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On Some New Topological Spaces

A Thesis

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By

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بسم الله الرحمن الرحيم

﴿هو الله الذي لا إله إلا هو عالم الغيب والشهادة هو الرحمن الرحيم ۝ هو الله الذي لا إله إلا هو الملك القدوس السلام المؤمن المهيمن العزيز الجبار المتكبر سبحان الله عما يشركون ۝ هو الله الخالق البارئ المصور له الأسماء الحسنى يسبح له مافي السموات والأرض وهو العزيز الحكيم۝﴾

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Conference Papers

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[2] N. A. Dawood and S. G. Gasim, **On Contractible J-Spaces** (2017), First International Conference for Science, College of Education for pure science – Ibn Al Haitham \ University of Baghdad, in13-14/ 12/ 2017.

ABSTRACT

The main aim of this thesis is to introduce and study the concept of $CJ-$ (respectively, strong $CJ-$, almost $CJ-$, almost strong $CJ-$ and contractible $J-$) spaces, some of them are considered a generalization of the concepts of J-space and strong $J-$space that are introduced by Michael [12]. On the other hand, some theorems about sufficient or necessary conditions for spaces to be $CJ-$spaces or almost CJ- spaces, are investigated. Also we gave the necessary condition that makes every $CJ-$ (respectively, almost $CJ-$) space is strong $CJ-$ (respectively, almost strong $CJ-$) space, that is, show that every strong $CJ-$space is a CJ- space and every almost strong $CJ-$space is an almost $CJ-$space. But the converse of these facts is not true unless the space is locally compact.

Furthermore, the concepts of semi-strong $CJ-$ (respectively, weak- $CJ-$, semi- weak $CJ-$, almost semi- strong $CJ-$, almost weak- $CJ-$, almost semi-weak $CJ-$) spaces are introduced, illustrated and show several properties of these spaces. Also study the relationship among all above concepts.

LIST OF ABBREVIATIONS

SYMBOL	DESCRIPTION
$\mathfrak{B}$	Base or subbase of a topology
$\mathrm{Cl}(A)$	The closure set of the set A
$\mathrm{In}(A)$	The interior set of the set A
p_2	The projection function
∂A	The boundary set of A
$\mathbb{N}$	The set of all natural numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^n$	The Euclidean space with n dimension
$\mathbb{R}^+$	$\{ x \in \mathbb{R}: x \geq 0\}$
$\mathbb{R}^-$	$\{ x \in \mathbb{R}: x \leq 0\}$
S^n	The unite sphere in $\mathbb{R}^{n+1}$
B^n	The unite ball in $\mathbb{R}^n$
$\pi_1(X, x_\circ)$	The fundamental group of X
e	The identity element of a group X
I_X	The identity function on X
I	The indiscrete topology
D	The discrete topology
$f\vert A$	The restriction of the function f on the set A
[x]	The equivalent class of the element x.
E^+	The set of all natural even numbers
O^+	The set of all natural odd numbers

CONTENTS

INTRODUCTION

Jordan curve theorem is one of the classical theorems of mathematics, this theorem was first formulated, at least in some form, by Bernand Bolzano (1781-1848), but it is named after by the French mathematician Camille Jordan (1838-1922) he was the first to publish a proof of the theorem in 1887 in [6]. Firstly, he gave the definition of arc as follow: If a given continuous function f from $[0,1]$ into a space X is a homeomorphism, then the curve f is said to be a Jordan arc in X. In the event that X is a Hausdorff space, then a curve f will automatically be a homeomorphism if it is one- one, and so for all common topological spaces a Jordan arc is simply defined to be a curve that does not pass through the same point twice. If f is a closed curve that is one-one on $[0,1]$ except for $f(0) = f(1)$, then f is said to be a Jordan curve [25].

The Jordan curve theorem states the following: If C is a graph of a simple closed curve in the complex plane the complement of C is the union of two regions, C being the common boundary of the two regions. One of the region is bounded and the other is unbounded [32].

The Jordan curve theorem is one of most important result about the topology of simple closed paths that also follows from the deformation theorem (for more details, see [15]).

For the great importance of Jordan curve theorem, any studies have been conducted about how proof this theorem. In [45] Hales talks about formal proofs in general and specially of the Jordan curve theorem, also he defends the original proof of Jordan curve theorem in [44].

In [26] Narens, gave a nonstandard proof of the Jordan curve theorem. According to [7] this is somewhat similar to Jordan's original proof.

In [13] a constructive proof is given, by Bery and others. Constructive in the sense that existence is not enough, the presentation is also essential.

Stoker [20] was used in discussing the Jordan theorem the winding number of an arc of a continuous curve with respect to a point not on the curve. an interesting notion in its own right.

Gamelin[48] gave the proof based on the jump theorem for the winding number because he was saw that the proof of Jordan curve theorem for piece wise smooth curves is substantially easier than the proof for arbitrary simple closed curves. This idea forms the basis for a proof in the general case.

On the other hand many generalizations of Jordan curve theorem are discussed by many researchers, for example not limited, we recall some of these generalizations.

In 1967, Kopperman, Khalimsky and Meyer stated a generalization in $\mathbb{Z}^2$ equipped with the khalimsky topology, [10].

In 1991, kong,et.al, introduced the following result: If Γ is an n-connected closed curve in $\mathbb{Z}^2$, then $\mathbb{Z}^2\backslash\Gamma$ has two and only two $\bar{\mathrm{n}}$-connectivity components $(\mathrm{n}+\bar{\mathrm{n}}=12, \mathrm{n}=4,8)$. This result is a kind of generalization of the classical Jordan curve theorem in $\mathbb{R}^2$, [49].

In 1999, Micael introduced and studied J-spaces and strong J-spaces which are considered to be generalizations of properties of Jordan curve theorem, [12].

In 2007, Nanjing introduced the concept of LJ-spaces exploited the common generalization of Lindelöf spaces and J-spaces, [56].

In 2007, Kornitowicz worked hard to mark crucial points in the proof of Jordan curve theorem,[2].

In 2008, Bouassida introduced a new proof of the Khalimsky's Jordan curve theorem using the specificity of the Khalimsky's plane as an Alexandroff topological space and the specific properties of connectivity on these spaces, [9].

In our thesis new types of generalizations of Jordan curve theorem are introduced , the concepts of countably compact and contractible are used to get many generalizations of this theorem.

By these generalizations, we get many new spaces, like CJ-space, strong CJ-space, semi-strong CJ-space, weak- CJ-space, semi-weak CJ-space, almost CJ-space, almost strong CJ-space, almost semi-strong CJ-space, almost weak- CJ-space, almost semi- weak CJ-space and contractible CJ-space.

Suitability with our thesis, we assumed all functions are continuous and all spaces are T_2, in spite of most of our results are useful wanting that presumption.

This thesis contains five chapters. Chapter one is preliminaries includes three sections; section one talks about compactness and countably compactness, while section two talks about connectedness and contractibility. In section three the review of the concepts of J-space, strong, semi-strong, weak and semi- weak J-space are introduced, in addition to some of their properties.

Chapter two consists of three sections. In section one we defined CJ-space and strong CJ-space, also we gave some examples of these concepts and we discussed the relationship between the two concepts and other known concepts as compact, countably compact, J-space and strong J-space. Some new theorems and propositions are given in this section.

In section two, we introduced three new spaces which are semi-strong CJ-space, weak CJ-space and semi- weak CJ- space. We found that strong CJ-space gives semi-strong CJ-space, and every semi-strong CJ-space is a CJ-space, while CJ-space is a semi-weak CJ-space which, in turn, weak CJ-space. But the converse is not true in general, so we gave many examples for the opposite directions. On the other hand, we found that if some spaces have special properties, then we can get semi- strong CJ- space, weak CJ-space and semi- weak CJ-space. The concepts of being CJ-space and weak CJ-space are equivalent in two ways.

In section three countably perfect and boundary countably perfect functions are given with some properties of them. Also, discussed the fact that these functions reservation property of being CJ-space or not.

Chapter three consists of three section. In section one we defined two new topological spaces which are almost CJ-space and almost strong CJ-space. We gave some properties of these spaces and established the relationship between them and with another known spaces as compact, countably compact, J-space, strong J-space, CJ-space and strong CJ-space.

The second section is dedicated to study of three new spaces which are almost semi- strong CJ-space, almost weak CJ-space and almost semi-weak CJ-space. During the study of the relationship between these spaces and another known spaces, we found that every almost strong CJ-space is an almost semi- strong CJ-space, and every almost semi- strong CJ-space is an almost CJ-space, while almost CJ-space is an almost semi- weak CJ-space which, in turn, almost weak CJ-space. The opposite directions of the Previous properties is not true in general.

In section three we used the same types of functions, that were used in the third section of chapter two, to prove some properties of almost CJ-space and almost strong CJ-space. Also discussed some theorems, that are considered equivalent to the definition of almost CJ- space and almost strong CJ- space, by using these types of functions.

In chapter four we used the notion of contractible space to define new topological space which called contractible J-space. This chapter includes two sections. In section one the definition of contractible J-space with its equivalent theorem are introduced. Also gave many varied examples, With a number of characteristics discussed.

In section two the study of functions that preserve the property of being contractible J-space are introduced and defined two new functions which are contractible function and boundary contractible function. Also gave some new properties.

Finally, chapter five consists of two sections. Section one reviews all the results obtained during this thesis. While in section two we suggest some future studies that can be obtain from this thesis.

CHAPTER ONE

PRELIMINARIES

INTRODUCTION

In this chapter, some preliminary concepts that are needed in our thesis are recalled. This chapter contains three sections, section one recalled some fundamental definitions, remarks and propositions about compactness and countably compactness.

In section two, many primary definitions, remarks and propositions of connectedness and contractibility are given.

Section three comprehends major definitions, remarks and propositions about J- spaces and strong J- spaces.

§1 COMPACTNESS

This section consists some important definitions such as: compact (respectively, countably compact and locally compact) space, also we give some properties of them. In addition, the relationships among these concepts are investigated.

Definition(1.1.1) [22]

A space X is said to be compact if every open cover of X contains a finite subcollection that also covers X.

Theorem(1.1.2) [43]

A subset of Euclidean space is compact if and only if it is closed and bounded.

Theorem(1.1.3) [42]

A compact subset of a Hausdorff space is closed.

Theorem(1.1.4) [22]

A finite union of compact subspaces of a space X is compact.

Proposition(1.1.5) [37]

Every closed subspace of a compact space is compact.

Definition(1.1.6) [24]

A topological space X is said to be countably compact if every countable open cover of X has a finite subcover.

Theorem(1.1.7) [46]

A topological space X is countably compact if and only if every infinite set having a cluster point.

Propositions(1.1.8) [18]

Any closed subspace of a countably compact space is again countably compact.

Proposition(1.1.9) [47]

Every compact space is a countably compact.

Remark(1.1.10) [18]

The converse of Proposition (1.1.9) is not true in general.

For example:

Let $X = \mathbb{N}$ and let $B_i = \{2i - 1, 2i\}; i = 1,2, \dots\dots$, let $\mathfrak{B} = \{B_i, i = 1,2, \dots\dots\}$ be a basis for a topology τ on X, then $(X, \tau)$is countably compact but not compact.

Theorem(1.1.11) [28]

A metric space is compact if and only if it is countably compact.

Definition (1.1.12) [22]

A function $f: A \to B$ is said to be injective if for each pair of distinct points of A, their images under f are distinct. It is said to be surjective if every element of B is the image of some element of A under the function f. If f is both injective and surjective, it is said to be bijective.

Proposition (1.1.13) [22]

Let $f: A \to B$ be a function, and let $A_\circ \subset A$, $A_1 \subset A$, $B_\circ \subset B$ and $B_1 \subset B$, then:

1. $A_\circ \subset f^{-1}(f(A_\circ))$ and that equality holds if f is injective.

2. $f\left(f^{-1}(B_\circ)\right) \subset B_\circ$ and that equality holds if f is surjective.

3. $f^{-1}(B_\circ \cap B_1) = f^{-1}(B_\circ) \cap f^{-1}(B_1)$.

4. $f^{-1}(B_\circ \cup B_1) = f^{-1}(B_\circ) \cup f^{-1}(B_1)$.

5. $f(A_\circ \cap A_1) \subset f(A_\circ) \cap f(A_1)$ and that equality holds if f is injective.

6. $f(A_\circ \cup A_1) = f(A_\circ) \cup f(A_1)$.

Definition(1.1.14) [19]

Let X and Y be two topological spaces. Let $f : X \to Y$ be a function from X into Y and x a point of X. If for every open neighbourhood V of f (x) in Y there is an open neighbourhood U of x such that $f(U) \subset V$, then the function f is said to be continuous at x. If f is continuous at every point of X, then it is called a continuous function.

Remark(1.1.15) [17]

Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then the composite function $g \circ f$ defined by $g \circ f(x) = g(f(x)), x \in X$, is also a continuous function from X into Z.

Remark (1.1.16) [19]

Every function from a discrete space into any space is continuous .

Remark (1.1.17) [11]

If $f : X \to Y$ is a continuous function and W is a subspace of X, then the restriction of f to W denoted by $f|W$ is a continuous function from W into Y.

Theorem(1.1.18) [19]

The following conditions are equivalent for a function $f : X \to Y$.

(a) f is a continuous function,

(b) for every open set V in Y, $f^{-1}(V)$ is an open set in X (see also [17]),

(c) for every closed set G in Y, $f^{-1}(G)$ is a closed set in X,

(d) for a base (or subbase) $\mathfrak{B}$ of Y , and for every $B \in \mathfrak{B}$, $f^{-1}(B)$ is an open set in X,

(e) for every subset A of X, $f(cl(A)) \subset cl(f(A))$ in Y, where $cl(A)$ denotes the closure of A.

Proposition (1.1.19) [22]

A continuous image of a compact space is compact.

Proposition (1.1.20) [24]

A continuous image of a countably compact space is countably compact.

Definition(1.1.21) [29]

A function $f : X \rightarrow Y$ is called a closed function if for every closed set $A \subseteq X$, the image $f(A)$ is closed set in Y.

Definition(1.1.22) [36]

A closed continuous function with compact preimages of points is called perfect. That is $f: X \rightarrow Y$ is a perfect function if f is closed, continuous and $f^{-1}(y)$ is compact for each $y \in Y$.

The point inverses $f^{-1}(y)$, for $y \in Y$, are sometimes called the fibers of the function f.

Theorem(1.1.23) [36]

If a function $f: X \rightarrow Y$ is a perfect function, then for any compact subset F of Y, the preimage $f^{-1}(F)$ is a compact subset of X.

Proposition(1.1.24) [19]

Any continuous function from a compact space X onto a Hausdorff space Y is a perfect function.

Proposition(1.1.25) [19]

If X is compact Hausdorff and Y is any space then the projection function $p_2 : X \times Y \to Y$ is a closed function with fibers all homeomorphic to X. So p_2 is a perfect map.

Definition(1.1.26) [12]

A function $f : X \to Y$ is called boundary perfect if it is closed and if $\partial f^{-1}(y)$ is compact for every $y \in Y$.

Definition(1.1.27) [19]

A continuous closed function with countably compact fibers is called a quasi-perfect function.

Definition(1.1.28) [14]

A function $f: (X, \tau) \to (Y, \tau')$ is said to be countably compact function if inverse image of each countably compact subset of Y is a countably compact subset of X.

Definition(1.1.29) [21]

A space X is locally compact if each point of X has a compact neighbourhood.

Definition(1.1.30) [24]

Let A be a subset of a space X, then A is said to be dense in X if $cl(A) = X$.

Theorem(1.1.31) [11]

A Hausdorff space X is locally compact if and only if X is an open dense subspace of a compact Hausdorff space.

Theorem(1.1.32) [11]

The product space of locally compact Hausdorff spaces is locally compact if and only if all but a finite number of factor spaces are compact.

Proposition(1.1.33) [35]

Every closed subspace of locally compact space is locally compact.

Proposition(1.1.34) [11]

If X is Hausdorff, then every open subspace is locally compact.

Proposition(1.1.35) [11]

Locally compactness is preserved by open continuous images.

Proposition(1.1.36) [11]

The preimages, under perfect maps, of locally compact spaces are locally compact.

Definition(1.1.37) [53]

Let M be a set and let $q_i: M_i \to M$ be functions from topological spaces $(M_i, \mu_i)_{i \in I}$ into M for some index I. The final topology $\boldsymbol{\mu}$ on M with respect to the functions $q_i: M_i \to M$ is the finest topology on M such that all the functions q_i are continuous. A subset $O \subseteq M$ is open if and only if $q_i^{-1}(O) \subseteq M_i$ is open $\forall\, i \in I$.

Definition(1.1.38) [19]

A Hausdorff space is said to be a k-space if it has the final topology with respect to all inclusions $C \to X$ of compact subspaces C of X, so that a set A in X is closed in X if and only if $A \cap C$ is closed in C for all compact subspaces C of X.

Remarks(1.1.39) [19]

1. All metric spaces are k-spaces.

2. A closed(respectively, open) subspace of a k-space is again k-space.

3. The product of k-spaces need not be a k-space.

Theorem(1.1.40) [11]

A Hausdorff space X is locally compact if and only if $X \times Y$ is a k-space for any k-space Y.

§2 CONNECTEDNESS AND CONTRACTIBILITY

In this section, the concept of connected (respectively, locally connected, path connected and simply connected) space is introduced, also we discuss the relationship among them.

In addition, the definition of contractible space with many properties and examples about this concept are introduced.

We begin with the following definition.

Definition(1.2.1) [40]

A topological space X is said to be connected if it cannot be represented as the union of two disjoint nonempty open sets.

Theorem(1.2.2) [33]

The continuous image of a connected space is connected.

Theorem(1.2.3) [38]

The finite product of connected spaces is connected.

Definition(1.2.4) [24]

Two subsets A and B of a space X are said to be separated if $cl\,(A) \cap B = \emptyset$ and $A \cap cl\,(B) = \emptyset$.

The condition separated sets is a little stronger than saying that A and B are disjoint. But it is weaker than saying that there closures are disjoint.

Theorem(1.2.5) [24]

Two subsets A and B of a topological space X are separated if and only if they are closed subsets of $A \cup B$ with relative topology.

Proposition(1.2.6) [24]

Let X be a space and C be a subset of X. Suppose $C \subset A \cup B$ where A, B are separated subsets of X. Then either $C \subset A$ or $C \subset B$.

Theorem(1.2.7) [24]

Let $\mathcal{C}$ be a collection of connected subsets of a space X such that no two members of $\mathcal{C}$ are separated. Then $\bigcup_{C \in \mathcal{C}} C$ is also connected.

Definition(1.2.8) [39]

A continuous function $f: X \rightarrow Y$ is monotone if all fibers $f^{-1}(y)$ are connected.

Theorem(1.2.9) [39]

If $f: X \rightarrow Y$ is a monotone function which is either closed or open, then for every connected subset C of the space Y the inverse image $f^{-1}(C)$ is connected.

Lemma(1.2.10) [12]

Let $f: X \rightarrow Y$ be monotone. If $\{A, B\}$ is a closed or open cover of X, then $f(A \cap B) = f(A) \cap f(B)$.

Definition(1.2.11) [40]

A maximal connected subspace of a topological space, that is, a connected subspace which is not properly contained in any larger connected subspace, is called a component of the space. A connected space clearly has only one component, namely, the space itself.

Theorem(1.2.12) [40]

If X is an arbitrary topological space, then we have the following:

(a) Each point in X is contained in exactly one component of X.

(b) Each connected subspace of X is contained in a component of X.

(c) A connected subspace of X which is both open and closed is a component.

(d) Each component of X is closed.

Definition(1.2.13) [27]

A space X is said to be a locally connected space if for each $x \in X$, and each neighbourhood U of x there is a connected neighbourhood V of x which is contained in U.

Theorem(1.2.14) [40]

A space X is locally connected if and only if the components of all open subspaces of X are open.

Proposition(1.2.15) [55]

If a topological space X is locally connected, then the connected components of X are open.

Propositions(1.2.16)

a) Local connectedness is hereditary with respect to open subsets, but not to closed subsets in general. **[11]**

b) Local connectedness is preserved by images under continuous and open functions.**[40]**

c) The product space of locally connected spaces is locally connected if and only if all but a finite number of factor spaces are connected. **[11]**

Definition(1.2.17) [50]

Let X be a topological space. A path in X is a continuous function from [0,1] to X.

Definition(1.2.18) [50]

Let X be a topological space and let $x_\circ, x_1 \in$ X. Then $x_\circ$and x_1 can be connected by a path in X if there is a path $\gamma : [0,1] \rightarrow$ X such that $\gamma(0) = x_\circ$ and $\gamma(1) = x_1$.

