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New Types Of Regular Fuzzy Modules And Pure Fuzzy Submodules

A Thesis

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BY

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بسم الله الرحمن الرحيم

﴿....يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ

وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ ....﴾

صدق الله العظيم

سورة المجادلة**الآية**(١١)

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SHAYMAA
2017

SUPERVISOR CERTIFICATION

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ABSTRACT

In this thesis, we introduce the concept of T-Pure fuzzy submodule as an extension of the concept of T-pure submodules. A fuzzy submodule A of a fuzzy module X is called T-Pure fuzzy submodule ,if for each fuzzy ideal I of R we have $I^2X \cap A = I^2A$. Also, we present the concept of T-pure fuzzy ideal which is similar to the concept of T-Pure fuzzy submodule and so, we study the concept of Quasi T-pure fuzzy submodule.

Based on that we also provide the concept of T-regular fuzzy module Which is defined by the following A fuzzy module X is called T-regular if every fuzzy submodule of X is fuzzy T-pure.

This idea leads us to introduce the concept of strongly pure fuzzy ideal, where a fuzzy ideal I of a ring R is called strongly pure fuzzy ideal if $x_t \subseteq I$. $\exists P_r \subseteq I$ s.t $x_t = x_t P_r$ where P_r is a prime fuzzy singleton of R.

Beside this we discuss the strongly regular fuzzy ring , means a ring R is called strongly regular fuzzy ring if and only if for all fuzzy singleton r_ℓ of R , $\exists$ a prime fuzzy singleton x_t of R such that $r_\ell = r_\ell x_t r_\ell, \forall\, t, \ell \in (0,1]$.

Among this study we present the concept of strongly pure fuzzy submodule and strongly regular fuzzy module .A is called Strongly pure fuzzy submodule denoted by S-pure fuzzy submodule ,if there exists a prime fuzzy ideal P of a ring R such that $PX \cap A = PA$. let X be a fuzzy module of an R-module M, X is called S-regular if every $x_t \subseteq X, \forall\, t \in (0,1]$ is S-regular fuzzy singleton.

The relationships a mong all these concepts and several other types of modules are studied.

CONTENTS

CHAPTER ONE

T-PURE FUZZY SUBMODULE

INTRODUCTION

In this thesis ,we introduction and study the concepts of T-pure fuzzy submodule (ideal), Quasi T-pure fuzzy submodule and weakly T-pure fuzzy submodules as a generalization of the ordinary concepts of T-pure submodule (ideal), Quasi T-pure submodule and weakly T-pure submodules ,see[2], where we call a fuzzy submodule A of fuzzy module X, T-pure fuzzy submodule of X if for each fuzzy ideal I of R such that $I^2X \cap A = I^2A$. see Definition (1.2.3), also a fuzzy ideal I of R is called T-pure fuzzy ideal if for each fuzzy ideal $J^2$of R , $J^2 \cap I = J^2I$ Definition(1.3.1). A is called a quasi T-pure fuzzy submodule of X denoted by QT-pure if for every $x_t \subseteq X$ and $x_t \nsubseteq A$, $\forall\, t \in (0,1]$ there exists a T-pure fuzzy submodule B of X such that $A \subseteq B$ and $x_t \nsubseteq B$ see Definition (1.4.2). and A is called weakly T-pure fuzzy submodule of X. if for each fuzzy singleton r_t of R , $(r_t)^2X \cap A = (r_t)^2A$. Definition (1.5.2).

As well as we have in this letter circulating the T-regular fuzzy module and weakly T-regular fuzzy module see [5],[16]. Where we call a fuzzy module X of an R-module M, T-regular fuzzy module if every fuzzy submodule of X is T-pure fuzzy submodule , Definition (2.1.3) also X is called weakly T-regular if every fuzzy submodule of X is weakly T-pure, Definition (2.3.2).

The principle aim of this research is to make a detailed study of various properties of T-pure fuzzy submodule (ideal), Quasi T-pure fuzzy submodule , Weakly T-pure fuzzy submodules , T-regular fuzzy module ,weakly T-regular fuzzy module, strongly pure fuzzy ideal (submodule) and strongly regular fuzzy ring (module).

This thesis consist of three chapter. Chapter one consist of five sections. In section one, we reviewed some basic definition and properties about fuzzy set, fuzzy module ,fuzzy submodule and fuzzy ideal of a ring R will be needed later on.

In sections two and three studied T-pure fuzzy submodule (ideal) and gives some important results of that study either,

In section four, we introduced the concept of Quasi T-pure fuzzy submodule , the last section in this chapter, we have explained the concept weakly T-pure fuzzy submodules.

Chapter two contains three sections. In section one, we introduce the concept of T-regular fuzzy module by extending the ordinary notion of T-regular module.

In section two, we examined T-regular fuzzy modules with related modules

In section there we study the behavior of weakly T-regular fuzzy module under homomorphism see,

1)Let $f : X \longrightarrow X^{\backslash}$ be a fuzzy epimorphism and $X, X^{\backslash}$ are two fuzzy modules of $M_{,}$ $M^{\backslash}$ respectively, if X is weakly T-regular fuzzy module, then $X^{\backslash}$ is weakly T-regular fuzzy module. Proposition (2.3.5)

2)Let $f : X \longrightarrow Y$ be a fuzzy epimorphism and X, Y are two fuzzy modules of $M_{1,}M_2$ respectively, and every fuzzy submodule of X is f-invariant , if Y is weakly T-regular fuzzy module, then X is weakly T-regular fuzzy module.

And other result, Proposition (2.3.6)

3)Let X and Y be two fuzzy modules of an R-module M_1 and M_2 respectively. If $X \oplus Y$ is weakly T-regular fuzzy module of $M_1 \oplus M_2$, then X and Y are weakly T-regular fuzzy module ,Proposition (2.3.7)

4) Let X be a fuzzy module of an R-module M. if every fuzzy submodule of X is fuzzy divisible , then X is weakly T-regular fuzzy module, (Proposition 2.3.8)

5) Let X be a divisible fuzzy module of an R-module M. then X is weakly T-regular fuzzy module $\Leftrightarrow$ every fuzzy submodules of X is divisible. Corollary (3.10)

Chapter three is devoted to study strongly pure fuzzy ideal (submodule) and strongly regular fuzzy ring (module).

This chapter consists of three sections. We present in section one the concept of strongly pure fuzzy ideal with some characterization as follows :

1)Let I be a fuzzy ideal of R then I is S-pure if and only if I_t is a S-pure ideal of R‹$\forall$ $t \in (0,1]$ proposition (3.1.4)

2)Let K be fuzzy ideal of R , and let R be a factorial fuzzy ring, such that $y_r \neq 0_1$ non unit fuzzy singleton of R is fuzzy irreducible. Then K is S-pure fuzzy ideal $\Leftrightarrow$ K is pure fuzzy ideal, Proposition(3.1.11).

3)Let K and H are two fuzzy ideal of a ring R, if K is S-pure fuzzy ideal of R then $K \cap H$ is S-pure fuzzy ideal of R, Proposition (3.1.12).

4)Let K and H are two fuzzy ideal of a ring R, such that $K \subseteq H$ if $K \cap H$ is S-pure fuzzy ideal of R, then K is S-pure fuzzy ideal of R, Proposition (3.1.13).

5)Let K and H are two fuzzy ideal of a ring R, if K $\oplus$ H is S-pure fuzzy ideal of R. then either K or H is S-pure fuzzy ideal of R, Proposition (3.1.16).

6) Let K is S-pure fuzzy ideal of R, such that K $\subseteq$ F-J(R), then K={0} Proposition (3.1.1).

Section two is studied the concepts strongly regular fuzzy ring and the following are some of the results proved in this section

1)Let R be a S-regular fuzzy ring $\Leftrightarrow$ R_t be S-regular ring ,$\forall t \in (0,1]$ Proposition (3.2.3).

2)Let R_1 and R_2 are two fuzzy ring , if $R_1 \oplus R_2$ is S-regular fuzzy ring. Then either R_1 or R_2 is S-regular fuzzy ring, Proposition (3.2.5).

In section three, we turn to fuzzify the concept strongly pure fuzzy submodule

a sufficient condition on strongly pure fuzzy submodule Proposition (3.3.4).

In section four is the end of chapter three we study the strongly regular fuzzy module and we prove some results about it. Also, we study its generalization.

Chapter One

T-pure Fuzzy Submodules

Introduction:

Recall that a submodule N of an R-module M is a T-pure submodule of M if for each ideal I of R, $I^2M \cap N = I^2N$. [2,Defenition 1.1.4]

Also, an ideal I of a ring R is called T-pure ideal of R if for each ideal J of R, $J^2 \cap I = J^2 I$.

Also, if every ideal of a ring R is T-pure ideal ¸then we say R is T-regular ring see [2,Definition (1.1.22)].

And so on, A submodule N of an R-module M is called a quasi T-pure submodule of M if for each $x \in M$ and $x \notin N$, there exists a T-pure submodule L of M such that $N \subseteq L$ and $x \notin L$.[2]

We fuzzify these concept such as: (T-pure submodule of M, T-pure ideal of R, Quasi T-pure submodule and weakly T-pure submodules) to T-pure fuzzy submodules , T-pure fuzzy ideal of R, quasi T-pure fuzzy submodule and weakly T-pure fuzzy submodule.

This chapter consist of five section. In section one we recall that many definitions and properties which are needed to prove the results in the next sections.

In section two we introduce the definition of T-pure fuzzy submodule and various basic properties about T-pure fuzzy submodule are discussed.

In section three included T-pure fuzzy ideal and basis properties about this concept.

In section four we introduce the definition of Quasi T-pure fuzzy submodule , some basic properties are studied .

In section five, we introduce the definition of weakly T-pure fuzzy submodules and we give some characterization of studied .

§1.1 Basic Concepts

In this section we recall that familiar concepts and some well-known results which are relevant in our work.

"**Definition 1.1.1:**

Let S be a non-empty set and I be the closed interval [0,1] of the real line (real numbers). A fuzzy set A in S (a fuzzy subset of S) is a function from S in to I. [21]"

"**Definition 1.1.2:**

Let $x_t : S \longrightarrow [0,1]$ be a fuzzy set in S, where $x \in S$, $t \in [0,1]$, define by $x_t(y) = t$ if $x=y$ and $x_t(y) = 0$ if $x \neq y$, for all $y \in S$. x_t is called a fuzzy singleton or fuzzy point in S. [23]"

"**Proposition 1.1.3:**

Let a_t, b_k be two fuzzy singletons of S. if $a_t = b_k$, then $a=b$ and $t=k$, where $t, k \in (0,1]$. [16]"

"**Definition 1.1.4**

Let A and B be two fuzzy sets in S, then:

1-$(A \cap B)(x) = \min\{A(x), B(x)\}$, for all $x \in S$

2-$(A \cup B)(x) = \max\{A(x), B(x)\}$, for all $x \in S$

$A \cap B$ and $A \cup B$ are fuzzy set in S.

In general. If $\{A_\alpha, \alpha \in \Lambda\}$ is a family of fuzzy set in S, then:

$[\bigcup_{\alpha \in \Lambda} A_\alpha](x) = \sup\{A_\alpha(x), \alpha \in \Lambda,\}$, for all $x \in S$.

$[\bigcap_{\alpha\in\Lambda} A_\alpha](x)=\inf\{A_\alpha(x), \alpha \in \Lambda,\}$, for all $x\epsilon S$.

Which are also fuzzy set. [23]"

"**Definition 1.1.5:**

Let A and B be two fuzzy sets in S, then:

1-A=B if and only if A(x) = B(x), for all x∈S

2-A⊆B if and only if A(x) ≤ B(x), for all x∈S

If A⊂ B and there exists x∈S such that A(x) <B(x), then A is called a proper fuzzy subset of B and written A ⊂B.

By part (2), we can deduce that x_t ⊆A if and only if A(x)≥t. [22]"

"**Definition 1.1.6:**

Let A be a fuzzy set in S, for all t∈[0,1], the set A_t ={x∈S, A(x)≥t} is called a level subset of A. [14]"

"**Remark 1.1.7:**

The following properties of level subsets hold for each t ∈ [0,1]

1-$(A \cap B)_t=A_t \cap B_t$

2-$(A \cup B)_t=A_t \cup B_t$

3- A= B if and only if $A_t = B_t$, for all t ∈ [0,1].[21]"

"**Definition 1.1.8:**

Let f be a function from a set M into a set N.A fuzzy subset A of M is called f-invariant if A(x)=A(y) whenever f(x)=f(y), where x, y ϵ M. [8]"

"**Pnroposition 1.1.9:**

If f is a function from a set M into a set N, A and B are fuzzy subsets of M, then f(A ∩ B)=f(A) ∩ f(B).[24]"

"**Proposition 1.1.10:**

Let f be a function from a set M into set N. A and B are fuzzy subsets of N then f^{-1}(A ∩ B)= f^{-1}(A) ∩ f^{-1}(B). [16]"

"**Proposition 1.1.11:**

If f is a function defined on a set M, A_1 and A_2 are fuzzy subsets of M, B_1 and B_2 are fuzzy subset of f(M). the following are true:

1-$A_1 \subseteq f^{-1}(f(A_1))$.

2- $A_1 = f^{-1}(f(A_1))$. whenever A_1 is f-invariant

3-$f(f^{-1}(B_1)) = B_1$

4- if $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$

5-if $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.[8]"

"**Definition 1.1.12:**

Let M be an R-module. A fuzzy set X of M is called a fuzzy module of an R-module M if,

1-X(0)=1.

2-X(x-y)≥min {X(x), X(y), for all x ,y∈M}.

3-X(rx)≥X(x), for all x∈M, r∈R.[23]"

"**Definition 1.1.13:**

Let X and A be two fuzzy modules of R-module M. A is called a fuzzy submodule of X if $A \subseteq X$. [14]"

"**Definition 1.1.14**:

Let f be a mapping from a set M into a set N, let A be a fuzzy set in M and B be a fuzzy set in N.

The image of A denoted by f(A) is the fuzzy set in N defined by:

$$f(A)_{(y)} = \begin{cases} \sup\{A(z) \mid z \in f^{-1}(y)\}, \text{if} \quad f^{-1}(y) \neq \emptyset, \text{for all } y \in N \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

And the inverse image of B denoted by $f^{-1}(B)$ is the fuzzy set in M defined by: $f^{-1}(B)(x)= B(f(x))$, for all $x \in M$ [21]."

"**Definition 1.1.15**:

If A is a fuzzy submodule of an R- module M, then the submodule A_t of M is called the level submodule of M where $t \in [0,1]$ [14]."

"**Proposition 1.1.16:**

A is a fuzzy submodule of fuzzy module X of an R- module M if and only if, A_t is a submodule of X_t, for each $t \in [0,1]$ [14]."

"**Definition 1.1.17:**

Let X be a fuzzy module of an R-module M. let $\{A_\alpha, \alpha \in \Lambda\}$ be a family of fuzzy submodules of X, then:

1-$\bigcap_{\alpha\in\Lambda} A_\alpha$ is a fuzzy submodules of X.

2- If $\{A_\alpha, \alpha \in \Lambda\}$ is a chain, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is a fuzzy submodule of X [14]."

"**Definition 1.1.18:**

Let A and B be two fuzzy subset of an R-module M. The addition A+B defined by $(A+B)(x)=\sup\{\min\{A(a), B(b), x=a+b\}$ for all $x, a, b \in M\}$.

A+B is fuzzy subset of M.[23]"

"**Proposition 1.1.19:**

Let A and B be two fuzzy submodules of a fuzzy module X, then A+B is fuzzy submodule of X.[23]"

"**Definition 1.1.20:**

Let X and Y be fuzzy modules of R-modules M_1 and M_2 respectively, $f: X \longrightarrow Y$ is called fuzzy homomorphism if $f: M_1 \longrightarrow M_2$ is R-homomorphism and $Y(f(x))=X(x)$ for each $x \in M_1$ [12]."

"**Proposition 1.1.21:**

Let X and Y be fuzzy modules of an R-modules M_1 and M_2 respectively, Let $f: X \longrightarrow Y$ be a fuzzy homomorphism.

If A and B are two fuzzy submodules of X and Y respectively, then

1-f(A) is a fuzzy submodule of Y.

2-$f^{-1}(B)$ is a fuzzy submodule of X [10]."

"**Remark 1.1.22:**

If X is a fuzzy module of an R- module M, A is a fuzzy submodule of X and r_t is a fuzzy singleton of R, then $r_tA \subseteq A$ and r_tX is a fuzzy submodules of X [5]."

"**Lemma 1.1.23:**

Let A be a fuzzy submodules of a fuzzy modules X of an R-module M and r_t be a fuzzy singleton of R , then for any $x \in M$.

$$(r_t A)(x) = \begin{cases} \sup\{\inf\{t, A(w)\}\} \text{ for all } x \in M \\ x = rw \\ 0 \qquad\qquad \text{otherwise} \end{cases} \qquad [21]."$$

"**Definition 1.1.24:**

A fuzzy subset K of a ring R is called a fuzzy ideal of R, if for each x, y∈R:

1-K(x-y)≥ min {K(x), K(y)}

2-K(x.y) ≥ max {K(x), K(y)} [9]."

"**Proposition 1.1.25:**

If f is a homomorphism from a ring R_1 onto a ring R_2. Then the following are true :

1- f(A) is a fuzzy ideal of R_2, for each fuzzy ideal A of R_1[22].

2- f^{-1}(B) is a fuzzy ideal of R_1, for each fuzzy ideal B of R_2.[8]

3-f(A_1 A_2)= f(A_1).f(A_2), where A_1 , A_2 are fuzzy ideals of R_1[22]."

"**Proposition 1.1.26:**

A fuzzy subset K of R is a fuzzy ideal of R if and only if K_t, $t \in [0,1]$ is an ideal of R [9]."

