac{1}{m} \mathcal{R}e\left\{1 + \frac{z q''(z)}{q'(z)}\right\}$, (1.25)

$z \in U$, $\xi \in \partial U, \zeta \in \bar{U}$,and $m \geq n$ .Assume $\Psi'_1[\Omega, q] = \Psi'[\Omega, q]$.

DEFINITION (1.1.46)[24]**.**The Bernardi integral operator is denoted by $J_{c,p}$ and defined as

$\left(J_{c,p} f\right)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \quad (f \in S_p(\alpha)\ ; c > -p; p \in N)$(1.26)

DEFINITION (1.1.47)[25].The fractional calculus operator is denoted by D_z^{μ} and defined as

$D_z^{\mu}(z^{\eta}) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\mu+1)} z^{\eta-\mu} \quad (\eta > -1; \mu \in R)$.(1.27)

DEFINITION (1.1.48)[26].For every $f \in S$, the convolutionoperator $W_{a,b,c}(f)(z)$can be defined below :

$$W_{a,b,c}(f)(z) = {}_2F_1(a, b, c; z) * f(z) = z - \sum_{n=2}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} a_n z^n$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric function given by (1.18) when

$\alpha_1 = a, \alpha_2 = b\ , \beta_1 = c \neq 0, -1, \dots, j = m = 1$.

1.2 FUNDAMENTAL RESULTS

The following lemmas and theorems will used to proves of the our results in the next chapters .

LEMMA (1.2.1)[27]**.**Let $\alpha \geq 0$. Then $\mathcal{R}e(w) > \alpha$ if and only if $|w-(1+\alpha)| < |w+(1-\alpha)|$, where w be any complex number .

LEMMA (1.2.2)[28].Let $\alpha \geq 0$. Then $\mathcal{R}e(w) > \alpha$ if and only if

$|w-1| < |w+(1-2\alpha)|$,where w be any complex number .

LEMMA (1.2.3)(*Schwarz Lemma*)[3]

Let f be analytic in the open unit disk U with $f(0) = 0$ and $|f(z)| < 1$ in U . Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ in U . Strict inequality holds in both estimates unless f is a rotation of the disk $f(z) = ze^{i\theta}$.

THEOREM (1.2.1)(*Distortion Theorem*)[3]

For each $f(z) \in \boldsymbol{\mathcal{D}}$, then

$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$,$|z| = r < 1$(1.28)

For each $z \in U\ , z \neq 0$ equality occurs if and only if f is a suitable rotation of the Koebe function .

We say upper and lower bounds for $|f'(z)|$ as distortion bounds .

THEOREM (1.2.2)(*Growth Theorem*) [3]

For each $f(z) \in \boldsymbol{\mathcal{D}}$,then

$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$,$|z| = r < 1$(1.29)

For each $z \in U\ , z \neq 0$ equality occurs if and only if f is a suitable rotation of the Koebe function .

THEOREM (1.2.3).(*Bieberbach conjecture*)[3]

The coefficients of each $f(z) \in \boldsymbol{\mathcal{D}}$ satisfy $|a_n| \leq n$for$n = 2,3, \dots$ .The

Strict inequality holds for all n unless f is the koebe function or one of its rotation .

THEOREM (1.2.4)(*Littlewoods Theorem*)[3]

For the constant e , the coefficients of each function $f(z) \in \boldsymbol{\mathcal{D}}$ satisfy $|a_n| \leq en$for$n = 2,3, \dots$.

THEOREM (1.2.5)(*Alexander's Theorem*)[3]

Let f be an analytic function in U , with $f(0) = f'(0) - 1 = 0$ .Then $f \in \mathcal{K}$ if and only if $zf' \in S^*$.

THEOREM (1.2.6)*(Maximum Modules Theorem*)[3]

Suppose that a function f is continuous on boundary of U (U any disk or region) .Then ,the maximum value of $|f(z)|$,which is always reached , occurs somewhere on the boundary of U and never in the interior .

LEMMA (1.2.4)[23]. Let $\psi \in \Psi_k[\Omega, q]$with $q(0) = a$. If $p \in \tau[a, n]$ satisfies

$\psi(p(z), zp'(z), z^2\, p''(z); z, \zeta) \in \Omega$, Then $p(z) \prec q(z)$.

LEMMA (1.2.5)[23]. Let $\psi \in \Psi_k[\Omega, q]$with $q(0) = a$. If $p \in Q(a)$ and

$\psi(p(z), zp'(z), z^2\, p''(z); z, \zeta)$ is univalent in U for $\zeta \in \overline{U}$,then

$\Omega \subset \{\psi(p(z), zp'(z), z^2\, p''(z); z, \zeta): z \in U, \zeta \in \overline{U}\}$, (1.30)

implies to $q(z) \prec p(z)$.

LEMMA (1.2.6)[23].Let $\psi \in \Psi'_k[\Omega, q]$with $q(0) = a$. If $p \in Q(a)$ and

$\psi(p(z), zp'(z), z^2\, p''(z); z, \zeta)$ is univalent in U for $\zeta \in \overline{U}$,then

$\Omega \subset \{\psi(p(z), zp'(z), z^2\, p''(z); z, \zeta): z \in U, \zeta \in \overline{U}\}$.

Implies to $q(z) \prec p(z)$.

THEOREM (1.2.7)[3]. Let $f, g \in \mathcal{H}(D)$,and suppose that g is univalent in U,then the subordination $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

THEOREM (1.2.8)[29].If the functions $f(z)$ and $g(z)$ are analytic in U with $f(z) \prec g(z)$, then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mathcal{S}} d\theta \le \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\mathcal{S}} d\theta$$

Where $\mathcal{S} \ge 0, z = re^{i\theta}$ and $0 < r < 1$.

CHAPTER 2

SOME RESULTS MEROMORPHIC FUNCTIONS

2.1 INTRODUCTION

Chapter two is devoted to the study of some properties of certain subclasses of univalent , meromorphic univalent and multivalent functions defined by subordination property with some operators.

This chapter consists of four sections. In section one ,we have introduced and studied some geometric properties of a certain subclass $S_n(p\,,q;,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ of multivalent functions defined by differential subordination property of the form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k (a_k \geq 0);$$

and satisfying the subordination condition :

$$\left| \frac{A\left\{\frac{f^{(q+2)}(z)}{\delta(p-2,q)z^{p-q-2}} - p(p-1)\right\}}{B\left\{\frac{f^{(q+2)}(z)}{\delta(p-2,q)z^{p-q-2}} - p(p-1)\right\} + \lambda(1-\alpha)} \right| < (l-\beta);$$

where $0 < B \leq 1\,, A > 0\,, \lambda > 0\,, 0 \leq \alpha < 1\,, 0 < \beta < l < 1, p \in \mathbb{N}$ and $p > q$.

We obtain coefficient inequalities, distortion theorems, radius of convexity, closure theorems and modified Hadamard products for functions in this class.

In section two , we have discussed some interesting properties of anew subclasses of univalent functions defined in the open unit disc involving Dziok Srivastava operator of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n\, z^n, (a_n \geq 0, n \in \mathbb{N}),$$

and satisfying the condition:

$$\left| \frac{\frac{zF_\lambda^{q+1}}{F_\lambda^q} - 1}{(B-A)\gamma\left[\frac{zF_\lambda^{q+1}}{F_\lambda^q} - \alpha\right] - B\left[\frac{zF_\lambda^{q+1}}{F_\lambda^q} - 1\right]} \right| < \beta, \quad z \in U$$

where

$$\frac{z^{q+1}F_{\lambda}^{q+1}(z)}{z^q F_{\lambda}^q(z)} = \frac{z^{q+1}\mathcal{H}f^{q+1}(z) + \lambda z^{q+2}\mathcal{H}f^{q+2}(z)}{(1-\lambda)z^q\mathcal{H}f^q(z) + \lambda z^{q+1}\mathcal{H}f^{q+1}(z)} \qquad 0 \le \lambda \le 1$$

For $0 \le \lambda \le 1$, $0 < \beta \le 1$, $-1 \le \mathrm{B} < A \le 1$, $0 \le \gamma \le 1$ and $\mathcal{H}f(z) = z^p + \sum_{n=p+1}^{\infty} \Gamma n a_n z^n$

We obtain some properties , such as , on modified Hadamard product ,Holder inequalities and closure properties underintegral transforms are discussed.

In section three , we have studied on certain new subclass of function with fixed second positive coefficients in the open unit disk., we have introduced the subclass $ZTw(k, \beta)$of univalent functionsof the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \, , (a_n \ge 0, n \in \mathbb{N}),$$

and satisfying the condition:

$$\left|\frac{(z-w)(I^k f(z))''}{(I^k f(z))'} + 2\right| < \left|\frac{(z-w)(I^k f(z))''}{(I^k f(z))'} + 2\beta\right|$$

$(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$, for some $\beta(0 \le \beta < 1)$

We obtain some properties , such as , coefficient estimates,distortion and covering theorem, radii of starlikeness and convexity and convex linear combination.

2.1 ON A CLASS OF ANALYTIC MULIVALENT FUNCTIONS INVOLVING HIGHER – ORDER DERIVATIVES

Let $\mathcal{D}_p$ denoted the class of analytic functions:

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \; ; (p, n \in \mathrm{N} = \{1, 2, 3, \dots\}) \qquad (2.1)$$

are p−valent in unit disc $U = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$. Let $T_p(n)$denote the subclass of $\mathcal{D}_p$ of the form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k (a_k \geq 0) \qquad (2.2)$$

We note that $T_p(1) = T_p$.

For all $f(z) \in \mathcal{D}\rho$, we have

$f^{(m)}(z) = \delta(p,m) z^{p-m} + \sum_{k=p+n}^{\infty} \delta(k,m) a_k z^{k-m}$ (2.3)

where

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} i(i-1)(i-2)\ldots\ldots(i-j+1) & j \neq 0 \\ 1 & j = 0 \end{cases} \qquad (2.4)$$

DEFINITION 2.1.1. Let $S_n(p\,,q;\,,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$be the subclass of $\mathcal{D}\rho$ consisting of functions $f(z)$ of the form (2.1) , and satisfying the analytic criterion:

$$\left| \frac{A\left\{\frac{f^{(q+2)}(z)}{\delta(p-2,q)z^{p-q-2}} - p(p-1)\right\}}{B\left\{\frac{f^{(q+2)}(z)}{\delta(p-2,q)z^{p-q-2}} - p(p-1)\right\} + \lambda(1-\alpha)} \right| < (l-\beta); \qquad (2.5)$$

where $0 < B \leq 1\,, A \geq 0\,, \lambda > 0$, $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $p > q$. And $T_n^*(p,q,A,B,\lambda,\alpha,l,\beta) = T_p(n) \cap S_n(p,q,A,B,\lambda,\alpha,l,\beta)$.

We assume that $0 < \beta \leq 1\,, 0 \leq \alpha < p, n \in \mathbb{N}, q \in \mathbb{N}_0, p > q$ and $\delta(i,j)(i > j)$ is defined by (2.4).

THEOREM 2.1.1. A function $f(z)$ of the form (2.2) is in the class $T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ if and only if

$$\sum_{k=p+n}^{\infty} [A + B(l-\beta)]k(k-1)\delta(k-2,q)a_k \leq \lambda(l-\beta)(1-\alpha)\delta(p-2,q) \qquad (2.6)$$

PROOF. Assume that the inequality (2.6) holds true , then

$$\begin{aligned} &\left|A\left\{f^{(q+2)}(z) - p(p-1)\delta(p-2,q)\, z^{p-q-2}\right\}\right| \\ &\quad - (l-\beta)\left|B\left\{f^{(q+2)}(z) - p(p-1)\delta(p-2,q)z^{p-q-2}\right\}\right. \\ &\quad \left. + \lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\right| \end{aligned}$$

$= \left|A\left\{\delta(p,q+2)\, z^{p-q-2} + \sum_{k=p+n}^{\infty} \delta(k,q+2) a_k z^{k-q-2} - p(p-1)\,\delta(p-2,q)\, z^{p-q-2}\right\}\right| - (l-\beta)\left|B\left\{\delta(p,q+2) z^{p-q-2} + \sum_{k=p+n}^{\infty} \delta(k,q+2)\, a_k\, z^{k-q-2} - p(p-1)\delta(p-2,q) z^{p-q-2}\right\} + \lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\right|$

We have $\delta(p,q+2) = p(p-1)\delta(p-2,q)$ then

$= \left|A\left\{p(p-1)\delta(p-2,q)z^{p-q-2} + \sum_{k=p+n}^{\infty} k(k-1,q)\delta(p-2,q)a_k z^{k-q-2} - p(p-1)\delta(p-2,q)z^{p-q-2}\right\}\right| - (l-\beta)\left|B\left\{p(p-1)\delta(p-2,q)z^{p-q-2} + \sum_{k=p+n}^{\infty} k(k-1)\delta(k-2,q)a_k z^{k-q-2} - p(p-1)\delta(p-2,q)z^{p-q-2}\right\} + \lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\right|$

then

$= \quad \left|A\sum_{k=p+n}^{\infty} k(k-1)\delta(k-2,q)a_k z^{k-q-2}\right| - (l-\beta)\left|B\sum_{k=p+n}^{\infty} k(k-1)\delta(k-2,q)a_k z^{k-q-2} + \lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\right|$

$$\leq \sum_{k=p+n}^{\infty} [A+B(l-\beta)](k(k-1)\delta(k-2,q)a_k|z|^{k-q-2} - \lambda(1-\alpha)(l-\beta)\delta(p-2,q)|z|^{k-q-2} \leq 0.$$

Conversely , assume that$f(z) \in T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ thus

$$\left|\frac{A\left[f^{(q+2)}(z)-p(p-1)\delta(p-2,q)z^{p-q-2}\right]}{B\left\{f^{(q+2)}(z)-p(p-1)\delta(p-2,q)z^{p-q-2}\right\}+\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}}\right|$$

$$= \left|\frac{\begin{array}{c}A\left[\delta(p,q+2)z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)a_k z^{k-q-2}\right]\\ -p(p-1)\delta(p-2,q)z^{p-q-2}\end{array}}{\begin{array}{c}B\left[\delta(p,q+2)z^{p-q-2}+\sum_{k=p+n}^{\infty}\delta(k,q+2)a_k z^{k-q-2}-p(p-1)\delta(p-2,q)z^{p-q-2}\right]\\ +\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\end{array}}\right| < l-\beta$$

We have $\delta(p,q+2) = p(p-1)\delta(p-2,q)$ then

$$\left|\frac{A\sum_{k=p+n}^{\infty} k(k-1)\delta(k-2,q)a_k z^{k-q-2}}{\begin{array}{c}B\sum_{k=p+n}^{\infty} k(k-1)\delta(k-2,q)a_k z^{k-q-2}\\ +\lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}\end{array}}\right| < l-\beta$$

Since$\mathcal{R}e(z) \leq |z|$ for all z , we have

$$\mathcal{R}e\left[\frac{A\sum_{k=p+n}^{\infty} k(k-1)\delta(k-2,q)a_k z^{k-q-2}}{B\sum_{k=p+n}^{\infty} k(k-1)\delta(k-2,q)a_k z^{k-q-2} + \lambda(1-\alpha)\delta(p-2,q)\, z^{p-q-2}}\right] < l-\beta \quad (2.7)$$

letting $z \to -1$ through real values , we obtain the desired result .

COROLLARY 2.1.1. Let the function $f(z)$ defined by (2.2) be in the class$T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ then

$$a_k \leq \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}(k \geq n+p\,, n \in \mathbb{N}) \quad (2.8)$$

The result is sharp for the function .

$$f(z) = z^p - \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}z^k (k \geq n+p\,, n \in \mathbb{N}) \qquad (2.9)$$

THEOREM 2.1.2. let the function $f(z)$ defined by (2.2) be in the class$T_n^*(p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$.Then for $|z| = r < 1$we have

$$\left\{\delta(\mathrm{p,m}) - \frac{\lambda(\mathrm{l}-\beta)(1-\alpha)\delta(p-2,q)\delta(\mathrm{p+n,m})}{[\mathrm{A}+\mathrm{B}(\mathrm{l}-\beta)]\delta(\mathrm{p+n,q+2})}\right\}\mathrm{r}^{\mathrm{p-m}} \leq \left|f^{(m)}(z)\right|$$
$$\leq \left\{\delta(\mathrm{p,m}) + \frac{\lambda(\mathrm{l}-\beta)(1-\alpha)\delta(p-2,q)\delta(\mathrm{p+n,m})}{[\mathrm{A}+\mathrm{B}(\mathrm{l}-\beta)]\delta(\mathrm{p+n,q+2})}\right\}\mathrm{r}^{\mathrm{p-m}} \qquad (2.10)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\lambda(\mathrm{l}-\beta)(1-\alpha)\delta(p-2,q)}{[\mathrm{A}+\mathrm{B}(\mathrm{l}-\beta)]\delta(\mathrm{p+n,q+2})}z^{p+n} \qquad (2.11)$$

PROOF . By *Theorem* 2.1.1 , we have

$$[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)\sum_{k=p+n}^{\infty} a_k$$
$$\leq \sum_{k=p+n}^{\infty}[A+B(l-\beta)]k(k-1)\delta(k-2,q)a_k$$
$$\leq \lambda(l-\beta)(1-\alpha)\delta(p-2,q) \tag{2.12}$$

That is

$$\sum_{k=p+n}^{\infty} a_k \leq \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)} \tag{2.13}$$
$$= \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]\delta(p+n,q+2)}$$

From (2.3) and (2.13) , we have

$$\left|f^{(m)}(z)\right| \geq \left\{\delta(p,m)r^{p-m}+r^{p+n-m}\delta(p+n,m)\sum_{k=p+n}^{\infty} a_k\right\}$$
$$\geq \left\{\delta(p,m)r^{p-m}+r^{p+n-m}\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)\delta(p+n,m)}{[A+B(l-\beta)]\delta(p+n,q+2)}\right\}$$
$$= \left\{\delta(p,m)+\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)\delta(p+n,m)}{[A+B(l-\beta)]\delta(p+n,q+2)}r^n\right\}r^{p-m} \tag{2.14}$$

and

$$\left|f^{(m)}(z)\right| \leq \left\{\delta(p,m)r^{p-m}+r^{p+n-m}\delta(p+n,m)\sum_{k=p+n}^{\infty} a_k\right\}$$
$$\leq \left\{\delta(p,m)r^{p-m}-r^{p+n-m}\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)\delta(p+n,m)}{[A+B(l-\beta)]\delta(p+n,q+2)}\right\}$$
$$= \left\{\delta(p,m)-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)\delta(p+n,m)}{[A+B(l-\beta)]\delta(p+n,q+2)}r^n\right\}r^{p-m} \tag{2.15}$$

Putting m=0 in *Theorem* 2.1.2 , we have the following corollary

COROLLARY 2.1.2. Let the function $f(z)$ defined by (2.2) be in the class $T_n^*(p,q;A,B,\lambda,\alpha,l,\beta)$.Then $|z|=r<1$ we have

$$|f(z)| \geq \left\{1-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[\mathrm{A}+\mathrm{B}(l-\beta)]\delta(\mathrm{p}+\mathrm{n},\mathrm{q}+2)}r^n\right\}r^p \tag{2.16}$$

and

$$|f(z)| \leq \left\{1+\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[\mathrm{A}+\mathrm{B}(l-\beta)]\delta(\mathrm{p}+\mathrm{n},\mathrm{q}+2)}r^n\right\}r^p \qquad (2.17)$$

The result is sharp.

Putting m = 1 in *Theorem* 2.1.2, we have the following *corollary*

COROLLARY 2.1.3. Let the function $f(z)$ defined by (2.1) be in the class $T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$.Then $|z| = r < 1$ we have

$$|f'(z)| \geq \left\{p-\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[\mathrm{A}+\mathrm{B}(l-\beta)]\delta(\mathrm{p}+\mathrm{n}-1,\mathrm{q}+1)}\mathrm{r}^{\mathrm{n}}\right\}\mathrm{r}^{\mathrm{p}-1}$$

and

$$|f'(z)| \leq \left\{p+\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[\mathrm{A}+\mathrm{B}(l-\beta)]\delta(\mathrm{p}+\mathrm{n}-1,\mathrm{q}+1)}\mathrm{r}^{\mathrm{n}}\right\}\mathrm{r}^{\mathrm{p}-1}$$

The result is sharp

THEOREM 2.1.3. Let the function $f(z)$ defined by (2.2) be in the class $T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$.Then $f(z)$ is $p-$ valent close– to–convex of order η $(0 \leq \eta < p)$ in $|z| \leq R_1$,where

$$R_3 = inf\left\{\frac{[A+B(l-\beta)](k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right\}^{\frac{1}{k-p}} \quad (k \geq n+p,p,n \in N) \qquad (2.18)$$

PROOF: we must show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq p-\eta \quad \text{for } |z| \leq R_1$$

where R_3 is given by (2.18). Indeed we find from (2.2) that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq \sum_{k=p+n}^{\infty} ka_k|z|^{k-p}$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq p-\eta$$

If

$$\sum_{k=p+n}^{\infty}\left(\frac{k}{p-\eta}\right)a_k|z|^{k-p} \leq 1 \qquad (2.19)$$

But by using *theorem* 2.1.1 , (2.19) will be true if

$$\left(\frac{k}{p-\eta}\right)|z|^{k-p} \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)$$

Then

$$|z| \leq \left\{\frac{[A+B(l-\beta)](k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right\}^{\frac{1}{k-p}} \tag{2.20}$$

THEOREM 2.1.4. Let the function $f(z)$ defined by (2.2) be in the class $T_n^*(p,q,A,B,\lambda,\alpha,l,\beta)$. Then $f(z)$ is $p-valent$ starlike of order η $(0 \leq \eta < p)$ in $|z| \leq R_1$where

$$R_1 = inf_{k\geq n+p}\left\{\frac{[A+B(l-\beta)](k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)(k-\eta)}\right\}^{\frac{1}{k-p}} \tag{2.21}$$

The result is sharp the extremal function

PROOF. We must show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \eta \quad \text{for} \quad |z| \leq R_1, \tag{2.22}$$

where $R_1$is given by (2.21).

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{k=n+p}^{\infty}(k-p)a_k|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}a_k|z|^{k-p}}$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \eta$$

If

$$\sum_{k=n+p}^{\infty}\left(\frac{k-\eta}{p-\eta}\right)a_k|z|^{k-p} \leq 1 \tag{2.23}$$

But by using *Theorem* 2.1.1 , (2.23) will be true if

$$\left(\frac{k-\eta}{p-\eta}\right)|z|^{k-p} \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right)$$

then

$$|z| \leq \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)(k-\eta)}\right)^{\frac{1}{k-p}} (k \geq n+p, n \in \mathbb{N}) \tag{2.24}$$

COROLLARY2.1.4 Let the function $f(z)$ defined by (2.2) be in the class $T_n^*(p,q,A,B,\lambda,\alpha,l,\beta)$.Then $f(z)$is in p-valent convex of order η (0 ≤ η < p) in $|z| \leq R_2$, where

$$R_2 = inf_{k\geq n+p}\left\{\frac{[A+B(l-\beta)]p(p-1)\delta(k-2,q)(p-\eta)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)(k-\eta)}\right\}^{\frac{1}{k-p}}$$

The result is sharp .

THEOREM 2.1.5. Let $\mu_j \geq 0\ for\ j = 1,2,\ldots.,m\ and\ \sum_{j=1}^{m}\mu_j \leq 1$, if function $f_j(z)$defined by

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \left(a_{k,j} \geq 0, j = 1,2, \dots, m\right) \qquad (2.25)$$

are in the class $T_n^*(p, q; A\,, B\,, \lambda\,, \alpha\,, l, \beta)$ for every $j = 1,2, \dots, m$,then the function $f(z)$defined by

$$f(z) = z^p - \sum_{k=p+n}^{\infty} \left(\sum_{j=1}^{m} (\mu_j a_{k,j}) \right) z^k \qquad (2.26)$$

is also in the class $T_n^*(p, q; A\,, B\,, \lambda\,, \alpha\,, l, \beta)$.

PROOF : Since $f_j(z)$ is in the class $T_n^*(p, q, A\,, B\,, \lambda\,, \alpha\,, l, \beta)$, then by*Theorem* 2.1.1 that

$$\sum_{k=p+n}^{\infty} [A + B(l - \beta)] k(k-1) \delta(k-2, q) a_{k,j} \leq \lambda(l - \beta)(1 - \alpha)\delta(p - 2, q) \qquad (2.27)$$

for every j=1,2,… m .Hence

$$\sum_{k=p+n}^{\infty} [A + B(l - \beta)] k(k-1) \delta(k-2, q) \left(\sum_{j=1}^{\infty} \mu_j a_{k,j} \right)$$

$$= \sum_{j=1}^{\infty} M_j \left(\sum_{k=p+n}^{\infty} [A + B(l - \beta)] k(k-1) \delta(k-2, q) a_{k,j} \right)$$

$$\leq \sum_{k=p+n}^{\infty} [A + B(l - \beta)] k(k-1) \delta(k-2, q) a_{k,j} \sum_{j=1}^{\infty} \mu_j = \lambda(l - \beta)(1 - \alpha)\delta(p - 2, q)$$

From *Theorem* 2.1.1 , it follows that $f(z) \epsilon\, T_n^*(p, q; A\,, B\,, \lambda\,, \alpha\,, l, \beta)$.

COROLLARY 2.1.5. The class $T_n^*(p, q, A\,, B\,, \lambda\,, \alpha\,, l, \beta)$ is closed under convex linear combination

PROOF : Let the function $f_j(z)(j = 1,2)$ be given by (2.27) be in the class $T_n^*(p, q, A\,, B\,, \lambda\,, \alpha\,, l, \beta)$. It is sufficient to show that the function $f(z)$ defined by

$$f(z) = \mu f_1(z) + (1 - \mu) f_2(z)$$

is in the class $T_n^*(p, q, A\,, B\,, \lambda\,, \alpha\,, l, \beta)$ .But , taking m=2 , $c_1 = \mu\,, c_2 = 1 - \mu$ in Theorem 2.1.5 , We have the corollary.

THEOREM 2.1.6. Let

$$f_{p+n-1}(z) = z^p$$

and

$$f_k(z) = z^p - \frac{\lambda(l - \beta)(1 - \alpha)\delta(p - 2, q)}{[A + B(l - \beta)] k(k-1) \delta(k-2, q)} z^k, k \geq P + n \qquad (2.28)$$

Then $f(z)$ is in the class $T_n^*(p, q;, A\,, B\,, \lambda\,, \alpha\,, l, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=p+n-1}^{\infty} \mu_k f_k(z) \qquad (2.29)$$

where $\mu_k \geq 0 \; and \; \sum_{k=p+n-1}^{\infty} \mu_k = 1$

PROOF :Assume that

$$f(z) = \sum_{k=p+n-1}^{\infty} \mu_k f_k(z)$$

$$= \mu_{p+n-1} f_{p+n-1+} \sum_{k=p+n}^{\infty} \mu_k f_k(z) \qquad (2.30)$$

$$= z^p - \sum_{k=p+n}^{\infty} \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \mu_k z^k \qquad (2.31)$$

$$= \mu_{p+n-1} z^p + \sum_{k=p+n}^{\infty} \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \mu_k z^k$$

Then it follows that

$$\sum_{k=p+n}^{\infty} \left(\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)} \right) \left(\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \mu_k z^k \right)$$

$$\leq \sum_{k=p+n}^{\infty} \mu_k = \left(1 - \mu_{p+n-1}\right) \leq 1.$$

Hence by *Theorem* 2.1.1 , we have $f(z) \in T_n^*(p,q;,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$.

Conversely , assume that the function $f(z)$defined by (2.2) belongs to the class $T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$, then

$$a_k \leq \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} z^k$$

Setting

$$\mu_k = \frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)} a_k,$$

Where

$\mu_{p+n-1} = 1 - \sum_{k=p+n}^{\infty} \mu_k.$

We can see that $f(z)$ can be express in the form (2.30).

COROLLARY 2.1.6. The extreme point of the class $T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)$ are the function $f_p(z) = z^p$ and

$$f_k(z) = z^p - \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} z^k, k \geq p+n$$

Let the function $f_j(z)(j=1,2)$ defined by (2.25) the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,1}a_{k,2}z^k = (f_1 * f_2)(z) \qquad (2.32)$$

THEOREM 2.1.7. Let the function $f_j(z)(j=1,2)$ defined by (2.25) be in the class $T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)\,, Then\ (f_1 * f_2)(z)$ be in the class $T_n^*(p,q,A\,,B\,,\lambda\,,\sigma\,,l,\beta)$, where

$$\sigma = 1 - \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+1)(p+n-1)\delta(p+n-2,q)}\ . \qquad (2.33)$$

The result is sharp for the function $f_j(z)$(j = 1, 2) defined by

$$f_j(z) = z^p + \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)](p+1)(p+n-1)\delta(p+n-2,q)} z^{p+n}\ . \qquad (2.34)$$

PROOF :We need to find the largest σ such that

$$\sum_{k=p+n}^{\infty} \frac{[A+B(l-\beta)]k(k-1))\delta(k-1,q)}{\lambda(l-\beta)(1-\sigma)\delta(p-2,q)} a_{k,1}a_{k,2} \leq 1 \qquad (2.35)$$

We have $f_j(z) \in T_n^*(p,q,A\,,B\,,\lambda\,,\alpha\,,l,\beta)\ (j=1,2)$ then

$$\sum_{k=p+n}^{\infty} \frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)} a_{k,1} \leq 1 \qquad (2.36)$$

and

$$\sum_{k=p+n}^{\infty} \frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)} a_{k,2} \leq 1 \qquad (2.37)$$

By using Cauchy Scharz inequality , we have

$$\sum_{k=p+n}^{\infty} \frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)} \sqrt{a_{k,1}a_{k,2}} \leq 1 \qquad (2.38)$$

It is sufficient to show that

$$\frac{1}{1-\sigma} a_{k,1}a_{k,2} \leq \frac{1}{(1-\alpha)} \sqrt{a_{k,1}a_{k,2}} \qquad (2.39)$$

or

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \qquad (2.40)$$

Hence in night of the inequality (2.40), it is sufficient to prove that

$$\frac{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \leq \frac{(1-\sigma)}{(1-\alpha)} (k \geq p+n) \qquad (2.41)$$

From (2.41) we have

$$\sigma \leq 1 - \frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \tag{2.42}$$

In the next , we defined the function $R(k)$by

$$R(k) = 1 - \frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \tag{2.43}$$

We note that $R(k)$ is an increasing function of k (k$\geq p+n$), therefore

$$\sigma \leq R(p+n) = 1 - \frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n-2,q)} \tag{2.44}$$

Putting β = 1 in *Theorem* 2.1.7, we obtain the following corollary.

<u>COROLLARY 2.1.7</u>.Let the functions $f_j(z)(j=1{,}2)$defined by (2.2) be in the class $\mathrm{T}_n^*(\mathrm{p},\mathrm{q};\mathrm{A}\,,\mathrm{B}\,,\lambda\,,\alpha,l)$Then $(f_1 * f_2)(z) \in \mathrm{T}_n^*(\mathrm{p},\mathrm{q};\mathrm{A}\,,\mathrm{B}\,,\lambda\,,\sigma\,,l)$ where

$$\sigma = 1 - \frac{\lambda(l-1)\delta(p-2,q)(1-\alpha)^2}{[A+B(l-1)](p+n)(p+n-1)\delta(p+n-2,q)}$$

Then result is sharp.

<u>COROLLARY 2.1.8</u>. For $f_1(z) and\ f_2(z)$ as in *Theorem* 2.1.7 , the function

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1}a_{k,2}}\, z^k$$

belongs to the class $T_n^*(p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$.

This result follows from the *Cauchy-Schwarz* inequality (2.28). It is sharp for the same functions as in *Theorem* 2.1.7.

<u>THEOREM 2.1.8</u>. Let the functions $f_j(z)(j=1{,}2)$ defined by (2.26) be in the class$T_n^*(p,q;A\,,B\,,\lambda\,,\alpha\,,l,\beta)$. Then the function

$h(z) = z^p - \sum_{k=p+n}^{\infty} \sqrt{\left({a^2}_{k,1+}{a^2}_{k,2}\right)}\, z^k$(2.45)

belongs to the class $T_n^*(p,q;A\,,B\,,\lambda\,,\varsigma\,,l,\beta), where$

$$\zeta \leq 1 - \frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \tag{2.46}$$

The result is sharp for the functions $f_j(z)(j=1{,}2)$ defined by (2.34)

<u>PROOF</u>. By *Theorem* 2.1.1, we obtain

$$\sum_{k=p+n}^{\infty}\left[\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)} a_{\ k,1}\right]^2 \leq 1 \text{(2.47)}$$

and

$$\sum_{k=p+n}^{\infty}\left[\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}a_{k,2}\right]^2 \leq 1 \quad (2.48)$$

It follows from (2.47) and (2.48) that

$$\sum_{k=p+n}^{\infty}\frac{1}{2}\left[\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right]^2\left(a^2{}_{k,1}+a^2{}_{k,2}\right) \leq 1. \quad (2.49)$$

Therefore, we need to find the largest ζ such that

$$\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)\delta(p-2,q)(1-\zeta)} \leq \frac{1}{2}\left[\frac{[A+B(l-\beta)]k(k-1)\delta(k-2,q)}{\lambda(l-\beta)(1-\alpha)\delta(p-2,q)}\right]^2, \quad (2.50)$$

that is, that

$$\zeta \leq 1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)]k(k-1)\delta(k-2,q)}, \quad (2.51)$$

since

$$D(k)=1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)]k(k-1)\delta(k-2,q)} \quad (2.52)$$

is an increasing function of $k(k \geq p+n)$, we readily have

$$\zeta \leq \mathrm{D(p+n)}=1-\frac{\lambda(l-\beta)\delta(p-2,q)(1-\alpha)^2}{2[A+B(l-\beta)](p+n)(p+n-1)\delta(p+n,q+2)}$$

2.2 A GENERALIZATION OF A SUBCLASS OF MULTIVALENT FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA LINEAR OPERATOR

Let $\mathcal{D}\rho$ denoted the class of analytic functions:

$$f(z) = z^p + \sum_{n=n+p}^{\infty} a_k z^k \,; (p, n \in \mathrm{N} = \{1, 2, 3, \ldots\})$$

which are analytic in the open unit disc $\mathrm{U} = \{\{\mathrm{z} : \mathrm{z} \in \mathbb{C} \text{ and } |\mathrm{z}| < 1\}$. Let$\mathcal{S}$ denote the class of all functions in $\mathcal{D}\rho$ which are normalized by $f(0) = 0 = f'(0) - 1$ and+ multivalent in $\mathcal{U}$.For given two functions $f \ and \ g \in \mathcal{D}\rho$where

$$f(z) = z^p + \textstyle\sum_{n=p+1}^{\infty} a_n z^n \text{ and } \quad g(z) = z^p + \textstyle\sum_{n=P+1}^{\infty} b_n z^n \,.$$

The Hadamard product $f(z) * g(z)$is defined

$f(z) * g(z) = (f * g)(z) = z^p + \sum_{n=P+1}^{\infty} a_n b_n z^n$ (2.53)

Also denote by $T_p(n)$the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, a_n \geq 0 \,, z \in \mathcal{U}$$

studied extensively by [30].

For positive real values of $a_1, \ldots., a_l$ and $\beta_1 \ldots.. \beta_m \,(\beta_j \neq 0, -1, \ldots.; j = 1,2, \ldots, m)$ the generalized hypergeometric function ${}_lF_m(a_1 \ldots \ldots. a_l \,;\, \beta_1 \ldots.. \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\mathrm{z}) = {}_lF_m(a_1 \ldots \ldots. a_l \,;\, \beta_1 \ldots.. \beta_m; z) := \textstyle\sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots\ldots (\alpha_l)_n z^n}{(\beta_1)_n \ldots\ldots (\beta_m)_n n!} \quad (2.54)$$

$(l \leq m + 1; m \in \mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}; z \in U)$,

Where $\boldsymbol{\mathcal{D}}$ denotes the set of all positive integers and $(\lambda)_k$is the Pochhammer symbol defined by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{Bmatrix} 1, & n = 0 \\ \lambda(\lambda+1)(\lambda+2) \ldots.. (\lambda + \mathrm{b} - 1), & \mathrm{n} \in \mathrm{N} \end{Bmatrix} \quad (2.55)$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial,the Bessel, the Laguerre polynomial.

Let $\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{D}\rho \rightarrow \mathcal{D}\rho$ be a linear operator defined by

$$\mathcal{H}(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) f(z) \coloneqq z \;{}_lF_m(a_1 \ldots \ldots. a_l \,;\, \beta_1 \ldots.. \beta_m; z) * f(z)$$

$$= z^p + \sum_{n=p+1}^{\infty} \Gamma n a_n z^n \qquad (2.56)$$

where

$$\Gamma n = \frac{(a_1)_{n-p} \ldots \ldots (a_l)_{n-p}}{(\beta_1)_{n-p} \ldots \ldots (\beta_m)_{n-p}} \frac{1}{(n-p)!} \qquad (2.57)$$

Fornotationalsimplicity,wecanuseashorter notation $H_m^l[a_1 \ldots\ldots a_l\,;\ \beta_1 \ldots\ldots \beta_m]$in the sequel. The linear operator $H_m^l[a_1]$is called Dziok-Srivastava operator, various other linear operators introduced and studied by[31],[32] and[33].Motivatedbyearlierworksof[34]and[33]. Wedefinethefollowingnew subclass of $T_p(n)$ involvinghypergeometricfunctions.

DEFINITION 2.2.1, A function $f \in T_p(n)$ is said to be in the class $HF_\gamma^\lambda(\mathrm{p},\mathrm{q},\alpha\,,\beta,A,B)$the analyticcondition.

$$\left|\frac{\frac{zF_\lambda^{q+1}}{F_\lambda^q}-1}{(B-A)\gamma\left[\frac{zF_\lambda^{q+1}}{F_\lambda^q}-\alpha\right]-B\left[\frac{zF_\lambda^{q+1}}{F_\lambda^q}-1\right]}\right|<\beta,\quad z\in U \qquad (2.58)$$

Where

$$\frac{z^{q+1}F_\lambda^{q+1}(z)}{z^qF_\lambda^q(z)}=\frac{z^{q+1}\mathcal{H}f^{q+1}(z)+\lambda z^{q+2}\mathcal{H}f^{q+2}(z)}{(1-\lambda)z^q\mathcal{H}f^q(z)+\lambda z^{q+1}\mathcal{H}f^{q+1}(z)}\qquad 0\le\lambda\le1$$

For $0 \le \lambda \le 1\,, 0 < \beta \le 1\,, -1 \le \mathrm{B} < A \le 1\,, 0 \le \gamma \le 1$and

$$\mathcal{H}f(z)=z^p+\sum_{n=p+1}^{\infty}\Gamma_n a_n z^n \qquad (2.59)$$

Where $\Gamma_n$is given by (2.57).

In our present investigation, we discuss some interesting properties of functions $f(z) \in HF_\gamma^\lambda(\mathrm{p},\mathrm{q},\alpha\,,\beta,A,B)$ based on convolution. Further we discuss certain closure properties under integral transformation.

thefollowingTheoremweobtainnecessaryandsufficientconditionsforfunctions $f(z) \in HF_\gamma^\lambda(\mathrm{p},\mathrm{q},\alpha\,,\beta,A,B)$.

THEOREM 2.2.1. A function $f(z)$of the form (2.52) is in the class $HF_\gamma^\lambda(\alpha\,,\beta,A,B)$ if and only if

$$\sum_{n=p+1}^{\infty}c_n a_n \le \frac{p!\,(\lambda p-\lambda+1)[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]}{(p-q)!} \qquad (2.60)$$

where

$$c_n=\frac{n!\,(\lambda n-\lambda+1)[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(n-q-\alpha)]\Gamma_n}{(n-q)!} \qquad (2.61)$$

andΓ_n is defined by (2.57) .

PROOF. For $|z|$ = 1 ,we have

$$\left|z^{q+1}F_\lambda^{q+1}(z)-z^qF_\lambda^q(z)\right| -\boldsymbol{\beta}\left|(B-A)\,\gamma\{z^{q+1}F_\lambda^{q+1}(z)-\alpha z^qF_\lambda^q(z)\}-B\left\{z^{q+1}F_\lambda^{q+1}(z)-z^qF_\lambda^q(z)\right\}\right|$$

$$
\begin{aligned}
&= |(1-\lambda)z^{q+1}Hf^{q+1}(z) + \lambda z^{q+2}Hf^{q+2}(z) - z^{q}Hf^{q}(z)| \\
&\qquad - \beta|(B-A)\gamma\{(1-\lambda\alpha)z^{q+1}Hf^{q+1}(z) + \lambda z^{q+2}Hf^{q+2}(z) \\
&\qquad - \alpha z^{q}Hf^{q}(z)\} \\
&\qquad - B\{(1-\lambda)z^{q+1}Hf^{q+1}(z) + \lambda z^{q+2}Hf^{q+2}(z) - z^{q}Hf^{q}(z)\}|
\end{aligned}
$$

$$= \left|\frac{(p-q-1)(\lambda p-\lambda+1)p!}{(p-q)(p-q-1)!}z^{p} + \sum_{n=p+1}^{\infty}\frac{(n-q-1)(\lambda n-\lambda+1)n!}{(p-q)(p-q-1)!}\Gamma n a_n z^n\right| - \beta\left|\frac{(\lambda p-\lambda+1)p!}{(p-q)(p-q-1)!}\{\gamma(B-A)(p-q-\alpha) - B(p-q-1)\} + \sum_{n=p+1}^{\infty}\frac{(\lambda n-\lambda+1)n!}{(p-q)(p-q-1)!}\Gamma n a_n z^n\{\gamma(B-A)(n-q-\alpha) - B(n-q-1)\}\right|$$

$$
\begin{aligned}
&\le \frac{(p-q-1)(\lambda p-\lambda+1)p!}{(p-q)!}|z|^{p} \\
&\qquad + \sum_{n=p+1}^{\infty}\frac{(n-q-1)(\lambda n-\lambda+1)n!}{(p-q)!}\Gamma n a_n|z|^n \\
&\qquad - \beta\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\{\gamma(B-A)(p-q-\alpha) - B(p-q-1)\}|z|^{p} \\
&\qquad + \sum_{n=p+1}^{\infty}\frac{\beta n!\,(\lambda n-\lambda+1)}{(p-q)!}\{\gamma(B-A)(n-q-\alpha) \\
&\qquad - B(n-q-1)\}\Gamma n a_n|z|^n
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=p+1}^{\infty}\frac{n!\,(\lambda n-\lambda+1)}{(n-q)!}[(n-q-1) + \beta\gamma(B-A)(n-q-\alpha) - \beta B(n-q-1)]\Gamma n a_n \\
&\qquad - \frac{p!\,(\lambda p-\lambda+1)}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha) + \beta B(p-q-1) - (p-q-1)]
\end{aligned}
$$

$$= \sum_{n=p+1}^{\infty}\frac{n!(\lambda n-\lambda+1)}{(n-q)!}[(n-q-1)(1-\beta B) + \beta\gamma(B-A)(n-q-\alpha)]\Gamma n a_n - \frac{p!(\lambda p-\lambda+1)}{(p-q)!}\{\beta\gamma(B-A)(p-q-\alpha) - (p-q-1)(1-\beta B)\}$$

≤ 0 , by hypothesis.Thus by maximum modulus Theorem $f \in HF_{\gamma}^{\lambda}(p, q, \alpha, \beta, A, B)$.

On the other hand suppose that

$$\left|\frac{\frac{zF_{\lambda}^{q+1}(z)}{F_{\lambda}^{q}(z)} - 1}{(B-A)\gamma\left[\frac{zF_{\lambda}^{q+1}(z)}{F_{\lambda}^{q}(z)} - \alpha\right] - B\left[\frac{zF_{\lambda}^{q+1}(z)}{F_{\lambda}^{q}(z)} - 1\right]}\right|$$

$$= \left|\frac{\dfrac{\frac{(\lambda p-\lambda+1)p!}{(p-q-1)!}\left(\frac{p-q-1}{p-q}\right)z^{p}\frac{(\lambda p-\lambda+1)p!}{(p-q-1)!}\left(\frac{p-q-1}{p-q}\right)z^{p}}{z^{q}F_{\lambda}^{q}(z)}}{\dfrac{\gamma(B-A)\left\{\frac{(\lambda p-\lambda+1)p!}{(p-q-1)!}\left(\frac{p-q-\alpha}{p-q}\right)z^{p} + \sum_{n=p+1}^{\infty}\frac{(\lambda n-\lambda+1)n!}{(n-q-1)!}\left(\frac{p-q-\alpha}{p-q}\right)\Gamma n a_n z^n\right\} - B\left\{\frac{(\lambda p-\lambda+1)p!}{(p-q-1)!}\left(\frac{p-q-1}{p-q}\right)z^{p} + \sum_{n=p+1}^{\infty}\frac{(\lambda n-\lambda+1)n!}{(n-q-1)!}\left(\frac{p-q-1}{p-q}\right)\Gamma n a_n z^n\right\}}{z^{q}F_{\lambda}^{q}(z)}}\right| < \beta$$

Since $\mathcal{R}e(z) < |z|$ for all z , we have

$$\mathcal{R}e\left\{\frac{\frac{(\lambda p-\lambda+1)(p-q-1)p!}{(p-q)!}|z|^p+\sum_{n=p+1}^{\infty}\frac{(\lambda n-\lambda+1)(n-q-1)n!}{(n-q)!}\Gamma n a_n|z|^n}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\gamma(B-A)(p-q-\alpha)^p-B(p-q-1)|z| \\ +\sum_{n=2}^{\infty}\frac{(\lambda n-\lambda+1)n!}{(n-q)!}(\gamma(B-A)(n-q-\alpha)-B(n-q-1)\Gamma n a_n|z|^n}\right\}$$

$$<\beta$$

the value of z on the real axis so that $f(z)$is real and letting $z\to\ -1$, we obtain

$$\sum_{n=p+1}^{\infty}\frac{n!\,(\lambda n-\lambda+1)}{(n-q)!}\begin{bmatrix}(n-q-1)(1-\beta B)\\+\beta\gamma(B-A)(n-q-\alpha)\end{bmatrix}\Gamma_n a_n \le \frac{p!\,(\lambda p-\lambda+1)[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]}{(p-q)!}$$

and hence the proof is complete.