Definition(1.2.19) [23]

A topological space X is called path connected if any two of its points can be connected by a path in X.

Examples(1.2.20) [22]

1. The unite sphere S^{n-1} in $\mathbb{R}^n$ is path connected $\forall\, n > 1$.
2. The unite ball B^n in $\mathbb{R}^n$ is path connected.
3. Every open ball and every closed ball in $\mathbb{R}^n$ is path connected.

Proposition(1.2.21) [34]

A path connected set is connected.

Proposition(1.2.22) [30]

The continuous image of path connected space is path connected.

Theorem(1.2.23) [23]

Two spaces X and Y are path connected if and only if $X \times Y$ is path connected.

Proposition(1.2.24) [23]

Let A and B be path connected subspaces of a space X. If $A \cap B \neq \emptyset$ is path connected, then $A \cup B$ is path connected.

Definition(1.2.25) [7]

Two continuous functions $f_0, f_1 : X \to Y$ are said to be homotopic if there is a continuous function $F: X \times I \to Y$ (I is the closed interval [0,1]), such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. This homotopic denoted by $f_0 \cong f_1$.

Definition(1.2.26) [7]

Two spaces X and Y are of the same homotopy type if there exist continuous functions $f: X \to Y$ and $g: Y \to X$ such that $gf \cong I: X \to X$ and $fg \cong I: Y \to Y$. The functions f and g are then called homotopy equivalences, we also say that X and Y are homotopy equivalent.

Remarks(1.2.27) [4]

1. Homotopy type defines an equivalence relation on the collection of all topological spaces.
2. Homotopy relation is an equivalence relation on the collection of all maps from X to Y.

Definition(1.2.28) [1]

The fundamental group of a space X is the set of all homotopy classes $[f]$ of loops $f : I \to X$ at the basepoint $x_\circ$, which is a group with respect to the product $[f][g] = [f\,g]$, and is denoted by $\pi_1(X, x_\circ)$.

Definition(1.2.29) [23]

A space X is simply connected if it is path connected and $\pi_1(X, x_\circ) = \{e\}, \forall x_\circ \in X$.

Definition(1.2.30) [52]

If Y is a subspace of a topological space X, a retraction from X to Y is a continuous function $r: X \to Y$ such that $r(p) = p, \forall\, p \in Y$. In this case Y is called a retract of X.

Proposition(1.2.31) [11]

A retract of a locally connected space is locally connected.

Definition(1.2.32) [52]

A subspace Y of a space X is called a deformation retract if there is a continuous retract $r: X \to Y$ such that the identity function from X to X homotopic to the function $i \circ r$, where i is the inclusion of Y in X.

Definition(1.2.33) [11]

A function from X to Y is said to be null- homotopic if it is homotopic to some constant function.

Definition(1.2.34) [22]

A space X is called contractible space if the identity function $I_X: X \to X$ is null- homotopic.

Example (1.2.35) [4]

The Euclidean space $\mathbb{R}^n$ is contractible.

Remark (1.2.36) [41]

A discrete space with more than one point is not contractible.

In the following we give some results about trivial spaces:

Remarks(1.2.37)

1. Any subspace (with more than one element) of a discrete space is not contractible since every subspace of a discrete space is also discrete.
2. A subset Y of $\mathbb{R}$ is contractible if Y is not discrete space (with more than one element).
3. An indiscrete space is a contractible space, this follows from the fact says that any function with indiscrete codomain is continuous.
4. Any subspace of an indiscrete space is contractible since every subspace of an indiscrete space is also indiscrete.

Definition (1.2.38) [16]

A set of points Ŋ is said to be convex if whenever two points x_1, x_2 belong to Ŋ all the points of the form $\lambda x_1 + \mu x_2$, where $\lambda \geq 0, \mu \geq 0$, $\lambda + \mu = 1$, also belong to Ŋ.

Propositions (1.2.39) [51]

1. Every convex subset of $\mathbb{R}^n$ is contractible.
2. Any ball in $\mathbb{R}^n$ is contractible.

Definition(1.2.40) [5]

Let X be a topological space and A the subset of $X \times [0,1]$ given by $X \times \{1\}$. The space $X \times [0,1]/A$ is called the cone over X, denoted by TX.

Theorem (1.2.41) [8]

A topological space X is a contractible space if and only if it is a homotopy equivalent to a point.

Theorem (1.2.42) [54]

A topological space X is a contractible space if and only if there exists a point $x_\circ \in X$ such that $\{x_\circ\}$ is a deformation retract of X.

Theorem (1.2.43) [29]

A topological space X is a contractible space if and only if it is a retract of any cone over it.

Theorem (1.2.44) [1]

A topological space X is a contractible space if and only if every function $f: X \to Y$, for arbitrary Y, is null-homotopic.

Theorem (1.2.45) [1]

A topological space X is a contractible space if and only if every map $f: Y \to X$, for arbitrary Y, is null-homotopic.

Proposition(1.2.46) [23]

Every contractible space is path connected space.

Proposition(1.2.47) [23]

Every contractible space is simply connected space.

Remark(1.2.48) [3]

The convers of Propositions (1.2.42) and (1.2.43) are not true in general.

For example:

S^n is path connected for every integer $n \geq 1$, and simply connected for every integer $n \geq 2$. Yet these spheres are not contractible.

Remark(1.2.49) [7]

The continuous image of a contractible space need not be contractible.

For example:

$f: [a, b] \to S^1$ is continuous and onto, since S^1 is a quotient space for $[a, b]$ by the relation $x{\sim}y$ if $x = a$ and $y = b$. Note that $[a, b]$ is contractible, but S^1 is not.

Theorem(1.2.50) [31]

The fundamental group of a contractible space X is trivial.

Proposition(1.2.51) [1]

A retract of a contractible space is contractible.

Proposition(1.2.52) [23]

If X is a contractible and Y is path connected, then any two continuous functions from X onto Y are homotopic (and each is null- homotopic).

Proposition(1.2.53) [8]

Two homeomorphic spaces are homotopy equivalent. Thus the classification of spaces up to homotopy equivalence is coarser than the homeomorphism classification.

§3 J-SPACES AND STRONG J-SPACES:

In this section two relevant concepts are given with simple definitions and interesting properties, these two concepts are J-space and strong J-space.

Definition(1.3.1) [12]

A space X is a J -space if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is compact.

Definition(1.3.2) [12]

A space X is a strong J -space if every compact $K \subset X$ is contained in a compact $L \subset X$ with $X \backslash L$ connected.

Note that Definitions (1.3.1) and (1.3.2) is a generalizations to Jordan curve theorem, since the Jordan curve theorem guarantees that if C is a simple closed curve in the plane $\mathbb{R}^2$, then $\mathbb{R}^2 \backslash C$ has precisely two components W_1 and W_2, of which C is the common boundary. Generalizing these properties, E. Michael [12] introduced and studied the following two definitions.

Now, introduce some propositions about strong J-space which is needed.

Proposition(1.3.3) [12]

Every strong J-space is a J -space. The converse holds if the space is locally connected (Corollary 3.2 in [12]), but not in general (Examples 9.1, 9.2 in [12]).

Proposition(1.3.4) [12]

Every compact space is a strong J –space.

Proposition(1.3.5) [12]

If X_1 and X_2 are connected and non- compact, then $X_1 \times X_2$ is a strong J- space.

Proposition(1.3.6) [12]

The space $\mathbb{R}^+$ as a subspace of the usual space $\mathbb{R}$ is a strong J-space.

Now, we recall the definition of semi-strong J -space, semi-weak J -space and weak J -space.

Definition(1.3.7) [12]

A space X is a semi-strong J -space if for every compact $K \subset X$ there is a compact $L \supset K$ in X and a connected $C \subset X\backslash K$ with $C \cup L = X$.

Definition(1.3.8) [12]

A space X is a semi-weak J-space if, whenever A and B are disjoint, closed subsets of X with ∂A and ∂B are compact, then A or B is compact.

Definition(1.3.9) [12]

A space X is a weak J-space if, whenever $\{A, B, K\}$ is a closed covering of X with K compact and $A \cap B = \emptyset$, then A or B is compact.

The following theorem illustrate the relationship among above spaces.

Theorem(1.3.10) [12]

Consider the following properties of a space X.

(a) X is a strong J-space.

(b) X is a semi-strong J-space.

(c) X is a J-space.

(d) X is a semi-weak J-space.

(e) X is a weak J-space.

Then (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e), and none of these implications is reversible (even for subsets of $\mathbb{R}^2$). However, (a) ⇔ (b) ⇔ (c) if X is locally connected, and (c) ⇔ (d) ⇔ (e) if X is locally compact.

Definition(1.3.11) [17]

Let L be any vector space over a non- discrete valuated field K and $\mathfrak{T}$ be a topology on L, the pair $(L, \mathfrak{T})$ is called a topological vector space (or topological linear space) over K if these two axioms are satisfied:

1. $(x, y) \to x + y$ is continuous on $L \times L$ into L.
2. $(\lambda, x) \to \lambda\, x$ is continuous on $K \times L$ into L.

Proposition(1.3.12)[12]

Every topological linear space $X \neq \mathbb{R}$ is a strong J- space.

CHAPTER TWO

CJ- SPACE
AND
STRONG CJ- SPACE

INTRODUCTION

In this chapter, the concept of " countably compact" is used to define a new topological spaces, called CJ- space and strong CJ- space.

This chapter consists of three sections, section one includes the above definitions with their properties and the relationship between them.

In section two, another new spaces which are called semi- strong CJ-space, weak CJ-space and semi- weak CJ-space are studied with the relationship among these spaces.

New types of functions are given in section three. The functions that transferred CJ- spaces to CJ-space are discussed. By using these types of functions, we prove theorems which give new definitions that equivalent to definitions of CJ-space and semi- weak CJ-space.

§1 CJ-SPACES AND STRONG CJ-SPACES

The main concern of this section is to introduce the concepts of CJ-space and strong CJ-space. These concepts are considered a generalization of Jordan curve theorem in different approach. In this approach we using the concept of countably compact to define CJ-space and strong CJ-space, and give some properties of them.

Now, we start with the following definitions.

Definition (2.1.1)

A topological space X is a CJ-space if, when $\{A, B\}$ is a closed cover of X such that $A \cap B$ countably compact, then A or B is countably compact.

Definition (2.1.2)

A topological space X is a strong CJ-space if every countably compact $K \subset X$ is contained in a closed countably compact $L \subset X$ with $X \setminus L$ is connected.

Remark(2.1.3)

We know that in any topological space "every compact subset of Hausdorff space is closed". This fact is not hold in general if we replace the concept of compact set by countably compact set. To ensure satisfies this fact in our work, we put the condition on the set L to be closed in Definition (2.1.2).

The following Proposition shows that every strong CJ-space is a CJ-space. While the converse is not true, as shown in Remark (2.1.5).

Proposition (2.1.4)

Every strong CJ-space is a CJ-space.

Proof:

Let X be a strong CJ- space and let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ countably compact, so there exists a closed countably compact $L \subset X$ such that $A \cap B \subset L$ and $X \setminus L$ is connected. Therefore $\{(A \cap X) \setminus L, (B \cap X) \setminus L\}$ is a disjoint closed cover of $X \setminus L$, but $X \setminus L$ is connected, so $X \setminus L$ must be in $(A \cap X) \setminus L$ or in $(B \cap X) \setminus L$, it follows that $X \setminus L \subset A$ or $X \setminus L \subset B$. By complementation we have $A^c \subset L$ or $B^c \subset L$, and since $A \cap B \subset L$, so $A \subset L$ or $B \subset L$. Then A or B is countably compact by Proposition (1.1.8). Hence X is CJ-space.

Remark (2.1.5)

The converse of proposition (2.1.4) is not true in general.

For example:

Let us take the topology defined on the set of natural numbers $\mathbb{N}$, this topology is generated by the partition $P = \{\{2k - 1, 2k\};\ k \in \mathbb{N}\}$ and called the $\mathrm{Odd} - \mathrm{Even}$ topology. The only countably compact subsets of $\mathbb{N}$ are the finite subsets, so if we take a closed cover $\{A, B\}$ of $\mathbb{N}$ with $A \cap B$ countably compact, that is mean $A \cap B$ is finite set and since the intersection of any two infinite sets in this space must be an infinite set, so A or B must be finite, that means A or B is countably compact. Hence ℕ is CJ-space.

But ℕ is not strong CJ-space since every countably compact subset of ℕ is finite and hence its complement is infinite and every infinite subset of ℕ is non-connected.

Remark (2.1.6)

Every finite space is a CJ-space.

Proposition (2.1.7)

Every countably compact space is a strong CJ-space.

Proof:

Let X be a countably compact space and let $K \subset X$ be a countably compact, then X is a closed countably compact with $K \subset X$ and $X \setminus X = \emptyset$ is connected.

Theorem(2.1.8)

A metric space X is a strong CJ-space if and if X is a strong J-space.

Proof:

The "if" part

Suppose that X is a strong CJ-space and let $K \subset X$ be a compact, then K is countably compact by Theorem(1.1.11). It follows by Definition (2.1.2) that there exists a countably compact subset L of X such that $K \subset L$ and $X \setminus L$ connected. Again by Theorem (1.1.11) L is compact. Hence X is strong J-space by Definition (1.3.2).

The "only if" part

Suppose that X is a strong J-space and let $K \subset X$ be a countably compact, then K is compact by Theorem(1.1.11). It follows by Definition (1.3.2) that there exists a compact subset L of X such that $K \subset L$ and $X \setminus L$ connected. Again by Theorem (1.1.11) L is countably compact. Hence X is strong CJ-space by Definition (2.1.2).

Remark (2.1.9)

The converse of Proposition (2.1.7) is not true in general.

For example:

Let us take $\mathbb{R}^+$ as a subspace of $\mathbb{R}$ with the usual topology which is not countably compact, but it is strong CJ-space by Proposition (1.3.6) and Theorem (2.1.8).

Corollary (2.1.10)

Every compact space is a strong CJ- space.

Proof:

Follows from Propositions (1.1.9) and (2.1.7).

Countably compact space is a CJ-space as shown in the following Proposition, but the converse is not true by Remark (2.1.12).

Proposition (2.1.11)

Countably compact space is a CJ-space.

Proof:

Follows from Propositions (2.1.7) and (2.1.4).

Remark (2.1.12)

The converse of Proposition (2.1.11) is not true in general.

For example:

Let us take the set of real numbers with the particular point topology τ, such that $\tau = \{U \subseteq \mathbb{R} | \ 0 \in U \text{ or } U = \emptyset\}$, note that every closed cover of $\mathbb{R}$ must contain $\mathbb{R}$. Let us take $\{\mathbb{R}, A\}$ as a closed cover of $\mathbb{R}$ with $\mathbb{R} \cap A$ countably compact, but $\mathbb{R} \cap A = A$, so A is countably compact. Hence $(\mathbb{R}, \tau)$ is CJ-space. But this space is not countably compact, since $C = \{(-n, n); n \in \mathbb{N}\}$ is a family of open subsets of $\mathbb{R}$, which covers $\mathbb{R}$, has no terminated open subcover.

Corollary (2.1.13)

Every compact space is a CJ- space.

Proof:

Follows from Propositions (1.1.9) and (2.1.11).

Now, some theorems, propositions, corollaries and important remarks are introduced in follow:

Proposition (2.1.14)

All topological linear spaces, excepting $\mathbb{R}$, are strong CJ-space.

Proof:

Take $X \neq \mathbb{R}$ as a topological linear space, and let $K \subset X$ be a countably compact and let $L = \{\alpha x : \alpha \in [0,1] \text{ and } x \in K\}$, then $K \subset L$ and L is a closed countably compact subset of X with $X \backslash L$ is connected.

Lemma (2.1.15)

If B is a closed non-countably compact subset of any topological space X and $C \subset B$ is countably compact, consequently there is a non-countably compact closed $D \subset B$ such that $D \cap C = \emptyset$.

Proof:

Let $\mathcal{G}$ be a countable open cover of B with no finite subcover, and let $C \subset B$ be a countably compact, then $\mathcal{G}$ is a countable open cover of C. Pick a finite $\mathfrak{J} \subset \mathcal{G}$ covering C. Then $D = B \setminus \bigcup \mathfrak{J}$ is a closed non– countably compact subset of B with $D \cap C = \emptyset$.

The following Theorem gives four different conditions that are equivalent to being CJ-space.

Theorem (2.1.16)

Let X be any topological space, then the following conditions are equivalent:

1. X is a CJ-space,

2. For any $A \subset X$ with countably compact boundary, $cl(A)$ or $cl(X\backslash A)$ is countably compact,

3. If A and B are closed in X with $A \cap B = \emptyset$ and ∂A or ∂B countably compact, then A or B is countably compact,

4. If $K \subset X$ is countably compact, and if $\mathcal{W}$ is an open cover of $X\backslash K$ with disjoint members, then there exists $W \in \mathcal{W}$ such that $X\backslash W$ is countably compact,

5. Same as (4), but with card $\mathcal{W} = 2$.

Proof

(1) $\Longrightarrow$ (2)

Let $A \subset X$ such that ∂A is countably compact, but $\partial A = cl(A) \cap cl(X\backslash A)$, now we have a closed cover $\{cl(A), cl(X\backslash A)\}$ of X with $cl(A) \cap cl(X\backslash A)$ is countably compact and X is CJ-space so $cl(A)$ or $cl(X\backslash A)$ is countably compact.

(2) $\Longrightarrow$ (3)

Let A and B be disjoint closed subsets of X with ∂A is countably compact. By (2) we can get $cl(A)$ or $cl(X\backslash A)$ is countably compact. But $cl(A) = A$, so A or $cl(X\backslash A)$ is countably compact and since $B \subset cl(X\backslash A)$, so A or B is countably compact (since B is closed and $cl(X\backslash A)$ is countably compact).

(3) ⟹ (1)

Let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ is countably compact. Suppose that B is non- countably compact and since $A \cap B \subset B$ is countably compact, so by lemma (2.1.15) there is a non- countably compact closed $D \subset B$ such that $D \cap (A \cap B) = \emptyset$, it follows that $D \cap A = \emptyset$,but ∂A countably compact since it is a closed subset of $A \cap B$. By (3) A or D is countably compact, but D is non- countably compact. Hence A must be countably compact.

(4) ⟹ (5)

Clear.

(5) ⟹ (4)

Let $K \subset X$ be a countably compact and let $\mathcal{W}$ be a disjoint open cover of $X \setminus K$. To show that $X \setminus W$ is countably compact for some $W \in \mathcal{W}$ we shall follow three demarches.

First, we prove that if U is open subset of X containing K, then $\mathcal{W}' = \{W \in \mathcal{W} : W \not\subseteq U\}$ is finite. Suppose that it is not finite, then $\mathcal{W} = W_1 \cup W_2$ with $W_1 \cap W_2 = \emptyset$ and $W_1 \cap \mathcal{W}'$ and $W_2 \cap \mathcal{W}'$ both finite. Let $V_1 = \cup W_1$ and $V_2 = \cup W_2$, then V_1, V_2 are two open subsets of X such that $V_1 \cap V_2 = \emptyset$ and $X \setminus K = V_1 \cup V_2$ so by (5) $X \setminus V_1$ or $X \setminus V_2$ is countably compact, but $V_1 \subseteq X \setminus V_2$ and $V_2 \subseteq X \setminus V_1$ since V_1 and V_2 are disjoint. It follows that $cl(V_1) \subseteq cl(X \setminus V_2) = X \setminus V_2$ and $cl(V_2) \subseteq cl(X \setminus V_1) = X \setminus V_1$, so we get $cl(V_1)$ or $cl(V_2)$ is countably compact by Proposition (1.1.8). Suppose that $cl(V_1)$ is countably compact, then $C = cl(V_1) \setminus U$ is countably compact. Now let $\mathcal{W}'_1 = W_1 \cap \mathcal{W}'$, then $\mathcal{W}_1'$ covers C and each $W \in \mathcal{W}'$

intersects C, so C is not countably compact since $\mathcal{w}_1{}'$ is infinite and disjoint, which is a contradiction. Hence $\mathcal{w}'$ is finite.

Second, we prove that if cl(W) is countably compact, $\forall$ W $\in \mathcal{w}$, consequently X is countably compact. Let $\mathcal{V}$ be a family of countably open subsets covers X, then $\mathcal{V}$ is a countably open cover of K, which is countably compact, so $\mathcal{V}$ has a finite subcover $\mathcal{F}$ covers K. Let $U = \bigcup \mathcal{F}$, by step one we get a finite family $\mathcal{w}' = \{W \in \mathcal{w}: W \not\subseteq U\}$, so $\bigcup\{cl(W): W \in \mathcal{w}'\}$ is countably compact and since $\mathcal{V}$ is an open cover of it therefore it is covered by some finite $\mathcal{E} \subset \mathcal{V}$. But $\bigcup\mathcal{E} \subset \mathcal{V}$ is finite and covers X, so X is countably compact.