"**Proposition 1.1.27:**

Let $\{A_\alpha, \alpha \in \Lambda\}$ be a family of fuzzy ideal of R, then:

1-$\bigcap_{\alpha\in\Lambda} A_\alpha$ is a fuzzy ideal of R.

2- If $\{A_\alpha, \alpha \in \Lambda\}$ is a chain, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is a fuzzy ideal of R [9]."

"**Definition 1.1.28**:

Let X be a fuzzy module of an R-module M. Let A be a fuzzy submodule of X and K be a fuzzy ideal of R. The product KA of K and A is defined by: KA(x) =

$$\begin{cases} \sup_{x=\sum_{i=1}^{n} r_i x_i}\{\inf\{k(r_1),\ldots k(r_n), A(x_1),\ldots\ A(x_n) \text{for some } r_i \in R, x_i, n \in N \\ 0 \qquad\qquad \text{otherwise} \end{cases} \quad [21]$$

<u>Note</u> $(KA)_t = K_t\, A_t$ for each $t \in [0,1]$, [10]."

"**<u>Proposition 1.1.29:</u>**

Let X be a fuzzy module of an R-module M. Let A be a fuzzy submodule of X and K be a fuzzy ideal of R, then KA is a fuzzy submodule of X [21]."

"**<u>Proposition 1.1.30:</u>**

Let X be a fuzzy module of an R-module M, let A and B be two fuzzy submodule of X and I, J are two fuzzy ideal of R, then :

1-IA ⊆ IB if A ⊆ B.

2-IA ⊆ JA if I ⊆ J [21]."

"**<u>Remark 1.1.31:</u>**

If X is a fuzzy module of an R-module M and $x_t \subseteq X$, then for all fuzzy singleton r_k of R, $r_k x_t = (rx)_\lambda$,where λ =min{k,t} [15]."

"**Proposition 1.1.32:**

Let X be a fuzzy ring of R_1 and Y is a fuzzy ring of R_2. Let A be a fuzzy ideal of R_1 such that $A \subseteq X$ and B is a fuzzy ideal of a ring R_2 such that $B \subseteq Y$. Then $A \oplus B$ is a fuzzy ideal of $R_1 \oplus R_2$.
where$(A\oplus B)(a,b)=\{A(a),B(b)\}$ for all $(a,b) \in R_1 \oplus R_2$ [1]."

"**Proposition 1.1.33:**

Let X be a fuzzy module of an R-module M, A be a fuzzy submodule of X and r_t be a fuzzy singleton of R, then $r_t \circ A = \langle r_t \rangle \circ A$ [23], from $r_t \circ A = r_t A$. Then $r_t A = \langle r_t \rangle A$ [15]. "

"**Definition 1.1.34:**

Let A and B be two fuzzy ideals of R. The product AB of A and B defined as:$(AB)(x) = \sup_{x=\sum_{i=1}^{n} a_i b_i}\{\min(\min(A(a_i), B(b_i)))\}$ $a_i,b_i \in R$, $n \in N$. [6]."

"**Proposition 1.1.35:**

Let A and B be two fuzzy ideals of R. then

1-AB is a fuzzy ideal of R[21]

2-$(AB)_t = A_t B_t$, for all $t \in [0,1]$ [6]."

"**Definition 1.1.36:**

Let A and B be two fuzzy submodules of a fuzzy module X of an R-module M .The residual quotient of A and B denoted by (A:B) is the fuzzy subset of R defined by: $(A:B)(r)=\sup\{t\in [0,1]: r_tB \subseteq A\}$, for all $r \in R$.

That is $(A:B) = \{r_t: r_t.B \subseteq A$; r_t is a fuzzy singleton of R$\}$.

If $B=<x_k>$, then $(A:<x_k>)=\{r_t : r_t x_k \subseteq A$, r_t is fuzzy singleton of R$\}$ [21]."

"**Proposition 1.1.37:**

Let A and B be two fuzzy submodules of a fuzzy module X of an R-module M. Then the residual quotient of A and B (A:B) is a fuzzy ideal of R [23]."

"**Definition 1.1.38:**

Let A be non- empty fuzzy submodule of a fuzzy module X. The fuzzy annihilator of A denoted by F-annA is defined by:

$(F\text{-ann } A)(r) = \sup\{t : t\in[0,1], r_t.A\subseteq 0_1\}$, for all $r \in R$.

That is F-ann $A=(0_1:A)$ [20]."

"**Definition 1.1.39:**

Let X be a fuzzy module of an R–module M is called fuzzy cyclic module if there exists $x_t \subseteq X$ such that $y_k \subseteq X$ written as $y_k = r_\ell x_t$ for some fuzzy singleton r_ℓ of R where $k,\ell,t \in (0,1]$ in this case, we shall write $X=(x_t)$ to denote the fuzzy cyclic module generated by x_t [5]."

"**Definition 1.1.40:**

A fuzzy ring R is said to be a fuzzy integral domain if R has no zero divisor.[8]"

"**Definition 1.1.41:**

A fuzzy set X of a ring R is called a fuzzy ring of R if for all $a, b \in R$

1-$X(a-b) \geq \min\{X(a), X(b)\}$

2-$X(ab) \geq \min\{X(a), X(b)\}$. [9]"

"**Definition 1.1.42:**

Let (x^{-1}) be the inverse element of x in R then $(x^{-1})_t$ is an inverse of a fuzzy singleton in A. and $x_t.(x^{-1})_t = (x.x^{-1})_t = 1_t = (x^{-1})_t .x_t$ where $1_t : R \rightarrow I$ such that

$$1_t(x) = \begin{cases} t & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases} \leq \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{otherwise} \end{cases} = \lambda_R(x) = 1 \text{ [4]. "}$$

§1.2 T-pure Fuzzy Submodule

"Recall that a submodule N of an R-module M is called a T-pure sub module of M if for each ideal I of R, $I^2M \cap N = I^2N$. see[2,Defenition 1.1.4]"

By the sequences of Definition(1.1.34) and Proposition(1.1.35) we define if I be a fuzzy ideal of R, then the product $I.I$ is denoted by I^2 is a fuzzy ideal.

"**Definition 1.2.1:**

Let X be a fuzzy module of an R-module M .let A be a fuzzy sub modules of X. A is called a pure fuzzy submodule if for each fuzzy ideal K of R, $KX \cap A = KA$. [16]"

"**Proposition 1.2.2:**

Let X be a fuzzy module of an R-module M and let A be fuzzy sub module of X. Then A is a pure fuzzy submodule if and only if A_t is a pure submodules of X_t, $\forall$ $t \in [0,1]$. [16]"

Now, we shall fazzify this concepts as follows:

Definition 1.2.3:

Let X be a fuzzy module of an R-module M and let C be a fuzzy sub modules of X. C is called T-pure fuzzy submodule of X if for each fuzzy ideal I of R such that $I^2X \cap C = I^2C$.

The following proposition specificates T-pure fuzzy submodule in terms of its level submodule.

Proposition 1.2.4:

Let X be a fuzzy module of an R-module M and let A be fuzzy sub modules of X. Then A is T-pure fuzzy submodule of X if and only if A_t is T-pure submodules of X_t, $\forall$ $t \in (0,1]$.

Proof:

($\Longrightarrow$)Let J be an ideal of ring R

Define $I^2: R \to [0,1]$ by $I^2(x) = \begin{cases} t & \text{if } x \in J \\ 0 & \text{otherwise} \end{cases} \quad \forall\ t \in (0,1]$

And let N be a submodule of an R-module M

Define A: M⟶[0,1] by $A(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases} \quad \forall\ t \in (0,1]$

It is clear that I^2 is fuzzy ideal of R and A is fuzzy submoduls of X .

Now, $A_t = N$, $I_t^2 = J$, $X_t = M$

Let A be T-pure fuzzy submodule of X . To prove A_t is T-pure submodules of X_t , $\forall$ t$\in$(0,1].

To show that $I_t^2X_t \cap A_t = I_t^2A_t$

$I_t^2X_t \cap A_t = (I^2X)_t \cap A_t$ by Proposition (1.1.28)

$= (I^2X \cap A)_t$ by Remark (1.1.7(1))

$=(I^2A)_t$ since A is T-pure

$=I_t^2A_t$ by Proposition (1.1.28)

Thus A_t is T-pure submodules of X_t , $\forall$ t$\in$(0,1].

Conversely**.** Let I^2 be a fuzzy ideal of R and A be a fuzzy submodules of X To prove A is T-pure fuzzy submodule of X

$(I^2X \cap A)_t=(I^2X)_t \cap A_t$ $\forall$ t$\in$(0,1]. by Remark (1.1.7(1))

$=I_t^2X_t \cap A_t$ by Proposition (1.1.28)

but A_t is T-pure submodules of X_t.

Then $I_t^2X_t \cap A_t = I_t^2A_t$

$= (I^2A)_t$ by Proposition (1.1.28)

Hence $(I^2X \cap A)_t=(I^2A)_t$

Implies that $I^2X \cap A = I^2A$ by Remark (1.1.7(3))

Therefore A is T-pure fuzzy submodule of X .

Remarks and Examples 1.2.5:

1- Let X be a fuzzy module of an R-module M and let C be a pure fuzzy submodule of X, then C is T-pure fuzzy submodule .

Proof: It is clear

The converse not true by

Example: Let M=Z_4 as Z-module and N=2Z_4

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define C: M⟶[0,1] by $C(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$ ∀ t∈(0,1]

Clearly X is fuzzy module, C is fuzzy submodule of X and X_t=M, C_t=N

C_t is T-pure submodules of X_t. by[2,Remarks and Examples (1.1.5 (1))]

Thus C is T-pure fuzzy submodule of X .by Proposition (1.2.4)

But C is not pure fuzzy submodule of X since if I_t=2Z where I:R→[0,1]

Such that I(x)=t if x∈2Z and I(x)=0 if x∉2Z

Now 2Z.Z_4 ∩ 2Z_4={$\bar{0}, \bar{2}$} but 2Z.2Z_4=2{$\bar{0}, \bar{2}$}={$\bar{0}$}

Thus C_t is not pure submodules of X_t

Therefore C is not pure fuzzy submodule of X. by Proposition (1.2.2)

2- Let X be a fuzzy module of an R-module M. It is clear that the fuzzy singleton {0_t} and X are always T-pure fuzzy submodule of X, ∀ t∈(0,1].

3-In the fuzzy module Z as Z-module. The only T-pure fuzzy submodule are fuzzy singleton {0_t} and Z.

Proof:

Let X: Z⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if } x \in Z \\ 0 & \text{otherwise} \end{cases}$

Let O_t: Z⟶[0,1] by $O_t(x) = \begin{cases} t & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$

X_t=Z and O_t=0

By(2) clear that X and O_t are T-pure fuzzy submodules

If there exists a fuzzy submodule A

Let nZ be a submodule of an Z-module and $\langle n\rangle^2$ be ideal of R

Let A: Z⟶[0,1] by $A(x) = \begin{cases} t & \text{if } x \in nZ \\ 0 & \text{otherwise} \end{cases} \quad \forall\, t\in(0,1]$

Let I^2:R⟶[0,1] by $I^2(x) = \begin{cases} t & \text{if } x \in \langle n\rangle^2 \\ 0 & \text{otherwise} \end{cases} \quad \forall\, t\in(0,1]$

Clearly $I^2_t=\langle n\rangle^2$, A_t=nZ and I is a fuzzy ideal

$n^2=n^2.\,1\in \langle n\rangle^2 Z\cap nZ$

But $n^2 \notin \langle n\rangle^2.nZ=n^3Z$

Thus A_t is not T-pure submodule

Therefore A is not T-pure fuzzy submodule by Proposition (1.2.4)

X_t=Z , 0_t=0 only two T-pure submodules

Hence 0_t ,X only T-pure fuzzy submodule of Z-module,$\forall$ t∈(0,1] by Proposition (1.2.4)

4-Let X be a fuzzy module of an Z-module Q, and let C be a non-empty fuzzy cyclic submodule of X, then C is not T-pure fuzzy submodule of X.
Proof:

Define X: Q⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{otherwise} \end{cases}$

Define C: Q⟶[0,1] by $C(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases} \quad \forall\, t\in(0,1]$

where N is cyclic submodule of Q , X_t=Q and C_t=N

N is not T- pure submodule of Q by [2,Remarks and Examples (1.1.5(4))]

Then C is not T-pure fuzzy submodule of X by Proposition (1.2.4)

5- Let X be a fuzzy module of an R-module M. let C be a T-pure fuzzy submodule of X such that C≅B where B is a fuzzy submodule of X, then B not need be T- pure fuzzy submodule of X for example.

Example: Let M=Z and N=Z, K=2Z

Let X: M⟶[0,1] by $X(x)=\begin{cases}1 & \text{if } x\in M\\ 0 & \text{otherwise}\end{cases}$

Let C: Z⟶[0,1] by $C(x)=\begin{cases}t & \text{if } x\in Z\\ 0 & \text{otherwise}\end{cases}$ $\forall\, t\in(0,1]$

Let B: Z⟶[0,1] by $B(x)=\begin{cases}t & \text{if } x\in 2Z\\ 0 & \text{otherwise}\end{cases}$ $\forall\, t\in(0,1]$

Clearly C and B are fuzzy submodules of X , Now C_t=Z and B_t=2Z and Z≅2Z but 2Z is not T- pure submodules [2,Remarks and Examples (1.1.5(6))]

Then B is not T-pure fuzzy submodule by Proposition (1.2.4)

The following proposition give some properties of T-pure fuzzy submodule .

Proposition 1.2.6:

Let A and B are two fuzzy submodules of a fuzzy module X. If A is T-pure fuzzy submodule of X ,B ⊆ A. and B is T-pure fuzzy submodule of A, then B is T-pure fuzzy submodule of X.

Proof:

Since A is a T-pure fuzzy submodule of X then $I^2X\cap A=I^2A$…..(1)

Where I^2 is fuzzy ideal of R and since B is a T-pure fuzzy submodule of A then

$I^2A\cap B=I^2B$….(2)

Now, we get $I^2B=\ I^2A\cap B$ by(2)

$=(I^2X\cap A)\cap B$ by(1)

$=I^2X\cap(A\cap B)$

$=I^2X\cap B$ since B ⊆A

Therefore B is T-pure fuzzy submodule of X .

Proposition 1.2.7:

Let X be a fuzzy module of an R-module M. and let C be a T-pure fuzzy submodule of X. If B is a fuzzy submodule of X contating C, then C is T-pure fuzzy submodule of B .

Proof :

Let I^2 be a fuzzy ideal of R and C be a T-pure fuzzy submodule of X.

Hence $I^2X \cap C = I^2C$

Now, $I^2B \cap C = (I^2B \cap I^2X) \cap C$ since $C \subseteq B \subseteq X$

$=I^2B \cap (I^2X \cap C)$

$=I^2B \cap I^2C$

$=I^2C$ since $C \subseteq B$

Thus C is T-pure fuzzy submodule of a fuzzy submodule B.

"**Definition 1.2.8:**

Let X and Y be two fuzzy modules of M_1 and M_2 respectively. Define

$X \oplus Y: M_1 \oplus M_2 \longrightarrow [0,1]$ by $(X \oplus Y)(a, b) = \min\{(X(a), Y(b))$ for all $(a, b) \in M_1 \oplus M_2\}$

$X \oplus Y$ is called a fuzzy external direct sum of X and Y . [5,Definition(3.5.1)]″

Lemma 1.2.9:

Let N_1 and N_2 be two submodules of R-module M_1 and M_2 .If $N_1 \oplus N_2$ is T-pure submodule of $M_1 \oplus M_2$ then N_1 and N_2 are T-pure submodules in M_1 and M_2.

Proof:

Let I be an ideal of a ring R

To prove $I^2M_1 \cap N_1 = I^2N_1$ and $I^2M_2 \cap N_2 = I^2N_2$ for each ideal I^2 of R.

Since $N_1 \oplus N_2$ is T-pure in $M_1 \oplus M_2$ we get:

$I^2(M_1 \oplus M_2) \cap (N_1 \oplus N_2) = I^2(N_1 \oplus N_2)$

$(I^2M_1 \oplus I^2M_2) \cap (N_1 \oplus N_2) = I^2N_1 \oplus I^2N_2$

Implies that $(I^2M_1 \cap N_1) \oplus (I^2M_2 \cap N_2) = (I^2N_1 \oplus I^2N_2)$

Hence $I^2M_1 \cap N_1 = I^2N_1$ and $I^2M_2 \cap N_2 = I^2N_2$

Thus N_1 and N_2 are T-pure.

The following proposition shows that the direct sum is closed under the concept of T-pure fuzzy submodule .

<u>Proposition 1.2.10:</u>

If A and B be are two fuzzy submodules of fuzzy module X_1 and X_2 respectively ,then A and B are T-pure fuzzy submodule of X_1 and X_2 iff $A \oplus B$ is T-pure fuzzy submodule of $X_1 \oplus X_2$.

<u>Proof:</u>

($\Longrightarrow$)Let A and B are T-pure fuzzy submodules of X_1 and X_2.

To prove $A \oplus B$ is T-pure fuzzy submodule of $X_1 \oplus X_2$.

Then A_t and B_t are T-pure submodules of $(X_1)_t$ and $(X_2)_t$ by Proposition (1.2.4)

$A_t \oplus B_t = (A \oplus B)_t$ and $(X_1)_t \oplus (X_2)_t = (X_1 \oplus X_2)_t$, $\forall t \in (0,1]$by [16,Lemma(2.2.4)]

Therefore $(A \oplus B)_t$ is T-pure submodule of $(X_1 \oplus X_2)_t$ by [2,Corollary (1.1.19)]

Thus $A \oplus B$ is T-pure fuzzy submodule of $X_1 \oplus X_2$. by Proposition (1.2.4)

($\Longleftarrow$)Let $A \oplus B$ is T-pure fuzzy submodule of $X_1 \oplus X_2$.