Putting $q=1$, in the above Theorem , to obtain

COROLLARY 2.2.1 A function $f(z)\in HF_{\gamma}^{\lambda}(p,q,\alpha\,,\beta,A,B)$ if and only if

$$\sum_{n=p+1}^{\infty}c_n a_n \le (1-\alpha)(B-A)\beta_{\gamma}$$

Where$c_n=(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(1-\alpha)]\Gamma_n$

We note that the result obtained by [35].

In the following theorem, using the techniques[36] we discuss some convolution properties for functions $f(z)\in HF_{\gamma}^{\lambda}(\mathrm{p},\mathrm{q},\alpha\,,\beta,A,B)$.

Let the function $f_j(z)(j=1{,}2)$ be defined by

$$f_j(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,j}z^n\big(a_{n,j}\ge 0\,;j=1{,}2\big) \tag{2.62}$$

then the modified Hadamard product of $f_1(z)$ *and* $f_2(z)$ is given by

$$(f_1*f_2)(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,1}a_{n,2}z^n \tag{2.63}$$

THEOREM 2.2.2 Let the function $f_1(z)$defined by (2.62) be in the class $HF_{\gamma}^{\lambda}(\mathrm{p},\mathrm{q},\xi 1,\beta,\mathrm{A},\mathrm{B})$and the function $f_2(z)$ defined by (2.62) be in the class HF_{γ}^{λ} (p,q,ξ_1, β, A, B). If the sequence $\{Cn\}$ is non-decreasing then $(f_1*f_2)(z)\in HF_{\gamma}^{\lambda}(p,q,\alpha^*\,,\beta,A,B)$ where

$$\alpha^*\le\left(\frac{\Lambda(\beta,\gamma,\xi_1,\mathrm{n})\Lambda(\beta,\gamma,\xi_2,\mathrm{n})(\lambda p-\lambda+1)\{\beta\gamma(B-A)(p-q)-(p-q-1)(1-\beta B)\}\Gamma_n}{\beta\gamma(B-A)\Lambda(\beta,\gamma,\xi_1,\mathrm{n})\Lambda(\beta,\gamma,\xi_2,\mathrm{n})(\lambda n-\lambda+1)\Gamma_n}\right.$$

$$\left.\frac{-(\psi_1)(\psi_2)\frac{(\lambda p-\lambda+1)p!n!}{(p-q)!(n-q)!}\{(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q)\}}{-\frac{(\lambda p-\lambda+1)p!n!}{(p-q)!(n-q)!}(\psi_1)(\psi_2)\beta\gamma(B-A)}\right) \tag{2.64}$$

PROOF: In view of *Theorem*(2.2.1) it is enough to show that

$$\sum_{n=p+1}^{\infty} \frac{(\lambda n-\lambda+1)\frac{n!}{(n-q)!}\begin{Bmatrix} n(n-q-1)(1-\beta B) \\ +\beta\gamma(B-A)(n-q-\alpha^*)\end{Bmatrix}}{(\lambda p-\lambda+1)\frac{p!}{(p-q)!}\begin{Bmatrix}\beta\gamma(B-A)(p-q-\alpha^*) \\ -(p-q-1)(1-\beta B)\end{Bmatrix}}\Gamma_n a_{n,1}a_{n,2} \leq 1 \quad (2.65)$$

Where α^* is defined by (2.62)

Since $f_1 \in HF_\gamma^\lambda$ (p,q,ξ_1, β, A, B) we have

$$\sum_{n=p+1}^{\infty} \frac{(\lambda n-\lambda+1)\Lambda(\beta,\gamma,\xi_1,\mathrm{n})}{(\lambda p-\lambda+1)\frac{p!}{(p-q)!}\begin{Bmatrix}\beta\gamma(B-A)(p-q-\xi_1) \\ -(p-q-1)(1-\beta B)\end{Bmatrix}}\Gamma_n a_{n,1} \leq 1 \qquad (2.66)$$

and for $f_2 \in HF_\gamma^\lambda$ (p.q,ξ_2, β, A, B) we have

$$\sum_{n=p+1}^{\infty} \frac{(\lambda n-\lambda+1)\Lambda(\beta,\gamma,\xi_2,\mathrm{n})}{(\lambda p-\lambda+1)\frac{p!}{(p-q)!}\left\{\begin{Bmatrix}\beta\gamma(B-A)(p-q-\xi_2) \\ -(p-q-1)(1-\beta B)\end{Bmatrix}\right\}}\Gamma_n a_{n,2} \leq 1 \qquad (2.67)$$

Where

$$\Lambda(\beta,\gamma,\xi_1,\mathrm{n}) = \left[\frac{n!}{(n-q)!}\{(n-q-1)(1-\beta B)+\beta\gamma(B-A)(n-q-\xi_1)\}\right]$$

$$\Lambda(\beta,\gamma,\xi_2,\mathrm{n}) = \left[\frac{n!}{(n-q)!}\{(n-q-1)(1-\beta B)+\beta\gamma(B-A)(n-q-\xi_2)\}\right]$$

$$\psi_1 = \{\beta\gamma(B-A)(p-q-\xi_1)-(p-q-1)(1-\beta B)\}$$

$$\psi_2 = \{\beta\gamma(B-A)(p-q-\xi_2)-(p-q-1)(1-\beta B)\}$$

On the other hand, under the hypothesis and by the Cauchy's-Schwarz inequality that

$$\sum_{n=p+1}^{\infty} \frac{[\Lambda(\beta,\gamma,\xi_1,\mathrm{n})]^{1/2}[\Lambda(\beta,\gamma,\xi_2,\mathrm{n})]^{1/2}}{\sqrt{(\psi_1)(\psi_2)}}(\lambda n-\lambda+1)\Gamma_n\sqrt{a_{n,1}a_{n,2}} \leq 1 \quad (2.68)$$

From (2.68) and (2.69), it follows that

$$\sum_{n=p+1}^{\infty} \frac{(\lambda n-\lambda+1)^2\Lambda(\beta,\gamma,\xi_1,\mathrm{n})\Gamma_\mathrm{n}\Lambda(\beta,\gamma,\xi_2,\mathrm{n})\Gamma_\mathrm{n}}{\left[(\lambda p-\lambda+1)\frac{p!}{(p-q)}\right]^2(\psi_1)(\psi_2)} a_{n,1} a_{n,2} \leq 1 \qquad (2.69)$$

Now we have to find largest α^* such that,

$$\sum_{n=p+1}^{\infty} \frac{(\lambda n-\lambda+1)\frac{n!}{(n-q)!}\begin{Bmatrix} (n-q-1)(1-\beta B) \\ +\beta\gamma(B-A)(p-q-\alpha^*)\end{Bmatrix}}{(\lambda p-\lambda+1)\frac{p!}{(p-q)!}\begin{Bmatrix}\beta\gamma(B-A)(p-q-\alpha^*) \\ -(p-q-1)(1-\beta B)\end{Bmatrix}}\Gamma_n a_{n,1}a_{n,2}$$

$$\leq \sum_{n=p+1}^{\infty} \frac{(\lambda n-\lambda+1)[\Lambda(\beta,\gamma,\xi_1,\mathrm{n})]^{1/2}[\Lambda(\beta,\gamma,\xi_2,\mathrm{n})]^{1/2}}{\left[(\lambda p-\lambda+1)\frac{p!}{(p-q)!}\right]\sqrt{(\psi_1)(\psi_2)}}\Gamma_n\sqrt{a_{n,1}a_{n,2}}$$

or, equivalently that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{\left(\frac{\{\beta\gamma(B-A)(p-q-\alpha^*)-(p-q-1)(1-\beta B1)\}}{[\Lambda(\beta,\gamma,\xi_1,\mathrm{n})]^{1/2}[\Lambda(\beta,\gamma,\xi_2,\mathrm{n})]^{1/2}}\right)}{\left(\sqrt{(\psi_1)(\psi_2)}\Lambda(\beta,\gamma,\alpha^*,\mathrm{n})\right)}, (n\geq 2)$$

where

$$\Lambda(\beta,\gamma,\alpha^*,\mathrm{n}) = \frac{n!}{(n-q)!}\{(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\alpha^*)\}$$

In view of (2.96) it is sufficient to find largest α^*such that

$$\frac{\left[(\lambda p-\lambda+1)\frac{p!}{(p-q)!}\right]\sqrt{(\psi_1)(\psi_2)}}{(\lambda n-\lambda+1)[\Lambda(\beta,\gamma,\xi_1,\mathrm{n})]^{1/2}[\Lambda(\beta,\gamma,\xi_2,\mathrm{n})]^{1/2}\Gamma_n}$$

$$\leq \frac{\{\beta\gamma(B-A)(p-q-\alpha^*)-(p-q-1)(1-\beta B)\}}{\sqrt{(\psi_1)(\psi_2)}}\frac{[\Lambda(\beta,\gamma,\xi_1,\mathrm{n})]^{1/2}[\Lambda(\beta,\gamma,\xi_2,\mathrm{n})]^{1/2}}{\Lambda(\beta,\gamma,\alpha^*,\mathrm{n})}$$

$$\alpha^* \leq \left(\frac{\Lambda(\beta,\gamma,\xi_1,\mathrm{n})\Lambda(\beta,\gamma,\xi_2,\mathrm{n})(\lambda p-\lambda+1)\{\beta\gamma(B-A)(p-q)-(p-q-1)(1-\beta B)\}\Gamma_n}{\beta\gamma(B-A)\Lambda(\beta,\gamma,\xi_1,\mathrm{n})\Lambda(\beta,\gamma,\xi_2,\mathrm{n})(\lambda n-\lambda+1)\Gamma_n}\right.$$

$$\left.\frac{-(\psi_1)(\psi_2)\frac{(\lambda p-\lambda+1)p!n!}{(p-q)!(n-q)!}\{(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q)\}}{-\frac{(\lambda p-\lambda+1)p!n!}{(p-q)!(n-q)!}(\psi_1)(\psi_2)\beta\gamma(B-A)}\right)$$

THEOREM 2.2.3 Let the function $f_j(z)(j=1,2)$ defined by (2.64) be in the class $\mathrm{HF}_\gamma^\lambda(\mathrm{p,q},\alpha,\beta,\mathrm{A},\mathrm{B})$. If the sequence {Cn} is non-decreasing. Then the function

$$h(z) = z^p - \sum_{n=2}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n \tag{2.70}$$

belongs to the class $HF_\gamma^\lambda(\mathrm{p,q},\alpha,\beta,A,B)$ where

$$\delta \leq \frac{\left\{\begin{array}{l}\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\{\beta\gamma(B-A)(p-q)-(p-q-1)(1-\beta B)\} \\ -\frac{2(\lambda p-\lambda+1)p!}{(p-q)!}\{(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q)\}\Gamma_n\end{array}\right\}}{\left\{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\beta\gamma(B-A)-\frac{2(\lambda p-\lambda+1)p!}{(p-q)!}\beta\gamma(B-A)\Gamma_n\right\}}$$

PROOF : By virtue of Theorem (2.2.1), it is sufficient prove that

$$\sum_{n=p+1}^{\infty}\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\begin{bmatrix}(n-q-1)(1-\beta B)\\+\beta\gamma(B-A)(p-q-\delta)\end{bmatrix}\Gamma_n}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\begin{bmatrix}\beta\gamma(B-A)(p-q-\delta)\\-(p-q-1)(1-\beta B)\end{bmatrix}}(a_{n,1}^2+a_{n,2}^2)\leq 1 \tag{2.71}$$

Since $f_j(z)(j=1,2) \in HF_\gamma^\lambda(\mathrm{p,q},\alpha,\beta,A,B)$ we have

$$\sum_{n=p+1}^{\infty}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\alpha)]\Gamma_n}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]}\right)^2 a_{n,1}^2$$

$$\leq \sum_{n=p+1}^{\infty}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\alpha)]\Gamma_n a_{n,1}}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]}\right)^2 \leq 1 \quad (2.72)$$

and

$$\sum_{n=p+1}^{\infty}\left(\frac{\frac{(\lambda n-\lambda+1)p!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\alpha)]\Gamma_n}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]}\right)^2 a_{n,2}^2$$

$$\leq \sum_{n=p+1}^{\infty}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\alpha)]\Gamma_n a_{n,2}}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]}\right)^2 \leq 1 \quad (2.73)$$

It follows from (2.72) and (2.73) that

$$\sum_{n=p+1}^{\infty}\frac{1}{2}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!]}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\alpha)]\Gamma_n}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]}\right)^2 \left(a_{n,1+}^2 a_{n,2}^2\right) \leq 1 \quad (2.74)$$

Therefore we need to find the largest δ, such that

$$\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\delta)]\Gamma_n}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\delta)-(p-q-1)(1-\beta B)]}$$

$$\leq \frac{1}{2}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B])+\beta\gamma(B-A)(p-q-\alpha)]\Gamma_n}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]}\right)^2 n \geq 2$$

$$\delta \leq \frac{\left\{\begin{array}{c}\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\{\beta\gamma(B-A)(p-q)-(p-q-1)(1-\beta B)\} \\ -\frac{2(\lambda p-\lambda+1)p!}{(p-q)!}\{(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q)\}\Gamma_n\end{array}\right\}}{\left\{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\beta\gamma(B-A)-\frac{2(\lambda p-\lambda+1)p!}{(p-q)!}\beta\gamma(B-A)\Gamma_n\right\}}$$

Recently, [37]have studied some results of Holder-type inequalities for a subclass of uniformly starlike functions. Now, we recall the generalization of the convolution [38] as given below,

$$\mathcal{H}_m(z) = z - \sum_{n=p+1}^{\infty}\left(\prod_{j=1}^{m} a_{n,j}^{Pj}\right) z^n \left(p_j > 0, j = 1,2,\dots,m\right) \quad (2.75)$$

Further for functions $f_j(z) \in HF_\gamma^\lambda(\mathrm{p}, \mathrm{q}, \alpha, \beta, A, B)$ (j = 1, 2, ..., m) given by (2.63), the familiar Holder inequality assumes the following form

$$\sum_{n=p+1}^{\infty}\left(\prod_{j=1}^{m} a_{n,j}\right) \leq \sum_{n=p+1}^{\infty}\left(\prod_{j=1}^{m} a_{n,j}^{Pj}\right)^{\frac{1}{p_j}} \quad (p_j > 1, j = 1,2, \dots, m: \sum_{j=1}^{m}\frac{1}{p_j} \geq 1) \qquad (2.76)$$

THEOREM2.2.12 If $f_j(z) \in HF_\gamma^\lambda(p, q, \xi_j, \beta, A, B), -1 \leq B < A \leq 1, 0 < \beta \leq 1$ „ $0 \leq \lambda \leq 1$, j = 1, 2, ..., m then $\mathcal{H}_m(z) \in HF_\gamma^\lambda(p, q, \xi, \beta, A, B)$ with

$$\begin{aligned}\xi \leq \Bigg\{ & \prod_{j=1}^{m}\left(\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\right)^{p_j}\Big[(n-q-1)(1\\ & -\beta B)+\beta\gamma(B-A)(p-q-\xi_j)\Gamma_n\Big]^{p_j}\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\{[\beta\gamma(B-A)(p-q)]\\ & -[(p-q-1)(1-\beta B)]\}\\ & -\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\big[\beta\gamma(B-A)(p-q-\xi_j)\right.\\ & \left.-(p-q-1)(1-\beta B)\big]\right]\{(n-q-1)(1-\beta B)-\beta\gamma(B-A)(p-q)\}\Bigg\}\\ & /\Bigg\{\prod_{j=1}^{m}\left(\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\right)^{p_j}\Big[(n-q-1)(1-\beta B)\\ & +\beta\gamma(B-A)(p-q-\xi_j)\Gamma_n\Big]^{p_j}\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\beta\gamma(B-A)\\ & -\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\prod_{j=1}^{m}\left[\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\big[\beta\gamma(B-A)(p-q-\xi_j)\right.\\ & \left.-(p-q-1)(1-\beta B)\big]\right]^{p_j}\beta\gamma(B-A)\Bigg\}\end{aligned}$$

where

(S=$\sum_{j=1}^{m} p_j \geq 1;\ p_j \geq \frac{1}{q_j}(j = 1,2 \dots, m), q_j > 1(j = 1,2, \dots, m): \sum_{j=1}^{m}\frac{1}{q_j} \geq 1$).

PROOF : Let $f_j(z) \in HF_\gamma^\lambda(\mathrm{p, q}, \xi_j, \beta, A, B)(j = 1,2 \dots, m)$Then we have

$$\sum_{n=p+1}^{\infty}\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\xi_j)\right]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)(p-q-\xi_j)-(p-q-1)(1-\beta B)\right]}\Gamma_n a_{n,j} \leq 1$$

which in turn, implies that

$$\left(\sum_{n=p+1}^{\infty}\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\xi_j)\right]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)(p-q-\xi_j)-(p-q-1)(1-\beta B)\right]}\Gamma_n a_{n,j}\right)^{\frac{1}{q_j}} \leq 1$$

$$(q_j > 1\ \ )(j = 1,2, \dots, m): \sum_{j=1}^{m}\frac{1}{q_j} = 1).$$

Applying the Holder inequality (2.76), we arrive at the following inequality

$$\sum_{n=p+1}^{\infty}\left[\sum_{j=1}^{m}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)\left(p-q-\xi_j\right)\right]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)\left(p-q-\xi_j\right)-(p-q-1)(1-\beta B)\right]}\Gamma_n\right)^{\frac{1}{q_j}} a_{n,j}^{\frac{1}{q_j}}\right]\leq 1$$

Thus, we have to determine the largest ξ such that

$$\sum_{n=p+1}^{\infty}\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\xi)]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\xi)-(p-q-1)(1-\beta B)]}\Gamma_n\left(\prod_{j=1}^{m}a_{n,j}^{Pj}\right)\leq 1$$

That is

$$\sum_{n=p+1}^{\infty}\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\xi)]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\xi)-(p-q-1)(1-\beta B)]}\Gamma_n\left(\prod_{j=1}^{m}a_{n,j}^{Pj}\right)$$
$$\leq\sum_{n=p+1}^{\infty}\left[\sum_{j=1}^{m}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)\left(p-q-\xi_j\right)\right]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)\left(p-q-\xi_j\right)-(p-q-1)(1-\beta B)\right]}\Gamma_n\right)^{\frac{1}{q_j}} a_{n,j}^{\frac{1}{q_j}}\right]$$

Therefore, we need to find the largest ξ such that

$$\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\xi)]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\xi)-(p-q-1)(1-\beta B)]}\Gamma_n\left(\prod_{j=1}^{m}a_{n,j}^{Pj}\right)$$
$$\leq\prod_{j=1}^{m}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)\left(p-q-\xi_j\right)\right]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)\left(p-q-\xi_j\right)-(p-q-1)(1-\beta B)\right]}\Gamma_n\right)^{p_j-\frac{1}{q_j}} a_{n,j}^{p_j-\frac{1}{q_j}}$$

Since ,

$$\prod_{j=1}^{m}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)\left(p-q-\xi_j\right)\right]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)\left(p-q-\xi_j\right)-(p-q-1)(1-\beta B)\right]}\Gamma_n\right)^{p_j-\frac{1}{q_j}} a_{n,j}^{p_j-\frac{1}{q_j}}\leq 1,\left(p_j-\frac{1}{q_j}\right.$$
$$\left.\geq 0, j=1,2,\dots,m\right)$$

We see that,

$$\prod_{j=1}^{m}a_{n,j}^{p_j-\frac{1}{q_j}}\leq\frac{1}{\prod_{j=1}^{m}\left(\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[\begin{array}{c}(n-q-1)(1-\beta B)\\+\beta\gamma(B-A)\left(p-q-\xi_j\right)\end{array}\right]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\begin{array}{c}\beta\gamma(B-A)\left(p-q-\xi_j\right)\\-(p-q-1)(1-\beta B)\end{array}\right]}\Gamma_n\right)^{p_j-\frac{1}{q_j}}}\qquad(2.77)$$

This last inequality implies that

$$\frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\xi)]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\xi)-(p-q-1)(1-\beta B)]}\Gamma_n$$
$$\leq\frac{\prod_{j=1}^{m}\left(\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\right)^{p_j}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)\left(p-q-\xi_j\right)\Gamma_n\right]^{p_j}}{\prod_{j=1}^{m}\left[\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)\left(p-q-\xi_j\right)-(p-q-1)(1-\beta B)\right]\right]^{p_j}}$$

which implies

$$\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\xi)]\prod_{j=1}^{m}\left[\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\xi_j)-(p-q-1)(1-\beta B)]\right]^{p_j}$$

$$\leq\prod_{j=1}^{m}\left(\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\right)^{p_j}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\xi_j)\Gamma_n]^{p_j}\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\xi)-(p-q-1)(1-\beta B)]$$

$$\xi\leq\left\{\prod_{j=1}^{m}\left(\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\right)^{p_j}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)\left(p-q-\xi_j\right)\Gamma_n\right]^{p_j}\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\{[\beta\gamma(B-A)(p-q)]-[(p-q-1)(1-\beta B)]\}-\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\left[\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)\left(p-q-\xi_j\right)-(p-q-1)(1-\beta B)\right]\right]\{(n-q-1)(1-\beta B)-\beta\gamma(B-A)(p-q)\}\right\}$$

$$/\left\{\prod_{j=1}^{m}\left(\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\right)^{p_j}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)\left(p-q-\xi_j\right)\Gamma_n\right]^{p_j}\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\beta\gamma(B-A)-\frac{(\lambda n-\lambda+1)n!}{(n-q)!}\prod_{j=1}^{m}\left[\frac{(\lambda p-\lambda+1)p!}{(p-q)!}\left[\beta\gamma(B-A)\left(p-q-\xi_j\right)-(p-q-1)(1-\beta B)\right]\right]^{p_j}\beta\gamma(B-A)\right\}$$

Integral transform of the class $HF_\gamma^\lambda(\mathrm{p},\mathrm{q},\alpha,\beta,A,B)$. For f ∈ S, we define the integral transform

$$V_\lambda(f)(z)=\int_0^1\lambda(t)\frac{f(tz)}{t}dt$$

where λ is a real value, non-negative weight function normalized so that

$$\int_0^1\lambda(t)dt=1.$$

Since special cases of λ(t) are particularly interesting such as

$$\lambda(t)=(1+c)^{t^c},\ \mathrm{c}>-1,$$

for which V_λ is known as[24], and

$$\lambda(t)=\frac{(1+c)^\delta}{\Gamma(\delta)}t^c\left(log\frac{1}{t}\right)^{\delta-1}\quad \mathrm{c}\ >\ -1,\delta\geq 0$$

which gives the [39].

THEOREM 2.2.5. Let $f(z) \in HF_{\gamma}^{\lambda}(\mathrm{p},\mathrm{q},\alpha,\beta,A,B)$ then $V_{\lambda}((f)(z)) \in HF_{\gamma}^{\lambda}(\mathrm{p},\mathrm{q},\alpha,\beta,A,B)$.

PROOF: By definition, we have

$$V_{\lambda}(f)(z) = \frac{(-1)^{\delta-1}(1+c)^{\delta}}{\Gamma(\delta)} \lim_{r \to o^{+}} \left[\int_{r}^{1} t^{c} \left(log \frac{1}{t} \right)^{\delta-1} \left[z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right] dt \right]$$

By simple computation, we get

$$V_{\lambda}(f)(z) = z - \sum\nolimits_{n=p+1}^{\infty} \left(\frac{c+1}{c+n} \right)^{\delta} a_n z^n$$

We need to prove that $V_{\lambda}((f)(z)) \in HF_{\gamma}^{\lambda}(\mathrm{p},\mathrm{q},\alpha,\beta,A,B)$, it is enough to prove

$$\sum_{n=p+1}^{\infty} \frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\alpha)]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]} \left(\frac{c+1}{c+n} \right)^{\delta} \Gamma_n a_n \leq 1 \tag{2.78}$$

on the other hand by Theorem (2.2.1), $f(z) \in HF_{\gamma}^{\lambda}(\mathrm{p},\mathrm{q},\alpha,\beta,A,B)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{\frac{(\lambda n-\lambda+1)n!}{(n-q)!}[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(p-q-\alpha)]}{\frac{(\lambda p-\lambda+1)p!}{(p-q)!}[\beta\gamma(B-A)(p-q-\alpha)-(p-q-1)(1-\beta B)]} \Gamma_n a_n \leq 1$$

Hence $\frac{c+1}{c+n} < 1$. *Therefore* (2.78) holds and the proof is complete. The above theorem yields the following two special cases.

THEOREM 2.2.6.

1) If $f(z)$is starlike of order γ then $V_{\gamma} f(z)$is also starlike of order α.

2) If $f(z)$ is convex of order γ then $V_{\gamma} f(z)$ is also convex of order α.

THEOREM 2.2.7. Let $f(z) \in HF_{\gamma}^{\lambda}(\mathrm{p},\mathrm{q},\alpha,\beta,A,B)$ Then $V_{\gamma} f(z)$ is starlike of order $0 \leq \xi < 1$ in $|z| < R_1$, where

$$R_1 = inf_n \left\{ \frac{(1-\xi)C_n}{\frac{n!(\lambda n-\lambda+1)(n-\xi)}{(n-q)!} \begin{bmatrix} (n-q-1)(1-\beta B) \\ +\beta\gamma(B-A)(n-q-\alpha) \end{bmatrix}} \right\}^{\frac{1}{n-1}} \tag{2.79}$$

Where C_n Is defined by (2.52)

PROOF. It is sufficient to prove

$$\left| \frac{z\left(V_{\gamma} f(z)\right)'}{V_{\gamma} f(z)} - 1 \right| < 1 - \xi \quad \text{for} \quad |z| < R_1$$

$$\left|\frac{z\left(V_\gamma f(z)\right)'}{V_\gamma f(z)}-1\right| \le \frac{\sum_{n=p+1}^{\infty}(n-1)\left(\frac{c+1}{c+n}\right)^{\delta} a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n|z|^{n-1}}$$

Thus

$$\left|\frac{z\left(V_\gamma f(z)\right)'}{V_\gamma f(z)}-1\right| < 1-\xi$$

if

$$\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta}\left(\frac{n-\xi}{1-\xi}\right) a_n|z|^{n-1} \le 1 \qquad (2.80)$$

where R_1 is given by (2. 81)

$$\sum_{n=p+1}^{\infty}\frac{C_n}{\frac{n!(\lambda n-\lambda+1)}{(n-q)!}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(n-q-\alpha)\right]}a_n \le 1 \qquad (2.81)$$

Comparing (2.82) and (2.83), we have

$$\left(\frac{c+1}{c+n}\right)^{\delta}\left(\frac{n-\xi}{1-\xi}\right)a_n|z|^{n-1} \le \frac{C_n}{\frac{n!(\lambda n-\lambda+1)}{(n-q)!}\left[(n-q-1)(1-\beta B)+\beta\gamma(B-A)(n-q-\alpha)\right]}$$

thus

$$|z| \le \left\{\frac{(1-\xi)C_n}{\left(\frac{c+1}{c+n}\right)^{\delta}\frac{n!(\lambda n-\lambda+1)(n-\xi)}{(n-q)!}\begin{bmatrix}(n-q-1)(1-\beta B)\\+\beta\gamma(B-A)(n-q-\alpha)\end{bmatrix}}\right\}^{\frac{1}{n-1}} \qquad (2.82)$$

<u>THEOREM 2.2.16</u> Let $f(z) \in HF_{\gamma}^{\lambda}(\mathrm{p},\mathrm{q},\alpha,\beta,A,B)$. Then $V_\gamma f(z)$ is convex of order $0 \le \xi < 1$ in $|z| < R_2$, where

$$R_2 = inf\left\{\frac{(1-\xi)C_n}{\left(\frac{c+1}{c+n}\right)^{\delta}\frac{n!(\lambda n-\lambda+1)n(n-\xi)}{(n-q)!}\begin{bmatrix}(n-q-1)(1-\beta B)\\+\beta\gamma(B-A)(n-q-\alpha)\end{bmatrix}}\right\}^{\frac{1}{n-1}} \qquad (2.83)$$

2.3 CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS CONVOLUTED WITH DIFFERENTIAL OPERATOR

Let $\mathcal{D}$ be the class of functions f normalized by

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$(2.81)

Which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S the subclass of $\mathcal{D}$, consisting of functions which are also univalentin D. Let w be a fixed point in D and $\mathcal{D}(w) = \{f \in H(D): f(w) = f'(w) - 1 = 0\}$. and S_w denote the subclass of $\mathcal{D}(w)$ consisting of the function of the form

$$f(z) = \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \; (a_n \geq 0) \qquad (2.82)$$

Where $\alpha = Res(z, w) with\ 0 < \alpha \leq 1$

For the function $f(z)$ in the class S_w , we define

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = (z-w) f'(z) + \frac{2\alpha}{(z-w)},$$

$$I^2 f(z) = (z-w)(I^1 f(z))' + \frac{2\alpha}{(z-w)}$$

and for $k = 2,3, ...$ We can write

$$I^k f(z) = (z-w)(I^{k-1} f(z))' + \frac{2\alpha}{(z-w)}$$

$$= \frac{\alpha}{(z-w)} + \sum_{n=1}^{\infty} n^k a_n (z-w)^n \qquad (2.83)$$

In the case w=0 , the differential operator I^k, given [40].

With the help of the differential operatorI^k we define the class$ST_w\ (k, \beta)$as follows :
DEFINITION 2.3.1 The function $f(z) \in D(w)$is said to be amember of the class $ST_w\ (k, \beta)$if it satisfies

$$\left|\frac{(z-w)(I^k f(z))''}{(I^k f(z))'} + 2\right| < \left|\frac{(z-w)(I^k f(z))''}{(I^k f(z))'} + 2\beta\right| \qquad (2.84)$$

$(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$, for some $\beta (0 \leq \beta < 1)$

Let us write

$$S_w^*(k, \beta) = ST_w\ (k, \beta) \cap S_w$$

WhereS_w is the class of functions of the form (2.82) that are analytic and univalent in D .

For the class$S_w\ (k, \beta)$, [41]gave the following:
THEOREM 2.3.1 Let the function f be defined by (2.87) and $\beta\ (0 \leq \beta < 1)$ if

$\sum_{n=1}^{\infty} n^{k+1}(n+\beta)|a_n| \leq \alpha(1-\beta) \quad k \in \mathbb{N}_0 \quad (2.83)$

$Then\ f(z) \in ST_w(k,\beta).$

In the view of Theorem (2.3.1), we can see that the function f given by (2.82) is in the class $ST_w\ (k,\beta)$ which satisfies

$a_n \leq \frac{\alpha(1-\beta)}{n^{k+1}(n+\beta)}$ (2.84)

Let $ST_w\ (k,\beta,c)$ denote the class of function f in $ST_w\ (k,\beta)$of the form

$$f(z) = \frac{\alpha}{(z-w)} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w) + \sum_{n=2}^{\infty} a_n(z-w)^n \ (2.85)$$

$with\ 0 \leq c \leq 1$

<u>THEOREM 2.3.2</u>A function f defined by (2.82) is in the class $ST_w\ (k,\beta,c)$if and only if

$$\sum_{n=1}^{\infty} n^{k+1}(n+\beta)|a_n| \leq \alpha(1-\beta)(1-c) \quad k \in \mathbb{N}_0 \qquad (2.86)$$

<u>PROOF</u>. Assume that the *inequality* (2.86) holds true , then

$$|(z-w)(I^k f(z))'' + 2(I^k f(z))'| - |(z-w)(I^k f(z))'' + 2\beta(I^k f(z))'|$$

$$= \left|\frac{2\alpha}{(z-w)^2} + \sum_{n=1}^{\infty} n^{k+1}(n+1)a_n(z-w)^{n-1} - \frac{2\alpha}{(z-w)^2} + 2\sum_{n=1}^{\infty} n^{k+1}a_n(z-w)^{n-1}\right|$$
$$- \left|\frac{2\alpha}{(z-w)^2} + \sum_{n=1}^{\infty} n^{k+1}(n-1)a_n(z-w)^{n-1} - \frac{2\alpha\beta}{(z-w)^2} + 2\beta\sum_{n=1}^{\infty} n^{k+1}a_n(z-w)^{n-1}\right|$$

$=|\sum_{n=1}^{\infty} n^{k+1}(n+1)a_n(z-w)^{n-1}| - \left|\frac{2\alpha}{(z-w)^2}(1-\beta) + \sum_{n=1}^{\infty} n^{k+1}a_n(z-w)^{n-1}(n-1+2\beta)\right|$

$$\leq \sum_{n=1}^{\infty} n^{k+1}(n+1)a_n - 2\alpha(1-\beta) + \sum_{n=1}^{\infty} n^{k+1}a_n(n-1+2\beta)$$

$=\sum_{n=1}^{\infty}(n+\beta)n^{k+1}a_n < \alpha(1-\beta)(1-C)$

Conversely , assume that $f(z) \in ST_w(k,\beta,c)$, thus

$$\left|\frac{(z-w)\left(I^k f(z)\right)'' + 2\left(I^k f(z)\right)'}{(z-w)\left(I^k f(z)\right)'' + 2\beta\left(I^k f(z)\right)'}\right| < 1$$

$$\mathcal{R}e\left(\frac{(z-w)\left(I^k f(z)\right)'' + 2\left(I^k f(z)\right)'}{(z-w)\left(I^k f(z)\right)'' + 2\beta\left(I^k f(z)\right)'}\right) < 1$$

$$\mathcal{R}e\left(\frac{\frac{2\alpha}{(z-w)^2} + \sum_{n=1}^{\infty} n^{k+1}(n+1)a_n(z-w)^{n-1} - \frac{2\alpha}{(z-w)^2} + 2\sum_{n=1}^{\infty} n^{k+1}a_n(z-w)^{n-1}}{\frac{2\alpha}{(z-w)^2} + \sum_{n=1}^{\infty} n^{k+1}(n-1)a_n(z-w)^{n-1} - \frac{2\alpha\beta}{(z-w)^2} + 2\beta\sum_{n=1}^{\infty} n^{k+1}a_n(z-w)^{n-1}}\right) < 1$$

$$\sum_{n=1}^{\infty}(n+\beta)n^{k+1}a_n < \alpha(1-\beta)$$

COROLLARY 2.3.1 Let the function f given by (2.85) be in the class $ST_w\ (k,\beta,c)$Then

$a_n \leq \frac{(1-c)\{\alpha(1-\beta)\}}{n^{k+1}(n+\beta)}$,$n \geq 2$

A distortion property for function f to be in the class $ST_w(k,\beta,c)$is given as follows:

THEOREM 2.3.3 If the function f defined by (2.85) is in the class $ST_w\ (k,\beta,c)$ for $0 < |z-w| = r < 1$, then ,we have

$$\frac{\alpha}{r} - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r - \frac{(1-c)\{\alpha(1-\beta)\}}{2(2+\beta)}r^2 \leq |f| \leq \frac{\alpha}{r} - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r + \frac{(1-c)\{\alpha(1-\beta)\}}{2(2+\beta)}r^2$$

with equality for

$$f_2 = \frac{\alpha}{(z-w)} - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w) - \frac{(1-c)\{\alpha(1-\beta)\}}{2(2+\beta)}(z-w)^2\big((z-w) = \mp r\big)$$

PROOF : Since $f \in ST_w(\mathrm{k},\beta,\mathrm{c})$, *Theorem*2.3.2 yields the inequality

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1-c)\{\alpha(1-\beta)\}}{n^{k+1}(n+\beta)} \qquad n \geq 2 \tag{2.87}$$

Thus , for $0 < |z-w| = r < 1$.

$$|f(z)| \leq \left|\frac{\alpha}{(z-w)}\right| + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}|z-w| + \sum_{n=2}^{\infty}|a_n||z-w|^n$$

$As\ |z-w| = r$

$$|f(z)| \leq \frac{\alpha}{r} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r - \ r^2\sum_{n=2}^{\infty}|a_n|$$

and

$$|f(z)| \geq \left|\frac{\alpha}{(z-w)}\right| - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}|z-w| - \sum_{n=2}^{\infty}|a_n||z-w|^n \; (|z-w| = r)$$

$$\geq \frac{\alpha}{r} - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r - r^2\sum_{n=2}^{\infty}|a_n|$$

$$\geq \frac{\alpha}{r} - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r - \frac{(1-c)\{\alpha(1-\beta)\}}{2(2+\beta)}r^2$$

<u>THEOREM 2.3.4</u>If the function f defined by (2.85) is in the class $ST_w\,(k,\beta,c)$ for $0 < |z-w| = r < 1$, then ,we have

$$\frac{\alpha}{r} - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)} - r\frac{(1-c)\{\alpha(1-\beta)\}}{2(2+\beta)} \leq |f'(z)|$$
$$\leq \frac{\alpha}{r} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)} + r\frac{(1-c)\{\alpha(1-\beta)\}}{(2+\beta)},$$

with equality for

$$f_2 = \frac{\alpha}{(z-w)} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w) + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w)^2 , \qquad \big((z-w) = \mp r\big)$$

<u>PROOF</u> : From *Theorem* (2.3.3) , it follows that

$$\sum_{n=2}^{\infty} na_n \leq \frac{(1-c)\{\alpha(1-\beta)\}}{n^k(n+\beta)} , \quad n \geq 2 \qquad (2.88)$$

Thus , for $0 < |z-w| = r < 1$, and making use of (2.88) ,we have

$$|f'(z)| \leq \left|\frac{-\alpha}{(z-w)}\right| + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)} + \sum_{n=2}^{\infty} n\,|a_n|(z-w)^n \; (|z-w| = r)$$

$$\leq \frac{\alpha}{r} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)} + r\sum_{n=2}^{\infty} n\,|a_n|$$

$$\leq \frac{\alpha}{r} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r + r\frac{(1-c)\{\alpha(1-\beta)\}}{2(2+\beta)}$$

and

$$f'(z) \geq \left|\frac{-\alpha}{(z-w)}\right| - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)} - \sum_{n=2}^{\infty} n\,|a_n|(z-w)^n \; (|z-w| = r)$$

$$\geq \frac{\alpha}{r} - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)} - r\sum_{n=2}^{\infty} n\,|a_n|$$

$$\geq \frac{\alpha}{r} - \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r - r\frac{(1-c)0\{\alpha(1-\beta)\}}{2(2+\beta)}$$

The radius of starlikeness and convexity for the class $ST_w\ (k,\beta,c)$ is given by the following theorem

THEOREM 2.3.5 If the function f defined by (2.85) in the class $ST_w\ (k,\beta,c)$ then f is starlikeness of order $\delta(0\le\delta<1)$ in the disk $|z-w|<R_1(k,\beta,c,\delta)$ is the Largest value for which

PROOF : It sufficient to show that

$$\left|\frac{(z-w)f'(z)}{f(z)}+1\right|\le 1-\delta$$

For $|z-w|=R_1$, we have

$$\left|\frac{(z-w)f'(z)}{f(z)}+1\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}(n+1)\ a_n(z-w)^n}{\frac{\alpha}{(z-w)}+\frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w)+\sum_{n=2}^{\infty}|a_n|(z-w)^n}\right|\le 1-\delta \quad (2.89)$$

$$=\frac{\left|\sum_{n=2}^{\infty}(n+1)\ a_n(z-w)^n\right|}{\frac{\alpha}{|z-w|}+\frac{c\{\alpha(1-\beta)\}}{(1+\beta)}|z-w|+\sum_{n=2}^{\infty}|a_n||z-w|^n}\le 1-\delta$$

$$\left(\sum_{n=2}^{\infty}(n-1)a_nr^n\le(1-\delta)\frac{\alpha}{r}-(1-\delta)\frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r-(1-\delta)\sum_{n=2}^{\infty}|a_n|r^n\right)\times r$$

$$\sum_{n=2}^{\infty}(n-1)|a_n|r^{n+1}\le(1-\delta)\alpha-(1-\delta)\frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r^2-(1-\delta)\sum_{n=2}^{\infty}|a_n|r^{n+1}$$

$$\sum_{n=2}^{\infty}(n-\delta)|a_n|r^{n+1}+\frac{c\{\alpha(1-\beta)\}(1-\delta)}{(1+\beta)}r^2\le\alpha(1-\delta)$$

and it follows that from (2.86) we may take

$$|a_n|\le\frac{(1-c)\{\alpha(1-\beta)\}}{n^{k+1}(n+\beta)}\lambda_n\,,n\ge 2$$

Where $\lambda_n\ \ge 0\ and\ \sum_{n=2}^{\infty}\lambda_n\ \le 1$

For each fixed r , we choose the positive integer $n_0=n_0(r)$ for which

$\frac{(n-\delta)}{n^{k+1}(n+\beta)}r^{n+1}$ is maxima

then it follows that

$$\sum_{n=2}^{\infty}(n-\delta)\left|a_{n_0}\right|r^{n_0+1}\le\frac{(1-c)\{\alpha(1-\beta)\}(n_0-\delta)}{{n_0}^{k+1}(n_0+\beta)}r^{n_0+1}\le\alpha(1-\delta)$$

We find the value $r_0=r_0(k,\alpha,\delta,\beta,c,n)$ and corresponding integer $n_0(r_0)$ so

$$\frac{c(1-\delta)\{\alpha(1-\beta)\}}{(1+\beta)}r_0^2+\frac{(n_0-\delta)(1-c)\alpha\{(1-\beta)\}}{{n_0}^{k+1}(n_0+\beta)}r_0^{n_0+1}\le\alpha(1-\delta)$$

Then this value is the radius of starlikeness of order δ for function f belong to class $ST_w(k,\beta,\alpha,c,q)$. .

THEOREM 2.3.6 If the function f defined by (2.85) in the class $ST_w\ (k,\beta,c)$ then f is convex of order $\delta(0\le\delta<1)$ in the disk $|z-w|<R_2(k,\beta,c,\delta)$ where $R_2(k,\beta,c,\delta)$ is the Largest value for which

$$\frac{c\{\alpha(1-\beta)\}(3-\delta)}{(1+\beta)}r^2+\frac{(1-c)\{\alpha(1-\beta)\}n(n+2-\delta)}{n^{k+1}(n+\beta)}r^{n+1}\le\alpha(1-\delta)$$

PROOF. It suffice to show that

$$\left|\frac{(z-w)f''(z)}{f'(z)}+2\right|\le 1-\delta$$

Note that

$$\left|\frac{(z-w)f''(z)}{f'(z)}+2\right|\le\frac{\frac{2c\{\alpha(1-\beta)\}}{(1+\beta)}+\sum_{n=2}^{\infty}n(n+1)a_n|z-w|^{n-1}}{\frac{\alpha}{|z-w|^2}\frac{c\{\alpha(1-\beta)\}}{(1+\beta)}-\sum_{n=2}^{\infty}na_n|z-w|^{n-1}}\le(1-\delta)\ (2.90)$$

Hence for $|z-w|<|z|<r$, (2.90) hold true if

$$\frac{2c\{\alpha(1-\beta)\}}{(1+\beta)}r^2+\sum_{n=2}^{\infty}n(n+1)a_nr^{n+1}\le(1-\delta)\left(\alpha-\frac{c\{\alpha(1-\beta)\}}{(1+\beta)}r^2-\sum_{n=2}^{\infty}na_nr^{n+1}\right)$$

or

$$\frac{c\{\alpha(1-\beta)\}(3-\delta)}{(1+\beta)}r^2+\sum_{n=2}^{\infty}n(n+2-\delta)a_nr^{n+1}\le\alpha(1-\delta)$$

Our next result involves a linear combination of function of the type (2.85).

THEOREM 2.3.7 If

$$f_1(z)=\frac{\alpha}{(z-w)}+\frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w)(2.91)$$

and

$$f_n(z)=\frac{\alpha}{(z-w)}+\frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w)+\frac{(1-c)\{\alpha(1-\beta)\}}{n^{k+1}(n+\beta)}(z-w)^n\quad n\ge 2\qquad(2.92)$$

Then $f\in ST_ww\ (k,\beta,c)$ if and only if it can expressed in the form

$$f(z)=\sum_{n=2}^{\infty}\lambda_n f_n\ (2.93)$$

where $\lambda_n\ \ge 0\ and\ \sum_{n=1}^{\infty}\lambda_n\ =1$

PROOF : It follows from (2.91),(2.92) and (2.93), we have

$$f(z)=\sum_{n=2}^{\infty}\lambda_n f_n\ (z)$$

So that

$$f(z) = \frac{\alpha}{(z-w)} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w) + \frac{(1-c)\{\alpha(1-\beta)\}}{n^{k+1}(n+\beta)}\lambda_n (z-w)^n \quad (2.94)$$

since

$$f(z) = \sum_{n=2}^{\infty}\left(\frac{(1-c)\{\alpha(1-\beta)\}}{n^{k+1}(n+\beta)}\right)\lambda_n \frac{n^{k+1}(n+\beta)}{(1-c)\{\alpha(1-\beta)\}}$$

$$=\sum_{n=2}^{\infty}\lambda_n = 1-\sigma_0 \le 1$$

Using *Theorem* (2.3.2) , we easily obtain $f \in ST_w\ (k,\beta,c)$, Conversely , let $f \in ST_w\ (k,\beta,c)$ since

$$a_n \le \frac{(1-c)\{\alpha(1-\beta)\}}{n^{k+1}(n+\beta)}, \quad n \ge 2$$

and Setting

$$\lambda_n = \frac{n^{k+1}(n+\beta)}{(1-c)\{\alpha(1-\beta)\}}a_n$$

and

$$\lambda_1 = 1-\sum_{n=2}^{\infty}\lambda_n$$

It follows that

$$f(z) = \sum_{n=2}^{\infty}\lambda_n f_n(z)$$

THEOREM 2.3.8 The class $ST_w\ (k,\beta,c)$ is closed under liner combination

PROOF Let the function defined by (2.85) and let the function g be defined by

$$g(z) = \frac{\alpha}{(z-w)} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w) + \sum_{n=2}^{\infty} b_n(z-w)^n \quad (b_n \ge 2)$$

Assuming that $f(z)$& $g(z)$ are in the class $ST_w\ (k,\beta,c)$, it is sufficient to prove that the function H defined by

$$H(z) = \lambda f(z) + (1-\lambda)g(z)(0 \le \lambda \le 1)$$

Is also in the class $ST_w\ (k,\beta,c)$, since

$$H(z) = \frac{\alpha}{(z-w)} + \frac{c\{\alpha(1-\beta)\}}{(1+\beta)}(z-w) + \sum_{n=2}^{\infty}|a_n\lambda + (1-\lambda)b_n|\,(z-w)^n \quad ,(b_n \ge 2)$$

We observe that

$$\sum_{n=2}^{\infty} n^{k+1}(n+\beta)\,|a_n\lambda + (1-\lambda)b_n| \le (1-c)\,\{\alpha(1-\beta)\}$$

With the aid of *Theorem*2.3.2 , thus $H \in ST_w\ (k,\beta,c)$ and hence the theorem is complete.

