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Variational Approach for Solving Linear Free and Moving Boundary Value Problems

A Thesis
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By

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

﴿قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ الْعَلِيمُ الْحَكِيمُ﴾

صَدَقَ الله العلي العَظيم

البقرة / الآية **32**

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PRAISE TO ALLAH, LORD OF THE WHOLE CREATION

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Isam Al-Safi

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List of Symbols

Phrase	Symbols
Moving Boundary Value Problems	(MBVP)
Moving Boundary Problem	(MBP)
Free Boundary Value Problem	(FBVP)
Ordinary Differential Equations	(ODEs)
Domain	D (L)
Range	R (L)
the velocity potential function	Φ
the slope of the rectum	M_0
the slope of this rectangle	M_1
total head charge	h
fluid speed through the porous medium at a certain point	v
Fluid pressure on the unit area.	p
Point higher the datum line.	z
Fluid weighted density.	γ
total charge difference between the two points A and B	Δh
acceleration	g
Average diameter of the soil granules.	d
mass density	ρ
The fluid viscosity coefficient.	M
the fluid velocity in the direction s	V_S
the velocity with n direction	V_n
the right reservoir side	R
left reservoir dam	L

Abstract

The first objective of this thesis is to study and the variational formulation of free and moving boundary value problems. The problems investigated are those in which the governing partial differential equation is the Laplace equation and the boundary conditions are of free type and to make this work of self-contents, as possible.

The second objective of this thesis is to derive the physical and mathematical formulation of the problem under consideration, which is the pond seepage problem that is considered as a free boundary value problem. This problem had been formulated and solved using variational approach. The direct Ritz method have been used to solve the problem approximately in which computer program is written in MATLAB 2016a which as used to solve the problem and find the numerical results.

Contents

Introduction

The topic of this thesis is closely related to differential equations and calculus of variation. This topic deals with the problem of maximizing or minimizing functional that is variable value, which depends on a variable running through a set of functions, or on a finite number of these variables, and which are completely determined by a definite choice of those variable functions. Problems that consist of finding maxima or minima of a functional are called variational problem, [1].

The phrase “variational formulation” had been used in recently and nowadays in connection with the generalized formulation of boundary or initial value problems. In boundary value problems, sometimes it happens that a part of the boundary is unknown and must be determined as a part of the solution. This unknown boundary occurs in two cases; the first one is called the moving boundary, which occurs mostly in heat-flow problems with phase changes and in certain diffusion processes. The second type is called a free boundary which does not move but its position has to be determined as a part of the solution of a steady-state problem, [3].

Historically, as a literature survey the essential features of variational methods goes back approximately for more than two centuries, when the first notions of the subject for variational of calculus began to be formulated. Actually, the most primitive ideas of variational theory had been presented first in Ariistothes writings on virtual velocities in 300 B.C., then they were reviewed by Galileo in the 16^{th} century. Later, they were formulated into a principle of virtual work by John Bernolli in 1717. The first step toward developing a general method for solving variational problems was given by Euler in 1732 through presenting "a general solution of the isoperimetric problem". In this work and subsequent writing of Euler, the variational concepts found an acceptance and enduring place in mechanics, [7], [9].

A more solid mathematical basis for variational theory was developed in the 19^{th} and the early of 20^{th} century. Necessary conditions for the existence of "minimizing curves" of a certain functional were studied during this period, and we found among the contributors in this area the familiar names of Legendre, Jacobi and Weirstrass. Legendre in 1786 gave a criteria for distinguishing between maximum and minimum, without considering criteria of existence. Jacobi in 1837 introduced sufficient conditions for existence of an extreme of a functionalism, [12]. It is notable that the main problem in calculus of variation is to find the extremum (maximum or minimum) values of a given functional J, this necessary condition is called the Euler-Lagrange equation. This problem is called, for simplicity, the direct problem of calculus of variation [7]. Roughly speaking, variable values which depends on variable function working through a set of functions or on a finite number of these variables, and which are completely determined by a specific selection of these variable functions, [9].

At the end of the 19^{th} century and in the early years of the 20^{th} century, several prominent contributions are found related to the subject of variational ideas, particularly, in the area of problems, in which Ritz, Galerkin and Hellinger are the pioneers.

Nowadays, variational concepts play a fundamental role in applied mathematics. As an example, the solution of any ordinary problems, such as partial differential equations, ordinary differential equations, integral equations, etc. are equivalent to the minimization of the functional J that corresponds to this ordinary equation, [13].

As it is well known, the initiation of the study of variational principles should be attributed to Euler and Lagrange and in a broader setting, to Poisson, Cauchy and Hamilton.

In recent years, the development of a unified theory for linear problems was given by several authors. Hussain in 1987 studied the solution of the boundary value problems using variational approach, [12]. Also, Mahlol in 1993 studied the solution of the direct and inverse of eigenvalue problems of Sturm-Lowville problem, and an applied it for localizing the size of Brain tumors, [15]. In addition, among other studies concerning the direct and inverse problems with application, the study given by Ali in 1994 for the mathematical inverse problem of acoustic wave scattering, [2]. Jabbar in 2001, considered the solution of the two-dimensional moving value problems of Hele-Show problem, which is solved using variational inequalities, [13]. Finally, Al-Ani In 2001 considered the study of two-dimensional inverse problem of the seepage problem in a simple rectangular dam, [1], in 2002 Zainab studied variational formulation for solving three dimensional moving boundary value problems and its application in laser ablation problem, [24], In 2010 Al-Mosawi considered the study about the direct and inverse variational formulation of three-dimensional dams, [3].

The main objective of this thesis is to study, in general, the seepage problems through porous media, which is an important source of the free boundary problems, for example the seepage through earth dams, seepage out of open channels such as rivers, canals, ponds and irrigation system, and then give the physical, mathematical and variational formulation of the pond seepage problem and then find its numerical solution.

The structure of this thesis consist of three chapters. In chapter one, we present the basic concepts related to the subject of calculas of variation, as well as, free and moving boundary value problems. In chapter two, the physical and mathematical derivation of the governing model related to the pond seepage problem is presented, as well as, it's free and boundary conditions are given. In chapter three, the variational formulation and the numerical solution of the problem under consideration is presented.

Conclusions and recommendations for future work are also presented in the end of this thesis.

Finally, computer program written in MATLAB 2016a is presented in the appendix.

Chapter One
Basic Concepts

Chapter
1
Basic Concepts

Introduction:

This chapter deals with the basic concepts related to this thesis. These concepts consist of the fundamentals of calculus of variation, free boundary value problem (FBVP) and moving boundary value problem (MBVP).

It is well known that, the subject of calculus of variation is that one driven from the classical extreme important branches of applied mathematics, since it has a great importance in solving many problems set using mathematical systems. The study may be so complicated, which is due to the governing equation and/or types of initial and boundary conditions and/or due to the domain of definitions, which may be so irregular, etc.

This chapter consisted of five sections. First, an introduction of this chapter have been introduced. Section (1.1) presents the elementary of calculus of variation, including the statement of the fundamental lemmas of calculus of variation, as well as, the derivation of the simplest variational problem. Basic definitions related to Magri's approach and its fundamental theorems are given in section (1.2).

The symmetry of the Laplace's operator have been proved in section (1.3), as well as, its variational formulation which is presented in section (1.4). Finally, in section (1.5) we presents the free and moving boundary value problems for completeness purpose.

1.1 Elementary of Calculus of Variation:

We start this section with an important two lemmas in calculus of variation. These results are necessary for the derivation of necessary condition that must be satisfied by the solution.

Lemma (1.1.1), [20]:

If a function g is continuous in an interval [a, b] $\subset \mathbb{R}$ and if $\int_a^b \eta(x)g(x)\,dx = 0$, for an arbitrary function η which is continuously and twice differentiable such that $\eta(a) = \eta(b) = 0$. Then $g(x) \equiv 0$, for all $x \in [a,b]$.

The generalization of lemma (1.1.1) for two dimensional problems may be stated as in the next lemma.

Lemma (1.1.2), [10]:

Let g(x,y) be a continuous function on D, which is a simply connected region in $\mathbb{R}^2$, and if:

$$\iint_D \eta(x,y)g(x,y)dxdy=0.$$

for an arbitrary function η which is twice continuously differentiable that vanishes on the boundary of D. Then $g(x,y) \equiv 0$, $\forall (x,y) \in D$.

The simplest variational problem that may be considered consists of finding an extremum (maximum or minimum) of the functional:

$$J(y(x)) = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx\,. \tag{1.1}$$

where the type of the accepted curves consists of all functions $y(x_0)=y_0$, $y(x_1)=y_1$, F is a function with continuous first and second partial derivatives with respect to all of its arguments.

Therefore, to find the necessary condition satisfied by y to be a solution for the varitional problem (1.1), we started by letting δJ to be the first variation of the functional J, which is defined by:

$$\delta J = J(y + \delta y) - J(y)\big|_{\text{Linear part in } \delta y}$$

where:

$$J(y + \delta y) = \int_{x_0}^{x_1} F(x, y+\delta y, y'+\delta y')\, dx\,.$$

which is equated to zero for the optimum solution y.

Now, suppose an extremum passes through the curve $y = y(x)$ alongside with all admissible solutions $y=y^*(x)$, $\forall\ x \in [x_0,x_1]$ and hence the variation of the solution y is defined to be $\delta y = y(x) - y^*(x)$ and since the first variation of y is a function on x, then it may be differentiated with by using the liner property to obtain $(\delta y)' = \delta y'$, and hence:

$$\delta J = J(y + \delta y) - J(y)$$

$$= \int_{x_0}^{x_1} F(x, y+\delta y, y'+\delta y')\ dx - \int_{x_0}^{x_1} F(x, y, y')\ dx$$

$$= \int_{x_0}^{x_1} \left[F(x, y+\delta y, y'+\delta y') - F(x, y, y') \right] dx$$

and upon using the first degree approximation of Taylor's series expansion of the incremental function F(x, y + δy, y′ + δy′) about (x, y, y′), one may get:

$$\delta J = \int_{x_o}^{x_1} \left[F_y \delta y + F_{y'} \delta y' \right] dx$$

Therefore, using the method of integration by parts and noting that $\delta J(x_0) = \delta J(x_1) = 0$ implies to:

$$\delta J = \int_{x_0}^{x_1} \left[F_y \delta y - \frac{d}{dx} F_{y'} \delta y \right] dx$$

$$= \int_{x_0}^{x_1} \left[F_y - \frac{d}{dx} F_{y'} \right] \delta y \; dx = 0 .$$

and for extrema, we set δJ =0 since δy is chosen as an arbitrary function, which is vanished at x_0 and x_1. Hence, by using the fundamental lemma of calculus of variation (1.2.1), getting:

$$F_y - \frac{d}{dx} F_{y'} = 0 . \qquad (1.2)$$

which is the necessary condition that must be satisfied by the solution curve y. This condition is also called the Euler-Lagrange equation (for simplicity Euler's equation).