To show that A and B are T-pure fuzzy submodules of X_1 and X_2 respectively.

By [16,Lemma (2.2.4)] and Proposition (1. 2 .4) we get :-

$(A\oplus B)_t = A_t \oplus B_t$ is T-pure submodule in $(X_1)_t \oplus (X_2)_t$

Thus A_t and B_t are T-pure submodule of $(X_1)_t$ and $(X_2)_t$ by Lemma (1.2.9)

Therefore A and B are two T-pure fuzzy submodules of a fuzzy modules X_1 and X_2 by Proposition (1.2.4).

Now ,we introduce the following proposition.

Proposition 1.2.11:

Let H be a direct summand of a fuzzy module X. Then H is T-pure fuzzy submodule of X.

Proof:

Let $X = H \oplus C$, where C is a fuzzy submodule of X and H is a direct summand of X.

Then $X=H+C$ and $H \cap C = 0$ by [Def of fuzzy direct summand]

To prove H is T-pure fuzzy (i.e $I^2X \cap H = I^2H$ for each fuzzy ideal I^2 of R)

$I^2X \cap H = I^2(H \oplus C) \cap H$

$= (I^2H \oplus I^2C) \cap (H \oplus 0)$

$= (I^2H \cap H) \oplus (I^2C \cap 0)$ by [16,Lemma (2.2.5)]

$= (I^2H \cap H) \oplus 0$

$= I^2H \cap H$

$= I^2H$ since $I^2H \subseteq H$

Therefore H is T-pure fuzzy submodule of X.

Proposition 1.2.12:

Let $f: X \longrightarrow Y$ be a fuzzy epimorphism and X, Y are two fuzzy modules of R-modules M_1, M_2 respectively, let B be a fuzzy submodule of X and X are f-invariant, if B is a T-pure fuzzy submodule of X, then f(B) is T-pure fuzzy submodule of Y.

Proof:

To prove $I^2Y \cap f(B) = I^2f(B)$ for each fuzzy ideal I^2 of R.

$I^2Y \cap f(B) = I^2f(X) \cap f(B)$ since f is epimorphism

$= f(I^2X) \cap f(B)$ by[16,Lemma (2.3.1)]

$= f(I^2X \cap B)$ by proposition (1.1.9)

$= f(I^2B)$ since B is T-pure

$= I^2f(B)$ by[16,lemma (2.3.1)]

Thus f(B) is T-pure fuzzy submodule of Y.

Proposition1.2.13:

Let $f : X \longrightarrow Y$ be a fuzzy epimorphism and X, Y are two fuzzy modules of R-modules $M_{1,}$ M_2 respectively, if C is a T-pure fuzzy submodule of Y, such that every fuzzy submodule of X is f-invariant ,then $f^{-1}(C)$ is T-pure fuzzy submodule of X.

Proof:

To prove $f^{-1}(C)$ is T-pure (i.e $I^2X \cap f^{-1}(C) = I^2\ f^{-1}(C)$ for each fuzzy ideal I^2 of R).

$f(I^2X \cap f^{-1}(C)) = f(I^2X) \cap f(f^{-1}(C))$ by Proposition (1.1.9)

$= f(I^2X) \cap C$ by Proposition (1.1.11)

$= I^2f(X) \cap C$ by[16,Lemma(2.3.1)]

$= I^2Y \cap C$ since f is epimorphism

$= I^2C$ since C is T-pure

Therefore $f(I^2X \cap f^{-1}(C)) = I^2C$ so $f^{-1}[f(I^2X \cap f^{-1}(C))] = f^{-1}(I^2C)$

But $f^{-1}[f(I^2X \cap f^{-1}(C))] = I^2X \cap f^{-1}(C)$

And other hand and $f^{-1}(I^2C) = I^2 f^{-1}(C)$ [16, Lemma(2.3.4)]

Thus $I^2X \cap f^{-1}(C) = I^2 f^{-1}(C)$

Lemma 1.2.14:

Let $\{A_i, i \in N\}$ be an ascending chain of T-pure fuzzy submodules of a fuzzy module X and let I be a fuzzy ideal of R, then $I^2[\bigcup_{i\in N} A_i] = \bigcup_{i\in N}[I^2A_i]$

proof:

Since $I^2A_i \subseteq I^2[\bigcup_{i\in N} A_i], \forall\, i \in N$

Implies that $\bigcup_{i\in N}[I^2A_i] \subseteq I^2[\bigcup_{i\in N} A_i]$……(1)

And by $I^2[\bigcup_{i\in N} A_i] \subseteq XI^2$ and $I^2[\bigcup_{i\in N} A_i] \subseteq \bigcup_{i\in N} A_i$

Since $\bigcup_{i\in N} A_i$ is fuzzy submodule of X

Then $I^2[\bigcup_{i\in N} A_i] \subseteq I^2X \cap [\bigcup_{i\in N} I_i]$

$= \bigcup_{i\in N}[I^2X \cap A_i]$ by [16,Lemma (2.4.1)]

$= \bigcup_{i\in N}[I^2A_i]$ since A_i T-pure fuzzy submodule

Thus $I^2[\bigcup_{i\in N} A_i] \subseteq \bigcup_{i\in N}[I^2A_i]$ by (2)

Therefore $I^2[\bigcup_{i\in N} A_i] = \bigcup_{i\in N}[I^2A_i]$

Proposition 1.2.15:

If $\{J_i, i \in N\}$ be an ascending chain of T-pure fuzzy submodule of fuzzy module X , then $\bigcup_{i\in N} J_i$ is T-pure fuzzy submodule of X .

proof:

We must prove that $\bigcup_{i\in N} J_i$ is T-pure
i.e $C^2X \cap [\bigcup_{i\in N} J_i] = C^2[\bigcup_{i\in N} J_i]$ for each fuzzy ideal C^2 of R

$C^2X \cap [\bigcup_{i\in N} J_i] = \bigcup_{i\in c}[C^2X \cap J_i]$ by[16,Lemma 2.4.1]

$= \bigcup_{i\in c}[C^2 J_i]$ since J_i T-pure fuzzy submodule of R

$= C^2[\bigcup_{i\in N} J_i]$ by Lemma (1.2.14)

Thus $\bigcup_{i\in N} J_i$ is T-pure fuzzy submodule of X .

§1.3 T-pure Fuzzy Ideal

"Recall that an ideal I of a ring R is called T-pure ideal of R if for each ideal J of R, $J^2 \cap I = J^2 I$.

If every ideal of a ring R is T-pure ideal ¸then we say R is T-regular ring see [2,Definition 1.1.22]"

We, shall fuzzify this concept as follows the Definitions

Definition 1.3.1:

An fuzzy ideal I of a ring R is called T-pure fuzzy ideal of R if for each fuzzy ideal $J^2$of R, then $J^2 \cap I = J^2 I$.

Definition 1.3.2:

If every fuzzy ideal of a ring R is T-pure fuzzy ideal ¸then we say R is T-regular fuzzy ring.

The following result characterizes T-pure fuzzy ideal in terms of it is level ideal.

Proposition 1.3.3:

Let I be a fuzzy ideal of R then I is T-pure if and only if I_t is a T-pure ideal of R،∀ t∈(0,1].

Proof:

(⟹)Let I be T-pure fuzzy ideal of R To prove I_t is a T-pure ideal of R, ∀ t∈(0,1].

Let J^2 be an ideal of R

Define K^2: R ⟶[0,1] by $K^2(x) = \begin{cases} t & \text{if} \quad x \in J^2 \\ 0 & \text{otherwise} \end{cases}$ ∀ t∈(0,1]

It is clear that K^2 is fuzzy ideal of R and $K_t^2 = J^2$

T.P $J^2 \cap I_t = J^2 I_t$

$J^2 \cap I_t = K_t^2 \cap I_t$

$= (K^2 \cap I)_t$ by Remark (1.1.7)

$= (K^2 I)_t$ since I is T-pure fuzzy ideal

$= K_t^2 I_t$ by Proposition (1.1.35(2))

$= J^2 I_t$ ∀ t∈(0,1]

Thus I_t is a T-pure ideal, ∀ t∈(0,1]

(⟸)Let I_t be a T-pure ideal To prove I is T-pure fuzzy ideal

Let K^2 be fuzzy ideal of R To prove $K^2 \cap I = K^2 I$

∀ t∈(0,1], $(K^2 \cap I)_t = K_t^2 \cap I_t$ by Remark (1.1.7)

$= K_t^2 I_t$ since I_t is T-pure ideal

$= (K^2 I)_t$ by Proposition (1.1.35(2))

Thus $(K^2 \cap I)_t = (K^2 I)_t$

Hence $K^2 \cap I = K^2 I$ by Remark (1.1.7(3))

Therefore I is T-pure fuzzy ideal of R.

<u>Remarks and Examples 1.3.4:</u>

1- Every regular fuzzy ring is T-regular fuzzy ring.

<u>Proof</u>: clearly

The converse not true by

<u>Example:</u> Let $R=Z_4$ be a ring and $K=\{\bar{0},\bar{2}\}$ be an ideal of R.

Define X: R⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if } x \in R \\ o & \text{otherwise} \end{cases}$

Define I: R⟶[0,1] by $I(x) = \begin{cases} t & \text{if } x \in K \\ o & \text{otherwise} \end{cases}$ $\forall$ $t \in (0,1]$

Clearly I is fuzzy ideal of R and $I_t=K$, $X_t=R$

Then X_t is T-regular ring by [2, Remarks and Examples(1.1.23(1))]

Hence X is T- regular fuzzy ring since every ideal of Z_4 is T-pure

By proposition (1.3.3) every ideal is fuzzy T-pure.

But X_t is not regular ring since ideal $\{\bar{0},\bar{2}\}$ is not pure by [2,Remarks and Examples(1.1.23(1))]

Thus I not pure fuzzy ideal by [16,Proposition (2.6.1)]

Then X is not regular fuzzy ring .

2-In any fuzzy ring there are two T-pure fuzzy ideals of R, R itself and $\{0_t\}$.

3- If R is a fuzzy field , then R is T- regular fuzzy ring.

<u>Proof</u>

Since every field has only one proper ideal {0} then by (2), R is T-regular fuzzy ring.

The converse of (3) is true if we gives the condition, R is a fuzzy integral domain.

4- Let R be a fuzzy integral domain. If R is T- regular fuzzy ring ,then R is a fuzzy field.

Proof:

Let I be a fuzzy ideal of R

Since R is T- regular fuzzy ring. Hence $J^2 \cap I = J^2 I$ for every fuzzy ideal $J^2$of R.

if we take I=J implies $I^2 = I^3$.

Thus for each fuzzy singleton $0 \neq r_\ell$ of R, $\forall\ \ell \in (0,1]$,$< r_\ell >^2 = < r_\ell >^3$

Hence $r_\ell^2 = x_t\, r_\ell^3$ for some fuzzy singleton x_t of R

Then $r_\ell^2(1\text{-}x_t r_\ell)=0$. but R be a fuzzy integral domain and $0 \neq r_\ell$

Implies $1\text{-}x_t r_\ell=0$. Thus $1=x_t r_\ell$

Therefore r_ℓ is invertible of R

Thus R is a fuzzy field.

First , we start with the following Proposition .

Proposition 1.3.5:

Let R_1 , R_2 be two rings and g be any epimorphism function from R_1 to R_2. If A be a T-pure fuzzy ideal of the ring R_1, then g(A) is a T-pure fuzzy ideal of R_2.

Proof:

Let I^2 be fuzzy ideal of R_2. To prove $g(A) \cap I^2 = I^2 g(A)$.

$g(A) \cap I^2 = g(A) \cap g(g^{-1}(I^2))$ by Proposition (1.1.11(3)).

$= g(A \cap g^{-1}(I^2))$ by Proposition (1.1.9)

But $g^{-1}(I^2)$ is fuzzy ideal of R_1 by Proposition (1.1.25)

And A is T-pure fuzzy ideal in R_1 , so that

$g(A \cap g^{-1}(I^2))= g(A.\, g^{-1}(I^2))$

$= g(A).g\,(g^{-1}(I^2))$ by Proposition (1.1.25)

$= g(A).\, I^2$ by Proposition (1.1.11(3))

Then g(A) is a T-pure fuzzy ideal of R_2.

Proposition 1.3.6:

Let R_1, R_2 be two rings and f be any epimorphism function from R_1 to R_2 ,if C is T-pure fuzzy ideal of R_2 and every fuzzy ideal of R_1 is f-invariant. Then $f^{-1}(C)$ is T-pure fuzzy ideal of R_1.

Proof:

Let C be a fuzzy ideal of R_2, then $f^{-1}(C)$ is fuzzy ideal of R_1.by(1.1.25)

Let J^2 be a fuzzy ideal of R_1. To Prove $f^{-1}(C) \cap J^2= f^{-1}(C)J^2$

And by $f(f^{-1}(C) \cap J^2)= f(f^{-1}(C)) \cap f(J^2)$ see Proposition (1.1.9)

$=C \cap f(J^2)$ see Proposition (1.1.11(3))

$=C.\, f(J^2)$ since C is T-pure fuzzy ideal

$= f(f^{-1}(C))f(J^2)$ by Proposition (1.1.11(3))

$= f(f^{-1}(C)\, J^2)$ by Proposition (1.1.23)

Hence $f^{-1}\,[f(f^{-1}(C)\, J^2)] =f^{-1}\,[\, f(f^{-1}(C) \cap J^2)]$ and by hyposse. We get

$f^{-1}(C) \cap J^2 = f^{-1}(C)\, J^2$

Therefore $f^{-1}(C)$ is T-pure fuzzy ideal of R_1.

Proposition 1.3.7:

Let K be a fuzzy ideal of a ring R_1 and let J be a fuzzy ideal of a ring R_2, then $K \oplus J$ is T-pure fuzzy ideal of $R_1 \oplus R_2$ if and only if K and J are T-pure fuzzy ideal in R_1 and R_2 respectively.

Proof:

($\Longrightarrow$)Let $K \oplus J$ is T-pure fuzzy ideal . To Prove K and J are T-pure fuzzy ideals

Let A^2 and B^2 be two fuzzy ideals of R_1 and R_2 respectively. Then $A^2 \oplus B^2$ is fuzzy ideal of $R_1 \oplus R_2$ see Proposition (1.1.32)

Hence $(K \oplus J) \cap (A^2 \oplus B^2) = (K \oplus J)\ (A^2 \oplus B^2)$ since $(K \oplus J)$ is T-pure fuzzy ideal

But $(K \oplus J) \cap (A^2 \oplus B^2) = (K \cap A^2) \oplus (J \cap B^2)$ see[16,Lemma (2.6.6(2))]

And $(K \oplus J)\ (A^2 \oplus B^2) = (KA^2) \oplus (J\, B^2)$ see[16,Lemma (2.6.6(1))]

Therefore $(K \cap A^2) = KA^2$ and $J \cap B^2 = J\, B^2$ see[16,Lemma (2.6.7(2))]

Thus K and J are T-pure fuzzy ideal of R_1 and R_2.

($\Longleftarrow$)Let K and J are T-pure fuzzy ideal of R_1 and R_2.

Let A^2 and B^2 be two fuzzy ideal of R_1 and R_2

Hence $A^2 \oplus B^2$ is fuzzy ideal in $R_1 \oplus R_2$ see Proposition (1.1.32)

T.p $(K \oplus J) \cap (A^2 \oplus B^2) = (K \oplus J)\ (A^2 \oplus B^2)$

$(K \oplus J) \cap (A^2 \oplus B^2) = (K \cap A^2) \oplus (J \cap B^2)$ see[16,Lemma (2.6.6(2))]

$= (KA^2) \oplus (J\, B^2)$ since K and J are T-pure

$= (K \oplus J)\ (A^2 \oplus B^2)$ see[16,Lemma (2.6.6(1))]

Hence $(K \oplus J) \cap (A^2 \oplus B^2) = (K \oplus J)\ (A^2 \oplus B^2)$

Thus $K \oplus J$ is T-pure fuzzy ideal in $R_1 \oplus R_2$.

Proposition 1.3.8:

Let I and J be pure fuzzy ideal of R, then $I \cap J$ is T-pure fuzzy ideal of R.

Proof:

To prove $I \cap J$ is T-pure fuzzy ideal of R,

To show that for each fuzzy ideal K^2 of R $(I \cap J) \cap K^2 = (I \cap J) K^2$

Now, $\forall\, t \in (0,1]$ $((I \cap J)K^2)_t = (I \cap J)_t K_t^2$ by Proposition (1.1.35)

$=(I_t \cap J_t)K_t^2$ by Remark (1.1.7(1))

$= (I_t J_t)K_t^2$ since J_t is pure by [16,Propo(2.6.1)]

$= I_t (J_t K_t^2)$

$= I_t(JK^2)_t$ by Proposition (1.1.35)

$= I_t(J \cap K^2)_t$ since J is pure

$= I_t \cap (J \cap K^2)_t$ since I_t is pure by [16]

$= I_t \cap (J_t \cap K_t^2)$ by Remark (1.1.7(1))

$=(I_t \cap J_t) \cap K_t^2$

$= (I \cap J)_t \cap K_t^2$ by (1.1.7(1))

$=((I \cap J) \cap K^2)_t$ by (1.1.7(1))

Therefore $(I \cap J) \cap K^2=(I \cap J)K^2$

Then $I \cap J$ is T-pure fuzzy ideal of R.