CHAPTER 3

ON RESULTS OF MEROMORPHIC UNIVALENT FUNCTIONS WITH FIXED POINTS DEFINED BY DIFFERENTIAL OPERATOR

INTRODUCTION

Chapter three introduced and studied some new subclasses of meromorphic univalent functions which are defined by means of a differential operator.

This chapter divided into two sections ,we have introduced and studied some new subclasses $A^*_{m,k}(\eta,\theta,\delta)$, $A^*_{m0,k}(\eta,\theta,\delta,z_0)$ and $A^*_{m1,k}(\eta,\theta,\delta,z_0)$ of meromorphic univalent functions of the form :

$$f(z)=\frac{a_0}{z}+\sum_{n=1}^{\infty}a_n z^n, \qquad a_0>0$$

satisfies the conditions:

$$\left|\frac{\frac{z^2\left(I^k f(z)\right)'+a_0}{z\left(I^k f(z)\right)}}{\frac{\delta z^2\left(I^k f(z)\right)'}{z\left(I^k f(z)\right)}-1+\eta(1+\delta)}\right|<\theta$$

and

$$\left|\frac{\frac{z^3\left(I^k f(z)\right)''+a_0}{z^2\left(I^k f(z)\right)'}}{\frac{\delta z^3\left(I^k f(z)\right)''}{z^2\left(I^k f(z)\right)'}-1+\eta(1+\delta)}\right|<\theta$$

For $0\le\eta<1$, $0<\theta\le1$, $0\le\delta\le1$, $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, and \ z\in U^*$.

For a given real number $z_0(0<z_0<1)$. Let $A_{mi}(i=0,1)$be a subclass of A^*_m satisfying the condition $z_0 f(z_0)=1$ and $-z_0^2 f'(z_0)=1$ respectively.

We have obtained numerous sharp results including coefficient conditions, extreme points, distortion bounds and convex combinations for the above classes of meromorphic univalent functions.

3.1 CERTAIN SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS INVOLVING DIFFERENTIAL OPERATOR

Let A denote the class of functions which are analytic in the punctured unit disk $U^*=\{z:0<|z|<1\}$ of the form:

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n, \qquad a_0 > 0 \quad . \tag{3.1}$$

Suppose that A^* denote the subclass of A consisting of functions that are univalent in U^*. Further A_m^* denote subclass of A^* consisting of functions f of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_{n+m} z^{n+m}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N} \quad . \tag{3.2}$$

<u>**DEFINITION 3.1.1:**</u> A function $f \in A_m^*$ is said to be meromorphic starlike of order α in U^*. if it satisfies the inequality

$$\mathcal{R}e\left\{\frac{zf''(z)}{f'(z)}\right\} > -\alpha\,, z \in U^*, 0 \leq \alpha < 1\,. \tag{3.3}$$

On the other hand, a function $f \in A_m^*$is said to be meromorphic convex of order α in U^*. if it satisfies the inequality

$$\mathcal{R}e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\alpha\,, z \in U^*, 0 \leq \alpha < 1 \;. \tag{3.4}$$

Various subclasses of A have been introduced and studied by many authors see [42], [25], [43], [43], [44], [5], [45], [46], [47], [37]and [48]. In recent years, some subclasses of meromorphic functions associated with several families of integral operators and derivative operators were introduced and investigated see [42] , [44], [49] and [50] . The first differential operator for meromorphic function was introduced by[40]. [51] introduced a differential operator:

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = zf'(z) + \frac{2a_0}{z},$$

$$I^2 f(z) = z\big(I^1 f(z)\big)' + \frac{2a_0}{z},$$

$$I^k f(z) = z\left(I^{(k-1)} f(z)\right)' + \frac{2a_0}{z},$$

where $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in U^*$.

For a function $f \in A_m^*$ in , from definition of the differential operator$I^k f(z)$, we easily see that

$$I^k f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} n^k a_{n+m} z^{n+m},$$

$$a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in U^*$$

(3.5)

By using the operator I^k , some authors have established many subclasses of meromorphic functions, for example[25],[51], [43] and[43]. With the help of the differential operator I^k , we define the following new class of meromorphic univalent functions and obtain some interesting results. Let $A^*_{m,k}(\eta,\theta,\delta)$ denote the family of meromorphic univalent functions f of the form (3.2) such that

$$\left|\frac{\frac{z^2\left(I^kf(z)\right)'+a_0}{z\left(I^kf(z)\right)}}{\frac{\delta z^2\left(I^kf(z)\right)'}{z\left(I^kf(z)\right)}-1+\eta(1+\delta)}\right|<\theta \tag{3.6}$$

For $0\le\eta<1\,,0<\theta\le1\,,0\le\delta\le1\,,k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, and\ \ z\in U^*.$

For a given real number $z_0(0<z_0<1)$. Let $A_{mi}(i=0,1)$be a subclass of A^*_m satisfying the condition $z_0f(z_0)=1$ and $-z_0^2f'(z_0)=1$ respectively.

Let

$$A^*_{mi,k}(\eta,\theta,\delta,z_0)=A^*_{m,k}(\eta,\theta,\delta)\cap A_{mi},\quad (i=1,2) \tag{3.7}$$

For other subclasses of meromorphic univalent functions, one may refer to the recent work of[25], [45], [42], [45] and [25].

COEFFICIENT INEQUALITIES

we provide a necessary and sufficient condition for a function f meromorphic univalent in $U^*$to be in $A^*_{m,k}(\eta,\theta,\delta)$, $A^*_{m0,k}(\eta,\theta,\delta,z_0)$and $A^*_{m1,k}(\eta,\theta,\delta,z_0)$.

<u>THEOREM 3.1.1:</u>

A function $f\in A^*_m$ defined by equation (3.2) is in the class $A^*_{m,k}(\eta,\theta,\delta)$ if and only if

$$\sum_{n=0}^{\infty}n^ka_{n+m}\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)$$
$$<\theta a_0(1-\eta)(1+\delta) \tag{3.8}$$

where $0\le\eta<1\,,0<\theta\le1\,,0\le\delta\le1\,,k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, and\ \ z\in U^*.$

The result is sharp for the function given by

$$f(z)=\frac{a_0}{z}+\frac{\theta a_0(1-\eta)(1+\delta)}{n^k(n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)}z^{n+m}, n\ge1. \tag{3.9}$$

<u>PROOF</u>: Assume that the condition (3.8 is true. We must show that $f\in A^*_{m,k}(\eta,\theta,\delta)$ or equivalently prove that

$$\left|\frac{\frac{z^2\left(I^k f(z)\right)'+a_0}{\left(I^k f(z)\right)}}{\frac{\delta z^2\left(I^k f(z)\right)'}{\left(I^k f(z)\right)}-1+\eta a_0(1+\delta)}\right|<\theta$$

$$\left|\frac{\frac{z^2\left(I^k f(z)\right)'+a_0}{\left(I^k f(z)\right)}}{\frac{\delta z^2\left(I^k f(z)\right)'}{\left(I^k f(z)\right)}-1+\eta a_0(1+\delta)}\right|=\left|\frac{z^2\left(I^k f(z)\right)'+a_0\left(I^k f(z)\right)}{\delta z^2\left(I^k f(z)\right)'-\left(zI^k f(z)\right)+\eta(1+\delta)\left(zI^k f(z)\right)}\right|$$

$$=\left|\frac{-a_0+\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}z^{n+m+1}-a_0+\sum_{n=0}^{\infty} n^k a_{n+m}z^{n+m+1}}{\begin{array}{c}\delta(-a_0+\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}z^{n+m+1})-(a_0+\sum_{n=0}^{\infty} n^k a_{n+m}z^{n+m+1})\\+\eta(1+\delta)(a_0+\sum_{n=0}^{\infty} n^k a_{n+m}z^{n+m+1})\end{array}}\right|$$

$$=\left|\frac{\sum_{n=0}^{\infty} n^k(n+m+1)a_{n+m}z^{n+m+1}}{-a_0\delta-a_0+\eta a_0(1+\delta)+\sum_{n=0}^{\infty} n^k a_{n+m}z^{n+m+1}(\delta(n+m)-1+\eta(1+\delta)}\right|$$

$$\leq\left|\frac{\sum_{n=0}^{\infty} n^k(n+m+1)a_{n+m}}{-a_0\delta-a_0+\eta a_0(1+\delta)+\sum_{n=0}^{\infty} n^k a_{n+m}(\delta(n+m)-1+\eta(1+\delta)}\right|<\theta$$

The last inequality is true by (3.8). Conversely, suppose that $f\in A^{*}_{m,k}(\eta,\theta,\delta)$. We must show that the condition (3.8) holds true. We have

$$\left|\frac{\frac{z^2\left(I^k f(z)\right)'+a_0}{\left(I^k f(z)\right)}}{\frac{\delta z^2\left(I^k f(z)\right)'}{\left(I^k f(z)\right)}-1+\eta a_0(1+\delta)}\right|<\theta$$

Thus

$$\left|\frac{\sum_{n=0}^{\infty} n^k(n+m+1)a_{n+m}}{-a_0\delta-a_0+\eta a_0(1+\delta)+\sum_{n=0}^{\infty} n^k a_{n+m}(\delta(n+m)-1+\eta(1+\delta)}\right|<\theta$$

Since $\mathcal{R}e(z)<|z|$ for all z , we have

$$\mathcal{R}e\left\{\frac{\sum_{n=0}^{\infty} n^k(n+m+1)a_{n+m}}{-a_0\delta-a_0+\eta a_0(1+\delta)+\sum_{n=0}^{\infty} n^k a_{n+m}(\delta(n+m)-1+\eta(1+\delta)}\right\}<\theta$$

Now, choosing values of z on the real axise and allowing $z\to 1$ from the left through real values, the last inequality immediately yields the desired condition in (3.8).

Finally, it is observed that the result is sharp for the function given by

$$f(z)=\frac{a_0}{z}+\frac{\theta a_0(1-\eta)(1+\delta)}{n^k[(n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)]}z^{n+m},n\geq 1.$$

THEOREM 3.1.2: A function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m,0,k}^*(\eta, \theta, \delta, z_0)$ if and only if

$$\sum_{n=0}^{\infty}\left[\frac{n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)}{\theta(1-\eta)(1+\delta)}+{z_0}^{n+m+1}\right]a_{n+m} \leq 1 \qquad (3.10)$$

where $0 \leq \eta < 1$, $0 < \theta \leq 1$, $0 \leq \delta \leq 1$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

$$f(z) = \frac{n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)-\theta(1-\eta)(1+\delta)z^{n+m}}{z\Big(n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)+\theta(1-\eta)(1+\delta){z_0}^{n+m}\Big)} \qquad (3.11)$$

PROOF: Assume that $f \in A_{m,0,k}^*(\eta, \theta, \delta, z_0)$ then

$$f(z_0) = \frac{a_0}{z_0} + \textstyle\sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m},\ \ a_0 > 0, a_{n+m} > 0, m \in \mathbb{N},$$

then

$$z_0 f(z_0) = a_0 + \sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}.$$

Hence

$$1 = a_0 + \textstyle\sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1},\ \ a_0 > 0, a_{n+m} > 0, m \in \mathbb{N},$$

therefore

$$a_{0=}1 - \sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}. \qquad (3.12)$$

Substituting equation (3.12) in inequality (3.8), we get

$$\sum_{n=0}^{\infty} n^k a_{n+m}\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big) \leq \theta\left(1-\sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1}\right)(1-\eta)(1+\delta)$$

Thus

$$\sum_{n=0}^{\infty}\left[n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)+{z_0}^{n+m+1}\right]a_{n+m} \leq \theta(1-\eta)(1+\delta)$$

$$\sum_{n=0}^{\infty}\left[\frac{n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)}{\theta(1-\eta)(1+\delta)}+{z_0}^{n+m+1}\right]a_{n+m} < 1$$

THEOREM 3.1.3: A function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m,1,k}^*(\eta, \theta, \delta, z_0)$ if and only if

$$\sum_{n=0}^{\infty}\left[\frac{n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)}{\theta(1-\eta)(1+\delta)}-(n+m){z_0}^{n+m+1}\right]a_{n+m} < 1 \qquad (3.13)$$

Where $0 \le \eta < 1\,, 0 < \theta \le 1\,, 0 \le \delta \le 1\,, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

PROOF: Assume that $f \in A_{m,0,k}^*(\eta, \theta, \delta, z_0)$ then

$$f(z_0) = \frac{a_0}{z_0} + \sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}\,.$$

hence

$$-{z_0}^2 f'(z_0) = a_0 - \sum_{n=0}^{\infty}(n+m)a_{n+m}{z_0}^{n+m}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}\,,$$

That mean

$1 = a_0 - \sum_{n=0}^{\infty}(n+m)a_{n+m}{z_0}^{n+m+1},\ a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}\,,$

therefore

$$a_{0=}1 + \sum_{n=0}^{\infty}(n+m)a_{n+m}{z_0}^{n+m+1}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N} \qquad (3.14)$$

substituting equation (3.14) in equation (3.8), we get

$$\sum_{n=0}^{\infty} n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)a_{n+m} \le \theta\left(1 + \sum_{n=0}^{\infty}(n+m)a_{n+m}{z_0}^{n+m+1}\right)(1-\eta)(1+\delta)$$

and

$$\sum_{n=0}^{\infty} n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)a_{n+m} + \sum_{n=0}^{\infty}\theta(n+m)(1-\eta)(1+\delta)a_{n+m}{z_0}^{n+m+1} \le \theta(1-\eta)(1+\delta)$$

Thus

$$\sum_{n=0}^{\infty}\left[\frac{n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)}{\theta(1-\eta)(1+\delta)}-(n+m){z_0}^{n+m+1}\right]a_{n+m} \le 1$$

From *Theorem* 3.1.2 and *Theorem* 3.1.3, we have the following results:

COROLLARY 3.1.1: Let a function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m0,k}^*(\eta, \theta, \delta, z_0)$ Then

$$a_{n+m} \leq \frac{\theta(1-\eta)(1+\delta)}{n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big) \atop +{z_0}^{n+m+1}\theta(1-\eta)(1+\delta)} \tag{3.15}$$

where $0 \leq \eta < 1, 0 < \theta \leq 1, 0 \leq \delta \leq 1, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

COROLLARY 3.1.2: Let a function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m1,k}^*(\eta, \theta, \delta, z_0)$ Then

$$a_{n+m} \leq \frac{\theta(1-\eta)(1+\delta)}{n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big) \atop -\theta(1-\eta)(1+\delta)(n+m){z_0}^{n+m+1}} \tag{3.16}$$

where $0 \leq \eta < 1, 0 < \theta \leq 1, 0 \leq \delta \leq 1, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

distortion theorems will be considered and covering property for functions in the classes $A_{m,0,k}^*(\eta, \theta, \delta, z_0)$ and $A_{m,1,k}^*(\eta, \theta, \delta, z_0)$ will also be given.

THEOREM 3.1.4: If a function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m0,k}^*(\eta, \theta, \delta, z_0)$, then

$$|f(z)| \geq \frac{(m+1)+\theta\delta(m+\eta)-\theta(1-\eta)-\theta(1-\eta)(1+\delta)r^{m+1}}{r\big((m+1)+\theta\delta(m+\eta)-\theta(1-\eta)+\theta(1-\eta)(1+\delta){z_0}^{m+1}\big)},$$

Where $0 \leq \eta < 1, 0 < \theta \leq 1, 0 \leq \delta \leq 1, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

The result is sharp with the extremal function given by

$$f(z) = \frac{(m+1)+\theta\delta(m+\eta)-\theta(1-\eta)-\theta(1-\eta)(1+\delta)r^{m+1}}{r\big((m+1)+\theta\delta(m+\eta)-\theta(1-\eta)+\theta(1-\eta)(1+\delta){z_0}^{m+1}\big)}.$$

PROOF : Since $f \in A_{m,0,k}^*(\eta, \theta, \delta, z_0)$, by *Theorem* 3.1.2 we have

$$\big((m+1)+\theta\delta(m+\eta)-\theta(1-\eta)+{z_0}^{m+1}\theta(1-\eta)(1+\delta)\big)\sum_{n=0}^{\infty} a_{n+m}$$
$$\leq \sum_{n=0}^{\infty} n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)$$
$$+\theta(1-\eta)(1+\delta){z_0}^{n+m+1}a_{n+m} \leq \theta(1-\eta)(1+\delta),$$

$$\sum_{n=0}^{\infty} a_{n+m} \leq \frac{\theta(1-\eta)(1+\delta)}{\big((m+1)+\theta\delta(m+\eta)-\theta(1-\eta)+\theta(1-\eta)(1+\delta){z_0}^{m+1}\big)}$$

also we have

$$a_{0=}1-\sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1} \quad ,a_0>0,a_{n+m}>0,m\in\mathbb{N}$$

$$\geq \frac{(m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})}{\left((m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})+\theta(1-\text{ŋ})(1+\delta){z_0}^{m+1}\right)}$$

Thus from the above equation we obtain

$$|f(z)|=\left|\frac{a_0}{z}+\sum_{n=1}^{\infty} a_{n+m}z^{n+m}\right| \quad ,a_0>0,a_{n+m}>0,m\in\mathbb{N}$$

$$\geq \frac{a_0}{r}-r^m\sum_{n=0}^{\infty} a_{n+m}$$

$$\geq \frac{(m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})-\theta(1-\text{ŋ})(1+\delta)r^{m+1}}{r\left((m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})+\theta(1-\text{ŋ})(1+\delta){z_0}^{m+1}\right)}$$

<u>THEOREM 3.1.5</u>: If a function $f\in A_m^*$ defined by equation (3.2) is in the class $A_{m,1,k}^*(\text{ŋ},\theta,\delta,z_0)$, then

$$|f(z)|\leq \frac{(m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})-\theta(1-\text{ŋ})(1+\delta)r^{m+1}}{r\left((m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})+\theta(1-\text{ŋ})(1+\delta){z_0}^{m+1}\right)},$$

where $0\leq \text{ŋ}<1$, $0<\theta\leq 1$, $0\leq\delta\leq 1$, $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, *and* $z\in U^*$.

The result is sharp with the extremal function given by

$$f(z)=\frac{(m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})-\theta(1-\text{ŋ})(1+\delta)r^{m+1}}{r\left((m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})+\theta(1-\text{ŋ})(1+\delta){z_0}^{m+1}\right)}$$

<u>PROOF</u> : Since $f\in A_{m,1,k}^*(\text{ŋ},\theta,\delta,z_0)$, by *Theorem* 3.1.3 ,we have

$$\left((m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})+{z_0}^{m+1}\theta(1-\text{ŋ})(1+\delta)\right)\sum_{n=0}^{\infty} a_{n+m}$$

$$\leq \sum_{n=0}^{\infty} n^k\left((n+m+1)+\theta\delta(n+m+\text{ŋ})-\theta(1-\text{ŋ})\right)$$
$$+\theta(1-\text{ŋ})(1+\delta){z_0}^{n+m+1}a_{n+m}\leq \theta(1-\text{ŋ})(1+\delta),$$

$$\sum_{n=0}^{\infty} a_{n+m}\leq \frac{\theta(1-\text{ŋ})(1+\delta)}{\left((m+1)+\theta\delta(m+\text{ŋ})-\theta(1-\text{ŋ})+\theta(1-\text{ŋ})(1+\delta){z_0}^{m+1}\right)}$$

Also we have

$$a_{0=}1-\sum_{n=0}^{\infty}a_{n+m}{z_0}^{n+m+1}\quad, a_0>0, a_{n+m}>0, m\in\mathbb{N}$$

$$\leq\frac{(m+1)+\theta\delta(m+\eta)-\theta(1-\eta)}{\left((m+1)+\theta\delta(m+\eta)-\theta(1-\eta)+\theta(1-\eta)(1+\delta){z_0}^{m+1}\right)}$$

Thus from the above equation we obtain

$$|f(z)|=\left|\frac{a_0}{z}+\sum_{n=1}^{\infty}a_{n+m}z^{n+m}\right|\quad, a_0>0, a_{n+m}>0, m\in\mathbb{N}$$

$$\leq\frac{a_0}{r}-r^m\sum_{n=0}^{\infty}a_{n+m}$$

$$\leq\frac{(m+1)+\theta\delta(m+\eta)-\theta(1-\eta)-\theta(1-\eta)(1+\delta)r^{m+1}}{r\left((m+1)+\theta\delta(m+\eta)-\theta(1-\eta)+\theta(1-\eta)(1+\delta){z_0}^{m+1}\right)}$$

COROLLARY 3.1.3: The disk $0<|z|<1$ is mapped onto a domain that contains the disk $|w|<\frac{(m+1)+\theta\delta(m+\eta)-\theta(1-\eta)-\theta(1-\eta)(1+\delta)r^{m+1}}{\left((m+1)+\theta\delta(m+\eta)-\theta(1-\eta)+\theta(1-\eta)(1+\delta){z_0}^{m+1}\right)}$ by any function $f\in A^*_{m,0,k}(\eta,\theta,\delta,z_0)$.

The extreme points of the class $A^*_{m,0,k}(\eta,\theta,\delta,z_0)$ and $A^*_{m1,k}(\eta,\theta,\delta,z_0)$ are given by the following theorem.

THEOREM 3.1.6 : Let $f_0(z)=\frac{1}{z}$,

and

$$f_{n+m}(z)=\sum_{n=0}^{\infty}\frac{\begin{gathered}n^k\left((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\right)\\+\theta(1-\eta)(1+\delta)z^{n+m+1}\end{gathered}}{z\begin{bmatrix}n^k\left((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\right)\\+\theta(1-\eta)(1+\delta){z_0}^{n+m+1}\end{bmatrix}}.$$

Then $f(z)$ is in the class $A^*_{m0,k}(\eta,\theta,\delta,z_0)$if and only if it can be expressed in the form $f(z)=\sum_{n=0}^{\infty}\gamma_n f_{n+m}(z)$ where $\gamma_n\geq 0$, $\gamma_i=0$ $(i=1,2,\dots,m-1, m\geq 2)$ and $\sum_{n=0}^{\infty}\gamma_n=1$

PROOF: Suppose

$$f(z)=\sum_{n=0}^{\infty}\gamma_n f_{n+m}(z)$$

$$= \frac{\gamma_0}{z} + \sum_{n=0}^{\infty} \frac{\begin{matrix} n^k\big((n+m+1)+\theta\delta(n+m+ŋ)-\theta(1-ŋ)\big) \\ +\theta(1-ŋ)(1+\delta)z^{n+m+1}\gamma_{n+m} \end{matrix}}{z\begin{bmatrix} n^k\big((n+m+1)+\theta\delta(n+m+ŋ)-\theta(1-ŋ)\big) \\ +\theta(1-ŋ)(1+\delta){z_0}^{n+m+1} \end{bmatrix}}$$

$$= \frac{1}{z}\left[\gamma_0 + \sum_{n=0}^{\infty} \frac{n^k\big((n+m+1)+\theta\delta(n+m+ŋ)(1-ŋ)\big)\gamma_{n+m}}{z\begin{bmatrix} n^k\big((n+m+1)+\theta\delta(n+m+ŋ)-\theta(1-ŋ)\big) \\ +\theta(1-ŋ)(1+\delta){z_0}^{n+m+1} \end{bmatrix}}\right]$$
$$+ \sum_{n=0}^{\infty} \frac{\theta(1-ŋ)(1+\delta)\gamma_{n+m}z^{n+m+1}}{\begin{bmatrix} n^k\big((n+m+1)+\theta\delta(n+m+ŋ)-\theta(1-ŋ)\big) \\ +\theta(1-ŋ)(1+\delta){z_0}^{n+m+1} \end{bmatrix}}$$

Then , we have

$$\sum_{n=0}^{\infty} \frac{\begin{bmatrix} n^k\big((n+m+1)+\theta\delta(n+m+ŋ)-\theta(1-ŋ)\big) \\ +\theta(1-ŋ)(1+\delta){z_0}^{n+m+1} \end{bmatrix}}{\theta(1-ŋ)(1+\delta)} \times \left(\frac{\theta(1-ŋ)(1+\delta)\gamma_{n+m}}{\begin{bmatrix} n^k\big((n+m+1)+\theta\delta(n+m+ŋ)-\theta(1-ŋ)\big) \\ +\theta(1-ŋ)(1+\delta){z_0}^{n+m+1} \end{bmatrix}}\right)$$

$$\sum_{n=0}^{\infty} \gamma_{n+m} = 1-\gamma_0 \le 1.$$

Now , we have

$$z_0 f_{n+m}(z_0) = 1$$

Thus

$$z_0 f(z_0) = \sum_{n=0}^{\infty} \gamma_{n+m} z_0 f_{n+m}(z_0) = \sum_{n=0}^{\infty} \gamma_{n+m} = 1.$$

This implies that $f \in A^*_{m0,k}$.

Therefore $f \in A^*_{m0,k}(ŋ, \theta, \delta, z_0)$.

Conversely, suppose $f \in A^*_{m0,k}(ŋ, \theta, \delta, z_0)$. Since

$$\sum_{n=0}^{\infty} a_{n+m} \le \frac{\theta(1-ŋ)(1+\delta)}{n^k\big((m+1)+\theta\delta(m+ŋ)-\theta(1-ŋ)+\theta(1-ŋ)(1+\delta){z_0}^{n+m+1}\big)}, \quad n \ge 0$$

Set

$$\gamma_{n+m} = \frac{n^k\big((m+1)+\theta\delta(m+ŋ)-\theta(1-ŋ)+\theta(1-ŋ)(1+\delta)z_0{}^{n+m+1}\big)}{\theta(1-ŋ)(1+\delta)} a_{n+m}\,, n \geq 0$$

and $\gamma_0 = 1 - \sum_{n=0}^{\infty} \gamma_{n+m}$.

Then

$$f(z) = \sum_{n=0}^{\infty} \gamma_n f_{n+m}(z).$$

THEOREM 3.1.7 : Let $f_0(z) = \frac{1}{z}$,

and

$$f_{n+m}(z) = \sum_{n=0}^{\infty} \frac{\begin{matrix} n^k\big((n+m+1)+\theta\delta(n+m+ŋ)-\theta(1-ŋ)\big) \\ +\theta(1-ŋ)(1+\delta)z^{n+m+1} \end{matrix}}{z\begin{bmatrix} n^k\big((n+m+1)+\theta\delta(n+m+ŋ)-\theta(1-ŋ)\big) \\ -\theta(n+m)(1-ŋ)(1+\delta)z_0{}^{n+m+1} \end{bmatrix}}.$$

Then $f(z)$ is in the class $A^*_{m1,k}(ŋ,\theta,\delta,z_0\ )$if and only if it can be expressed in the form

$f(z) = \sum_{n=0}^{\infty} \gamma_n f_{n+m}(z)$ where $\gamma_n \geq 0\,, \gamma_i = 0\ (i = 1,2,\dots,m-1, m \geq 2)$ and $\sum_{n=0}^{\infty} \gamma_n = 1$

CLOSURE THEOREMS

THEOREM 3.1.8: The class $A^*_{m0,k}(ŋ,\theta,\delta,z_0\ )$ is closed under convex linear combination

PROOF : Suppose that the functions $f, g \in A^*_{m0,k}(ŋ,\theta,\delta,z_0\ )$ defined by

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_{n+m} z^{n+m}, \qquad a_0 > 0, a_{n+m} > 0, z \in U^*$$

and

$$g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_{n+m} z^{n+m}, \qquad a_0 > 0, b_{n+m} > 0, z \in U^*$$

respectively, it is sufficient to prove that the function $\mathcal{H}$ defined by

$$\mathcal{H}(z) = \omega f(z) + (1-\omega) g(z), \ \ (0 \leq \omega \leq 1)$$

is also in the class $A^*_{m0,k}(ŋ,\theta,\delta,z_0\ )$.Since

$\mathcal{H}(z) = \frac{\omega a_0 + (1-\omega) b_0}{z} + \sum_{n=0}^{\infty} (\boldsymbol{\omega} a_{n+m} + (1-\omega) b_{n+m})\, z^{n+m}$, $a_0 > 0$, $a_{n+m} > 0$, $z \in U^*$.

We observe that

$$\sum_{n=0}^{\infty} \left[n^k \big((n+m+1) + \theta\delta(n+m+\eta) - \theta(1-\eta) \big) + {z_0}^{n+m+1} \right] (\boldsymbol{\omega} a_{n+m} + (1 - \omega) b_{n+m}) \leq \theta(1-\eta)(1+\delta)$$

with the aid of *Theorem* (3.1.2).Thus $\mathcal{H}(z) \in A^*_{m0,k}(\eta, \theta, \delta, z_0\ )$.

THEOREM 3.1.9: The class $A^*_{m1,k}(\eta, \theta, \delta, z_0\ )$ is closed under convex linear combination

PROOF: The proof is similar to that of theorem (3.1.8).

THEOREM 3.1.10: Let the function $f_l(z), l = 0,1,2, \dots, q$ defined by

$$f_l(z) = \frac{a_{0,l}}{z} + \sum_{n=1}^{\infty} a_{n+m,l} z^{n+m}, \qquad a_0 > 0, a_{n+m,l} > 0, z \in U^*$$

be in the class $A^*_{m0,k}(\eta, \theta, \delta, z_0\ )$. Then the function

$$\vartheta(z) = \sum_{l=0}^{q} c_l f_l(z), \quad (c_l \geq 0)$$

is also in the class $A^*_{m0,k}(\eta, \theta, \delta, z_0\ )$, where $\sum_{l=0}^{q} c_l = 1$

PROOF : By *Theorem* (3.1.2) and for every $l = 0,2,3, \dots, q$ we have

$$\sum_{n=0}^{\infty} \left[n^k \big((n+m+1) + \theta\delta(n+m+\eta) - \theta(1-\eta) \big) + {z_0}^{n+m+1} \right] a_{n+m} \leq \theta(1-\eta)(1+\delta)$$

Then

$$\vartheta(z) = \sum_{l=0}^{q} c_l \left(\frac{a_{0,l}}{z} + \sum_{n=1}^{\infty} a_{n+m,l} z^{n+m} \right), \quad (c_l \geq 0)$$

$$= \frac{c_l a_{0,l}}{z} + \sum_{n=0}^{\infty} \left(\sum_{l=0}^{q} c_l a_{n+m,l} \right) z^{n+m}$$

Since

$$\sum_{n=0}^{\infty}\left[n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)+{z_0}^{n+m+1}\right]\left(\sum_{l=0}^{q}c_l a_{n+m,l}\right)$$

$$=\sum_{l=0}^{q}c_l\left(\sum_{l=0}^{q}\left[n^k\big((n+m+1)+\theta\delta(n+m+\eta)-\theta(1-\eta)\big)+{z_0}^{n+m+1}\right]a_{n+m,l}\right);$$

$$\leq\left(\sum_{l=0}^{q}c_l\right)\theta(1-\eta)(1+\delta),$$

$$=\theta(1-\eta)(1+\delta)$$

Then , $\vartheta(z)\in A^*_{m0,k}(\eta,\theta,\delta,z_0\ )$.

THEOREM 3.1.11: Let the function $f_l(z), l=0,1,2,\dots,q\ defined\ by$

$$f_l(z)=\frac{a_{0,l}}{z}+\sum_{n=1}^{\infty}a_{n+m,l}z^{n+m},\qquad a_0>0, a_{n+m,l}>0, z\in U^*$$

be in the class $A^*_{m1,k}(\eta,\theta,\delta,z_0\ )$. Then the function

$$\vartheta(z)=\sum_{l=0}^{q}c_l f_l(z),\quad (c_l\geq 0)$$

is also in the class $A^*_{m1,k}(\eta,\theta,\delta,z_0\ )$, where $\sum_{l=0}^{q}c_l=1$

PROOF :The proof is similar to that of theorem (3.1.10).

CONVEX FAMILY

DEFINITION 3.1.2: The family $A^*_{m0,k}(\eta,\theta,\delta,c\ \ )$ is defined by $A^*_{m0,k}(\eta,\theta,\delta,c$ $)=u_{zr\in c}A^*_{m0,k}(\eta,\theta,\delta,z_r\ )$,where is a nonempty subset of the real interval $[0,1]$ and $A^*_{m0,k}(\eta,\theta,\delta,c\ )$ is defined by a convex family if the subset C consists of one element only by *Theorems* (3.1.8) and (3.1.10).

Now, we have the following results:

LEMMA 3.1.1: Let $z_1,z_2\in C$ be two distinct positive numbers and $f\in A^*_{m0,k}(\eta,\theta,\delta,z_0)\cap A^*_{m0,k}(\eta,\theta,\delta,z_1)$, then $f(z)=\frac{1}{z}$.

PROOF: Suppose that $f\in A^*_{m0,k}(\eta,\theta,\delta,z_1)\cap A^*_{m0,k}(\eta,\theta,\delta,z_2)$

We have

$$a_{0=}1-\sum_{n=0}^{\infty}a_{n+m}{z_1}^{n+m+1}$$

$$= 1 - \sum_{n=0}^{\infty} a_{n+m} {z_2}^{n+m+1}$$

also

$$f(z) = \frac{a_0}{z} + \sum_{n=0}^{\infty} a_{n+m} z^{n+m} \; a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}.$$

Thus , $a_{n+m} = 0$, $\forall n \geq 0$, because $a_{n+m} \geq 0$, $z_1 > 0, z_2 > 0$, hence

$$f(z) = \frac{1}{z}$$

This complete the proof of the Lemma.

<u>THEOREM 3.1.12:</u> Suppose that $c \subset [0,1], A^*_{m0,k}(\eta, \theta, \delta, c)$ is a convex family if and only if C is connected .

<u>PROOF</u>: Assume that C is connected and $z_1, z_2 \in C$ with $z_1 < z_2$.

$$a_{0=} 1 - \sum_{n=0}^{\infty} a_{n+m} {z_0}^{n+m+1}$$

$$= 1 - \sum_{n=0}^{\infty} a_{n+m} {z_1}^{n+m+1}$$

Suppose that the function $f \in A^*_{m0,k}(\eta, \theta, \delta, z_0)$ defined by

$f(z) = \frac{a_0}{z} - \sum_{n=0}^{\infty} a_{n+m} z^{n+m+1} \; a_0 > 0, a_{n+m,l} > 0, m \in \mathbb{N}$, z$\in U$

and

$$g \in A^*_{m0,k}(\eta, \theta, \delta, z_1)$$

$$g(z) = \frac{b_0}{z} - \sum_{n=0}^{\infty} a_{n+m} z^{n+m+1} \; b_0 > 0, b_{n+m,l} > 0, m \in \mathbb{N}, \mathrm{z} \in U^*$$

it is sufficient to prove that the function $\boldsymbol{\mathcal{H}}$ defined by

$$\mathcal{H}(z) = \boldsymbol{\omega f(z)} + (\mathbf{1} - \boldsymbol{\omega})\boldsymbol{g(z)}, \;\; (\mathbf{0} \leq \boldsymbol{\omega} \leq \mathbf{1})$$

That there exists a $z_2 (z_0 \leq z_2 \leq z_1)$ is also in the calss $A^*_{m0,k}(\eta, \theta, \delta, z_2)$

Then $\mathrm{ķ}(\omega) = z\mathcal{H}(z)$

$$\mathrm{ķ}(\omega) = \omega a_0 + (1-\omega) b_0 + \sum_{n=0}^{\infty} (\omega a_{n+m} + (1-\omega) b_{n+m}) z^{n+m}, a_0 > 0 \, , a_{n+m} > 0 \, , z \in U^*$$

$$= 1 + \boldsymbol{\omega} \sum_{n=0}^{\infty} (z^{n+m} - z_0^{n+m}) a_{n+m} + (1 - \boldsymbol{\omega}) \sum_{n=0}^{\infty} (z^{n+m} - z_1^{n+m}) b_{n+m} , a_0 > 0 , a_{n+m} > 0 , z \in U^*$$

Since z is real number , then $ꞔ(z)$ is also number also we have $ꞔ(z_0) \leq 1$ and $ꞔ(z_1) \geq 1$, there exists $z_2 \in [z_0, z_1]$ such that $ꞔ(z_2) = 1$.

Therefore ,

$$z_2 \mathcal{H}(z_2) = z_2, (z_0 \leq z_2 \leq z_1)$$

This implies that $\mathcal{H}(z) \in A^*_{m0,k}$

We observe that

$$\sum_{n=0}^{\infty} \left[n^k \big((n+m+1) + \theta\delta(n+m+ŋ) - \theta(1-ŋ) \big) + {z_2}^{n+m+1} \right] (\boldsymbol{\omega} a_{n+m} + (1 - \omega) b_{n+m})$$

$$= \boldsymbol{\omega} \sum_{n=0}^{\infty} \left[n^k \big((n+m+1) + \theta\delta(n+m+ŋ) - \theta(1-ŋ) \big) + {z_0}^{n+m+1} \right] a_{n+m} + (1-\omega) \left[n^k \big((n+m+1) + \theta\delta(n+m+ŋ) - \theta(1-ŋ) \big) + {z_1}^{n+m+1} \right] b_{n+m} + \theta(1-ŋ)(1+\delta)\omega \sum_{n=0}^{\infty} ({z_2}^{n+m+1} - {z_0}^{n+m+1})\, a_{n+m} + \theta(1-ŋ)(1+\delta)(1-\omega) \sum_{n=0}^{\infty} ({z_2}^{n+m+1} - {z_1}^{n+m+1})\, b_{n+m}$$

$$= \omega \sum_{n=0}^{\infty} \left[n^k \big((n+m+1) + \theta\delta(n+m+ŋ) - \theta(1-ŋ) \big) + {z_0}^{n+m+1} \right] a_{n+m} + (1-\omega) \left[n^k \big((n+m+1) + \theta\delta(n+m+ŋ) - \theta(1-ŋ) \big) + {z_1}^{n+m+1} \right] b_{n+m} \leq \theta(1-ŋ)(1+\delta) + (1-\omega)\, \theta(1-ŋ)(1+\delta) = \theta(1-ŋ)(1+\delta)$$

With the aid of *theorem* (3.1.2).

Thus , $\mathcal{H}(z) \in A^*_{m0,k}(ŋ, \theta, \delta, z_2)$. Since z_1 *and* z_2 are arbitrary numbers , the family $A^*_{m0,k}(ŋ, \theta, \delta, c)$ is convex.Conversely , if the set C is not connected , then there exists $z_0, z_1 \text{ and } z_2 \text{ such that } z_0, z_1 \in C \text{ and } z_2 \notin C \text{ and } z_0 < z_2 < z_1$.

Now , let $f(z) \in A^*_{m0,k}(ŋ, \theta, \delta, z_0)$ and $f(z) \in A^*_{m0,k}(ŋ, \theta, \delta, z_1)$ therefore

$$ꞔ(\omega) = ꞔ(z_2, \omega)$$

$=$
$1 + \omega \sum_{n=0}^{\infty} ({z_2}^{n+m+1} - {z_0}^{n+m+1})\, a_{n+m} +$
$(1-\omega) \sum_{n=0}^{\infty} ({z_2}^{n+m+1} - {z_1}^{n+m+1})\, b_{n+m}, a_0 > 0\,, a_{n+m} > 0\,, z \in U^*.$

For fixed z_2 and $0 \le \omega < 1$.

Since$k(z_2$,0)<1 and $k(z_2$,1) >1, there exists $\boldsymbol{\omega}_0$, $0 \le \omega_0 < 1$ such that $k(z_2$,ω_0)=1 or $z_2 k(z_2$)=1, where $k(z) = \omega_0 f(z) + (1-\omega_0) g(z)$. Therefore $k(z) \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_0)$.

Also $k(z) \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, C)$ using *Lemma* (3.1.1) Since $z_2 \in C$ and $k(z)$ ∉z.Thus the family $A^*_{m0,k}(\text{ŋ}, \theta, \delta, C)$ is not convex which is a contradiction .

3.2 A NEW SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS ASSOCIATED WITH A DIFFERENTIAL OPERATOR

Let A^* denote the class of functions which are analytic in the punctured unit disk $U^* = \{z: 0 < |z| < 1\}$ of the form (3.1).

Suppose that A^* denote the subclass of A consisting of functions that are univalent in U^*. Further A_m^* denote subclass of A^* consisting of functions f of the form (3.2).

Let $A_{m,k}^*(\eta, \theta, \delta)$ denote the family of meromorphic univalent functions f of the form (3.18) such that

$$\left| \frac{\frac{z^3\left(I^k f(z)\right)'' + a_0}{z^2\left(I^k f(z)\right)'}}{\frac{\delta z^3\left(I^k f(z)\right)''}{z^2\left(I^k f(z)\right)'} - 1 + \eta(1+\delta)} \right| < \theta \tag{3.17}$$

For $0 \le \eta < 1, 0 < \theta \le 1, 0 \le \delta \le 1, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ z \in U^*$.

For a given real number $z_0 (0 < z_0 < 1)$. Let $A_{mi}(i = 0,1)$ be a subclass of A_m^* satisfying the condition $z_0 f(z_0) = 1$ and $-z_0^2 f'(z_0) = 1$ respectively.

Let $A_{mi,k}^*(\eta, \theta, \delta, z_0) = A_{m,k}^*(\eta, \theta, \delta) \cap A_{mi}(i = 1,2)$

we provide a necessary and sufficient condition for a function f meromorphic univalent in U^* to be in $A_{m,k}^*(\eta, \theta, \delta)$, $A_{m0,k}^*(\eta, \theta, \delta, z_0)$ and $A_{m1,k}^*(\eta, \theta, \delta, z_0)$.

THEOREM 3.2.1:

A function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m,k}^*(\eta, \theta, \delta)$ if and only if

$$\sum_{n=0}^{\infty} n^k (n+m)\big((n+m) + (1+\theta\delta) - \theta(1+\delta)(1-\eta)\big) a_{n+m} < \theta a_0 (1-\eta)(1+\delta) \tag{3.18}$$

where $0 \le \eta < 1, 0 < \theta \le 1, 0 \le \delta \le 1, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ z \in U^*$.

The result is sharp for the function given by

$$f(z) = \frac{a_0}{z} + \frac{\theta a_0 (1-\eta)(1+\delta)}{n^k (n+m)\big((n+m) + (1+\theta\delta) - \theta(1+\delta)(1-\eta)\big)} z^{n+m}, n \ge 1 \tag{3.19}$$

PROOF: Assume that the condition (3.18) is true. We must show that $f \in A_{m,k}^*(\eta, \theta, \delta)$ or equivalently prove that

$$\left|\frac{\frac{z^3\left(I^k f(z)\right)''+a_0}{z^2\left(I^k f(z)\right)'}}{\frac{\delta z^3\left(I^k f(z)\right)''}{z^2\left(I^k f(z)\right)'}-1+\eta(1+\delta)}\right|<\theta$$

$$\left|\frac{\frac{z^3\left(I^k f(z)\right)''+a_0}{z^2\left(I^k f(z)\right)'}}{\frac{\delta z^3\left(I^k f(z)\right)''}{z^2\left(I^k f(z)\right)'}-1+\eta(1+\delta)}\right|<\theta=\left|\frac{z^3\left(I^k f(z)\right)''+a_0 z^2\left(I^k f(z)\right)'}{\delta z^3\left(I^k f(z)\right)''-\left(z^2\left(I^k f(z)\right)'\right)+\eta(1+\delta)\left(z^2\left(I^k f(z)\right)'\right)}\right|$$

$$=\frac{a_0+\sum_{n=0}^{\infty} n^k(n+m)(n+m-1)a_{n+m}z^{n+m+1}-a_0+\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}z^{n+m+1}}{\delta(a_0+\sum_{n=0}^{\infty} n^k(n+m)(n+m-1)a_{n+m}z^{n+m+1})+a_0-\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}z^{n+m+1})+\eta(1+\delta)(-a_0+\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}z^{n+m+1})}$$

$$=\left|\frac{\sum_{n=0}^{\infty} n^k(n+m)^2 a_{n+m}z^{n+m+1}}{a_0\delta+a_0-\eta a_0(1+\delta)+\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}z^{n+m+1}(\delta(n+m-1)-1+\eta(1+\delta)}\right|$$

$$\leq\left|\frac{\sum_{n=0}^{\infty} n^k(n+m)^2 a_{n+m}}{a_0(1-\eta)(1+\delta)+\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}(\delta(n+m-1)-1+\eta(1+\delta)}\right|$$

$$<\theta$$

Conversely, suppose that $f\in A^*_{m,k}(\eta,\theta,\delta)$. We must show that the condition (3.18) holds true. We have

$$\left|\frac{\frac{z^3\left(I^k f(z)\right)''+a_0}{z^2\left(I^k f(z)\right)'}}{\frac{\delta z^3\left(I^k f(z)\right)''}{z^2\left(I^k f(z)\right)'}-1+\eta(1+\delta)}\right|<\theta.$$

Thus

$$\left|\frac{\sum_{n=0}^{\infty} n^k(n+m)^2 a_{n+m}}{a_0\delta+a_0-\eta a_0(1+\delta)+\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}z^{n+m+1}(\delta(n+m-1)-1+\eta(1+\delta)}\right| < \theta.$$

Since $\mathcal{R}e(z)<|z|$ for all z , we have

$$\mathcal{R}e\left\{\frac{\sum_{n=0}^{\infty} n^k(n+m)^2 a_{n+m}}{a_0(1-\eta)(1+\delta)+\sum_{n=0}^{\infty} n^k(n+m)a_{n+m}(\delta(n+m-1)-1+\eta(1+\delta)}\right\}<\theta.$$

Now, choosing values of z on the real axis and allowing $z\to 1$ from the left through real values, the last inequality immediately yields the desired condition in (3.18).