Lastly, we show that, X\W is countably compact for some W $\in \mathcal{w}$. If cl(W) is countably compact for all W $\in \mathcal{w}$, then X is countably compact by step (2) and since X\W is a closed subset of X, so X\W is countably compact. Suppose that there exists $W_0 \in \mathcal{w}$ such that $cl(W_0)$ is not countably compact. Let $W^* = \cup \{W \in \mathcal{w}: W \neq W_0\}$, then, $\{W_0, W^*\}$ is a disjoint open cover of $X\backslash K$, so $X\backslash W_0$ or $X\backslash W^*$ is countably compact, by (5). If $X\backslash W^*$ is countably compact, and since $cl(W_0)$ is a closed subset of $X \backslash W^*$, so $cl(W_0)$ is countably compact which is a contradiction, so $X\backslash W^*$ is not countably compact, it follows that $X\backslash W_0$ is countably compact.

(5)$\Longrightarrow$ (1)

Let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ countably compact, then $\{X\backslash A, X\backslash B\}$ is an open cover of $X\backslash A \cap B$ with $X\backslash A \cap X\backslash B = \emptyset$.

By (5) $X\backslash(X\backslash A)$ or $X\backslash(X\backslash B)$ is countably compact, that is A or B is countably compact. Hence X is CJ-space.

(1)⟹ (5)

Let K be a countably compact subset of X and let $\{W_1, W_2\}$ be an open cover of $X\backslash K$ with $W_1 \cap W_2 = \emptyset$, so $\{X\backslash W_1, X\backslash W_2\}$ is a closed cover of X with $X\backslash W_1 \cap X\backslash W_2 = X\backslash(W_1 \cup W_2)$ which is a closed subset of K since $X\backslash K \subset W_1 \cup W_2$, then $X\backslash W_1 \cap X\backslash W_2$ is countably compact by Proposition (1.1.8). But X is CJ-space, so $X\backslash W_1$ or $X\backslash W_2$ is countably compact.

Theorem (2.1.17) ensure that the concepts of being CJ- space and strong CJ- space, are equivalent if the space is locally connected.

Theorem (2.1.17)

A locally connected space X is CJ-space if and only if it is a strong CJ-space.

Proof:

A strong CJ-space is CJ-space by Proposition (2.1.4). If X is CJ-space we must prove that X is strong CJ-space, let $K \subset X$ be countably compact. Now we have an open cover $\mathcal{W}$ of $X\backslash K$ with disjoint members and each $W \in \mathcal{W}$ connected since X is locally connected. By theorem (2.1.16), there exists a $W_0 \in \mathcal{W}$ where $X\backslash W_0$ is countably compact. Taking $L = X\backslash W_0$, then L is a closed countably compact containing K and $X\backslash L$ is connected.

It is known, that some concepts of general topology are hereditary properties, for example being T_2-space is a hereditary property, while compact or connected is not. Compactness, for example, is weakly hereditary property, when the subspace is closed. In our case CJ-spaces are not hereditary, nor weakly hereditary, as shown in Remark (2.1.18) below. But if the subspace is clopen (closed and open), then it is CJ- space, if the space is so, as seen in Proposition (2.1.19).

The following Remark shows that the property of being CJ-space is not weak hereditary property and therefore not hereditary property.

Remark (2.1.18)

A closed subset of CJ-space need not be CJ-space.

For example:

Let us take the same topological space $\mathbb{R}$ in the example of Remark (2.1.12), this space is CJ-space, but the closed subspace $\mathbb{N}$ with the induced topology, which is the discrete topology, is not CJ-space since $\{O^+, E^+\}$ is a closed cover of $\mathbb{N}$ with $O^+ \cap E^+ = \emptyset$ is countably compact, but neither O^+ nor E^+ is countably compact.

Proposition (2.1.19)

A clopen subset of a CJ-space is CJ-space.

Proof:

Let X be any CJ-space and let W be a clopen subspace of it, we have to show that W is CJ-space, let $\{A, B\}$ be a closed cover of W with $A \cap B$ countably compact, then $\{A \cup (X \setminus W), B\}$ is a closed cover of X with $(A \cup (X \setminus W)) \cap B = A \cap B$ countably compact, so $A \cup (X \setminus W)$ or B is countably compact. If B is countably compact, then we have got the proof. If B is not countably compact, then $A \cup (X \setminus W)$ is countably compact, but A is a closed subset of $A \cup (X \setminus W)$, so A is countably compact, and hence W is CJ-space.

Theorem (2.1.20) below describes a condition which ensures transmission the property of being (strong) CJ- space from a space X to each component of its closed cover and vice versa.

Theorem (2.1.20)

Let $\{X_1, X_2\}$ be a closed cover of a topological space X with $X_1 \cap X_2$ countably compact. Then X is a (strong) CJ-space if and only if X_1 and X_2 are (strong) CJ-spaces and X_1 or X_2 is countably compact.

Proof:

(1) CJ-space

The "if" part

Assume that X is CJ-space, then X_1 or X_2 is countably compact by definition of CJ-space. Suppose that X_1 is countably compact, it follows that X_1 is CJ-space. Now to show that X_2 is CJ-space. Let A, B be two closed subsets of X which cover X_2 with $A \cap B$ countably compact, therefore $\{A, B \cup X_1\}$ is a closed cover of X such that $A \cap (B \cup X_1)$ countably compact, so A or $B \cup X_1$ is countably compact since X is CJ-space. But B is a closed subset of $B \cup X_1$, so A or B is countably compact.

The "only if" part

Assume that X_1 and X_2 are CJ- spaces and suppose that X_2 is countably compact, we have to show that X is CJ-space. Let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ countably compact. Now let $A_i = A \cap X_i$ and $B_i = B \cap X_i$ $(i = 1,2)$, consequently $\{A_1, B_1\}$ is a closed cover of X_1, which is CJ-space, with $A_1 \cap B_1$ countably compact, so A_1 or B_1 is countably compact. If A_1 is countably compact, then $A = A_1 \cup A_2$ is countably compact since A_2 is a closed subset of countably compact X_2. By the same way, if B_1 is countably compact, then so is B.

(2) Strong CJ-space

The "if" part

Assume that X is a strong CJ-space, it follows by Proposition (2.1.4) that X is CJ-space, and thus X_1 or X_2 is countably compact. Suppose that X_1 is countably compact, it follows by Proposition (2.1.7) that X_1 is strong CJ-space, so it remains to show that X_2 is strong CJ-space. Let $K_2 \subseteq X_2$ be countably compact. Define $K = K_2 \cup X_1$, then K is a countably compact subset of X which is strong CJ-space, so there exists a closed countably compact subset L of X such that $K \subset L$ and $X \backslash L$ is connected. Let $L_2 = L \cap X_2$, then $L_2 \subseteq X_2$ is countably compact since L_2 is a closed subset of countably compact set L. Also $K_2 \subseteq L_2$ since $K_2 \subseteq K \subset L$ implies that

$K_2 \cap X_2 \subset L \cap X_2$. Also note that $X_1 \subset K \subset L$ which implies $X_2 \backslash L_2 = X \backslash L$, and hens $X_2 \backslash L_2$ is connected.

The "only if" part

Assume that X_1 and X_2 are strong CJ-space and suppose that X_1 is a countably compact and let $K \subset X$ be countably compact. Define $K_2 = (K \cup X_1) \cap X_2$, so K_2 is countably compact since it is a closed subset of the countably compact set $K \cup X_1$, so K_2 is countably compact subset of the strong CJ-space X_2, then there exists a closed countably compact subset L_2 of X_2 such that $K_2 \subseteq L_2$ and $X_2 \backslash L_2$ is connected. Now let $L = L_2 \cup X_1$, then L is a closed countably compact subset of X and $K \subset L$ and $X \backslash L = X_2 \backslash L_2$ is connected. Hence X is strong CJ-space.

Corollary (2.1.21)

Let A a closed subset of a topological space X with ∂A countably compact. If X is a (strong) CJ-space, then so is A.

Proof:

Let $A, cl(X \backslash A)$ be two closed subsets of X with $A \cup cl(X \backslash A) = X$ and $A \cap cl(X \backslash A) = \partial A$ which is countably compact, but X is (strong) CJ-space by hypothesis, it follows by Theorem (2.1.20) that A is (strong) CJ-space.

Corollary (2.1.22)

Let $X = E \cup U$, with E (strong) CJ-space and U is open with countably compact closure. Then X is a (strong) CJ-space.

Proof:

Let $A = X\backslash U$, then A is a closed subset of E with countably compact boundary since $\partial A = \partial U$ which is closed subset of cl(U) which is countably compact by hypothesis, but E is (strong) CJ-space, it follows by Corollary (2.1.21) that A is also (strong) CJ-space. Now we have a closed cover $\{A, cl(U)\}$ of X with $A \cap cl(U) = \partial U$ which is countably compact with A and cl(U) are (strong) CJ-spaces and cl(U) is countably compact, so X is (strong) CJ-space by Theorem (2.1.20).

A space is CJ- space if the intersection of any closed cover of this space is not countably compact, as shown in the following Proposition , while this Proposition is not true in general without the assumption condition, as we see in Remark (2.1.24).

Proposition (2.1.23)

Let $\{X_1, X_2\}$ be a closed cover a topological space X with $X_1 \cap X_2$ non- countably compact. If X_1 and X_2 are CJ-spaces, then X is also CJ-space.

Proof:

Take A, B as two closed subsets of X which are cover X such that $A \cap B$ countably compact, we have to show that A or B is countably compact. Let $A_i = A \cap X_i$ and $B_i = B \cap X_i$ for $(i = 1,2)$, then $\{A_i, B_i\}$ is a closed cover of CJ-space X_i with $A_i \cap B_i = (A \cap B) \cap X_i$ which is a closed subset of $A \cap B$, and thus countably compact, it follows by definition of CJ-space that A_i or B_i

is countably compact. Now we will show that if A_1 is countably compact, then so is A. Note that,

$$X_1 \cap X_2 = (A_1 \cup B_1) \cap (A_2 \cup B_2) \subset (A \cap B) \cup B_2 \cup A_1.$$

Since $A \cap B$ and A_1 are countably compact, so B_2 cannot be countably compact, for if B_2 is countably compact, then the closed subset $X_1 \cap X_2$ must be countably compact which is a contradiction with hypothesis, so A_2 is countably compact, and thus $A = A_1 \cup A_2$ is countably compact. Similarly we can show that, if B_1 is countably compact, then so is B.

Remark (2.1.24)

Proposition (2.1.23) is false without the assumption that $X_1 \cap X_2$ non-countably compact and illustrated in the following example.

For example:

Let us take the usual topological space $\mathbb{R}$ and $\{\mathbb{R}^+, \mathbb{R}^-\}$ as a closed cover of $\mathbb{R}$ with $\mathbb{R}^+ \cap \mathbb{R}^- = \{0\}$ which is countably compact. We know that $\mathbb{R}$ is not CJ-space, therefor Proposition (2.1.23) is not true.

Remark (2.1.25)

If we replace the concept of CJ-space in Proposition (2.1.23) by the concept strong CJ-space, then this Proposition need not be true and the following is the counter example.

For example:

Let us take the usual topological space $\mathbb{R}^2$ and let $E_{n,i}$ be the closed segment of the plane $\mathbb{R}^2$ joining $(n, 0)$ to $(n + 1, 1/i)$. Let

$E_n = \left(\bigcup_{i=1}^{\infty} E_{n,i}\right) \cup ([n, n+1] \times \{0\})$, and let $Y = \bigcup_{n=0}^{\infty} E_n$. Then Y is not strong CJ-space for if $K \subset Y$ is countably compact, then $Y \backslash K$ is not connected. Now let

$$A = (\mathbb{R}^+ \times \{0\}) \cup \bigcup \{E_n : n \geq 0, n \text{ even}\},$$

and

$$B = (\mathbb{R}^+ \times \{0\}) \cup \bigcup \{E_n : n \geq 0, \ n \text{ odd}\}.$$

Then A, B are two closed subsets of $\mathbb{R}^2$ which are cover Y where $A \cap B = \mathbb{R}^+ \times \{0\}$ which is non- countably compact. Now we have to prove that A is strong CJ-space, let

$$A_n = \{(y_1, y_2) \in A : \ y_1 \leq 2n + 1\}.$$

Hence A_n is countably compact and $A \backslash A_n$ is connected for each n, moreover every countably compact $K \subset A$ is a subset of A_n for some n. The proof is similar for B.

Corollary (2.1.26)

Let $\{X_1, X_2\}$ be a closed cover of a topological space X with X_1 and X_2 both CJ-space. Then X is CJ-space if and only if X_1 or X_2 is countably compact or $X_1 \cap X_2$ is non- countably compact.

Proof:

The "if" part

Assume that X is CJ-space and $X_1 \cap X_2$ is countably compact, it follows by definition of CJ-space that X_1 or X_2 is countably compact.

The "only if" part

Assume that X_1 or X_2 is countably compact and $X_1 \cap X_2$ is countably compact, but X_1 and X_2 both CJ-space by hypothesis, it follows by Theorem (2.1.20) that X is CJ-space. If $X_1 \cap X_2$ is non- countably compact, then X is CJ-space by Proposition (2.1.23).

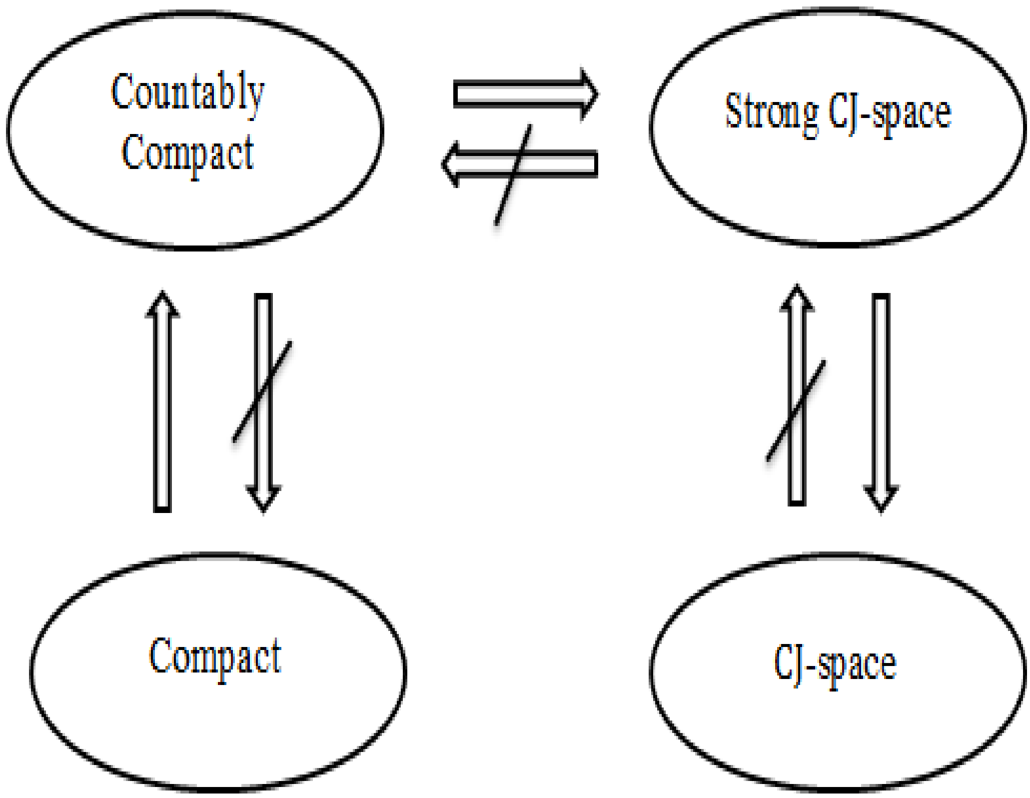
Proposition (2.1.27)

Let A be a closed subset of a strong CJ-space X, then A is a strong CJ-space if it is union of components of X.

Proof:

Let $K \subset A$ be a countably compact, so K is a countably compact subset of X which is strong CJ-space, then there exists a closed countably compact $L \subset X$ such that $K \subset L$ and $X \backslash L$ is connected. If $A \subset L$, then A is countably compact space which in turn strong CJ-space. If $A \not\subset L$, so $X \backslash L \cap A \neq \emptyset$, thus $X \backslash L \subset A$. Now let $L' = L \cap A$, so L' is a countably compact subset of A since it is a closed subset of the countably compact set L, also $K \subset L'$ since $K \subset A$ and $K \subset L$, and $A \backslash L' = X \backslash L$ is connected. Hence A is strong CJ-space.

The following diagram illustrate the relationship among compact, countably compact, strong CJ-space and CJ-space.


Countably
Compact
Strong CJ-space
Compact
CJ-space

§2 SEMI-STRONG CJ-SPACES, WEAK CJ-SPACES AND SEMI-WEAK CJ-SPACES

The purpose of this section is to introduce new types of topological spaces called semi-strong CJ-space, weak CJ-space and semi-weak CJ-space. Many new illustration examples and properties are given. We start with the following definitions.

Definition (2.2.1)

A topological space X is a semi- strong CJ- space if there is a closed countably compact subset L of X for each countably compact $K \subset X$, such that $K \subset L$ and there exists a connected subset C of X with $C \subset X \backslash K$ and $C \cup L = X$.

Definition (2.2.2)

A topological space X is a weak CJ-space if, whenever $\{A, B, K\}$ is a closed covering of X with K countably compact and $A \cap B = \emptyset$, then A or B is countably compact.

Definition (2.2.3)

A topological space X is said to be semi-weak CJ-space if, whenever A and B are closed in X with $A \cap B = \emptyset$ and ∂A, ∂B are countably compact , then A or B is countably compact.

The following Theorem says that the concept of strong CJ-space gives the concepts of semi-strong CJ-space, semi-weak CJ-space and weak CJ-space. While the converse is not true in general, see Remarks (2.2.5) and (2.2.6).

Theorem (2.2.4)

Consider the following properties of a space X.

1. X is a strong CJ-space.

2. X is a semi- strong CJ-space.

3. X is a CJ- space.

4. X is a semi- weak CJ-space.

5. X is a weak CJ-space.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Rightarrow (5)$

Proof:

(1) ⟹ (2)

Let X be a strong CJ- space and let $K \subset X$ be countably compact, then there exists a closed countably compact subset L of X such that $K \subset L$ and $X \backslash L$ is connected by definition of strong CJ-space. Now let $C = X \backslash L$, then C is connected and $C \subset X \backslash K$ since $K \subset L$, and $C \cup L = X$. Hence X is a semi-strong CJ-space.

(2) ⟹ (3)

Let X be a semi- strong CJ- space and let $\{A, B\}$ be a family of closed subset of X covers X with $A \cap B$ countably compact, so there exists a closed countably compact $L \subset X$ such that $A \cap B \subset L$ and there exists a connected subset C of X with $C \subset X \setminus A \cap B$ and $C \cup L = X$ by definition of semi-strong CJ-space. Note that $(A \cap C) \cap (B \cap C) = (A \cap B) \cap C = \emptyset$ since $C \subset X \setminus A \cap B$, and that $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C = X \cap C = C$, so we get a disjoint closed cover $\{A \cap C, B \cap C\}$ of C which is connected, therefor C must be in $A \cap C$ or in $B \cap C$, and thus $C \subset A$ or $C \subset B$. If $C \subset A$, then $C \cap B = \emptyset$, it follows that $B \subset X \setminus C \subset L$ which is countably compact, so B is countably compact. Similarly if $C \subset B$, then A is countably compact. Hence X is CJ-space.

(3) $\Leftrightarrow$ (4)

Let X be any CJ-space and let A, B be two closed sets in X with $A \cap B = \emptyset$ and $\partial A, \partial B$ are countably compact, then A or B is countably compact by Theorem (2.1.16) part 3. Thus X is semi-weak CJ-space.

(4) $\Longrightarrow$ (5)

Suppose that X is a semi-weak CJ-space and let $\{A, B, K\}$ be a closed cover of X with K countably compact and $A \cap B = \emptyset$. Note that $A^c = B \cup (K \setminus K \cap A)$ and $\partial A = \partial A^c$, then $\partial A = \partial\big(B \cup (K \setminus K \cap A)\big)$, so $\partial A \subset K \cap A \subset K$, similarly we can prove that $\partial B \subset K$, and thus ∂A and ∂B are countably compact, it follows by (4) that A or B is countably compact. Hence X is weak CJ- space.

Remark (2.2.5)

A semi- strong CJ-space need not be strong CJ-space.