Remark (1.1.3):

The above ideas may be generalized for more complicated cases, such as with higher order functional, functional of extra than one dependent variable, or for functional of more than one independent variable, etc. (for more details, see [Elsgolc, 1962], [Gelfand, 1963]).

1.2 The Inverse Problem of Calculus of Variation, [14]:

The main difficulty in the calculus of variation occurs when the necessary condition to represent the Euler-Lagrange equation, represents a real life problem under consideration, which may be rewritten in operator form as Ly = f, where L is a liner operator and f is any given function. The problem is to find an equivalent variational formulation corresponding this problem. This topic is called the inverse problem of calculus of variation, which is aimed for deriving the related functional J, which should be minimized.

Before indulging this subject and its main theorem, some additional basic concepts must be introduced at first.

Definition (1.2.1), [16]:

Let U and V be two normed linear spaces, a bilinear form defined on U and V is a functional $L: U\times V \longrightarrow \mathbb{R}$, which is linear in both u and v, where u and v are elements of U and V respectively, and the following properties are fulfilled:

1. $L(u_1+u_2,v)=L(u_1,v)+L(u_2,v)$, u_1, $u_2 \in U$, $v\in V$

2. $L(u,\alpha v)=\alpha L(u,v)$, $u\in U$, $v\in V$, $\alpha \in \mathbb{R}$

3. $L(u,v_1+v_2)=L(u,v_1)+L(u,v_2)$, $u \in U$, $v_1, v_2 \in V$

4. $L(u,\alpha v)=\alpha L(u,v)$, $u \in U$, $v \in V$, $\alpha \in \mathbb{R}$

this functional is usually denoted by the symbol $<u,v>$.

Definition (1.2.2), [21]:

Let $<u,v>$ be a bilinear form over $U \times V$, then:

1. $<u,v>$ is said to be symmetric if $<u,v> = <v,u>$, for all $u \in U$, $v \in V$

2. $F[u] = \frac{1}{2}<u,u>$ is a quadratic form of u.

3. The bilinear form $<u,v>$ is non-degenerate on U and V if

   i. $<u, \bar{v}> = 0 \rightarrow \bar{v} = 0 \;\; ,\forall u \in U$.

   ii. $<\bar{u}, v> = 0 \rightarrow \bar{u} = 0 \;\; ,\forall v \in V$.

Among the most usual examples of non-degenerate bilinear forms, are the following:

$$<u,v> = \int_0^\tau u(t)\, v(t)\, dt$$

where u, v : $C[0, \tau] \longrightarrow \mathbb{R}$, $\tau > 0$.

$$<u,v> = \int_0^\tau \sum_n u_n(t)\, v_n(t)\, dt$$

where u, v : $C[0, \tau] \longrightarrow \mathbb{R}^n$, $\tau > 0$.

$$<u,v> = \int_0^\tau u(t)v(\tau\text{-}t)\,dt$$

where u, v : C[0, τ] $\longrightarrow \mathbb{R}$, τ > 0.

$$<u,v> = \int_0^\tau u(t)\left[\int_{s=0}^{s=\tau\text{-}t} K(t,s)v(s)ds\right]dt$$

where u, v : C[0, τ] $\longrightarrow \mathbb{R}$, and K : C[0, τ]×C[0, τ] $\longrightarrow \mathbb{R}$ is a preassigned function that is denoted to the kernel of the bilinear form; the above example may be written for higher dimensions as follows:

$$<u,v> = \int_0^\tau\int_0^\tau u(x,t)\,v(x,t)dt\,dx$$

where u, v : C[0, τ]×[C[0, τ] $\longrightarrow \mathbb{R}$, τ > 0.

$$<u,v> = \int_0^\tau\int_0^\tau \sum_n u_n(x,t)\,v_n(x,t)dt\,dx$$

where u, v : C[0, τ]×C[0, τ] $\longrightarrow \mathbb{R}^n$, τ > 0.

$$<u,v> = \int_0^\tau\int_0^\tau u(x,t)v(x,\tau\text{-}t)dtdx$$

where u, v : C[0, τ]×C[0, τ] $\longrightarrow \mathbb{R}$, τ > 0.

$$<u,v> = \int_0^\tau\int_0^\tau u(x,t)\left[\int_{s=0}^{s=\tau\text{-}t} K(t,s)\,v(x,s)ds\right]dt\,dx\,.$$

where u, v : C[0, τ]×C[0, τ] $\longrightarrow \mathbb{R}$, τ > 0 and K : C[0, τ]×C[0, τ] $\longrightarrow \mathbb{R}$ is a pre-assigned function called the kernel of the bilinear form.

Definition (1.2.3), [21]:

A given linear operator L: D (L) →R (L) is called symmetric with respect to the chosen bilinear form $<u,v>$ if it satisfies:

$$<Lu,v>=<Lv,u>.$$

Theorem (1.2.4), [21]:

There is a variational problem $\mathbf{J(u)=\frac{1}{2}<Lu,u>-<f,u>}$ corresponding to initial boundary value problem Lu = f, if and only if the operator L is symmetric relative to the chosen bilinear form which is non-degenerate.

1.3 Symmetry of Laplace's Operator, [21]:

Because of the impotency of the operator used in the derivation of the varitional problem of the Laplace's operator used to the seepage problem, we will prove next the symmetry of the Laplace's operator, i.e., to prove:

$$<Lu,v>=<Lv,u>.$$

for this purpose, the following form is considered:

$$L=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Hence

$$<Lu, v>=\iint_D \left(u_{xx}+u_{yy}\right) v\, dx\, dy$$

$$= \iint_D \left(u_{xx}\, v+u_{yy}\, v\right) dx\, dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}(u_x v)-u_x v_x + \frac{\partial}{\partial y}(u_y v)-u_y v_y\right] dx\, dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}(u_x v)+\frac{\partial}{\partial y}(u_y v)\right] dx\, dy-\iint_D \left[u_x v_x+u_y v_y\right] dx\, dy$$

Therefore, upon using Green’s theorem for the fist integral we have:

$$<Lu, v> = -\iint_D \left[u_x v_x+u_y v_y\right] dx\, dy.$$

Similarly:

$$<Lv,u> = \iint_D uv dx\, dy.$$

$$= -\iint_D \left[u_x v_x+u_y v_y\right] dx\, dy$$

i.e.

$$<Lu, v> = <Lv,u> .$$

Hence, the linear operator L related to the Laplace's operation is symmetric relative to the chosen non-degenerate bilinear form defined by:

$$<u,v>=\iint_D uv dx\, dy.$$

1.4 Varitional Formulation of Laplace Equation, [21]:

Theorem (1.2.4) may be applied to find the functional corresponding to the Laplace equation:

$$u_{xx} + u_{yy} = 0 \ , (x,y) \in D$$

where D is a simply connected domain in R^2, in which the operator L is given by:

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

and the corresponding functional is given by:

$$J(u) = \tfrac{1}{2} \langle Lu,u \rangle - \langle f,u \rangle .$$

Now, since f = 0, hence upon expanding the bilinear form, then the functional J (u), will take the form:

$$\begin{aligned} J(u) &= \tfrac{1}{2} \iint_D \left(u_{xx} + u_{yy} \right) u \ dxdy \\ &= \tfrac{1}{2} \iint_D \left(u_{xx} u + u_{yy} u \right) dx\,dy \end{aligned}$$

and recalling that:

$$\frac{\partial}{\partial x}(uu_x) = uu_{xx} + u_x^2$$

$$\frac{\partial}{\partial y}(uu_y) = uu_{yy} + u_y^2$$

Hence:

$$uu_{xx} = \frac{\partial}{\partial x}(uu_x) - u_x^2$$

Therefore:

$$J(u)=\frac{1}{2}\iint_D\left[\left(-u_x{}^2-u_y{}^2\right)+\left(\frac{\partial}{\partial x}\left(uu_x\right)+\frac{\partial}{\partial y}\left(uu_y\right)\right)\right]dxdy$$

$$=-\frac{1}{2}\iint_D\left(u_x^2+u_y{}^2\right)dx\,dy+\frac{1}{2}\iint_D\left(\frac{\partial}{\partial x}\left(uu_x\right)+\frac{\partial}{\partial y}\left(uu_y\right)\right)dxdy$$

Finally, upon applying Green's theorem, the functional J will take the following final form:

$$J(u)=\iint_D\left(u_x^2+u_y^2\right)dx\,dy. \tag{1.3}$$

1.5 Initial and Boundary Value Problems, [5]:

Differential equations can be also classified according to the conditions associated with the differential equation under consideration, which may be of initial type (i.e., the conditions are given at a single initial time $t = 0$) and the problem in this state is denoted as an initial value problem, while "boundary value problems" are related to the problems at which the case are given at several times or the conditions are given about the boundary at the range of solution.

1.5.1 Free Boundary Value Problems, [12], [20]:

A FBVP consisted of a partial differential equation of elliptic type, which satisfied into a bounded region together with the necessary boundary conditions in which one part or more of the boundary (the free boundary) is unknown and should be determined as a part of the solution. To make this possible, additional conditions has to be specified on that free boundary, as an example, problems related to earth dams. In addition, flow through permeable media is an important source of free boundary value problems that appear most frequently in seepage nature phenomena. Examples are seepage through earth dams, seepage out of an open channel such as reveres, canals, ponds and irrigation system, or into wells.

Practical interest in FBVPs, in whatever way, is not restricted to natural leakage but extended also to another subjects in plasma physics, semiconductors and electrochemical machining, [4].

1.5.2 Moving Boundary Value Problems, [5]:

The MBVP is commonly used when the boundary is associated with the time-dependent problems and it is already known that the boundaries of the domain must be selected as a part of the problem. However, moving boundaries are functions of time and space.

An MBVP will be taken to mean a time-dependent problem governed by a parabolic partial differential equation with a prescribed initial and boundary conditions. In all cases, two conditions are needed on the moving boundary; one for determining the boundary itself and the other for completing the definition of the problem to be well posed.

Large classes of MBVPs are usually concerned with fluid flow in porous media, and with diffusion and heat flow or chemical reactions. Moving boundary value problems are often called Stefan problems. Since 1890, Stefan was interested in the melting of the polar ice cap as there is no exact analytical solution available for general MBVPs, [5].