Finally, it is observed that the result is sharp for the function given by

$$f(z)=\frac{a_0}{z}+\frac{\theta a_0(1-\eta)(1+\delta)}{n^k(n+m)\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)}z^{n+m}, n\geq 1$$

THEOREM 3.2.2: A function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m,0,k}^*(\eta,\theta,\delta,z_0)$ if and only if

$$\sum_{n=0}^{\infty}\left[\frac{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-\eta)(1+\delta)\big)}{\theta(1-\eta)(1+\delta)}+{z_0}^{n+m+1}\right]a_{n+m} \leq 1 \qquad (3.20)$$

where $0 \leq \eta < 1$, $0 < \theta \leq 1$, $0 \leq \delta \leq 1$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

Then result is sharp for the function given by

$$f(z) = \frac{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-\eta)(1+\delta)\big)-\theta(1-\eta)(1+\delta)z^{n+m+1}}{z\big(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)+\theta(1-\eta)(1+\delta){z_0}^{n+m+1}\big)} \qquad (3.21)$$

PROOF: Assume that $f \in A_{m,0,k}^*(\eta,\theta,\delta,z_0)$ then

$$f(z_0) = \frac{a_0}{z_0} + \sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}$$

hence

$$z_0 f(z_0) = a_0 + \sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}$$

therfore

$$1 = a_0 + \sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N}$$

therfore

$$a_{0=}1 - \sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1}, \qquad a_0 > 0, a_{n+m} > 0, m \in \mathbb{N} \qquad (3.22)$$

Substituting equation (3.22) in inequality (3.18), we get

$$\sum_{n=0}^{\infty} n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)a_{n+m} \leq \theta\left(1-\sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1}\right)(1-\eta)(1+\delta).$$

Thus

$$\sum_{n=0}^{\infty} n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)+\theta(1+\delta)(1-\eta){z_0}^{n+m+1} \leq \theta(1-\eta)(1+\delta)$$

$$\sum_{n=0}^{\infty}\left[\frac{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-\eta)(1+\delta)\big)}{\theta(1-\eta)(1+\delta)}+{z_0}^{n+m+1}\right]a_{n+m}\le 1.$$

THEOREM 3.2.3: A function $f\in A_m^*$ defined by equation (3.18) is in the class $A_{m,1,k}^*(\eta,\theta,\delta,z_0)$ if and only if

$$\sum_{n=0}^{\infty}\left[\frac{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)}{\theta(1-\eta)(1+\delta)}-(n+m){z_0}^{n+m+1}\right]a_{n+m} \le 1 \qquad (3.23)$$

where $0\le\eta<1\,,0<\theta\le 1\,,0\le\delta\le 1\,,k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, and\;\; z\in U^*$.

The result is sharp for the function given by

$$f(z)=\frac{\begin{gathered}n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-\eta)(1+\delta)\big)\\+\theta(1-\eta)(1+\delta)z^{n+m+1}\end{gathered}}{z\begin{pmatrix}n^k(n+m)\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)\\-\theta(1-\eta)(1+\delta)(n+m){z_0}^{n+m+1}\end{pmatrix}}\qquad m\in\mathbb{N},n\ge 1. \qquad (3.24)$$

PROOF: Assume that $f\in A_{m,0,k}^*(\eta,\theta,\delta,z_0)$ then

$$f(z_0)=\frac{a_0}{z_0}+\sum_{n=0}^{\infty}a_{n+m}{z_0}^{n+m},\qquad a_0>0,a_{n+m}>0,m\in\mathbb{N}$$

hence

$$-{z_0}^2f'(z_0)=a_0-\sum_{n=0}^{\infty}(n+m)a_{n+m}{z_0}^{n+m},\qquad a_0>0,a_{n+m}>0,m\in\mathbb{N}$$

$$1=a_0-\sum_{n=0}^{\infty}(n+m)a_{n+m}{z_0}^{n+m+1},\qquad a_0>0,a_{n+m}>0,m\in\mathbb{N}$$

then

$$a_{0=}1+\textstyle\sum_{n=0}^{\infty}(n+m)a_{n+m}{z_0}^{n+m+1},\;\; a_0>0,a_{n+m}>0,m\in\mathbb{N} \qquad (3.25)$$

substituting equation (3.25) in equation (3.18), we get

$$\sum_{n=0}^{\infty}n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)a_{n+m} \le\theta\left(1+\sum_{n=0}^{\infty}(n+m)a_{n+m}{z_0}^{n+m+1}\right)(1-\eta)(1+\delta)$$

and

$$\sum_{n=0}^{\infty} n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)a_{n+m}$$
$$-\sum_{n=0}^{\infty}\theta(1-\eta)(1+\delta)(n+m)a_{n+m}{z_0}^{n+m+1} \le \theta(1-\eta)(1+\delta).$$

Thus

$$\sum_{n=0}^{\infty}\left[\frac{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)}{\theta(1-\eta)(1+\delta)}-(n+m){z_0}^{n+m+1}\right]a_{n+m}$$
$$\le 1.$$

From *Theorem* (3.2.2) and *Theorem* (3.2.3), we have the following results:

COROLLARY 3.2.1: Let a function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m0,k}^*(\eta,\theta,\delta,z_0\ )$then

$$a_{n+m} \le \frac{\theta(1-\eta)(1+\delta)}{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-\eta)(1+\delta)\big) \atop +\theta(1-\eta)(1+\delta){z_0}^{n+m+1}} \tag{3.26}$$

where $0 \le \eta < 1\ , 0 < \theta \le 1\ , 0 \le \delta \le 1\ , k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

COROLLARY 3.2.2: A function $f \in A_m^*$ defined by equation (3.18) is in the class $A_{m1,k}^*(\eta,\theta,\delta,z_0\ )$then

$$a_{n+m} \le \frac{\theta(1-\eta)(1+\delta)}{(n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big) \atop +\theta(1-\eta)(1+\delta)(n+m){z_0}^{n+m+1}} \tag{3.27}$$

Where $0 \le \eta < 1\ , 0 < \theta \le 1\ , 0 \le \delta \le 1\ , k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

distortion theorems will be considered and covering property for functions in the classes $A_{m,0,k}^*(\eta,\theta,\delta,z_0\ )$ and $A_{m,1,k}^*(\eta,\theta,\delta,z_0\ )$ will also be given.

THEOREM 3.2.4: If a function $f \in A_m^*$ defined by equation (3.2) is in the class $A_{m0,k}^*(\eta,\theta,\delta,z_0\ )$, then

$$|f(z)| \ge \frac{m[m(1+\theta\delta)-\theta(1-\eta)(1+\delta)]-\theta(1-\eta)(1+\delta)r^{m+1}}{r(m[m(1+\theta\delta)-\theta(1-\eta)(1+\delta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1})}$$

where $0 \le \eta < 1\ , 0 < \theta \le 1\ , 0 \le \delta \le 1\ , k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \ z \in U^*$.

The result is sharp with the extremal function given by

$$f(z) = \frac{m[m(1+\theta\delta)-\theta(1-\eta)(1+\delta)]+\theta(1-\eta)(1+\delta)r^{m+1}}{r(m[m(1+\theta\delta)-\theta(1-\eta)(1+\delta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1})}$$

PROOF : Since $f \in A_{m,0,k}^*(\eta,\theta,\delta,z_0\ )$, by *Theorem* (3.2.2) we have

$$m[m(1+\theta\delta)-(\eta-\theta)(1+\delta)]$$
$$+\theta(1-\eta)(1+\delta){z_0}^{m+1}\sum_{n=0}^{\infty}a_{n+m}$$
$$\leq \sum_{n=0}^{\infty} n^k(n+m)\big((n+m)(1+\theta\delta)-(1+\delta)(\eta-\theta)\big)$$
$$+\theta(1+\delta)(1-\eta){z_0}^{n+m+1} \leq \theta(1-\eta)(1+\delta)$$

$$\sum_{n=0}^{\infty} a_{n+m} \leq \frac{\theta(1-\eta)(1+\delta)}{m[m(1+\theta\delta)+(\eta-\theta)(1+\delta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1}}$$

also we have

$$a_{0=}1-\sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1} \ , a_0>0, a_{n+m}>0, m\in\mathbb{N}$$

$$\geq \frac{m[m(1+\theta\delta)-\theta(1-\eta)(1+\delta)]}{m[m(1+\theta\delta)-\theta(1-\eta)(1+\delta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1}}$$

Thus from the above equation we obtain

$$|f(z)| = \left|\frac{a_0}{z}+\sum_{n=1}^{\infty} a_{n+m}z^{n+m}\right| \ , a_0>0, a_{n+m}>0, m\in\mathbb{N}$$

$$\geq \frac{a_0}{r}-r^m\sum_{n=0}^{\infty} a_{n+m}$$

$$\geq \frac{m[m(1+\theta\delta)-\theta(1-\eta)(1+\delta)]-\theta(1-\eta)(1+\delta)r^{m+1}}{r(m[m(1+\theta\delta)-\theta(1-\eta)(1+\delta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1})}$$

Hence the proof is complete.

THEOREM 3.2.5: If a function $f\in A_m^*$ defined by equation (3.18) is in the class $A_{m,1,k}^*(\eta,\theta,\delta,z_0)$, then

$$|f(z)| \leq \frac{m(m(1+\theta\delta)-\theta(1+\delta)(1-\eta))+\theta(1-\eta)(1+\delta)r^{m+1}}{r[m(m(1+\theta\delta)-\theta(1+\delta)(1-\eta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1}}$$

Where $0\leq \eta<1$, $0<\theta\leq 1$, $0\leq\delta\leq 1$, $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, and \ \ z\in U^*$.

The result is sharp with the extremal function given by

$$f(z) = \frac{m(m(1+\theta\delta)-\theta(1+\delta)(1-\eta))+\theta(1-\eta)(1+\delta)r^{m+1}}{r[m(m(1+\theta\delta)-\theta(1+\delta)(1-\eta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1}}$$

PROOF : Since $f\in A_{m,1,k}^*(\eta,\theta,\delta,z_0)$, by *theorem* (3.2.15) we have

$\mathrm{m}\big([m(1+\theta\delta)-(1+\delta)(1-\eta)]-\theta(1-\eta)(1+\delta)\big)\sum_{n=0}^{\infty} a_{n+m} \leq \sum_{n=0}^{\infty} n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-\eta)(1+\delta)\big)-\theta(1-\eta)(1+\delta)(n+m){z_0}^{n+m+1}a_{n+m} \leq \theta(1-\eta)(1+\delta),$

$$\sum_{n=0}^{\infty} a_{n+m} \leq \frac{\theta(1-\eta)(1+\delta)}{\mathrm{m}[m(1+\theta\delta)-\theta(1+\delta)(1-\eta)]-\theta(1-\eta)(1+\delta){z_0}^{m+1}}$$

Also we have

$$a_{0=}1+\sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1} \quad , a_0>0, a_{n+m}>0, m \in \mathbb{N}$$

$$\leq \frac{m[m(1+\theta\delta)-\theta(1+\delta)(1-\eta)]}{m[m(1+\theta\delta)-\theta(1+\delta)(1-\eta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1}}$$

Thus from the above equation we obtain

$$|f(z)|=\left|\frac{a_0}{z}+\sum_{n=1}^{\infty} a_{n+m}z^{n+m}\right| \quad , a_0>0, a_{n+m}>0, m \in \mathbb{N}$$

$$\leq \frac{a_0}{r}-r^m\sum_{n=0}^{\infty} a_{n+m}$$

$$\leq \frac{m(m(1+\theta\delta)-\theta(1+\delta)(1-\eta))+\theta(1-\eta)(1+\delta)r^{m+1}}{r[m(m(1+\theta\delta)-\theta(1+\delta)(1-\eta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1}}$$

Hence the proof is complete.

COROLLARY 3.2.8: The disk $0<|z|<1$ is mapped onto a domain that contains the disk $|w|<\frac{m(m(1+\theta\delta)-\theta(1+\delta)(1-\eta))-\theta(1-\eta)(1+\delta)r^{m+1}}{[m(m(1+\theta\delta)-\theta(1+\delta)(1-\eta)]+\theta(1-\eta)(1+\delta){z_0}^{m+1}}$ by any function $f \in A^*_{m,0,k}(\eta,\theta,\delta,z_0)$

The extreme points of the class $A^*_{m,0,k}(\eta,\theta,\delta,z_0)$ and $A^*_{m1,k}(\eta,\theta,\delta,z_0)$ are given by the following theorem.

THEOREM 3.2.6 : Let $f_0(z)=\frac{1}{z}$,

and

$$f_{n+m}(z)=\sum_{n=0}^{\infty}\frac{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-\eta)(1+\delta)\big)-\theta(1-\eta)(1+\delta)z^{n+m+1}}{z\big(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)+\theta(1-\eta)(1+\delta){z_0}^{n+m+1}\big)}, \quad n \geq 0$$

Then $f(z)$ is in the class $A^*_{m0,k}(\eta,\theta,\delta,z_0)$if and only if it can be expressed in the form $f(z)=\sum_{n=0}^{\infty}\gamma_n f_{n+m}(z)$ where $\gamma_n \geq 0$, $\gamma_i=0$ $(i=1,2,\ldots,m-1, m\geq 2)$ and $\sum_{n=0}^{\infty}\gamma_n=1$

PROOF: Suppose

$$f(z)=\sum_{n=0}^{\infty}\gamma_n f_{n+m}(z)$$

$$=\frac{\gamma_0}{z}+\sum_{n=0}^{\infty}\frac{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-ŋ)(1+\delta)\big)+\theta(1-ŋ)(1+\delta)z^{n+m+1}}{z\left(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-ŋ)\big)+\theta(1-ŋ)(1+\delta){z_0}^{n+m+1}\right)}$$

$$=\frac{1}{z}\left[\gamma_0+\sum_{n=0}^{\infty}\frac{n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1-ŋ)(1+\delta)\big)\gamma_{n+m}}{\left(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-ŋ)\big)+\theta(1-ŋ)(1+\delta){z_0}^{n+m+1}\right)}\right]$$
$$+\sum_{n=0}^{\infty}\frac{\theta(1-ŋ)(1+\delta)\gamma_{n+m}z^{n+m+1}}{\left(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-ŋ)\big)+\theta(1-ŋ)(1+\delta){z_0}^{n+m+1}\right)}$$

Then , we have

$$\sum_{n=0}^{\infty}\frac{\left[\left(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-ŋ)\big)+\theta(1-ŋ)(1+\delta){z_0}^{n+m+1}\right)\right]}{\theta(1-ŋ)(1+\delta)}$$
$$\times\left(\frac{\theta(1-ŋ)(1+\delta)\gamma_{n+m}}{\left[\left(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-ŋ)\big)+\theta(1-ŋ)(1+\delta){z_0}^{n+m+1}\right)\right]}\right)$$

$$\sum_{n=0}^{\infty}\gamma_{n+m}=1-\gamma_0\leq 1.$$

Now , we have

$$z_0 f_{n+m}(z_0)=1$$

Thus

$$z_0 f(z_0)=\sum_{n=0}^{\infty}\gamma_{n+m}z_0 f_{n+m}(z_0)=\sum_{n=0}^{\infty}\gamma_{n+m}=1$$

This implies that $f\in A^*_{m0,k}(ŋ,\theta,\delta,z_0\ )$ .Conversely, suppose $f\in A^*_{m0,k}(ŋ,\theta,\delta,z_0\ )$. Since

$$\sum_{n=0}^{\infty}a_{n+m}\leq\frac{\theta(1-ŋ)(1+\delta)}{\left[\left(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-ŋ)\big)+\theta(1-ŋ)(1+\delta){z_0}^{n+m+1}\right)\right]}\quad n\geq 0$$

Set

$$\gamma_{n+m}=\frac{\left[\left(n^k\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-ŋ)\big)+\theta(1-ŋ)(1+\delta){z_0}^{n+m+1}\right)\right]}{\theta(1-ŋ)(1+\delta)}a_{n+m}\ n\geq 0$$

and $\gamma_0=1-\sum_{n=0}^{\infty}\gamma_{n+m}$

Then

$$f(z)=\sum_{n=0}^{\infty}\gamma_n f_n(z).$$

THEOREM 3.2.7 : Let $f_0(z) = \frac{1}{z}$,

and

$$f_{n+m}(z) = \sum_{n=0}^{\infty} \frac{\begin{matrix} n^k(n+m)\big((n+m)(1+\theta\delta) - \theta(1-\text{ŋ})(1+\delta)\big) \\ +\theta(1-\text{ŋ})(1+\delta)z^{n+m+1} \end{matrix}}{z\begin{pmatrix} n^k(n+m)\big((n+m)+(1+\theta\delta)-\theta(1+\delta)(1-\text{ŋ})\big) \\ -\theta(1-\text{ŋ})(1+\delta)(n+m){z_0}^{n+m+1} \end{pmatrix}}$$

Then $f(z)$ is in the class $A^*_{m1,k}(\text{ŋ}, \theta, \delta, z_0\ )$if and only if it can be expressed in the form

$f(z) = \sum_{n=0}^{\infty} \gamma_n f_{n+m}(z)$ where $\gamma_n \geq 0\ , \gamma_i = 0\ (i = 1,2, \ldots, m-1, m \geq 2)$ and $\sum_{n=0}^{\infty} \gamma_n = 1$

THEOREM 3.2.8: The class $A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_0\ )$ is closed under convex linear combination

PROOF : Suppose that the functions $f, g \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_0\ )$ defined by

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_{n+m} z^{n+m}, \qquad a_0 > 0, a_{n+m} > 0, z \in U^*$$

and

$$g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_{n+m} z^{n+m}, \qquad a_0 > 0, b_{n+m} > 0, z \in U^*$$

respectively, it is sufficient to prove that the function $\mathcal{H}$defined by
$\mathcal{H}(z) = \omega f(z) + (1-\omega)g(z),\ \ (0 \leq \omega \leq 1)$is also in the class$A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_0\ )$.

Since

$\mathcal{H}(z) = \frac{\omega a_0 + (1-\omega)b_0}{z} + \sum_{n=0}^{\infty} (\boldsymbol{\omega} a_{n+m} + (1-\omega)b_{n+m})\, z^{n+m}\ ,\ a_0 > 0\ , a_{n+m} > 0\ , z \in U^*$

We observe that

$$\sum_{n=0}^{\infty} \left[n^k(n+m)\big((n+m)(1+\theta\delta) - \theta(1-\text{ŋ})(1+\delta)\big) + {z_0}^{n+m+1}\right](\boldsymbol{\omega} a_{n+m} + (1 - \omega)b_{n+m}) \leq \theta(1-\text{ŋ})(1+\delta)$$

with the aid of *theorem* (3.2.14).Thus $\mathcal{H}(z) \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_0\ )$.

This completes the proof of the theorem.

THEOREM 3.2.9: The class $A^*_{m1,k}(\eta,\theta,\delta,z_0\ )$ is closed under convex linear combination

PROOF.the proof is similar on that of theorem(3.2.8)

THEOREM 3.2.10: Let the function $f_l(z), l=0,1,2,\dots,q\ defined\ by$

$$f_l(z)=\frac{a_{0,l}}{z}+\sum_{n=1}^{\infty}a_{n+m,l}z^{n+m},\qquad a_0>0, a_{n+m,l}>0, z\in U^*$$

be in the class $A^*_{m0,k}(\eta,\theta,\delta,z_0\ )$. Then the function

$$\vartheta(z)=\sum_{l=0}^{q}c_l f_l(z),\quad (c_l\geq 0)$$

is also in the class $A^*_{m0,k}(\eta,\theta,\delta,z_0\ )$, where $\sum_{l=0}^{q}c_l=1$

PROOF : By *theorem* (3.2.14) and for every $l=0,2,3,..,q$ we have

$$\sum_{n=0}^{\infty}\left[n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)+{z_0}^{n+m+1}\right]a_{n+m}$$
$$\leq\theta(1-\eta)(1+\delta)$$

Then

$$\vartheta(z)=\sum_{l=0}^{q}c_l\left(\frac{a_{0,l}}{z}+\sum_{n=1}^{\infty}a_{n+m,l}z^{n+m}\right),\quad (c_l\geq 0)$$
$$=\frac{c_l a_{0,l}}{z}+\sum_{n=0}^{\infty}\left(\sum_{l=0}^{q}c_l a_{n+m,l}\right)z^{n+m}$$

$$\sum_{n=0}^{\infty}\left[n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)+{z_0}^{n+m+1}\right]a_{n+m}$$

Since

$$\sum_{n=0}^{\infty}\left[n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)+{z_0}^{n+m+1}\right]\left(\sum_{l=0}^{q}c_l a_{n+m,l}\right)$$
$$=\sum_{l=0}^{q}c_l\left(\sum_{l=0}^{q}\left[n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)+{z_0}^{n+m+1}\right]a_{n+m,l}\right);$$
$$\leq\left(\sum_{l=0}^{q}c_l\right)\theta(1-\eta)(1+\delta),$$

$$= \theta(1-\eta)(1+\delta)$$

Then , $\vartheta(z) \in A^*_{m0,k}(\eta, \theta, \delta, z_0\ )$.

THEOREM 3.2.11: Let the function $f_l(z), l = 0,1,2, \dots, q\ defined\ by$

$$f_l(z) = \frac{a_{0,l}}{z} + \sum_{n=1}^{\infty} a_{n+m,l} z^{n+m}, \qquad a_0 > 0, a_{n+m,l} > 0, z \in U^*$$

be in the class $A^*_{m1,k}(\eta, \theta, \delta, z_0\ )$. Then the function

$$\vartheta(z) = \sum_{l=0}^{q} c_l f_l(z), \quad (c_l \geq 0)$$

is also in the class $A^*_{m1,k}(\eta, \theta, \delta, z_0\ )$, where $\sum_{l=0}^{q} c_l = 1$

PROOF.the proof is similar on that of theorem(3.2.10)

LEMMA 3.2.1: Let $z_1, z_2 \in C$ be two distinct positive numbers and $f \in A^*_{m0,k}(\eta, \theta, \delta, z_0) \cap (\eta, \theta, \delta, z_1)$, then $f(z) = \frac{1}{z}$.

PROOF: Suppose that $f \in A^*_{m0,k}(\eta, \theta, \delta, z_1) \cap A^*_{m0,k}(\eta, \theta, \delta, z_2)$.We have

$$a_{0=}1 - \sum_{n=0}^{\infty} a_{n+m} {z_1}^{n+m+1}$$

$$= 1 - \sum_{n=0}^{\infty} a_{n+m} {z_2}^{n+m+1}$$

also

$$f(z) = \frac{a_0}{z} - \sum_{n=0}^{\infty} a_{n+m} z^{n+m+1}\ a_0 > 0, a_{n+m,l} > 0, m \in \mathbb{N}$$

Thus , $a_{n+m} = 0\ , \forall n \geq 0$, because $a_{n+m} \geq 0\ , z_1 > 0, z_2 > 0$, hence

$$f(z) = \frac{1}{z}$$

This complete the proof of the Lemma.

THEOREM 3.2.12: Suppose that $c \subset [0,1], A^*_{m0,k}(\eta, \theta, \delta, c)$ is a convex family if and only if C is connected .

PROOF: Assume that C is connected and $z_1, z_2 \in C$ with $z_1 < z_2$.

$$a_{0=}1-\sum_{n=0}^{\infty} a_{n+m}{z_0}^{n+m+1}$$

$$=1-\sum_{n=0}^{\infty} a_{n+m}{z_1}^{n+m+1}$$

Suppose that the function $f\in A^*_{m0,k}(\eta,\theta,\delta,z_0)$ defined by

$f(z)=\frac{a_0}{z}-\sum_{n=0}^{\infty} a_{n+m}z^{n+m+1}\; a_0>0, a_{n+m,l}>0, m\in\mathbb{N}$, z$\in U$

and

$$g\in A^*_{m0,k}(\eta,\theta,\delta,z_1)$$

$$g(z)=\frac{b_0}{z}-\sum_{n=0}^{\infty} a_{n+m}z^{n+m+1}\; b_0>0, b_{n+m,l}>0, m\in\mathbb{N}, \mathrm{z}\in U^*$$

it is sufficient to prove that the function $\boldsymbol{\mathcal{H}}$ defined by

$$\mathcal{H}(z)=\boldsymbol{\omega f(z)}+(\mathbf{1}-\boldsymbol{\omega})\boldsymbol{g(z)},\;\; (\mathbf{0}\le\boldsymbol{\omega}\le\mathbf{1})$$

That there exists a $z_2(z_0\le z_2\le z_1)$ is also in the class$A^*_{m0,k}(\eta,\theta,\delta,z_2)$

Then

$$\c{k}(\omega)=z\mathcal{H}(z)$$

$$\c{k}(\omega)=\omega a_0+(1-\omega)b_0+\sum_{n=0}^{\infty}(\omega a_{n+m}+(1-\omega)b_{n+m})z^{n+m}, a_0>0\;, a_{n+m}>0\;, z\in U^*$$

$$=1+\boldsymbol{\omega}\sum_{n=0}^{\infty}(z^{n+m}-z_0^{n+m})a_{n+m}+(1-\boldsymbol{\omega})\sum_{n=0}^{\infty}(z^{n+m}-z_1^{n+m})b_{n+m}\;, a_0>0\;, a_{n+m} >0\;, z\in U^*$$

Since z is real number , then $\c{k}(z)$ is also number also we have $\c{k}(z_0)\le 1$ and $\c{k}(z_1)\ge 1$, there exists $z_2\in[z_0,z_1]$ such that $\c{k}(z_2)=1$.

Therefore , $z_2\mathcal{H}(z_2)=z_2, (z_0\le z_2\le z_1)$ This implies that $\mathcal{H}(z)\in A^*_{m0,k}$

We observe that

$$\sum_{n=0}^{\infty}\left[n^k(n+m)\big((n+m)(1+\theta\delta)-\theta(1+\delta)(1-\eta)\big)+{z_2}^{n+m+1}\right](\boldsymbol{\omega} a_{n+m}+(1-\omega)b_{n+m})$$

$$= \boldsymbol{\omega} \sum_{n=0}^{\infty} \left[n^k (n+m)\big((n+m)(1+\theta\delta) - \theta(1+\delta)(1-\text{ŋ})\big) + {z_0}^{n+m+1} \right] a_{n+m}$$
$$+ (1-\omega)\left[n^k (n+m)\big((n+m)(1+\theta\delta) - \theta(1+\delta)(1-\text{ŋ})\big) + {z_1}^{n+m+1} \right] b_{n+m} + \theta(1-\text{ŋ})(1+\delta)\omega \sum_{n=0}^{\infty} \left({z_2}^{n+m+1} - {z_0}^{n+m+1}\right) a_{n+m}$$
$$+ \theta(1-\text{ŋ})(1+\delta)(1-\omega) \sum_{n=0}^{\infty} \left({z_2}^{n+m+1} - {z_1}^{n+m+1}\right) b_{n+m}$$

$$= \omega \sum_{n=0}^{\infty} \left[n^k (n+m)\big((n+m)(1+\theta\delta) - \theta(1+\delta)(1-\text{ŋ})\big) + {z_0}^{n+m+1} \right] a_{n+m}$$
$$+ (1-\omega)\left[n^k (n+m)\big((n+m)(1+\theta\delta) - \theta(1+\delta)(1-\text{ŋ})\big) + {z_1}^{n+m+1} \right] b_{n+m} \le \theta(1-\text{ŋ})(1+\delta) + (1-\omega)\,\theta(1-\text{ŋ})(1+\delta)$$
$$= \theta(1-\text{ŋ})(1+\delta)$$

With the aid of *theorem* (3.2.14).

Thus , $\mathcal{H}(z) \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_2)$. Since z_1 and z_2 are arbitrary numbers , the family $A^*_{m0,k}(\text{ŋ}, \theta, \delta, c)$ is convex.

Conversely , if the set C is not connected , then there exists z_0, z_1 and z_2 $such\ that$ $z_0, z_1 \in C$ and $z_2 \notin C$ and $z_0 < z_2 < z_1$.

Now , let $f(z) \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_0)$ and $f(z) \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_1)$ therefore

$$\text{ķ}(\omega) = \text{ķ}(z_2, \omega)$$

$$= 1 + \omega \sum_{n=0}^{\infty} \left({z_2}^{n+m+1} - {z_0}^{n+m+1}\right) a_{n+m} + (1-\omega) \sum_{n=0}^{\infty} \left({z_2}^{n+m+1} - {z_1}^{n+m+1}\right) b_{n+m}, a_0 > 0\,, a_{n+m} > 0\,, z \in U^*$$

For fixed z_2 and $0 \le \omega < 1$.

Since$k(z_2$,0)<1 and $k(z_2$,1)<1, there exists $\boldsymbol{\omega}_0$, $0 \le \omega_0 < 1$ such that $k(z_2$,ω_0)=1 or $z_2 k(z_2$)=1, where $k(z) = \omega_0 f(z) + (1-\omega_0) g(z)$. Therefore $k(z) \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, z_0)$.

Also $k(z) \in A^*_{m0,k}(\text{ŋ}, \theta, \delta, C)$ using *Lemma* (3.22) Since $z_2 \in C$ and $k(z)$ ∉z.

Thus the family $A^*_{m0,k}(\text{ŋ}, \theta, \delta, C)$ is not convex which is a contradiction .

CHAPTER 4

SOME RESULTS OF SUBCLASSES ON MULTIVALENT HARMONIC FUNCTIONS

INTRODUCTION

Chapter four is fully devoted to study of some results of certain classes of harmonic multivalent functions.

Several mathematicians have contributed a lot in the study of harmonic functions. Ahuja and Jahangiri have defined and investigated a family of Noshiro-type complex valued harmonic functions of the form : $f = h + \overline{g}$,where h and g are analytic in the unit disk.

A lot of interesting properties leading to distortion theorem , extreme points , convolution conditions and convex combinations for the family of harmonic functions have been obtained by several researchers .Ahuja Jahangiri and Silverman[52]have contributed a lot about the contraction of harmonic univalent mappings.These authors have results i.e. for example by considering second fixed coefficients , a number of noted also researched for the subclass of harmonic univalent functions.Expressing Taylor series expansion under different conditions , these authors have obtained a lot of good results are found out.

At the same time a comprehensive class of complex valued harmonic univalent functions with varying arguments is introduced by Jahangiri(2002) [53] ,Jahangiri and Silverman [52].We also mention the contribution in the study of univalent harmonic functions by Rosy , Stephan ,Subramanian and Jahangiri(200)[54].Atshan and Wanas(2013)[55] discussed a new class of harmonic univalent functions

Chapter four is divided into four sections .In , section one and two , we have introduced and studied certain subclasses $SC_H(b,\gamma,\lambda)$and $\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ of harmonic multivalent functions of complex order consisting function of the form :

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad , \qquad g(z) = \sum_{n=p}^{\infty} b_n z^n \ |b_p| < 1$$

and satisfy the condition

$$\mathcal{R}\left\{1 + \frac{1}{b}\left(\frac{\Phi(z)}{\Psi(z)} - 1\right)\right\} \geq \gamma, \ \ z \in U,$$

where

$$\Phi(z) = \lambda\left(z^{q+2}(h(z))^{q+2} - \overline{z^{q+2}(g(z))^{q+2}}\right) + (2\lambda+1)\left(z^{q+1}(h(z))^{q+1}\right)$$
$$+ (1-4\lambda)\left(\overline{z^{q+1}(g(z))^{q+1}}\right) + \left(z^{q}(h(z))^{q}\right) + (1-2\lambda)\overline{z^{q}(g(z))^{q}}$$
$$\Psi(z) = \lambda\left(z^{q+2}(h(z))^{q+2} + \overline{z^{q+2}(g(z))^{q+2}}\right) + (2\lambda-1)\overline{\left(z^{q}(g(z))^{q}\right)}$$
$$+ \left(z^{q}(h(z))^{q}\right)$$

We investigate coefficient conditions, extreme points and distortion bounds. we also examine their convolution and convex combination properties and the closure property of this class under integral operator.

An attempt is also made in undertaking study of multivalent harmonic meromorphic functions in section three, we have introduced a subclass of multivalent harmonic meromorphic functions defined in the exterior of the unit disk and obtain the several geometric results which are routine in character.

4.1 CERTAIN SUBCLASSES OF HARMONIC MULTIVALENT FUNCTIONS OF COMPLEX ORDER

Let $S_H$the family of functions $f = h + \bar{g}$, which are harmonic, univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus , for $f = h + \bar{g} \in S_{H,}$ that are harmonic , orientation preserving and multivalentin the open disk U with the normalization

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad , \qquad g(z) = \sum_{n=p}^{\infty} b_n z^n \ |b_p| < 1$$

and $f(z)$ is then is given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n + \overline{\sum_{n=p}^{\infty} b_n z^n} \, |b_p| < 1 \tag{4.1}$$

Also, we denote by TS_H the subfamily of S_H consisting of harmonic function $f = h + g$ such that h and g are in form

$$h(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n \quad , \qquad g(z) = \sum_{n=p}^{\infty} |b_n| z^n \tag{4.2}$$

DEFINITION 4.1.1

For $0 \le \gamma < 1$, $0 \le \lambda \le \frac{\gamma}{(1+\gamma)}$ or $\lambda \ge \frac{1}{1+\gamma}$, $b \in C \setminus \{0\}$ $with$ $|b| < 1$ and let the $SC_H(b, \gamma, \lambda)$ denote the family of harmonic functions $f \in S_H$ of the form (4.1) which satisfy the condition

$$\mathcal{Re}\left\{1 + \frac{1}{b}\left(\frac{\Phi(z)}{\Psi(z)} - 1\right)\right\} \ge \gamma, \;\; z \in U, \tag{4.3}$$

where

$$\Phi(z) = \lambda\left(z^{q+2}(h(z))^{q+2} - \overline{z^{q+2}(g(z))^{q+2}}\right) + (2\lambda + 1)\left(z^{q+1}(h(z))^{q+1}\right)$$
$$+ (1 - 4\lambda)\left(\overline{z^{q+1}(g(z))^{q+1}}\right) + \left(z^{q}(h(z))^{q}\right) + (1 - 2\lambda)\overline{z^{q}(g(z))^{q}}$$

$$\Psi(z) = \lambda\left(z^{q+2}(h(z))^{q+2} + \overline{z^{q+2}(g(z))^{q+2}}\right) + (2\lambda - 1)\overline{\left(z^{q}(g(z))^{q}\right)}$$
$$+ \left(z^{q}(h(z))^{q}\right)$$

We begin with a sufficient coefficient bounds for the class $SC_H(b, \gamma, \lambda)$. These conditions are shown to be necessary for the functions in $TSC_H(b, \gamma, \lambda)$.where

$$TSC_H(b, \gamma, \lambda) = SC_H(b, \gamma, \lambda) \cap TS_H$$

THEOREM 4.1.1. Let $f = h + \bar{g}$ with h and g are given by (4.1) . if

$$\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}(\lambda n - \lambda q + 1)\left((n-q) + |b|(1-\gamma)\right)}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}} |a_n|$$
$$+ \sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(n-q)(\lambda(n-q-1) - (1-5\lambda)) - 2(1-2\lambda) - |b|(1-\gamma)(\lambda n - \lambda q + 1)\}}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}} |b_n|$$
$$\le 1 \tag{4.4}$$

where $\alpha_1 = 1$, $0 \le \gamma < 1$, $(|b| < 1)$ is a non – zero complex number, $0 \le \lambda \le \frac{\gamma}{1+\gamma}$ or $\lambda \le \frac{1}{1+\gamma}$ then $f \in SC_H(b, \gamma, \lambda)$ and f is sense preserving multivalent harmonic in U.

PROOF. We show that f is multivalent in U, we show that $f(z_1) \ne f(z_2)$ whenever $z_1 \ne z_2$. Since U is simply connected and convex we have

$z(t) = (1-t)z_1 + tz_2 \in U$, where $0 \le t \le 1$ and if $z_1, z_2 \in U$ *so that* $z_1 \neq z_2$. Then we write

$$f(z_2) - f(z_1) = \int_0^1 \left\{ (z_2 - z_1)h'\big(z(t)\big) + \overline{(z_2 - z_1)g'\big(z(t)\big)} \right\} dt.$$

Dividing the above equation by $(z_1 - z_2) \neq 0$ and taking the real part, we obtain

$$\mathcal{R}e\left\{\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right\} = \int_0^1 \mathcal{R}e\left\{h'\big(z(t)\big) + \frac{\overline{(z_2 - z_1)}}{(z_2 - z_1)} g'\big(z(t)\big)\right\} dt.$$

$$> \left\{\mathcal{R}e\, h'\big(z(t)\big) + \left|g'\big(z(t)\big)\right|\right\} dt$$

On the other hand, for $|b| < 1$, $0 \le \lambda \le \frac{\gamma}{(1+\gamma)}$ *or* $\lambda \ge \frac{1}{1+\gamma}$ we have

$$\mathcal{R}eh'(z) - |g'(z)| \ge \mathcal{R}eh'(z) - \sum_{n=p}^{\infty} \frac{n!}{(n-q)!}|b_n|$$

$$\ge \frac{p!}{(p-q)!} - \sum_{n=p+1}^{\infty} \frac{n!}{(n-q)!}|a_n| - \sum_{n=p}^{\infty} \frac{n!}{(n-q)!}|b_n|$$

$$\ge \frac{p!}{(p-q)!}$$

$$- \sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}(\lambda n - \lambda q + 1)\big((n-q) + |b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}}|a_n|$$

$$- \sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(n-q)\big(\lambda(n-q-1) - (1-5\lambda)\big) - 2(1-2\lambda) - |b|(1-\gamma)(\lambda n - \lambda q + 1)\}}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}}|b_n| \ge 0 \tag{4.5}$$

This along with inequality (4.5) leads to the *p*-valent of f Note that f is sense preserving in U, for for $|b| < 1$, $0 \le \lambda \le \frac{\gamma}{(1+\gamma)}$ *or* $\lambda \ge \frac{1}{1+\gamma}$. This is because

$$|h'(z)| \ge \frac{p!}{(p-q)!} - \sum_{n=p+1}^{\infty} \frac{n!}{(n-q)!}|a_n||z|^{n-p} > \frac{p!}{(p-q)!} - \sum_{n=p+1}^{\infty} \frac{n!}{(n-q)!}|a_n|$$

$$\ge \frac{p!}{(p-q)!} - \sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}(\lambda n - \lambda q + 1)\big((n-q) + |b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}}|a_n|$$

$$\ge \sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\left\{\begin{array}{c}(n-q)\big(\lambda(n-q-1) - (1-5\lambda)\big) - 2(1-2\lambda) \\ -|b|(1-\gamma)(\lambda n - \lambda q + 1)\end{array}\right\}}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}}|b_n|$$

$$> \sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\left\{\begin{array}{c}(n-q)\big(\lambda(n-q-1) - (1-5\lambda)\big) - 2(1-2\lambda) \\ -|b|(1-\gamma)(\lambda n - \lambda q + 1)\end{array}\right\}}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}}|b_n||z|^{n-p}$$

$$\ge \sum_{n=p}^{\infty} \frac{n!}{(n-q)!}|b_n||z|^{n-p}$$

$$\ge |g'(z)|$$

The function

$$f(z)=z^p+\sum_{n=p+1}^{\infty}\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}x_n z^n$$

$$+\sum_{n=p}^{\infty}\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{n!}{(n-q)!}\left\{\begin{matrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{matrix}\right\}}\overline{y_n z^n} \qquad (4.6)$$

Where $\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|y_n|=1$, shows that the coefficient bound given by (4.4) is sharp. The functions of the form (4.6) are in $SC_H(b,\gamma,\lambda)$ because

$$\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|a_n|$$

$$+\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\left\{\begin{matrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|b_n|$$

$$=1+\sum_{n=p+1}^{\infty}|x_n|+\sum_{n=p}^{\infty}|y_n|=2$$

The next theorem shows that the condition (4.4) is necessary for $f\in TSC_H(b,\gamma,\lambda)$.

Putting q=1, in the above theorem , to obtain

<u>COROLLARY 4.1.1</u>. Let $f=h+\bar{g}$ with h *and* g are given by (4.2) if

$$\sum_{n=p+1}^{\infty}\frac{(\lambda n-\lambda+1)\big((n-1)+|b|(1-\gamma)\big)}{|b|(1-\gamma)}|a_n|+\sum_{n=p}^{\infty}\frac{n(\lambda n+\lambda-1)\big((n+1)-|b|(1-\gamma)\big)}{|b|(1-\gamma)}|b_n|$$
$$\le 2$$

Where $a_1=1$,$0\le\gamma<1$, $b(|b|<1)$ is a non-zero complex number, $0\le\lambda\le\frac{\gamma}{(1+\gamma)}$ *or* $\lambda\ge\frac{1}{1+\gamma}$. then $f\in SC_H(b,\gamma,\lambda)$ and f is sense preserving , univalent harmonic in U.

We note that the result obtained by [56].

<u>THEOREM 4.1.2</u> Let $f=h+\bar{g}$ with h *and* g are given by (4.2) . Then $f\in TSC_H(b,\gamma,\lambda)$ if and only if

$$\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|a_n|$$

$$+\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\left\{\begin{matrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|b_n|\le 1,$$

(4.6)

where $a_1=1$,$0\le\gamma<1$, $0\le\lambda\le\frac{\gamma}{(1+\gamma)}$ *or* $\lambda\ge\frac{1}{1+\gamma}$ *and* $b\in\mathbb{C}\backslash\{0\}$.

PROOF.The ’if part’ follows from *Theorem* (4.1.1) upon noting that $TSC_H(b,\gamma,\lambda) \subset SC_H(b,\gamma,\lambda)$. For the ’only if’ part, we show that $f \in TSC_H(b,\gamma,\lambda)$. Then for $z = re^{i\theta} in\ U$, we obtain

$$\mathcal{R}e\left\{1+\frac{1}{b}\left(\frac{\begin{array}{c}\lambda\left(z^{q+2}\left(h(z)\right)^{q+2}-\overline{z^{q+2}\left(g(z)\right)^{q+2}}\right)+(2\lambda+1)\left(z^{q+1}\left(h(z)\right)^{q+1}\right)\\+(1-4\lambda)\left(\overline{z^{q+1}\left(g(z)\right)^{q+1}}\right)+\left(z^{q}\left(h(z)\right)^{q}\right)+(1-2\lambda)\overline{z^{q}\left(g(z)\right)^{q}}\end{array}}{\begin{array}{c}\lambda\left(z^{q+2}\left(h(z)\right)^{q+2}+\overline{z^{q+2}\left(g(z)\right)^{q+2}}\right)\\+(2\lambda-1)\overline{z^{q}\left(g(z)\right)^{q}}+\left(z^{q}\left(h(z)\right)^{q}\right)\end{array}}-1\right)-\gamma\right\}$$

$$=\mathcal{R}e\left\{(1-\gamma)+\frac{1}{b}\left(\frac{\begin{array}{c}\lambda z^{q+2}\left(h(z)\right)^{q+2}+(\lambda+1)\left(z^{q+1}\left(h(z)\right)^{q+1}\right)-\lambda\left(\overline{z^{q+1}\left(g(z)\right)^{q+1}}\right)\\+(1-5\lambda)\left(\overline{z^{q+1}\left(g(z)\right)^{q+1}}\right)+2(1-2\lambda)\overline{z^{q}\left(g(z)\right)^{q}}\end{array}}{\begin{array}{c}\lambda\left(z^{q+2}\left(h(z)\right)^{q+2}+\overline{z^{q+2}\left(g(z)\right)^{q+2}}\right)\\+(2\lambda-1)\left(\overline{z^{q}\left(g(z)\right)^{q}}\right)+\left(z^{q}\left(h(z)\right)^{q}\right)\end{array}}\right)\right\}$$

$$=\mathcal{R}e\left\{(1-\gamma)+\frac{1}{b}\left(\frac{\begin{array}{c}\frac{\lambda p!}{(p-q-2)!}z^p+\sum_{n=p+1}^{\infty}\frac{\lambda n!}{(n-q-2)!}a_nz^n+\frac{(\lambda+1)p!}{(p-q-1)!}z^p+\sum_{n=p+1}^{\infty}\frac{(\lambda+1)n!}{(n-q-1)!}a_nz^n\\-\sum_{n=p}^{\infty}\frac{\lambda n!}{(n-q-2)!}\overline{b_nz^n}+\sum_{n=p}^{\infty}\frac{(1-5\lambda)n!}{(n-q-1)!}\overline{b_nz^n}\\+\sum_{n=p}^{\infty}\frac{2(1-2\lambda)n!}{(n-q)!}\overline{b_nz^n}\end{array}}{\begin{array}{c}\frac{\lambda p!}{(p-q-1)!}z^p+\sum_{n=p+1}^{\infty}\frac{\lambda n!}{(n-q-1)!}a_nz^n+\frac{p!}{(p-q)!}z^p+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q)!}a_nz^n+\\ \sum_{n=p}^{\infty}\begin{array}{c}\frac{\lambda n!}{(n-q-1)!}\overline{b_nz^n}\\+\sum_{n=p}^{\infty}\frac{(2\lambda-1)n!}{(n-q)!}\overline{b_nz^n}\end{array}\end{array}}\right)\right\}$$

$$=\mathcal{R}e\left\{(1-\gamma)+\frac{1}{b}\left(\frac{\begin{array}{c}\frac{p!}{(p-q-1)!}\{\lambda(p-q-1)+(\lambda+1)\}z^p+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q-1)!}\{\lambda(n-q-1)+(\lambda+1)\}a_nz^n\\-\sum_{n=p}^{\infty}\frac{n!}{(n-q)!}\{\lambda(n-q)(p-q-1)-(n-q)(1-5\lambda)-2(1-2\lambda)\}\overline{b_nz^n}\end{array}}{\begin{array}{c}\frac{p!}{(p-q)!}\{\lambda(p-q)+1\}z^p+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q)!}\{\lambda(n-q)+1\}a_nz^n\\+\sum_{n=p}^{\infty}\frac{n!}{(n-q)!}\{\lambda(n-q)+(2\lambda-1\}b_nz^n\end{array}}\right)\right\}$$

$$\mathcal{R}e\left\{(1-\gamma)+\left(\frac{\begin{array}{c}\frac{p!}{b(p-q-1)!}\{\lambda p-\lambda q+1\}z^p+\sum_{n=p+1}^{\infty}\frac{n!}{b(n-q-1)!}\{\lambda n-\lambda q+1\}a_nz^n\\-\sum_{n=p}^{\infty}\frac{n!}{b(n-q)!}\left\{(n-q)\left(\lambda(n-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\right\}\overline{b_nz^n}\end{array}}{\begin{array}{c}\frac{p!}{(p-q)!}\{\lambda p-\lambda q+1\}z^p+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q)!}\{\lambda n-\lambda q+1\}a_nz^n\\+\sum_{n=p}^{\infty}\frac{n!}{(n-q)!}\{\lambda n-\lambda q+2\lambda-1\}\overline{b_nz^n}\end{array}}\right)\right\}$$

$$=\mathcal{R}e\left\{\frac{\begin{array}{c}\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}z^p+\\ \sum_{n=p+1}^{\infty}\frac{n!}{|b|(n-q)!}(\lambda n-\lambda q+1)\left((n-q)+|b|(1-\gamma)\right)|a_n|z^n-\\ \sum_{n=p}^{\infty}\frac{n!}{|b|(n-q)!}\left\{\begin{array}{c}(n-q)\left(\lambda(n-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\\-|b|(1-\gamma)(\lambda n-\lambda q+1)\end{array}\right\}|b_n|\overline{z^n}\end{array}}{\begin{array}{c}\frac{p!}{(p-q)!}\{\lambda p-\lambda q+1\}z^p+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q)!}\{\lambda n-\lambda q+1\}|a_n|z^n\\+\sum_{n=p}^{\infty}\frac{n!}{(n-q)!}\{\lambda n-\lambda q+2\lambda-1\}\overline{|b_n|z^n}\end{array}}\right\}$$

$$\geq \mathcal{R}e\left\{\frac{\begin{array}{c}\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\} \\ -\sum_{n=p+1}^{\infty}\frac{n!}{|b|(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)|a_n|r^{n-p} \\ +\sum_{n=p}^{\infty}\frac{n!}{|b|(n-q)!}\left\{\begin{array}{c}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big) \\ -2(1-2\lambda)-|b|(1-\gamma)(\lambda n-\lambda q+1)\end{array}\right\}|b_n|r^{n-p}\end{array}}{\begin{array}{c}\frac{p!}{(p-q)!}\{\lambda p-\lambda q+1\}z^p+\sum_{n=p+1}^{\infty}\frac{n!}{(n-q)!}\{\lambda n-\lambda q+1\}|a_n|r^{n-p} \\ +\sum_{n=p}^{\infty}\frac{n!}{(n-q)!}\{\lambda n-\lambda q+2\lambda-1\}|b_n|r^{n-p}\end{array}}\right\}$$

$> o$

The above inequality must hold for all z ∈ U. In particular, letting z = r → $^{-}1$ yields the required condition.