For example:

Let us take the usual topological space $\mathbb{R}^2$ and let $E_{n,i}$; $i \geq 1$ and $n \geq 0$ be the closed segment of the plane $\mathbb{R}^2$ joining $(n,0)$ to $(n+1,1/i)$. Let $E_n = \left(\bigcup_{i=1}^{\infty} E_{n,i}\right) \cup ([n,n+1] \times \{0\})$, and let $Y = \bigcup_{n=0}^{\infty} E_n$. Then Y is not strong CJ-space for if $K \subset Y$ is closed countably compact, then $Y\backslash K$ is not connected. But Y is semi-strong CJ-space, to prove that, let $L_n = \{(y_1,y_2) \in Y : y_1 \leq n\}$ and $C_n = \mathrm{cl}(Y\backslash L_n)$. Not that L_n is closed countably compact and C_n is connected and $L_n \cup C_n = Y$ (for each n). Now let K be a countably compact subset of Y and pick n such that $K \subset L_{n-1}$, then $K \subset L_n$ and $C_n \subset Y\backslash K$.

Remark (2.2.6)

A CJ- space need not be semi- strong CJ-space.

For example:

Consider the $\mathrm{Odd} - \mathrm{Even}$ topology defined on the set of natural numbers $\mathbb{N}$. The only countably compact subsets of $\mathbb{N}$ are the finite subsets, so if we take a closed cover $\{A, B\}$ of $\mathbb{N}$ with $A \cap B$ countably compact, that is mean $A \cap B$ is finite set and since the intersection of any two infinite sets in this space must be an infinite set, so A or B must be finite, that is mean A or B is countably compact. Hence $\mathbb{N}$ is CJ-space.

But $\mathbb{N}$ is not semi-strong CJ-space since every countably compact subset of $\mathbb{N}$ is finite and every infinite subset of $\mathbb{N}$ is non- connected, so if we take a countably compact subset K of $\mathbb{N}$ and a countably compact subset

L of $\mathbb{N}$ such that $K \subset L$ and a connected subset C of $\mathbb{N}$ with $C \subset \mathbb{N} \backslash K$ and $C \cup L = \mathbb{N}$, then C must be infinite which is a contradiction.

We can get a semi-weak CJ-space, weak CJ-space and semi strong CJ-space from given spaces that have some given properties as shown in Propositions (2.2.7), (2.2.8) and (2.2.10). We can use these Propositions to show that the converse of last directions of Theorem (2.2.4) is not true in general.

Proposition (2.2.7)

If X is a CJ- space (semi- weak CJ- space) and $Z = X \cup \{z_\circ\}$, then Z is a CJ- space (semi- weak CJ- space).

Proof:

Let A, B be two closed sets in Z such that $A \cap B = \emptyset$ and $\partial A, \partial B$ are countably compact, then $z_\circ \notin A$ or $z_\circ \notin B$. Suppose that $z_\circ \notin B$ and let $E = cl(X \backslash B)$, then $\{B, E\}$ is a closed cover of X with $E \cap B = \partial B$ which is countably compact, so B or E is countably compact since X is CJ-space (semi-weak CJ-space). But $A \subset E \cup \{z_\circ\}$, so A or B is countably compact, and thus X is a CJ- space (semi-weak CJ-space).

Proposition (2.2.8)

Let $\{X_1, X_2\}$ be a closed cover of a topological space X such that $X_1 \cap X_2$ non- countably compact. If X_1 and X_2 are weak CJ- spaces, then so is X.

Proof:

Let $\{A, B, K\}$ be a closed cover of X with $A \cap B = \emptyset$ and K is countably compact. To prove A or B is countably compact, let $A_i = A \cap X_i$ and $B_i = B \cap X_i$ and $K_i = K \cap X_i$, for $(i = 1,2)$. Then $\{A_i, B_i, K_i\}$ is a closed cover of X_i with $A_i \cap B_i = \emptyset$ and K_i is countably compact. Now by using the fact saying that X_1 is weak CJ-space, we get A_1 or B_1 is countably compact. Suppose that B_1 is countably compact, we claim that B_2 is also countably compact, for if B_2 is not countably compact, so A_2 must be countably compact since X_2 is weak CJ-space, it follows that $C = A_2 \cup B_1 \cup K$ is countably compact, but $X_1 \cap X_2$ is a closed subset of C, so $X_1 \cap X_2$ must be countably compact which is a contradiction. Thus $B = B_1 \cup B_2$ is countably compact. Similarly we can prove that A is countably compact whenever A_1 is countably compact.

Remark (2.2.9)

A weak CJ- space need not be semi- weak CJ-space.

For example:

Let $X = \mathbb{R} \times [0,1)$ and let $Z = X \cup \{(-1,1), (1,1)\}$. To see that Z is a weak CJ-space, let $Z_1 = \{(s,t) \in Z: s \leq 0\}$ and $Z_2 = \{(s,t) \in Z: s \geq 0\}$, then $\{Z_1, Z_2\}$ is a closed cover of Z, and $Z_1 \cap Z_2 = \{0\} \times [0,1)$ which is non- countably compact, but Z_1 and Z_2 are both semi- weak CJ- space since they are homeomorphic to the space Z of Remark (2.2.8), and thus they are weak CJ-spaces, therefor Z is weak CJ- space from Proposition (2.2.8). We have to show that Z is not a semi-weak CJ-space, let

$A = \{(s,t) \in Z: s \leq -1\}$ and $B = \{(s,t) \in Z: s \geq 1\}$, then A and B are disjoint closed subsets of Z with countably compact boundaries, but neither A nor B is countably compact.

Proposition (2.2.10)

Let $\{X_1, X_2\}$ be a closed cover of a topological space X such that $X_1 \cap X_2$ non- countably compact. If X_1 and X_2 are semi-strong CJ-spaces, then so is X.

Proof:

Let $K \subset X$ be a countably compact and let $K_i = K \cap X_i$, then K_i is a closed subset of K, it follows by Proposition (1.1.8) that K_i is a countably compact subset of the semi- strong CJ-space X_i, for $i = 1,2$, so there exists a closed countably compact subset L_i of X_i such that $K_i \subset L_i$ and there exists a connected subset C_i of X_i such that $C_i \subset X_i \backslash K_i$ and $L_i \cup C_i = X_i$, (for $i = 1,2$)by definition of semi- strong CJ-space. Now let $L = L_1 \cup L_2$ and $C = C_1 \cup C_2$, so L is a closed countably compact subset of X with $K \subset L$ and $C \cup L = X$ and $C \subset X \backslash K$. It remains to show that C is connected, we need only cheek that $C_1 \cap C_2 \neq \emptyset$ since C_1 and C_2 are connected. Note that $X_1 \cap X_2 \backslash L \neq \emptyset$, for if $X_1 \cap X_2 \backslash L = \emptyset$, then $X_1 \cap X_2$ is a closed subset of L which is countably compact, so $X_1 \cap X_2$ is countably compact which is a contradiction. Also we have $X_i \backslash L \subseteq X_i \backslash L_i \subseteq C_i$, , so $(X_1 \cap X_2) \backslash L \subseteq C_1 \cap C_2$, and thus $C_1 \cap C_2 \neq \emptyset$. Hence $C = C_1 \cup C_2$ is connected. Therefor X is a semi- strong CJ-space.

Theorem (2.2.11)

The concepts of CJ-space and weak CJ-space are equivalent if the space X is locally compact.

Proof:

A CJ- space is a weak CJ-space from Theorem (2.2.4). Now if X is a locally compact weak CJ- space, we must prove that X is CJ- space. Let $\{A, B\}$ be a closed cover of X with $A \cap B$ countably compact. But X is locally compact so $A \cap B \subset \mathrm{Int}(k)$, for some compact $K \subset X$. Let $A^* = A \setminus \mathrm{Int}(K)$ and $B^* = B \setminus \mathrm{Int}(K)$, then $\{A^*, B^*, K\}$ is a closed cover of X with K compact, and thus countably compact, and $A^* \cap B^* = \emptyset$, it follows by definition of weak CJ-space, that A^* or B^* is countably compact, then $A^* \cup K$ or $B^* \cup K$ is countably compact. But A and B are closed subsets of $A^* \cup K$ and $B^* \cup K$ respectively, so A or B is countably compact by Proposition (1.1.8). Hence X is a CJ-space.

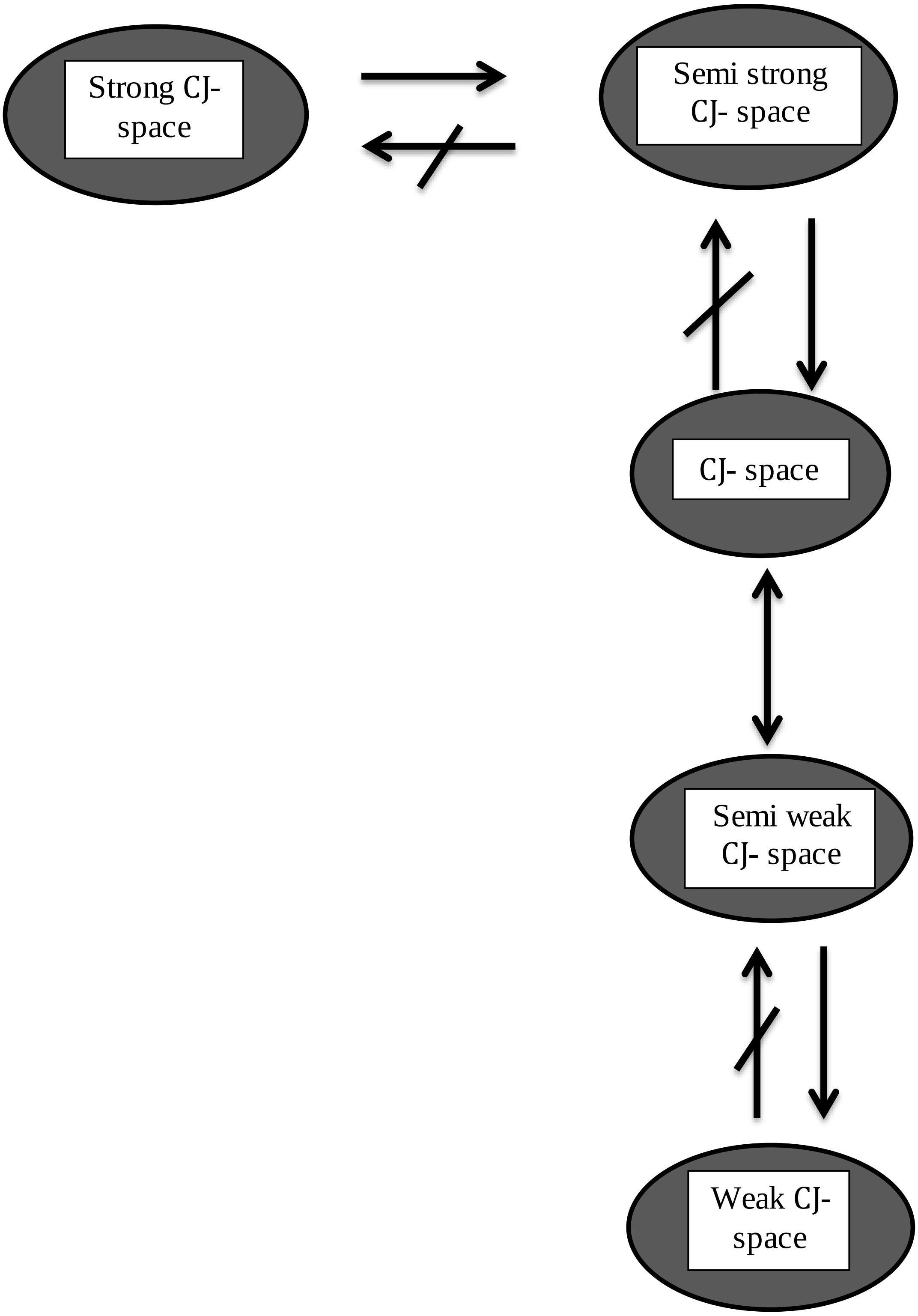
Theorem (2.2.12)

If X is a topological space and $X \times Y$ is a k-space for each k-space Y, then X is a weak CJ-space if and only if it is a CJ-space.

Proof:

Follows from Theorems (1.1.40) and (2.2.11).

The following diagram illustrate the relationship among all types of new spaces given in the previous sections.



§3 FUNCTIONAL CHARACTERIZATIONS OF CJ-SPACES

In this section we discuss some types of functions and indicate which ones save that the property of being CJ-space and which does not save. Also we introduce new types of functions, we begin with the definitions of these types, which are used to get a characterization for CJ-spaces as in Theorem(2.3.9).

Definition (2.3.1)

A continuous function $f:(X,\tau) \to (Y,\tau')$ is said to be countably perfect if it is closed and $f^{-1}(B)$ is countably compact subset of X for every countably compact subset B of Y. That is a function f is countably perfect if it is closed and countably compact function.

Remark(2.3.2)

The concepts of perfect function and countably perfect function are independent.

Definition (2.3.3)

A function $f:(X,\tau) \to (Y,\tau')$ is said to be boundary countably perfect if it is closed and $\partial(f^{-1}(y))$ is countably compact subset of X for every $y \in Y$.

Remark(2.3.4)

Every boundary perfect function is boundary countably perfect since every compact space is countably compact.

It is known that continuity preserves compactness, connectedness and the property of being countably compact. While the property of being CJ-space does not preserved by continuous image, as shown in the following remark.

Remark (2.3.5)

The continuous image of CJ-space is not CJ-space in general.

For example:

Let $f: (\mathbb{N}, \tau) \to (\mathbb{N}, \tau')$ such that $f(2k) = f(2k-1) = k;\ k \in \mathbb{N}$, where τ is the $\mathrm{Odd} - \mathrm{Even}$ topology, (see example of Remark (2.1.5)), and $\tau' = D$ the discrete topology. Clear that f is continuous and onto function and $(\mathbb{N}, \tau)$ is CJ-space, but $(\mathbb{N}, \tau')$ is not CJ-space.

The following proposition gives the condition of function which guaranties that the image of a CJ-space is CJ-space.

Proposition (2.3.6)

If $f: (X, \tau) \rightarrow (Y, \tau')$ is countably compact function from a CJ-space X onto a topological space Y, then Y is CJ-space.

Proof:

Let A, B be two closed subsets of Y with $Y = A \cup B$ and $A \cap B$ countably compact, then $\{f^{-1}(A), f^{-1}(B)\}$ is a closed cover of X since f is continuous and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$ by Proposition (1.1.13(3)). But $f^{-1}(A \cap B)$ is countably compact since f is countably compact, so $f^{-1}(A)$ or $f^{-1}(B)$ is countably compact by Definition (2.1.1). It follows that $f\left(f^{-1}(A)\right)$ or $f\left(f^{-1}(B)\right)$ is countably compact since f is continuous and by Proposition (1.1. 20), then A or B is countably compact since f is surjective. Hence Y is CJ-space.

The following proposition guaranties that the inverse image of a CJ-space is a CJ-space also, if the function is countably perfect and monotone function.

Proposition (2.3.7)

Let $f : X \rightarrow Y$ be a countably perfect, monotone function from a topological space X onto a CJ-space Y, then X is also CJ-space.

Proof:

Let $\{A, B\}$ be a closed cover of X with $A \cap B$ countably compact, then $\{f(A), f(B)\}$ is a closed cover of Y since f is closed, and $f(A) \cap f(B) = f(A \cap B)$ by Lemma (1.2.10). But $f(A \cap B)$ is countably compact by Proposition (1.1.20), so $f(A)$ or $f(B)$ is countably compact since Y is CJ-space. Then $f^{-1}f(A)$ or $f^{-1}f(B)$ is countably compact since f is countably perfect, it follows by Proposition (1.1.8), that A or B is countably compact since A and B are closed subsets of $f^{-1}f(A)$ and $f^{-1}f(B)$ respectively by Proposition (1.1.13(1)). Hence X is CJ-space.

From previous Proposition, we can get the following Proposition.

Proposition (2.3.8)

The property of being "CJ-space" is a topological property.

Proof:

Let X and Y be two homeomorphic spaces. First suppose that X is CJ-space, to prove Y is CJ-space. Let $f: X \to Y$ be a homeomorphism function, and let A, B be two closed subsets of Y with $Y = A \cup B$ and $A \cap B$ countably compact, then $\{f^{-1}(A), f^{-1}(B)\}$ is a closed cover of X since f is continuous. From number three of Proposition (1.1.13) we have $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$ which is countably compact since f^{-1} is continuous and by Proposition (1.1.8). It follows by Definition (2.1.1), that $f^{-1}(A)$ or $f^{-1}(B)$ is countably compact. Again by Proposition (1.1.8) we can get $f(f^{-1}(A))$ or $f(f^{-1}(B))$ is countably compact since f is continuous.

From number two of Proposition (1.1.13) we have A or B is countably compact since f is surjective. Thus Y is a CJ- space. Similarly we can prove that X is CJ-space, when Y is so.

Proposition (2.3.9)

If a topological space X is CJ-space, then every boundary countably perfect function from X onto a non- countably compact space Y is a quasi-perfect.

Proof:

Suppose that (X,τ) is a CJ-space and (Y,τ') is a non- countably compact and $f:(X,\tau)\rightarrow(Y,\tau')$ is a closed boundary countably perfect function. We have to show that f is quasi -perfect, let $y\in Y$, then $f^{-1}(y)$ is a subset of the CJ-space X with countably compact boundary, it follows by Theorem (2.1.16) that either $f^{-1}(y)$ or $cl(X\backslash f^{-1}(y))$ is countably compact. But $cl(X\backslash f^{-1}(y))$ is not countably compact, for if $cl(X\backslash f^{-1}(y))$ is countably compact, then $Y=\{y\}\cup f(cl(X\backslash f^{-1}(y))$ is countably compact which is a contradiction, so $f^{-1}(y)$ is countably compact. Hence f is a quasi-perfect.

Proposition (2.3.10)

If every boundary countably perfect function from a topological space X onto a non- countably compact space Y is a countably perfect, then X is CJ-space.

Proof:

Suppose that every boundary countably perfect function from X onto a non-countably compact space Y is countably perfect. To prove that X is CJ-space, let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ countably compact. Let $Y = X/B$ and $f: X \to Y$ be the quotient function and let $y_\circ = f(B)$, then f is boundary countably perfect since it is closed and $\partial\left(f^{-1}(y)\right)$ is countably compact for each $y \in Y$, since if $y = y_\circ$,then $\partial\left(f^{-1}(y)\right)$ is a closed subset of $A \cap B$, and if $y \neq y_\circ$, then $\partial\left(f^{-1}(y)\right)$ is a one-element set. Now if Y is non- countably compact, then f is countably perfect by hypothesis and thus $B = f^{-1}(y_\circ)$ is countably compact. If Y is countably compact, so f(A) is countably compact since it is closed subset of Y. On the other hand we have $f\big|A: A \to f(A)$ is countably perfect since it is closed function and its fibers are either one-element sets or equal to $A \cap B$, so $A = f^{-1}(f(A))$ is countably compact. Therefor X is CJ-space.

Proposition(2.3.11) guaranties that the image of a strong CJ- space is strong CJ- space if the function is open countably perfect.

Proposition (2.3.11)

Let $f: (X, \tau) \to \left(Y, \tau'\right)$ be an open countably perfect function from a strong CJ-space X onto a topological space Y, then Y is also strong CJ-space.

Proof:

To prove Y is strong CJ-space, let $K \subset Y$ be a countably compact. Let $K' = f^{-1}(K)$ then $K' \subset X$ is countably compact, so there exist a closed countably compact $L' \subset X$ such that $K' \subset L'$ and $X \backslash L'$ is connected. Let $L = Y \backslash f(X \backslash L')$. Since L' is a closed in X, so $X \backslash L'$ is open, but f is open function, then $f(X \backslash L')$ is open subset of Y, then L is a closed subset of Y such that $K \subset L$ and $Y \backslash L = f(X \backslash L')$ is connected by Theorem (1.2.2). Also $f^{-1}(L)$ is a closed subset of X since f is continuous and $f^{-1}(L) \subset L'$, so $f^{-1}(L)$ is countably compact, and thus L is also countably compact.

The following proposition gives a characterization for semi-weak CJ- spaces, boundary countably perfect function.

Theorem (2.3.12)

For any space X, the following conditions are equivalent:

a) X is a semi- weak CJ-space

b) If $f: X \to Y$ is boundary countably perfect, then the fiber $f^{-1}(y)$ is non-countably compact for at most one $y \in Y$.