Among the foremost free and moving boundary value problems, are as jolt waves in gas dynamics, as in cracks through solid mechanics or optimal stoppage problem in decision theory.

Chapter Two

Physical and Mathematical Derivation of the Pond Seepage Problem

Chapter

2

Physical and Mathematical Derivation of the Pond Seepage Problem

Introduction:

In this chapter, we will discuss the problem of the free surface then address the problem of the seepage in a natural lake from two sides and the base of non-permeable water in a realistic application of the free surface. After that, we will turn to the physical derivation and mathematical modeling of the problem of the pond seepage problem. Therefore this chapter consists of two sections, in sections (2.1) the physical derivation of the problem will be considered which based on the classical laws of physics, such as Darcy’s law, Bernoulli equation velocity potentials, stream lines, etc. In section (2.2), the mathematical modeling of the seepage problem have been presented, as well as, its initial and boundary condition including the free surface.

2.1 Physical Derivation of the Problem:

In this section, the physical derivation of the water seepage problem in soil sands will be given and presented for the generalized dam problem. First of all, some concept of fundamental fluid dynamic are presented.

2.1.1 Darcy's law, [9],[23]:

Darcy's law is considered when a water flow in a porous medium in one direction. In proper real world, problems such as leaks happen in all three dimensional dams, underground water flow, seepage through reservoirs, etc. In many situation, two and three dimensional problems are simplified into a leakage flow, which will be calculated as accordingly.

The analysis of the water flow happens when a flux occurs in the porous medium from one point to another because there is a difference in the total pressure charge resulting from an increase in the kinetic energy at a certain point (this will be abbreviated as the total charge). Hence, the total charge is a kinetic energy stored in the dam’s body that is represented usually as an equivalent height, which is the same of the velocity pressure charge and height difference. Thus, the total charge may be represented at any point as in the following:

$$h = \frac{v^2}{2g} + \frac{p}{\gamma} + z \qquad , \qquad g \neq 0,\ \gamma \neq 0 \tag{2.1}$$

where

h: total head charge.

v: fluid speed through the porous medium at a certain point.

p: fluid pressure over per the unit area.

z: point higher the datum line.

γ: fluid weighted density.

g: acceleration.

It is remarkable that equation (2-1) is modeled and developed by the Swiss Mathematica Daniela Bernoulli in 1738, and often called

Bernoulli's equation. The velocity of the flow within the porous medium is proportional to the rate of change of the total pressure relative to the distance. If we compare the total charge between the two points A and B (see fig. 2-1), it seems clear that:

$$h_A = z_A + \frac{p_A}{\gamma_w} + \frac{v_A^2}{2g} \tag{2.2}$$

as well

$$h_B = z_B + \frac{p_B}{\gamma_w} + \frac{v_B^2}{2g} \tag{2.3}$$

which will give

$$h_A = h_B + \Delta h \tag{2.4}$$

where Δh represents the total charge difference between the two points A and B. Let $i = -\lim_{\Delta s \to 0} \frac{\Delta h}{\Delta s}$ which will be called the hydraulic gradient.

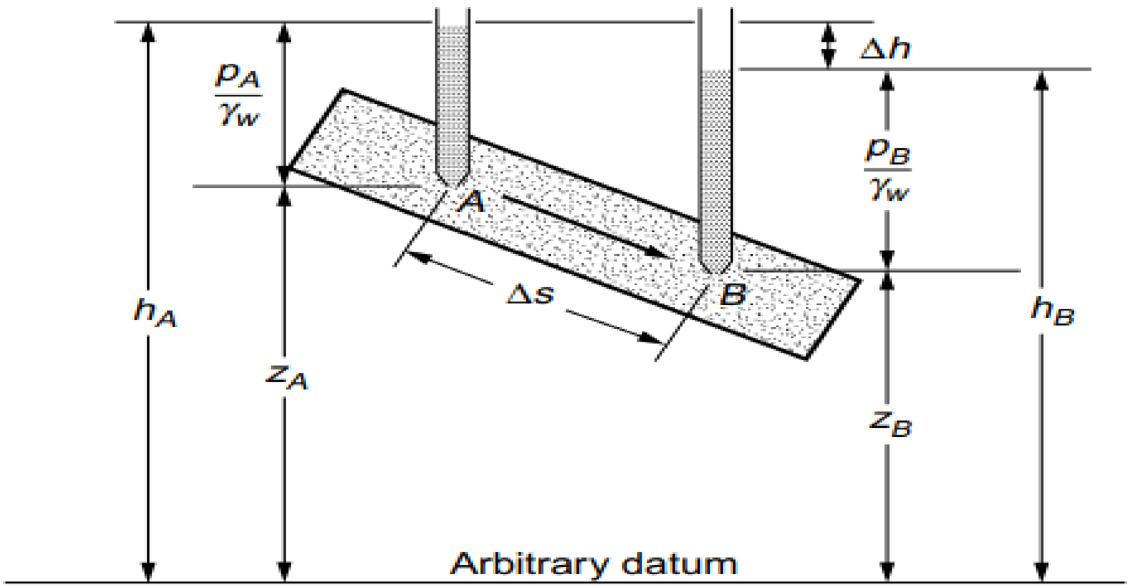

Fig. (2-1) Heads in Bernoulli's equation.

In 1856. Henry Darcy, derived the following formula:

$$v = Ki = -k\frac{dh}{ds}. \tag{2.5}$$

Equation (2-5) is a labeled as the Darcy's equation, which represent the linear relationship between the hydraulic gradient and velocity. In this case, K is called the coefficient of the hydraulic conductivity, which is a common property between the fluid and the porous medium that is called the coefficient of permeability. The permeability coefficient may be expressed as:

$$K = cd^2. \tag{2.6}$$

where

c: Constant.

d: Average diameter of the soil granules.

The relationship between K and k may be given as follows:

$$K = \frac{k\gamma}{M},\ M \neq 0 \tag{2.7}$$

where

γ : The fluid weighted quality.

M: The fluid viscosity coefficient.

In the present problem of this work, the soil will be considered as the porous medium, which is considered to be homogenous and isotropic, i.e., the permeability coefficient is considered to be uncharged for every point of the medium.

2.1.2 The Continuity Equation, [23]:

The fluid flow amount which is usually called the discharge that represent the amount of the fluid volume passing through the time unit. Hence, in order to derive the continuity equation, which states that "the difference between the entering mass flow rate average and the outcome mass flow rate average which permit the area are assumed to be equal". For illustration consider an element of the liquid as it is shown in Fig. (2-2).

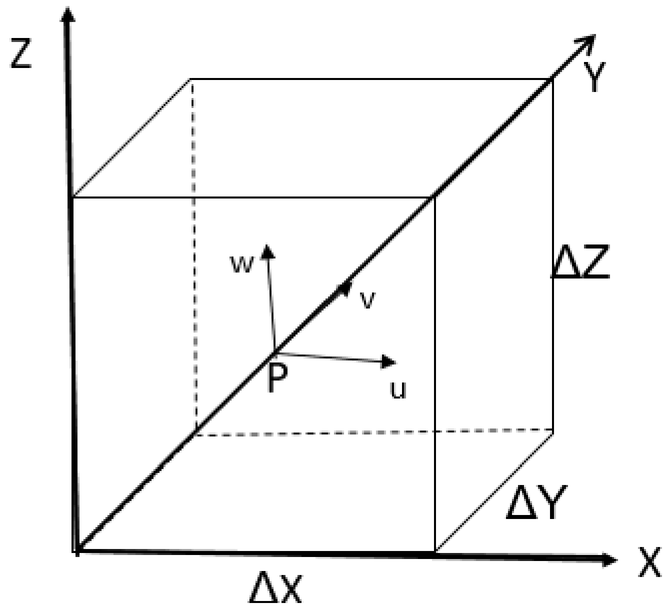

Fig. (2-2) Element of the liquid.

In Fig. (2-2) it is assumed that; p is a point located in the center of the middle of the element; u, v and w are the velocity point decompositions with respect to time t.

Hence the average unit time mass flow rate at a point p = mass density × velocity × the area of the vertical section on the direction of velocity.

The three fluid flux may be considered as follows:

1. ***Flux in x-direction:***

Average of mass p flux for a unit time equals to $\rho u\Delta y\Delta z$ where ρ is the mass density. Therefore, the rate of mass that flowing and entering the cube from the left hand side with respect to the unit time is given by:

$$\rho u\Delta y\Delta z-\frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\Delta y\Delta z \tag{2.8}$$

hence, the cube out rate flow from the right hand side with respect to the unit time is:

$$\rho u\Delta y\Delta z+\frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\Delta y\Delta z \tag{2.9}$$

thus, subtracting (2.8) from (2.9) yields to:

$$-\frac{\partial(\rho u)}{\partial x}\Delta x\Delta y\Delta z \tag{2.10}$$

2. ***Flux in y-direction:***

Similarly, as in the flux in the x-direction, we may note that the difference between the entering and out coming rate flow to be:

$$-\frac{\partial(\rho v)}{\partial y}\Delta x\Delta y\Delta z\,. \tag{2.11}$$

3. ***Flux in z-direction:***

Also, in the z-direction

$$-\frac{\partial(\rho w)}{\partial z}\Delta x\Delta y\Delta z\ .$$

Thus, form the flux rate in the x, y and z-direction, it will implies that the rate of mass flowing through the cube element in a unit time is given by:

$$-\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]\Delta x\Delta y\Delta z\,. \tag{2.12}$$

Now, since

mass = volume × density

$= \rho\Delta x\Delta y\Delta z\,.$

Hence, the mass flowing in a unit time is:

$$\frac{\partial(\rho\Delta x\Delta y\Delta z)}{\partial t}\,. \tag{2.13}$$

Therefore, for equations (2.12) and (2.13), one may get:

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=-\frac{\partial\rho}{\partial t}\,. \tag{2.14}$$

Since the water is incompressible, i.e., $\frac{\partial\rho}{\partial t}=0$ and for steady state flow (which means that the following property are unchanged with respect to time, such as velocity and pressure). Shown in Fig. (2-3).