The next theorem gives the extreme points of the closed convex hulls of $TSC_H(b,\gamma,\lambda)$

THEOREM 4.1.3.A function $f = h + \bar{g}$ belongs $\mathrm{TSC_H}(b,\gamma,\lambda)$ if and only if f can be expressed as

$$f(z) = \sum_{n=p}^{\infty}\big(X_n h_n(z) + Y_n g_n(z)\big)\,, \tag{4.8}$$

where

$$h_p(z) = z^p, h_n(z) = z^p - \frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}z^n, \quad (n = p+1, p+2, \ldots.)$$

and

$$g_n(z) = z^p + \frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{n!}{(n-q)!}\left\{\begin{array}{c}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda) \\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{array}\right\}}z^n$$

$$(n = p, p+1, \ldots\ldots), \sum_{n=p}^{\infty}(X_n + Y_n) = 1, \;\; X_n \geq 0 \quad and \quad Y_n \geq 0$$

In particular the extreme points of $\mathrm{TSC_H}(b,\gamma,\lambda)$ are $\{h_n\} and \{g_n\}$.

PROOF : Let f be written as (4.8). Then ,we have

$$f(z) = \sum_{n=p}^{\infty}\big(X_n h_n(z) + Y_n g_n(z)\big)$$

$$= \sum_{n=p}^{\infty}(X_n+Y_n)\,z^p - \sum_{n=p+1}^{\infty}\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}(\,\lambda\,)}X_n z^n$$

$$-\sum_{n=p}^{\infty}\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{n!}{(n-q)!}\left\{\begin{array}{c}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda) \\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{array}\right\}}Y_n(\bar{z})^n$$

then,

$$\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}} \times \left(\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}\right) X_n$$

$$+\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\left\{\begin{matrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}$$

$$\times\left(\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{n!}{(n-q)!}\left\{\begin{matrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{matrix}\right\}}\right) Y_n(\bar{z})^n$$

$$=\sum_{n=p+1}^{\infty} X_n+\sum_{n=p}^{\infty} Y_n = 1-X_p \le 1.$$

Then $f \in$ clco $TSC_H(b,\gamma,\lambda)$. Conversely, assume that $f \in$ clco $TSC_H(b,\gamma,\lambda))$.

Letting

$$X_p = 1-\sum_{n=p+1}^{\infty} X_n-\sum_{n=p}^{\infty} Y_n$$

where

$$X_n=\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|a_n|, \quad n=p+1,p+2,\ldots$$

$$Y_n=\frac{\frac{n!}{(n-q)!}\left\{\begin{matrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}(1-\alpha)}|b_n|, \quad n=p,p+1,\ldots$$

we obtain the required representation, since

$$f(z)=z^p-\sum_{n=p+1}^{\infty}|a_n|z^n+\sum_{n=p+1}^{\infty}|b_n|(\bar{z})^n$$

$$=z^p-\sum_{n=p+1}^{\infty}\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}X_n}{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}z^n$$

$$+\sum_{n=p}^{\infty}\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}Y_n}{\frac{n!}{(n-q)!}\left\{\begin{matrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{matrix}\right\}}(\bar{z})^n$$

$=z^p-\sum_{n=p+1}^{\infty}(z^p-h_n(z))X_n-\sum_{n=p}^{\infty}(z^p-g_n(z))y_n$

$=\big(1-\sum_{n=p+1}^{\infty}X_n-\sum_{n=p}^{\infty}y_n\big)z^p+\sum_{n=p+1}^{\infty}h_n(z)X_n-\sum_{n=p}^{\infty}g_n(z)y_n$

$=\sum_{n=p}^{\infty}X_nh_n(z)+y_ng_n(z)$

The following theorem gives the distortion bounds for functions inwhich $TSC_H(b,\gamma,\lambda)$yields a covering result for this family.

THEOREM 4.1.4. If $f \in TSC_H(b,\gamma,\lambda)$then for $z=re^{i\theta}$, we have

$$|f(z)| \geq (1+|b_p|)r^p + \left(\frac{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda - \lambda q + 1\}\{(p+1-q) + |b|(1-\gamma)\}} - \frac{\frac{p!}{(p-q)!}\left\{ \begin{matrix} (p-q)\big(\lambda(p-q-1) - (1-5\lambda)\big) - 2(1-2\lambda) \\ -|b|(1-\gamma)(\lambda p - \lambda q + 1) \end{matrix} \right\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda - \lambda q + 1\}\{(p+1-q) + |b|(1-\gamma)\}} \right) r^{p+1}$$

and

$$|f(z)| \leq (1+|b_p|)r^p + \left(\frac{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda - \lambda q + 1\}\{(p+1-q) + |b|(1-\gamma)\}} - \frac{\frac{p!}{(p-q)!}\left\{ \begin{matrix} (p-q)\big(\lambda(p-q-1) - (1-5\lambda)\big) - 2(1-2\lambda) \\ -|b|(1-\gamma)(\lambda p - \lambda q + 1) \end{matrix} \right\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda - \lambda q + 1\}\{(p+1-q) + |b|(1-\gamma)\}} \right) r^{p+1}$$

PROOF: We have

$$|f(z)| \leq (1+|b_p|)r^p + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n$$

$$\leq (1+|b_p|)r^p + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^{p+1}$$

$$= (1+|b_p|)r^p + \frac{1}{\frac{\frac{p!}{(p+1-q)!}\{(p+1)\lambda - \lambda q + 1\}\{(p+1-q) + |b|(1-\gamma)\}}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}}}$$

$$\times \sum_{n=p+1}^{\infty} \left(\frac{\frac{p!}{(p+1-q)!}\{(p+1)\lambda - \lambda q + 1\}\{(p+1-q) + |b|(1-\gamma)\}}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}} |a_n| + \frac{\frac{p!}{(p+1-q)!}\{(p+1)\lambda - \lambda q + 1\}\left\{ \begin{matrix} (p+1-q) \\ +|b|(1-\gamma) \end{matrix} \right\}}{\frac{p!}{(p-q)!}(\lambda p - \lambda q + 1)\{|b|(1-\gamma) - (p-q)\}} |b_n| \right) r^{p+1}$$

$$\leq \left(1+|b_p|\right)r^p + \frac{1}{\frac{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\left\{\begin{matrix}(p+1-q)\\+|b|(1-\gamma)\end{matrix}\right\}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}}$$

$$\times \sum_{n=p+1}^{\infty}\left(\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\left((n-q)+|b|(1-\gamma)\right)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|a_n|\right.$$

$$\left.-\frac{\frac{n!}{(n-q)!}\left\{\begin{matrix}(n-q)\left(\lambda(n-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\\-|b|(1-\gamma)(\lambda n-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|b_n|\right)r^{p+1}$$

$$\leq \left(1+|b_p|\right)r^p + \frac{1}{\frac{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}}$$

$$\times\left(1-\frac{\frac{p!}{(p-q)!}\left\{\begin{matrix}(p-q)\left(\lambda(p-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\\-|b|(1-\gamma)(\lambda p-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}\right)r^{p+1}$$

$$\leq \left(1+|b_p|\right)r^p + \left(\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}\right.$$

$$\left.-\frac{\frac{p!}{(p-q)!}\left\{\begin{matrix}(p-q)\left(\lambda(p-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\\-|b|(1-\gamma)(\lambda p-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}\right)r^{p+1}$$

Similarly

$$|f(z)| \geq \left(1+|b_p|\right)r^p$$

$$+\left(\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}\right.$$

$$\left.-\frac{\frac{p!}{(p-q)!}\left\{\begin{matrix}(p-q)\left(\lambda(p-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\\-|b|(1-\gamma)(\lambda p-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}|b_p|\right)r^{p+1}$$

The upper and lower bounds given in *Theorem* (4.1.4) are respectively attained for the following functions

$$f(z) = z^p + |b_p|\overline{(z)}^p + \frac{1}{\Gamma(p+1)}\left(\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}} - \frac{\frac{p!}{(p-q)!}\begin{Bmatrix}(p-q)\big(\lambda(p-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda p-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}|b_p|\right)\overline{(z)}^{p+1}$$

and

$f(z) =$

$$(1-|b_p|)z^p - \frac{1}{\Gamma(p+1)}\left(\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}} - \frac{\frac{p!}{(p-q)!}\begin{Bmatrix}(p-q)\big(\lambda(p-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda p-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}|b_p|\right)z^{p+1}.$$

We show that the class $TSC_H(b,\gamma,\lambda)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For

$$f(z) = z^p + \sum_{n=p+1}^{\infty}|a_n|z^n + \sum_{n=p}^{\infty}|b_n|\bar{z}^n$$

and

$$F(z) = z^p + \sum_{n=p+1}^{\infty}|A_n|z^n + \sum_{n=p}^{\infty}|B_n|\bar{z}^n$$

we define the convolution of two harmonic functions f and F as

$$(f*F)(z) = f(z)*F(z) = z^p - \sum_{n=p+1}^{\infty}|a_n||A_n|z^n + \sum_{n=p}^{\infty}|b_n||B_n|\overline{(z)}^n \qquad (4.9)$$

Using the definition, we show that the class $TSC_H(b,\gamma,\lambda)$ is closed under convolution

THEOREM4.1.5 For $0 \le \delta < \gamma < 1$, *let* $f \in TSC_H(b,\gamma,\lambda)$ *and* $F \in TSC_H(b,\delta,\lambda)$.then$f * F \in TSC_H(b,\gamma,\lambda) \subset TSC_H(b,\delta,\lambda)$

PROOF.

Let

$$f(z) = z^p + \sum_{n=p+1}^{\infty}|a_n|z^n + \sum_{n=p}^{\infty}|b_n|\bar{z}^n$$

and

$$F(z) = z^p + \sum_{n=p+1}^{\infty}|A_n|z^n + \sum_{n=p}^{\infty}|B_n|\bar{z}^n$$

be in $TSC_H(b,\gamma,\lambda)$. Then $f * F \in TSC_H(b,\delta,\lambda)$. We note that $|A_n| \le 1$ and $|B_n| \le 1$. In view of Theorem 4.1.2 and the inequality$0 \le \delta < \gamma < 1$, we have

$$\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\delta)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\delta)-(p-q)\}}|a_n||A_n|$$
$$+\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\begin{Bmatrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\delta)(\lambda n-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\delta)-(p-q)\}}|b_n||B_n|$$
$$\leq\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\delta)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\delta)-(p-q)\}}|a_n|$$
$$+\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\begin{Bmatrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\delta)(\lambda n-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\delta)-(p-q)\}}|b_n|$$

$$\leq\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|a_n|$$
$$+\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\begin{Bmatrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|b_n|\leq 1$$

by *Theorem* (4.1.2), $f\in TSC_H(b,\gamma,\lambda)$. By the same token, we then conclude that $f*F\in TSC_H(b,\gamma,\lambda)\subset TSC_H(b,\delta,\lambda)$.

Next, we show that the class $TSC_H(b,\gamma,\lambda)$ is closed under convex combination of its members

THEOREM 4.1.6 The class $TSC_H(b,\gamma,\lambda)$ is closed under convex combinations

PROOF. Suppose $f_i\in TSC_H(b,\gamma,\lambda)$(i = 1, 2,· · ·) are defined by

$$f_i(z)=z^p-\sum_{n=p+1}^{\infty}|a_{i,n}|z^n+\sum_{n=p}^{\infty}|b_{i,n}|\bar{z}^n,$$

for $\sum_{i=1}^{\infty}t_i=1, 0\leq t_i\leq 1$, the convex combination of $f_i$may be written as

$$\sum_{i=1}^{\infty}t_if_i=z^p-\sum_{n=p+1}^{\infty}\left(\sum_{i=1}^{\infty}t_i\,|a_{i,n}|\right)z^n-\sum_{n=p}^{\infty}\left(\sum_{i=1}^{\infty}t_i\,|b_{i,n}|\right)\bar{z}^n$$

Since

$$\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\big((n-q)+|b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|a_{i,n}|$$
$$+\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\begin{Bmatrix}(n-q)\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}|b_{i,n}|\leq 1$$

From the above equation we obtain

$$\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\left((n-q)+|b|(1-\gamma)\right)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}\left(\sum_{i=1}^{\infty}t_i\left|a_{i,n}\right|\right)$$

$$+\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\begin{Bmatrix}(n-q)\left(\lambda(n-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\\-|b|(1-\gamma)(\lambda n-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}\left(\sum_{i=1}^{\infty}t_i\left|b_{i,n}\right|\right)$$

$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\left((n-q)+|b|(1-\gamma)\right)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}\left|a_{i,n}\right|\right.$$

$$\left.++\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\begin{Bmatrix}(n-q)\left(\lambda(n-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\\-|b|(1-\gamma)(\lambda n-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}\left|b_{i,n}\right|\right\}$$

$$\le\sum_{i=1}^{\infty}t_i=1$$

Then $\sum_{i=1}^{\infty}t_i\,f_i\in TSC_H(b,\gamma,\lambda)$.

We consider the closure property of the class $TSC_H(b,\gamma,\lambda)$under the Bernardi integral operator$\mathcal{L}_c[f(z)]$ which is defined by

$$\mathcal{L}_c[f(z)]=\frac{c+1}{z^c}\int_0^z\zeta^{c-1}f(\zeta)d\zeta\qquad c>-1$$

THEOREM 3.2.10.. Let $f\in TSC_H(b,\gamma,\lambda).Then\ \mathcal{L}_c[f(z)]\in TSC_H(b,\gamma,\lambda)$

PROOF. From the representation of $\mathcal{L}_c[f(z)]$, it follows that

$$\mathcal{L}_c[f(z)]=\frac{c+1}{z^c}\int_0^z\zeta^{c-1}f(\zeta)d\zeta+\overline{\frac{c+1}{z^c}\int_0^z\zeta^{c-1}f(\zeta)d\zeta}$$

$$\frac{c+1}{z^c}\int_0^z\zeta^{c-1}\left(\zeta^p-\sum_{n=p+1}^{\infty}|a_n|\zeta^n\right)d\zeta+\overline{\frac{c+1}{z^c}\int_0^z\zeta^{c-1}\left(\sum_{n=p}^{\infty}|b_n|\zeta^n\right)d\zeta}$$

$$=z^p-\sum_{n=p+1}^{\infty}A_nz^n+\sum_{n=p}^{\infty}B_nz^n,$$

Where $A_n=\frac{c+1}{c+n}|a_n|$ and $B_n=\frac{c+1}{c+n}|b_n|$. Hence

$$\sum_{n=p+1}^{\infty}\frac{\frac{n!}{(n-q)!}(\lambda n-\lambda q+1)\left((n-q)+|b|(1-\gamma)\right)}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}\times\left(\frac{c+1}{c+n}|a_n|\right)$$

$$+\sum_{n=p}^{\infty}\frac{\frac{n!}{(n-q)!}\begin{Bmatrix}(n-q)\left(\lambda(n-q-1)-(1-5\lambda)\right)-2(1-2\lambda)\\-|b|(1-\gamma)(\lambda n-\lambda q+1)\end{Bmatrix}}{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}$$

$$\times\left(\frac{c+1}{c+n}|b_n|\right)$$

$$\leq \sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}(\lambda n-\lambda \mathrm{q}+1)\big((n-q)+|b|(1-\gamma)\big)}{\frac{p!}{(p-q)!}(\lambda p-\lambda \mathrm{q}+1)\{|b|(1-\gamma)-(p-q)\}}|a_n|$$

$$+\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\begin{Bmatrix}(\mathrm{n}-\mathrm{q})\big(\lambda(n-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda n-\lambda \mathrm{q}+1)\end{Bmatrix}}{\frac{p!}{(p-q)!}(\lambda p-\lambda \mathrm{q}+1)\{|b|(1-\gamma)-(p-q)\}}|b_n| \leq 1$$

Since $\mathrm{f} \in TSC_H(b,\gamma,\lambda)$, Therefore by *theorem* (4.1.2) , $\mathcal{L}_c[f(z)] \in TSC_H(b,\gamma,\lambda)$.

4.2 A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY THE DZIOK-RAINA OPERATOR

Denote by $S_H$the family of functions $f = h + \bar{g}$, which are harmonic, multivalent and orientation preserving in the open unit disc U = {z : |z| < 1} so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus , for $f = h + \bar{g} \in S_{H,}$ that are harmonic , orientation preserving and multivalent in the open disk U with the normalization

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad , \qquad g(z) = \sum_{n=p}^{\infty} b_n z^n \; |b_p| < 1$$

and $f(z)$ is then is given by (4.1)

We note that the family S_H of orientation preserving, normalized harmonic multivalent functions reduces to the well known class S of normalized multivalent functions if the co-analytic part of f is identically zero ($g\equiv 0$).

Also, we denote by TS_H the subfamily of S_H consisting of harmonic functions of the form$f = h + \bar{g}$ such that h and g are of the form (4.2), defined the subclass $G_H(\gamma) \subset S_H$ consisting of harmonic multivalent functions $f(z)$satisfying the following condition

$$\mathcal{R}e\left\{(1 + e^{i\alpha})\frac{zf'(z)}{z'f(z)} - e^{i\alpha}\right\} \geq \gamma, \quad 0 \leq \gamma < 1\,, \quad \alpha \in R \tag{4.10}$$

Where

$z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right), f'(z) = \frac{\partial}{\partial\theta}\left(f(z) = f(re^{i\theta})\right), 0 \leq \gamma < 1,$ and θ is real. The Hadamard product (or convolution) of two power series

$$\varphi(z) = z + \textstyle\sum_{n=p+1}^{\infty} \varphi_n z^n \text{ and } \Psi(z) = z + \sum_{n=p+1}^{\infty} \Psi_n z^n \tag{4.11}$$

in S is defined by

$$(\varphi * \Psi)(z) = \varphi(z) * \Psi(z) = z + \sum_{n=p+1}^{\infty} \Psi_n \varphi_n z^n \;. \tag{4.12}$$

For positive real parameters $\alpha_1, A_1, \dots\dots, \alpha_l, A_l$ *and* $\beta_1, B_1, \dots\dots, \beta_m, B_m (l, m \in \mathbb{N} = 1,2,3, \dots)$such that $l\Psi m$

$$1 + \sum_{i=1}^{m} B_i - \sum_{j=1}^{t} A_j \geq 0, \tag{4.13}$$

the Wrights generalization is given by

$$ {}_l\Psi_m[(\alpha_1, A_1), .., (\alpha_l, A_l);\ (\beta_1, B_1), \dots, (\beta_m, B_m); z] = {}_l\Psi_m\left[(\alpha_i, A_i)_{1,l};\ (\beta_j, B_j)_{1,m}; z\right] $$

of the generalized hypergeometric function $pFq(\alpha_1, \dots\dots \alpha_p;\ \beta_1, \dots\dots, \beta_q; z)$is defined by

$$ {}_l\Psi_m\left[(\alpha_t, A_t)_{1,l};\ (\beta_t, B_t)_{1,m}; z\right] = \sum_{n=0}^{\infty}\left(\prod_{t=0}^{l}\Gamma(\alpha_t + nA_t\right)\left(\prod_{t=0}^{m}\Gamma(\beta_t + nB_t)^{-1}\right)\frac{z^n}{n!}, z \in U $$

if $A_t = 1\ (t = 1, \dots, l)\ and\ B_t = 1(t = 1, \dots, m)$, we have the relationship

$$ \Omega\ {}_l\Psi_m\left[(\alpha_t, 1)_{1,l}(\beta_t, 1)_{1,m}; z\right] \equiv {}_l\Psi_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots\dots, \beta_m; z) $$

$$ = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n \dots\dots (\alpha_l)_n}{(\beta_1)_n \dots\dots (\beta_m)_n}\frac{z^n}{n!}, \tag{4.14} $$

where ${}_lF_m$ is the generalized hypergeometric function such that $l \leq m + 1\ (l, m \in N_0 = N \cup \{0\})\ z \in U$; N denotes the set of all positive integers, $(\lambda)_n = \lambda(\lambda + 1) \dots (\lambda + n - 1)$is the Pochhammer symbol, and $\mathbf{\Omega}$ is given by

$$ \mathbf{\Omega} = \left(\prod_{t=0}^{l}\Gamma(\alpha_t)\right)^{-1}\left(\prod_{t=0}^{m}\Gamma(\beta_t)\right) \tag{4.15} $$

By using the generalized hypergeometric function (4.14), Dziok Srivastava introduced a linear operator which was subsequently extended by using the Wright's generalized hypergeometric function (defined above). For the purpose of this section, we recall the Dziok-Raina linear operator as follows $\Theta\left[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}\right]: S \to S$is a linear operator which is defined (in terms of the convolution) by

$$ \Theta\left[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}\right]f(z) \coloneqq z\ {}_l\Psi_m\left[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z\right] * f(z). $$

We observe for the function $f \in S$ of the form $f(z) = z + \sum_{n=p+1}^{\infty} a_n z^n$

That

$$ \Theta[\alpha_l]f(z) = \Theta\left[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}\right]f(z) = z + \sum_{n=p+1}^{\infty}\sigma_n(\alpha_1)a_n z^n \tag{4.16} $$

where $\sigma_n(\alpha_1)$ is defined by

$$ \sigma_n(\alpha_1) = \frac{\mathbf{\Omega}\Gamma\left(\alpha_1 + A_1(n-1)\right) \dots .\Gamma\left(\alpha_t + A_t(n-1)\right)}{(n-1)!\,\Gamma\left(\beta_1 + B_1(n-1)\right) \dots .\Gamma\left(\beta_m + B_m(n-1)\right)} \tag{4.17} $$

and Ω is given by (4.15).

In view of the relationship (4.14), the linear operator (4.16) includes (as its special cases) various other linear operators like Bernardi-Libera -Livingston integral operator, Carlson and Shaffer linear operator, Cho-Kwon- Srivastava operator, Choi-Saigo-Srivastava operator, Ruscheweyh derivative operator and for more details on these operators.

We now define the Wright generalized hypergeometric harmonic function $f = h + \bar{g}$of the form (4.1) as

$$\Theta[\alpha_t]f(z) = \Theta[\alpha_1]h(t) + \overline{\Theta[\alpha_1]g(t)} \quad (4.18)$$

DEFINITION 4.2.1. let $\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$denote the subclass of $S_H$consisting of functions $f = h + \bar{g}$ of form (4.1) that satisfy the condition

$$\Re\left\{1 + \frac{1}{b}(1+\beta e^{i\alpha})\left(\frac{z^{q+1}(\Theta[\alpha_1]h(z))^{q+1} - \overline{z^{q+1}(\Theta[\alpha_1]g(z))^{q+1}}}{z^q(\Theta[\alpha_1]h_t(z))^q - \overline{z^q(\Theta[\alpha_1]g_t(z))^q}}\right) - 1 - \beta e^{i\alpha}\right\} \geq \gamma, \quad (4.19)$$

Where $z \in U, b \in C\backslash\{0\}, 0 \leq \gamma < 1, \alpha \in R, \beta \geq 0, h_t(z) = (1-t)z + th(z), g_t(z) = tg(z)$ and $0 \leq t \leq 1$. Further, we define the subclass $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ of $\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ consisting of functions $f = h + \bar{g}$of the form (4.2).

Also, we observe that, by specializing the parameters $l, m, \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_m, \alpha, \beta, \gamma$ and t the class $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ leads to various subclasses. As for illustrations, we present some examples for the cases.

Our first theorem gives a sufficient condition for functions in $\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.

THEOREM4.2.1. Let $f = h + \bar{g}$ be given (4.1). if

$$\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t) + (1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|a_n| + \sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t) - (1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|b_n| \leq 1 \quad (4.20)$$

where $b \in C\backslash\{0\}, 0 \leq \gamma < 1, \beta \geq 0,$ and $0 \leq t < 1$ then $f \in \mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.

PROOF . To prove that, $f = h + g \in \mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$, we only need to show that if (4.20) holds, then the required condition (4.19) is satisfied. For (4.19), we can write

$$\mathcal{R}e\left\{1+\frac{1}{b}\left(1+\beta e^{i\alpha}\right)\left(\frac{z^{q+1}(\Theta[\alpha_1]h(z))^{q+1}-\overline{z^{q+1}(\Theta[\alpha_1]g(z))^{q+1}}}{z^{q}(\Theta[\alpha_1]h_t(z))^{q}-\overline{z^{q}(\Theta[\alpha_1]g_t(z))^{q}}}\right)-1-\beta e^{i\alpha}\right\}\geq\gamma.$$

Using the fact that $\mathcal{R}(\omega)\geq\gamma$ if and only if $|1-\gamma+\omega|\geq|1+\gamma-\omega|$, it suffices to show that

$$\begin{aligned}
&\Bigg|\frac{p!}{(p-q)!}\{(2-\gamma)|b|t+(1+\beta)(p-q-1)\}-\left(1+\beta e^{i\alpha}\right)\left(z^{q}\left(\Theta[\alpha_1]h_t(z)\right)^{q}-\overline{z^{q}\left(\Theta[\alpha_1]g_t(z)\right)^{q}}\right)\\
&\qquad+\left(1+\beta e^{i\alpha}\right)\left(z^{q+1}(\Theta[\alpha_1]h(z))^{q+1}-\overline{z^{q+1}(\Theta[\alpha_1]g(z))^{q+1}}\right)\Bigg|\\
&\qquad-\Bigg|\frac{p!}{(p-q)!}\{\gamma|b|t+(1+\beta)(p-q-1)\}\\
&\qquad+\left(1+\beta e^{i\alpha}\right)\left(z^{q}\left(\Theta[\alpha_1]h_t(z)\right)^{q}-\overline{z^{q}\left(\Theta[\alpha_1]g_t(z)\right)^{q}}\right)\\
&\qquad-\left(1+\beta e^{i\alpha}\right)\left(z^{q+1}(\Theta[\alpha_1]h(z))^{q+1}-\overline{z^{q+1}(\Theta[\alpha_1]g(z))^{q+1}}\right)\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\geq\frac{p!}{(p-q)!}2\{(1-\gamma)|b|t+(1+\beta)(p-q-1)\}|z|\\
&\qquad-\sum_{n=p+1}^{\infty}2\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}\,\sigma_n(\alpha_1)|a_n||z|^n\\
&\qquad-\sum_{n=p}^{\infty}2\{(\beta+1)(n-q-t)-(1-\gamma)|b|t\}\,\sigma_n(\alpha_1)|b_n||z|^n
\end{aligned}$$

$$\begin{aligned}
&\geq\frac{p!}{(p-q)!}2\{(1-\gamma)|b|t+(1+\beta)(p-q-1)\}\left\{1-\sum_{n=p+1}^{\infty}\frac{\sum_{n=p+1}^{\infty}\left\{\begin{matrix}(\beta+1)(n-q-t)\\+(1-\gamma)|b|t\end{matrix}\right\}}{\frac{p!}{(p-q)!}\left\{\begin{matrix}(1-\gamma)|b|t\\+(1+\beta)(p-q-1)\end{matrix}\right\}}\sigma_n(\alpha_1)|a_n||z|^{n-1}\right.\\
&\qquad\left.-\frac{\{(\beta+1)(n-q-t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|t+(1+\beta)(p-q-1)\}}\sigma_n(\alpha_1)|b_n||z|^{n-1}\right\}
\end{aligned}$$

$$\begin{aligned}
&\geq\frac{p!}{(p-q)!}2\{(1-\gamma)|b|t+(1+\beta)(p-q-1)\}\left\{1\right.\\
&\qquad-\sum_{n=p+1}^{\infty}\frac{\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|t+(1+\beta)(p-q-1)\}}\sigma_n(\alpha_1)|a_n|\\
&\qquad\left.-\frac{\{(\beta+1)(n-q-t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|t+(1+\beta)(p-q-1)\}}\sigma_n(\alpha_1)|b_n|\right\}
\end{aligned}$$

$\geq 0,$

which implies that $f\in\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.. The harmonic function

$$\begin{aligned}
f(z)&=z^p+\sum_{n=p+1}^{\infty}\frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|t+(1+\beta)(p-q-1)\}}{\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}\sigma_n(\alpha_1)}x_nz^n\\
&\qquad+\sum_{n=p}^{\infty}\frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|t+(1+\beta)(p-q-1)\}}{\{(\beta+1)(n-q-t)-(1-\gamma)|b|t\}\sigma_n(\alpha_1)}\overline{y_nz^n}
\end{aligned}$$

where $\sum_{n=p+1}^{\infty}|x_n| + \sum_{n=p}^{\infty}|y_n| = 1$ shows that the coefficient bound given by (4.20) is sharp.

Next, we show that the bound (4.20) is also necessary for functions $f \in \mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.

Putting q=1, in the above theorem , to obtain

<u>COROLLARY 4.2.2</u> : Let $f = h + \bar{g}$ be given (4.1). If

$$\sum_{n=2}^{\infty} \frac{\{(\beta+1)(n-t)+(1-\gamma)|b|t\}}{(1-\gamma)|b|}\sigma_n(\alpha_1)|a_n| + \sum_{n=1}^{\infty} \frac{\{(\beta+1)(n+t)-(1-\gamma)|b|t\}}{(1-\gamma)|b|}\sigma_n(\alpha_1)|b_n| \leq 1$$

where $b \in C\backslash\{0\}, 0 \leq \gamma < 1 , \beta \geq 0,\ and\ \ 0 \leq t < 1,\ then\ \ f \in \mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.

We note that the result obtained by [57].

<u>THEOREM 4.2.2</u>. Let $f = h + \bar{g}$be so that h and g are given by (4.1). Then $f \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$. if and only if

$$\sum_{n=p+1}^{\infty} \frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}\sigma_n(\alpha_1)|a_n| + \sum_{n=p}^{\infty} \frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}\sigma_n(\alpha_1)|b_n| \leq \frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\} \quad (4.21)$$

<u>PROOF</u>. The ’if part’ follows from *Theorem* (4.2.1) upon noting that the functions $h(z)$ and $g(z)$ in $f \in \mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ are of the form (4.1), then $f \in \mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$. For the ’only if’ part, we wish to show that $f \notin T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$if the condition (4.21) does not hold. Note that, a necessary and sufficient condition for $f(z) = h(z) + \overline{g(z)}$ given by (4.2) be in $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ is that

$$\mathcal{R}e\left\{\frac{\begin{array}{c} b\left(z^q\left(\Theta[\alpha_1]h_t(z)\right)^q + \overline{z^q\left(\Theta[\alpha_1]g_t(z)\right)^q}\right) + (1+\beta e^{i\alpha}) \times \\ \left(z^{q+1}\left(\Theta[\alpha_1]h(z)\right)^{q+1} - \overline{z^{q+1}\left(\Theta[\alpha_1]g(z)\right)^{q+1}}\right) \\ -(1+\beta e^{i\alpha}) \times \left(z^q(\Theta[\alpha_1]h_t(z))^q - \overline{z^q(\Theta[\alpha_1]g_t(z))^q}\right) \end{array}}{b\left(z^q(\Theta[\alpha_1]h_t(z))^q + \overline{z^q(\Theta[\alpha_1]g_t(z))^q}\right)} - \gamma\right\}$$

$$= \mathcal{R}e\left\{\frac{\begin{array}{c}\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}z^p \\ -\sum_{n=p+1}^{\infty}\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}\sigma_n(\alpha_1)|a_n|z^n \\ -\sum_{n=p}^{\infty}\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}\sigma_n(\alpha_1)|b_n|z^n\end{array}}{b\begin{pmatrix}z^p-\sum_{n=p+1}^{\infty}t\sigma_n(\alpha_1)|a_n|z^n+ \\ \sum_{n=p}^{\infty}t\sigma_n(\alpha_1)|b_n|z^n\end{pmatrix}}\right\}$$

$$= \mathcal{R}e\left\{\frac{\begin{array}{c}\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\} \\ -\sum_{n=p+1}^{\infty}\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}\sigma_n(\alpha_1)|a_n|z^{n-p} \\ -\sum_{n=p}^{\infty}\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}\sigma_n(\alpha_1)|b_n|z^{n-p}\end{array}}{b\begin{pmatrix}1-\sum_{n=p+1}^{\infty}t\sigma_n(\alpha_1)|a_n|z^{n-p}+ \\ \sum_{n=p}^{\infty}t\sigma_n(\alpha_1)|b_n|z^{n-p}\end{pmatrix}}\right\} \geq 0$$

If we choose z to be real $z \to 1^-$ and since $\mathcal{R}(-e^{i\alpha}) - |e^{i\alpha}| = -1$ the above inequality reduces to

$$\frac{\begin{array}{c}\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\} \\ -\sum_{n=p+1}^{\infty}\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}\sigma_n(\alpha_1)|a_n|r^{n-p} \\ \sum_{n=p}^{\infty}\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}\sigma_n(\alpha_1)|b_n|r^{n-p}\end{array}}{b\begin{pmatrix}1-\sum_{n=p+1}^{\infty}t\,\sigma_n(\alpha_1)|a_n|r^{n-p}+ \\ \sum_{n=p}^{\infty}t\sigma_n(\alpha_1)|b_n|r^{n-p}\end{pmatrix}} \geq 0 \qquad (4.22)$$

If the condition (4.21) does not hold then the numerator in (4.22) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in (0,1) for which the quotient in (4.22) is negative. This contradicts the condition or $f \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$. Hence the proof is complete.

Next, we determine the extreme points of the closed convex hulls of $f \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.

THEOREM 4.2.3 A function $f = h + \overline{g}$ belongs $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$if and only if f can be expressed as

$$f(z) = \sum_{n=p}^{\infty}\left(X_n h_n(z) + Y_n g_n(z)\right) \qquad (4.23)$$

where

$$h_p(z) = z^p,$$

$$h_n(z) = z^p - \frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}z^n, (n = p+1, p+2, \ldots.)$$

and

$$g_n(z) = z^p + \frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}}{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t) - (1-\gamma)|b|t\}} z^n$$

$$(n = p, p+1, \dots \dots), \sum_{n=p}^{\infty}(X_n + Y_n) = 1, \;\; X_n \geq 0 \quad and \quad Y_n \geq 0$$

In particular the extreme points of $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ are $\{h_n\} and \; \{g_n\}$.

PROOF : Let f be written as (4.23). Then we have

$$f(z) = \sum_{n=p}^{\infty}\big(X_n h_n(z) + Y_n g_n(z)\big)$$

$$= \quad \sum_{n=p}^{\infty}(X_n + Y_n)\, z^p - \sum_{n=p+1}^{\infty} \frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}\sigma_n(\alpha_1)} X_n z^n +$$
$$\sum_{n=p}^{\infty} \frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}\sigma_n(\alpha_1)} Y_{n(}\overline{z})^n$$

then,

$$\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t) + (1-\gamma)|b|t\}\sigma_n(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}} \times \left(\frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}}{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t) + (1-\gamma)|b|t\}\sigma_n(\alpha_1)}\right) X_n$$
$$+ \sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t) - (1-\gamma)|b|t\}\sigma_n(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}}$$
$$\times \left(\frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}}{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t) - (1-\gamma)|b|t\}\sigma_n(\alpha_1)}\right) Y_n$$
$$= \sum_{n=p+1}^{\infty} X_n + \sum_{n=p}^{\infty} Y_n = 1 - X_p \leq 1.$$

Then $f \in$ clco $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$. Conversely, assume that $f \in$clco $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$). Letting

$$X_p = 1 - \sum_{n=P+1}^{\infty} X_n - \sum_{n=P}^{\infty} Y_n$$

where

$$X_n = \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t) + (1-\gamma)|b|t\}\sigma_n(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}} |a_n|, \quad n = p+1, p+2, \dots$$

$$Y_n = \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t) - (1-\gamma)|b|t\}\sigma_n(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}} |b_n|, \quad n = p, p+1, \dots$$

Where$\sum_{n=p}^{\infty}(X_n + Y_n) = 1$

Then note that by *Theorem* (4.2.3), $0 \leq X_n \leq 1, (n = 2,3, \dots)$ and $0 \leq Y_n \leq 1, (n = 1,2,3, \dots)$, We define $X_p = 1 - \sum_{n=p+1}^{\infty} X_n - Y_n$ and note that by Theorem 3.1.2, $X_p \geq 0$, Consequenty , we obtain

$$f(z)=\sum_{n=p}^{\infty}X_nh_n(z)+Y_ng_n(z)$$

Using *Theorem*(4.2.4), it is easily seen that $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ is convex and closed , so *clco* $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)=T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.In other words, the statement of *Theorem*(4.2.3) is really for $f\in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.

THEOREM 4.2.4 If $f\in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$,then for $z=re^{i\theta}$, we have

$$|f(z)|\le(1+|b_p|)r^p+\left(\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}-\frac{\frac{p!}{(p-q)!}\left\{\begin{matrix}(p-q)\big(\lambda(p-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda p-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}\right)r^{p+1}$$

and

$$|f(z)|\le(1+|b_p|)r^p+\left(\frac{\frac{p!}{(p-q)!}(\lambda p-\lambda q+1)\{|b|(1-\gamma)-(p-q)\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}-\frac{\frac{p!}{(p-q)!}\left\{\begin{matrix}(p-q)\big(\lambda(p-q-1)-(1-5\lambda)\big)-2(1-2\lambda)\\ -|b|(1-\gamma)(\lambda p-\lambda q+1)\end{matrix}\right\}}{\frac{p!}{(p+1-q)!}\{(p+1)\lambda-\lambda q+1\}\{(p+1-q)+|b|(1-\gamma)\}}\right)r^{p+1}.$$

PROOF: We have

$$|f(z)|\le(1+|b_p|)r^p+\sum_{n=p+1}^{\infty}(|a_n|+|b_n|)r^n$$

$$\le(1+|b_p|)r^p+\sum_{n=p+1}^{\infty}(|a_n|+|b_n|)r^{p+1}$$

$$=(1+|b_p|)r^p+\frac{1}{\frac{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}\sigma_{p+1}(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}}$$

$$\times\sum_{n=p+1}^{\infty}\left(\frac{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}\sigma_{p+1}(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}|a_n|+\frac{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}\sigma_{p+1}(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}|b_n|\right)r^{p+1}$$

$$\leq \left(1+|b_p|\right)r^p + \frac{1}{\frac{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}\sigma_{p+1}(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}}$$

$$\times \sum_{n=p+1}^{\infty}\left(\frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|a_n| - \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|b_n|\right)r^{p+1}$$

$$\leq \left(1+|b_p|\right)r^p + \frac{1}{\frac{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}\sigma_{p+1}(\alpha_1)}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}}$$

$$\times\left(1-\frac{\frac{p!}{(p-q)!}\{(\beta+1)(p-q+t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}|b_p|\right)r^{p+1}$$

$$\leq \left(1+|b_p|\right)r^p + \frac{1}{\sigma_{p+1}(\alpha_1)}\left(\frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}} - \frac{\frac{p!}{(p-q)!}\{(\beta+1)(p-q+t)-(1-\gamma)|b|t\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}}|b_p|\right)r^{p+1}.$$

Similarly

$$|f(z)| \geq \left(1+|b_p|\right)r^p + \frac{1}{\sigma_{p+1}(\alpha_1)}\left(\frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}} - \frac{\frac{p!}{(p-q)!}\{(\beta+1)(p-q+t)-(1-\gamma)|b|t\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}}|b_p|\right)r^{p+1}$$

The upper and lower bounds given in *Theorem* (4.2.4) are respectively attained for the following functions

$$f(z) = z^p + |b_p|\overline{(z)}^p + \frac{1}{\sigma_{p+1}(\alpha_1)}\left(\frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}} - \frac{\frac{p!}{(p-q)!}\{(\beta+1)(p-q+t)-(1-\gamma)|b|t\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}}|b_p|\right)(\bar{z})^{p+1}$$

and

$$f(z) = \left(1-|b_p|\right)z^p - \frac{1}{\sigma_{p+1}(\alpha_1)}\left(\frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}} - \frac{\frac{p!}{(p-q)!}\{(\beta+1)(p-q+t)-(1-\gamma)|b|t\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t)+(1-\gamma)|b|t\}}|b_p|\right)\bar{z}^{p+1}.$$

The following covering result follows from the left hand inequality in *Theorem* (3.2.4).

COROLLARY 4.2.2. Let f of the form (4.1) be so that $f \in TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$

Then

$$\left\{\omega: |\omega| < 1 - \frac{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t) + (1-\gamma)|b|t\}\sigma_{p+1}(\alpha_1)} - \left[1 - \frac{\frac{p!}{(p-q)!}\{(\beta+1)(p-q+t) - (1-\gamma)|b|t\}}{\frac{(p+1)!}{(p+1-q)!}\{(\beta+1)(p+1-q-t) + (1-\gamma)|b|t\}}|b_p|\right]\right\}.$$

We show that the class $TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For

$$f(z) = z^p + \sum_{n=p+1}^{\infty} |a_n| z^n + \sum_{n=p}^{\infty} |b_n| (\bar{z})^n$$

and

$$F(z) = z^p + \sum_{n=p+1}^{\infty} |A_n| z^n + \sum_{n=p}^{\infty} |B_n| \overline{(z)}^n$$

we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z^p - \sum_{n=p+1}^{\infty} |a_n||A_n| z^n + \sum_{n=p}^{\infty} |b_n||B_n| \bar{z}^n \quad (4.24)$$

Using the definition, we show that the class $TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ is closed under convolution

THEOREM 4.2.5 For $0 \le \delta < \gamma < 1$, *let* $f \in TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ *and* $F \in TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\delta;t)$.

Then. $f * F \in TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t) \subset TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\delta;t)$.

PROOF. Let

$$f(z) = z^p + \sum_{n=p+1}^{\infty} |a_n| z^n + \sum_{n=p}^{\infty} |b_n| \bar{z}^n$$

and

$$F(z) = z^p + \sum_{n=p+1}^{\infty} |A| z^n + \sum_{n=p}^{\infty} |B_n| \bar{z}^n$$

be in $TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\delta;t)$. Then $f * F \in TW_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\delta;t)$. We note that $|A_n| < 1$ and $|B_n| < 1$. In view of *Theorem* (4.2.2) and the inequality $0 \le \delta < \gamma < 1$, we have

$$\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t) + (1-\delta)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}} \sigma_n(\alpha_1)|a_n||A_n|$$
$$+ \sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t) - (1-\delta)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b| + (\beta+1)(p-q-1)\}} \sigma_n(\alpha_1)|b_n||B_n|$$

$$\leq \sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\delta)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|a_n|$$
$$+\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\delta)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|b_n|$$

$$\leq \sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|a_n|$$
$$+\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|b_n| \leq 1$$

by *Theorem* (4.2.2), $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$. By the same token, we then conclude that $f * F \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t) \subset T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\delta;t)$.