Proof:

(a) $\Longrightarrow$ (b)

Suppose that X is a semi-weak CJ-space and $y_1 \neq y_2$ in Y, and let $A_i = f^{-1}(y_i)$ (for $i = 1,2$). Then A_1 and A_2 are closed subsets of X with $A_1 \cap A_2 = \emptyset$ and $\partial A_1, \partial A_2$ are countably compact since f is boundary countably perfect, so A_1 or A_2 is countably compact by definition of semi-weak CJ- space.

(b) ⟹ (a)

Assume that A_1 and A_2 are two closed subsets of X with $A_1 \cap A_2 = \emptyset$ $\partial A_1, \partial A_2$ are countably compact. Define a relation R on X such that $x\,R\,y \Leftrightarrow x, y \in A_1$ or $x, y \in A_2$. Then

$$[x] = \begin{Bmatrix} A_1 & \text{if } x \in A_1 \\ A_2 & \text{if } x \in A_2 \\ \{x\} & \text{if } x \notin A_1 \text{ and } x \notin A_2 \end{Bmatrix}.$$

Let Y be the quotient space of X with respect to the relation R, and let $f: X \to Y$ be the quotient function, so f is a closed, continuous and onto map. Now to show that f is boundary- countably perfect, it is sufficient to prove that $\partial\left(f^{-1}(y)\right)$ is countably compact for each $y \in Y$. Let $y \in Y$, then

$$f^{-1}(y) = \begin{Bmatrix} A_1 & \text{if } x \in A_1 \\ A_2 & \text{if } x \in A_2 \\ \{y\} & \text{if } x \notin A_1 \text{ and } x \notin A_2 \end{Bmatrix}.$$

But $\partial A_1, \partial A_2$ are countably compact by hypothesis and $\partial\{y\}$ is also countably compact, so f is boundary-countably perfect and hence A_1 or A_2 is countably compact by (b).

CHAPTER THREE

ALMOST CJ- SPACE
AND
ALMOST STRONG CJ- SPACE

INTRODUCTION

There are three sections in this chapter. In section one we introduced two new concepts called almost CJ- space and almost strong CJ-space, where a topological space X is an almost CJ- space if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is countably compact. And it is an almost strong CJ-space if each compact $K \subset X$ is contained in a countably compact $L \subset X$ such that $X \setminus L$ is connected. Also we give the properties of these new spaces and their relationship with each other, as in Proposition **(3.1.5)** and Remark **(3.1.6)**. Also, we have established the relations between these two spaces and other known spaces, see Propositions **(3.1.3), (3.1.4), (3.1.7), (3.1.8)** and **(3.1.10)**.

The second section is concerned with the notions of almost semi-strong CJ- space, almost weak CJ- space and almost semi- weak CJ- space. The relation between these spaces is investigated, see Theorem **(3.2.7)**. We gave diverse examples about the opposite directions of Theorem **(3.2.7)**, see Remarks **(3.2.8), (3.2.9), (3.2.13)** and **(3.2.15)**. Also, we gave a necessary condition to make every almost CJ-space is an almost weak CJ- space, see Theorem **(3.2.10)**.

In section three we used the concepts of boundary- perfect, countably perfect and boundary- countably perfect function to discuss some properties and theorems concerning spaces studied in the previous sections.

§1 ALMOST CJ-SPACE AND ALMOST STRONG CJ-SPACE

As a generalization of the concepts of CJ-space and strong CJ-space, we introduce the concepts of almost CJ-space and almost strong CJ-space. We give many characterizations and many properties of these concepts.
We begin with the following definition.

Definition (3.1.1)

A space X is said to be almost CJ-spaces if, whenever $\{A, B\}$ is a closed cover of X such that $A \cap B$ compact, then A or B is countably compact.

Definition (3.1. 2)

A space X is said to be almost strong CJ-space if each compact $K \subset X$ is included in a countably compact $L \subset X$ such that $X \backslash L$ is connected.

Proposition(3.1.3)

Every CJ-space is an almost CJ-space.

Proof:

Let X be any CJ-space and let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ compact, so $A \cap B$ is countably compact, it follows by definition of CJ-space that A or B is countably compact. Hence X is an almost CJ-space.

Proposition(3.1.4)

Every strong CJ- space is an almost strong CJ-space.

Proof:

Let X be any strong CJ-space and let $K \subset X$ be compact, and thus countably compact, it follows by definition of strong CJ-space that $K \subset L$ for some countably compact subset L of X with $X \setminus L$ is connected. Hence X is an almost strong CJ- space.

Proposition (3.1. 5)

Every almost strong CJ- space is an almost CJ-space.

Proof:

Suppose that X is an almost strong CJ-space, and let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ compact, so there exists a countably compact $L \subset X$ such that $A \cap B \subset L$ and $X \setminus L$ is connected. Note that $\{A \cap X \setminus L, B \cap X \setminus L\}$ is a disjoint closed cover of $X \setminus L$ which is connected, so $X \setminus L$ must be in $A \cap X \setminus L$ or in $B \cap X \setminus L$, it follows that $X \setminus L \subset A$ or $X \setminus L \subset B$, then $X \setminus A \subset L$ or $X \setminus B \subset L$, it follows that $A \subset L$ or $B \subset L$ because $A \cap B \subset L$, but A and B are closed sets and L is countably compact set, therefore A or B is countably compact. Hence X is an almost CJ-space.

Remark(3.1.6)

The converse of Proposition (3.1.5) is not true in general.

For example:

The topological space $\mathbb{N}$ with the $\mathrm{Odd} - \mathrm{Even}$ topology is CJ-space, as we saw in the example of Remark (2.1.9), so it is an almost CJ-space by Proposition (3.1.3). But $\mathbb{N}$ is not almost strong CJ-space since every countably compact subset of $\mathbb{N}$ is finite and hence its complement is infinite and every infinite subset of $\mathbb{N}$ is non-connected.

Proposition (3.1. 7)

Every J- space is an almost CJ- space.

Proof:

Let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ compact, so from definition of J-space we can get that A or B is compact, so A or B is countably compact by Proposition (1.1.9). Hence X is an almost CJ-space.

Proposition (3.1. 8)

Every strong J-space is an almost strong CJ-space.

Proof:

Let $K \subset X$ be compact, it follows by definition of strong J-space that there exists a compact $L \subset X$ such that $K \subset L$ and $X \backslash L$ is connected, but every compact set is countably compact, so L is countably compact. Hence X is an almost strong CJ-space.

Remark (3.1.9)

The converses of Propositions (3.1.3), (3.1.4), (3.1.7) and (3.1.8) are not true in general.

Proposition (3.1.10)

Every countably compact space is an almost strong CJ-space.

Proof:

Let X be a countably compact space and let $K \subset X$ be compact, then K is contained in a countably compact set (X itself) with connected complement. Hence X is an almost strong CJ-space.

Remark(3.1.11)

The converse of Proposition (3.1.10) is not true in general.

For example:

$\mathbb{R}^+$ as a subspace of $\mathbb{R}$ with the usual topology is an almost strong CJ-space by Propositions (1.3.6) and (3.1.8). But it is not countably compact.

Corollary (3.1. 12)

Every compact space is an almost strong CJ-space.

Proof:

Follows from Propositions (1.1.9) and (3.1.10).

Corollary (3.1. 13)

Every countably compact space is an almost CJ- space.

Proof:

Follows from Propositions (3.1.10) and (3.1.5).

Corollary (3.1. 14)

Every compact space is an almost CJ-space.

Proof:

Follows from Corollary (3.1. 12) and Proposition (3.1. 5).

Remark (3.1.15)

The converses of Corollaries (3.1.12), (3.1.13) and (3.1.14) are not true in general.

For example:

$\mathbb{R}^+$ as a subspace of $\mathbb{R}$ with the usual topology is an almost CJ-space and almost strong CJ-space, but it is neither compact nor countably compact.

Examples (3.1. 16)

A. The real line with usual topology is not almost CJ-space, since $\{\mathbb{R}^+, \mathbb{R}^-\}$ is a closed cover of $\mathbb{R}$ with $\mathbb{R}^+ \cap \mathbb{R}^- = \{0\}$ which is compact, but neither $\mathbb{R}^+$nor $\mathbb{R}^-$ is countably compact.

B. The discrete topology defined on any infinite set X, is not almost CJ-space in general. For example, let us take the space $(\mathbb{N}, D)$, where $\mathbb{N}$ is the set of all natural numbers, this space is not almost CJ-space, since $\{E^+, O^+\}$ is a closed cover of $\mathbb{N}$ with $E^+ \cap O^+ = \emptyset$ which is compact, but neither E^+ nor O^+ is countably compact.

C. The plane $\mathbb{R}^2$ with the usual topology is an almost strong CJ- space, for if $K \subset \mathbb{R}^2$ is compact, then a closed ball $L \subset \mathbb{R}^2$ such that $K \subset L$ is compact, and thus countably compact and $\mathbb{R}^2 \backslash L$ is connected.

D. The indiscrete topology defined on any nonempty set X, is an almost Strong CJ-space, since it is countably compact.

For any closed non- countably compact subset of any topological space, we can obtain two disjoint subsets one of them compact and the other non-countably compact as shown in the next Lemma.

Lemma(3.1.17)

If B is a closed non-countably compact subset of any topological space X and $C \subset B$ is compact, consequently there is a non-countably compact closed $D \subset B$ with $D \cap C = \emptyset$.

Proof:

Let $\mathcal{G}$ be a countably open cover of B with no finite subcover, and let $C \subset B$ be a compact, then $\mathcal{G}$ is an open cover of C. Pick a finite $\mathfrak{J} \subset \mathcal{G}$ covering C. Then $D = B \setminus \cup\mathfrak{J}$ is a closed non-countably compact subset of B with $D \cap C = \emptyset$.

Theorem (3.1.18)

Let X be any topological space, then the following conditions are equivalent:

1. X is an almost CJ-space,

2. For any closed set $A \subset X$ with compact boundary, $cl(A)$ or $cl(X \setminus A)$ is countably compact,

3. If A and B are closed sets in X with $A \cap B = \emptyset$ and ∂A or ∂B compact, then A or B is countably compact,

4. If $K \subset X$ is compact, and if $\mathcal{W}$ is a disjoint open cover of $X \setminus K$,then there exists $W \in \mathcal{W}$ such that $X \setminus W$ is countably compact.

5. Same as (4), but with card $\mathcal{W} = 2$.

Proof:

(1)$\Longrightarrow$ (2)

Let $A \subset X$ such that ∂A is compact. Note that $\{cl(A), cl(X\backslash A)\}$ is a closed cover of X with $\partial A = cl(A) \cap cl(X\backslash A)$ is compact, so $cl(A)$ or $cl(X\backslash A)$ is countably compact by definition of almost CJ-space.

(2)$\Longrightarrow$ (3)

Let A and B be disjoint closed subsets of X and suppose that ∂A is compact, it follows by (2) that $cl(A)$ or $cl(X\backslash A)$ is countably compact. But $cl(A) = A$, and B is a closed subset of $cl(X\backslash A)$, so A or B is countably compact.

(3) $\Longrightarrow$ (1)

Let $\{A, B\}$ be a closed cover of X wit $A \cap B$ is compact, we have to show that A or B is countably compact. Suppose that B is non-countably compact and since $A \cap B \subset B$ is compact, so by lemma (3.1.17) there exists a non-countably compact closed $D \subset B$ such that $D \cap (A \cap B) = \emptyset$, it follows that $D \cap A = \emptyset$, and ∂A compact since it is a closed subset of $A \cap B$, so we get by (3) A or D is countably compact, but D is non- countably compact. Hence A must be countably compact.

(4) $\Longrightarrow$ (5)

Clear.

(5) ⟹ (4)

Let $K \subset X$ be compact and let $\mathcal{W}$ be a disjoint open cover of $X \setminus K$. To show that $X\backslash W$ is countably compact for some $W \in \mathcal{W}$ we shall follow three demarches.

First, we prove that if U is open subset of X containing K, then $\mathcal{W}' = \{W \in \mathcal{W} : W \nsubseteq U\}$ is finite. Suppose that it is not finite, then $\mathcal{W} = W_1 \cup W_2$ with $W_1 \cap W_2 = \emptyset$ and $W_1 \cap \mathcal{W}'$ and $W_2 \cap \mathcal{W}'$ both finite. Let $V_1 = \cup W_1$ and $V_2 = \cup W_2$, then $\{V_1, V_2\}$ is a disjoint open cover of $X\backslash K$, so by (5) $X\backslash V_1$ or $X\backslash V_2$ is countably compact but $V_1 \subseteq X\backslash V_2$ and $V_2 \subseteq X\backslash V_1$ since V_1 and V_2 are disjoint. It follows that $cl(V_1) \subseteq cl\,(X\backslash V_2) = X\backslash V_2$ and $cl(V_2) \subseteq cl\,(X\backslash V_1) = X\backslash V_1$, so we get $cl(V_1)$ or $cl(V_2)$ is countably compact by Proposition (1.1.8). Suppose that $cl(V_1)$ is countably compact, then $C = cl(V_1)\backslash U$ is countably compact. Now let $\mathcal{W}'_1 = W_1 \cap \mathcal{W}'$, then $\mathcal{W}_1'$ covers C and each $W \in \mathcal{W}'$ intersects C, so C is not countably compact since $\mathcal{W}_1'$ is infinite and disjoint, which is a contradiction. Hence $\mathcal{W}'$ is finite.

Second, we prove that if $cl(W)$ is countably compact, $\forall\, W \in \mathcal{W}$, then X is countably compact. Let $\mathcal{V}$ be a countably open cover of X, consequently $\mathcal{V}$ is a countably open cover of K, which is compact, so $\mathcal{V}$ has a finite subcover $\mathcal{F}$ covers K. Let $U = \bigcup \mathcal{F}$, by step one we get a finite family $\mathcal{W}' = \{W \in \mathcal{W} : W \nsubseteq U\}$, so $\bigcup\{cl(W) : W \in \mathcal{W}'\}$ is countably compact and since $\mathcal{V}$ is a countably open cover of it therefore it is covered by some finite $\mathcal{E} \subset \mathcal{V}$. But $\bigcup \mathcal{E} \subset \mathcal{V}$ is finite and covers X, so X is countably compact.

Finally, let us show that $X\backslash W$ is countably compact for some $W \in \mathcal{W}$. If $cl(W)$ is countably compact for all $W \in \mathcal{W}$, then X is countably compact by step (2) and since $X\backslash W$ is a closed subset of X, so $X\backslash W$ is countably compact. Suppose that there exists $W_0 \in \mathcal{W}$ such that $cl(W_0)$ is not countably compact.

Let $W^* = \cup\{W \in \mathcal{W} : W \neq W_0\}$, then, $\{W_0, W^*\}$ is a disjoint open cover of $X\backslash K$, so $X\backslash W_0$ or $X\backslash W^*$ is countably compact, by (5). If $X\backslash W^*$ is countably compact, and since $cl(W_0)$ is a closed subset of $X\backslash W^*$, so $cl(W_0)$ is countably compact which is a contradiction, so $X\backslash W^*$ is not countably compact, it follows that $X\backslash W_0$ is countably compact.

(5)⟹ (1)

Let $\{A, B\}$ be a closed cover of X with $A \cap B$ compact , then $X\backslash A, X\backslash B$ are open subsets of X such that $X\backslash A \cap B \subseteq X\backslash A \cup X\backslash B$ and $X\backslash A \cap X\backslash B = \emptyset$, then by (5) we get $X\backslash(X\backslash A)$ or $X\backslash(X\backslash B)$ is countably compact, that is A or B is countably compact. Hence X is CJ-space.

(1)⟹ (5)

Let $K \subset X$ be compact and let W_1, W_2 be two open subsets of X such that $X\backslash K \subseteq W_1 \cup W_2$ and $W_1 \cap W_2 = \emptyset$, then $X\backslash W_1, X\backslash W_2$ are closed subsets of X such that $X\backslash W_1 \cup X\backslash W_2 = X$ and $X\backslash W_1 \cap X\backslash W_2 = X\backslash(W_1 \cup W_2)$ compact, because $X\backslash K \subset W_1 \cup W_2$,and so $X\backslash(W_1 \cup W_2) \subset K$ which is compact and by Proposition (1.1.5). But X is almost CJ-space, so $X\backslash W_1$ or $X\backslash W_2$ is countably compact.

Theorem (3.1.19)

The concepts of almost CJ-space and almost strong CJ-space are equivalent if the space X is locally connected.

Proof:

If X is an almost strong CJ-space, so it is an almost CJ-space by Proposition (3.1.5). So we suppose that X is almost CJ-space, and let $K \subset X$ be compact, so there exists an open cover $\mathcal{W}$ of $X \backslash K$ with disjoint members such that each $W \in \mathcal{W}$ is connected since X is locally connected. It follows by Theorem (3.1.18) that there is $W_0 \in \mathcal{W}$ such that $X \backslash W_0$ is countably compact. Pick $L = X \backslash W_0$, then L is countably compact and $K \subset L$ and $X \backslash L$ is connected. Hence X is almost strong CJ- space.

The following theorem illustrate whether the intersection of two almost CJ- space is an almost CJ-space.

Theorem (3.1. 20)

Let $\{X_1, X_2\}$ be a closed cover of a topological space X such that $X_1 \cap X_2$ compact. Consequently X is an almost CJ-space if and only if X_1 and X_2 are almost CJ-spaces and X_1 or X_2 is countably compact.

Proof:

The "if" part

Assume that X is an almost CJ-space, then X_1 or X_2 is countably compact by definition of almost CJ-space. Suppose X_2 is countably compact, then X_2 is certainly almost CJ-space, so it remains to show that X_1 is an almost CJ-space. Let $\{A, B\}$ be a closed cover of X_1 with $A \cap B$ compact, then $\{A, B \cup X_2\}$ is a closed cover of X with $A \cap (B \cup X_2) = (A \cap B) \cup A \cap X_2$ which is compact, so A or $B \cup X_1$ is countably compact since X is almost CJ-space. But B is a closed subset of $B \cup X_1$, so A or B is countably compact.

The "only if" part

Assume that X_1 and X_2 are almost CJ- spaces and suppose that X_2 is countably compact, we have to show that X is an almost CJ-space. Let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ compact. Now let $A_i = A \cap X_i$ and $B_i = B \cap X_i$ $(i = 1,2)$, therefore $\{A_1, B_1\}$ is a closed cover of X_1 with $A_1 \cap B_1 = (A \cap X_1) \cap (B \cap X_1) = (A \cap B) \cap X_1 \subset A \cap B$ which is compact, so A_1 or B_1 is countably compact because X_1 is almost CJ-space. If A_1 is countably compact, then $A = A_1 \cup A_2$ is countably compact since A_2 is a closed subset of countably compact X_2. Similarly, if B_1 is countably compact, then so is B.

Corollary (3.1.21)

Let A be a closed subset of a topological space X with ∂A is compact. If X is an almost CJ-space, then A is also almost CJ-space.

Proof:

Let $A, cl(X \backslash A)$ be two closed subsets of X with $X = A \cup B$ and $A \cap cl(X \backslash A) = \partial A$ which is compact, but X is an almost CJ-space by hypothesis, it follows by Theorem (3.1.20) that A is almost CJ-space.

Corollary (3.1. 22)

If $X = E \cup U$, with E is an almost CJ-space, U open in X, and cl(U) compact, then X is an almost CJ-space.

Proof:

Let $A = X\setminus U$, then A is a closed subset of X with $\partial A = \partial A^c \subseteq cl(U)$ which is compact, and since every boundary set is closed, so ∂A is compact, and thus A is a closed subset of E with compact boundary, it follows by Corollary (3.1.21) that A is an almost CJ- space. Now we have $\{A, cl(U)\}$ is a closed cover of X with $cl(U) \cap A = \partial A$ compact and $cl(U)$ and A are almost CJ-spaces and cl(U) countably compact, so X is an almost CJ-space by Theorem (3.1.20).

Proposition (3.1.23)

Let $\{X_1, X_2\}$ be a closed cover a topological space X with $X_1 \cap X_2$ non-countably compact. If X_1 and X_2 are almost CJ-spaces, then X is also almost CJ-space.

Proof:

Let $\{A, B\}$ be a closed cover of X with $A \cap B$ compact, we have to show that A or B is countably compact. For $i = 1,2$, let $A_i = A \cap X_i$ and $B_i = B \cap X_i$, then $\{A_i, B_i\}$ is a closed cover of the Almost CJ-space X_i with $A_i \cap B_i = (A \cap X_i) \cap (B \cap X_i) = (A \cap B) \cap X_i$ which is compact since it is a closed subset of $A \cap B$, it follows by definition of almost CJ-space that A_i or B_i is countably compact. Now if B_1 is countably compact we can show that B is also countably compact. Note that,

$$X_1 \cap X_2 = (A_1 \cup B_1) \cap (A_2 \cup B_2) \subset (A \cap B) \cup B_1 \cup A_2.$$

Since $A \cap B$ and B_1 are countably compact, so A_2 cannot be countably compact, for if A_2 is countably compact, then the closed subset $X_1 \cap X_2$ must be countably compact which is a contradiction with hypothesis, so B_2 is countably compact, and thus $B = B_1 \cup B_2$ is countably compact. By the same way we can prove that if A_1 is countably compact, then A is also. Hence X is an almost CJ-space.