Lemma 1.3.9:

Let $\{J_i, i \in N\}$ be an ascending chain of T-pure fuzzy ideal of R. let C be a fuzzy ideal of R, then $C^2[\bigcup_{i\in N} J_i] = \bigcup_{i\in N}[C^2 J_i]$.

proof:

$C^2 J_i \subseteq C^2[\bigcup_{i\in N} J_i], \forall\, i \in N$

Implies that $\bigcup_{i\in N}[\ C^2 J_i] \subseteq C^2[\bigcup_{i\in N} J_i]$……(1)

But $C^2[\bigcup_{i\in N} J_i] \subseteq C^2$ and $C^2[\bigcup_{i\in N} J_i] \subseteq \bigcup_{i\in N} J_i$

Since $\bigcup_{i\in N} J_i$ is fuzzy ideal

On other side $C^2[\bigcup_{i\in N} J_i] \subseteq\ C^2 \cap [\bigcup_{i\in N} J_i]$

$=\bigcup_{i\in N}[\ C^2 \cap J_i]$ by [16,Lemma (2.4.1)]

$=\bigcup_{i\in N}[\ C^2 J_i]$ since J_i T-pure fuzzy ideal of R

Thus $C^2[\bigcup_{i\in N} J_i] \subseteq \bigcup_{i\in N}[\ C^2 J_i]$

Therefore $C^2[\bigcup_{i\in N} J_i] = \bigcup_{i\in N}[C^2 J_i]$

Proposition 1.3.10:

If $\{I_i, i \in N\}$ be an ascending chain of T-pure fuzzy ideal of R, then $\bigcup_{i\in N} I_i$ is T-pure fuzzy ideal of R.

proof:

We must show that for each fuzzy ideal $C^2$of R
$C^2 \cap [\bigcup_{i\in N} I_i] = C^2[\bigcup_{i\in N} I_i]$

$C^2 \cap [\bigcup_{i\in N} I_i] =\bigcup_{i\in N}\ [\ C^2 \cap I_i]$ by[16,Lemma (2.4.1)]

$=\bigcup_{i\in N}\ [\ C^2 I_i]$ since I_i T-pure fuzzy ideal of R

$= C^2[\bigcup_{i\in N} I_i]$ by Lemma(1.3.9)

Hence $\bigcup_{i\in N} I_i$ is T-pure fuzzy ideal of R.

§1.4 Quasi T-pure Fuzzy Submodules

"Recall that a submodule N of an R-module M is called a quasi pure sub module of M if for each $x \in M$ and $x \notin N$, there exists a pure submodule L of M such that $N \subseteq L$ and $x \notin L$. [17]

And a submodule N of an R-module M is called a quasi T-pure submodule of M if for each $x \in M$ and $x \notin N$, there exists a T-pure submodule L of M such that $N \subseteq L$ and $x \notin L$.[2]

And an ideal I of a ring R is called quasi T-pure ideal if it is quasi T-pure submodule of the R-module M. [2]"

We fuzzify this concepts as follows:-

Definition 1.4.1:

Let X be a fuzzy module of an R-module M and A be a fuzzy submodule of X. A is called a quasi-pure fuzzy submodule of X denoted by Q-pure if for every $x_t \subseteq X$ and $x_t \nsubseteq A$, $\forall\, t \in (0,1]$ there exists a pure fuzzy submodule B of X such that $A \subseteq B$ and $x_t \nsubseteq B$.

Definition 1.4.2:

Let X be a fuzzy module of an R-module M and A be a fuzzy submodule of X. A is called a quasi T-pure fuzzy submodule of X denoted by QT-pure if for every $x_t \subseteq X$ and $x_t \nsubseteq A$, $\forall\, t \in (0,1]$ there exists a T- pure fuzzy submodule B of X such that $A \subseteq B$ and $x_t \nsubseteq B$.

Definition 1.4.3:

Let I be a fuzzy ideal of a ring R then I is called quasi T-pure fuzzy ideal of R if it is quasi T-pure fuzzy submodule of X, if X is a fuzzy module of an R-module M .

Proposition 1.4.4:

Let A be a fuzzy submodule of a fuzzy module X, then A is a QT-pure fuzzy submodule of X if and only if A_t is QT-pure submodule of X_t , $\forall$ t $\in$ (0,1].

Proof:

Let K, J be two submodule of R-module M

Define X: M⟶[0,1] s.t $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define A: M⟶[0,1] s.t $A(x) = \begin{cases} t & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t$\in$(0,1]

Define B: M⟶[0,1] s.t $B(x) = \begin{cases} t & \text{if } x \in J \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t$\in$(0,1]

Clearly X is fuzzy module ,A and B are two fuzzy submodules of X and X_t=M ,A_t=K ,$B_t = J$, $\forall$ t$\in$(0,1]

Let A be a QT-pure fuzzy submodule of X To prove A_t is QT-pure submodule of X_t ,$\forall$ t$\in$(0,1].

Let $x \in X_t$ and $x \notin A_t$. So $X(x) \geq t$ and $A(x) < t$.

Hence $x_t \subseteq X$ and $x_t \nsubseteq A$

Since A is QT-pure fuzzy submodule then there exists a T-pure fuzzy submodule B of X such that $A \subseteq B$

Then $B_t = J$ is a T-pure submodule of X_t by Proposition(1.2.4)

Therefore $A(x) \leq B(x)$ by Definition(1.1.5(2))

and $x_t \nsubseteq B$ imply $B(x) \ngeq t$. Then $x \notin B_t$, $\forall$ t$\in$(0,1]

Then A_t is QT-pure submodule of X_t .

Conversely let A_t be a QT-pure submodule of X_t .To show that A is a QT-pure fuzzy submodule of X .

Let $x_t \subseteq X$ and $x_t \nsubseteq A$. Since $x_t \subseteq X$ imply $X(x) \geq t$.

Then $x \in X_t$ and $x_t \nsubseteq A$ see(Definition (1.1.5))

Hence $A(x) < t$. implies that $x \notin A_t$ $\forall\, t \in (0,1]$

Since A_t is a QT-pure submodule of X_t, then there exists a T-pure submodule B_t of X_t=M such that $A(x) \leq B(x)$

Then B is T-pure fuzzy submodule of X , by Proposition (1.2.4)

Therefore $A \subseteq B$,by Definition(1.1.5 (2)) and $x \notin B_t$

Then $x_t \nsubseteq B$ $\forall\, t \in (0,1]$ by Definition (1.1.5)

Then A is a QT-pure fuzzy submodule of X.

Remarks and Examples 1.4.5:

1- Let X be a fuzzy module of an R-module M. If C is a T- pure fuzzy submodule of X, then C is QT-pure fuzzy submodule of X.

Proof: It is clear

The converse not true by

Example: Let $M = Z_8 \oplus Z_2$ as Z-module and $N = <(\bar{2}, \bar{0})>$

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define C: M⟶[0,1] by $C(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$ $\forall\, t \in (0,1]$

It is clear that X is fuzzy module, C is fuzzy submodules of X and X_t=M ,C_t=N

C_t is QT-pure submodules of X_t. by[2,Remarks and Examples (3.3.2 (1))]

Thus C is QT-pure fuzzy submodule of X by Proposition (1.4.4)

But C_t is not T-pure submodule of X_t by[2,Remarks and Examples (3.3.2 (1))]

Therefore C is not T-pure fuzzy submodule of X. by Proposition (1.2.4)

2- Let X be a fuzzy module of an R-module M. It is clear that the submodule X and $< 0_1 >$ are always QT-pure fuzzy submodule of X.

3- Let X be a fuzzy module of an R-module M and let C be a Q-pure fuzzy submodule of X, Then C is QT-pure fuzzy submodule of X.

Proof: It is clear

The converse not true by

Example: Let M=Z_4 as Z-module and N=2Z_4

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if} \quad x \in M \\ 0 & \text{otherwise} \end{cases}$

Define C: M⟶[0,1] by $C(x) = \begin{cases} t & \text{if} \quad x \in N \\ 0 & \text{otherwise} \end{cases}$ ∀ t∈(0,1]

Clearly X is fuzzy module, C is fuzzy submodule of X and X_t=M, C_t=N

C is T-pure submodules of X. by Remarks and Examples (1.2.5(1))

Thus C is QT-pure fuzzy submodule of X by(1)

But C is not pure fuzzy submodule of X by Remarks and Examples(1.2.5(1))

Therefore C is not Q-pure fuzzy submodule of X.

4- Let X be a fuzzy module of an R-module M. and A be a QT-pure fuzzy submodule of X such that A≅B where B is a fuzzy submodule of X, then B need not be QT- pure fuzzy submodule of X , for example.

Example: Let M=Z and N=Z, K=2Z

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if} \quad x \in M \\ 0 & \text{otherwise} \end{cases}$

Define A: M⟶[0,1] by $A(x) = \begin{cases} t & \text{if} \quad x \in N \\ 0 & \text{otherwise} \end{cases}$ ∀ t∈(0,1] and

Define B: M⟶[0,1] by $B(x) = \begin{cases} t & \text{if} \quad x \in K \\ 0 & \text{otherwise} \end{cases}$ ∀ t∈(0,1]

Clearly X is fuzzy module ,A and B are fuzzy submodules of X, and $X_t=M$, $A_t=Z$, $B_t=2Z$ and $Z\cong 2Z$

But $2Z$ is not QT- pure submodule of X_t [2,Remarks and Examples (3.3.2(3))]

Thus B is not QT-pure fuzzy submodule of X, by Proposition (1.4.4)

Proposition 1.4.6:

Let X be a fuzzy module of an R-module M , and let A be a T-pure fuzzy submodule of X . If B is QT-pure fuzzy submodule of A, then B is QT-pure fuzzy submodule of X.

Proof:-

Let $x_t \subseteq X$ with $x_t \nsubseteq B$, then either $x_t \subseteq A$ or $x_t \nsubseteq A$. assume that $x_t \subseteq A$

Since B is a QT-pure fuzzy submodule of A, so there exists a T-pure fuzzy submodule C of A such that $B \subseteq C$ and $x_t \nsubseteq C$,

Thus we have C is T-pure in A and A is T-pure of X .

By Proposition (1.2.6), so C is T-pure of X.

Therefore B is QT-pure fuzzy submodule of X.

If $x_t \nsubseteq A$,then nothing to prove since A is a T-pure fuzzy submodule of X containing B and $x_t \nsubseteq A$.

Proposition 1.4.7:

Let X be a fuzzy module of an R-module M , and C be a T-pure fuzzy submodule of X ,if $C \subseteq B$ where B is fuzzy submodule of X, then C is QT-pure fuzzy submodule of B.

Proof:-

Since C is T-pure fuzzy submodule of X and $C \subseteq B$

C is T-pure fuzzy submodule of B by Proposition (1.2.7)

Implies that C is QT-pure fuzzy submodule of B.

Proposition 1.4.8:

Let A and B are QT-pure fuzzy submodule of fuzzy module X, then $A \cap B$ is QT-pure fuzzy submodule of X.

Proof:-

Let $x_t \subseteq X$ and $x_t \nsubseteq A \cap B$, then either $x_t \nsubseteq A$ or $x_t \nsubseteq B$

Assume that $x_t \nsubseteq A$ since A is QT-pure fuzzy submodule of X,

Then there exists a T-pure fuzzy submodule C of X such that $A \subseteq C$ and $x_t \nsubseteq C$.

This implies that $A \cap B \subseteq C$ and $x_t \nsubseteq C$.

That is $A \cap B$ is QT-pure fuzzy submodule of X.

Proposition1.4.9:

Let A be a fuzzy submodule of a fuzzy module X . Then A is QT-pure fuzzy submodule of X if and only if there exists a collection of a fuzzy submodule $\{A_\alpha\}_{\alpha\in\Lambda}$,where Λ is an index set, such that for each $\alpha\in\Lambda$, A_α is T-pure fuzzy submodule of X and $A=\cap_{\alpha\in\Lambda} A_\alpha$

Proof:-

($\Longrightarrow$)Let A is QT-pure fuzzy submodule of X .To prove $A=\cap_{\alpha\in\Lambda} A_\alpha$

If A is T-pure fuzzy submodule of X, then nothing to prove.

If A is not T-pure fuzzy submodule of X.

Since A is QT-pure fuzzy submodule of X

Then there exists a collection of T-pure fuzzy submodule $\{A_\alpha\}_{\alpha\epsilon\Lambda}$ such that $A \subseteq \cap_{\alpha\in\Lambda} A_\alpha$ where $\wedge$ is an index set,

To show that $\cap_{\alpha\in\Lambda} A_\alpha \subseteq A$

Let $x_t \subseteq \cap_{\alpha\in\Lambda} A_\alpha$ then $x_t \subseteq A_\alpha$ for each $\alpha \in \Lambda$

Suppose $x_t \nsubseteq A$ since A is QT-pure fuzzy submodule of X

Then x_t is not contained in any T-pure fuzzy submodule that contains A

So $x_t \nsubseteq A_\alpha$ which is a contradiction

Therefore $x_t \subseteq A$ and hence $\cap_{\alpha\in\Lambda} A_\alpha \subseteq A$

That is $A=\cap_{\alpha\in\Lambda} A_\alpha$

($\Leftarrow$)Let $A=\cap_{\alpha\in\Lambda} A_\alpha$ where A_α is T-pure fuzzy submodule of X for each $\alpha \in \Lambda$ and A_α containing A

T.p A is QT-pure fuzzy submodule of X

Let $x_t \subseteq X$ and $x_t \nsubseteq A$ since $A= \cap_{\alpha\in\Lambda} A_\alpha$ so there exists $\alpha_i\epsilon\Lambda$ such that

$x_t \nsubseteq A_\alpha$ .Thus $A \subseteq A_\alpha$ and $x_t \nsubseteq A_\alpha$.

Therefore A is QT-pure fuzzy submodule of X .

<u>Proposition 1.4.10:</u>

Let X and Y be two fuzzy module of an R-module M_1, M_2. Let A and B be a fuzzy submodule of X and Y , then A is QT-pure fuzzy submodule of X and B is QT-pure fuzzy submodule of Y if and only if $A\oplus B$ is QT-pure fuzzy submodule of $X\oplus Y$.

<u>Proof:-</u>

($\Rightarrow$)If A and B are a QT-pure fuzzy submodules of X and Y

To prove $A\oplus B$ is QT-pure fuzzy submodule of $X\oplus Y$

Let $(x_t, y_t) \subseteq X \oplus Y$ with $(x_t, y_t) \nsubseteq A \oplus B$ then either $x_t \nsubseteq A$ or $y_t \nsubseteq B$

Assume that $x_t \nsubseteq A$ since A is QT-pure fuzzy submodule of X, so there exists a T-pure fuzzy submodule C of X such that $A \subseteq C$ and $x_t \nsubseteq C$,

But C is T-pure of X, so by (Proposition 1.2.10) $C \oplus Y$ is T-pure in $X \oplus Y$ also $A \oplus B \subseteq C \oplus Y$ and $(x_t, y_t) \nsubseteq C \oplus Y$

Similarly if $y_t \nsubseteq B$, then there exists a T-pure fuzzy submodule of $X \oplus Y$ containing $A \oplus B$ and does not contain (x_t, y_t).

Therefore $A \oplus B$ is QT-pure fuzzy submodule of $X \oplus Y$.

($\Longrightarrow$)Let $A \oplus B$ is QT-pure fuzzy submodule of $X \oplus Y$.

T.p A is QT-pure fuzzy submodule of X and B is QT-pure fuzzy submodule of Y.

Let $x_t \subseteq X$ and $x_t \nsubseteq A$, then $(x_t, 0) \nsubseteq A \oplus B$

Since $A \oplus B$ is QT-pure fuzzy submodule of $X \oplus Y$ so there exists a T-pure fuzzy submodule $D = C \oplus K$ of $X \oplus Y$ such that $A \oplus B \subseteq D$ and $(x_t, 0) \nsubseteq D$

It is follows that C is T-pure fuzzy submodule of X and K is T-pure fuzzy submodule of Y by (proposition 1.2.10)

Since $A \oplus B \subseteq C \oplus K$ so $A \subseteq C$ and $B \subseteq K$ but $(x_t, 0) \nsubseteq D = C \oplus K$ then $x_t \nsubseteq C$

Therefore A is QT-pure fuzzy submodule of X

Similarly B is QT-pure fuzzy submodule of Y .

"**Definition 1.4.11:**

Let X be a fuzzy module of an R-module M . X is called a finitely generated fuzzy module if there exists $x_{1,}x_2,x_3\ldots\ldots\subseteq X$ such that $X=\{a_1(x_1)_{t1}+a_2(x_2)_{t2}+\ldots..+a_n(x_n)_{tn}$, where $a_i\in R$ and $a(x)_t=(ax)_t$, $\forall t\in(0,1]$

Where $(ax)_{t(y)}=\begin{cases} t & \text{if } \quad y=ax \\ 0 & \text{otherwise} \end{cases}$.[7,Definition(2.11)]"

"**Definition 1.4.12:**

Let X be a fuzzy module of an R-module M . Then X is said to be faithful if F-ann$X=0_1$ where; F-ann$X=\{x_t: r_\ell x_t=0_1$ for all $x_t\subseteq X$ and r_ℓ be a fuzzy singleton of R , $\forall t, \ell\in(0,1]$ }. [20,Definition (3.2.6)]"

"**Definition 1.4.13:**

A fuzzy module X of an R-module M is called fuzzy multiplication module if for each non-empty fuzzy submodule A of X, there exists a fuzzy ideal I of R , such that A=IX. [5,Definition (2.2.1)]"

Proposition 1.4.14:

Let X be a fuzzy module of an R-module M, and let C be a fuzzy submodule of X, if X is finitely generated faithful multiplication fuzzy module and C is QT-pure fuzzy submodule of X. Then (C:X) is QT-pure fuzzy ideal of R.