Next, we show that the class $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ is closed under convex combination of its members

THEOREM 4.2.6. The class $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$ is closed under convex combinations

PROOF Suppose $f_i \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$(i = 1, 2, · · ·) are defined by

$f_i(z) = z^p - \sum_{n=p+1}^{\infty}|a_{i,n}|z^n + \sum_{n=p}^{\infty}|b_{i,n}|\overline{(z)}^n$,

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i = z^p - \sum_{n=p+1}^{\infty}\left(\sum_{i=1}^{\infty} t_i\,|a_{i,n}|\right)z^n - \sum_{n=p}^{\infty}\left(\sum_{i=1}^{\infty} t_i\,|b_{i,n}|\right)\bar{z}^n$$

Since

$$\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|a_{i,n}|$$
$$+\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|b_{i,n}| \leq 1$$

From the above equation we obtain

$$\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)\left(\sum_{i=1}^{\infty} t_i\,|a_{i,n}|\right)$$
$$+\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)\left(\sum_{i=1}^{\infty} t_i\,|b_{i,n}|\right)$$

$=$

$\sum_{i=1}^{\infty} t_i \left\{\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|a_{i,n}| + \right.$

$\left.\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|b_{i,n}|\right\} \leq \sum_{i=1}^{\infty} t_i = 1.$

Then $\sum_{i=1}^{\infty} t_i f_i \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.The proof is complete.

We consider the closure property of the class $T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$under the Bernardi integral operator$\mathcal{L}_c[f(z)]$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c}\int_0^z \zeta^{c-1} f(\zeta)d\zeta \ , \qquad c > -p$$

THEOREM 4.2.7. Let $f \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$then$\mathcal{L}_c[f(z)] \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$

PROOF. From the representation of $\mathcal{L}_c[f(z)]$, it follows that

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c}\int_0^z \zeta^{c-1} f(\zeta)d\zeta + \overline{\frac{c+1}{z^c}\int_0^z \zeta^{c-1} f(\zeta)d\zeta}$$

$$= \frac{c+1}{z^c}\int_0^z \zeta^{c-1}\left(\zeta^p - \sum_{n=p+1}^{\infty}|a_n|\zeta^n\right)d\zeta + \overline{\frac{c+1}{z^c}\int_0^z \zeta^{c-1}\left(\sum_{n=p}^{\infty}|b_n|\zeta^n\right)d\zeta}$$

$$= z^p - \sum_{n=p+1}^{\infty} A_n z^n + \sum_{n=p}^{\infty} B_n z^n$$

where $A_n = \frac{c+1}{c+n}|a_n|$ and $B_n = \frac{c+1}{c+n}|b_n|$. Hence

$$\sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)\times\left(\frac{c+1}{c+n}|a_n|\right)$$

$$+\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)\ \times\ \left(\frac{c+1}{c+n}|b_n|\right)$$

$$\le \sum_{n=p+1}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q-t)+(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|a_n|$$

$$+\sum_{n=p}^{\infty} \frac{\frac{n!}{(n-q)!}\{(\beta+1)(n-q+t)-(1-\gamma)|b|t\}}{\frac{p!}{(p-q)!}\{(1-\gamma)|b|+(\beta+1)(p-q-1)\}}\sigma_n(\alpha_1)|b_n| \le 1$$

Since $f \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$, Therefore by *theorem* (4.2.12) , $\mathcal{L}_c[f(z)] \in T\mathcal{W}_{\mathcal{H}}^{l,m}(b,\alpha,\beta,\gamma;t)$.

4.3 ON A CERTAIN CLASS OF MULTIVALENTLY HARMONIC MEROMORPHIC FUNCTIONS

Considered harmonic sense preserving univalent mappings defined on U = {z : |z| >1}that map ∞ to ∞ and represented by

$$f(z) = h(z) + \overline{g(z)} + A\log|z|,$$

where

$h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}$, $g(z) = \beta z + \sum_{n=0}^{\infty} b_n z^{-n}$

are holomorphic in U and $|\alpha| > |\beta| \ge 0$, $A \in \mathrm{C}$.

Now, let us denote the family Σp(s) consisting of all harmonic sense preserving multivalent meromorphic mapping

$f(z) = h(z) + \overline{g(z)}$, (4.25)

where

$$h(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{(n+p-1)},$$

$$g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{-(n+p-1)}, |b_p| < 1, |z| > 1 \quad (4.26)$$

For $0 \leq \alpha < 2(1-k), 0 \leq \lambda \leq 1, \frac{1}{2} \leq k < 1, 0 \leq \theta < 1, 0 \leq \xi < 1$ and $z = re^{i\beta}$. $1 < r < \infty$; β, ξ, θ and α are real , we introduce the subclass $AJ_s(\alpha, \lambda, k, p)$ consisting of all functions f satisfying

$$\mathcal{R}e\left\{(1+e^{i\theta})\frac{\frac{f'(z)}{z^{p-1}}}{\frac{\lambda f'(z)}{z^{p-1}}+(1-p\lambda)} - pk(1+e^{i\theta})\right\} \geq p\alpha \quad (4.27)$$

Also the subclass of multivalent meromorphic harmonic functions with

$$f(z) = z^p + \sum_{n=1}^{\infty}|a_{n+p-1}|z^{-(n+p-1)} - \sum_{n=1}^{\infty}|b_{n+p-1}|z^{-(n+p-1)}|b_p| < 1 \quad (4.28)$$

that satisfies (4.27) is denoted by $\overline{AJ_s}(\alpha, \lambda, k, p)$.

<u>THEOREM 4.3.1</u>: let $f = h + \bar{g}$ with h and g given by (4.26). If

$$\sum_{n=1}^{\infty}(n+p-1)(|a_{n+p-1}| + |b_{n+p-1}|) \leq p \quad (4.29)$$

then f is harmonic, sense preserving and multivalent in U and $f \in$Σp(s).

<u>THEOREM 4.3.2</u>: let $f = h + \bar{g}$ where h and g given by (4.26). Furthermore , let

$$\sum_{n=1}^{\infty}(n+p-1)(2-p\lambda(2k+\alpha))|a_{n+p-1}| + \sum_{n=1}^{\infty}(n+p-1)(2-p\lambda(2k+\alpha))|b_{n+p-1}|$$
$$\leq p(2-2k-\alpha) \quad (4.30)$$

Where $0 \leq \alpha < 2(1-k), 0 \leq \lambda \leq 1$ where $\frac{1}{2} \leq k < 1$. Then f is sense preserving multivalent meromorphic harmonic functions with $f \in AJ_s(\alpha, \lambda, k, p)$.

<u>PROOF</u>: If the inequality (4.30) holds for coefficients of $f = h + \bar{g}$, $then$ f is sense preserving and harmonic multivalent in U. Now we want to show that $f \in AJ_s(\alpha, \lambda, k, p)$ .According to (4.27), we have

$$\mathcal{R}e\left\{\frac{(zh'(z) - \overline{zg'(z)})}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-p\lambda)z^p}(1+e^{i\theta}) - pk(1+e^{i\varsigma})\right\} \geq p\alpha,$$

Where

$z = re^{i\varsigma}$, $0 \leq \gamma \leq 2\pi$, $0 \leq r < 1$, $0 \leq \alpha < 2(1-k)$, $0 \leq \theta < 1$, $0 \leq \xi < 1$, $\frac{1}{2} \leq k < 1$ and $0 \leq \lambda \leq 1$. Let

$$N(\lambda, z) = (1+e^{i\theta})(zh'(z) - \overline{zg'(z)}) - pk(1 + e^{i\varsigma})[\lambda(zh'(z) - \overline{zg'(z)} + (1-p\lambda)z^p]$$

and

$$M(\lambda, z) = \lambda(zh'(z) - \overline{zg'(z)} + (1-p\lambda)z^p$$

By using the fact $\mathcal{R}e\,(w) > p\alpha$ if and only if $|p(1-\alpha)+w| > |p(1+\alpha)-w|$, it is enough to show that

$$|p(1-\alpha)M(\lambda,z)+N(\lambda,z)| - |N(\lambda,z)-p(1+\alpha)M(\lambda,z)| \geq 0.$$

Therefore

$$\begin{aligned}|p(1-\alpha)M(\lambda,z)+N(\lambda,z)| &= \Big|p(1-\alpha)\{\lambda\big(zh'(z)-\overline{zg'(z)}\big)+(1-p\lambda)z^{p}\}\\ &\quad+\{(1+e^{i\theta})\big(zh'(z)-\overline{zg'(z)}\big)\}\\ &\quad-pk(1+e^{i\varsigma})\big[\lambda\big(zh'(z)-\overline{zg'(z)}\big)+(1-p\lambda)z^{p}\big]\Big|\end{aligned}$$

$= \Big|\{\lambda p(1-\alpha)+(1+e^{i\theta})-\lambda pk(1+e^{i\varsigma})\}\big(zh'(z)\big)-\{\lambda p(1-\alpha)+(1+e^{i\theta})-\lambda pk(1+e^{i\varsigma})\}\big(\overline{zg'(z)}\big)-\{(1-p\lambda)pk(1+e^{i\varsigma})-p(1-\alpha)(1-p\lambda)\}z^{p}\Big|$

$= \Big|\{\lambda p(1-\alpha)+(1+e^{i\theta})-\lambda pk(1+e^{i\varsigma})\}\{pz^{p}-\sum_{n=1}^{\infty}(n+p-1)a_{n+p-1}z^{-(n+p-1)}\}-\{\lambda p(1-\alpha)+(1+e^{i\theta})-\lambda pk(1+e^{i\varsigma})\}\{\overline{-\sum_{n=1}^{\infty}(n+p-1)b_{n+p-1}z^{-(n+p-1)}}\}-\{(1-p\lambda)pk(1+e^{i\varsigma})-p(1-\alpha)(1-p\lambda)\}z^{p}\Big|$

$\geq (3p-2kp-p\alpha)|z|^{p}$ - $\sum_{n=1}^{\infty}(n+p-1)(2-p\lambda(1-2k-\alpha))|a_{n+p-1}||z|^{-(n+p-1)}-\sum_{n=1}^{\infty}(n+p-1)\big(2-p\lambda(1-2k-\alpha)\big)\,\overline{|b_{n+p-1}||z|^{-(n+p-1)}}$

also we have

$$\begin{aligned}|N(\lambda,z)-p(1+\alpha)M(\lambda,z)| &= \Big|(1+e^{i\theta})\big(zh'(z)-\overline{zg'(z)}\big)\\ &\quad-pk(1+e^{i\varsigma})\big[\lambda\big(zh'(z)-\overline{zg'(z)}\big)+(1-p\lambda)z^{p}\big]\\ &\quad-p(1+\alpha)\{\lambda\big(zh'(z)-\overline{zg'(z)}\big)+(1-p\lambda)z^{p}\}\Big|\\ &= \Big|\{(1+e^{i\theta})-pk\lambda(1+e^{i\varsigma})-p\lambda(1+\alpha)\}\big(zh'(z)\big)\\ &\quad-\{pk(1-p\lambda)(1+e^{i\varsigma})+p(1+\alpha)(1-p\lambda)\}(z^{p})\\ &\quad-\{(1+e^{i\theta})-\lambda pk(1+e^{i\varsigma})-\lambda p(1+\alpha)\}\big(\overline{zg'(z)}\big)\Big|\end{aligned}$$

$= \Big|\{(1+e^{i\theta})-pk\lambda(1+e^{i\varsigma})-p\lambda(1+\alpha)\}\big(pz^{p}-\sum_{n=1}^{\infty}(n+p-1)a_{n+p-1}z^{-(n+p-1)}\big)-\{pk(1-p\lambda)(1+e^{i\varsigma})+p(1+\alpha)(1-p\lambda)\}(z^{p})-\{(1+e^{i\theta})-\lambda pk(1+e^{i\varsigma})-\lambda p(1+\alpha)\}\{\overline{-\sum_{n=1}^{\infty}(n+p-1)b_{n+p-1}z^{-(n+p-1)}}\}\Big|$

$$\geq (p\alpha-p+2kp)|z|^{p}\sum_{n=1}^{\infty}(n+p-1)\big(2-p\lambda(-1-2k-\alpha)\big)\,|a_{n+p-1}||z|^{-(n+p-1)}$$
$$-\sum_{n=1}^{\infty}(n+p-1)\big(2-p\lambda(-1-2k-\alpha)\big)\,\overline{|b_{n+p-1}||z|^{-(n+p-1)}}$$

Thus

$$|p(1-\alpha)M(\lambda,z)+N(\lambda,z)| - |N(\lambda,z)-p(1+\alpha)M(\lambda,z)|$$

$\geq 2p(2-2k-\alpha)-2\sum_{n=1}^{\infty}(n+p-1)\big(2-p\lambda(2k+\alpha)\big)\,|a_{n+p-1}|-2\sum_{n=1}^{\infty}(n+p-1)\big(2-p\lambda(2k+\alpha)\big)\,|b_{n+p-1}| \geq 0$(by (4.30)).

So $f \in AJ_{s}(\alpha,\lambda,k,p)$.

THEOREM 4.3.3: $let\ f = h + \bar{g}$where h and g have the form given by (4.26). then $f \in \overline{AJ}_s$(α,λ, k, p) if and only if

$$\sum_{n=1}^{\infty} \frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)|a_{n+p-1}|}{p(2-2k-\alpha)} + \frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)|b_{n+p-1}|}{p(2-2k-\alpha)} \le 1. \tag{4.31}$$

PROOF: The "if" part is clear, since $\overline{AJ}_s$ (α, λ, k, p) ⊆AJ_s(α, λ, k, p), for "only if" part, we show that $f \notin \overline{AJ}_s$(α, λ, k, p) if the inequality (4.31) does not hold. So, we must show that

$$\mathcal{R}e\left\{\left(\frac{[zh'(z)-\overline{zg'(z)}]}{\lambda\big(zh'(z)-\overline{zg'(z)}\big)+(1-p\lambda)z^p}\right)(1+e^{i\theta}) - pk\big(1+e^{i\varsigma}\big) - p\alpha\right\}.$$

$= \mathcal{R}e\left\{\frac{C(z)}{D(z)}\right\} \ge 0,$

where

$$C(z) = \left[\big(zh'(z)-\overline{zg'(z)}\big)\big(1+e^{i\theta}\big) - p\big(k(1+e^{i\varsigma})+\alpha\big)\left\{\lambda\big(zh'(z)-\overline{zg'(z)}\big)+(1-p\lambda z^p\right\}\right]$$

$$= \left\{\big(1+e^{i\theta}\big) - \lambda p\big(k\big(1+e^{i\varsigma}\big)+\alpha\big)\right\}\big(zh'(z)\big) - \left\{\big(1+e^{i\theta}\big) - \lambda p\big(k\big(1+e^{i\varsigma}\big)+\alpha\big)\right\}\big(zg'(z)\big) - \Big(p(1-p\lambda)\big(k\big(1+e^{i\varsigma}\big)+\alpha\big)\Big)z^p$$

$$= \left\{\big(1+e^{i\theta}\big) - \lambda p\big(k\big(1+e^{i\varsigma}\big)+\alpha\big)\right\}\left\{pz^p - \sum_{n=1}^{\infty}(n+p-1)a_{n+p-1}z^{-(n+p-1)}\right\}$$

$$- \left\{\big(1+e^{i\theta}\big) - \lambda p\big(k\big(1+e^{i\varsigma}\big)+\alpha\big)\right\}\left\{\overline{-\sum_{n=1}^{\infty}(n+p-1)b_{n+p-1}z^{-(n+p-1)}}\right\}$$

$$- \Big(p(1-p\lambda)\big(k\big(1+e^{i\varsigma}\big)+\alpha\big)\Big)z^p$$

$$= p(2-2k-\alpha)|z|^p - \sum_{n=1}^{\infty}(n+p-1)\big(2-p\lambda(2k+\alpha)\big)\,|a_{n+p-1}||z|^{-(n+p-1)} - \sum_{n=1}^{\infty}(n+p-1)\big(2-p\lambda(2k+\alpha)\big)\,|b_{n+p-1}||z|^{-(n+p-1)}$$

also

$$D(z) = \lambda\big(zh'(z)-\overline{zg'(z)}\big)+(1-p\lambda)z^p$$

$$= \lambda\left\{pz^p - \sum_{n=1}^{\infty}(n+p-1)a_{n+p-1}z^{-(n+p-1)}\right\} - \lambda\left\{\overline{-\sum_{n=1}^{\infty}(n+p-1)b_{n+p-1}z^{-(n+p-1)}}\right\} + (1-p\lambda)z^p$$

$$= z^p - \sum_{n=1}^{\infty}\lambda(n+p-1)|a_{n+p-1}||z|^{-(n+p-1)} + \sum_{n=1}^{\infty}\lambda(n+p-1)|b_{n+p-1}||z|^{-(n+p-1)}$$

Upon choosing the values of z on the positive real axis, where |z| = r < 1, then we must show that

$$\frac{p(2-2k-\alpha) - \sum_{n=1}^{\infty}(n+p-1)\big(2-p\lambda(2k+\alpha)\big)\,|a_{n+p-1}| + (n+p-1)\big(2-p\lambda(2k+\alpha)\big)|b_{n+p-1}|}{1 - \sum_{n=1}^{\infty}\lambda(n+p-1)|a_{n+p-1}||r|^{-(n+2p-1)} + \lambda(n+p-1)|b_{n+p-1}||r|^{-(n+2p-1)}} \ge 1\ . \tag{4.32}$$

We note that the last inequality is negative for r sufficiently close to 1, then the inequality (3.25) does not hold, therefore $\mathcal{R}e\left\{\frac{C(z)}{D(z)}\right\}$is negative. This contradicts the required condition for f $\in \overline{AJ}_s$(α, λ, k, p). This completes the proof of theorem.

Next we obtain the distortion bounds and extreme points.

THEOREM 4.3.4:let $f(z) \in \overline{AJ}_s(\alpha, \lambda, k, p)$.Then

$$r^p - p(2-2k-\alpha)r^{-p} \leq |f(z)| \leq r^p + p(2-2k-\alpha)r^{-p}$$

PROOF: Let f $\in \overline{AJ}_s(\alpha, \lambda, k, p)$. Then for $|z| = r > 1$,we have

$$|f(z)| = \left| z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1} \bar{z}^{-(n+p-1)} \right|$$

$$\leq r^p + \sum_{n=1}^{\infty} (a_{n+p-1} + b_{n+p-1}) r^{-(n+p-1)}$$

$$\leq r^p + r^{-p} \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|)$$

$$\leq r^p + r^{-p} \sum_{n=1}^{\infty} (n+p-1)(2 - p\lambda(2k+\alpha)) |a_{n+p-1}| r^{-(n+p-1)}$$
$$+ (n+p-1)(2 - p\lambda(2k+\alpha)) |b_{n+p-1}| r^{-(n+p-1)}$$
$$\leq r^p + p(2-2k-\alpha) r^{-p}$$

The left hand inequality can be proved by using similar arguments. This completes the proof of theorem.

THEOREM 4.3.5: The function

$f(z) = h(z) + \overline{g(z)} \in \overline{AJ}_s(\alpha, \lambda, k, p)$

if and only if

$$f(z) = \sum_{n=0}^{\infty} \left(S_{n+p-1} h_{n+p-1}(z) + T_{n+p-1} g_{n+p-1}(z) \right), \quad z \in U, p \geq 1 \qquad (4.33)$$

where

$$h_{p-1}(z) = z^p, h_{n+p-1}(z) = z^p + \frac{p(2-2k-\alpha)}{(n+p-1)(2 - p\lambda(2k+\alpha))} z^{-(n+p-1)},$$

$$g_{p-1}(z) = z^p, g_{n+p-1}(z) = z^p - \frac{p(2-2k-\alpha)}{(n+p-1)(2 - p\lambda(2k+\alpha))} \bar{z}^{-(n+p-1)}$$

For $n \geq 1)$, $\sum_{n=0}^{\infty} (S_{n+p-1} + T_{n+p-1}) = 1, S_{n+p-1} \geq 0$ *and* $T_{n+p-1} \geq 0$. In particular the extreme points of $\overline{A}J_s(\alpha, \lambda, k, p)$ are $\{h_{n+p-1}\}$ and $\{g_{n+p-1}\}$.

PROOF: Suppose that f can be written of the form (4.33), then

$$f(z) = S_{p-1} h_{p-1}(z) + T_{p-1} g_{p-1}(z)$$
$$+ \sum_{n=1}^{\infty} S_{n+p-1} \left(z^p + \frac{p(2-2k-\alpha)}{(n+p-1)(2 - p\lambda(2k+\alpha))} z^{-(n+p-1)} \right)$$
$$+ \sum_{n=1}^{\infty} T_{n+p-1} \left(z^p - \frac{p(2-2k-\alpha)}{(n+p-1)(2 - p\lambda(2k+\alpha))} z^{-(n+p-1)} \right)$$

$$= \sum_{n=0}^{\infty} \left(S_{n+p-1} + T_{n+p-1}\right) z^p$$
$$+ \sum_{n=1}^{\infty} \left[\frac{p(2-2k-\alpha)}{(n+p-1)\left(2-p\lambda(2k+\alpha)\right)} S_{n+p-1} z^{-(n+p-1)} - \frac{p(2-2k-\alpha)}{(n+p-1)\left(2-p\lambda(2k+\alpha)\right)} S_{n+p-1} \overline{z^{-(n+p-1)}} \right]$$

Since

$$\sum_{n=1}^{\infty} (n+p-1)\left(2-p\lambda(2k+\alpha)\right) \left(\frac{p(2-2k-\alpha)}{(n+p-1)\left(2-p\lambda(2k+\alpha)\right)} \right) + (n+p-1)\left(2-p\lambda(2k+\alpha)\right) \left(\frac{p(2-2k-\alpha)}{(n+p-1)\left(2-p\lambda(2k+\alpha)\right)} \right)$$

$= p(2-2k-\alpha) \sum_{n=0}^{\infty} \left(S_{n+p-1} + T_{n+p-1}\right) \leq p(2-2k-\alpha)$

So by theorem (4.3.20) $f \in \bar{A}J_S(\alpha, \lambda, k, p)$.

Conversely, if $f \in \bar{A}J_S(\alpha, \lambda, k, p)$, then by Theorem (4.3.20) we have

$$\sum_{n=1}^{\infty} \frac{1}{p(2-2k-\alpha)} (n+p-1)\left(2-p\lambda(2k+\alpha)\right) |a_{n+p-1}| + (n+p-1)\left(2-p\lambda(2k+\alpha)\right) |b_{n+p-1}| \leq 1$$

Putting

$$S_{n+p-1} = \frac{(n+p-1)\left(2-p\lambda(2k+\alpha)\right)}{p(2-2k-\alpha)}$$
$$T_{n+p-1} = \frac{(n+p-1)\left(2-p\lambda(2k+\alpha)\right)}{p(2-2k-\alpha)}$$
$$0 \leq S_{p-1} \leq 1 \ \ and \ \ T_{p-1} = 1 - S_{p-1} - \sum_{n=1}^{\infty} \left(S_{n+p-1} + T_{n+p-1}\right)$$

we get the required result.

<u>DEFINITION 4.3.1</u>: Let

$$f(z) = z^p + \sum_{n=1}^{\infty} |a_{n+p-1}| z^{-(n+p-1)} - \sum_{n=1}^{\infty} |b_{n+p-1}| \overline{z^{-(n+p-1)}} \quad .(4.34)$$
$$g(z) = z^p + \sum_{n=1}^{\infty} |c_{n+p-1}| z^{-(n+p-1)} - \sum_{n=1}^{\infty} |d_{n+p-1}| \overline{z^{-(n+p-1)}} \quad (4.35)$$

be in $\bar{A}J_S(\alpha, \lambda, k, p)$, for real number β, we define the β-convolution of f and g as follows.

$$(f \otimes_\beta g)(z) = z + \sum_{n=1}^{\infty} \frac{|a_{n+p-1} c_{n+p-1}|}{(n+b-1)^\beta} z^{-(n+p-1)} - \sum_{n=1}^{\infty} \frac{|b_{n+p-1} d_{n+p-1}|}{(n+b-1)^\beta} \overline{z^{-(n+p-1)}}$$

The 0-convolution of f and g is the familiar Hadamard product, also the 1-convolution of f and g is named integral convolution and is defined by

$(f \otimes_1 g)(z) =$

$z + \sum_{n=1}^{\infty} \frac{|a_{n+p-1} c_{n+p-1}|}{n+b-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} \frac{|b_{n+p-1} d_{n+p-1}|}{n+b-1} \overline{z^{-(n+p-1)}}$.

<u>THEOREM 4.3.6</u> : Let $f(z), g(z)$ defined as (4.34), (4.35) respectively be in $AJ_S(\alpha, \lambda, k, p)$. Then the β-convolution of f and g where $\beta \geq \frac{p(2-2k-\alpha)}{2p-p\lambda(2k+\alpha)(n+p-1)}$

$$\beta \geq max_{n\neq 1}\left\{\log(n+p-1)^{-1}\ log\frac{(2-2k-\alpha)}{(2-(\alpha+2k)(\lambda+\lambda p-1))};\ \log(n+p-1)^{-1}\ log\frac{(2-2k-\alpha)}{(2-(\alpha+2k)(\lambda+\lambda p-1))}\right\}$$

belongs to $\overline{AJ_s}(\alpha, \lambda, k, p)$, if furthermore of β, then, we have

PROOF: By assumption, we have $f, g \in \overline{AJ_s}(\alpha, \lambda, k, p)$,therefore

$$\sum_{n=1}^{\infty}\frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)}\left|a_{n+p-1}\right|+\frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)}\left|b_{n+p-1}\right| \leq 1, \qquad (4.36)$$

and

$$\sum_{n=1}^{\infty}\frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)}\left|c_{n+p-1}\right|+\frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)}\left|d_{n+p-1}\right| \leq 1, \qquad (4.37)$$

also ,we have

$$\left|c_{n+p-1}\right| \leq \frac{p(2-2k-\alpha)}{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}$$

$$\left|d_{n+p-1}\right| \leq \frac{p(2-2k-\alpha)}{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}$$

we have to show that

$$\sum_{n=1}^{\infty}\frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)}\frac{\left|a_{n+p-1}c_{n+p-1}\right|}{(n+p-1)^{\beta}}+\frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)}\frac{\left|b_{n+p-1}d_{n+p-1}\right|}{(n+p-1)^{\beta}} \leq 1 \qquad (4.38)$$

For this purpose, we have

$$\sum_{n=1}^{\infty}\frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)}\frac{\left|a_{n+p-1}c_{n+p-1}\right|}{(n+p-1)^{\beta}}+\frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)}\frac{\left|b_{n+p-1}d_{n+p-1}\right|}{(n+p-1)^{\beta}} \leq \sum_{n=1}^{\infty}\frac{\left|a_{n+p-1}\right|+\left|b_{n+p-1}\right|}{(n+p-1)^{\beta}}$$

Therefore the inequality in (4.38) holds true if

$$(n+p-1)^{\beta} \geq \frac{p(2-2k-\alpha)}{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}$$

holds true or

$$\beta \geq max_{n\neq 1}\left\{\log(n+p-1)^{-1}\ log\frac{(2-2k-\alpha)}{(2-(\alpha+2k)(\lambda+\lambda p-1))};\ \log(n+p-1)^{-1}\ log\frac{(2-2k-\alpha)}{(2-(\alpha+2k)(\lambda+\lambda p-1))}\right\}$$

THEOREM 4.3.7 Suppose $0 \leq \alpha_1 \leq \alpha_2 < 2(1-k), \frac{1}{2} \leq k < 1\ \ and\ \ f, g \in \overline{AJ_s}(\alpha_2, \lambda, k, p)$, then

$f * g \in \overline{AJ}_s(\alpha_2, \lambda, k, p) \subset \overline{AJ}_s(\alpha_1, \lambda, k, p)$.

PROOF: By assumption it is clear that $\overline{AJ}_s(\alpha_2, \lambda, k, p) \subset \overline{AJ}_s(\alpha_1, \lambda, k, p)$. Further

$$\frac{\sum_{n=1}^{\infty}(n+p-1)\left(2-p\lambda(2k+\alpha_2)\right)\left|a_{n+p-1}\right| + (n+p-1)\left(2-p\lambda(2k+\alpha_2)\right)\left|b_{n+p-1}\right|}{p(2-2k+\alpha_2)} \leq 1$$

$$\frac{\sum_{n=1}^{\infty}(n+p-1)\left(2-p\lambda(2k+\alpha_1)\right)\left|c_{n+p-1}\right| + (n+p-1)\left(2-p\lambda(2k+\alpha_1)\right)\left|d_{n+p-1}\right|}{p(2-2k+\alpha_1)} \leq 1$$

and consequently, we have

$$\left|c_{n+p-1}\right| \leq \frac{p(2-2k-\alpha_1)}{(n+p-1)\left(2-p\lambda(2k+\alpha_1)\right)}$$

$$\left|d_{n+p-1}\right| \leq \frac{p(2-2k-\alpha_1)}{(n+p-1)\left(2-p\lambda(2k+\alpha_1)\right)}.$$

Therefore,

$$\frac{\sum_{n=1}^{\infty}(n+p-1)\left(2-p\lambda(2k+\alpha_2)\right)\left|a_{n+p-1 c_{n+p-1}}\right| + (n+p-1)\left(2-p\lambda(2k+\alpha_2)\right)\left|b_{n+p-1}d_{n+p-1}\right|}{p(2-2k+\alpha_2)} \leq 1$$

$$\leq \frac{(n+p-1)\left(2-p\lambda(2k+\alpha_2)\right)p(2-2k+\alpha_1)}{p(2-2k-\alpha_2)(n+p-1)\left(2-p\lambda(2k+\alpha_1)\right)}\left|a_{n+p-1}\right| + \sum_{n=1}^{\infty}\frac{(n+p-1)\left(2-p\lambda(2k+\alpha_2)\right)p(2+2k+\alpha_1)}{p(2-2k-\alpha_2)(n+p-1)\left(2-p\lambda(2k+\alpha_1)\right)}\left|b_{n+p-1}\right| \leq 1$$

then by *Theorem* (4.3.6), we have $f * g \in \overline{\mathrm{A}}\mathrm{J}_s(\alpha_2, \lambda, \mathrm{k}, \mathrm{p})$.

THEOREM 4.3.8: Let $f \in \bar{A}Js\ (\alpha, \lambda, k, p)$. Then f is closed under convex combination

PROOF: Let $f_j(z) \in \bar{A}Js\ (\alpha, \lambda, k, p)$ for $j \geq 1$ be defined by the following form

$$f_j(z) = z^p + \sum_{n=1}^{\infty}\left|a_{n+p-1,j}\right| z^{-(n+p-1)} - \sum_{n=1}^{\infty}\left|b_{n+p-1,j}\right| \bar{z}^{-(n+p-1)}$$

$$\left(a_{n+p-1,j} \geq 0 \text{ and } b_{n+p-1,j} \geq 0\right).$$

Therefore by *Theorem* (4.3.20), we have

$$\sum_{n=1}^{\infty}\left(\frac{(n+p-1)\left(2-p\lambda(2k+\alpha)\right)}{p(2-2k-\alpha)} a_{n+p-1,j} + \frac{(n+p-1)\left(2-p\lambda(2k+\alpha)\right)}{p(2-2k-\alpha)} b_{n+p-1,j}\right).$$

$$\leq 1 \tag{4.39}$$

Then ,we can write for $\sum_{n=1}^{\infty} S_j = 1$, $0 \leq S_j \leq 1$the convex combination of $f_j$as

$\sum_{n=1}^{\infty} S_j f_j(z) = z^p + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} s_j a_{n+p-1,j}\right) z^{-(n+p-1)} - \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} s_j b_{n+p-1,j}\right) \bar{z}^{-(n+p-1)}$.

From (4.39) we get

$$\sum_{n=1}^{\infty} \frac{\left[(n+p-1)\big(2-p\lambda(2k+\alpha)\big)\right]}{p(2-2k-\alpha)} \left(\sum_{j=1}^{\infty} s_j a_{n+p-1,j}\right) + \frac{\left[(n+p-1)\big(2-p\lambda(2k+\alpha)\big)\right]}{p(2-2k-\alpha)} \left(\sum_{j=1}^{\infty} s_j b_{n+p-1,j}\right)$$

$$= \sum_{j=1}^{\infty} \mathrm{s_j} \left(\sum_{n=1}^{\infty} \left[\frac{\left[(n+p-1)\big(2-p\lambda(2k+\alpha)\big)\right]}{p(2-2k-\alpha)} \mathrm{a_{n+p-1,j}} + \frac{(n+p-1)\big(2-p\lambda(2k+\alpha)\big)}{p(2-2k-\alpha)} \mathrm{b_{n+p-1,j}}\right]\right) \leq \sum_{j=1}^{\infty} \mathrm{s_j} = 1.$$

$therefore$ $\sum_{j=1}^{\infty} \mathrm{s_j f_j(z)} \in AJ\mathrm{s}(\alpha, \lambda, \mathrm{k}, \mathrm{p})$. This completes the proof.

CHAPTER 5

DIFFERENTIALSUBORDINATION AND STRONGDIFFERENTIAL SUBORDINATION OF SUBCLASSES OF MULTIVALENT FUNCTIONS

INTRODUCTION

Chapter five is fully devoted for the study of differential subordination properties of classes of univalent and multivalent functions defined by Ruscheweyh derivative operator , have Taylor series expansion .

Here, we have studied differential subordinations and their properties to univalent and multivalent functions. Actually the differential subordination topic originates from the article by the authors entitled "Differential subordinations and univalent functions . This paper laid a function for the subsequent development. This concept has been used in various fields , such as , differential equations , partial differential equations , meromorphic functions , harmonic functions , integral operators , Banach spaces and functions of several complex variables. The term differential subordination in field of complex plane can be looked upon as the generalization of differential inequality on the real line . The growth of the differential inequalities is a development of the last fifty years .

M. Kamali[58], W. Walter[59] and M. H. Protter [60] have also contributed a lot in theory of differential inequalities . First order differential implications were presented in 1935 by G. M. Goluzin[61] and R. M. Robinson[62].

This chapter is divided into four sections. The first section is concerned with strong differential subordination properties for multivalent functions defined by Ruscheweyh derivative operator .

We obtain some applications of differential subordination and superordination results involving a Rescheweyh derivative operator for certain normalized analytic functions.

The second section deals with some applications of differential subordination Involving hadamard product. We obtain some subordination results for univalent functions in the open unit disk U. *The three section* has been fully dealt with study of Certain family of multivalent functions Associated with subordination and The *four section* we have introduced new classes by using subordination and we have obtained coefficient estimates and properties which contains Distortion and Growth theorems , radius of starlikeness and radius of convexity, and other related results for $K\mathcal{M}(A, B, \alpha, \delta, p)$ and $\mathcal{M}(A, B, \alpha, \delta, p)$.

5.1 STRONG DIFFERENTIAL SUBORDINATION PROPERTIES FOR MULTIVALENT FUNCTIONS DEFINED BY INTEGRAL OPERATOR

Let $\mathcal{D}\rho$ denoted the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \ , (p, n \in N = \{1,2, \dots\}) z \in U \qquad (5.1)$$

For the function $f \in \mathcal{D}\rho$ given by (5.1) and $g \in \mathcal{D}\rho$ defined by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$$

The Hadamard product (or convolution) of f *and* g is given by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z) \qquad (5.2)$$

Motivated essentially by Jung et al.[2], Shams et al.[10] introduced the operator $I_p^\alpha : \mathcal{D}\rho \to \mathcal{D}\rho$as follows:

$$I_p^\alpha f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+1}{k+p+1}\right)^\alpha a_{k+p} z^{k+p}, \ \ \alpha \in \mathbb{R} \qquad (5.3)$$

Using the above definition relation , it is easy verify that the operator becomes an integral operator

$$I_p^\alpha f(z) = \frac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left(log \frac{z}{t}\right)^{\alpha-1} f(t)dt, \text{ for } \alpha > 0 \qquad (5.4)$$

$$I_p^\alpha f(z) = f(z), for\ \alpha = 0,$$

and , moreover

$$(p+1) I_p^{\alpha-2} f(z) = z\left(I_p^{\alpha-1} f(z)\right)' + I_p^{\alpha-1} f(z), \quad for\ \alpha \in \mathbb{R}$$

We mention that the one- parameter family of integral operator $I^\alpha \equiv I_1^\alpha$ was defined by Jung et al. [2].

DEFINITION(5.1.1): Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mu[0,p]$ and $\lambda > -p$. The class of admissible functions $\Phi_k[\Omega, q]$ consists of those functions $\emptyset: \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\emptyset(u, v, w; z, \xi) \notin \Omega, \tag{5.5}$$

whenever

$$u = q(\xi), v = p + 1 ,$$

and

$$\mathcal{R}e\left\{\frac{(p+1)\{w(p+1) - 2v + u\}}{(p+1)v - u}\right\} \geq k\mathcal{R}e\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\}, \tag{5.6}$$

$z \in U, \zeta \in \partial U \backslash E(q), \xi \in \bar{U}$ and $k \geq p$.

THEOREM(5.1.1): Let $\emptyset \in \Phi_k[\Omega, q]$. If $f \in \mathcal{D}\rho$ satisfies

$$\{\emptyset\left(I_p^{\alpha-2} f(z), I_p^{\alpha-1} f(z), f(z); z \in U, \xi \in \bar{U}\right\} \subset \Omega, \tag{5.7}$$

then $I_p^{\alpha-2} f(z) \prec q(z)$.

PROOF: By using (5.3) and (5.4), we get the equivalent relation

$$I_p^{\alpha-2} f(z) = \frac{z\left(I_p^{\alpha-1} f(z)\right)' + I_p^{\alpha-1} f(z)}{(p+1)} \tag{5.8}$$

$$I_p^{\alpha-1} f(z) = \frac{z\left(I_p^{\alpha} f(z)\right)' + I_p^{\alpha} f(z)}{(p+1)} \tag{5.9}$$

$$\left(I_p^{\alpha-1} f(z)\right)' = \frac{z\left(I_p^{\alpha} f(z)\right)'' + 2\left(I_p^{\alpha} f(z)\right)'}{(p+1)} \tag{5.10}$$

Assume that $F(z) = I_p^{\alpha} f(z)$. Then

$$I_p^{\alpha-1} f(z) = \frac{z\big(F'(z) + F(z)\big)}{(p+1)}$$

Therefore

$$I_p^{\alpha-2}f(z)=\frac{z}{(p+1)}\left\{\frac{z\left(I_p^{\alpha}f(z)\right)''+2\left(I_p^{\alpha}f(z)\right)'}{(p+1)}\right\}+\frac{z(I_p^{\alpha}f(z))'+I_p^{\alpha}f(z)}{(p+1)}$$

Then we have by (5.8)

$$I_p^{\alpha-2}f(z)=\frac{z}{(p+1)}\left\{\frac{zF''(z)+zF'(z)}{(p+1)}\right\}+\frac{z\big(F'(z)\big)+F(z)}{(p+1)}$$

$$=\frac{1}{(p+1)}\left\{\frac{z^2F''(z)+(z+1)zF'(z)+F(z)}{(p+1)}\right\} \qquad (5.11)$$

Let $u=r \quad ,v=\dfrac{s+r}{(p+1)}\ , \quad w=\dfrac{t+(z+1)s+r}{(p+1)^2}.$

Assume that

$$\psi(r,s,t;z,\xi)=\emptyset(u,v,w;z,\xi)=\emptyset\left(r,\frac{s+r}{(p+1)},\frac{t+(z+1)s+r}{(p+1)^2};z,\xi\right). \qquad (5.12)$$

By using (5.8) and (5.9) , we obtain

$$\psi(F(z),zF'(z),z^2F''(z);z,\xi)=\emptyset\left(I_p^{\alpha}f(z),I_p^{\alpha-1}f(z),I_p^{\alpha-2}f(z);z,\xi\right). \quad (5.13)$$

Therefore, by making use (5.7), we get

$$\psi(F(z),zF'(z),z^2F''(z);z,\xi)\in\Omega. \qquad (5.14)$$

Also, by using

$$w=\frac{t+(z+1)s+r}{(p+1)^2}$$

and by simple calculations ,we get

$$\frac{(p+1)\{w(p+1)-zv\}+u(z-1)}{(p+1)v-u}=\frac{t}{s}+1. \qquad (5.15)$$

and the admissibility condition for $\emptyset\in\Phi_k[\Omega,q]$ is equivalent to the admissibility condition for ψ then , F(z)≺q(z). Hence , we get $I_p^{\alpha-2}f(z)\prec q(z)$.

If we assume that $\Omega \neq \mathbb{C}$ is a simply connected domain. So, $\Omega = h(U)$, for some conformal mapping h of U onto Ω. Assume the class is written as $\Phi_k[h, q]$.Therefore, we conclude immediately the following Theorem.

THEOREM(5.1.2): Let $\emptyset \in \Phi_k[h, q]$. If $f \in \mathcal{D}\rho$ satisfies

$$\emptyset(I_p^{\alpha-2}f(z), I_p^{\alpha-1}f(z), I_p^{\alpha}f(z); z \in U, \xi) \prec\prec \mathrm{h}(z), \tag{5.16}$$

then $I_p^{\alpha-2}f(z) \prec q(\mathrm{z})$.

The next result is an extension of Theorem (5.1.1) to the case where the behavior of q on ∂U is not known.

COROLLARY(5.1.1): Let $\Omega \subset \mathbb{C}$, q be univalent in U and $q(0)=0$. Let $\emptyset \in \Phi_k[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{D}\rho$ satisfies

$$\emptyset\left(I_p^{\alpha-2}f(z), I_p^{\alpha-1}f(z), I_p^{\alpha}f(z); z, \xi\right) \in \Omega, \tag{5.17}$$

then $I_p^{\alpha-2}f(z) \prec q(\mathrm{z})$.

THEOREM (5.1.3): Let h and q be univalent in U , with $q(0)=0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\emptyset: \mathbb{C}^3 \times \mathrm{U} \times \overline{\mathrm{U}} \to \mathbb{C}$ satisfy one of the following conditions:

(1) $\emptyset \in \Phi_k[\Omega, q_\rho]$ for some $\rho \in (0,1)$ or

(2) there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_\mathrm{k}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{D}\rho$ satisfies (5.16) , then $I_p^{\alpha-2}f(z) \prec q(z)$.

PROOF: Case (1): By using Theorem (5.1.1) , we get $I_p^{\alpha-2}f(z) \prec q_\rho$. Since $q_\rho(z) \prec q(z)$, then we get the result.

Case (2): Assume that $F(z)=I_p^{\alpha}f(z)$ and $F_\rho(z) = F(\rho z)$. So,

$$\emptyset\left(F_\rho(z), zF_\rho'(z), z^2F_\rho''(z); \rho z\right) = \emptyset(F(\rho z), \rho zF'(\rho z), \rho^2 z^2 F''(\rho z); \rho z) \in h_\rho(U).$$

By using Theorem (5.1.1) with associated

$\emptyset(F(z), zF'(z), z^2F''(z); w(z)) \in \Omega$, where w is any function mapping from

U onto U, with $w(z)=\rho z$, we obtain $F_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get

$$I_p^{\alpha-2} f(z) f(z) \prec q(z).$$

The next theorem gives the best dominant of the differential subordination (5.13).

THEOREM (5.1.4): Let h be univalent in U and let $\emptyset: \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation

$$\emptyset\left(q(z), \frac{z\left(q'(z)\right) + q(z)}{(p+1)}, \frac{1}{(p+1)}\left\{\frac{z^2 q''(z) + (z+1)zq'(z) + q(z)}{(p+1)}\right\}; z, \xi\right)$$
$$= h(z), \qquad (5.18)$$

has a solution q with $q(0)=0$ and satisfy one of the following conditions:

(1) $q \in Q_0$ and $\emptyset \in \Phi_k[h, q]$.

(2) q is univalent in U and $\emptyset \in \Phi_k[h, q_\rho]$ for some $\rho \in (0,1)$.

(3) q is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$. If $f \in \mathcal{D}\rho$ satisfies (5.16), then $I_p^{\alpha-2} f(z) \prec q(z)$ and q is the best dominant.

PROOF: By using Theorem (5.1.2) and Theorem (5.1.3), we get that q is a dominant of (5.16). Since q satisfies (5.18), it is also a solution of (5.16) and therefore q will be dominant by all dominants of (5.16). Hence, q is the best dominant of (5.16).

DEFINITION(5.1.2) : Let Ω be a set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\Phi_k[\Omega, M]$ consists of those functions $\emptyset: \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ such that

$$\emptyset\left(Me^{i\theta}, \frac{(k+1)Me^{i\theta}}{(p+1)}, \frac{(L+k+1)Me^{i\theta}}{(p+1)^2}; z, \xi\right) \notin \Omega, \quad (5.19)$$

whenever z$\in U, \xi \in \overline{U}$,$k \geq 1$.