Proposition (3.1.24)

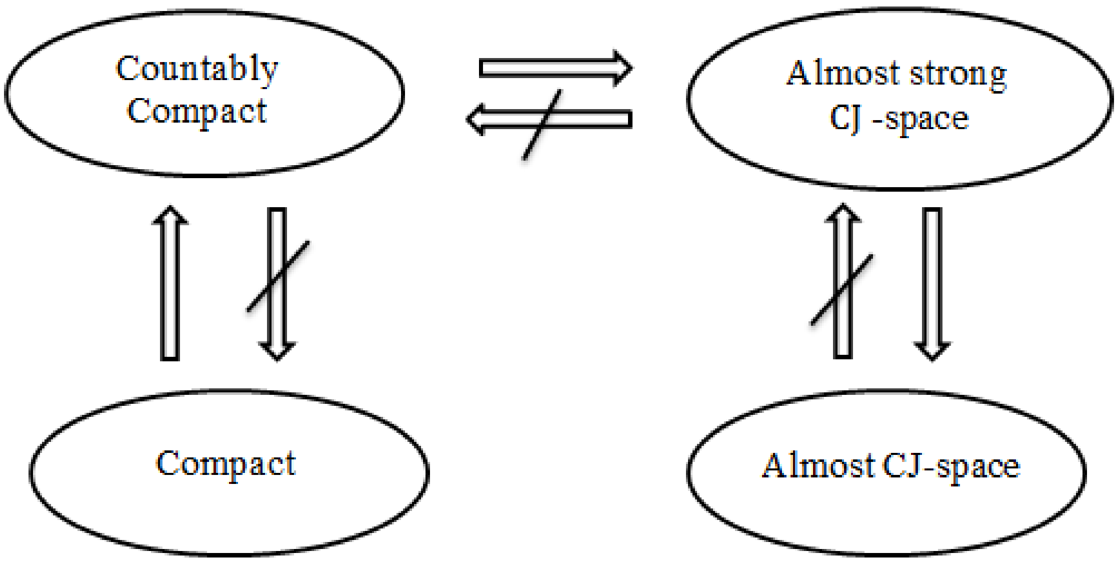
Let A be a closed subset of an almost strong CJ-space X, then A is an almost strong CJ-space if it is union of components of X.

Proof:

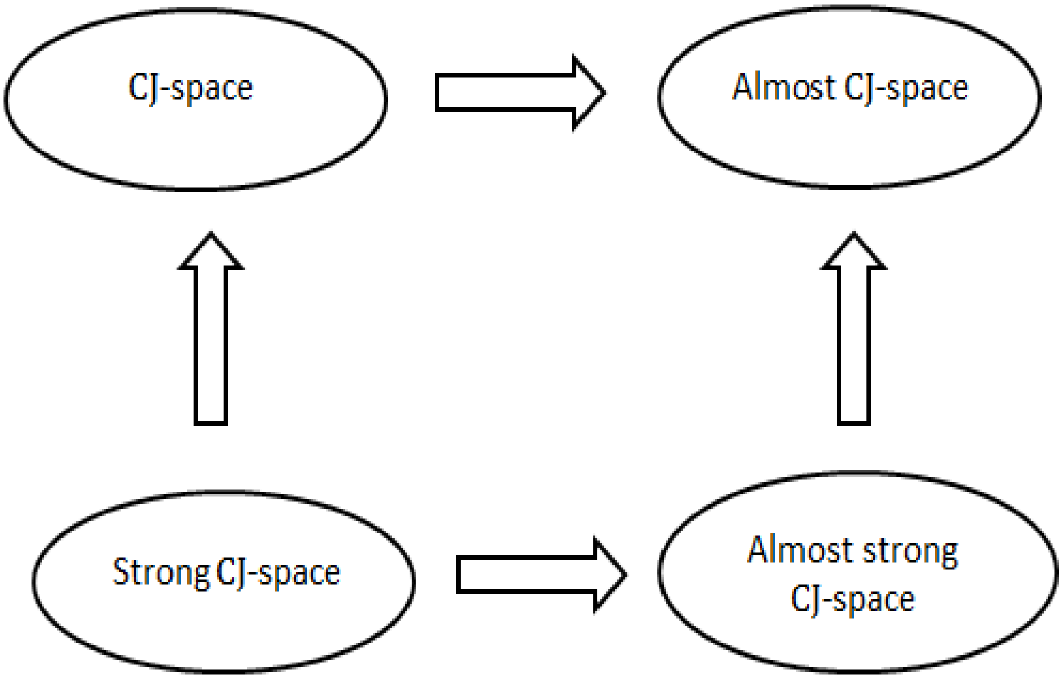
Let $K \subset A$ be compact, so K is a compact subset of X which is an almost strong CJ-space, then there exists a countably compact $L \subset X$ such that $K \subset L$ and $X \backslash L$ is connected. Now if $A \subset L$, then A is countably compact and thus almost strong CJ-space. If $A \not\subset L$, then the connected set $X \backslash L$ intersects A which is union of components of X, thus $X \backslash L \subset A$ since $X \backslash L$ must be one of these component.

Now let $L' = L \cap A$, so L' is a countably compact subset of A since it is a closed subset of the countably compact set L, also $K \subset L'$ since $K \subset A$ and $K \subset L$, and since $X \backslash L \subset A$, therefore $A \backslash L' = X \backslash L$ which is connected, so $A \backslash L'$ is connected. Hence A is an almost strong CJ-space.

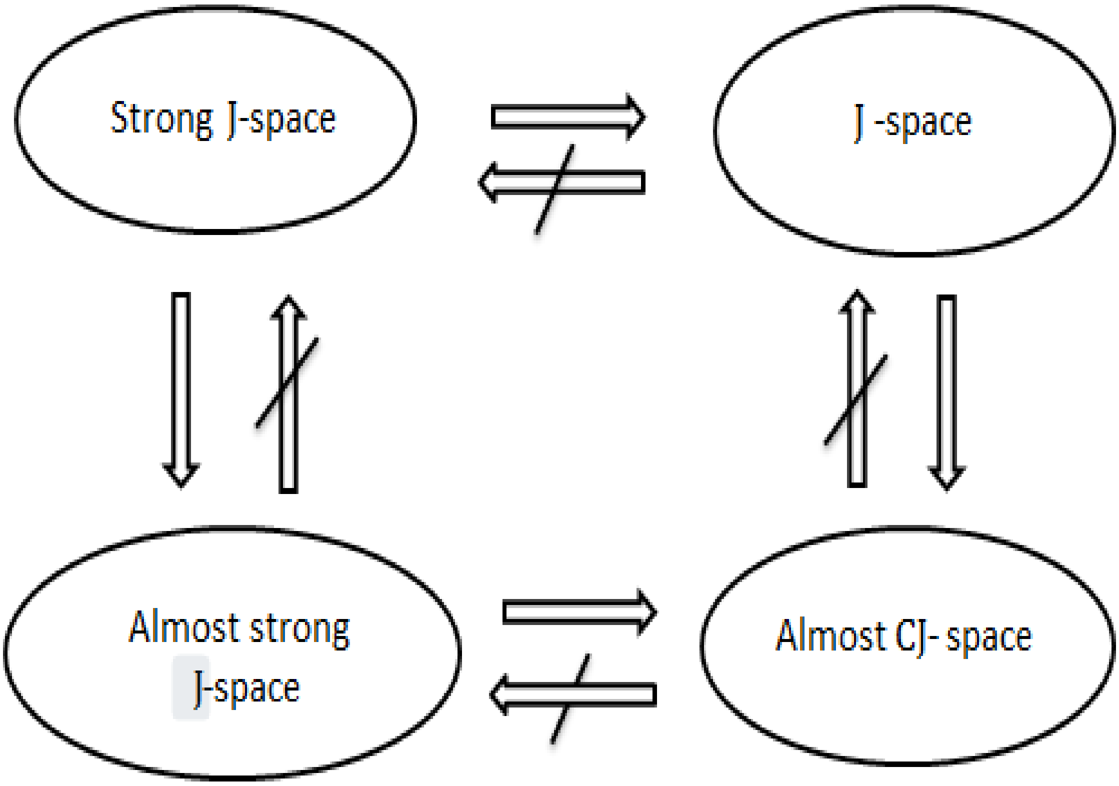
The following diagram illustrate the relationship among compact, countably compact, almost strong CJ-space and almost CJ-space:



The relationship among the new spaces CJ-space, almost CJ-space, strong CJ-space and almost strong CJ-space is illustrated in the following diagram:



While the relationship among J-space, strong J-space, almost CJ-space and almost strong CJ-space is given in the following diagram:


Strong J-space
J -space
Almost strong J-space
Almost CJ- space

§2 ALMOST SEMI-STRONG CJ-SPACE, ALMOST WEAK CJ-SPACE AND ALMOST SEMI-WEAK CJ-SPACE:

This section is appropriated for the studying of what we call almost semi-strong CJ-space, almost weak CJ-space and almost semi-weak CJ-space. We give many interesting characterizations of these radicals.

First, we introduce the following concept.

Definition (3.2.1)

A topological space X is an almost semi strong CJ-space if every compact $K \subset X$ contained in a countably compact $L \subset X$ such that $L \cup C = X$ for some connected $C \subset X \backslash K$.

Definition (3.2.2)

A topological space X is said to be an almost weak CJ-space if, whenever $\{A, B, K\}$ is a closed cover of X with K compact and $A \cap B = \emptyset$, then A or B is countably compact.

Definition (3.2.3)

A topological space X is said to be an almost semi weak CJ-space if, whenever A and B are disjoint closed subsets of X with compact boundaries, then A or B is countably compact.

Proposition(3.2.4)

Every semi-strong CJ-space is an almost semi- strong CJ-space.

Proof:

Let X be a semi-strong CJ-space and let K be a compact subset of X, then K is countably compact by Proposition (1.1.9), it follows by Definition (2.2.1) that there is a countably compact subset L of X such that $K \subset L$ and there exists a connected subset C of X with $C \subset X \backslash K$ and $C \cup L = X$. Hence X is an almost strong CJ-space.

Proposition(3.2.5)

Every weak CJ-space is an almost weak CJ-space.

Proof:

Let X be a weak CJ-space and let $\{A, B, K\}$ be a family of closed subsets of X covers X such that K is compact and $A \cap B = \emptyset$, it follows by Proposition (1.1.9) that K is countably compact, then we can get from Definition (2.2.2) A or B is countably compact. Thus X is an almost weak CJ-space.

Proposition(3.2.6)

Every semi-weak CJ-space is an almost semi-weak CJ-space.

Proof:

Let X be a semi-weak CJ-space and let A and B be disjoint closed subsets of X with ∂A and ∂B are compact, and thus countably compact, it follows by Definition (2.2.3) that A or B is countably compact. Hence X is an almost semi-weak CJ-space.

Theorem (3.2.7)

Let X be any topological space, consider the following conditions:

1. X is an almost strong CJ-space.
2. X is an almost semi strong CJ-space.
3. X is an almost CJ-space.
4. X is an almost semi weak CJ-space.
5. X is an almost weak CJ-space.

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5)$

Proof:

(1) ⟹ (2)

Suppose that X is an almost strong CJ-space and let $K \subset X$ be compact, then there exists a countably compact subset L of X such that $K \subset L$ and $X\backslash L$ is connected by definition of almost strong CJ-space. Pick $C = X\backslash L$, then C is connected and $C \subset X\backslash K$ since $K \subset L$, and $C \cup L = X$. Hence X is an almost semi strong CJ-space.

(2) ⟹ (3)

Let X be an almost semi strong CJ- space and let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ compact, so there exists a countably compact $L \subset X$ such that $A \cap B \subset L$ and there exists a connected subset C of X with $C \subset X\backslash A \cap B$ and $C \cup L = X$ by definition of almost semi strong CJ-space. Note that

$$(A \cap C) \cap (B \cap C) = (A \cap B) \cap C = \emptyset \text{ since } C \subset X\backslash A \cap B,$$

and that

$$(A \cap C) \cup (B \cap C) = (A \cup B) \cap C = X \cap C = C.$$

So we get a disjoint closed cover $\{A \cap C, B \cap C\}$ of C which is connected, therefor C must be in $A \cap C$ or in $B \cap C$, then $C \cap B = \emptyset$ or $C \cap A = \emptyset$, it follows that $B \subset X \backslash C \subset L$ or $A \subset X \backslash C \subset L$ which is countably compact, so A or B is countably compact.

(3) ⟹ (4)

Suppose X is an almost CJ-space and let A, B be two closed sets in X such that $A \cap B = \emptyset$ and $\partial A, \partial B$ are compact, then A or B is countably compact by Theorem (3.1.17). Thus X is an almost semi weak CJ-space.

(4) ⟹ (5)

Assume that X is an almost semi weak CJ-space and let $\{A, B, K\}$ be a family of closed subsets of X which covers X with K compact and $A \cap B = \emptyset$. But ∂A and ∂B are closed subsets of K since $A^c = B \cup (K \backslash K \cap A)$, and $\partial A = \partial A^c$ so $\partial A = \partial\big(B \cup (K \backslash K \cap A)\big)$, so $\partial A \subset K \cap A \subset K$, similarly we can prove that $\partial B \subset K$, and thus ∂A and ∂B are compact, it follows by (4) that A or B is countably compact. Hence X is almost weak CJ- space.

Remark (3.2.8)

An almost semi- strong CJ-space need not be almost strong CJ-space.

For example:

The space Y in the example of Remark (2.2.5) is a semi-strong CJ-space, so it follows by Proposition (3.2.4) that Y is an almost semi-strong CJ-space. But Y is not almost strong CJ-space since $Y \backslash L$ is not connected for any countably compact subset L of Y.

Remark (3.2.9)

An almost CJ-space need not be almost semi-strong CJ-space.

For example:

Let us take the same topological space $\mathbb{N}$ in the example of Remark (3.1.6) which is an almost CJ-space as we have noted. But $\mathbb{N}$ is not almost semi-strong CJ-space since every countably compact subset of $\mathbb{N}$ is finite and hence its complement is infinite and every infinite subset of $\mathbb{N}$ is non-connected.

Theorem (3.2.10)

The concepts of almost weak CJ- space and almost CJ-space are equivalent if the space X is locally compact.

Proof:

An almost CJ- space is an almost weak CJ-space from Theorem (3.2.7), suppose, then, X is an almost weak CJ- locally compact space, and let A, B be two closed subsets of X such that $X = A \cup B$ and $A \cap B$ compact. But X is locally compact so $A \cap B \subset \mathrm{Int}(k)$, for some compact $K \subset X$. Let $A^* = A \setminus \mathrm{Int}(K)$ and $B^* = B \setminus \mathrm{Int}(K)$ then $\{A^*, B^*, K\}$ is a closed cover of X with K compact and $A^* \cap B^* = \emptyset$, it follows by definition of almost weak CJ-space, that A^* or B^* is countably compact, then $A^* \cup K$ or $B^* \cup K$ is countably compact since K is compact, and thus countably compact. But A and B are closed subsets of $A^* \cup K$ and $B^* \cup K$ respectively, so A or B is countably compact. Hence X is an almost CJ-space.

Theorem (3.2.11)

If X is a topological space and $X \times Y$ is a k-space for each k- space Y, then X is an almost weak CJ-space if and only if it is an almost CJ-space.

Proof:

Follows from Theorems (1.1.38) and (3.2.10).

Proposition (3.2.12)

If X is an almost CJ-space and $Z = X \cup \{z_\circ\}$, then Z is an almost semi-weak CJ-space.

Proof:

Let A, B be two closed subsets of Z such that $A \cap B = \emptyset$ and $\partial A, \partial B$ are compact, then $z_\circ \notin A$ or $z_\circ \notin B$. Suppose that $z_\circ \notin B$ and let $E = cl(X \backslash B)$, then $\{B, E\}$ is a closed cover of X with $E \cap B = \partial B$ which is compact, so B or E is countably compact since X is almost CJ-space. But $A \subset E \cup \{z_\circ\}$, so A or B is countably compact, and thus X is an almost semi weak CJ-space.

Remark(3.2.13)

An almost semi-weak CJ-space need not be almost CJ-space.

For example:

The space Z in the example of Remark (2.2.8) is an almost semi-weak CJ-space since it is semi-weak CJ-space, (as we saw in the same example), and by Proposition (3.2.6). But it is not almost CJ-space since $A = \{(s,t) \in Z: s \leq 0\}$ and $B = \{(s,t) \in Z: s \geq 0\}$ form a closed cover of Z with $A \cap B$ is the closed segment joining $(0,0)$ to $(0,1)$ which is compact, but neither A nor B is countably compact.

Proposition (3.2.14)

Let $\{X_1, X_2\}$ be a closed cover of a topological space X such that $X_1 \cap X_2$ non-countably compact. If X_1 and X_2 are almost weak CJ-spaces, then so is X.

Proof:

Let $\{A, B, K\}$ be a family of closed subsets of X which covers X such that $A \cap B = \emptyset$ and K is compact. To prove A or B is countably compact, let $A_i = A \cap X_i$ and $B_i = B \cap X_i$ and $K_i = K \cap X_i$, for $i = 1,2$. Then $\{A_i, B_i, K_i\}$ is a closed cover of X_i with $A_i \cap B_i = \emptyset$ and K_i is compact. Now by using the fact saying that X_1 is almost weak CJ-space, we get A_1 or B_1 is countably compact. Suppose that B_1 is countably compact, we claim that B_2 is also countably compact, for if B_2 is not countably compact, so A_2 must be countably compact since X_2 is an almost weak CJ-space, it follows that $C = A_2 \cup B_1 \cup K$ is countably compact since K is compact, and thus countably compact, but $X_1 \cap X_2$ is a closed subset of C, so $X_1 \cap X_2$ must be countably compact which is a contradiction. Thus $B = B_1 \cup B_2$ is countably compact. Similarly we can prove that A is countably compact whenever A_1 is countably compact.

Remark (3.2.15)

An almost weak CJ- space need not be almost semi- weak CJ-space.

For example:

Let $X = \mathbb{R} \times [0,1)$ and let $Z = X \cup \{(-1,1),(1,1)\}$. To see that Z is an almost weak CJ- space, let $Z_1 = \{(s,t) \in Z: s \leq 0\}$ and $Z_2 = \{(s,t) \in Z: s \geq 0\}$, then $\{Z_1, Z_2\}$ is a closed cover of Z, and $Z_1 \cap Z_2 = \{0\} \times [0,1)$ which is non- countably compact, but Z_1 and Z_2 are both almost semi-weak CJ-space since they are homeomorphic to the space Z of Remark (3.2.13), and thus they are almost weak CJ-spaces by Theorem (3.2.7). Hence Z is an almost weak CJ-space by Proposition (3.2.14).

To see that Z is not almost semi-weak CJ-space, let

$$A = \{(s,t) \in Z: s \leq -1\} \text{ and } B = \{(s,t) \in Z: s \geq 1\},$$

then A and B are closed sets in Z such that $A \cap B = \emptyset$ and $\partial A, \partial B$ are compact, but neither A nor B is countably compact.

Proposition (3.2.16)

Let $\{X_1, X_2\}$ be a closed cover of a topological space X such that $X_1 \cap X_2$ non- countably compact. If X_1 and X_2 are almost semi strong CJ-spaces, then X is also almost semi strong CJ-space.