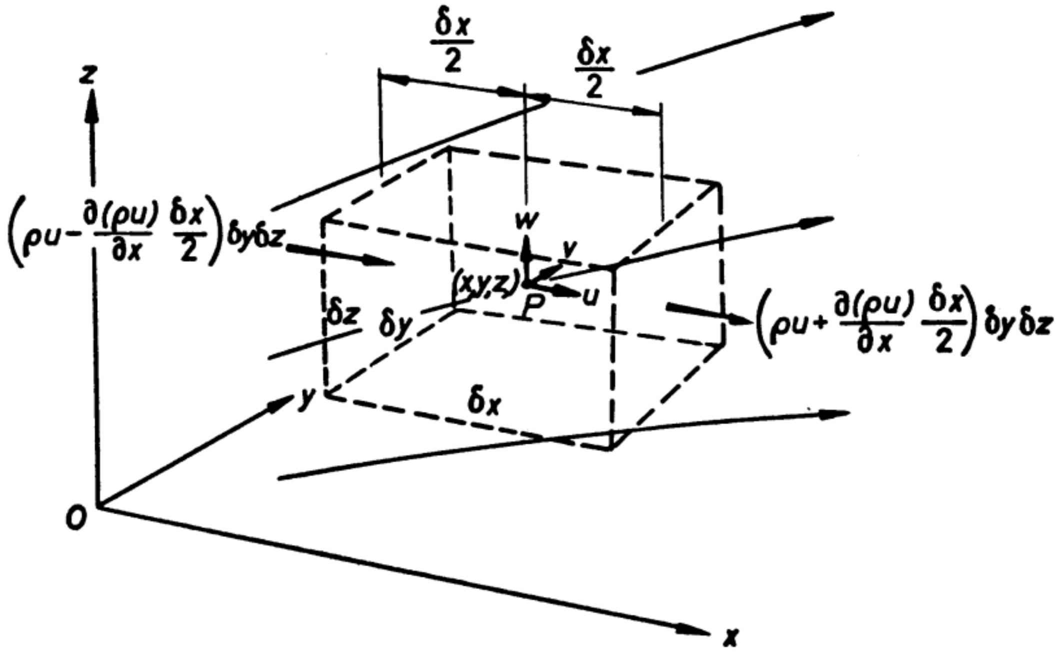

Fig. (2-3) Equation of continuity, mass flows in the x-direction across the faces of a parallelepiped in three-dimensional flow.

Therefore, the continuity equation resulting from (2.14) will take the following form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad (2.15)$$

2.1.3 The Velocity Potential Function, [9]:

Let the velocity potential function Φ is a function of state variables x, y, z, and time t. Its first order partial derivatives in any direction represents the partial velocity in that direction. Hence, if V_s is the fluid velocity in the direction s then following:

$$V_S=\frac{\partial \Phi}{\partial s}.$$

The function V_s satisfies two conditions really, namely:

1. The function V_s is the continuity equation; where if $u=\frac{\partial \Phi}{\partial x}$ and $v=\frac{\partial \Phi}{\partial y}$, then by substituting in equation (2.15) for two-dimensional cross section will give:

$$\Phi_{xx}+\Phi_{yy}=0. \qquad (2.16)$$

i.e., Φ satisfies the Laplace equation.

2. The fluid flow is laminar, because if the flow is not laminar, i.e., turbulent or rotational, then it will implies that Φ is not a harmonic function.

Now, from the definition of Φ, $V_S = \frac{\partial \Phi}{\partial s}$ and from Darcy's equation, we have:

$$\Phi(x, y, z) = -Kh + C. \tag{2.17}$$

where h is the total charge which is equals to the pressure charge $\frac{p}{\gamma}$ is a height plus its potential energy for a certain height Z (in which $\frac{v^2}{2g}$ may be neglected as a small value because the fluid velocity in the porous medium is relatively small), i.e.

$$\Phi = -K\left(\frac{p}{\gamma} + Z\right) + C \ . \tag{2.18}$$

For completeness purpose, we introduce the following definitions as a basic concepts:

1. *The stream lines:*

The stream lines are that curves in which at every point and for a certain time the tangent lines determines the velocity driven and therefore it is not possible that liquid flows through the stream line. The stream line equation may be described as

$$\psi = C \ . \tag{2.19}$$

where ψ is the stream function, [20].

2. The Equipotential lines:

They are not real curves which representing the line joint between that points with equal charge in which they are vertical to the stream lines.

3. The Flow net:

It is that net produced from the stream lines and equipotential lines. This net has its utility for describing the velocity at each point of the domain.

2.1.4 The boundary conditions:

For steady state flow with, incompressible fluid and homogenous soil, there will be four types of non-compressible fluid and homogeneous soil. In addition, there are four types of boundary conditions for the irregular region:

1. The impervious boundary:

The fluid cannot pass through this boundary and hence it is a non-permeable layer. If we consider this layer is attached in placed in the t-direction and n be the normal direction on the layer, the velocity with n direction equals zero, i.e., there is no breach through the layer, and hence:

$$V_n = \frac{\partial \Phi}{\partial n} = 0 .$$

which is the stream line, [8].

2. The reservoir boundary:

Along this boundary the pressure distribution is hydrostatic, let M be any point in this boundary, the pressure will equals to, (see Fig (2-4)):

$$p = \gamma(h - y). \tag{2.20}$$

where p is the pressure and γ is the weighted density of the fluid. Hence substituting equation (2.17) will yields to:

$$\Phi = -Kh + C$$

where c, h and k are constants.

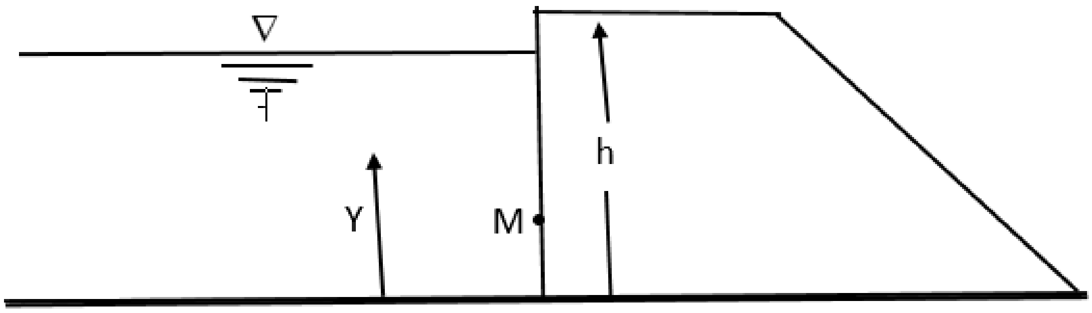

Fig. (2-4) Reservoir boundary.

3. Seepage surface boundary condition:

The surface in where the fluid emerge out of the porous medium and the flow is free. A pressure on this surface is atmospheric and will be constant along the seepage surface, and therefore:

$$\Phi = -K(\frac{p}{\gamma} - y) + C_0.$$

whereas $\frac{p}{\gamma}$ is a constant say C_1 ,i.e.:

$$\Phi = -K(C_1 - y) + C_0. \tag{2.21}$$

if Φ will depend on y, it will be not a constant, not an equipotential line, not a stream line. Since it will intersects with the other stream lines.

As an illustration for the above, we consider Fig (2-5).

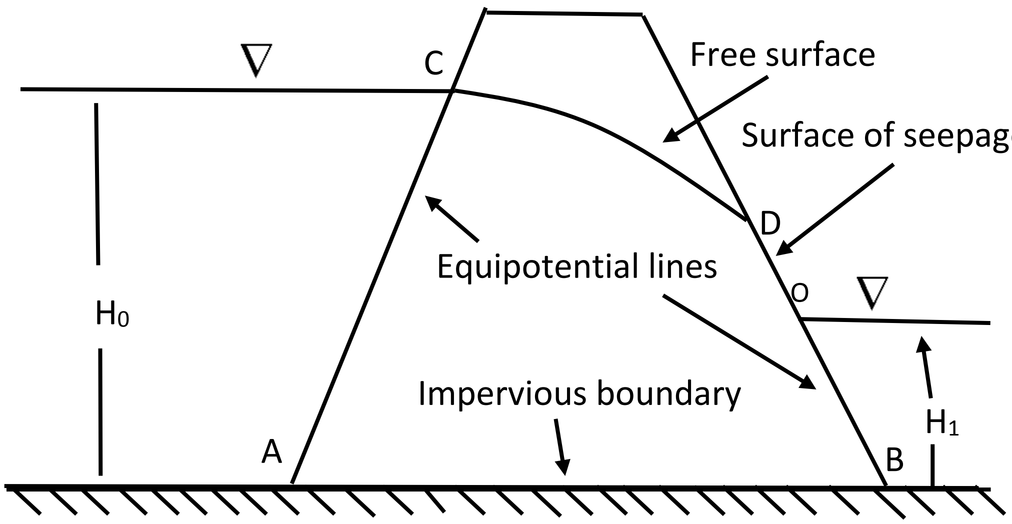

Fig. (2-5)Two dimensional cross section.

4. ***Free surface:***

The free surface is the first stream line from the flow network in the flowing reign, which may considered as an interface between the saturated and unsaturated soil areas, the leakage line is a stream line and so ψ will be constant along this line. Similarly, for the leakage surface, Φ will satisfy:

$$\Phi + ky = \text{Constant} \tag{2.22}$$

the boundary condition on the free surface are:

$$y = H(x) \quad , \quad x_0 \le x \le x_1$$

of the lateral fact:

$$H(x_0) = H_0 \tag{2.23.a}$$

$$H'(x_0) = -\frac{1}{M_0} \quad , \quad M_0 \neq 0 \tag{2.23.b}$$

$$H(x_1) = E_2(x_1) \tag{2.23.c}$$

$$H'(x_1) = E_2'(x_1) \tag{2.23.d}$$

Remark:

In this chapter, we will use the velocity potential function of Φ so that:

$$\Phi = -Kh + C \tag{2.24}$$

In the next chapter, we will select C = 0 by choosing a given level and use a function equal to the voltage of ϕ. So that the ϕ uses in the therefore, ϕ is not depend on k in the next chapter, which is:

$$\Phi = -\frac{1}{k}\phi \qquad , k \neq 0 \tag{2.25}$$

2.2 The Governing Equations:

Crank and Gupta in their book entitled "Free and Moving Boundary Value Problems", [5] described an accurate description and overview of the most analytical and numerical methods used to solve some of the real life problems of free and moving boundary value problems, in which the evaluation of the free or the moving surface is considered as a part of the problem. Therefore, in this section, we do not need to study such methods, which are dealt with in a number of research papers that are appeared recently (see for example [7], [11], and [21].

The proposed approach can be described in the following steps

1. Gonverting the boundary problem into given extreme variational problem.
2. Using the direct Ritz method to solve the equivalent variational problem, where it is supposed that the solution when to be found has the following form:

$$u = \sum_{n=1}^{N} a_n \Phi_n \quad , \quad \forall \ n = 1, 2 \ldots N \qquad (2.26)$$

where a_n parameters to be found and Φ_n is a complete set of functions.

What is the different between other methods and the variational approach is the assumptions that the unknown part of the free surface can be expressed as an approximated function of other linearly independent functions, more generally, or complete set of function. The determination of this function is considered as a part of the problem under consideration.