<u>COROLLARY(5.1.2):</u> Let $\emptyset \in \Phi_k[\Omega, M]$. If $f \in A$ satisfies that

$$\emptyset\left(I_p^{\alpha-2}f(z), I_p^{\alpha-1}f(z), I_p^{\alpha}f(z); z, \xi\right) \in \Omega \text{ , then } I_p^{\alpha-2}f(z) \prec Mz.$$

<u>COROLLARY(5.1.3):</u> Let $\emptyset \in \Phi_k[\Omega, M]$. If $f \in \mathcal{D}\rho$ satisfies that

$$\left|\emptyset\left(I_p^{\alpha-2}f(z), I_p^{\alpha-1}f(z), I_p^{\alpha}f(z); z, \xi\right)\right| < M, \text{then} \left|I_p^{\alpha-2}f(z)\right| < M$$

<u>COROLLARY(5.1.4):</u>Let M>0 , and Let $C(\xi)$ be an analytic function in $\overline{U}$ with Re$\{\xi C(\xi)\} \geq 0$ for $\xi \in \partial U$. If $f \in \mathcal{D}\rho$ satisfies

$$|(p+1)^2 I_p^{\alpha-2}f(z) - (p+1)I_p^{\alpha-1}f(z) - \lambda^2 I_p^{\alpha}f(z) + C(\xi)| < M,$$

then $$|I_p^{\alpha-2}f(z)| < M.$$

<u>PROOF:</u> From Corollary (5.1.2) by taking $\emptyset(u, v, w, z, \xi) = (p+1)^2 w - (p+1)v - \lambda^2 u + C(\xi)$ and $\Omega = h(U)$, where h(z)= Mz. By using Corollary (5.1.2), we need to show that

$\emptyset \in \Phi_k[\Omega, M]$, that is ,the admissible condition (5.19) is satisfied .We get

$$\left|\emptyset\left(Me^{i\theta}, \frac{(k+1)Me^{i\theta}}{(p+1)}, \frac{(L+k+1)Me^{i\theta}}{(p+1)^2}; z, \xi\right)\right|$$

$$= |(L+K+1)Me^{i\theta} - (k+1)Me^{i\theta} - \lambda^2 Me^{i\theta} + \mathrm{C}(\xi)|$$

$$= |(L-\lambda^2)Me^{i\theta} + \mathrm{C}(\xi)| \geq (L-\lambda^2)M + \mathcal{R}e\{Le^{-i\theta}\} + \mathcal{R}e\{\mathrm{C}(\xi)e^{-i\theta}\}$$

$\geq \lambda M$. Hence by Corollary(5.1.3), we get the result.

<u>DEFINITION(5.1.3):</u>Let Ω be a set in $\mathbb{C}$ and $q \in \mu[0, p]$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_k[\Omega, q]$ consists of those functions $\emptyset: \mathbb{C}^3 \times \overline{U} \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\emptyset(u, v, w; \zeta, \xi) \notin \Omega, \tag{5.20}$$

whenever

$$u = q(z), v = \frac{\frac{1}{m} zq'(z) + q(z)}{1+p},$$

and

$$\mathcal{R}e\left\{\frac{(p+1)\{w(p+1) - zv\} + u(z-1)}{(p+1)v - u}\right\} \geq \frac{1}{m}\mathcal{R}e\left\{\frac{zq''(z)}{q'(z)} + 1\right\}, \quad (5.21)$$

$z \in U, \zeta \in \partial U \backslash E(q), \xi \in \bar{U}$ *and* $m \geq p$.

<u>THEOREM(5.1.5):</u> Let $\emptyset \in \Phi'_k[h, q]$. If $f \in \mathcal{D}\rho$, $I_p^{\alpha-2} f(z) \in Q_0$ and $\emptyset(I_p^{\alpha-2} f(z), I_p^{\alpha-1} f(z), I_p^{\alpha} f(z) ; z, \xi)$ is univalent in U , then

$\Omega \subset \{\emptyset\big(I_p^{\alpha-2} f(z), I_p^{\alpha-1} f(z), I_p^{\alpha} f(z); z \in U, \xi \in \bar{U}\big)\}$, implies that

$$q(z) \prec I_p^{\alpha-2} f(z).$$

<u>PROOF:</u> By (5.13) and $\Omega \subset \{\emptyset(I_p^{\alpha-2} f(z)); z \in U, \xi \in \bar{U})$,

we have $\Omega \subset \{\psi(F(z), zF'(z), z^2F''(z); z, \xi); z \in U, \xi \in \bar{U})\}$. From

$$u = r \quad , v = \frac{s+r}{(p+1)}, \quad w = \frac{t + (z+1)s + r}{(p+1)^2}$$

we see that the admissibility for $\emptyset \in \Phi'_k[\Omega, q]$ is equivalent to admissibility condition for ψ. Hence , $\psi \in \Psi'[\Omega, q]$ and so we have $q(z) \prec I_p^{\alpha-2} f(z)$.

The following Theorem is an immediate consequence of Theorem(5.1.5).

<u>THEOREM(5.1.6):</u> Let $q \in \mu[0, p], h$ be analytic in U and $\emptyset \in \Phi'_k$ $[h, q]$. If f(z) $\in \mathcal{D}\rho$, $I_p^{\alpha-2} f(z) \in Q_0$ and$\{\emptyset\big(I_p^{\alpha-2} f(z), I_p^{\alpha-1} f(z), I_p^{\alpha} f(z); z, \xi\}$ is univalent in U, then

$$h(z) \prec\prec \emptyset\big(I_p^{\alpha-2} f(z), I_p^{\alpha-1} f(z), I_p^{\alpha} f(z); z, \xi\}, \quad (5.22)$$

implies that $\qquad q(z) \prec I_p^{\alpha-2} f(z).$

<u>THEOREM(5.1.7):</u> Let h be analytic in U and $\emptyset$: $\mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$. Suppose that the differential equation

$$\emptyset\left(q(z),\frac{z\left(q'(z)\right)+q(z)}{(p+1)},\frac{1}{(p+1)}\left\{\frac{z^2q''(z)+(z+1)zq'(z)+q(z)}{(p+1)}\right\};z,\xi\right)=h(z),$$

has a solution $q\in Q_0$. If $\emptyset\in\Phi'_k[h,q]$, $f\in\mathcal{D}\rho$, $I_p^{\alpha-2}f(z)\in Q_0$ and

$\{\emptyset\left(I_p^{\alpha-2}f(z),I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);z,\xi\right\}$ is univalent in U , then

$$h(z)\prec\prec\emptyset\left(I_p^{\alpha-2}f(z),I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);z,\xi\right\},\qquad(5.23)$$

implies that $q(z)\prec I_p^{\alpha-2}f(z)$, and q is the best dominant.

PROOF: The proof of this Theorem is the same of proof Theorem (5.1.4).

Theorem (5.1.2) and Theorem (5.1.6), we obtained the following Theorem.

THEOREM(5.1.8): Let h_1 and q_1 be analytic functions in U,h_2 be a univalent functions in U , $q_2\in Q_0$ with $q_1(0)=q_2(0)=0$ and $\emptyset\in\Phi_k[h_2,q_2]\cap\Phi'_k[h_1,q_1]$. If $f\in\mathcal{D}\rho$, $I_p^{\alpha-2}f(z)\in\mu[0,p]\cap Q_0$ and

$\{\emptyset\left(I_p^{\alpha-2}f(z),I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);z,\xi\right\}$is univalent in U, then

$$h_1(z)\prec\prec\emptyset\left(I_p^{\alpha-2}f(z),I_p^{\alpha-1}f(z),I_p^{\alpha}f(z);z,\xi\right\}\prec\prec h_2,\qquad(5.24)$$

implies that $q_1(z)\prec I_p^{\alpha-2}f(z)\prec q_2(z)$.

5.2 SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION INVOLVING HADAMARD PRODUCT

Let $\mathcal{D}(p,1)$ represents the class of functions as given below

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} (a_n \geq 0\,; p \in \mathbb{N}) \qquad (5.25)$$

These functions are analytic in open disk U defined as $U = \{z: |z| < 1\}$ .Let $f(z), g(z) \in D(p,1)$, where

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$$

and

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$$

Then the convolution

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} \quad (5.26)$$

Let A, B, σ, η, ξ and $\varepsilon\,, \delta, \tau$ be fixed real numbers . $f(z) \in \mathcal{D}(p,1)$ Contained in $\mathcal{L}_{\sigma,\eta,\xi,\varepsilon\,,\delta,\tau,}(p; A, B)$ gives

$$\mathcal{L}_{\varepsilon\,,\delta,\tau,p}(f) \prec \frac{1+2Az}{1+2Bz} \qquad z \in U \qquad (5.27)$$

$$\mathcal{L}_{\varepsilon\,,\delta,\tau,p}(f) = [1 - \sigma(\eta + \xi)] \frac{\mathcal{H}_p^{\mu+p-1} f(z)}{z^p} + \sigma(\eta + \xi) \frac{\mathcal{H}_p^{\mu+p} f(z)}{z^p}$$

Where

$$\mathcal{H}_p^{\mu+p-1} f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(\mu+p+n)}{\Gamma(\mu+p)n!} a_{n+p} z^{n+p} \qquad (5.27)$$

$$\left(\begin{array}{c} \alpha_{2k} \geq 0, n \geq 0\,, m \geq 0\,, 0 \leq \gamma \leq 1 \\ , \xi \geq 0, \tau \geq 0, \delta \geq 0, 0 < \varepsilon \leq \frac{1}{2}, 0 \leq \alpha < 1, \eta \geq 0, 0 < \sigma \leq \frac{1}{2}, -\frac{1}{2} \leq B \leq A \leq \frac{1}{2} \end{array} \right)$$

Hence from above relation , we have been obtain

$$z\left(H_p^{\mu+p-1} f(z)\right)' = (\mu + p) H_p^{\mu+p} f(z) - \mu H_p^{\mu+p-1} f(z) \qquad (5.28)$$

This work is due to the [63] and [22].where we have used the techniques of differential subordination to obtain several interesting properties.

A holomorphic function f is said to be close-to-convex of order α $(0 \leq \alpha < 1)$ if there exists a convex function $h \in \mathcal{D}(1,1)$ and a real β such that $\mathcal{R}e\left(\frac{f'(z)}{e^{i\beta}h'(z)}\right) > \alpha \; for \; z \in U$.

THEOREM 5.2.1.Let the function $f(z) \in \mathcal{D}(p,1)$. Then

$$z\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)'' = (\mu+p)\left(z^{1-p}\mathcal{H}_p^{\mu+p}f(z)\right)' - \mu\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)' \qquad (5.29)$$

PROOF: we know that

$$z\left(z^{1-p}\mathcal{H}_p^{\mu+p}f(z)\right)' = (\mu+p)\mathcal{H}_p^{\mu+p}f(z) - \mu\mathcal{H}_p^{\mu+p-1}f(z)$$

since $z\left(\mathcal{H}_p^{\mu+p-1}f(z)\right)' + (1-\mathrm{p})\mathcal{H}_p^{\mu+p-1}f(z) = (\mu+p)\mathcal{H}_p^{\mu+p}f(z) + (1-\mu+p)\mathcal{H}_p^{\mu+p-1}f(z)$

But owing to

$$z\left(\mathcal{H}_p^{\mu+p-1}f(z)\right)' + (1-\mathrm{p})\mathcal{H}_p^{\mu+p-1}f(z) = z^p\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)',$$

We obtain

$$z\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)' = (\mu+p)\left(z^{1-p}\mathcal{H}_p^{\mu+p}f(z)\right) + (1-\mu+p)\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)$$

differentiating both sides of above equation we get

$$z\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)'' = (\mu+p)\left(z^{1-p}\mathcal{H}_p^{\mu+p}f(z)\right)' - \mu\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)'$$

COROLLARY 5.2.1: Let $f(z) \in \mathcal{D}(p,1)$ and $z^{1-p}\mathcal{H}_q^{\mu+p-1}f(z)$ is convex univalent function. Then $z^{1-p}\mathcal{H}_p^{\mu+p}f(z)$is close-to-convex of order $\frac{(\mu+p)-1}{|(\mu+p)|}$with respect to $z^{1-p}\mathcal{H}_p^{(\mu+p)-1}f(z)$.

PROOF. Since

$$z\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)'' = (\mu+p)\left(z^{1-p}\mathcal{H}_p^{\mu+p}f(z)\right)' - (\mu+p)\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)'.$$

We obtain

$$\frac{\left(z^{1-p}\mathcal{H}_p^{\mu+p}f(z)\right)'}{\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)'} = \frac{z\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)''}{(\mu+p)\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)''} + 1$$

Since $z^{1-p}\mathcal{H}_q^{\mu+p-1}f(z)$ is a convex function ,

$$\mathcal{R}e\left\{\frac{(\mu+p)}{|\mu+p|}\frac{\left(z^{1-p}\mathcal{H}_p^{\mu+p}f(z)\right)'}{\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)'}\right\}=\mathcal{R}e\left\{\frac{z}{|\mu+p|}\frac{\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)''}{\left(z^{1-p}\mathcal{H}_p^{\mu+p-1}f(z)\right)'}+\frac{\mu+p}{|\mu+p|}\right\}$$

$$>Re\left(\frac{\mu+p-1}{|\mu+p|}\right)$$

Therefore, by definition of close-to-convex we get the required result.

THEOREM 5.2.2 Let $f_1(z), f_2(z) \in D(p,1), \mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z)) \prec h_1(z)$ and $\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z)) \prec h_2(z)$, where $h_1(z), h_2(z)$ are convex univalent in U and if $\frac{\mu+p}{\lambda} \geq 0$, $\mu+p>\lambda>0$, then

$$\mathcal{L}_{\varepsilon,\delta,\tau,p}\left(H_q^{\mu+p-1}(f_1*f_2)\right)\prec\frac{\mu+p}{\lambda}z^{-\frac{\mu+p}{\lambda}}\int_0^z t^{\frac{\mu+p}{\lambda}}h_1(t)*h_2(t)dt\prec h_1(t)*h_2(t).$$

PROOF: Since $\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z)) \prec h_1(z)\ and\ \mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z)) \prec h_2(z)$ then we have $\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z)) * \mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z)) \prec h_1(z)*h_2(z)$ and, the convolution of convex univalent functions is also the convex univalent function .Now, let

$$p(z)=\mathcal{L}_{\varepsilon,\delta,\tau,p}\left(\mathcal{H}_p^{\mu+p-1}((f_1*f_2))(z)\right)$$

$$=(1-\lambda)\frac{\left(\mathcal{H}_p^{\mu+p-1}\left(\mathcal{H}_p^{\mu+p-1}(f_1*f_2)\right)(z)\right)}{z^p}+\lambda\frac{\left(\mathcal{H}_p^{\mu+p}\left(\mathcal{H}_p^{\mu+p}(f_1*f_2)\right)(z)\right)}{z^p}$$

Then $p(z)$ is holomorphic function and $p(0) = 1\ in\ U$.

By using (5.28), we have

$$p(z)+\frac{\lambda p}{\mu+p}p'(z)=\mathcal{L}_{\varepsilon,\delta,\tau,p}\left(\mathcal{H}_p^{\mu+p-1}(f_1*f_2)\right)(z)+\frac{\lambda z}{\mu+p}\left(\mathcal{L}_{\varepsilon,\delta,\tau,p}\left(\mathcal{H}_p^{\mu+p-1}(f_1*f_2)\right)(z)\right)'$$

$$=\left(1-\tfrac{\lambda p}{\mu+p}\right)z^{-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)+\tfrac{\lambda}{\mu+p}z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_q^{\mu+p-1}f_2(z)\right)'+$$
$$\tfrac{\lambda z}{\mu+p}\Big[\left(1-\tfrac{\lambda p}{\mu+p}\right)z^{-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)+$$
$$\tfrac{\lambda}{\mu+p}z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)'\Big]'$$

$$=\left(1-\tfrac{\lambda p}{\mu+p}\right)z^{-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)+\tfrac{\lambda}{\mu+p}z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)'+$$
$$\tfrac{\lambda z}{\mu+p}\times$$
$$\Big[\left(1-\tfrac{\lambda p}{\mu+p}\right)\left(-pz^{-p-1}\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)+z^{-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)'\right)+$$
$$\tfrac{\lambda}{\mu+p}\left((1-p)z^{-p}\times\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)\right)'+z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)''\Big]$$

$=\left[1-\frac{\lambda p}{\mu+p}-\frac{\lambda p}{\mu+p}+\frac{\lambda^2p^2}{(\mu+p)^2}\right]z^{-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)+\left[\frac{\lambda}{\mu+p}+\frac{\lambda}{\mu+p}\left(1-\frac{\lambda p}{\mu+p}\right)+\frac{\lambda^2}{(\mu+p)^2}(1-p)\right]\times z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)'+\frac{\lambda^2}{(\mu+p)^2}z^{2-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)''.$

Now

$$\begin{aligned}\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z))*\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z))&=\left[\left(1-\frac{\lambda p}{\mu+p}\right)z^{-p}\mathcal{H}_p^{\mu+p-1}f_1(z)+\frac{\lambda}{\mu+p}z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)\right)'\right]\\&*\left[\left(1-\frac{\lambda p}{\mu+p}\right)z^{-p}\mathcal{H}_p^{\mu+p-1}f_2(z)+\frac{\lambda}{\mu+p}z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_2(z)\right)'\right]\end{aligned}$$

$$\begin{aligned}=\left(1-\frac{\lambda p}{\mu+p}\right)^2z^{-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)+2\left(1-\frac{\lambda p}{\mu+p}\right)\\\times\frac{\lambda}{\mu+p}z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)'+\left(\frac{\lambda}{\mu+p}\right)^2\\\times z^{1-p}\left[z\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)'\right]'\end{aligned}$$

$$\begin{aligned}=\left(1-\frac{\lambda p}{\mu+p}\right)^2z^{-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)+\left[2\left(1-\frac{\lambda p}{\mu+p}\right)\frac{\lambda}{\mu+p}+\left(\frac{\lambda p}{\mu+p}\right)^2\right]\\\times z^{1-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)'+\left(\frac{\lambda}{\mu+p}\right)^2\\\times z^{2-p}\left(\mathcal{H}_p^{\mu+p-1}f_1(z)*\mathcal{H}_p^{\mu+p-1}f_2(z)\right)''.\end{aligned}$$

Then we get

$p(z)+\frac{\lambda z}{\mu+p}p'(z)=\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1(z))*\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_2(z))\prec h_1(z)*h_2(z)$

$$p(z)\prec\frac{\mu+p}{\lambda}z^{-\frac{\mu+p}{\lambda}}\int_0^z t\frac{\mu+p}{\lambda}h_1(t)*h_2(t)dt\prec h_1(t)*h_2(t).$$

<u>THEOREM 5.2.3</u> Let $f_1(z)\in\mathcal{L}_{\sigma,\mu,\xi,\varepsilon,\delta,\tau}(p;A_1,B_1)$ and $f_2(z)\in\mathcal{L}_{\sigma,\mu,\xi,\varepsilon,\delta,\tau}(p;A_2,B_2)$ Where $\mathcal{L}_{\varepsilon,\delta,\tau}(f_1(z))\prec\frac{1+A_1z}{1+B_1z}$and $\mathcal{L}_{\varepsilon,\delta,\tau}(f_2(z))\prec\frac{1+A_2z}{1+B_2z}$ where $-1\le B_1<A_1\le 1;\ -1\le B_2<A_2\le 1$ and $\frac{\mu+p}{\lambda}\ge 0$, $\varepsilon(\delta+\tau)+q>\sigma(\eta+\xi)>0$. Then $\mathcal{L}_{\varepsilon,\delta,\tau,p}\left(\mathcal{H}_q^{\mu+p-1}(f_1*f_2)(z)\right)\prec 1+(A_1-B_1)(A_2-B_2)\frac{\mu+p}{\lambda}z^{-\frac{\mu+p}{\lambda}}\int_0^z\frac{t^{\frac{\mu+p}{\lambda}}}{1-B_1B_2t}dt=q(z)$

where

$$\mathrm{q}(z)=1+\frac{(\mu+p)(A_1-B_1)(A_2-B_2)z}{\mu+p+\lambda}[1-B_1B_2]^{-1}\,{}_2F_1\left(1,1;2+\frac{\mu+p}{\lambda};\frac{B_1B_2z}{B_1B_2z-1}\right)\qquad(5.30)$$

PROOF :Since $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are univalent convex function,

$$\frac{1+A_1z}{1+B_1z}*\frac{1+A_2z}{1+B_2z}=1+(A_1-B_1)\frac{z}{1+B_1z}*1+(A_2-B_2)\frac{z}{1+B_2z}$$

$$=1+(A_1-B_1)(A_2-B_2)\frac{z}{1+B_1B_2z}.$$

Thus, by *Theorem* (5.2.10), we have

$$\mathcal{L}_{\varepsilon,\delta,\tau}\left(\mathcal{H}_q^{\mu+p-1}(f_1*f_2)(z)\right)\prec 1+(A_1-B_1)(A_2-B_2)\frac{\mu+p}{\lambda}z^{-\frac{\mu+p}{\lambda}}\int_0^z\frac{t^{\frac{\mu+p}{\lambda}-1}}{1-B_1B_2t}dt.$$

Now, in order to prove (5.30), we write

$$p(z)=\frac{\mu+p}{\lambda}z^{-\frac{\mu+p}{\lambda}}\int_0^z t^{\frac{\mu+p}{\lambda}-1}\left(1+\frac{(A_1-B_1)(A_2-B_2)t}{1-B_1B_2t}\right)dt$$

$$=1+(A_1-B_1)(A_2-B_2)z\left(\frac{\mu+p}{\lambda}\right)\int_0^1 s^{\frac{\mu+p}{\lambda}}(1-B_1B_2sz)^{-1}ds.$$

Hence we obtained the required result . Putting $A_1=A_2=B_1=B_2=1$ in Theorem 5.2.3, we have next corollary.

COROLLARY 5.2.2 Let $f_1(z),f_2(z)\in\mathcal{D}(p,1)$.and $\mathcal{L}_{\varepsilon,\delta,\tau,p}\left(f_1(z)\right)\prec\frac{1+z}{1-z}$ and $\mathcal{L}_{\varepsilon,\delta,\tau,p}\left(f_2(z)\right)\prec\frac{1+z}{1-z}$ then

$$\mathcal{L}_{\varepsilon,\delta,\tau,p}(f_1*f_2)(z)\prec 1+4\frac{\mu+p}{\lambda}z^{-\frac{\mu+p}{\lambda}}\int_0^z\frac{t^{\frac{\mu+p}{\lambda}-1}}{1+t}dt.$$

Putting λ=1 , $\mu=0$ in *Corollary* (5.2.2) we have

COROLLARY 5.2.7 Let $f_1(z),f_2(z)\in\mathcal{D}(p,1)$.and let $\mathcal{L}_{0,0,0,p}\left(f_1(z)\right)\prec\frac{1+z}{1-z}$ and $\mathcal{L}_{0,0,0,p}\left(f_2(z)\right)\prec\frac{1+z}{1-z}$ then

$$\mathcal{L}_{0,p}(f_1*f_2)(z)\prec 1+4pz^{-p}\int_0^1\frac{t^{p-1}}{1+t}dt.$$

Consider the following integral transform[16]

$$F_c(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt=\sum_{n=p}^{\infty}\frac{c+p}{c+n}z^n*f(z) \qquad (5.31)$$

Where , $f(z)\in D(p,1)$ and $c+p>0$. Now since

$$\mathcal{H}_p^{\mu+p-1}\text{f(z)}=\frac{z^p}{(1-z)^{\mu+p}}*f(z)=z^p+\sum_{n=1}^{\infty}\frac{\Gamma(\mu+p+n)}{\Gamma(\mu+p)n!}a_{n+p}z^{n+p}$$

Where $f(z)\in A(p,1)$ and $c+p>0$. Now since

$$\mathcal{H}_p^{\mu+p-1}f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(\mu+p+n)}{\Gamma(\mu+p)n!} a_{n+p} z^{n+p}$$

We have

$$z\left(\mathcal{H}_p^{\mu+p-1}F_c(z)\right)' = (\mathrm{c}+\mathrm{p})\mathcal{H}_p^{\mu+p}f(z) - c\mathcal{H}_p^{\mu+p-1}F_c(z) \qquad (5.32)$$

THEOREM 5.2.12 Let μ, c be real number $(\mu \geq 0)$ such that $c+p > 0$ if $f_1(z), f_2(z) \in \mathcal{D}(p,1)$ satisfy

$$\frac{\mathcal{H}_q^{\mu+p-1}(f_1 * f_2)(z)}{z^p} \prec 1 + \frac{(A_1-B_1)(A_2-B_2)z}{1-B_1B_2z},$$

Then

$$\frac{\mathcal{H}_q^{\mu+p-1}(F_c(z) * G_c(z))}{z^p} \prec q(z) \prec 1 + \frac{(A_1-B_1)(A_2-B_2)z}{1-B_1B_2z}$$

Where $F_c(z)$ is defined as

$$F_c(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt = \sum_{n=p}^{\infty}\frac{c+p}{c+n}z^n * f(z)$$

and

$$\mathcal{H}_q^{\mu+p-1}f(z) = \frac{z^p}{(1-z)^{\lambda}} * f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(\mu+p+n)}{\Gamma(\mu+p)n!} a_{n+p} z^{n+p}$$

$G_c(z)$ is defined as follows

$$G_c(z) = \sum_{n=p}^{\infty}\frac{(c+p)}{(c+n)}z^n * f_2(z)$$

and

$$q(z) = 1 + (1-B_1B_2z)^{-1}\frac{c+p}{c+n+1}(A_1-B_1)(A_2-B_2)z\ {}_2F_1\left(1,1;2+c+p;\frac{B_1B_2z}{B_1B_2z-1}\right)$$

PROOF: Let $p(z) = \frac{\mathcal{H}_p^{\mu+p-1}(F_c(z) * G_c(z))}{z^p}$

Then $p(z)$ is holomorphic in the disk U such that $p(0) = 1$ .since we know

$$z\left(\mathcal{H}_p^{\mu+p-1}f_c(z)\right)' = (\mathrm{c}+\mathrm{p})\mathcal{H}_p^{\mu+p}f(z) - c\mathcal{H}_p^{\mu+p-1}f_c(z)$$

Then

$p(z)+\frac{zp'}{c+p}=\frac{\mathcal{H}_p^{\mu+p-1}(f_1*f_2)(z)}{z^p}\prec 1+\frac{(A_1-B_1)(A_2-B_2)z}{1-B_1B_2z}$

Then

$$\frac{\mathcal{H}_p^{\mu+p-1}(F_c(z)*G_c(z))}{z^p}\prec \mathrm{p(z)}$$

$$=(c+p)z^{-(c+p)}\int_0^z t^{c+p-1}\frac{(1+A_1t)}{(1+B_1t)}*\frac{(1+A_2t)}{(1+B_2t)}dt\prec 1+\frac{(A_1-B_1)(A_2-B_2)z}{1-B_1B_2z}$$

Finally we obtain

$$q(z)=1+(1-B_1B_2z)^{-1}\frac{c+p}{c+n+1}(A_1-B_1)(A_2-B_2)z\ {}_2F_1\left(1,1;2+c+p;\frac{B_1B_2z}{B_1B_2z-1}\right)$$

If we put $\delta=\tau=0, p=A_1=A_2=1, B_1=B_2=-1$. In above *theorem* 5.2.4 , we have next result.

COROLLARY 5.2.4 Let $c+1>0$ where c a real number . If $f_1(z), f_2(z)\in\mathcal{D}(p,1)$ and $\frac{(f_1*f_2)(z)}{z}\prec 1+\frac{4z}{1-z}$ then $\frac{(F_c(z)*G_c(z))}{z^p}\prec p(z)\prec 1+\frac{4z}{1-z}$,

where it is ,

$$F_c(z)=\sum_{n=p}^{\infty}\frac{c}{c+n}z^n*f_1(z), G_c(z)=\sum_{n=p}^{\infty}\frac{c}{(c+n)}z^n*f_2(z)$$

and

$$q(z)=1+4(1-z)^{-1}\frac{c+1}{c+n}\ {}_2F_1\left(1,1;3+c;\frac{z}{z-1}\right)$$

5.3 SUBORDINATION OF CERTAIN FAMILY OF MULTIVALENT FUNCTIONS

Let $\mathcal{D}\rho$ be the set of all function $f(z)$ having the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \qquad a_n \geq 0 \tag{5.33}$$

Where $p \in \mathbb{N}$, a set of natural numbers which are p-valent in $\mathcal{U}$ for $p \in \mathbb{N}$

DEFINITION 5.3.1: A function $f(z) \in \mathcal{D}\rho$ is in the subclass $\mathcal{H}(\propto)$ of starlike function if $\mathcal{R}\left(\frac{zf'(z)}{f(z)}\right) > \propto$, $z \in \mathcal{U}$, $0 \leq \propto \leq 1$.

DEFINITION 5.3.2: A function $f(z) \in \mathcal{D}\rho$ is in the subclass $G(\propto)$ of convex function if $\mathcal{R}\left(1 + \frac{zf'(z)}{f(z)}\right) > \propto$, $z \in \mathcal{U}$.

DEFINITION 5.3.3: A function $f(z) \in \mathcal{D}\rho$ is in the subclass $\mathcal{M}(A, B, \alpha, \delta, p)$ if it satisfy

$$1 + \frac{1}{\alpha}\left\{\frac{\frac{z^2 f''(z)}{zf'(z} + 1 - p}{\frac{z^2 f''(z)}{zf'(z} + 1 + p - 2\delta}\right\} \prec \frac{1+Az}{1+Bz} \tag{5.34}$$

For $0 < \mathcal{R}e(\alpha), 0 < \delta \leq 1 , -1 \leq B < A \leq 1 , z \in \mathcal{U}$.

Furthermore a function $f(z) \in \mathcal{D}\rho$ is in the class $K\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ if $zf'(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

THEOREMD 5.3.1:A function given by (5.33) is in $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$if and only if

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1$$

PROOF: Let $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

Therefore from (5.34) we have

$$P(z) = 1 + \frac{1}{\alpha}\left\{\frac{\frac{z^2 f''(z)}{zf'(z)} + (1-p)}{\frac{z^2 f''(z)}{zf'(z} + (1 + p - 2\delta)}\right\} \prec \frac{1 + Az}{1 + Bz}$$

$$P(z) = \frac{1 + Ak(z)}{1 + Bk(z)},$$

where k(z) is Schwarz function

$$P\text{(z)=}(1 + Bk(z)) = 1 + Ak(z)$$

$$k(z)(BP(z)-A)=1-P(z)$$

$$k(z)=\frac{P(z)-1}{A-BP(z)}$$

$$|k(z)|<1$$

$$\left|\frac{\frac{1}{\alpha}\left[\frac{\left\{\frac{z^2f''(z)}{zf'(z)}+(1-p)\right\}}{\frac{z^2f''(z)}{zf'(z)}+1+p-2\delta}\right]}{A-B\left\{1+\frac{1}{\alpha}\left[\frac{\left\{\frac{z^2f''(z)}{zf'(z)}+(1-p)\right\}}{\frac{z^2f''(z)}{zf'(z)}+(1+p-2\delta)}\right]\right\}}\right|<1$$

$$\left|\frac{z^2f''(z)+(1-p)zf'(z)}{\alpha(A-B)\{z^2f''(z)+(1+p-2\delta)zf'(z)\}-B\{z^2f''(z)+(1-p)zf'(z)\}}\right|<1 \qquad (5.35)$$

$$z^2f''(z)+(1-p)zf'(z)=-\sum_{n=p+1}^{\infty}n(n-p)a_nz^n$$

$$z^2f''(z)+(1+p-2\delta)zf'(z)=2p(p-\delta)z^p-\sum_{n=p+1}^{\infty}n(n+p-2\delta)a_nz^n.$$

From (5.35) we have

$$\left|\frac{-\sum_{n=p+1}^{\infty}n(n-p)a_nz^n}{\begin{gathered}\alpha(A-B)\left\{2p(p-\delta)z^p-\sum_{n=p+1}^{\infty}n(n+p-2\delta)a_nz^n\right\}\\ +B\left\{\sum_{n=p+1}^{\infty}n(n-p)a_nz^n\right\}\end{gathered}}\right|<1$$

$$\left|\frac{-\sum_{n=p+1}^{\infty}n(n-p)a_nz^n}{2\alpha p(A-B)(p-\delta)z^p-\sum_{n=p+1}^{\infty}\{n(n+p-2\delta)-Bn(n-p)\}a_nz^n}\right|<1$$

Since $\mathcal{R}e(z)<|z|$. We obtain after considering on real axis and letting z→ 1 we get

$$\sum_{n=p+1}^{\infty}n(n-p)a_n$$

$$\leq 2|\alpha|p(A-B)(p-\delta)-\sum_{n=p+1}^{\infty}|n(n+p-2\delta)-Bn(n-p)|a_nz^n$$

$$\sum_{n=p+1}^{\infty}n(n-p)+|n\{(n+p-2\delta)-B(n-p)\}|\leq 2|\alpha|p(A-B)(p-\delta)$$

That is

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1$$

where

$$k(n) = \frac{2|\alpha|p(A-B)(p-\delta)}{\sum_{n=p+1}^{\infty} n(n-p) + |n\{(n+p-2\delta) - B(n-p)\}|}$$

COROLLARY 5.3.1: If $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ then

$$a_n \leq k(n)$$

and the equality holds for

$$f(z) = z^p - k(n)z^n$$

THEOREMD 5.3.2 : $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0$ is in $\boldsymbol{k\mathcal{M}}(A, B, \alpha, \delta, p)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{n}{k(n)} a_n \leq p$$

PROOF: Suppose $f(z) \in k\mathcal{M}(A, B, \alpha, \delta, p)$.If $zf'(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

Let $g(z) = zf'(z)$Therefore from (5.34) we have

$$P(z) = 1 + \frac{1}{\alpha}\left\{\frac{\frac{z^2f''(z)}{zf'(z)} + (1-p)}{\frac{z^2f''(z)}{zf'(z} + (1+p-2\delta)}\right\} \prec \frac{1+Az}{1+Bz}$$

This is equivalent to (since $|k(z)| < 1$)

$$\left|\frac{\frac{1}{\alpha}\left[\frac{\left\{\frac{z^2g''(z)}{zg(z)}+(1-p)\right\}}{\frac{z^2g''(z)}{zg'(z)}+1+p-2\delta}\right]}{A - B\left\{1 + \frac{1}{\alpha}\left[\frac{\left\{\frac{z^2g''(z)}{zg'(z)}+(1-p)\right\}}{\frac{z^2g''(z)}{zg(z)}+(1+p-2\delta)}\right]\right\}}\right| < 1$$

$$\left|\frac{z^2g''(z) + (1-p)zg'(z)}{\alpha(A-B)\left\{\begin{matrix} z^2g''(z) \\ +(1+p-2\delta)zg'(z) \end{matrix}\right\} - B\left\{\begin{matrix} z^2g''(z) \\ +(1-p)zg'(z) \end{matrix}\right\}}\right| < 1$$

$$z^2g''(z) + (1-p)zg'(z) = -\sum_{n=p+1}^{\infty} n^2(n-p)a_n z^n$$

$$z^2g''(z) + (1+p-2\delta)zg'(z) = 2p^2(p-\delta)z^p - \sum_{n=p+1}^{\infty} n^2(n+p-2\delta)a_n z^n$$

we have

$$\left|\frac{-\sum_{n=p+1}^{\infty} n^2(n-p)a_n z^n}{\alpha(A-B)\left\{2p^2(p-\delta)z^p - \sum_{n=p+1}^{\infty} n^2(n+p-2\delta)a_n z^n\right\} \atop +B\left\{-\sum_{n=p+1}^{\infty} n^2(n-p)a_n z^n\right\}}\right| < 1$$

$$=\left|\frac{-\sum_{n=p+1}^{\infty} n^2(n-p)a_n z^n}{(2\alpha p^2(A-B)(p-\delta))z^p - \sum_{n=p+1}^{\infty}(n^2(n+p-2\delta) - Bn^2(n-p))a_n z^n}\right| < 1$$

Since $\mathcal{Re}(z) < |z|$. We obtain after considering on real axis and letting

z$\rightarrow$ 1 we get

$$\sum_{n=p+1}^{\infty} n^2(n-p)a_n \leq 2|\alpha|p^2(A-B)(p-\delta) - \sum_{n=p+1}^{\infty} n^2\big((n+p-2\delta) - B(n-p)\big)a_n$$

$$\sum_{n=p+1}^{\infty}\left\{n^2(n-p) + \left|n^2\big((n+p-2\delta) - B(n-p)\big)\right|\right\}a_n \leq 2|\alpha|p^2(A-B)(p-\delta)$$

$$\sum_{n=p+1}^{\infty} \frac{n}{k(n)} a_n \leq p$$

<u>**COROLLARY 5.3.2**</u>: If $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$a_n \leq \frac{pk(n)}{n}$$

and the equality holds for

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \frac{pk(n)}{n} z^n$$

<u>**THEOREM 5.3.3**</u>**:** $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$|z|^p - |z|^{p+1}k(p+1) \leq |f(z)| \leq |z|^p + |z|^{p+1}k(p+1)$$

with equality hold for

$$f(z) = z^p - z^{p+1}k(p+1)$$

<u>**PROOF**</u>: $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$

Therefore from *theorem* (5.3.2)

$$\sum_{n=p+1}^{\infty} a_n \leq k(n)$$

$$|f(z)| \geq |z|^p - \sum_{n=p+1}^{\infty} |a_n||z|^n \geq |z|^p - |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n| \geq |z|^p - |z|^{p+1}k(p+1)$$

Similarly

$$|f(z)| \leq |z|^p + \sum_{n=p+1}^{\infty} |a_n||z|^n \leq |z|^p + |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n| \leq |z|^p + |z|^{p+1}k(p+1)$$

Therefore

$$|z|^p - |z|^{p+1}k(p+1) \leq |f(z)| \leq |z|^p + |z|^{p+1}k(p+1)$$

THEOREM 5.3.4 : $f(z) \in \boldsymbol{k\mathcal{M}}(A, B, \alpha, \delta, p)$ then

$$|z|^p - |z|^{p+1} \frac{pk(p+1)}{(p+1)} \leq |f(z)| \leq |z|^p + |z|^{p+1} \frac{pk(p+1)}{(p+1)}$$

with equality hold for

$$f(z) = z^p - z^{p+1} \frac{pk(p+1)}{(p+1)}$$

PROOF: $f(z) \in \boldsymbol{k\mathcal{M}}(A, B, \alpha, \delta, p)$ Therefore from *theorem* (5.3.2)

$$\sum_{n=p+1}^{\infty} \frac{n}{k(n)} a_n \leq p$$

$$|f(z)| \geq |z|^p - \sum_{n=p+1}^{\infty} |a_n||z|^n \geq |z|^p - |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n| \geq |z|^p - |z|^{p+1} \frac{pk(p+1)}{(p+1)}$$

Similarly

$$|f(z)| \leq |z|^p - \sum_{n=p+1}^{\infty} |a_n||z|^n \leq |z|^p + |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n| \leq |z|^p + |z|^{p+1} \frac{pk(p+1)}{(p+1)}$$

Therefore

$$|z|^p - |z|^{p+1} \frac{pk(p+1)}{(p+1)} \leq |f(z)| \leq |z|^p + |z|^{p+1} \frac{pk(p+1)}{(p+1)}$$

THEOREM 5.3.5: $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ then

$$p|z|^{p-1} - (p+1)|z|^p k(p+1) \leq |f'(z)| \leq p|z|^{p-1} + (p+1)|z|^p k(p+1)$$

PROOF: $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

Therefore from *Theorem* (3.3.1)

$$\sum_{n=p+1}^{\infty} a_n \leq k(n)$$

$$f'(z) = pz^{p-1} - \sum_{n=p+1}^{\infty} n\, a_n z^{n-1}$$

$$|f'(z)| \geq p|z|^{p-1} - \sum_{n=p+1}^{\infty} n\, |a_n||z|^{n-1}$$

$$\geq p|z|^{p-1} - (p+1)|z|^p \sum_{n=p+1}^{\infty} |a_n| \geq p|z|^{p-1} - (p+1)|z|^p k(p+1)$$

Similarly

$$|f'(z)| \leq p|z|^{p-1} + \sum_{n=p+1}^{\infty} n\, |a_n||z|^{n-1}$$

$$\leq p|z|^{p-1} + (p+1)|z|^p \sum_{n=p+1}^{\infty} |a_n| \leq p|z|^{p-1} + (p+1)|z|^p k(p+1)$$

Therefore

$$p|z|^{p-1} - (p+1)|z|^p k(p+1) \leq |f'(z)| \leq p|z|^{p-1} + (p+1)|z|^p k(p+1)$$

THEOREM 5.3.6: $f(z) \in \boldsymbol{K\mathcal{M}}(A, B, \alpha, \delta, p)$ then

$$p|z|^{p-1} - |z|^p pk(p+1) \leq |f'(z)| \leq p|z|^{p-1} + |z|^p pk(p+1)$$

PROOF:$f(z) \in \boldsymbol{K\mathcal{M}}(A, B, \alpha, \delta, p)$ then

Therefore from *theorem* 4.1.2

$$\sum_{n=p+1}^{\infty} \frac{n}{k(n)} a_n \leq p$$

$$f'(z) = pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1}$$

$$|f'(z)| \geq p|z|^{p-1} - \sum_{n=p+1}^{\infty} n|a_n||z|^{n-1}$$

$$\geq p|z|^{p-1} - (p+1)|z|^p \sum_{n=p+1}^{\infty} |a_n| \geq p|z|^p - |z|^{p+1}pk(p+1)$$

Similarly

$$|f'(z)| \leq p|z|^{p-1} + \sum_{n=p+1}^{\infty} n|a_n||z|^{n-1}$$

$$\leq p|z|^{p-1} + (p+1)|z|^p \sum_{n=p+1}^{\infty} |a_n| \leq p|z|^p + |z|^{p+1}pk(p+1)$$

Therefore

$$p|z|^{p-1} - |z|^p pk(p+1) \leq |f'(z)| \leq p|z|^p + |z|^{p+1}pk(p+1)$$

$f(z)$ is function in $\mathcal{D}\rho$ is called close to convex of order $\propto$ $(0 \leq \propto < 1)$ if $\mathcal{R}e(z)\{f'(z)\} > \propto$ for all $z \in \mathcal{U}$.

A function $f(z) \in \mathcal{D}\rho$is starlike of order $\propto (0 \leq \propto < 1)$ if $\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > \propto$ for all $z \in \mathcal{U}$.

A function $f(z) \in \mathcal{D}\rho$ is convex of order $\propto (0 \leq \propto < 1)$ if $zf'(z)$ is starlike of order $\propto$, that is $\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > \propto$ for all $z \in \mathcal{U}$.

<u>THEOREM 5.3.7</u> : IF $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$,then $f \in K(\propto)$if

$$|z| \leq r_1(A, B, \alpha, \delta, p) = \inf_n \left(\frac{p-\propto}{nk(n)}\right)^{\frac{1}{n-p}}$$

<u>PROOF</u>: We need to show that $\left|\frac{zf'(z)}{z^{p-1}} - p\right| < p-\propto$

That is

$$\left|\frac{zf'(z)}{z^{p-1}} - p\right| \leq \sum_{n=p+1}^{\infty} n|a_n||z|^{n-p} < p-\propto \qquad (5.36)$$

From *theorem* (5.3.1) we have

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1$$

Note that (5.36) is true if

$$\frac{n|z|^{n-p}}{p-\propto} \leq \frac{1}{k(n)}$$

Therefore

$$|z| \leq \left(\frac{p-\propto}{nk(n)}\right)^{\frac{1}{n-p}}$$

$(p \neq n \,, p, n \in \mathbb{N})$, thus we get required result.

THEOREM 5.3.8 : IF $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$,then $f \in \boldsymbol{S}^{*}(\propto)$if

$$|z| \leq r_2(A, B, \alpha, \delta, p) = \inf_n \left(\left(\left(\frac{p-\propto}{n-\propto}\right)\frac{1}{k(n)}\right)^{\frac{1}{n-p}}\right)$$

PROOF: We must show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p-\propto$$

We have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{n=p+1}^{\infty}(n-p)|a_n||z|^{n-p}}{1-\sum_{n=p+1}^{\infty}|a_n||z|^{n-p}} < p-\propto \tag{5.37}$$

Hence (5.37) holds true if

$$\sum_{n=p+1}^{\infty} \frac{(n-\propto)}{(p-\propto)} |a_n||z|^{n-p} \leq 1 \tag{5.38}$$

From *theorem* (5.3.1) we have

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1 \tag{5.39}$$

Hence by using (5.38) and (5.39) we can obtain required result.

THEOREM 5.3.9 : IF $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$,then $f \in \boldsymbol{C}(\propto)$if

$$|z| \leq r_3(A, B, \alpha, \delta, p) = \inf_n \left(\left(\left(\frac{p(p-\propto)}{n(n-\propto)}\right)\frac{1}{k(n)}\right)^{\frac{1}{n-p}}\right)$$

PROOF: We know that f is convex if and only if zf' is starlike

We must show that

$$\left|\frac{zg'(z)}{g(z)} - p\right| p-\propto$$

Where $g(z) = zf'(z)$

Therefore we have

$$\sum_{n=p+1}^{\infty} \frac{n(n-\propto)}{n(n-\propto)} |a_n||z|^{n-p} \leq 1 \qquad (5.40)$$

From T*heorem*5.3.1 we have

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1 \qquad (5.41)$$

Hence by using (5.40) and (5.41) we get

$$\left(\frac{p(p-\propto)}{n(n-\propto)}\right) |z|^{n-p} \leq \frac{1}{k(n)}$$

$$|z| \leq \left(\left(\left(\frac{p(p-\propto)}{n(n-\propto)}\right) \frac{1}{k(n)}\right)^{\frac{1}{n-p}}\right)$$

Which complete the proof.

THEOREM 5.3.10: Let $f_1(z) = z^n$ and $f_n(z) = z^p - k(n)z^p, for\ n \geq p+1$ then $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ if and only if $f(z)$ can be express in the form $f(z) = \lambda_1 f_1(z) + \sum_{n=p+1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0\ and\ \lambda_1 + \sum_{n=p+1}^{\infty} \lambda_n = 1$

PROOF: Let $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

We have

$$a_n \leq k(n)$$

If we take

$$\lambda_n = \frac{1}{k(n)} a_n$$

$n \geq p+1$and $\sum_{n=p+1}^{\infty} \lambda_n = 1 - \lambda_1$

Then we get required result.