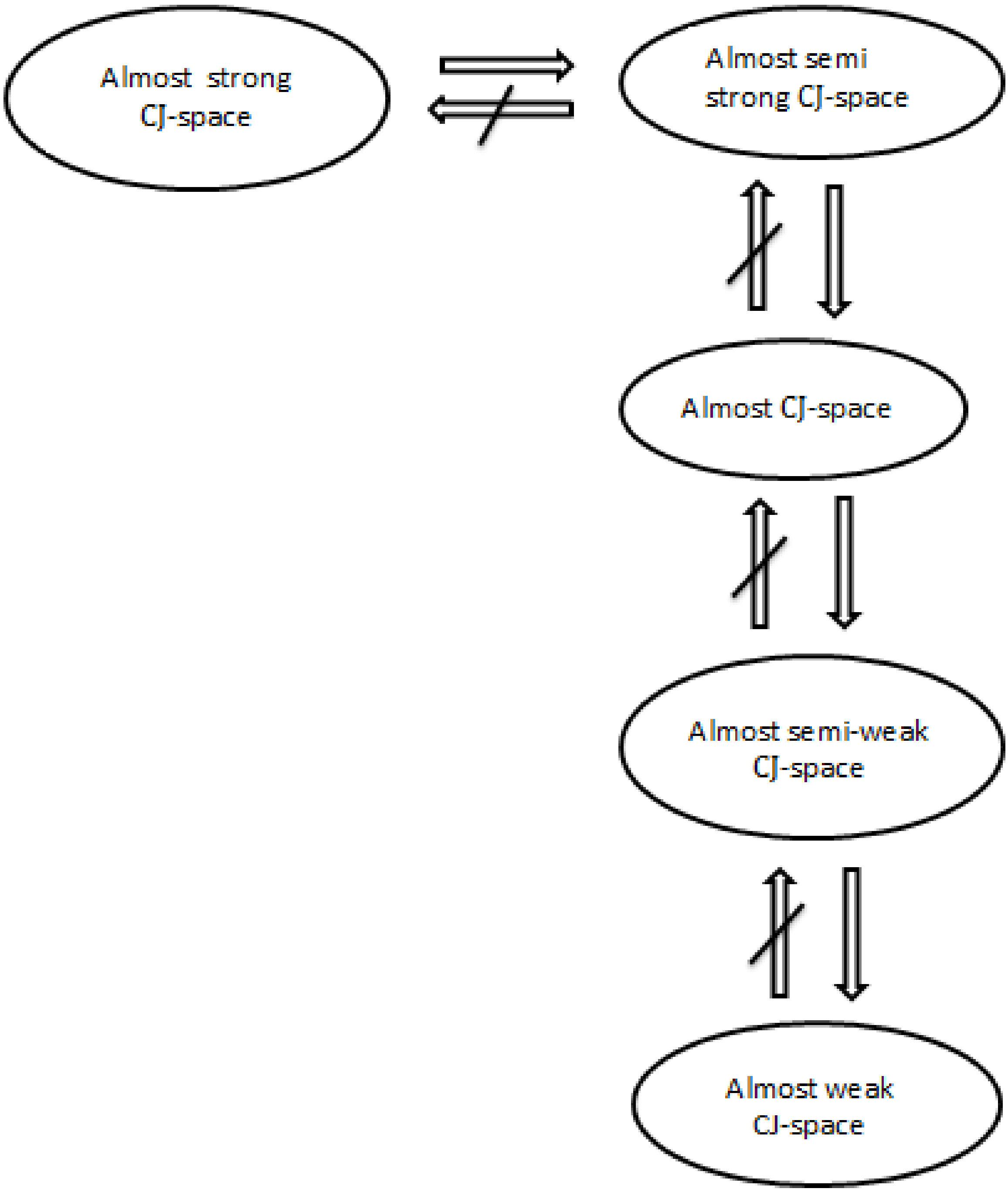
Proof:

Let $K \subset X$ be a compact and let $K_i = K \cap X_i$, then K_i is a closed subset of K, and thus compact subset of the almost semi strong CJ-space X_i, so there exists a countably compact subset L_i of X_i such that $K_i \subset L_i$ and there exists a connected subset C_i of X_i such that $C_i \subset X_i \backslash K_i$ and $L_i \cup C_i = X_i$ for $i = 1,2$, by definition of almost semi strong CJ-space. Now let $L = L_1 \cup L_2$ and $C = C_1 \cup C_2$, so L is a countably compact subset of X with $K \subset L$ and $C \cup L = X$ and $C \subset X \backslash K$. It remains to show that C is connected, we need only cheek that $C_1 \cap C_2 \neq \emptyset$ since C_1 and C_2 are connected. Note that $X_1 \cap X_2 \backslash L \neq \emptyset$, for if $X_1 \cap X_2 \backslash L = \emptyset$, then $X_1 \cap X_2$ is a closed subset of L which is countably compact, so $X_1 \cap X_2$ is countably compact which is a contradiction. Also we have

$$X_i \backslash L \subseteq X_i \backslash L_i \subseteq C_i, \text{so } (X_1 \cap X_2) \backslash L \subseteq C_1 \cap C_2,$$

and thus $C_1 \cap C_2 \neq \emptyset$. Hence $C = C_1 \cup C_2$ is connected. Therefor X is an almost semi strong CJ- space.

Follow, we introduced the diagram which represents the relationship among all new spaces given in sections one and two of this chapter.


Almost strong CJ-space
Almost semi strong CJ-space
Almost CJ-space
Almost semi-weak CJ-space
Almost weak CJ-space

§3 FUNCTIONAL CHARACTERIZATIONS OF ALMOST CJ-SPACES

In this section we talk about the functions that preserve property of being almost CJ- space and almost strong CJ-space, where we have employed this type of functions to deduce new properties of almost CJ-space and almost strong CJ- space.

Proposition (3.3.1)

If a space X is an almost CJ- space, then every closed, boundary perfect function $f: X \to Y$ onto a non- countably compact space Y is quasi-perfect.

Proof:

Assume that X is an almost CJ-space, and let $f: X \to Y$ be closed, boundary-perfect function from X onto Y. We have to show that f is quasi-perfect, let $y \in Y$, then $f^{-1}(y)$ is a subset of the almost CJ-space X with compact boundary since f is boundary- perfect, it follows by Theorem (3.1.17) that either $cl(f^{-1}(y))$ or $cl(X\backslash f^{-1}(y))$ is countably compact, and since $\{y\}$ is closed in the Hausdorff space Y, so $f^{-1}(y)$ is closed subset of X since f is continuous, then $f^{-1}(y)$ or $cl(X\backslash f^{-1}(y))$ is countably compact. But $cl(X\backslash f^{-1}(y))$ is not countably compact, for if $cl(X\backslash f^{-1}(y))$ is countably compact, then $Y = \{y\} \cup f(cl(X\backslash f^{-1}(y))$ is countably compact which is a contradiction, thus $f^{-1}(y)$ is countably compact. Hence f is quasi-perfect.

Proposition (3.3.2)

If every closed, boundary perfect function $f: X \rightarrow Y$ from a topological space X onto a non-countably compact space Y is countably perfect, then X is an almost CJ- space.

Proof:

Suppose that every closed boundary-perfect function from X onto a non-countably compact space Y is countably perfect. To prove that X is an almost CJ-space, let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ compact. Let $Y = X/B$ and $f: X \rightarrow Y$ be the quotient map, so f is closed and continuous function. Let $y_{\circ} = f(B)$, note that for each $y \in Y$ $\partial\left(f^{-1}(y)\right)$ is compact since if $y \neq y_{\circ}$, then $\partial\left(f^{-1}(y)\right)$ is one-element set, and if $y = y_{\circ}$, then $\partial\left(f^{-1}(y)\right)$ is a closed subset of $A \cap B$. Therefor f is boundary perfect. Now if Y is non-countably compact, then f is countably perfect by hypothesis, and thus $B = f^{-1}(y_{\circ})$ is countably compact. If Y is countably compact, so f (A) is countably compact because it is closed subset of Y. On the other hand we have $f|A: A \rightarrow f(A)$ is countably perfect since it is closed function and its fibers are either one-element sets or equal to $A \cap B$ which is compact, and thus countably compact. Hence $A = f^{-1}(f(A))$ is countably compact.

Proposition (3.3.3)

Let $f: X \rightarrow Y$ be a countably perfect function from a CJ-space X onto Y, then Y is an almost CJ- space.

Proof:

To prove Y is an almost CJ- space, let A, B be two closed subsets of Y with $Y = A \cup B$ and $A \cap B$ compact, and thus countably compact. Then $\{f^{-1}(A), f^{-1}(B)\}$ is a closed cover of X since f is continuous, and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$ which is countably compact since f is countably perfect, it follows by Definition (2.1.1) that $f^{-1}(A)$ or $f^{-1}(B)$ is countably compact, and thus $f(f^{-1}(A))$ and $f(f^{-1}(B))$ are countably compact since f continuous, it follows by Proposition (1.1.13(2)) that A or B is countably compact since f is surjective. Hence Y is an almost CJ-space.

Proposition (3.3.4)

Let $f: X \to Y$ be a countably perfect monotone function from X onto Y. Then, if Y is an almost CJ-space, so is X.

Proof:

Let A, B be two closed subsets of X with $X = A \cup B$ and $A \cap B$ compact, then $\{f(A), f(B)\}$ is a closed cover of Y since f continuous, with $f(A) \cap f(B) = f(A \cap B)$ since f is monotone. But $f(A \cap B)$ is compact by Proposition (1.1.18), and since Y is an almost CJ-space, so $f(A)$ or $f(B)$ is countably compact, therefore $f^{-1}(f(A))$ or $f^{-1}(f(B))$ is countably compact because f is countably perfect. But A, B are closed subsets of $f^{-1}(f(A))$ and $f^{-1}(f(B))$ respectively by Proposition (1.1.13(1)), so A or B is countably compact. Thus X is an almost CJ- space.

Proposition (3.3.5)

Let $f: X \to Y$ be a countably perfect monotone function from X onto Y. Then, if Y is an almost strong CJ-space, so is X.

Proof:

Let $K \subset X$ be compact, then $f(K) \subset Y$ is compact since f is continuous, so there exists a countably compact $L' \subset Y$ such that $f(K) \subset L'$ and $Y \backslash L'$ is connected by definition of almost strong CJ-space. Let $f^{-1}(L') = L$, then $L \subset X$ is also countably compact because f is countably perfect, and $K \subset f^{-1}(f(K)) \subset f^{-1}(L) = L$ and $X \backslash L = f^{-1}(Y \backslash L')$ which is connected since f is closed and monotone and by Theorem (1.2.8). Hence X is an almost strong CJ-space.

Proposition (3.3.6)

Let $f: X \to Y$ be a perfect function from X onto Y. Then, if X is an almost CJ- space, so is Y.

Proof:

Let $\{A, B\}$ be a closed cover of Y with $A \cap B$ compact, then $\{f^{-1}(A), f^{-1}(B)\}$ is a closed cover of X with $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$ which is compact since f is continuous and perfect. But X is an Almost CJ-space, it follows by Definition (3.1.1) that $f^{-1}(A)$ or $f^{-1}(B)$ is countably compact, then $f(f^{-1}(A))$ or $f(f^{-1}(B))$ is countably compact since f is continuous and by Proposition (1.1.20). It follows by Proposition (1.1.13(2)) A or B is countably compact since f is surjective. Hence Y is an almost CJ-space.

Proposition (3.3.7)

Let $f: X \rightarrow Y$ be an injective perfect function from X onto Y. Then, if X is an almost strong CJ-space, so is Y.

Proof:

Let $K \subset Y$ be compact, then $f^{-1}(K) \subset X$ is compact since f is perfect, but X is an almost strong CJ- space, so there exists a countably compact set $L' \subseteq X$ where $f^{-1}(K) \subset L'$ and $X \backslash L'$ is connected. Let $L = f(L')$, then L is a countably compact subset of Y since f is continuous and by Proposition (1.1.20), also $K \subset L$ since f is surjective and $Y \backslash L = f(X \backslash L')$ since f is injective, so $Y \backslash L$ is connected since f is continuous. Hence Y is an almost strong CJ-space.

Theorem (3.3.8)

Let Y be any space, then the following conditions are equivalent.

1. Y is an almost CJ-space.

2. $Y \times Z$ is an almost CJ-space for every connected and compact space Z.

3. $Y \times Z$ is an almost CJ-space for some compact space Z.

Proof:

(1)$\Longrightarrow$ (2)

Consider the projection function $f: Y \times Z \rightarrow Y$ which is a closed, surjective, continuous, monotone and perfect, and thus countably perfect. It follows by Proposition (3.3.4) that $Y \times Z$ is an almost CJ-space since Y is an almost CJ-space.

$(2) \Rightarrow (3)$

Clear.

$(3) \Rightarrow (1)$

Again we take the projection function $f: Y \times Z \to Y$, which is surjective, continuous and perfect function. It follows by Proposition (3.3.3) that Y is an almost CJ-space since $Y \times Z$ is an almost CJ-space.

Theorem (3.3.9)

Let Y be any space, then the following conditions are equivalent.

1. Y is an almost strong CJ-space.

2. $Y \times Z$ is an almost strong CJ-space for every connected and compact space Z.

3. $Y \times Z$ is an almost strong CJ-space for some compact space Z.

Proof:

$(1) \Rightarrow (2)$

Consider the projection function $f: Y \times Z \to Y$ which is a closed, surjective, continuous, monotone and perfect, and thus countably perfect. It follows by Proposition (3.3.5) that $Y \times Z$ is an almost strong CJ-space since Y is an almost strong CJ-space.

$(2) \Rightarrow (3)$

Clear.

$(3) \Longrightarrow (1)$

Again we take the projection function $f: Y \times Z \rightarrow Y$, which is surjective, injective, perfect and continuous function. It follows by Proposition (3.3.7) that Y is an almost strong CJ-space since $Y \times Z$ is an almost strong CJ-space.

Proposition (3.3.10)

Let $f: X \rightarrow Y$ be an injective perfect function onto Y. Then, if X is an almost semi- strong CJ- space, so is Y.

Proof:

Let $K \subset Y$ be compact, then $K' = f^{-1}(K)$ is a compact subset of X since f is perfect. But X is an almost semi- strong CJ-space, so there exists a countably compact $L' \subset X$ such that $K' \subset L'$ and a connected $C' \subset X \backslash K'$ with $C' \cup L' = X$ by definition of almost semi- strong CJ-space. Now let $L = f(L')$ and $C = f(C')$, then L is countably compact and C is connected since f is continuous, moreover $K \subset L$ since f is surjective and $C \subset Y \backslash K$ since f is injective and clear that $L \cup C = Y$. Hence Y is an almost semi- strong CJ-space.

Theorem (3.3.11)

A topological space X is an almost semi weak CJ-space if and only if for any boundary- perfect function $f: X \rightarrow Y$, $f^{-1}(y)$ is non- countably compact for at most one $y \in Y$.

Proof:

The "if" part

Suppose that X is an almost semi weak CJ-space and $y_1 \neq y_2$ in Y, and let $A_i = f^{-1}(y_i)$ (for $i = 1, 2$). Then A_1 and A_2 are closed subsets of X with $A_1 \cap A_2 = \emptyset$ and $\partial A_1, \partial A_2$ are compact since f is boundary-perfect, so A_1 or A_2 is countably compact by definition of almost semi weak CJ-space.

The "only if" part

Suppose A_1 and A_2 are closed sets in X such that $A_1 \cap A_2 = \emptyset$ and $\partial A_1, \partial A_2$ are compact .Define a relation R on X such that $x \, R \, y \Leftrightarrow x, y \in A_1$ or $x, y \in A_2$.Then

$$[x] = \begin{Bmatrix} A_1 & \text{if } x \in A_1 \\ A_2 & \text{if } x \in A_2 \\ \{x\} & \text{if } x \notin A_1 \text{ and } x \notin A_2 \end{Bmatrix}.$$

Let Y be the quotient space of X with respect to the relation R, and let $f: X \rightarrow Y$ be the quotient function, so f is a closed, continuous and onto map. Now to show that f is boundary- perfect, it is sufficient to prove that $\partial\left(f^{-1}(y)\right)$ is compact for each $y \in Y$. Let $y \in Y$, then

$$f^{-1}(y) = \begin{Bmatrix} A_1 & \text{if } x \in A_1 \\ A_2 & \text{if } x \in A_2 \\ \{y\} & \text{if } x \notin A_1 \text{ and } x \notin A_2 \end{Bmatrix}$$

But $\partial A_1, \partial A_2$ are compact by hypothesis and $\partial\{y\}$ is also compact, so f is boundary-perfect, and thus A_1 or A_2 is countably compact by (b). Hence X is an almost semi weak CJ-space.

CHAPTER FOUR

CONTRACTIBLE J- SPACE

INTRODUCTION

A space X is called contractible space if the identity map $I_X : X \to X$ is null- homotopic, equivalently if it a homotopy equivalent to a point. We used this concept to define other new topological space called contractible J-space.

This chapter consists of two sections, section one includes definition of contractible J-space with its properties and its relationship with contractible space. We gave many miscellaneous examples about this space. Several equivalents for contractible J-space are given in Theorem **(4.1.11)**.

In section two, new types of functions are given, like contractible function and contractible perfect function. We prove that a contractible perfect function maintains the property of being contractible J-space, see Proposition **(4.2.9)**. While a contractible perfect function transfers the inverse image of contractible J- space to a contractible J- space, see Proposition **(4.2.15)**.

§1 CONTRACTIBLE J-SPACE

As a generalization of the concepts of CJ- space and almost CJ-space, we introduce the concept of contractible J-space. We give many characterizations and many properties of these concepts.

We begin with the following definition.

Definition (4.1.1)

A topological space X is said to be contractible J-space if for every proper closed cover $\{E, F\}$ of X with $E \cap F$ compact, either E or F is contractible.

Remark (4.1.2)

If the closed cover in Definition (4.1.1) is not proper, then every contractible J- space must be contractible.

Remark (4.1.3)

If a topological space X has no proper closed cover, then X is a contractible J-space.

The following remark, follows from Remark (4.1.3).

Remark (4.1.4)

Every indiscrete space is a contractible J-space.

Remark (4.1.5)

If X is any topological space with every subspace of it is contractible, then X is a contractible J- space.

The following example illustrate the contractible J-space.

Example (4.1.6)

The usual space $\mathbb{R}$ is a contractible J-space, for if $\{E, F\}$ is a closed cover of $\mathbb{R}$ with $E \cap F$ compact, then E and F can not be both discrete, thus E or F is contractible (see Remark (1.2.33(2)).

Remark (4.1.7)

A discrete space with more than two points is not contractible J-space, follows from Example (1.2.32).

The following remark indicating when the subspace of $\mathbb{R}$ be contractible J-space.

Remark (4.1.8)

A subspace Y of $\mathbb{R}$ is a contractible J-space if it is not discrete space (with more than two elements) follows from (Remark (1.2.33(2)) and Example (1.2.32).

Example (4.1.9)

Let X be a non empty set and A is a proper subset of X. Define a topology on X by $\tau = \{X, \emptyset, A\}$, then (X, τ) is a contractible J-space since X has no proper closed cover.

Remark (4.1.10)

A contractible space need not be contractible J-space.

For example

Let X be a subspace of Euclidian space $\mathbb{R}^2$ such that $X = \{(x,y) \in \mathbb{R}^2, (x-1)^2 + y^2 \leq 1\} \cup \{(x,y) \in \mathbb{R}^2, (x+1)^2 + y^2 \leq 1\}$.

Let E, F be two subsets of X such that

$E = \{(x,y) \in \mathbb{R}^2, (x-1)^2 + y^2 \leq 1\} \cup \{(x,y) \in \mathbb{R}^2, (x+1)^2 + y^2 = 1\}$,

$F = \{(x,y) \in \mathbb{R}^2, (x-1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2, (x+1)^2 + y^2 \leq 1\}$,

then $\{E, F\}$ is a closed cover of X with

$E \cap F = \{(x,y) \in \mathbb{R}^2, (x-1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2, (x+1)^2 + y^2 = 1\}$

which is compact subset of X, but neither E nor F is contractible. Hence X is not contractible J-space, but X is contractible since X is closed ball in $\mathbb{R}^2$.

Remark (4.1.11)

A contractible J-space need not be contractible.

For example

The unite circle S^1 as a subspace of $\mathbb{R}^2$ is not contractible, but it is contractible J-space since every proper subset of S^1 is contractible.

Theorem (4.1.12)

The following conditions are equivalent for any space X.

1. X is a contractible J- space.
2. For every proper closed cover $\{E, F\}$ with $E \cap F$ compact E or F is homotopy equivalent to a point.
3. For every proper closed cover $\{E, F\}$ with $E \cap F$ compact, there exists $x_\circ \in E$(or $x_\circ \in F$) such that $\{x_\circ\}$ is a deformation retract of E (or of F).
4. For every proper closed cover $\{E, F\}$ with $E \cap F$ compact, E or F is a retract of any cone over it.
5. For every proper closed cover $\{E, F\}$ with $E \cap F$ compact, every function f from E (or F) to an arbitrary space Y, is null- homotopic.
6. For every proper closed cover $\{E, F\}$ with $E \cap F$ compact, every function f from an arbitrary space Y to E (or F) is null- homotopic.

Proof

Follows from Theorems (1.2.37), (1.2.38), (1.2.39), (1.2.40) and (1.2.41), and Definition (4.1.1).

Proposition (4.1.13)

If X is a contractible J- space, then for every proper closed cover $\{E, F\}$ with $E \cap F$ compact, E or F is path connected.

Proof

Follows from Proposition (1.2.42) and Definition (4.1.1).

Remark (4.1.14)

The converse of Proposition (4.1.12) is not true in general.