Proof:-

Let $r_\ell\subseteq R$ and $r_\ell\nsubseteq(C:X)$, then $r_\ell X\nsubseteq C$ so there exists $x_t\subseteq X$, such that $r_\ell x_t\nsubseteq C$.

But C is QT-pure fuzzy submodule of X, then there exists a T- pure fuzzy submodule K of X such that $C\subseteq K$ and $r_\ell\, x_t\nsubseteq K$

Since X is finitely generated faithful multiplication so it is clear that if K is T- pure fuzzy submodule of X

Then (K:X) is T- pure fuzzy ideal I of R. by [4]

Also $r_\ell\, x_t \nsubseteq K$ then $r_\ell \nsubseteq (K:X)$

Hence (K:X) is T- pure fuzzy ideal I of R

Then $(C:X) \subseteq (K:X)$ and $r_\ell \nsubseteq (K:X)$

That is (C:X) is QT-pure fuzzy ideal of R.

Proposition 1.4.15:

Let X be a fuzzy module of an R-module M, and let A be a fuzzy submodule of X, if X is finitely generatedm faithful multiplication fuzzy module and (A:X) is QT-pure fuzzy ideal of R, then A=IX for some QT-pure fuzzy ideal I of R.

Proof:- It is clear

§1.5 Weakly T-pure Fuzzy Submodules

In this section we define the concept of weakly T-pure fuzzy sub modules and we give some characterizations of study.

"**Definition 1.5.1:**

Let X be a fuzzy module of an R-module M and A be a fuzzy sub module of X. A is called weakly pure fuzzy submodule of X. if for each fuzzy singleton r_t of R , $r_t X \cap A = r_t A$. [20]"

Definition 1.5.2:

Let X be a fuzzy module of an R-module M and C be a fuzzy sub modules of X. C is called weakly T-pure fuzzy submodule of X. if for each fuzzy singleton r_t of R, $(r_t)^2X \cap C = (r_t)^2C$.

Proposition 1.5.3:

Let X be a fuzzy module of an R-module M and A be a fuzzy sub modules of X, then A is weakly T-pure fuzzy submodule of X if and only if A_t is weakly T-pure submodule of X_t, $\forall\, t \in (0,1]$.

Proof:

Let A is weakly T-pure fuzzy submodule of X, To prove A_t is weakly T-pure submodule of X_t, $\forall\, t \in (0,1]$.

To show that $r^2X_t \cap A_t = r^2A_t$, $\forall\, t \in (0,1]$.

Let $y \in r^2X_t \cap A_t$, then $y=r^2m$ for some $m \in X_t$, $y \in A_t$

Thus $m_t \subseteq X$ and $y_t \subseteq A$. But $y=r^2m$

Implies $y_t=(r^2m)_t=(r_t)^2m_t$

Hence $y_t \subseteq (r_t)^2X \cap A$

But A is weakly T-pure fuzzy submodule, so $(r_t)^2X \cap A=(r_t)^2A$

Hence $y_t \subseteq (r_t)^2A$

It is follows that there exists $a_s \subseteq A$ such that $y_t = (r_t)^2a_s$

So that $s \geq t$, $\forall\, s \in (0,1]$.

Hence $a_t \subseteq a_s \subseteq A$, That is $a \in A_t$

Thus $y=r^2a \in r^2A_t$. so $r^2X_t \cap A_t \subseteq r^2A_t$

On the other hand, $r^2A_t \subseteq r^2X_t \cap A_t$

Thus $r^2X_t \cap A_t = r^2A_t$

Conversely if A_t is weakly T-pure submodule of X_t , $\forall$ t $\in$ (0,1].

To prove A is weakly T-pure fuzzy submodule of X

We must prove that $(r_\ell)^2X \cap A=(r_\ell)^2A$ for each fuzzy singleton r_ℓ of R, $\forall \ell \in(0,1]$.

Let $y_t \subseteq (r_\ell)^2X \cap A$, then $y_t \subseteq A$ and $y_t=(r_\ell)^2m_k$ for some $m_k \subseteq X$, $\forall k \in(0,1]$.

Thus $y \in A_t$ and $m\in X_K$, but $y_t=(r_\ell)^2m_k=(r^2m)_\lambda$ where $\lambda = \min\{\ell, k\}$

Thus $y=r^2m$ and $t= \lambda = \min\{\ell, k\}$, so $t\leq k$

Which implies $X_k \subseteq X_t$ and hence $m\in X_t$

Thus $y=r^2m \in r^2X_t \cap A_t = r^2A_t$

Since A_t is weakly T-pure submodule of X_t,

Hence $y=r^2m$ and $m \in A_t$

Which implies $y_t = (r_\ell)^2m_t$ and $m_t \subseteq A$.

So $y_t \subseteq (r_\ell)^2A$, then $(r_\ell)^2X \cap A \subseteq (r_\ell)^2A$

But $(r_\ell)^2A \subseteq (r_\ell)^2X \cap A$

Thus $(r_\ell)^2X \cap A=(r_\ell)^2A$

Hence A is weakly T-pure fuzzy submodule of X

Remark 1.5.4:

Every T-pure fuzzy submodule is weakly T-pure fuzzy submodule

Proof: It is clear

Remark 1.5.5:

Every weakly pure fuzzy submodule is weakly T-pure fuzzy submodule

Proof: It is clear

"**Definition 1.5.6:**

Let X be a fuzzy module of an R-module M, X is called torsion free if F-annx_t=0_1 for all $x_t \subseteq X$, $x_t \neq 0$ where F-annx_t={r_k : r_k is fuzzy singleton of R; $r_k\, x_t \subseteq 0_1$} equvilent T(x)=0.[20]"

"**Definition 1.5.7:**

Let X be a fuzzy module of an R-module M. A fuzzy submodule A of X is called prime submodule if and only if A≠ X and whenever $r_\ell x_t \subseteq A$. for a fuzzy singleton r_ℓ of R and $x_t \subseteq X$, implies either $r_\ell \subseteq (A\colon X)$ or $x_t \subseteq A$.[5]"

Proposition 1.5.8:

Let X be a fuzzy torsion free of an R-module M, where every fuzzy singleton of R is idempotent ,if A is weakly T-pure fuzzy submodule of X ,then A is a prime fuzzy submodule of X.

Proof:

Let $r_\ell x_t \subseteq A$, for fuzzy singleton r_ℓ of R and $x_t \subseteq X$. If $r_\ell \nsubseteq (A\colon X)$, then $r_\ell \neq 0_\ell$ implies $(r_\ell)^2 \neq 0_\ell$ and so $r^2 \neq 0$.

But A is weakly T-pure fuzzy submodule , hence $(r_\ell)^2X \cap A$=$(r_\ell)^2A$

Thus $r_\ell x_t \subseteq (r_\ell)^2X \cap A$. which implies that there exists $y_s \subseteq A$,such that $r_\ell x_t$=$r_\ell y_s$

Then $r_\ell(x_t\text{-}y_s) = 0_\lambda$ where $\lambda = \min\{\ell, t, s\}$.

Consequently : $r_\ell(x-y)_a = 0_\lambda$ where $a = \min\{t, s\}, r_\ell \neq 0_\ell$

So $(x-y)_a \subseteq T(x)$. but X is torsion free

Hence $(x-y)_a \subseteq 0_1$. thus x_t-$y_s \subseteq 0_1$

Which implies $x_t \subseteq y_s$.

Thus $x_t \subseteq A$.

Then A is a prime fuzzy submodule of X.

"**Definition 1.5.9:**

Let X be a fuzzy module of an R-module M . A proper fuzzy submodule A of X is called fuzzy quasi-prime submodule if whenever $a_\ell x_t b_h \subseteq A$. for fuzzy singleton a_ℓ ,b_h of R and $x_t \subseteq X$, implies that either $a_\ell x_t \subseteq A$ or $x_t b_h \subseteq A$. [5]"

"**Proposition 1.5.10:**

Every fuzzy prime submodule of a fuzzy module is fuzzy quasi prime submodule. [5]"

Lemma 1.5.11:

Let M be a torsion free R-module and $N \subseteq M$, where every element of R is idempotent . If N is weakly T-pure submodule in M , then N is a prime submodule in M.

Proof:

Let $rx \in N$ for some $r \in R$ and $x \in M$.

If $r \notin (N:M)$, then $r \neq 0$ so $r^2 \neq 0$

But N is weakly T-pure , hence $r^2M \cap N = r^2N$

Thus $rx \in r^2M \cap N$. which implies that there exists $y \in N$

Such that $rx = ry$, then $r(x-y)=0$, $r \neq 0$

Consequently $(x-y) \in T(M)$.

But M is torsion free ,

Thus $x-y=0$ which implies $x=y$

Then $x \in N$

Therefore N is prime submodule in M

Proposition 1.5.12:

Let X be a fuzzy a torsion free of an R-module M ,and let B be a proper fuzzy submodule of X, then B is weakly T-pure fuzzy submodule of X if and only if B is a prime fuzzy submodule with $(B{:}X)= 0_1$

Proof:

Let B be weakly T-pure fuzzy submodule of X ,then B_t is weakly T-pure submodule of X_t. $\forall\, t \in (0,1]$. by Proposition(1.5.3).

But X is a fuzzy torsion free so X_t is a torsion free, $\forall\, t \in (0,1]$ by [20,Lemma (2.2.23)]

Hence B_t is a prime submodule of X_t by Lemma (1.5.11) and $(B_t{:}\, X_t)=\{0\}, \forall$ $t \in (0,1]$ by [20]

But $(B{:}\, X)_t \subseteq (B_t{:}\, X_t)$, $\forall\, t \in (0,1]$ by [20, Lemma (2.1.5)]

So $(B{:}\, X)_t=\{0\}, \forall\, t \in (0,1]$. Hence $(B{:}\, X)= 0_1$

Now ,to prove B is prime fuzzy submodule .

Let $r_\ell x_t \subseteq B$, for $x_t \subseteq X$, $x_t \not\subseteq B$, r_ℓ is a fuzzy singleton of R .

Then $r_\ell x_t \subseteq (r_\ell)^2 X \cap B=(r_\ell)^2 B$

Since B is a weakly T-pure fuzzy submodule ,so $r_\ell x_t = r_\ell y_s$ for some $y_s \subseteq B$

But x_t-$y_s = (x-y)_\lambda$ where $\lambda = \min\{t, s\}$.

Hence $r_\ell (x-y)_\lambda \subseteq 0_1$. thus $r_\ell \subseteq$F-ann$(x-y)_\lambda$

But F-ann$(x-y)_\lambda$=0_1 since X is torsion free

Hence $r_\ell \subseteq 0_1$

On the other hand (B:X)$= 0_1$ implies that $r_\ell \subseteq$ (B:X)

Therefore B is a prime fuzzy submodule of X.

Conversely if B is a prime fuzzy submodule and (B:X)$= 0_1$ T.P B is weakly T-pure fuzzy submodule of X

Since B is a prime fuzzy submodule of X. Then B_t is a prime submodule of X_t, $\forall$ t $\in$(0,1] by [20,Theorem (2.1.10)]

But (B:X)$= 0_1$, so $(B{:}\,X)_t$=$\{0\}$, $\forall$ t $\in$(0,1]

Thus $(B_t{:}\,X_t)$=$\{0\}$ by [20, Proposition (2.1.8)]

Therefore B_t is weakly T-pure submodule in X_t, $\forall$ t $\in$(0,1]. by [20]

Hence B is weakly T-pure fuzzy submodule of X . by Proposition (1.5.3)

Corollary 1.5.13:

Let X be a fuzzy module of an R-module M, if C is T-pure fuzzy submodule then C is prime fuzzy submodule of X.

Proof: It is clear by Remark (1.5.4) and Proposition (1.5.8)

Corollary 1.5.14:

Let X be a fuzzy module of an R-module M, if A is weakly T-pure fuzzy submodule then A is quasi prime fuzzy submodule of X.

Proof: It is clear.

CHAPTER TWO

T-REGULAR FUZZY MODULES

Chapter two

T-regular Fuzzy Modules

Introduction:

First , recall that R-module M is called T-regular module if every submodule of M is T-pure . [2]

In this chapter we shill fuzzify of this concepts

This chapter consists of three section .our main aim in section one is definition the T-regular fuzzy module and we give some characterization for a fuzzy module to be a T-regular.

In section two we study the T-regular fuzzy module with related modules

In section three we define the concept of Weakly T-regular fuzzy modules and study the property of this concept.

§2.1 T-regular Fuzzy Modules

"**Definition 2.1.1:**

Let X be a fuzzy module of an R-module M. X is called regular if every fuzzy submodule of X is pure. [16]"

"**Proposition 2.1.2:**

Let X be a fuzzy module of an R-module M .Then X is regular fuzzy module if and only if X_t is regular modules , $\forall$ t$\in$(0,1] .[16]"

"Recall that R-module M is called T-regular module if every submodule of M is T-pure .see [2]"

Definition 2.1.3:

Let X be a fuzzy module of an R-module M. X is called T-regular fuzzy module if every fuzzy submodule of X is T-pure fuzzy submodule .

Proposition 2.1.4:

Let X be a fuzzy module of an R-module M .Then X is T-regular fuzzy module if and only if X_t is T-regular modules , $\forall$ t∈(0,1].

Proof:

(⟹)Let X be a T-regular fuzzy module To prove X_t is T-regular module , $\forall$ t∈(0,1].

Let K be a submodule of X_t , $\forall$ t∈(0,1] To prove K is T-pure in X_t

Define A: M⟶[0,1] by $A(x) = \begin{cases} t & \text{if} \quad x \in k \\ 0 & \text{otherwise} \end{cases} \quad \forall\, t \in (0,1]$

It is clear that A is a fuzzy submodule of X and A_t=K.

Hence A is T-pure fuzzy submodule since X is T-regular fuzzy module it

follows A_t is T-pure submodule of X_t . by Proposition (1.2.4)

Thus X_t is T-regular module

(⟸)Let X_t be T-regular modules,$\forall$ t∈(0,1]. To prove X is a T-regular fuzzy module.

Let A be fuzzy submodule of X. Then A_t is submodule of X_t , $\forall$ t∈(0,1] by (Proposition 1.1.16)

Hence A_t is T-pure submodule of X_t since X_t is T-regular

That is A is T-pure fuzzy submodule of X, by Proposition (1.2.4)

Thus X is T- regular fuzzy module.

Remarks and Examples 2.1.5:

1- Let X be a fuzzy module of an R-module M. If X is regular module then X is T-regular fuzzy module .

Proof: It is clear

The converse not true in general by

Example: Let M=Z_4 as Z-module and N=$2Z_4$

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if} \quad x \in M \\ 0 & \text{otherwise} \end{cases}$

Define A: M⟶[0,1] by $A(x) = \begin{cases} t & \text{if} \quad x \in N \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t ∈ (0,1]

It is clear that X_t=M, A_t=N and A is fuzzy submodule of X. Then

A is T-pure fuzzy submodule of X (Remarks and Examples (1.2.5(1))

Thus X is T- regular fuzzy module. But A is not pure fuzzy submodule of X Remarks and Examples (1.2.5(1))

Thus X is not regular fuzzy module.

2-Let X be a fuzzy module of an Z-module Z and Q . Then X is not T-regular fuzzy module.

Proof: See Remarks and Examples (1.2.5(3),(4))

3-Let M=Z_9 and N= $3Z_9$

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if} \quad x \in M \\ 0 & \text{otherwise} \end{cases}$

Define A: M⟶[0,1] by $A(x) = \begin{cases} t & \text{if} \quad x \in N \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t ∈ (0,1]

It is clear that A is fuzzy submodule of X and X_t=M , A_t=N

X_t=M= Z_9 is T-regular module by[2,Remarks and Examples (1.1.13(3))]

Thus X is T-regular fuzzy module by Proposition (2.1.4)

But X_t=M= Z_9 is not regular since the submodule A_t is not pure [2, Remarks and Examples (1.1.5(5))]

Hence A is not pure fuzzy submodule by Proposition (1.2.2)

Then X is not regular fuzzy submodule.

4-Let M=Z_{12} , N=$< 2 >$

Define X: Z_{12} ⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if} \quad x \in Z_{12} \\ 0 & \text{otherwise} \end{cases}$

Define A: Z_{12} ⟶[0,1] by $A(x) = \begin{cases} t & \text{if} \quad x \in < 2 > \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1]

It is clear that X_t=M, A_t=N and A is fuzzy submodules of X.

A_t is T-pure submodule of X_t, by[2, Remarks and Examples(1.1.5(3))]

Thus A is T-pure fuzzy submodule of X by Proposition (1.2.4)

Similarly $< 3 >, < 4 >, < 6 >$ are T-pure submodules

Therefore X is T- regular fuzzy module

5-Let M=Z_8 , N=$< 4 >$

Define X: Z_8 ⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if} \quad x \in Z_8 \\ 0 & \text{otherwise} \end{cases}$

Define A: Z_8 ⟶[0,1] by $A(x) = \begin{cases} t & \text{if} \quad x \in < 4 > \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1]

It is clear that X_t=Z_8, A_t=N and A is fuzzy submodule of X.

A_t is not T-pure submodule of X_t. by[2, Remarks and Examples (1.1.5(3))]

Hence A is not T-pure fuzzy submodule of X by Proposition (1.2.4)

Therefore Z_8 is not T-regular fuzzy module .

Proposition 2.1.6:

Let X be a T-regular fuzzy module, then every fuzzy submodule of X also T-regular fuzzy module.

Proof:

Let A be fuzzy submodule of X.

To Prove A is T-regular fuzzy module.

Let I^2 be a fuzzy ideal of R and let B be a fuzzy submodule of A then

$I^2A \cap B=(I^2X \cap A) \cap B$ since A is T-pure of X

$=I^2X \cap (A \cap B)$

$=I^2X \cap B$ since $B\subseteq A$

$=I^2B$ since B is T-pure of X

Therefore B is fuzzy T-pure of A implies A is T-regular fuzzy module.