THEOREM 5.3.11: Let $f_i(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n, a_{n,i} \geq 0\ (i = 1,2,3, \ldots .m)$ be the functions in the class $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$, $(i = 1,2,3, \ldots .m)$ then the function $G(z) = z^p - \frac{1}{m} \sum_{n=p+1}^{\infty} \sum_{i=1}^{m} a_{n,i} z^n$ is also in $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ where $\delta = \min_{1 \leq i \leq m} \{\delta_i\}$ with $0 \leq \delta_i < 1$

PROOF: since $f_i(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n$, $a_{n,i} \geq 0$ is in $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

So by *theorem* (5.3.2) we have

$$\sum_{n=p+1}^{\infty} a_n \leq k(n,\delta)$$

$$k(n,\delta) = \frac{2|\alpha|p(A-B)(p-\delta)}{\{n(n-p) + |(n(n+p-2\delta) - Bn(n-p))|\}}$$

We have

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n,\delta_i)} \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right)$$

$$= \frac{1}{m} \sum_{i=1}^{m} \sum_{n=p+1}^{\infty} \frac{1}{k(n,\delta_i)} a_{n,i} \leq \left(\frac{1}{m} \sum_{i=1}^{m} 1 \right) < 1$$

Hence by *theorem* (5.3.13) , $G(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$

THEOREM 5.3.12: Let the function $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n and\ \ g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n$ be in the class $\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$. Then the function $F(z)$defined by

$$F(z) = (1-y)f(z) + yg(z) = z^p - \sum_{n=p+1}^{\infty} c_n z^n$$

Where $c_n = (1-y)a_n + yb_n, 0 \leq y \leq 1$ is also in $\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$.

PROOF: we have

$$F(z) = (1-y)f(z) + yg(z)$$

$$= (1-y)\left(z^p - \sum_{n=p+1}^{\infty} a_n z^n \right) + y\left(z^p - \sum_{n=p+1}^{\infty} b_n z^n \right)$$

$$= z^p - \sum_{n=p+1}^{\infty} \big((1-y)a_n + yb_n\big) z^n$$

Since $f, g \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ so by theorem 5.3.1 we have

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1$$

and

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} b_n \leq 1$$

Therefore

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} \big((1-y)a_n + yb_n\big) = (1-y) \sum_{n=p+1}^{\infty} \frac{1}{k(x)} a_n + y \sum_{n=p+1}^{\infty} \frac{1}{k(x)} b_n$$

$$\leq (1-y) \sum_{n=p+1}^{\infty} \frac{1}{k(x)} + y \sum_{n=p+1}^{\infty} \frac{1}{k(x)} = 1$$

Therefore

$$\boldsymbol{c_n} \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$$

Let $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$, $\tau \geq 0$ $then\ a(t, \tau)-$ neighborhood of the function $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ is defined by

$$\aleph_\tau^t(z) = \left\{g \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p): g(z) = z^n - \textstyle\sum_{n=p+1}^{\infty} b_n z^n \ \ and\ \ \sum_{n=p+1}^{\infty} |a_n - b_n| n^{t+1} \leq \tau\right\} \tag{5.42}$$

For the identity function if $e(z) = z^n, q \in \mathbb{N}$, then

$$\aleph_\tau^t(e) = \left\{ g \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p): g(z) \right.$$
$$\left. = z^n - \sum_{n=p+1}^{\infty} b_n z^n \ \ and\ \ \sum_{n=p+1}^{\infty} |b_n| n^{t+1} \leq \tau \right\} \tag{5.43}$$

DEFINITION 5.3.4: A function $f(z) = z^n - \sum_{n=p+1}^{\infty} a_n z^n, a_n \geq 0$ is in the class $\boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A, B, \alpha, \delta, p)$ if there exist $g(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < p - \pi\ , z \in \mathcal{U}, 0 \leq \pi < 1 \tag{5.44}$$

THEOREM 5.3.13 : If $g(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ and

$$\pi = p - \frac{\tau}{n^{t+1}} \left[\frac{1}{1-k(n)}\right] \tag{5.45}$$

Then $\aleph_\tau^t(g) \subset \boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A, B, \alpha, \delta, p)$

PROOF: Let $f \in \aleph_\tau^t(g)$, then by (4.12)

$$\sum_{n=p+1}^{\infty} n^{n+1} |a_n - b_n| \leq \tau$$

This implies that

$$\sum_{n=p+1}^{\infty}|a_n - b_n| \leq \frac{\tau}{n^{t+1}} \tag{5.46}$$

Therefore

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum_{n=p+1}^{\infty}|a_n - b_n|}{1 - \sum_{n=p+1}^{\infty} b_n} \leq \frac{\tau}{n^{t+1}}\left[\frac{1}{1-k(n)}\right] < \frac{\tau}{n^{t+1}}\left[\frac{1}{1-k(n)}\right] = p - \pi$$

Then by *definition* 5.3.4, we get $f \in \boldsymbol{\mathcal{M}}^{\pi}(A, B, \alpha, \delta, p)$ Thus $\aleph_{\tau}^{t}(g) \subset \boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A, B, \alpha, \delta, p)$.

The generalized Bernardi integral operator is given by

$$\mathcal{L}_c[f(z)] = \frac{c+p}{z^c}\int_0^z f(\zeta)\zeta^{c-1}d\zeta \qquad (c > -p, z \in \mathcal{U})$$

$$\mathcal{L}_c[f(z)] = z^p - \sum_{n=p+1}^{\infty} d a_n z^n \tag{5.47}$$

Where $d = \left(\frac{c+p}{c+n}\right)$

<u>THEOREM 5.3.14</u>: Let $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$then$\mathcal{L}_c[f(z)] \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

<u>PROOF</u> : We need to prove that

$$\sum_{n=p+1}^{\infty} \frac{d}{k(n)} a_n \leq 1$$

Since $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ then from *Theorem* (5.3.1)

$$\sum_{n=p+1}^{\infty} k(n) a_n \leq 1$$

But $d < 1$ therefore *theorem* (5.3.14) holds and the proof is over.

<u>THEOREM 5.3.15</u>: Let $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$then$\mathcal{L}_c[f(z)]$is starlice of order $\sigma, 0 \leq \sigma < 1$ in $|z| < r_1$

Where

$$|z| \leq \left(\left(\frac{p-\sigma}{n-\sigma}\right)\left(\frac{1}{dk(n)}\right)\right)^{\frac{1}{n-p}}$$

<u>PROOF</u>: $\mathcal{L}_c[f(z)] = z^n - \sum_{n=p+1}^{\infty} d a_n z^n$

It is enough to prove

$$\left|\frac{z(\mathcal{L}_c[f(z)])'}{\mathcal{L}_c[f(z)]} - p\right| < p - \sigma$$

$$\left|\frac{z(\mathcal{L}_c[f(z)])'}{\mathcal{L}_c[f(z)]}-p\right|=\left|\frac{\sum_{n=p+1}^{\infty} d(n-p)a_n z^{n-p}}{1-\sum_{n=p+1}^{\infty} da_n z^{n-p}}\right|$$

$$\leq \frac{\sum_{n=p+1}^{\infty} d(n-p)a_n |z|^{n-p}}{1-\sum_{n=p+1}^{\infty} da_n |z|^{n-p}} < p-\sigma$$

$$\sum_{n=p+1}^{\infty} d(n-p)a_n |z|^{n-p} < p-\sigma\left(1-\sum_{n=p+1}^{\infty} da_n |z|^{n-p}\right)$$

$$\sum_{n=p+1}^{\infty} \frac{(n-\sigma)}{(p-\sigma)} da_n |z|^{n-p} \leq 1 \tag{5.48}$$

From *Theorem* (5.3.1)

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1 \tag{5.49}$$

Hence by using (5.48) and (5.49) we get

$$\frac{(n-\sigma)}{(p-\sigma)} d|z|^{n-p} \leq \frac{1}{k(n)}$$

$$|z|^{n-p} \leq \left(\frac{p-\sigma}{n-\sigma}\right)\left(\frac{1}{dk(n)}\right)$$

Therefore

$$|z| \leq \left(\left(\frac{p-\sigma}{n-\sigma}\right)\left(\frac{1}{dk(n)}\right)\right)^{\frac{1}{n-p}}$$

DEFINITION 5.3.5:For a function $f(z)$ which is analytic function in $w-$ plane containing the origin which is a simply connected region , we define the fractional integral of order μ as

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi \ where\ \mu > 0$$

DEFINITION 5.3.6:For a function $f(z)$ which is analytic function in $w-$ plane containing the origin which is a simply connected region , we define the fractional integral of order μ as

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(\mu)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\mu}} d\xi \ where\ \ 1 > \mu \geq 0$$

THEOREM 5.3.16: Let $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$then

$$\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1-\frac{(p+1)k(n)}{(p+\mu+1)}|z|\right)\leq\left|D_z^{-\mu}f(z)\right|$$
$$\leq\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\Bigg(1$$
$$+\frac{(p+1)k(n)}{(p+\mu+1)}|z|\Bigg) \tag{5.50}$$

PROOF: From definition (5.3.5) we have

$$D_z^{-\mu}f(z)=\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}z^{p+\mu}-\sum_{n=p+1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\mu+1)}a_n z^{n+\mu} \tag{5.51}$$

$$\mu>0\,,n\geq p+1\,;p,n\in\mathbb{N}$$

Let $\phi(n)=\frac{\Gamma(n+1)}{\Gamma(n+\mu+1)}$

Clearly $\phi(n)$ is non – increasing function of n , $0<\phi(n)\leq\phi(p+1)=\frac{\Gamma(p+2)}{\Gamma(p+\mu+2)}$

From *theorem* (5.3.1) we have

$$\sum_{n=p+1}^{\infty}|a_n|\leq k(n) \tag{5.52}$$

From (5.51) and (5.52) it follows that

$$\left|D_z^{-\mu}f(z)\right|\leq|z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}+\phi(p+1)|z|\sum_{n=p+1}^{\infty}|a_n|\right)$$
$$\leq\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1+\frac{(p+1)k(n)}{(p+\mu+1)}|z|\right)$$

Similarly

$$\left|D_z^{-\mu}f(z)\right|\geq|z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}-\phi(p+1)|z|\sum_{n=p+1}^{\infty}|a_n|\right)$$
$$\geq\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1-\frac{(p+1)k(n)}{(p+\mu+1)}|z|\right)$$

This proves the theorem

THEOREM 5.3.17: Let $f\in\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$then

$$\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1-\frac{(p+1)k(n)}{(p-\mu+1)}|z|\right) \le \left|D_z^{\mu} f(z)\right|$$
$$\le \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\Bigg(1$$
$$+\frac{(p+1)k(n)}{(p-\mu+1)}|z|\Bigg) \qquad (5.53)$$

PROOF: From *definition* (5.3.6) we have

$$D_z^{\mu} f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)} a_n z^{n-\mu} \qquad (5.54)$$

$$1 > \mu \ge 0\,, n \ge p+1\,; p, n \in \mathbb{N}$$

Let $\psi(n) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}$

Clearly $\psi(n)$ is non – increasing function of n , $0 < \psi(n) \le \psi(p+1) = \frac{\Gamma(p+2)}{\Gamma(p+\mu+2)}$

From *theorem* (5.3.1) we have

$$\sum_{n=p+1}^{\infty} |a_n| \le k(n) \qquad (5.55)$$

From (5.54) and (5.55) it follows that

$$\left|D_z^{\mu} f(z)\right| \le |z|^{p-\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \psi(p+1)|z| \sum_{n=p+1}^{\infty} |a_n|\right)$$

$$\le \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1+\frac{(p+1)k(n)}{(p-\mu+1)}|z|\right)$$

Similarly

$$\left|D_z^{\mu} f(z)\right| \ge |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \psi(p+1)|z| \sum_{n=p+1}^{\infty} |a_n|\right)$$

$$\ge \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1-\frac{(p+1)k(n)}{(p-\mu+1)}|z|\right)$$

5.4 CERTAIN FAMILY OF MULTIVALENT FUNCTIONS ASSOCIATED WITH SUBORDINATION

Let $\mathcal{D}\rho$ be the set of all function $f(z)$ having the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \qquad a_n \geq 0 \tag{5.56}$$

Where $p \in \mathbb{N}$, a set of natural numbers which are p-valent in $\mathcal{U}$ for $p \in \mathbb{N}$

DEFINITION 5.4.1: A function $f(z) \in \mathcal{D}\rho$ is in the subclass $\mathcal{H}(\propto)$ of starlike function if $\mathcal{R}\left(\frac{zf'(z)}{f(z)}\right) > \propto$, $z \in \mathcal{U}$, $0 \leq \propto \leq 1$.

DEFINITION 5.4.2: A function $f(z) \in \mathcal{D}\rho$ is in the subclass $G(\propto)$ of convex function if $\mathcal{R}\left(1 + \frac{zf'(z)}{f(z)}\right) > \propto$, $z \in \mathcal{U}$.

DEFINITION 5.4.3: A function $f(z) \in \mathcal{D}\rho$is in the subclass $\mathcal{M}(A, B, \alpha, \delta, p)$ if it satisfy

$$1 + \frac{1}{\alpha}\left\{\frac{\frac{zf'(z)}{z^p} - p}{\frac{zf'(z)}{z^p} + p - 2\delta}\right\} \prec \frac{1+Az}{1+Bz} \tag{5.57}$$

For $0 < \mathcal{R}e(\alpha), 0 < \delta \leq 1, -1 \leq B < A \leq 1, z \in \mathcal{U}$.

Furthermore a function $f(z) \in \mathcal{D}\rho$ is in the class $K\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ if $zf'(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

THEOREMD 5.4.1:A function given by (4.1) is in $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$if and only if

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1$$

PROOF: Let $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

Therefore from (5.57) we have

$$P(z) = 1 + \frac{1}{\alpha}\left\{\frac{\frac{zf'(z)}{z^p} - p}{\frac{zf'(z)}{z^p} + p - 2\delta}\right\} \prec \frac{1 + Az}{1 + Bz}$$

$$P(z) = \frac{1 + Ak(z)}{1 + Bk(z)}$$

Where $k(z)$ is Schwarz function

$$P(\mathrm{z}) = (1 + Bk(z)) = 1 + Ak(z)$$

$$k(z)(BP(z) - A) = 1 - P(z)$$

$$k(z)=\frac{P(z)-1}{A-BP(z)}$$

$$|k(z)|<1$$

$$\left|\frac{\frac{1}{\alpha}\left[\frac{\left\{\frac{zf'(z)}{z^p}-p\right\}}{\frac{zf'(z)}{z^p}+p-2\delta}\right]}{A-B\left\{1+\frac{1}{\alpha}\left[\frac{\frac{1}{\alpha}\left\{\frac{zf'(z)}{z^p}-p\right\}}{\frac{zf'(z)}{z^p}+p-2\delta}\right]\right\}}\right|<1$$

$$\left|\frac{zf'(z)-pz^p}{\alpha(A-B)\{zf'(z)+z^p(p-2\delta)\}-B\{zf'(z)-pz^p\}}\right|<1 \tag{5.58}$$

$$zf'(z)-pz^p=-\sum_{n=p+1}^{\infty}na_nz^n$$

$$zf'(z)+z^p(p-2\delta)=2(p-\delta)z^p-\sum_{n=p+1}^{\infty}na_nz^n$$

From (5.58) we have

$$\left|\frac{-\sum_{n=p+1}^{\infty}na_nz^n}{\alpha(A-B)\left\{2(p-\delta)z^p-\sum_{n=p+1}^{\infty}na_nz^n\right\}+B\left\{\sum_{n=p+1}^{\infty}na_nz^n\right\}}\right|<1$$

Since $\mathcal{R}e(z)<|z|$. We obtain after considering on real axis and letting

z→ 1 we get

$$\sum_{n=p+1}^{\infty}na_n\leq 2|\alpha|(p-\delta)(A-B)-\sum_{n=p+1}^{\infty}a_n|n\alpha(A-B)-Bn|$$

That is

$$\sum_{n=p+1}^{\infty}(n+|n\alpha(A-B)-Bn|)\,a_n\leq 2|\alpha|(p-\delta)(A-B)$$

$$\sum_{n=p+1}^{\infty}\frac{1}{k(n)}a_n\leq 1$$

Where

$$k(n)=\frac{2|\alpha|(p-\delta)(A-B))}{(n+|n\alpha(A-B)-Bn|)}$$

COROLLARY 5.4.1: If $f(z)\in\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$a_n\leq k(n)$$

and the equality holds for

$$f(z) = z^p - k(n)z^n$$

THEOREMD 5.4.2 : $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad a_n \geq 0$ is in $\boldsymbol{k\mathcal{M}}(A, B, \alpha, \delta, p)$ if and only if

$$\sum_{n=p+1}^{\infty} n^2(1 + |(\alpha(A - B) - B)|)a_n \leq |\alpha|(A - B)(p^2 + p - 2\delta) - p(p - 1)(B + 1)$$

PROOF: Suppose $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$, If $zf'(z) \in \boldsymbol{k\mathcal{M}}(A, B, \alpha, \delta, p)$

Let $g(z) = zf'(z)$

Therefore from (5.57) we have

$$P(z) = 1 + \frac{1}{\alpha}\left\{\frac{\frac{zg'(z)}{z^p} - p}{\frac{zg'(z)}{z^p} + p - 2\delta}\right\} \prec \frac{1 + Az}{1 + Bz}$$

This is equivalent to (since $|k(z)| < 1$)

$$\left|\frac{\frac{1}{\alpha}\left[\frac{\left\{\frac{zg'(z)}{z^p} - p\right\}}{\frac{zg'(z)}{z^p} + p - 2\delta}\right]}{A - B\left\{1 + \frac{1}{\alpha}\left[\frac{\left\{\frac{zg'(z)}{z^p} - p\right\}}{\frac{zg'(z)}{z^p} + p - 2\delta}\right]\right\}}\right| < 1 \qquad (5.59)$$

$$\left|\frac{zg'(z) - pz^p}{\alpha(A - B)\{zg'(z) + z^p(p - 2\delta)\} - B\{zg'(z) - pz^p\}}\right| < 1$$

$$zg'(z) - pz^p = p(p - 1)z^p - \sum_{n=p+1}^{\infty} n^2 a_n z^n$$

$$zg'(z) + z^p(p - 2\delta) = (p^2 + p - 2\delta)z^p - \sum_{n=p+1}^{\infty} n^2 a_n z^n$$

From (5.59) we have

$$\left|\frac{p(p - 1)z^p - \sum_{n=p+1}^{\infty} n^2 a_n z^n}{\begin{array}{c}\alpha(A - B)\left\{(p^2 + p - 2\delta)z^p - \sum_{n=p+1}^{\infty} n^2 a_n z^n\right\} \\ -B\left\{p(p - 1)z^p - \sum_{n=p+1}^{\infty} n^2 a_n z^n\right\}\end{array}}\right| < 1$$

$$= \left|\frac{p(p - 1)z^p - \sum_{n=p+1}^{\infty} n^2 a_n z^n}{\{\alpha(A - B)(p^2 + p - 2\delta) - Bp(p - 1)\}z^p - \sum_{n=p+1}^{\infty} n^2(\alpha(A - B) - B)a_n z^n}\right|$$

Since $\mathcal{R}e(z) < |z|$. We obtain after considering on real axis and letting

$z \to 1$ we get

$$p(p-1)z^p+\sum_{n=p+1}^{\infty} n^2 a_n \leq \{|\alpha|(A-B)(p^2+p-2\delta)-Bp(p-1)\} - \sum_{n=p+1}^{\infty} |n^2(\alpha(A-B)-B)|a_n$$

$$\sum_{n=p+1}^{\infty} n^2 a_n + \sum_{n=p+1}^{\infty} |n^2(\alpha(A-B)-B)|a_n \leq \{|\alpha|(A-B)(p^2+p-2\delta)-Bp(p-1)\}-p(p-1)$$

$$\sum_{n=p+1}^{\infty} (n^2+|n^2(\alpha(A-B)-B)|)a_n \leq |\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)$$

COROLLARY5.4.2: If $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$a_n \leq \frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{(n^2+|n^2(\alpha(A-B)-B)|)}$$

And the equality holds for

$$f(z)=z^p-\sum_{n=p+1}^{\infty} \frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{(n^2+|n^2(\alpha(A-B)-B)|)} z^n$$

THEOREM 5.4.2 : $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$|z|^p-|z|^{p+1}k(p+1) \leq |f(z)| \leq |z|^p+|z|^{p+1}k(p+1)$$

With equality hold for

$$f(z)=z^p-z^{p+1}k(p+1)$$

PROOF: $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$, Therefore from *theorem* (5.4.31)

$$\sum_{n=p+1}^{\infty} a_n \leq k(n)$$

$$|f(z)| \geq |z|^p - \sum_{n=p+1}^{\infty} |a_n||z|^n \geq |z|^p-|z|^{p+1}\sum_{n=p+1}^{\infty} |a_n| \geq |z|^p-|z|^{p+1}k(p+1)$$

Similarly

$$|f(z)| \leq |z|^p + \sum_{n=p+1}^{\infty} |a_n||z|^n \leq |z|^p+|z|^{p+1}\sum_{n=p+1}^{\infty} |a_n| \leq |z|^p+|z|^{p+1}k(p+1)$$

Therefore

$$|z|^p-|z|^{p+1}k(p+1) \leq |f(z)| \leq |z|^p+|z|^{p+1}k(p+1)$$

THEOREM 5.4.3: $f(z) \in \boldsymbol{k\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$|z|^p - |z|^{p+1}\frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)} \le |f(z)|$$
$$\le |z|^p + |z|^{p+1}\frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)}$$

With equality hold for

$$f(z) = z^p - z^{p-1}$$

PROOF: $f(z) \in \boldsymbol{k\mathcal{M}}(A,B,\alpha,\delta,p)$, Therefore from *theorem* (5.4.2)

$$\sum_{n=p+1}^{\infty} a_n \le \frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{n^2(1+|(\alpha(A-B)-B)|)}$$

$$|f(z)| \ge |z|^p - \sum_{n=p+1}^{\infty} |a_n||z|^n$$
$$\ge |z|^p$$
$$- |z|^{p+1}\sum_{n=p+1}^{\infty} |a_n| \ge |z|^p - |z|^{p+1}\frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)}$$

Similarly

$$|f(z)| \le |z|^p + \sum_{n=p+1}^{\infty} |a_n||z|^n$$
$$\le |z|^p$$
$$+ |z|^{p+1}\sum_{n=p+1}^{\infty} |a_n|$$
$$\le |z|^p + |z|^{p+1}\frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)}$$

Therefore

$$|z|^p - |z|^{p+1}\frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)} \le |f(z)|$$
$$\le |z|^p + |z|^{p+1}\frac{|\alpha|(A-B)(p^2+p-2\delta)-p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)}$$

THEOREM 5.4.4: $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then

$$p|z|^{p-1} - (p+1)|z|^p k(p+1) \le |f'(z)| \le p|z|^{p-1} + (p+1)|z|^p k(p+1)$$

PROOF: $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$

Therefore from *Theorem* (5.4.1)

$$\sum_{n=p+1}^{\infty} a_n \le k(n)$$

$$f'(z) = pz^{p-1} - \sum_{n=p+1}^{\infty} n\, a_n z^{n-1}$$

$$|f'(z)| \geq p|z|^{p-1} - \sum_{n=p+1}^{\infty} n\,|a_n||z|^{n-1}$$

$$\geq p|z|^{p-1} - (p+1)|z|^{p} \sum_{n=p+1}^{\infty} |a_n| \geq p|z|^{p-1} - (p+1)|z|^{p} k(p+1)$$

Similarly

$$|f'(z)| \leq p|z|^{p-1} + \sum_{n=p+1}^{\infty} n\,|a_n||z|^{n-1}$$

$$\leq p|z|^{p-1} + (p+1)|z|^{p} \sum_{n=p+1}^{\infty} |a_n| \leq p|z|^{p-1} + (p+1)|z|^{p} k(p+1)$$

Therefore

$$p|z|^{p-1} - (p+1)|z|^{p} k(p+1) \leq |f'(z)| \leq p|z|^{p-1} + (p+1)|z|^{p} k(p+1)$$

THEOREM 5.4.5: $f(z) \in \boldsymbol{K\mathcal{M}}(A, B, \alpha, \delta, p)$ then

$$p|z|^{p} - p|z|^{p+1} \frac{|\alpha|(A-B)(p^2+p-2\delta) - p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)} \leq |f'(z)|$$
$$\leq p|z|^{p} + p|z|^{p+1} \frac{|\alpha|(A-B)(p^2+p-2\delta) - p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)}$$

PROOF: $f(z) \in \boldsymbol{K\mathcal{M}}(A, B, \alpha, \delta, p)$ then

Therefore from theorem (5.4.2)

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{|\alpha|(A-B)(p^2+p-2\delta) - p(p-1)(B+1)}{n^2(1+|(\alpha(A-B)-B)|)}$$

$$f'(z) = pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1}$$

$$|f'(z)| \geq p|z|^{p-1} - \sum_{n=p+1}^{\infty} n|a_n||z|^{n-1} \geq p|z|^{p-1} - (p+1)|z|^{p} \sum_{n=p+1}^{\infty} |a_n|$$
$$\geq p|z|^{p-1} - p|z|^{p+1} \frac{|\alpha|(A-B)(p^2+p-2\delta) - p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)}$$

Similarly

$$|f'(z)| \le p|z|^{p-1} + \sum_{n=p+1}^{\infty} n|a_n||z|^{n-1}$$
$$\le p|z|^{p-1} + (p+1)|z|^{p} \sum_{n=p+1}^{\infty} |a_n|$$
$$\le p|z|^{p-1} + p|z|^{p+1} \frac{|\alpha|(A-B)(p^2+p-2\delta) - p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)}$$

Therefore

$$p|z|^{p} - p|z|^{p+1} \frac{|\alpha|(A-B)(p^2+p-2\delta) - p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)} \le |f'(z)|$$
$$\le p|z|^{p} + p|z|^{p+1} \frac{|\alpha|(A-B)(p^2+p-2\delta) - p(p-1)(B+1)}{(p+1)^2(1+|(\alpha(A-B)-B)|)}$$

<u>THEOREM5.4.6</u> : IF $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$,then $f \in K(\propto)$if

$$|z| \le r_1(A, B, \alpha, \delta, p) = \inf_{n} \left(\frac{p-\propto}{nk(n)}\right)^{\frac{1}{n-p}}$$

<u>PROOF</u>: We need to show that $\left|\frac{zf'(z)}{z^{p-1}} - p\right| < p-\propto$

That is

$$\left|\frac{zf'(z)}{z^{p-1}} - p\right| \le \sum_{n=p+1}^{\infty} n|a_n||z|^{n-p} < p-\propto \qquad (5.60)$$

From *theorem* (5.4.1) we have

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \le 1 \qquad .(5.61)$$

Note that (5.61) is true if

$$\frac{n|z|^{n-p}}{p-\propto} \le \frac{1}{k(n)}$$

Therefore

$$|z| \le \left(\frac{p-\propto}{nk(n)}\right)^{\frac{1}{n-p}}$$

$(p \ne n\ , p, n \in \mathbb{N})$, thus we get required result.

<u>THEOREM5.4.7</u>: IF $f(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$,then $f \in \boldsymbol{S}^{*}(\propto)$if

$$|z| \le r_2(A, B, \alpha, \delta, p) = \inf_{n} \left(\left(\left(\frac{p-\propto}{n-\propto}\right)\frac{1}{k(n)}\right)^{\frac{1}{n-p}}\right)$$

<u>PROOF</u>: We must show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p-\propto$$

We have

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leq \frac{\sum_{n=p+1}^{\infty}(n-p)|a_n||z|^{n-p}}{1-\sum_{n=p+1}^{\infty}|a_n||z|^{n-p}} < p{-}\propto \tag{5.62}$$

Hence (5.62) holds true if

$$\sum_{n=p+1}^{\infty}\frac{(n-\propto)}{(p-\propto)}|a_n||z|^{n-p} \leq 1 \tag{5.63}$$

From *Theorem* (5.4.1) we have

$$\sum_{n=p+1}^{\infty}\frac{1}{k(n)}a_n \leq 1 \tag{5.64}$$

Hence by using (5.63) and (5.64) we can obtain required result.

THEOREM5.4.10 : IF $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$,then $f \in \boldsymbol{C}(\propto)$if

$$|z| \leq r_3(A,B,\alpha,\delta,p) = \inf_{n}\left(\left(\left(\frac{p(p{-}\propto)}{n(n{-}\propto)}\right)\frac{1}{k(n)}\right)^{\frac{1}{n-p}}\right)$$

PROOF: We know that f is convex if and only if zf' is starlike

We must show that

$$\left|\frac{zg'(z)}{g(z)}-p\right| p{-}\propto$$

Where $g(z) = zf'(z)$

Therefore we have

$$\sum_{n=p+1}^{\infty}\frac{n(n-\propto)}{n(n-\propto)}|a_n||z|^{n-p} \leq 1 \tag{5.65}$$

From *theorem*5.4.30 we have

$$\sum_{n=p+1}^{\infty}\frac{1}{k(n)}a_n \leq 1 \tag{5.66}$$

Hence by using (5.65) and (5.66) we get

$$\left(\frac{p(p{-}\propto)}{n(n{-}\propto)}\right)|z|^{n-p} \leq \frac{1}{k(n)}$$

$$|z| \leq \left(\left(\left(\frac{p(p{-}\propto)}{n(n{-}\propto)}\right)\frac{1}{k(n)}\right)^{\frac{1}{n-p}}\right)$$

THEOREM 5.4.10: Let $f_1(z) = z^n$ and $f_n(z) = z^p - k(n)z^p, for\ n \geq p+1$ then $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ if and only if $f(z)$ can be express in the form $f(z) = \lambda_1 f_1(z) + \sum_{n=p+1}^{\infty}\lambda_n f_n(z)$ where $\lambda_n \geq 0\ and\ \lambda_1 + \sum_{n=p+1}^{\infty}\lambda_n = 1$

PROOF: Let $f(z) \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$

We have

$$a_n \leq k(n)$$

If we take

$$\lambda_n = \frac{1}{k(n)} a_n$$

$n \geq p + 1$and $\sum_{n=p+1}^{\infty} \lambda_n = 1 - \lambda_1$, Then we get required result.

THEOREM 5.4.11: Let $f_i(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n, a_{n,i} \geq 0 \ (i = 1,2,3, \dots .m)$ be the functions in the class $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$, $(i = 1,2,3, \dots .m)$then the function $G(z) = z^p - \frac{1}{m} \sum_{n=p+1}^{\infty} \sum_{i=1}^{m} a_{n,i} z^n$ is also in $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ where $\delta = {}_{1 \leq i \leq m}^{min}\{\delta_i\}$ with $0 \leq \delta_i < 1$

PROOF: since $f_i(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n$, $a_{n,i} \geq 0$ is in $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

So by *theorem* (5.4.1) we have

$$\sum_{n=p+1}^{\infty} a_n \leq k(n, \delta)$$

$$k(n, \delta) = \frac{2|\alpha|(p - \delta)(A - B))}{n^2(1 + |\alpha(A - B) - Bn|)}$$

We have

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n, \delta_i)} \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right)$$

$$= \frac{1}{m} \sum_{i=1}^{m} \sum_{n=p+1}^{\infty} \frac{1}{k(n, \delta_i)} a_{n,i} \leq \left(\frac{1}{m} \sum_{i=1}^{m} 1 \right) < 1$$

Hence by *theorem* (5.4.1) , $G(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$

THEOREM 5.4.12: Let the function $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n and\ \ g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n$ be in the class $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$. Then the function $F(z)$ defined by

$$F(z) = (1 - y)f(z) + yg(z) = z^p - \sum_{n=p+1}^{\infty} c_n z^n$$

Where $c_n = (1 - y)a_n + yb_n, 0 \leq y \leq 1$ is also in $\boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$.

PROOF: we have

$$F(z) = (1 - y)f(z) + yg(z)$$

$$= (1 - y) \left(z^p - \sum_{n=p+1}^{\infty} a_n z^n \right) + y \left(z^p - \sum_{n=p+1}^{\infty} b_n z^n \right)$$

$$= z^p - \sum_{n=p+1}^{\infty} \big((1 - y)a_n + yb_n \big) z^n$$

Since $f, g \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ so by *theorem* (5.4.1) we have

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \leq 1$$

and

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} b_n \leq 1$$

Therefore

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} \big((1-y)a_n + yb_n\big) = (1-y) \sum_{n=p+1}^{\infty} \frac{1}{k(x)} a_n + y \sum_{n=p+1}^{\infty} \frac{1}{k(x)} b_n$$

$$\leq (1-y) \sum_{n=p+1}^{\infty} \frac{1}{k(x)} + y \sum_{n=p+1}^{\infty} \frac{1}{k(x)} = 1$$

Therefore

$$\boldsymbol{c_n} \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$$

Let $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$, $\tau \geq 0$ $then\ a(t,\tau) -$ neighborhood of the function $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ is defined by

$$\aleph_\tau^t(z) = \left\{g \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p) : g(z) = z^n - \textstyle\sum_{n=p+1}^{\infty} b_n z^n \ and\ \sum_{n=p+1}^{\infty} |a_n - b_n| n^{t+1} \leq \tau\right\} \quad (5.67)$$

For the identity function if $e(z) = z^n, q \in \mathbb{N}$, then

$$\aleph_\tau^t(e) = \left\{g \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p) : g(z) = z^n - \sum_{n=p+1}^{\infty} b_n z^n \ and\ \sum_{n=p+1}^{\infty} |b_n| n^{t+1} \leq \tau\right\} \qquad (5.68)$$

DEFINITION5.4.4: A function $f(z) = z^n - \sum_{n=p+1}^{\infty} a_n z^n, a_n \geq 0$ is in the class $\boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A, B, \alpha, \delta, p)$ if there exist $g(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \pi , z \in \mathcal{U}, 0 \leq \pi < 1 \qquad (5.69)$$

THEOREM 5.4.13 : If $g(z) \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$ and

$$\pi = p - \frac{\tau}{n^{t+1}} \left[\frac{1}{1 - k(n)}\right]$$

Then $\aleph_\tau^t(g) \subset \boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A, B, \alpha, \delta, p)$

PROOF: Let $f \in \aleph_\tau^t(g)$, then by (5.67)

$$\sum_{n=p+1}^{\infty} n^{n+1} |a_n - b_n| \leq \tau$$

This implies that

$$\sum_{n=p+1}^{\infty}|a_n - b_n| \le \frac{\tau}{n^{t+1}} \tag{5.70}$$

Therefore

$$\left|\frac{f(z)}{g(z)} - 1\right| \le \frac{\sum_{n=p+1}^{\infty}|a_n - b_n|}{1 - \sum_{n=p+1}^{\infty} b_n} \le \frac{\tau}{n^{t+1}}\left[\frac{1}{1-k(n)}\right] < \frac{\tau}{n^{t+1}}\left[\frac{1}{1-k(n)}\right] = p - \pi$$

Then by *definition* 5.4.8 , we get $f \in \boldsymbol{\mathcal{M}}^{\pi}(A,B,\alpha,\delta,p)$ Thus $\aleph_{\tau}^{t}(g) \subset \boldsymbol{\mathcal{M}}^{\boldsymbol{\pi}}(A,B,\alpha,\delta,p)$.

The generalized Bernardi integral operator is given by

$$\mathcal{L}_c[f(z)] = \frac{c+p}{z^c}\int_0^z f(\zeta)\zeta^{c-1}d\zeta \qquad (\, c > -p, z \in \mathcal{U})$$

$$\mathcal{L}_c[f(z)] = z^p - \sum_{n=p+1}^{\infty} d a_n z^n \tag{5.71}$$

Where $d = \left(\frac{c+p}{c+n}\right)$

<u>THEOREM 5.4.14</u>: Let $f \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$then$\mathcal{L}_c[f(z)] \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$

<u>PROOF</u> : We need to prove that

$$\sum_{n=p+1}^{\infty}\frac{d}{k(n)}a_n \le 1$$

Since $f \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ then from *Theorem* (5.4.1)

$$\sum_{n=p+1}^{\infty} k(n)a_n \le 1$$

But $d < 1$ therefore *theorem* (5.4.14) holds .

<u>THEOREM 5.4.15</u>: Let $f \in \boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$then$\mathcal{L}_c[f(z)]$is starlice of order $\sigma, 0 \le \sigma < 1$ in $|z| < r_1$

Where

$$|z| \le \left(\left(\frac{p-\sigma}{n-\sigma}\right)\left(\frac{1}{dk(n)}\right)\right)^{\frac{1}{n-p}}$$

<u>PROOF</u>: $\mathcal{L}_c[f(z)] = z^n - \sum_{n=p+1}^{\infty} d a_n z^n$

It is enough to prove

$$\left|\frac{z(\mathcal{L}_c[f(z)])'}{\mathcal{L}_c[f(z)]} - p\right| < p - \sigma$$

$$\left|\frac{z(\mathcal{L}_c[f(z)])'}{\mathcal{L}_c[f(z)]} - p\right| = \left|\frac{\sum_{n=p+1}^{\infty} d(n-p)a_n z^{n-p}}{1 - \sum_{n=p+1}^{\infty} d a_n z^{n-p}}\right|$$

$$\le \frac{\sum_{n=p+1}^{\infty} d(n-p)a_n |z|^{n-p}}{1 - \sum_{n=p+1}^{\infty} d a_n |z|^{n-p}} < p - \sigma$$

$$\sum_{n=p+1}^{\infty} d(n-p)a_n|z|^{n-p} < p-\sigma\left(1-\sum_{n=p+1}^{\infty} da_n|z|^{n-p}\right)$$

$$\sum_{n=p+1}^{\infty} \frac{(n-\sigma)}{(p-\sigma)} da_n|z|^{n-p} \le 1 \qquad (5.72)$$

From *Theorem* (5.4.1)

$$\sum_{n=p+1}^{\infty} \frac{1}{k(n)} a_n \le 1 \qquad (5.73)$$

Hence by using (5.72) and (5.73) we get

$$\frac{(n-\sigma)}{(p-\sigma)} d|z|^{n-p} \le \frac{1}{k(n)}$$

$$|z|^{n-p} \le \left(\frac{p-\sigma}{n-\sigma}\right)\left(\frac{1}{dk(n)}\right)$$

Therefore

$$|z| \le \left(\left(\frac{p-\sigma}{n-\sigma}\right)\left(\frac{1}{dk(n)}\right)\right)^{\frac{1}{n-p}}$$

Therefore theorem (5.4.15) holds and the proof is over.

<u>DEFINITION 5.4.5</u>:For a function $f(z)$ which is analytic function in $w-$ plane containing the origin which is a simply connected region , we define the fractional integral of order μ as

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi \ where\ \mu > 0$$

<u>DEFINITION 5.4.6</u>:For a function $f(z)$which is analytic function in $w-$ plane containing the origin which is a simply connected region , we define the fractional integral of order μ as

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(\mu)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\mu}} d\xi \ where\ 1 > \mu \ge 0$$

<u>THEOREM 5.4.45</u>: Let $f \in \mathcal{M}(A,B,\alpha,\delta,p)$then

$$\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1-\frac{(p+1)k(n)}{(p+\mu+1)}|z|\right) \le \left|D_z^{-\mu} f(z)\right|$$
$$\le \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1+\frac{(p+1)k(n)}{(p+\mu+1)}|z|\right) \qquad (5.74)$$

<u>PROOF</u>: From *definition* (5.4.4) we have

$$D_z^{-\mu} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} z^{p+\mu} - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\mu+1)} a_n z^{n+\mu} \qquad .(5.75)$$

$$\mu > 0\,, n \ge p+1\,; p,n \in \mathbb{N}$$

Let $\phi(n) = \frac{\Gamma(n+1)}{\Gamma(n+\mu+1)}$

Clearly $\phi(n)$ is non – increasing function of n , $0 < \phi(n) \leq \phi(p+1) = \frac{\Gamma(p+2)}{\Gamma(p+\mu+2)}$

From *theorem* (5.4.1) we have

$$\sum_{n=p+1}^{\infty} |a_n| \leq k(n) \tag{5.76}$$

From (5.75) and (5.76) it follows that

$$\left|D_z^{-\mu} f(z)\right| \leq |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + \phi(p+1)|z| \sum_{n=p+1}^{\infty} |a_n|\right)$$

$$\leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu}\left(1 + \frac{(p+1)k(n)}{(p+\mu+1)}|z|\right)$$

Similarly

$$\left|D_z^{-\mu} f(z)\right| \geq |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \phi(p+1)|z| \sum_{n=p+1}^{\infty} |a_n|\right)$$

$$\geq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu}\left(1 - \frac{(p+1)k(n)}{(p+\mu+1)}|z|\right)$$

THEOREM 5.4.17: Let $f \in \boldsymbol{\mathcal{M}}(A, B, \alpha, \delta, p)$then

$$\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu}\left(1 - \frac{(p+1)k(n)}{(p-\mu+1)}|z|\right) \leq \left|D_z^{\mu} f(z)\right|$$
$$\leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu}\left(1 + \frac{(p+1)k(n)}{(p-\mu+1)}|z|\right) \tag{5.77}$$

PROOF: From definition 6 we have

$$D_z^{\mu} f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)} a_n z^{n-\mu} \tag{5.78}$$

$$1 > \mu \geq 0\,, n \geq p+1\,; p, n \in \mathbb{N}$$

Let $\psi(n) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}$

Clearly $\psi(n)$ is non – increasing function of n , $0 < \psi(n) \leq \psi(p+1) = \frac{\Gamma(p+2)}{\Gamma(p+\mu+2)}$

From theorem 1 we have

$$\sum_{n=p+1}^{\infty} |a_n| \leq k(n) \tag{5.79}$$

From (5.78) and (5.79) it follows that

$$\left|D_z^{\mu} f(z)\right| \leq |z|^{p-\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \psi(p+1)|z| \sum_{n=p+1}^{\infty} |a_n|\right)$$

$$\leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1+\frac{(p+1)k(n)}{(p-\mu+1)}|z|\right)$$

Similarly

$$\left|D_z^{\mu} f(z)\right| \geq |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}-\psi(p+1)|z|\sum_{n=p+1}^{\infty}|a_n|\right)$$

$$\geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1-\frac{(p+1)k(n)}{(p-\mu+1)}|z|\right)$$

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الخلاصــــــــــة

الغرض من هذه الاطروحة هو دراسة استقصائية لبعض الخصائص التحليلية والهندسية للفئات المشتملة على وظائف غير متكافئة ومتعددة . يتكون هذا العمل من خمسة فصول ومراجع.

الفصل الاول ، يشكل مقدمة عامة لموضوع الاطروحة . كما أنه يتكون من تعريفات ومفاهيم أساسية مثل التكافؤ والتحدي والنجم والمفاهيم ذات الصلة. الخصائص المعروفة والمهمة للعديد من الفئات الفرعية للوظائف (ذات الصلة بالعمل الحالي) مذكورة في شكل نظريات بدون برهان.

في الفصل الثاني ، يتم تعريف فئة فرعية جديدة من الوظائف غير التكافلية والدالة متعددة التكافؤ. وتناقش خاصية التوصيف التي تظهرها الوظائف في الفئة ونصف قطر اللمعان والتحدب. يتم أيضًا تضمين النتائج المتعلقة بنظريات النمو والتشويه ، خاصية الإغلاق ، النقاط القصوى ، عوامل التشغيل الأساسية للحفاظ على الفئة ، منطقة الوفرة وغيرها من الخصائص المهمة للفئة.

في الفصل الثالث ، قدمنا ودرسنا بعض الفئات الفرعية الجديدة , $A^*_{m,k}(\eta,\theta,\delta)$ $A^*_{m0,k}(\eta,\theta,\delta,z_0)$ و $A^*_{m1,k}(\eta,\theta,\delta,z_0)$ لوظائف أحادية الشكل التي يتم تعريفها عن طريق عامل تفاضلي. لقد حصلنا على العديد من النتائج الحادة بما في ذلك ظروف المعامل ، والنقاط القصوى ، وحدود التشوه ومجموعات محدبة للفئات المذكورة أعلاه من وظائف ثنائية التكافؤ.

في الفصل الرابع ، تم تعريف أربع فئات فرعية من الوظائف التوافقية ذات الترتيب المعقد مع المعاملات السلبية. بالنسبة للوظائف في هذه الفئات ، يتم تحديد النتائج على ظروف المعامل والنقاط المتطرفة وحدود التشوه وخصائص الالتواء والمحدبة وخاصية إغلاق الفئة تحت عامل التشغيل المتكامل. تجدر الإشارة إلى أن النتائج التي تمت مناقشتها تعمل على تحسين النتائج التي تم الحصول عليها بواسطة Yalcin و .Ozturk (2006; 2006a).

في الفصل الخامس ناقشنا خواص التبعية التفاضلية القوية للدوال المتعددة التكافؤ والمعرفة بواسطة مؤثر مشتقة رشاوية . حصلنا على بعض التطبيقات لنتائج التبعية التفاضلية التي تتضمن منتج . وحصلنا على بعض نتائج التبعية للدوال غير المتكافئة في قرص الوحدة المفتوحة U. وقد تم التعامل معها بشكل كامل مع دراسة مجموعة معينة من الوظائف متعددة التكافؤ المرتبطة بالتبعية ، وقد أدخلنا فئات جديدة باستخدام التبعية وحصلنا على تقديرات وخصائص معاملات تحتوي على نظريات التشويه والنمو ، دائرة نصف قطرها لامعة و نصف قطر التحدب ، والنتائج الأخرى ذات الصلة ب $K\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$ و $\boldsymbol{\mathcal{M}}(A,B,\alpha,\delta,p)$.

توجد العديد من تطبيقات العالم الواقعي في مجال نظرية الوظائف المتكافئة للفيزياء الرياضية الحديثة ، وديناميات الموائع (الطبقات الخاصة للوظائف غير المتكافئة تقر بتفسير هندسي واضح لتمييز شكل الواجهة الحرة) ، غير الخطية نظرية النظم القابلة للتكامل ونظرية المعادلات التفاضلية الجزئية (Vasil’ev, 2001; Abdhulhadi et al., 2005; Prokhorov و Vasil’ev 2002).

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بغداد

كلية التربية للعلوم الصرفه / ابن الهيثم

دراسة استقصائية على بعض الخصائص التحليلية والهندسية لاصناف دوال احادية التكافؤ ومتعددة التكافؤ

أطروحة

مقدمة إلى مجلس كلية التربية للعلوم الصرفه - ابن الهيثم /جامعة بغداد

وهي جزء من متطلبات نيل شهادة الدكتوراه فلسفة في علوم الرياضيات

من قبل

زينب هادي محمود

أشراف

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