For example

Let us take the example of Remark (4.1.9), as we saw in this example X is not contractible J-space, but for every proper closed cover $\{E, F\}$ of X with $E \cap F$ compact, E or F is path connected.

Proposition (4.1.15)

If X is a contractible J- space, then for every proper closed cover $\{E, F\}$ with $E \cap F$ compact, E or F is simply connected.

Proof

Follows from Proposition (1.2.43) and Definition (4.1.1).

Remark (4.1.16)

The converse of Proposition (4.1.14) is not true in general.

For example

Let X be a subspace of $\mathbb{R}^3$ such that $X = E \cup F$, where

$$E = \{(x, y, z) \in \mathbb{R}^3, (x-1)^2 + y^2 + z^2 = 1\}$$

and

$$F = \{(x, y, z) \in \mathbb{R}^3, (x-3)^2 + y^2 + z^2 = 1\},$$

then $\{E, F\}$ is a closed cover of X with $E \cap F = \{(2,0,0)\}$ which is compact, but neither E nor F is contractible. Hence X is not contractible J-space, but E and F are simply connected since both of them homotopic equivalent to S^2.

Remark (4.1.17)

The property of being contractible J-space is not a weak hereditary property, and thus not hereditary property.

For example:

The usual space $\mathbb{R}$ is a contractible J-space, but the natural numbers $\mathbb{N}$ as a subspace of $\mathbb{R}$ is not contractible J-space since the induced topology of the usual topology with respect to $\mathbb{N}$ is the discrete topology.

Proposition (4.1.18)

If A is a subset of a contractible J-space with compact boundary, then $\mathrm{cl}(A)$ or $\mathrm{cl}(X\backslash A)$ is contractible.

Proof

Consider the closed cover $\{\mathrm{cl}(A), \mathrm{cl}(X\backslash A)\}$ of X, such that $\mathrm{cl}(A) \cap \mathrm{cl}(X\backslash A) = \partial A$ which is compact, it follows by definition of contractible J- space that $\mathrm{cl}(A)$ or $\mathrm{cl}(X\backslash A)$ is contractible.

Remark (4.1.19)

The converse of Proposition (4.1.18) is not true in general.

For example

Let us take the finite set $X = \{1,2,3\}$ with the discrete topology, and let $A = \{1,2\}$ be a subset of X, then A has a compact boundary since it is finite, moreover $cl(A) = A$ and $cl(X\setminus A) = \{3\}$ which are contractible sets, but X is not contractible J-space.

Remark (4.1.20)

If X and Y are two contractible J-spaces, then $X \times Y$ need not be so.

For example

Let $X = \{1,2\}$ and $\tau = D$, then X is contractible J-space since $\{\{1\},\{2\}\}$ is the only proper closed cover of X with $\{1\}\cap\{2\} = \emptyset$ which is compact and $\{1\}$ and $\{2\}$ are contractible.

But $X \times X = \{(1,1),(1,2),(2,1),(2,2)\}$, is not contractible J-space since it has more than two elements and by Remark (4.1.7).

§2 FUNCTIONAL CHARACTERIZATIONS OF CONTRACTIBLE J-SPACE:

This section consists of new types of functions which are contractible function and contractible perfect function, and we discussed how to save the property of being contractible J-space under the effect of these functions. We'll start with the definition of first type.

Definition (4.2.1)

A function $f: X \to Y$ is said to be contractible function if it preserves the property of being contractible space. That is the image of any contractible subspace of X is a contractible subspace of Y.

Proposition (4.2.2)

The identity function on any topological space is a contractible function.

Proof:

Let X be any topological space and let $I_X: X \to X$ be the identity function on X, let S be a contractible subset of X, then $I_X(S) = S$ is also contractible subset of X. Hence I_X is a contractible function.

Proposition (4.2.3)

Any constant function is a contractible function.

Proof:

Let $k\colon X \to Y$ be a constant function from a topological space X into a topological space Y, that is, $k(x) = c$, $\forall\ x \in X$, where $c \in Y$. To show that k is a contractible function, let S be a contractible subset of X, then $k(S) = c$ is a contractible subset of Y by Remark (1.2.33(3)).

Proposition (4.2.4)

Any function defined from any topological space to an indiscrete space is contractible function.

Proof:

Follows from Remark (1.2.33(3)).

Proposition(4.2.5)

The composition of two contractible functions is contractible.

Proof:

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two contractible functions. To prove that $g \circ f : X \to Z$ is contractible, let A be a contractible subset of X, then $f(A)$ is a contractible subset of Y since f is contractible function. On the other hand g is also contractible function, so $g(f(A))$ is contractible subset of Z. But $g\big(f(A)\big) = g \circ f(A)$. Hence $g \circ f$ is a contractible function.

Example (4.2.6)

Let $\mathbb{R}$ be the usual topological space and $(\mathbb{R}, I)$ be the indiscrete space, a function $f: (\mathbb{R}, I) \to \mathbb{R}$ such that $f(x) = x, \forall x \in \mathbb{R}$, is not contractible function since $\mathbb{N}$ is a contractible subspace of $\mathbb{R}$ with the indiscrete topology, but $f(\mathbb{N}) = \mathbb{N}$ is not contractible subset of $\mathbb{R}$ with the usual topology.

Remark (4.2.7)

A continuous function need not be contractible function.

For example:

Let $f: [a,b] \to S^1$ such that $f(x) = e^{2ix}, \forall x \in [a,b]$, clear that f is continuous onto function, but not contractible function since $[a,b]$ is a contractible set while S^1 is not.

Remark (4.2.8)

A contractible function need not be continuous function.

For example: Let $X = \{1,2,3\}$, and $\tau = \{X, \emptyset, \{1\}\}$, and let $f: X \to X$ such that $f(2) = f(3) = 1$ and $f(1) = 2$, then f is a contractible function since every subset of X is contractible, and thus $f(A) \subseteq X$ is contractible for each contractible $A \subseteq X$. But f is not continuous function since $\{1\} \in \tau$ while $f^{-1}(\{1\}) = \{2,3\} \notin \tau$.

Proposition (4.2.9)

Let $f: X \to Y$ be a perfect and contractible function from X onto Y. If X is a contractible J-space, then so is Y.

Proof:

Let $\{E, F\}$ be a closed cover of Y with $E \cap F$ compact, then $\{f^{-1}(E), f^{-1}(F)\}$ is a closed cover of X since f is continuous, and $f^{-1}(E) \cap f^{-1}(F) = f^{-1}(E \cap F)$ which is compact since f is perfect, but X is contractible J-space, so $f^{-1}(E)$ or $f^{-1}(F)$ is contractible, it follows by definition of contractible function that $f\big(f^{-1}(E)\big)$ or $f\big(f^{-1}(F)\big)$ is contractible, but f is surjective, so E or F is contractible. Hence Y is contractible J- space.

Proposition (4.2.10)

Every homeomorphism function is a contractible function.

Proof:

$f: X \rightarrow Y$ be a homeomorphism function, and let A be a contractible subset of X, we have to show that f(A) is contractible subset of Y. Note that A and f(A) are homeomorphic spaces, it follows by Proposition (1.2.49) that A and f(A) are homotopy equivalent. But A is contractible, so A is homotopy equivalent to a point by Theorem (1.2.37), it follows by Remark (1.2.50(1)), that f(A) is homotopy equivalent to a point, and thus contractible.

Remark (4.2.11)

A contractible function need not be a homeomorphism function.

For example:

See example of Remark (4.2.8).

Definition (4.2.12)

A function $f: X \to Y$ is said to be contractible perfect function if it is closed and the inverse image of any contractible subspace of Y is a contractible subspace of X.

Example (4.2.13)

Any closed function from indiscrete space to any topological space is a contractible perfect function.

Proposition (4.2.14)

The composition of two contractible functions is contractible.

Proof:

Let $f: X \to Y$ and $g: Y \to Z$ be two contractible perfect functions. To prove that $g \circ f : X \to Z$ is contractible perfect, let B be a contractible subset of Z, then $g^{-1}(B)$ is a contractible subset of Y since g is contractible perfect function. On the other hand f is also contractible perfect function, so $f^{-1}(g^{-1}(A))$ is contractible subset of X.

But $f^{-1}\left(g^{-1}(A)\right) = f^{-1} \circ g^{-1}(A) = (g \circ f)^{-1}(A)$. Hence $g \circ f$ is a contractible perfect function.

Proposition (4.2.15)

Let $f: X \to Y$ be a contractible perfect function from X onto Y. If Y is a contractible J-space, then so is X.

Proof:

Let $\{E, F\}$ be a closed cover of X with $E \cap F$ compact, then $\{f(E), f(F)\}$ is a closed cover of Y since f is closed, and $f(E) \cap f(F) = f(E \cap F)$ which is compact since f is continuous, but Y is contractible J-space, so $f(E)$ or $f(F)$ is contractible, it follows by definition of contractible perfect function that $f^{-1}(f(E))$ or $f^{-1}(f(F))$ is contractible, but f is surjective, so E or F is contractible. Hence X is contractible J- space.

Proposition (4.2.16)

Every homeomorphism function is a contractible perfect function.

Proof:

$f: X \rightarrow Y$ be a homeomorphism function, and let B be a contractible subset of Y, we have to show that $f^{-1}(B)$ is contractible subset of X.

Note that B and $f^{-1}(B)$ are homeomorphic spaces, it follows by Proposition (1.2.49) that B and $f^{-1}(B)$ are homotopy equivalent. But B is contractible, so B is homotopy equivalent to a point by Theorem (1.2.37), it follows by Remark (1.2.50(1)), that $f^{-1}(B)$ is homotopy equivalent to a point, and thus contractible.

Corollary (4.2.17)

The property of being contractible J-space is a topological property.

Proof:

Follows from Propositions (4.2.9), (4.2.10), (4.2.15) and (4.2.16).

CHAPTER FIVE

CONCLUSIONS
AND
FUTURE STUDIES

INTRODUCTION

This chapter consists of two sections, section one includes most important results obtained during this thesis.

In section two, we suggest some proposals concerning our thesis, for use in future studies on the subject.

§1 CONCLUSIONS

In the following we review main results we have obtained:

1. Every countably compact space is a strong CJ-space, but not conversely, see Proposition (2.1.5) and Remark (2.1.6).

2. Every strong CJ-space is a CJ-space, but not conversely, see Proposition (2.1.8) and Remark (2.1.9).

3. Every countably compact space is a CJ-space, but not conversely, see Proposition (2.1.10) and Remark (2.1.11).

4. The concepts CJ-space and strong CJ-space are equivalent, when the space is locally connected, see Theorem (2.1.17).

5. The following conditions equivalent to property of being CJ-space.

- For any $A \subset X$ with countably compact boundary, cl(A) or cl(X\A) is countably compact,
- If A and B are closed sets in X such that $A \cap B = \emptyset$ and ∂A or ∂B is countably compact, then A or B is countably compact,
- If $K \subset X$ is countably compact, and if $\mathcal{w}$ is a disjoint open cover of X\K ,then there exists $W \in \mathcal{w}$, such that X\W is countably compact.
- Same as (4), but with card $\mathcal{w} = 2$.

6. Every strong CJ-space is a semi-strong CJ-space, but not conversely, see Theorem (2.2.4) and Remark (2.2.5).

7. Every semi-strong CJ-space is a CJ-space, but not conversely, see Theorem (2.2.4) and Remark (2.2.6).

8. Every CJ-space is a semi-weak CJ-space, but not conversely, see Theorem (2.2.4) and Remark (2.2.8).

9. Every semi- weak CJ- space is a weak CJ- space, but not conversely, see Theorem (2.2.4) and Remark (2.2.10).

10. The concepts CJ-space and weak CJ-space are equivalent, when the space is locally compact, see Theorem (2.2.12).

11. The continuous image of CJ-space is not CJ-space in general, see example of Remark (2.3.5).

12. A countably compact function preserves the property of being CJ-space, see Proposition (2.3.6).

13. The property of being "CJ-space" is a topological property, see Proposition (2.3.8).

14. We used the concept boundary countably perfect function to conclude the equivalent theorem of definition of CJ-space, see Theorem (2.3.9).

15. We used the concept boundary countably perfect function to conclude the equivalent theorem of definition of semi-weak CJ-space, see Theorem (2.3.12).

16. Every almost strong CJ-space is an almost CJ-space, but not conversely, see Proposition (3.1.5) and Remark (3.1.6).

17. Every countably compact space is an almost strong CJ-space, but not conversely, see Proposition (3.1.9) and Remark (3.1.10).

18. Every CJ-space is an almost CJ-space, see Proposition (3.1.3).

19. Every strong CJ-space is an almost strong CJ-space, see Proposition (3.1.4).

20. The concepts almost CJ-space and almost strong CJ- space are equivalent if the space is locally connected, see Theorem (3.1.18).

21. Every semi-strong CJ-space is an almost semi-strong CJ-space, see Proposition (3.2.4).

22. Every weak CJ-space is an almost weak CJ-space, see Proposition (3.2.5).

23. Every semi-weak CJ-space is an almost semi-weak CJ-space, see Proposition (3.2.6).

24. The following conditions are equivalent to definition of almost strong CJ- space.

- ❖ $Y \times Z$ is an almost strong CJ-space for every connected and compact space Z.
- ❖ $Y \times Z$ is an almost strong CJ-space for some compact space Z, see Theorem (3.3.8).

25. The following conditions are equivalent to definition of almost CJ-space.

- ❖ $Y \times Z$ is an almost CJ-space for every connected and compact space Z.
- ❖ $Y \times Z$ is an almost CJ-space for some compact space Z, see Theorem (3.3.7).

26. A continuous and perfect function transfers an almost CJ-space to almost CJ-space, see Proposition (3.3.5).

27. An injective continuous perfect function transfers an almost strong CJ-space to almost strong CJ-space, see Proposition (3.3.6).

28. An injective perfect function transfers an almost semi-strong CJ-space to almost semi-strong CJ-space, see Proposition (3.3.9).

29. The following conditions are equivalent to definition of contractible J-space.

- ❖ For every proper closed cover $\{E, F\}$ with $E \cap F$ compact E or F is homotopy equivalent to a point.
- ❖ For every proper closed cover $\{E, F\}$ with $E \cap F$ compact, there exists $x_\circ \in E$(or $x_\circ \in F$) such that $\{x_\circ\}$ is a deformation retract of E (or of F).
- ❖ For every proper closed cover $\{E, F\}$ with $E \cap F$ compact, E or F is a retract of any cone over it.
- ❖ For every proper closed cover $\{E, F\}$ with $E \cap F$ compact, every function f from E (or F) to an arbitrary space Y, is null-homotopic.
- ❖ For every proper closed cover $\{E, F\}$ with $E \cap F$ compact, every function f from an arbitrary space Y to E (or F) is null-homotopic. See Theorem (4.1.11).

30. A perfect and contractible function transfers a contractible J-space to contractible J-space, see Proposition (4.2.11).

31. The property of being contractible J-space is a topological property, see Corollary (4.2.19).

§2 FUTURE STUDIES

In the finally of thesis, we suggest some problems to future studies.

1. We can use the concept pseudocompact to define new topological space, and can be call **pseudo J-space**,

" A space X is pseudo J- space if for every closed cover $\{N, M\}$ of X with $N \cap M$ countably compact, then N or M is pseudocompact". Or **strong pseudo *J*- space**,

" A space X is strong pseudo J- space if every countably compact $E \subset X$ is contained in a pseudocompact $W \subset X$ with $X \setminus W$ connected".

2. Another concept can be used to deduce a new space, which is connectedness. New space can be named **united J- space,**

" A space X is united J- space if for every closed cover $\{N, M\}$ of X with $N \cap M$ connected, then N or M is compact". Or **strong united J-space,**

" A space X is strong united J- space if every connected $E \subset X$ is contained in a compact $W \subset X$ with $X \setminus W$ connected".

3. Contractibility and retraction can be used together to define a **reco J- space**,

" A space X is reco J- space if, whenever $\{N, M\}$ is a closed cover of X with $N \cap M \neq \emptyset$ and a retract of N, then M is contractible".

Similarly, we can use many known concepts as: path connectedness, locally compactness, limit point compactness and manifold space, to define new topological spaces.

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المستخلص

الهدف الرئيسي من الأطروحة هو دراسة الفضاءات التبولوجية من النوع CJ- ، strong CJ- ، almost CJ- ، almost strong CJ- و contractible J-.

والتي تعتبر تعميمات لفضاءات تبولوجية من النوع J- والنوع strong J- التي درست من قبل الباحث Michael [12].

من ناحية أخرى نبحث عن الشروط الضرورية والكافية التي بتوافرها يكون الفضاء التبولوجي من النوع CJ- أو من النوع almost CJ-. وأثبات الشرط الضروري الذي يجعل كل فضاء CJ(almost CJ)- هو فضاء strong CJ(almost strong CJ)-، حيث وجد أن كل فضاء strong CJ(almost strong CJ)- هو فضاء CJ(almost CJ)-، لكن العكس غير صحيح إلا إذا كان الفضاء متراص موضعيا. فضلا عن ذلك، العديد من المفاهيم الجديدة قد قدمت مع توضيحها بأمثلة وخواص، وهذه المفاهيم هي الفضاء من النوع semi-strong CJ- و الفضاء من النوع semi-weak CJ- و الفضاء من النوع weak CJ-و الفضاء من النوع almost semi-strong CJ- و الفضاء من النوع almost semi-weak CJ- و الفضاء من النوع almost weak CJ- .

شكر وتقدير

الحمد لله الذي بنعمته تتم الصالحات

شكر الإله لكم جميل صنيعكم   وجزاكم الفردوس أرفع منزلة

بكل لمسات الوفاء والإخلاص .... وبكل بارقة ولاء وعرفان .... أتقدم بالشكر والتقدير والعرفان إلى مشرفتي وأستاذتي **الأستاذ المساعد الدكتورة *نرجس عبد الجبار داود*** لتفضلها بالأشراف على هذه الأطروحة، والتي لم تألو جهداً في التوجيه والمساعدة بكل ما هو مفيد، ولكل ما أبدته من ملاحظات وإرشادات قيمة كان لها الأثر الكبير في إنجاز هذا العمل وإخراجه على أكمل وجه، فجزاها الله خير الجزاء. كما أتقدم بجزيل الشكر إلى أستاذتي **الأستاذ الدكتورة *لمى ناجي توفيق*** لدعمها المتواصل لي.

كما أشكر وأثمن دور رئيس قسم الرياضيات **الأستاذ المساعد الدكتور *مجيد أحمد ولي*** في تذليل العقبات التي واجهتنا أثناء فترة العمل وأشكر وأثمن دور أستاذتي **الأستاذ المساعد الدكتورة *سلوى سلمان عبد*** في تذليل العقبات التي واجهتنا أثناء فترة الدراسة حيث كانت رئيسة القسم آنذاك. كما أشكر أستاذي **الأستاذ *عباس نجم سلمان*** والذي كان مقرراً للدراسات العليا أثناء فترة دراستي وكان لنا نعم الأخ والناصح والمرشد.

تتسابق الكلمات وتتزاحم العبارات لتنظم عقد الشكر الذي يستحقه من كان لهن قدم السبق في ركب العلم والتعليم إليكن يامن بذلتن ولم تنتظرن العطاء أستاذتي الفاضلة ***الأستاذ دكتورة ليلى سلمان محمود*** و أستاذتي الفاضلة ***الأستاذ ا لدكتورة أنعام محمد علي*** إليكن أهدي عبارات الشكر والتقدير.

ولا أنسى أن أشكر جميع زميلاتي وزملائي في قسم الرياضيات لما أبدوه من تعاون في مجال العمل وأخص بالذكر الصديقة ***م. سوسن جواد كاظم*** وأشكر الصديقة ***الأستاذ المساعد نيران صباح جاسم*** لما قدمته لي من مساعدة في طباعة الأطروحة ولن أنسى رفيقة دربي ***المدرس الدكتورة مي محمد هلال***.

كما أتقدم بجزيل شكري وعظيم إمتناني ***لأفراد عائلتي*** لما قدموه لي من دعم ومساعدة ولتهيئتهم الظروف الملائمة لإتمام دراستي، أسأل الله أن يديمهم ويحفظهم ويوفقهم.

سعاد جدعان جاسم   تموز ٢٠١٧

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بغداد
كلية التربية للعلوم الصرفة / ابن الهيثم

حول بعض الفضاءات التبولوجية الجديدة

رســـــالة

مقدمة إلـى كلية التربية للعلوم الصرفة / ابن الهيثم، جامعـة بغداد وهي جزء من متطلبات نيل درجة الدكتوراه فـي فلسفة علوم الرياضيـــات

مــن قبل

سعاد جدعان جاسم

بإشــراف

أ.م.د. نرجس عبد الجبار داود

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