Therefore, when compensating for the unknown surface in this way, the approximate solution of the variational problem will be turned into an equivalent problem, which is of finding the unknown parameters defining the solution and/or the free surface.

The rest of this chapter, we will explain how to find the solution of the water seepage problem in soil porous medium using series expansion in terms of polynomial basis function.

The problem of seepage in an earth dam in which its base ground is not permeable will be modeled in this section. The governing equation for this problem takes the form:

$$\Phi_{xx} + \Phi_{yy} = 0.$$

where $\Phi = -\frac{1}{k}\phi$, $k \neq 0$.

where ϕ is the velocity potential functional and k is a constant.

To determine the boundary conditions of the problem, let us try to draw the following illustration figure for the two dimensional cross section of the earth dam (see Fig. (2-6)):

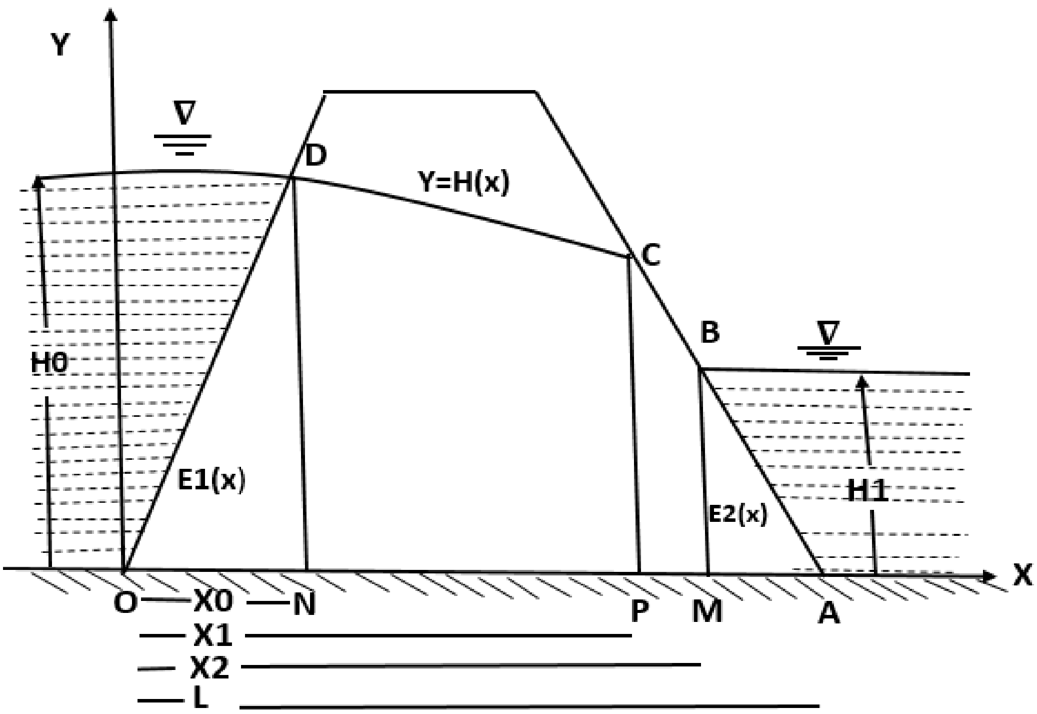

Fig. (2-6) Cross sectional through two-dimensional dam.

where

H_0: The water level in the reservoir.

H_1: The seepage water level through the dam.

OA: The length of the dam base which are equivalent mathematically to y = 0, and $\Phi_y(x,0)=0$.

OD: The height of the dam which passes through the origin (0, 0). If the slope of the rectum is M_0, then M_0 is given with the following relationship:

$$M_0 = \frac{DN}{ON} \quad . \tag{2.27}$$

and if ON = x_0, then

$$M_0 = \frac{H_0}{x_0} \quad , \quad x_0 \neq 0. \tag{2.28}$$

Thus, if y = E_1 (x) represents the equation of the straight line OD, then it is given by the following relationship $E_1(x) = M_0\, x$ and so $\Phi(x, E_1(x)) = H_0$.

AB: is the dam reservoir side, which passes through the point (L, 0), where L is the length of the base of the dam. If the slope of this rectangle is M_1, then M_1 may be given as in the following relationship:

$$AM = L - x_2 \tag{2.29}$$

$$OM = x_2 \tag{2.30}$$

$$M_1 = \frac{MB}{AM} \quad , \quad AM \neq 0. \tag{2.31}$$

therefore:

$$M_1 = \frac{H_1}{L - x_2} \quad , \quad L \neq x_2 . \tag{2.32}$$

If $y = E_2(x)$ represents the equation of the straight line with slope M_1 and AB passes through (L, 0), then E_2 may be written as:

$$E_2(x) = M_1(L - x) \quad , \quad x_2 \leq x \leq L \tag{2.33}$$

Therefore, from (2-32) and (2-33) we have:

$$\Phi(x, E_2(x)) = H_1 \tag{2.34}$$

DC: represents the free surface and which may written using the curve y = H (x). Because the evaluation of this curve is considered as a part of the problems, then it must be:

$$\Phi(x,H(x))=H(x) \quad , \quad x_0 \le x \le x_1 \tag{2.35}$$

where $x_1 = OP$

CB: The line of the seepage surface which is the extension of the AB line, and then its equation has the form:

$$y=E_2(x) \quad , \quad x_1 \le x \le x_2 \tag{2.36}$$

and therefore

$$\Phi(x,E_2(x))=E_2(x)\,. \tag{2.37}$$

Thus, the governing equations and the boundary conditions of the underground water seepage problem are as follows:

$$\Phi_{xx}+\Phi_{yy}=0 \tag{2.38}$$

$$\left.\begin{array}{lcl}
\Phi_y(x,0)=0 & , & 0 \le x \le L \\
\Phi(x,E_1(x))=H_0 & , & 0 \le x \le x_0 \\
\Phi(x,E_2(x))=H_1 & , & x_2 \le x \le L \\
\Phi(x,H(x))=H(x) & , & x_0 \le x \le x_1 \\
\Phi(x,E_2(x))=E_2(x) & , & x_1 \le x \le x_2
\end{array}\right\} \tag{2.39}$$

Now, for the problem under consideration of this thesis, the above mathematical derivation of the system of governing equations (2.38) and (2.39) may be extended to the real life problem of the pond seepage problem (see Fig. (2-7)).

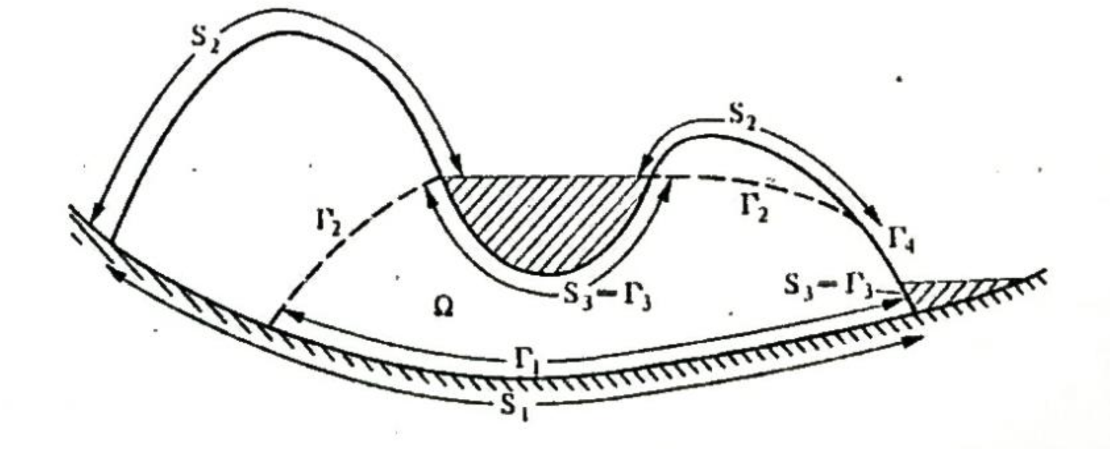

Fig. (2-7) Pond seepage

The boundary conditions related to the governing equation of the duplicated (one for the right side and the other for the left side), which will take the following final from:

$$\Phi_{xx}+\Phi_{yy}=0 \ , \ (x,y)\in \Omega R \text{ or } \Omega L$$

with initial and boundary condition for the right reservoir side:

$$\Phi R_y(x,0)=0 \qquad , \qquad 0\le x\le LR$$

$$\Phi R(x,ER_1(x)) = HR_0 \qquad , \qquad 0\le x\le xR_0$$

$$\Phi R(x,ER_2(x)) = HR_1 \qquad , \qquad xR_2 \le x\le LR$$

$$\Phi R\ (x,HR(x)) = HR(x) \qquad , \qquad xR_0 \le x\le xR_1$$

$$\Phi R\ (x,ER_2(x)) = ER_2(x) \qquad , \qquad xR_1 \le x\le xR_2$$

For illustration and simplicity, consider the above formulation for right reservoir as is at given in Fig. (2-8).

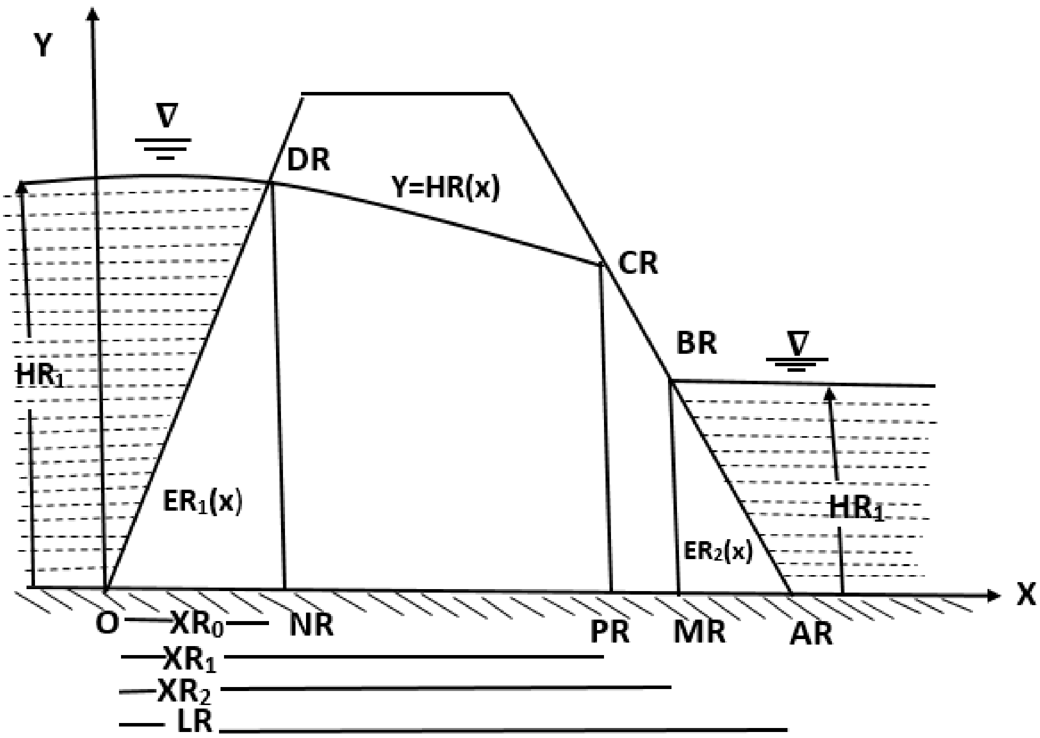

Fig. (2-8)One side of the right dam.