§2.2 T-regular Fuzzy Modules With Related Modules

Proposition 2.2.1:

Let X be a fuzzy module of an R-module M .Then X is T-regular fuzzy module if and only if every fuzzy cyclic submodule of X is T-pure fuzzy submodule of X.

Proof:

($\Longrightarrow$) It abovis by (Def T-regular fuzzy module)

($\Longleftarrow$) Let A be fuzzy submodule of X and I^2 fuzzy ideal of R

To show that $I^2X \cap A = I^2A$.

Let $x_t \subseteq I^2X \cap A$ $\forall x_t\subseteq X$, $\forall t\in(0,1]$

Then $x_t \subseteq A$.

But the cyclic fuzzy submodule $< x_t >$ is T-pure fuzzy of X by hyp.

Therefor $I^2 X \cap < x_t > = I^2 < x_t >$

So $x_t \subseteq I^2 X \cap < x_t > = I^2 < x_t > \subseteq I^2 A$.

Thus $x_t \subseteq I^2 A$.

Hence $I^2 X \cap A = I^2 A$.

Proposition 2.2.2:

Let X be a fuzzy module of an R-module M. Then X is T- regular fuzzy module if and only if every finitely generated fuzzy submodule of X is T-pure fuzzy submodule.

Proof:

($\Longrightarrow$) It abovias by (Def T-regular fuzzy module)

($\Longleftarrow$)Since every finitely generated fuzzy submodule of X is T-pure fuzzy submodule of X

Then every fuzzy cyclic submodule of X is T-pure of X.

Thus by proposition (2.2.1)

Then X is T-regular fuzzy module.

"**Definition 2.2.3:**

A fuzzy ideal I of a ring R is called a principle fuzzy ideal if there exists $x_t \subseteq I$ such that $I=(x_t)$ then for each $m_s \subseteq I$, there exists a fuzzy singleton a_ℓ of R such that $m_s=a_\ell x_t$ where s, ℓ ,t $\in [0,1]$, that is $I=(x_t)=\{m_s \subseteq I,\ m_s=a_\ell x_t$ for some fuzzy singleton a_ℓ of $R\}$. [14]"

Proposition 2.2.4

Let X be a fuzzy module of an R-module M, if X is T-regular fuzzy module, then for each fuzzy singleton r_ℓ of R s.t $r_\ell^2 x_t = r_\ell^2 c_s r_\ell^2 x_t$ for each $x_t \subseteq X$ and for some fuzzy singleton c_s of R . $\forall s,\ell,t \in (0,1]$.

Proof:

Suppose that $x_t \subseteq X$ and for each fuzzy singleton r_ℓ of R.

Since $r_\ell^2 x_t \subseteq r_\ell^2 X$ and $r_\ell^2 x_t \subseteq < r_\ell^2 x_t >$ implies $r_\ell^2 x_t \subseteq r_\ell^2 X \cap < r_\ell^2 x_t >$

But X is T-regular then $r_\ell^2 X \cap < r_\ell^2 x_t > = r_\ell^2 < r_\ell^2 x_t >$

Thus $r_\ell^2 x_t \subseteq r_\ell^2 < r_\ell^2 x_t >$

Implies $r_\ell^2 x_t = r_\ell^2 c_s r_\ell^2 x_t$ for some fuzzy singleton c_s of R.

The converse of Proposition(2.2.4) cannot be a chieved unless we add the requirement follows when X is a cyclic fuzzy module or R is a principle fuzzy ideal we give in Proposition(2.2.5) and Proposition(2.2.6).

Proposition 2.2.5:

Let X be a fuzzy module of an R-module M, if R is a principle fuzzy ideal ring, and for every fuzzy singleton r_ℓ of R, and $x_t \subseteq X$ s.t $r_\ell^2 x_t = r_\ell^2 s_r r_\ell^2 x_t$ for some fuzzy singleton s_r of R implies that X is T-regular module. $\forall s, \ell, t \in (0,1]$.

Proof:

Suppose that A is fuzzy submodule of X and I be a fuzzy principle ideal of R.

To show that $r_\ell^2 X \cap A = r_\ell^2 A$ for every fuzzy singleton r_ℓ of R.

Let $x_t \subseteq r_\ell^2 X \cap A$. Then $x_t \subseteq r_\ell^2 X$ and $x_t \subseteq A$

Therefore $x_t=r_\ell^2 c_n$ for some $c_n \subseteq X$, $\forall n \in(0,1]$.

By hypothesis we get : $x_t= r_\ell^2 s_r r_\ell^2 c_n$ for some fuzzy singleton s_r of R

Thus $x_t \subseteq r_\ell^2 A$.

Then $I^2X \cap A = I^2A$ for each fuzzy ideal I of R by (hypothesis).

Proposition 2.2.6:

Let X be a cyclic fuzzy module of an R-module M, If $x_t \subseteq X$ and for each fuzzy singleton r_ℓ of R such that $r_\ell^2 x_t= r_\ell^2 c_s r_\ell^2 x_t$ for some fuzzy singleton c_s of R implies X is T-regular fuzzy module.

Proof:

Let $X=Rx_t$ be a fuzzy cyclic module for some $x_t \subseteq X$.

Let A be a fuzzy submodule of X and I is fuzzy ideal of R .let $y_t \subseteq I^2X \cap A$

Then $y_t \subseteq I^2X$ and $y_t \subseteq A$.

Thus $y_t= r_\ell^2 x_t=r_\ell^2 c_s r_\ell^2 x_t \subseteq r^2A$ for some fuzzy singleton c_s of R and $r_\ell \subseteq I$.

Therefore $y_t \subseteq I^2A$.

Implies X is T-regular

"**Proposition 2.2.7:**

Every cyclic fuzzy module is fuzzy multiplication .[5]"

Corollary 2.2.8:

Let X be a multiplication fuzzy module of an R-module M, if $x_t \subseteq X$ and for each fuzzy singleton r_ℓ of R such that $r_\ell^2 x_t= r_\ell^2 c_s r_\ell^2 x_t$ for some fuzzy singleton c_s of R then X is T-regular fuzzy module.

Corollary 2.2.9;

If R is a T-regular fuzzy ring ,then every fuzzy singleton a_t of R $a_t^2 = a_t^2 s_t a_t^2$ for some fuzzy singleton s_R of R and the converse is true if R is a principle fuzzy ideal ring .

"**Definition 2.2.10:**

A fuzzy submodule A of a fuzzy module X is called fuzzy semi-prime submodule, if for every fuzzy singleton r_ℓ of R and $x_t \subseteq X$, $k \in Z_+$ such that $r_\ell^k x_t \subseteq A$ then $r_\ell x_t \subseteq A$.[5, Definition (3.2.4)]"

Lemma 2.2.11:

Let X be a nonempty fuzzy module over a principle fuzzy ideal ring R. Then X is a regular fuzzy module iff every proper fuzzy submodule of X is fuzzy semi-prime.

Proof:

($\Longrightarrow$)Let X be a regular fuzzy module.

Let A be a proper fuzzy submodules of X.

Suppose that r_ℓ be a fuzzy singleton of R and $x_t \subseteq X$, $\forall \ell, t \in (0,1]$.

Such that $r_\ell^2 x_t \subseteq A$. It is clear that $r_\ell x_t \subseteq (r_\ell x_t) = (rx)_\lambda$ $\lambda = \min\{\ell, t\}$

Where $(rx)_\lambda$ be a cyclic fuzzy submodule of X generated by $r_\ell x_t$

Then $r_\ell x_t \subseteq r_\ell X$

Therefore $r_\ell x_t \subseteq r_\ell X \cap (rx)_\lambda$

But X is regular fuzzy module .

Hence $r_\ell X \cap (rx)_\lambda = r_\ell (rx)_\lambda$

We get $r_\ell x_t \subseteq r_\ell (rx)_\lambda = r_\ell (r_\ell x_t) = r_\ell^2 x_t \subseteq A$

Thus $r_\ell x_t \subseteq A$

Therefore A is fuzzy semi-prime submodule

(⟸)Suppose that B be a fuzzy semi-prime submodule of X and $X \neq B$.

To prove $m_s X \cap B = m_s B$ for all m_s a fuzzy singleton of R ,$\forall$ s $\in(0,1]$.

It is clear that $m_s B \subseteq m_s X \cap B$.

Now, let $y_n \subseteq m_s X \cap B$. $\forall n, s \in (0,1]$.

Then there exists $x_t \subseteq X$, $\forall t \in (0,1]$.

Such that $y_n = m_s x_t$, but $m_s y_n = m_s^2 x_t \subseteq m_s B$

Therefore $m_s x_t \subseteq m_s B$

Hence $y_n \subseteq m_s B$

Thus $m_s X \cap B = m_s B$ for all fuzzy singleton m_s of R.

Therefore X is regular fuzzy module .

<u>Proposition 2.2.12:</u>

Let X be a fuzzy module of an R-module M over a fuzzy principle ideal ring R, if every proper fuzzy submodule of X is fuzzy semi-prime .Then X is T-regular fuzzy module.

<u>Proof:</u> it directly for a Lemma (2.2.11)

The converse of above proposition is not true for example:

<u>Example:</u> Let $M=Z_4$ and N=(0)

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define A: Z⟶[0,1] by $A(x) = \begin{cases} t & \text{if } x = (0) \\ 0 & \text{otherwise} \end{cases}$ $\forall\, t \in (0,1]$

It is clear that A is fuzzy submodule of X, $A_t = (0)$ and $X_t = M$.

Then X is T- regular fuzzy module by Remarks and Examples (2.1.5(1))

But $A_t = (0)$ is not semi-prime See [2,Proposition (1.2.2)]

Hence A is not fuzzy semi-prime by[5, Proposition (3.2.6)]

X is called fuzzy semi-prime if (0) is semi-prime fuzzy submodules of X.

The converse of proposition(2.2.12) is true if we give the condition X is semi-prime fuzzy modules.

Proposition 2.2.13:

Let X be a fuzzy module of an R-module M. If X is T-regular and semi-prime fuzzy module, then every fuzzy submodule of X is semi-prime.

Proof:

Suppose that A is a fuzzy submodule of X and $r_\ell^2 x_t \subseteq A$ where r_ℓ is fuzzy singleton of R and $x_t \subseteq X$, $\forall \ell, t \in (0,1]$.

Implies $r_\ell^2 x_t \subseteq r_\ell^2 X \cap A = r_\ell^2 A$ since X is T-regular fuzzy module

Then $r_\ell^2 x_t = r_\ell^2 c_n$ for some $c_n \subseteq A$, $\forall$ n $\in (0,1]$.

Hence $r_\ell^2 (x_t - c_n) \subseteq 0_1$

But 0_1 is fuzzy semi-prime.

Hence $r_\ell (x_t - c_n) \subseteq 0_1$

Thus $r_\ell x_t = r_\ell c_n \subseteq A$.

Therefore A is fuzzy semi-prime submodule of X .

"**Definition 2.2.14:**

A fuzzy submodule A of a fuzzy module X is called divisible if for every fuzzy singleton r_ℓ of R, $r_\ell \neq 0$ $r_\ell A = A$, $\forall \ell \in (0,1]$. [16]"

"**Definition2.2.15:**

Let X be a fuzzy module of an R-module M. X is called fuzzy prime module if F-annX=F-ann A for every non –constant fuzzy submodule A of X .[5]"

Proposition 2.2.16:

Let X be a fuzzy module of an R-module M. If R is a fuzzy principle ideal and let A be a divisible fuzzy submodule of X, then A is T-pure fuzzy submodule of X.

Proof:

Let A be a fuzzy divisible submodule of X. then for each fuzzy singleton r_ℓ of R such that r_ℓ^2A=A

Therefore $r_\ell^2 X \cap A = r_\ell^2 A \;\; \forall\, \ell \in (0,1]$.

Remark 2.2.17:

The converse of the proposition(2.2.16) is not true for example:

Example: Let M=Z_4 and N=$\{\bar{0},\bar{2}\}$

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if } \; x \in M \\ 0 & \text{otherwise} \end{cases}$

Define A: M⟶[0,1] by $A(x) = \begin{cases} t & \text{if } \; x \in N \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, t \in (0,1]$

It is clear that A is fuzzy submodule of X and X_t=M , A_t=N

Then A is fuzzy T-pure by Remarks and Examples(1.2.5(1))

But A_t is not divisible submodule see [2,Remark (1.2.9)]

Hence A is not fuzzy divisible submodule see [5, Proposition (2.2.12)]

The converse is true if we give the condition X is divisible.

Proposition 2.2.18:

Let X be a fuzzy divisible module and R a principle fuzzy ideal , if C is T-pure fuzzy submodule of X , then C is divisible fuzzy submodule of X.

Proof:

Suppose that C is fuzzy T-pure of X.

To show that C is a fuzzy divisible of X.

Let $x_t \subseteq C$ and r_ℓ be a fuzzy singleton of R $\forall \ell$,t $\in$ (0,1].

Since X is a divisible fuzzy module ,then $x_t = r_\ell^2 R_n$ for some $y_n \subseteq X$

, $\forall$n $\in$(0,1]. But $x_t = r_\ell^2 y_n \subseteq r_\ell^2 X \cap C = r_\ell^2 C \subseteq r_\ell C$

Which implies that $C = r_\ell C$

Thus C is a fuzzy divisible submodule of X .

Corollary 2.2.19:

Let X be a fuzzy module of an R-module M if R is a fuzzy principle ideal and every proper fuzzy submodule of X is fuzzy divisible, then X T-regular fuzzy module.

Proof: Clearly of Proposition (2.2.16)

Remark 2.2.20:

The converse is true if X is divisible .

Proof: Clearly of Proposition (2.2.18)

Corollary 2.2.21:

Let X be a fuzzy module of an R-module M if R is a fuzzy principle ideal and X is T-regular and fuzzy divisible, then X is prime fuzzy module.

Proof:

By Proposition (2.2.18) every fuzzy submodule A of X is divisible .

Thus $r_\ell A = A$ for every fuzzy singleton r_ℓ of R , $\forall\ \ell \in (0,1]$.

Therefore F-ann(A)= F-ann(X) by[5]

Hence X is prime fuzzy module.

"Recall that an R-module M is said to be I-multiplication module if each submodule N of M of the form JM for some idempotent ideal J of R ,[11,Definition (2.8) p.75]"

Now, we shall fuzzify this concept as follows:

Definition 2.2.22:

An fuzzy singleton a_t of R is called fuzzy idempotent if $(a_t)^2 = a_t$ $\forall$ t∈(0,1].

Definition 2.2.23:

Let I be a fuzzy ideal of a ring R ,I is called fuzzy idempotent if $I^2 = I$

Definition 2.2.24:

Let X be a fuzzy module of an R-module M. X is said to be I-multiplication fuzzy module, denoted by ID-multiplication if A=BX where A is fuzzy submodule of X and B is fuzzy idempotent ideal of R.

Remark 2.2.25:

It is clear that every ID-multiplication fuzzy module is multiplication fuzzy module.

Proposition 2.2.26:

If X is an ID-multiplication T-regular fuzzy modules , then X is regular fuzzy modules.

proof:

Let A is a fuzzy submodule of X and I be a fuzzy ideal of R, and

A=BX where B is fuzzy idempotent ideal of R.

To prove $IX \cap A = IA$

$IX \cap A = IX \cap BX$ but $B = B^2$

Thus $IX \cap A = IX \cap B^2X$

$=B^2(IX)$ since X is T-regular and IX is fuzzy submodule

$=I(B^2X)$

$=I(BX)$

$=IA$

Proposition 2.2.27:

If X is an ID-multiplication T-regular fuzzy module .Then every fuzzy submodule of X is ID-multiplication fuzzy submodule.

Proof:

Let A be a fuzzy submodule of X and B be any fuzzy submodule of A.

such that $B \subseteq A$. Then B is fuzzy submodule of X, by Proposition (1.1.113) and B=IX   but $I=I^2$

Thus $B=I^2X$ for some fuzzy idempotent ideal I of R.

$B=A\cap B$ since $B\subseteq A$

$=A\cap I^2X$

$=I^2A$ since A is T-pure fuzzy module

$=IA$

Therefore A is ID-multiplication fuzzy module.

§2.3 Weakly T-regular Fuzzy Modules

"**Definition 2.3.1**:

A fuzzy module X is called weakly regular if every fuzzy submodule of X is weakly pure .[5]"

Now, we shall fazzify this concepts as follows:

Definition 2.3.2:

Let X be a fuzzy module of an R-module M. X is called weakly T-regular if every fuzzy submodule of X is weakly T-pure.

Proposition 2.3.3:

Let X be a fuzzy module of an R-module M, X is weakly T-regular fuzzy module if and only if X_t is weakly T-regular module, $\forall\, t \in (0,1]$.

Proof:

Let X is weakly T-regular fuzzy module, to prove X_t is weakly T-regular module

Let K be a submodules of X_t, $\forall$ t $\in$ (0,1].

Define A: M$\longrightarrow$ [0.1] by $A(x)=\begin{cases} t & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t $\in$ (0,1]

It is clear that A is fuzzy submodule of X and A_t=K, $\forall$ t $\in$ (0,1].

But A is weakly T-pure fuzzy submodule of X, since X is weakly T-regular

So A_t is weakly T-pure submodule of X_t, $\forall$ t $\in$ (0,1].by Proposition (1.5.3)

Thus X_t is weakly T-regular module

Conversely To prove X is weakly T-regular fuzzy module .

Let A be a fuzzy submodule of X.,we must prove that A is weakly T-pure fuzzy submodule of X.