Also, the initial and boundary conditions for left reservoir dam (see Fig. (2-9)) is given by:

$$\Phi L_y(x,0)=0 \quad , \quad 0 \le x \le LL$$

$$\Phi L(x,EL_1(x)) = HL_0 \quad , \quad 0 \le x \le xL_0$$

$$\Phi L(x,EL_2(x)) = HL_1 \quad , \quad xL_2 \le x \le LL$$

$$\Phi L\,(x,HL(x)) = HL(x) \quad , \quad xL_0 \le x \le xL_1$$

$$\Phi L\,(x,EL_2(x)) = EL_2(x) \quad , \quad xL_1 \le x \le xL_2$$

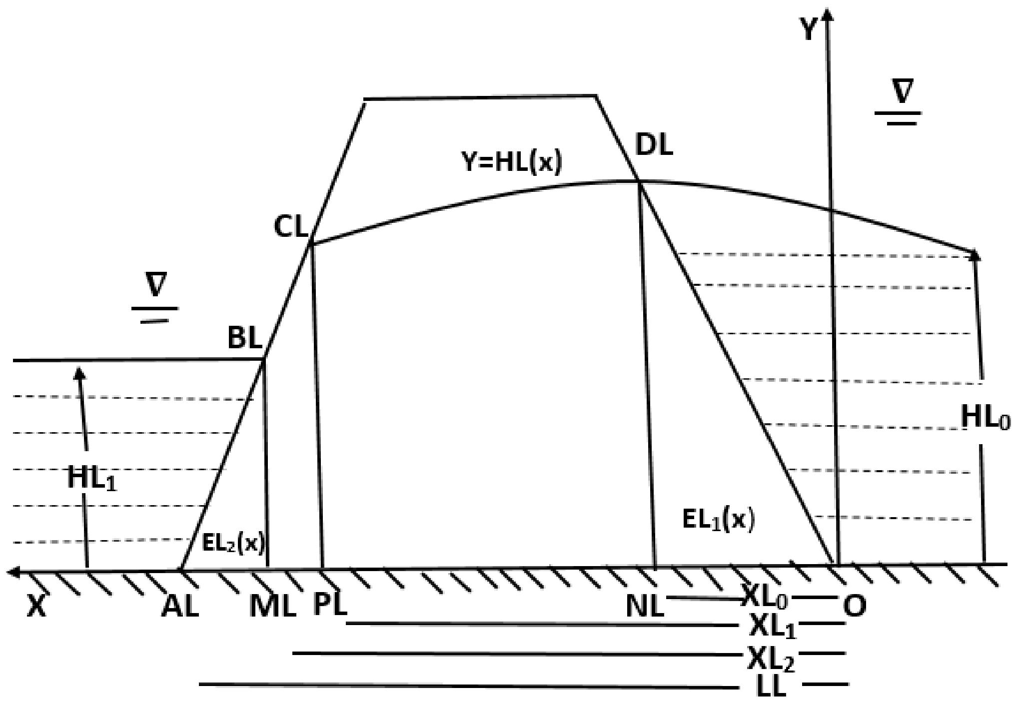

Fig. (2-9)One side of the left dam.

Figures (2-8) and (2-9) may be collected for our problem as it is shown in Fig (2-10).

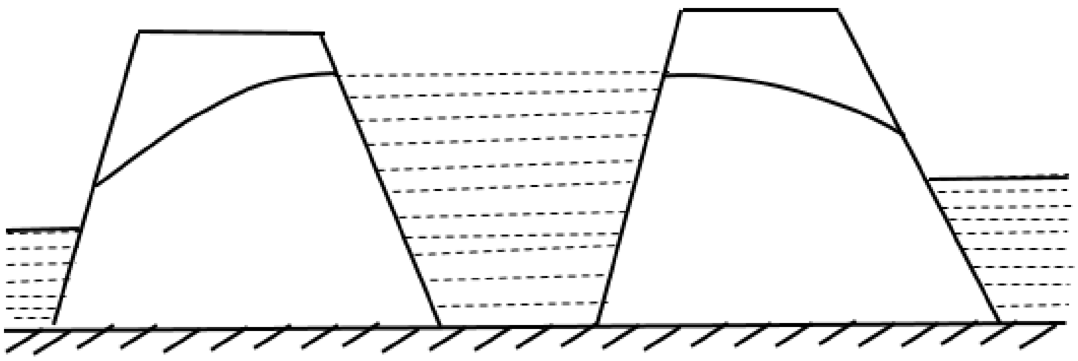

Fig. (2-10)Two – sided dams.

Chapter three

The Variational Formulation and the Numerical Solution of the Pond Seepage Problem

Chapter

3

The Variational Formulation and the Numerical Solution of the Pond Seepage Problem

Introduction:

The organization of this chapter as follows. Where consisted of three section. In section 3.1, the mathematical formulation of the problem using Magri's approach which was given in chapter one is presented, while in section 3.2, variational formulation of the problem to solve pond seepage problem. Finally, numerical simulation for the considered problem is presented with certain dam's dimension, in section 3.4.

3.1 Mathematical Formulation of the Problem:

The mathematical formulation and modeling the pond seepage problem must pass through the physical derivation of the problem, which will not present here using by Darcy's low for deriving the continuity equation and velocity potential function of the problem, [20].

The mathematical modeling of the problem is formulated as a free boundary value problem and have the governing equation with initial and boundary condition of the pond seepage problem, which will has the form (see Fig. 3-1).

$$\Phi_{xx}+\Phi_{yy}=0 \quad ,(x,y)\in \Omega R \text{ or } \Omega L \tag{3.1}$$

with initial and boundary condition for the right dam side

$$\left.\begin{array}{ll}
\Phi R_y(x,0)=0 & , \quad 0\le x\le LR \\
\Phi R(x,ER_1(x))=HR_0 & , \quad 0\le x\le xR_0 \\
\Phi R(x,ER_2(x))=HR_1 & , \quad xR_2\le x\le LR \\
\Phi R\ (x,HR(x))=HR(x) & , \quad xR_0\le x\le xR_1 \\
\Phi R\ (x,ER_2(x))=ER_2(x) & , \quad xR_1\le x\le xR_2
\end{array}\right\} \tag{3.2}$$

Also, the initial and boundary conditions for the left dam side one given by:

$$\left.\begin{array}{ll}
\Phi L_y(x,0)=0 & , \quad 0\le x\le LL \\
\Phi L(x,EL_1(x))=HL_0 & , \quad 0\le x\le xL_0 \\
\Phi L(x,EL_2(x))=HL_1 & , \quad xL_2\le x\le LL \\
\Phi L\ (x,HL(x))=HL(x) & , \quad xL_0\le x\le xL_1 \\
\Phi L\ (x,EL_2(x))=EL_2(x) & , \quad xL_1\le x\le xL_2
\end{array}\right\} \tag{3.3}$$

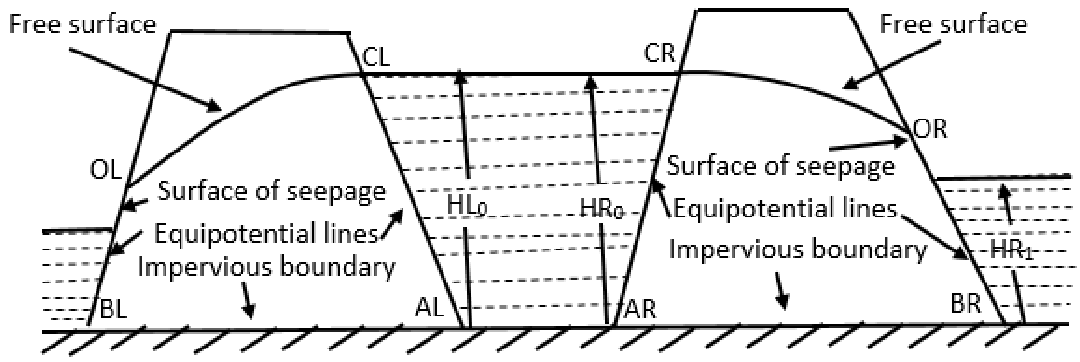

Fig. (3-1) The free surface of the pond seepage problem.