A_t is submodule of X_t, $\forall$ t $\in$ (0,1]. By Proposition (1.1.16)

Hence A_t is weakly T-pure submodule of X_t, since X_t is weakly T-regular module, $\forall$ t $\in$ (0,1]

Then A is weakly T-pure fuzzy submodule of X. by Proposition (1.5.3)

Therefore X is weakly T-regular fuzzy module.

Remark 2.3.4:

Every weakly regular fuzzy module is weakly T-regular fuzzy module

Proof: It is clear

Proposition 2.3.5:

Let f : X $\longrightarrow$ Y be a fuzzy epimorphism and X, Y are two fuzzy modules of M, $M^\backslash$ respectively, if X is weakly T-regular fuzzy module, then Y is weakly T-regular fuzzy module.

Proof:

Let A be a fuzzy submodule of Y . To prove that A is weakly T-pure fuzzy submodule of Y.

That is $(r_\ell)^2 Y \cap A=(r_\ell)^2 A$ for each fuzzy singleton r_ℓ of R , $\forall \ell \in (0,1]$.

Since $f^{-1}(A)$ is fuzzy submodule of X by Proposition (1.1.21(2)) and X is weakly T-regular, then $f^{-1}(A)$ is weakly T-pure fuzzy submodules of X

Hence $(r_\ell)^2 X \cap f^{-1}(A)=(r_\ell)^2 f^{-1}(A)$.

Which implies that : $f((r_\ell)^2 X \cap f^{-1}(A))=f((r_\ell)^2 f^{-1}(A))$

$f((r_\ell)^2 X) \cap f(f^{-1}(A))=f((r_\ell)^2 f^{-1}(A))$ by (1.1.9)

$(r_\ell)^2 f(X) \cap f(f^{-1}(A))=(r_\ell)^2 f(f^{-1}(A))$ by [16,Lemma 2.3.4]

It follows that $(r_\ell)^2 Y \cap A=(r_\ell)^2 A$ since f is fuzzy epimorphism

Thus Y is weakly T-regular fuzzy module.

Proposition 2.3.6:

Let $f : X \longrightarrow Y$ be a fuzzy epimorphism and X, Y are two fuzzy modules of $M_{1,}M_2$ respectively, and every fuzzy submodule of X is f-invariant , if Y is weakly T-regular fuzzy module, then X is weakly T-regular fuzzy module.

Proof:

Let H be a fuzzy submodule of X. prove that $(r_\ell)^2 X \cap H=(r_\ell)^2 H$ for each fuzzy singleton r_ℓ of R , $\forall \ell \in (0,1]$.

$f((r_\ell)^2 X \cap H) = f((r_\ell)^2 X) \cap f(H)$ by Proposition (1.1.9)

$= (r_\ell)^2 f(X) \cap f(H)$ by [16,Lemma 2.3.1]

$=(r_\ell)^2 Y \cap f(H)$ since f is fuzzy epimorphism

$=(r_\ell)^2 f(H)$ since Y is weakly T-regular fuzzy module

$=f((r_\ell)^2H)$

But $(r_\ell)^2H$ and $(r_\ell)^2X \cap H$ are fuzzy submodule of X and every fuzzy submodule of X is f-invariant

Hence $f^{-1}(f((r_\ell)^2X \cap H)) = f^{-1}(f((r_\ell)^2H)$

Therefore $(r_\ell)^2X \cap H=(r_\ell)^2H$

Thus X is weakly T-regular fuzzy module.

Proposition 2.3.7:

Let X and Y be two fuzzy modules of an R-module M_1 and M_2 respectively. If $X\oplus Y$ is weakly T-regular fuzzy module of $M_1 \oplus M_2$, then X and Y are weakly T-regular fuzzy module .

Proof:

Let H be a fuzzy submodule of X. Then $H\oplus 0_1$ is weakly T-pure fuzzy submodule of $X\oplus Y$.

Hence $(r_\ell)^2(X\oplus Y) \cap (H\oplus 0_1)= (r_\ell)^2(H \oplus 0_1)$

Which implies that $(r_\ell)^2X\cap H= (r_\ell)^2H$

So H is weakly T-pure fuzzy submodule of X.

Thus X is weakly T-regular fuzzy module.

Proposition 2.3.8:

Let X be a fuzzy module of an R-module M. If every fuzzy submodule of X is fuzzy divisible , then X is weakly T-regular fuzzy module.

Proof:

Let H be a fuzzy submodule of X. Let r_ℓ be fuzzy singleton of R $\forall \ell \in (0,1]$,such that $r_\ell \nsubseteq 0_1$. To show that $(r_\ell)^2X \cap H=(r_\ell)^2H$

Since H is fuzzy divisible $H \cap (r_\ell)^2X=(r_\ell)^2H \cap (r_\ell)^2X=(r_\ell)^2H$

Therefore X is weakly T-regular fuzzy module.

<u>Remark 2.3.9:</u>

The converse of the Proposition (2.3.8) not true as shown by following example

<u>Example:</u> Let $M=Z_6$

Define $X: M\longrightarrow[0,1]$ by $X(x) = \begin{cases} 1 & \text{if} \quad x \in (\bar{2}) \\ 1/2 & \text{otherwise} \end{cases}$

It is clear that X is fuzzy module

Hence X is weakly regular fuzzy module by[16,Remark(3.5.9)]

Therefore X is weakly T-regular fuzzy module by(Remark 2.3.4)

Let $A: M\longrightarrow [0,1]$ by $A(x)=\begin{cases} 1 & \text{if} \quad x = 0 \\ 1/2 & \text{if} \quad x \in (\bar{2}) - \{0\} \\ 0 & \text{otherwise} \end{cases}$

It is clear that A is fuzzy submodule of X and $A_{1/2}=(\bar{2})$ which is not divisible submodules of X_t.

Since $3(\bar{2})= (\bar{0}) \neq (\bar{2})$

Therefore A is not fuzzy divisible, by [5,(2.2.12)]

<u>Proposition 2.3.10:</u>

Let X be a divisible fuzzy module of an R-module M, then X is weakly T-regular fuzzy module if and only if every fuzzy submodule of X is divisible.

Proof:

If X is weakly T-regular fuzzy module, let H be a fuzzy submodule of X.

To prove H is divisible.

Since X is fuzzy divisible, then $(r_\ell)^2 X = X$ for each fuzzy singleton r_ℓ of R $\forall \ell \in (0,1]$, $r_\ell \nsubseteq 0_1$

Thus $(r_\ell)^2 X \cap H = H$ but X is weakly T-regular fuzzy module

S0 $(r_\ell)^2 X \cap H = (r_\ell)^2 H$ for all fuzzy singleton r_ℓ of R, $r_\ell \nsubseteq 0_1$

Therefore $(r_\ell)^2 H = H$

That is H is divisible fuzzy submodule of X.

The converse by Proposition (2.3.8)

CHAPTER THREE

STRONGLY PURE FUZZY IDEALS AND STRONGLY PURE FUZZY SUBMODULES

Chapter Three

Strongly Pure Fuzzy Ideals And Strongly Pure Fuzzy Submodules

Introduction :

This chapter consists of four sections .In section one we give the basic properties about strongly pure fuzzy ideal , also we give some characterizations of strongly pure fuzzy ideal.

In section two is devoted for studying the strongly regular fuzzy ring

In section three we study the strongly pure fuzzy submodule
Next , section four included strongly regular fuzzy module with some fundamental properties .

§3.1 Strongly Pure Fuzzy Ideals

"Recall that an ideal I of a ring R is said to be pure if for each $x \in I$ there exists $y \in I$ such that $x=x.y$[3].

And ideal I of a ring R is called strongly pure if for each $x \in I$ there exists a prime element $p \in I$ such that $x=xp$.[18]"

Now. We shill fazzify this concepts as follows :

<u>Definition 3.1.1:</u>

Let I be a fuzzy ideal of a ring R ,I is called pure fuzzy ideal if for each $x_t \subseteq I$, there exists $r_\ell \subseteq I$ such that $x_t=x_t\, r_\ell$, $\forall$ t, $\ell \in(0,1]$.

Definition 3.1.2:

Let x_t: R→[0,1] such that $x_t(p)= \begin{cases} t & \text{if} \quad x = p \\ 0 & \text{otherwise} \end{cases}$ Where p is prime number of R, x_t is prime fuzzy singleton.

Definition 3.1.3:

Let K be a fuzzy ideal of a ring R ,K is called strongly pure fuzzy ideal denoted by S-pure fuzzy ideal if for each $r_\ell \subseteq$ K, there exists a prime $x_t \subseteq$ K such that $r_\ell = r_\ell x_t$, ∀ t, ℓ ∈(0,1].

Proposition 3.1.4:

Let I be a fuzzy ideal of R then I is S-pure if and only if I_t is a S-pure ideal of R،∀ t∈(0,1].

Proof:

(⟹)Let I is S-pure fuzzy ideal and $r_\ell \subseteq$ I

By Def (3.1.3)$r_\ell = r_\ell x_t$ ،∀ℓ,t∈(0,1]. To show that r=rx

$r_\ell = (rx)_\ell$ where ℓ=min{ℓ, t} by Proposition (1.1.31)

r=rx by Proposition (1.1.7(3))

Then I_t is S-pure ideal of R ،∀ t∈(0,1].

(⟸)I_t is a S-pure ideal of R and let r∈ I_t

Then there exists a prime element x ∈ I_t such that r=rx

Let r=rx to prove $r_\ell = r_\ell x_t$ ∀ℓ, t∈(0,1]

r=rx implies $r_\ell = (rx)_\ell$ by Proposition (1.1.7(3))

$r_\ell = r_\ell x_t$ where ℓ=min{ ℓ, t} by Proposition (1.1.31)

Therefore I is a S-pure fuzzy ideal of R.

Remarks and Examples 3.1.5:

1-Let K be a fuzzy ideal of a ring R, if K is S-pure fuzzy ideal then K is pure fuzzy ideal.

Proof: it is clear

The converse not true by

Example: A ring Z_6 and N=($\bar{3}$)={$\bar{0}, \bar{3}$} ,H=($\overline{2}$)={$\bar{0}, \bar{2}, \bar{4}$}

Define K: $Z_6 \longrightarrow [0,1]$ by $K(x) = \begin{cases} t & \text{if} \quad x \in N \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ t \in (0,1]$

Define J: $Z_6 \longrightarrow [0,1]$ by $J(x) = \begin{cases} t & \text{if} \quad x \in H \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ t \in (0,1]$

It is clear that K and J are fuzzy ideal of Z_6 and J_t=H ,K_t=N

K_t is S-pure ideal of Z_6 by [18]

Thus K is S-pure fuzzy ideal of a ring Z_6 , by Proposition (3.1.4)

But J_t is not S-pure ideal of a ring Z_6 by [18]

Hence J is not S-pure fuzzy ideal of Z_6 , by Proposition (3.1.4)

But J is pure fuzzy ideal.

2- Let K be a fuzzy ideal of a ring R. If K is S-pure fuzzy ideal of R, then JK=J∩ K for each fuzzy ideal J of R.

3- Let K be a fuzzy ideal of a ring R. If K generated by prime idempotent fuzzy singleton , then K is S-pure fuzzy ideal

Proof:

Let K=(p_s) be a fuzzy ideal generated by prime idempotent fuzzy singleton p_s, ∀ s∈(0,1]

Such that p_s=p_s^2. If $x_t \subseteq K$ there exists fuzzy singleton r_ℓ of R such that

$x_t = r_\ell\, p_s$ implies $x_t = r_\ell\, p_s = r_\ell p_s^2 = r_\ell\, p_s\, p_s = x_t\, p_s$

Therefore K is S-pure fuzzy ideal of a ring R.

4-Let K be a fuzzy ideal of a ring R, if K is S-pure fuzzy ideal then K is idempotent .

Proof:

Let K be a S-pure fuzzy ideal of R, and $r_\ell \subseteq K, \forall \ell \in (0,1]$

Then there exists a prime $x_t \subseteq K$.

Such that $r_\ell = r_\ell x_t \ \ \forall \ell, t \in (0,1]$ but $r_\ell x_t \subseteq K.K$

Hence $r_\ell \subseteq K^2$. Thus $K \subseteq K^2$ and it is clear that $K^2 \subseteq K$. implies $K = K^2$.

Therefore K is idempotent fuzzy ideal of R.

Definition 3.1.6:

Let R be a fuzzy ring, then there exists 1_t of R such that $a_t.\, 1_t = (a.1)_t = a_t$ for all fuzzy singleton a_t of R , a_t is called unit fuzzy singleton.

Definition 3.1.7:

Let x_t be a fuzzy singleton of R is called fuzzy irreducible if $x_t = r_\ell\, y_s$. where $r_\ell \neq 0_1 \neq y_s$ is a fuzzy singleton of R ،$\forall \ell$,s, t$\in$(0,1] it is non unit fuzzy singleton of R then either r_ℓ or y_s is unity of R.

Now, we introduce the concept of fuzzy factorial ring

Definition 3.1.8:

Let S be a non empty fuzzy subset of R, and has no fuzzy singleton unit of integral domain of R ,then R is called fuzzy factorial if every non empty fuzzy singleton of R ,written uniquely form $y_r\, x_{t1} \ldots\ldots x_{tk}$ where y_r is unit of R and $x_{t1} \ldots\ldots x_{tk} \subseteq S$ ،$\forall r, t \in (0,1]$.

Lemma 3.1.9:

Let R be a factorial fuzzy ring. Then every irreducible fuzzy singleton y_r of R is fuzzy prime, every $x_t \subseteq$ S is prime fuzzy singleton and every prime fuzzy singleton of set S is the product of unit of R ،$\forall\, t \in (0,1]$.

Proof:

Let y_r irreducible fuzzy singleton of R

Thus y_r is non unit and if $a_s\, b_r \subseteq (\, y_r)$

Then $a_s\, b_r = x_t\, y_r$ with $x_t \subseteq$ S. we write $a_s, b_r,\ x_t$ as product of irreducible

$a_s = y_{r1} \ldots\ldots y_{ri}$ $\quad b_r = q_{k1} \ldots\ldots q_{km}$ $\quad x_t = r_{\ell 1} \ldots\ldots r_{\ell n}$ ،$\forall\, r, k, t \in (0,1]$.

Here , one of those first two product may be empty

$y_{r1} \ldots\ldots y_{ri}\, q_{k1} \ldots\ldots q_{km} = r_{\ell 1} \ldots\ldots r_{\ell n}\, y_r$

It is mean that either $a_s \subseteq (\, y_r)$ or $b_r \subseteq (\, y_r)$

Thus $(\, y_r)$ is prime fuzzy ideal of R and it is generated by prime

Proposition 3.1.10:

Let K be fuzzy ideal of R , and R be a factorial fuzzy ring, such that $x_t \neq 0_1$ non unit fuzzy singleton of R is fuzzy irreducible. Then K is S-pure fuzzy ideal if and only if K is pure fuzzy ideal.

Proof:

Let K be a pure fuzzy ideal of R, and $r_\ell \subseteq K$, there exists $x_t \subseteq K$, such that $r_\ell = r_\ell x_t$. since x_t fuzzy singleton of R is irreducible.

Hence x_t is fuzzy prime of K , by Lemma (3.1.9)

Therefore K is S-pure fuzzy ideal of R.

The converse is clear .

Proposition 3.1.11:

Let K and H are two fuzzy ideals of a ring R, if K is S-pure fuzzy ideal of R then $K \cap H$ is S-pure fuzzy ideal of R.

Proof: obviously.

Proposition 3.1.12:

Let K and H are two fuzzy ideals of a ring R, such that $K \subseteq H$, if $K \cap H$ is S-pure fuzzy ideal of R, then K is S-pure fuzzy ideal of R.

Proof: it is clear

Corollary 3.1.13:

Let K and H are two S-pure fuzzy ideal of a ring R, then $K \cap H$ is S-pure fuzzy ideal of R.

Corollary 3.1.14:

Let K and H are two fuzzy ideal of a ring R, then K is S-pure fuzzy ideal of R if and only if $K \cap H$ is S-pure fuzzy ideal of R.

Proposition 3.1.15:

Let K and H are two fuzzy ideals of a ring R, if $K \oplus H$ is S-pure fuzzy ideal of R, then either K or H is S-pure fuzzy ideal of R.

Proof:

Let $x_t \subseteq K$ and $y_t \subseteq H$, implies $x_t + y_t \subseteq K \oplus H$

Since $K \oplus H$ is S-pure fuzzy ideal of R. there exists a prime $r_\ell \subseteq K \oplus H$

Where $r_\ell = r_\ell + 0 \subseteq K \oplus H$

Such that $x_t + y_t = (x_t + y_t) r_\ell = (x_t + y_t)(r_\ell + 0) = x_t r_\ell + y_t r_\ell \subseteq K \oplus H$

Since $y_t r_\ell \subseteq K \cap H$ and $K \cap H = \{0\}$

Hence $y_t r_\ell = 0$. Thus $x_t = x_t r_\ell \subseteq K$.

Therefore K is S-pure fuzzy ideal of R.

And if $r_\ell = 0 + r_\ell \subseteq K \oplus H$, then we can get H is S-pure fuzzy ideal of R.

Corollary 3.1.16:

Let K and H are two fuzzy ideals of R and R be a factorial fuzzy ring , such that $K \oplus H$ is S-pure fuzzy ideal of R, then K and H are S-pure fuzzy ideal of R.

"**Definition 3.1.17:**

The fuzzy Jacobson radical of a ring R denoted by F-J(R) is the intersection of all fuzzy maximal ideal of R.[13]"

Proposition 3.1.18:

Let K be S-pure fuzzy ideal of R, such that $K \subseteq F\text{-}J(R)$, then $K=\{0\}$.

Proof:

Let $r_\ell \subseteq K$, since K is S-pure fuzzy ideal of R there exists a prime $x_t \subseteq K$, such that $r_\ell = r_\ell x_t$

Implies $r_\ell(1-x_t)=0$. And since $K \subseteq$ F-J(R), then $x_t \subseteq$ F-J(R).

Hence $r_\ell=0$, so K={0}.

§3.2 Strongly Regular Fuzzy Ring

"Recall that a ring R is regular if and only if for each $x \in R$, there exists $y \in R$ such that x=x.y.x. [19]

And a ring R is called strongly regular if and only if for each $x \in R$, there exists a prime element $p \in R$ such that x=xpx .Also a ring R is called strongly regular if each element of R is strongly regular .[18]"

Now. We shall fazzify this concepts as follows :

Definition 3. 2.1:

A fuzzy ring R is called regular if and only if for each fuzzy singleton x_t of R , there exists fuzzy singleton r_ℓ of R such that $x_t=x_t.r_\ell.x_t$, $\forall$ t, $\ell \in(0,1]$.

Definition 3. 2.2:

Let R be a fuzzy ring ,R is called strongly regular denoted by S-regular if and only if for each fuzzy singleton r_ℓ of R , there exists a prime fuzzy singleton p_t of R such that $r_\ell = r_\ell.p_t.r_\ell$, $\forall$ t, $\ell \in(0,1]$.

Equivalent a ring R is S-regular fuzzy ring if for each fuzzy singleton of R is S-regular.

Proposition 3.2.3:

Let R be a S-regular fuzzy ring if and only if R_t be S-regular ring ,∀t∈(0,1].

Proof:

(⟹)Let R be a S-regular fuzzy ring and r∈ R, to prove R_t is S-regular ring

$r_\ell = r_\ell x_t r_\ell$ ،∀ℓ,t∈(0,1].

Implies $r_\ell = (rxr)_\ell$ where ℓ=min{ℓ, t} by Proposition (1.1.31)

r=r x r by Proposition (1.1.7(3))

Then R_t is S-regular ring , ∀t∈(0,1].

(⟸)Let R_t be S-regular ring to show that R is S-regular fuzzy ring

Let r=r x r to prove $r_\ell = r_\ell x_t\, r_\ell$ ∀ℓ, t∈(0,1]

r=rxr implies $r_\ell = (rxr)_\ell$ where ℓ=min{ ℓ, t} by Proposition (1.1.7(3))

$r_\ell = r_\ell x_t r_\ell$ by Proposition (1.1.31)

Therefore R is S-regular fuzzy ring .

Remarks and Examples 3.2.4:

1-Let R: Z_4 →[0,1] define by

$$R(r) = \begin{cases} t & \text{if} \quad r \in Z_4 \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ t\in(0,1]$$

It is clear that $R_t = Z_4$ and Z_4 is S-regular ring [18]

Thus R is S-regular fuzzy ring. by Proposition(3.2.3)

By the same method we can to show that if R_t= Z_6 is not S-regular ring and we get R is not S-regular fuzzy ring .

2-Let R be a fuzzy ring , if R is S-regular ring , then R is regular fuzzy ring.

Proof: it is clear

The converse not true for example

Example: Let R: $Z_6 \to [0,1]$ define by

$$R(r) = \begin{cases} t & \text{if } \ r \in Z_6 \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, t \in (0,1]$$

It is clear that $R_t = Z_6$ and Z_6 is regular ring ,but Z_6 is not S-regular ring by[18]

Then R is not S-regular fuzzy ring . by Proposition (3.2.3)

3-Every fuzzy ideal of R is irreducible and let R be a factorial fuzzy ring, then R is S-regular fuzzy ring if and only if every fuzzy ideal of R is S-pure.

Proposition 3.2.5:

Let R_1 and R_2 are two fuzzy ring , if $R_1 \oplus R_2$ is S-regular fuzzy ring. Then either R_1 or R_2 is S-regular fuzzy ring.

Proof:

Let fuzzy singleton $r_{\ell 1}$ of R_1 and $r_{\ell 2}$ of R_2, $\forall \ell \in (0,1]$

Implies $r_{\ell 1} + r_{\ell 2} \subseteq R_1 \oplus R_2$.Put $x_t = r_{\ell 1} + r_{\ell 2}$

Since $R_1 \oplus R_2$ is S-regular fuzzy ring, there exists a prime fuzzy singleton $y_t = y_t + 0 \subseteq R_1 \oplus R_2$, such that $x_t = x_t y_t x_t = (r_{\ell 1} + r_{\ell 2}) y_t (r_{\ell 1} + r_{\ell 2})$ $\forall \ell, t \in (0,1]$ where $0_t \leq 0_1$

$= r_{\ell 1} y_t r_{\ell 1} + r_{\ell 1} y_t r_{\ell 2} + r_{\ell 2} y_t r_{\ell 1} + r_{\ell 2} y_t r_{\ell 2}$

But $r_{\ell 1} y_t r_{\ell 2}, r_{\ell 2} y_t r_{\ell 1} \subseteq R_1 \cap R_2$ and $R_1 \cap R_2 = (0)$

Thus $x_t = r_{\ell 1} + r_{\ell 2} = r_{\ell 1} y_t r_{\ell 1} + r_{\ell 2} y_t r_{\ell 2}$.

Implies that $r_{\ell 1} = r_{\ell 1} y_t r_{\ell 1} \subseteq R_1$

Therefore R_1 is S-regular fuzzy ring.

And if $0 + y_t \subseteq R_1 \oplus R_2$, by same method we get $r_{\ell 2} = r_{\ell 2} y_t r_{\ell 2} \subseteq R_2$

Hence R_2 is S-regular fuzzy ring.

§3.3 Strongly Pure Fuzzy Submodules

"**Definition 3.3.1:**

A submodule N of an R-module M is called is called strongly pure , if there exists prime ideal P of a ring R such that $N \cap MP = NP$.[18]"

Definition 3.3.2:

Let X be a fuzzy module of an R-module M and A be a fuzzy sub module of X. A is called strongly pure fuzzy submodule denoted by S-pure fuzzy submodule ,if there exists a prime fuzzy ideal P of a ring R such that $PX \cap A = PA$.

Proposition 3.3.3:

Let B be fuzzy submodule of a fuzzy module X. Then B is S-pure fuzzy submodule of X if and only if B_t is S-pure submodule of X_t, $\forall\ t \in (0,1]$.

Proof:

($\Longrightarrow$)Let I be a prime ideal of ring R

Define $P: R \rightarrow [0,1]$ by $P(x) = \begin{cases} t & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases} \quad \forall\ t \in (0,1]$

And let N be a submodule of an R-module M.

Define $B: M \longrightarrow [0,1]$ by $B(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases} \quad \forall\ t \in (0,1]$

It is clear that P is a prime fuzzy ideal of R by [6] since I is prime .and B is fuzzy submodule of X .

Now, It is clear that $B_t = N$, $P_t = I$, $X_t = M$

Let B be S-pure fuzzy submodule of X . To prove B_t is S-pure submodule of X_t , $\forall$ t$\in$(0,1].

To show that $P_tX_t \cap B_t = P_tB_t$

$PX \cap B = PB$

$(PX \cap B)_t = (PB)_t$

$(PX)_t \cap B_t = (PB)_t$

$P_tX_t \cap B_t = P_tB_t$

Thus B_t is S-pure submodule of X_t. $\forall$ t$\in$(0,1].

Conversely let P be a prime fuzzy ideal of R and B be a fuzzy submodule of X
T.p B is S-pure fuzzy submodule of X

$P_tX_t \cap B_t = P_tB_t$ $\quad \forall$ t$\in$(0,1].

$(PX)_t \cap B_t = (PB)_t$ by Proposition (1.1.7(1))

$(PX \cap B)_t = (PB)_t$ by Definition (1.1.28)

$PX \cap A = PB$

Therefore B is S-pure fuzzy submodule of X .

Remarks and Examples 3.3.4:

1- Let $M=Z_6$ as Z-module and $N=(\bar{3})$,$K=(\bar{2})$

Define $X: Z_6 \longrightarrow [0,1]$ by $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define $A: Z_6 \longrightarrow [0,1]$ by $A(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$ $\quad \forall$ t$\in$(0,1]

Define $B: Z_6 \longrightarrow [0,1]$ by $B(x) = \begin{cases} t & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases}$ $\quad \forall$ t$\in$(0,1]

It is clear that X is fuzzy module and A, B are fuzzy submodules of X and X_t=M , A_t=N, B_t=K

A_t is S-pure submodule of X_t. by[18]

Then A is S-pure fuzzy submodule of X, by Proposition (3.3.3)

But B is not S-pure fuzzy submodule of X since K is not S-pure submodule of X_t , by Proposition (3.3.3)

2-Let X be a fuzzy module of an R-module M ,and let C be a S-pure fuzzy submodule of X, then C is pure fuzzy submodule of X.

Proof: it is clear

The converse is not true in general for example

Example: Let M=Z_{12} as Z-module and N= $(\bar{3})$

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define C: M⟶[0,1] by $A(x) = \begin{cases} t & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$ $\forall$ t∈(0,1]

It is clear that X is fuzzy module, C is fuzzy submodule of X and X_t=M ,C_t=N

C_t is pure submodule of X_t. by[18]

Thus C is pure fuzzy submodule of X by Proposition (1.2.2)

But C_t is not S-pure submodule of X_t , by [18]

Therefore C is not S-pure fuzzy submodule of X. by Proposition (3.3.3)

Proposition 3.3.5:

Let A be fuzzy submodule of a fuzzy module X and B be fuzzy submodule of A. if A is pure fuzzy submodule of X and B is S-pure fuzzy submodule of A, then B is S-pure fuzzy submodule of X.

Proof:

Let B is S-pure fuzzy submodule of A, there exists a prime fuzzy ideal P of R such that $PA \cap B = PB$

Since A is pure fuzzy submodule of X ,then $PX \cap A = PA$.

Implies that $B \cap A \cap XP = BP$, and since $B \subseteq A$, then $B \cap A = B$

Implies $XP \cap B = PB$

Therefore B is S-pure fuzzy submodule of X.

Corollary 3.3.6

Let A and B are two fuzzy submodules of a fuzzy module X. if A is pure fuzzy submodule of X and $A \cap B$ is S-pure fuzzy submodule of A. then $A \cap B$ is S-pure fuzzy submodule of X.

§3.4 Strongly Regular Fuzzy Module

"Recall that an element $x \in M$ is called strongly regular if there exists an R-module homomorphism $\theta: M \longrightarrow R$, such that $\theta(x)x = x$ where $\theta(x)$ is strongly regular element in a ring R[18].

And an R-module M is called strongly regular if every element of M is strongly regular.[18]"

Now. We shill fazzify this concepts as follows :

Definition 3.4.1:

Let X be a fuzzy module of an R-module M, $x_t \subseteq X, \forall\, t \in (0,1]$ is called strongly regular fuzzy singleton denoted by S-regular fuzzy if there exists a homomorphism $\theta: M \longrightarrow R$, such that $\theta(x_t)\, x_t = x_t$ where $\theta(x_t)$ is S-regular fuzzy singleton in a ring R.

Definition 3.4.2:

Let X be a fuzzy module of an R-module M, X is called S-regular if every $x_t \subseteq X, \forall\ t \in (0,1]$ is S-regular fuzzy singleton.

Proposition 3.4.3:

Let X be S-regular fuzzy module and A be a fuzzy submodule of X, then A is S-pure fuzzy submodule of X.

Proof:

Let A be a fuzzy submodule of X and let I be a prime fuzzy ideal of R

To show that $IX \cap A = IA$

It is clear that $IA \subseteq IX \cap A$

Now, to show that $IX \cap A \subseteq IA$

Let $x_t \subseteq IX \cap A$, then $x_t = \sum_{i=1}^{n} r_{\ell i}\, x_{ti}$ $\forall$ t, $\ell \in (0,1]$. Where $r_{\ell i} \subseteq I$ and $x_{ti} \subseteq X$

Since X is S-regular fuzzy modules ,hence x_t S-regular fuzzy singleton

Thus there exists a homomorphism $\theta: M \longrightarrow R$, such that $x_t = \theta(x_t)\, x_t$,

So $\theta(x_t) = \sum_{i=1}^{n} r_{\ell i}\, \theta(x_{ti})$ and $x_t = \theta(x_t)\, x_t = \sum_{i=1}^{n} r_{\ell i}\, \theta(x_{ti}) x_t$

And since $x_t \subseteq A$, hence $x_t = \sum_{i=1}^{n} r_{\ell i}\, \theta(x_{ti}) x_t \subseteq IA$

Thus $IX \cap A \subseteq IA$

Then $IX \cap A = IA$

Therefore A is S-pure fuzzy submodule of X.

Proposition 3.4.4:

Let X be a fuzzy module of an R-module M ,then R is S-regular fuzzy module if and only if R is a S-regular fuzzy ring .

Proof:

Let R is S-regular fuzzy module to prove R is S-regular fuzzy ring .

Let $x_t \subseteq R$, there exists a homomorphism $\theta: R \longrightarrow R$, such that $x_t = \theta(x_t)\, x_t$,

where $\theta(x_t)$ is S-regular fuzzy singleton of R.

Since $\theta(x_t)= \theta(1.x_t)= \theta(1)x_t$,

Hence $x_t p_s x_t = x_t \theta(1) x_t$,then $x_t p_s x_t = x_t$

Therefore R is S-regular fuzzy ring .

Conversely Let R be a S-regular fuzzy ring to prove R is S-regular fuzzy module.

Let $x_t \subseteq R$, thus there exists a prime fuzzy singleton p_s of R such that $x_t = x_t p_s x_t$, $\forall\, t, \ell \in (0,1]$.

Now, define a function $\theta: R \longrightarrow R$, by $\theta(x_t) = x_t p_s$ for each fuzzy singleton x_t of R

Then $\theta(x_t)\, x_t = x_t p_s x_t$, so $\theta(x_t)\, x_t = x_t$

Thus R is S-regular fuzzy module.

Proposition 3.4.5:

Let X is S-regular fuzzy module and fuzzy divisible over an fuzzy integral domain R, then every fuzzy submodule of X is fuzzy divisible.

Proof:

Let A be a fuzzy submodule of X , and let r_ℓ be a non zero fuzzy singleton of R. To prove $r_\ell A = A\ \forall\, \ell \in (0,1]$.

By Remark(3.4.3) A is S-pure fuzzy submodules of X.

So $< r_\ell > A = A \cap < r_\ell > X$

Claim that $r_\ell A = A \cap r_\ell X$, let $x_t \subseteq A \cap r_\ell X$, then $x_t = r_\ell s_r$, $s_r \subseteq X$

Since x_t is S-regular fuzzy singleton of X there exists a homomorphism

$\theta: M \longrightarrow R$ such that $x_t = \theta(x_t)\, x_t$ and hence $x_t = \theta(x_t)\, x_t = r_\ell\, \theta(s_r)\, x_t$, as $x_t \subseteq A$

This implies that $x_t \subseteq r_\ell A$, hence $A \cap r_\ell X \subseteq r_\ell A$,

Hence $r_\ell A = A \cap r_\ell X$.

As $r_\ell X = X$. thus $A \cap X = r_\ell A$

So $r_\ell A = A$

Therefore A is fuzzy divisible.

FUTURE WORK

And continued to the subject of the thesis will provide ideas for future

Work is the following :

***** T-prime fuzzy submodule .

***** T-semi prime fuzzy submodule .

***** The relationship between some types of fuzzy submodule (T-pure , T-prime and T-semi prime)

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المستخلص

في هذه الرسالة تم تضبيب مفهوم الموديولات الجزئية النقية من النوع Tحيث يكون الموديول الجزئي الضبابي A موديلاً جزئياً نقياً من النوع T أذا كان لكل مثال ضبابي غير خال في الحلقة R يحقق $I^2X \cap A = I^2A$ ودراسة الموديول الجزئي الضبابي شبة النقي من النوع T كذلك تم تقديم مفهوم المثالي النقي من نوع T وبنفس مفهوم الموديول الجزئي الضبابي النقي من نوعT

استناداً على ذلك قدمنا كذلك مفهوم الموديول الضبابي المنتظم من النوع Tوتعريفه يكون الموديول الضبابي منتظم

اذا كان كل موديولا جزئيا فيه نقي من نوع T.

وهذه الفكرة تقودنا الى تقديم مفهوم المثال الضبابي النقي القوي (حيث يكون المثالي الضبابي I نقياً قوياً في الحلقة R اذا كانت كل نقطة مفردة$x_t \subseteq I$ يوجد نقطة مفردة اولية $P_r \subseteq I$ حيث تحقق $x_t = x_t P_r$ بالإضافة إلى ذلك نناقش الحلقة الضبابية المنتظمة القوية (حيث تكون الحلقة الضبابية المنتظمة قوية اذا وفقط اذا لكل نقطة مفردة r_ℓ في الحلقة الضبابية R توجد نقطة مفردة اولية في R حيث ان r_ℓ $r_\ell = r_\ell x_t$)

ومن بين هذه الدراسة نقدم مفهوم الموديول الجزئي الضبابي النقي القوي والموديول الضبابي المنتظم القوي

وكذلك يتم دراسة العلاقات بين كل هذا المفاهيم والعديد من الأنواع أخرى .

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بغداد
كلية التربية أبن الهيثم

أنواع جديدة من الموديولات الضبابية المنتظمة والموديولات الجزئية الضبابية النقية

رسالـــة

مقدمة إلى كلية التربية للعلوم الصرفة /أبن الهيثم جامعة بغداد كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قبل
شيماء عبد الكريم محي الدين

بأشراف
أ.م.د. حاتم يحيى خلف

١٤٣٩هـ **٢٠١٧م**