3.2 Variational Formulation of the Problem:

To solve pond seepage problem, we have to solve equation with its relevant boundary and initial conditions, as well as the evaluation of the free surface as a part of solution of the problem. In this section, we turn to the variational formulation problem. The related functional derived using Magri's approach is given by:

$$J(\Phi) = \iint_{\Omega} \left[\Phi_x^2 + \Phi_y^2 \right] dxdy \qquad (3.4)$$

where $\Omega = \Omega_1 \cup \Omega_2$, and Ω_1 is the right sided seepage region which is decomposed for computation purpose into the following sub regions:

$RR_1 = \{(x, y): 0 \le x \le xR_0,\ 0 \le y \le ER_1(x)\}$

$RR_2 = \{(x, y): xR_0 \le x \le xR_1,\ 0 \le y \le HR(x)\}$

$RR_3 = \{(x, y): xR_1 \le x \le xR_2,\ 0 \le y \le ER_2(x)\}$

$RR_4 = \{(x, y): xR_2 \le x \le LR,\ 0 \le y \le ER_2(x)\}$

While Ω_2 is the left sided seepage region and also for computation purpose is decomposed into the following sub regions:

$RL_1 = \{(x, y): 0 \le x \le sL_0,\ 0 \le y \le ML_1(x)\}$

$RL_2 = \{(x, y): sL_0 \le x \le sL_1,\ 0 \le y \le GL(x)\}$

$RL_3 = \{(x, y): sL_1 \le x \le sL_2,\ 0 \le y \le ML_2(x)\}$

$RL_4 = \{(x, y): sL_2 \le x \le KL,\ 0 \le y \le ML_2(x)\}$

Hence, the functional J may be rewritten for Ω_1 and Ω_2 respectively as follows:

$$J(\Phi) = \int_0^{xR_0}\int_0^{ER_1(x)} \left[\Phi_x^2 + \Phi_y^2\right] dydx + \int_{xR_0}^{xR_1}\int_0^{HR(x)} \left[\Phi_x^2 + \Phi_y^2\right] dydx \ +$$

$$\int_{xR_1}^{xR_2}\int_0^{ER_2(x)} \left[\Phi_x^2 + \Phi_y^2\right] dydx + \int_{xR_2}^{LR}\int_0^{ER_2(x)} \left[\Phi_x^2 + \Phi_y^2\right] dydx \qquad (3.5)$$

$$J(\Phi) = \int_0^{sL_0}\int_0^{ML_1(x)} \left[\Phi_x^2 + \Phi_y^2\right] dydx \ + \int_{sL_0}^{sL_1}\int_0^{GL(x)} \left[\Phi_x^2 + \Phi_y^2\right] dydx +$$

$$\int_{sL_1}^{sL_2}\int_0^{ML_2(x)} \left[\Phi_x^2 + \Phi_y^2\right] dydx + \int_{sL_2}^{KL}\int_0^{ML_2(x)} \left[\Phi_x^2 + \Phi_y^2\right] dydx \quad (3.6)$$

also, Φ_1, Φ_2 , Φ_3 and Φ_4 on the region Ω_1 or Ω_2 respectively in the following form:

$$\begin{aligned}
\Phi_1 &= y^2(y\text{-}E_1(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm} x^n y^m + H_0 \qquad , \qquad 0 \le x \le x_0 \\
\Phi_2 &= y^2(y\text{-}H(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm} x^n y^m + H(x) \qquad , \qquad x_0 \le x \le x_1 \\
\Phi_3 &= y^2(y\text{-}E_2(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm} x^n y^m + E_1(x) \qquad , \qquad x_1 \le x \le x_2 \\
\Phi_4 &= y^2(y\text{-}E_2(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm} x^n y^m + H_1 \qquad , \qquad x_2 \le x \le L_1
\end{aligned} \qquad (3.7)$$

and functions Φ_X, Φ_Y take the following formulas:

$$\Phi_x = \frac{\partial \Phi}{\partial x} \quad , \quad \Phi_y = \frac{\partial \Phi}{\partial y}$$

When the derivation of the x variable:

$$\Phi_1 x = y^2(y-E_1(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + y^2(-E_1'(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m \quad , \quad 0 \le x \le x_0$$

$$\Phi_2 x = y^2(y-H(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + y^2(-H'(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + H'(x) \quad , \quad x_0 \le x \le x_1$$

$$\Phi_3 x = y^2(y-E_2(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + y^2(-E_2'(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + E_1'(x) \quad , \quad x_0 \le x \le x_2$$

$$\Phi_4 x = y^2(y-E_2(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + y^2(-E_2'(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m \quad , \quad x_2 \le x \le L$$

When the derivation of the y variable:

$$\Phi_1 y = (3y^2-2yE_1(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + (y^3-y^2E_1(x))(\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m)y \quad , \quad 0 \le x \le x_0$$

$$\Phi_2 y = (3y^2-2yH(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + (y^3-y^2H(x))(\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m)y \quad , \quad x_0 \le x \le x_1$$

$$\Phi_3 y = (3y^2(y-2yE_2(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + (y^3-y^2E_2(x))(\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m)y \quad , \quad x_0 \le x \le x_2$$

$$\Phi_4 y = (3y^2-2yE_2(x))\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m + (y^3-y^2E_2(x))(\sum_{m=0}^{M}\sum_{n=0}^{N} a_{nm}x^n y^m)y, \quad x_2 \le x \le L$$

let us assume that the function H (x) is given:

$$H(x)=\sum_{k=0}^{K} b_k x^k \qquad x_0 \le x \le x_1$$

on the free surface, as well as, functions definition, as follows:

$$H(x)=b_0+b_1(x-x_0)+b_2(x-x_0)^2+b_3(x-x_0)^3$$

and

$$H'(x)=b_1+2b_2(x-x_0)+3b_3(x-x_0)^2$$

3.3 Numerical Simulation of the Problem:

In order to solve the problem of this work, numerical simulation was carried using computer program written in MATLAB 2016a. For this purpose, suppose for Ω_1 the following parameters are considered x_0=0.5, x_1=1, x_2=1.5, L=2, H_0=1.0, H_1=0.5. Also, the free surface is assumed to be.

$$H(x)=b_0+b_1(x-x_0)+b_2(x-x_0)^2+b_3(x-x_0)^3 \qquad (3.8)$$

for applying the boundary conditions, the formula becomes as follows:

$$H(x)=H_0-\frac{1}{M_0}(x-x_0)+B(x-x_0)^2+b_3(x-x_0)^3$$

So,

$$b_2=B$$

$$b_3=\frac{M_1+\frac{1}{M_0}+2B(x_1-x_0)}{3(x_1-x_0)^2} \quad , \; x_1 \neq x_0 \qquad (3.9)$$

The approximation of the function Φ using Ritz method over the four sub reigns R_1, R_2, R_3 and R_4, respectfully are:

$$\begin{aligned}
\Phi_1 &= y^2(y-E_1(x))(A+CX+DY)+H_0 \quad , \quad 0 \le x \le x_0 \\
\Phi_2 &= y^2(y-H(x))(A+CX+DY)+H(x) \quad , \quad x_0 \le x \le x_1 \\
\Phi_3 &= y^2(y-E_2(x))(A+CX+DY)+E_1(x) \quad , \quad x_1 \le x \le x_2 \\
\Phi_4 &= y^2(y-E_2(x))(A+CX+DY)+H_1 \quad , \quad x_2 \le x \le L
\end{aligned} \qquad (3.10)$$

In order to find the coefficients A, B, C and D which minimized the functional (3.10), we evaluate the partial derivations of J with respect to those constants and everting to zero will leads to a liner system ,i.e., if:

$$\begin{aligned}
&\Phi_{1x} = y^2(y-E_1(x))C + y^2(-E_1'(x))(A+CX+DY) &&, \quad 0 \le x \le x_0 \\
&\Phi_{2x} = y^2(y-H(x))C + y^2(-H'(x))(A+CX+DY) + H'(x) &&, \quad x_0 \le x \le x_2 \\
&\Phi_{3x} = y^2(y-E_2(x))C + y^2(-H'(x))(A+CX+DY) + E_1'(x) &&, \quad x_1 \le x \le x_2 \\
&\Phi_{4x} = y^2(y-E_2(x))C + Y^2(-E_2'(x))(A+CX+DY &&, \quad x_2 \le x \le L
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\Phi_{1y} = 3y^2 - 2yE_1(x)(A+CX+DY) + y^3 - y^2E_1(x))D &&, \quad 0 \le x \le x_0 \\
&\Phi_{2y} = 3y^2 - 2yH(x)(A+CX+DY) + (y^3 - y^2H(x))D &&, \quad x_0 \le x \le x_1 \\
&\Phi_{3y} = 3y^2 - 2yE_2(x)(A+CX+DY) + (y^3 - y^2E_2(x))D &&, \quad x_1 \le x \le x_2 \\
&\Phi_{4y} = 3y^2 - 2yE_2(x)(A+CX+DY) + (y^3 - y^2E_2(x))D &&, \quad x_2 \le x \le L
\end{aligned} \tag{3.12}$$

then

$$\begin{aligned}
\frac{\partial I}{\partial A} = &\int_0^{x_0}\int_0^{E_1(x)} \left[2\Phi_{1x}(y^2(-E_1'(x)) + 2\Phi_{1y}(3y^2 - 2yE_1(x))\right] dydx \\
&+ \int_{x_0}^{x_1}\int_0^{H(x)} \left[2\Phi_{2x}(y^2(-H'(x)) + 2\Phi_{2y}(3y^2 - 2yH(x))\right] dydx \\
&+ \int_{x_1}^{x_2}\int_0^{E_2(x)} \left[2\Phi_{3x}(y^2(-E_2'(x)) + 2\Phi_{3y}(3y^2 - 2yE_2(x))\right] dydx \\
&+ \int_{x_2}^{L}\int_0^{E_2(x)} \left[2\Phi_{4x}(y^2(-E_2'(x)) + 2\Phi_{4y}(3y^2 - 2yE_2(x))\right] dydx
\end{aligned} \tag{3.13}$$

and

$$\frac{\partial I}{\partial B}=\int_{x_0}^{x1}\left[((x\text{-}x_0)^2+\frac{2(x\text{-}x_0)^3}{3(x_1\text{-}x_0)})(\Phi_{2x}^2+\Phi_{2y}^2)\Big|_{y=H(x)}\right]dx$$
$$+\int_{x_0}^{x1}\int_0^{H(x)}2\Phi_{2x}(y^2(-(x\text{-}x_0)^2-\frac{2(x\text{-}x_0)^3}{3(x_1\text{-}x_0)})C+y^2(-2(x\text{-}x_0)$$
$$-\frac{2(x\text{-}x_0)^2}{x_1\text{-}x_0})(A+Cx+Dy)+2\Phi_{2y}(-2y((x\text{-}x_0)^2 \tag{3.14}$$
$$+\frac{2(x\text{-}x_0)^3}{3(x_1\text{-}x_0)})(A+Cx+Dy)-y^2D(x\text{-}x_0)+\frac{2(x\text{-}x_0)^3}{3(x_1\text{-}x_0)}))dydx$$

and

$$\frac{\partial I}{\partial C}=\int_0^{x_0}\int_0^{E_1(x)}\left[2\Phi_{1x}((y^2(y\text{-}E_1(x))+y^2(-E_1(x))x)+2\Phi_{1y}((3y^2\_2yE_1(x))x)\right]dydx$$
$$+\int_{x_0}^{x_1}\int_0^{H(x)}\left[2\Phi_{2x}((y^2(y\text{-}H(x))+y^2(-H'(x)x)+2\Phi_{2y}(3y^2\text{-}2yH(x))x\right]dydx$$
$$+\int_{x_1}^{x_2}\int_0^{E_2(x)}\left[2\Phi_{3x}(y^2(y\text{-}E_2(x))+y^2(-E_2'(x)x)+2\Phi_{3y}(3y^2\text{-}2yE_2(x))x\right]dydx \tag{3.15}$$
$$+\int_{x_2}^{L}\int_0^{E_2(x)}\left[2\Phi_{4x}(y^2(y\text{-}E_2(x))+y^2(-E_2'(x)x)+2\Phi_{4y}(3y^2\text{-}2yE_2(x))x\right]dydx$$

and

$$\frac{\partial I}{\partial D}=\int_0^{x_0}\int_0^{E_1(x)}\left[2\Phi_{1x}(y^3(-E_1'(x))+2\Phi_{1y}((3y^2\text{-}2y^2E_1(x))y)+(y^3\text{-}y^2E_1(x)))\right]dydx$$
$$+\int_{x_0}^{x_1}\int_0^{H(x)}\left[2\Phi_{2x}((y^3(-H'(x))+2\Phi_{2y}(3y^3\text{-}2y^2H(x))y+y^3\text{-}y^2H(x))\right]dydx$$
$$+\int_{x_1}^{x_2}\int_0^{E_2(x)}\left[2\Phi_{3x}(y^3(-E_2'(x))+2\Phi_{3y}(3y^3\text{-}2y^2E_2(x))y+y^3\text{-}y^2E_2(x))\right]dydx \tag{3.16}$$
$$+\int_{x_2}^{L}\int_0^{E_2(x)}\left[2\Phi_{4x}(y^3(-E_2'(x))+2\Phi_{4x}(3y^3\text{-}y^2E_2(x))y+y^3\text{-}y^2E_2(x))\right]dydx$$

and upper carry the computer program, we get the following results:

H_0=1.00, H_1=0.5, x_0=0.5, x_1=1, x_2=1.5, L=2:

$$H(x)=b_0+b_1(x-x_0)+b_2(x-x_0)^2+b_3(x-x_0)^3 \qquad (3.17)$$

then A= -0.0706, B= -0.0037, C= -0.0677, D= -0.0403

and b_0=1.0000 , b_1=0.5000 , b_2=-1.0000 , b_3= -0.6667

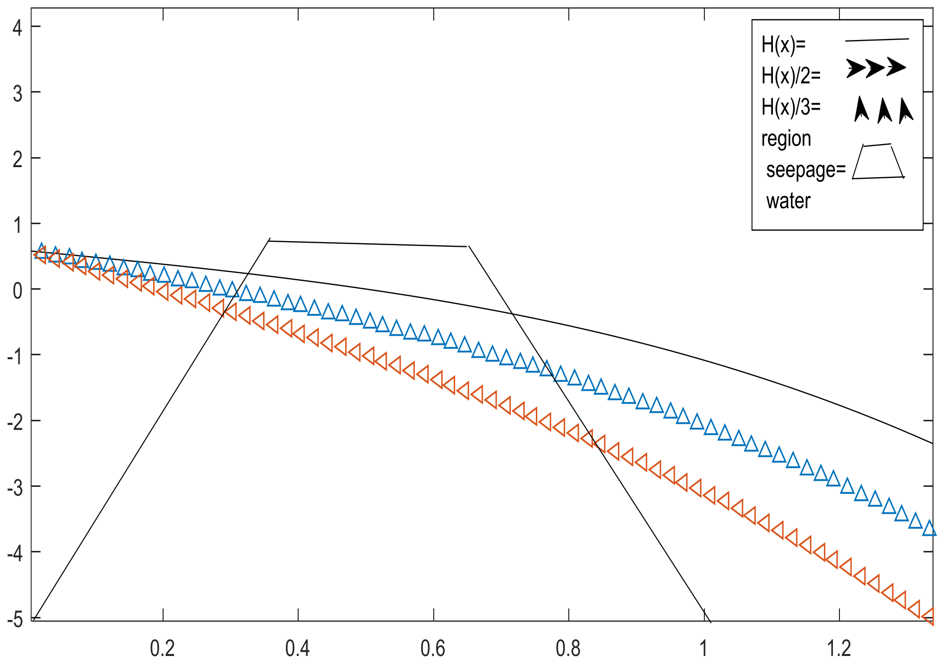

Fig. (3-2) The right-hand side free surface of the two-dimensional pond seepage.

Similarly, we carry out the simulation for the left dam region Ω_2 with the free surface is assumed to be:

Using MATLAB R2016a, the following results have emerged when:

H_0=1.00 , H_1=0.5 , s_0=-0.5 , s_1=-1 , s_2=-1.5, K=2:

$$G(x)=r_0+r_1(x-s_0)+r_2(x-s_0)^2+r_3(x-s_0)^3 \qquad (3.18)$$

A= 0.1475, B= 0.0000, C= -0.2040, D= 0.0527

and

$r_0 = 0.5000$, $r_1 = 1.0000$, $r_2 = 0.5000$, $r_3 = 2.0000$

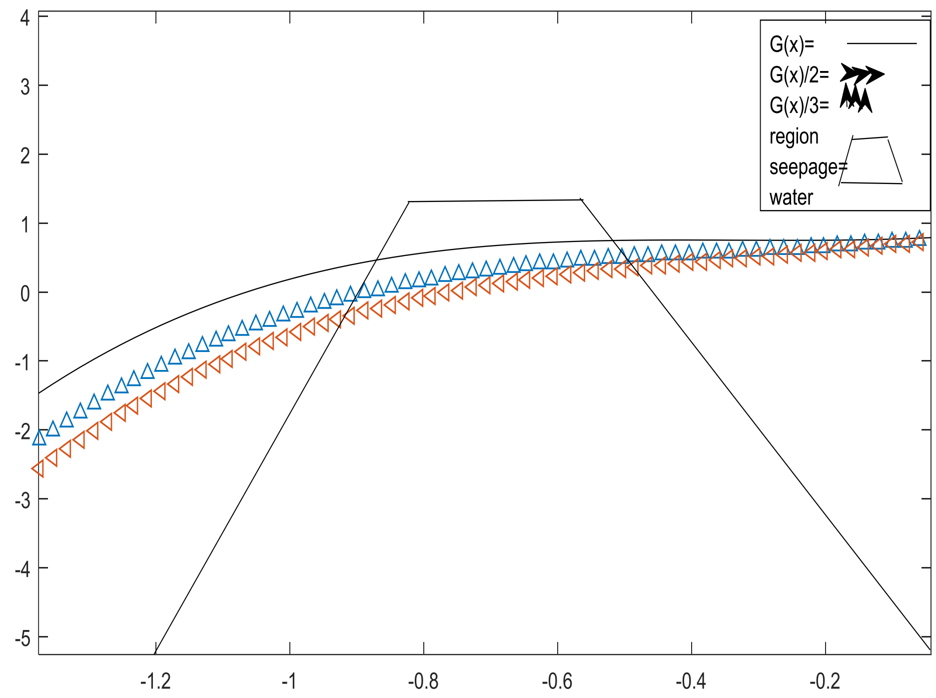

Fig. (3-3)The left-hand side free surface of the two-dimensional pond seepage.

Conclusions and Recommendations for Future Work:

From the present work, we may conclude that the variational approach that may be used to formulate and solve many real life problems especially those problems which have so many initial and/or boundary condition and/or those problems which consists boundary condition of free or moving boundaries which must be determined as a part of the solution.

Also, we may recommend some problems for future work concerning to topic of this thesis, such as:

1. Studying the three-dimensional pond seepage problems.
2. Study the physical and mathematical formulation of the invers problem of pond seepage problem.
3. Study and solve the underground water or oil or gas reservoirs.
4. Use other numerical methods to solve the pond seepage problem, such the methods of lines and finite difference methods.

References

1. Al-Ani O. H., "Inverse Variational Formulation of the Seepage Problem with Singularity", M.Sc. Thesis, Al-Nahrain University, 2001.
2. Ali J. A., "On the Mathematical Inverse Problem with Application to Acoustic Wave Scattering". M.Sc. Thesis, Al-Nahrain University, 1994.
3. Al-Mosawi, A. K., “About the Direct and Inverse Variational Formulation of Three-Dimensional Dams”, M.Sc. Thesis, College of Science of Al-Mustansiriya, 2010.
4. Baiocchi C. and Capelo A., “Variational and Quasi Variational Inequalities: Applications to Free Boundary Problems”, John Wiley and Sons Ltd., 1984.
5. Crank J., “Free and Boundary Value Problems”, Clarendon Press, 1984.
6. Courant R. and Hilbert D., "Methods of Mathematical Physics", Intr. Science Pub., Inc., New York, 1953
7. Elliott, C. M., “On a Variational Inequality Formulation of an Electrochemical Machining Moving Boundary Problem and it’s Approximation by the Finite Element Method”, J. Inst. Math. Applics., Vol. 25, pp.121 – 131, 1980.
8. Elsgolic, L. E., “Calculus of Variation”, Pergamon Press Ltd., 1962.
9. Eman, A. H., “Extremum Variational Principles for Boundary Value Problems with Application”, M.Sc. Thesis, College of Science, Baghdad University, 1987.
10. Gelfand I. M. and Fomin S. V., “Calculus of variations”, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1963.

11. Goodman, T. R., “The Heat Balance and its applications to Problem Involving Change of Phase”, Trans., Asme., Vol. 80, pp. 335 – 342, 1985.
12. Hussain E. A., “Extremum Variational Principles for Boundary Value Problems with Application”, M.Sc. Thesis, College of Science, Baghdad University, 1987.
13. Jabbar S. M., “Variational Approached for Solving Two-Dimensional Free and Moving Boundary Value Problems”, M.Sc. Thesis, College of Education, Babylon University, 2001.
14. Louuis K., “Applied calculus of Variations for Engineers”, CRC Press Taylor & Francis Group 6000 Broken Sound Parkway NW, Suite 300Boca Raton, FL 33487-2742, 2012.
15. Mahlol S. M., “Inverse Problems in Differential Equation with an Application to Localizing Brain Tumors”, Ph.D. Thesis, University of Baghdad, 1993.
16. Magri F., “Variational Formulation for Every Linear Equation”, Int. J. Eng. Sci., Vol.12, pp.537-549, 1974.
17. Meyer, G. H., “The Method of Lines and Invariant Imbedding for Elliptic and Parabolic Free Boundary Problems”, SIAM J. Numerical Anal., Vol. 18, pp.150 – 164, 1981.
18. Milton E. Harr., “Groundwater and seepage”, Purdue University, 2003.
19. Myskis A. D., "Advanced Mathematics for Engineers", Special Courses, MIR Publishers, Moscow, English Translation, 1975.
20. Peter J. Olver.,”Introduction to the Calculus of Variations” University of Minnesota, 2016.

21. Sahar, M. J., “Variational Approached for Solving Two-Dimensional Free and Moving Boundary Value Problems”, M.Sc. Thesis, College of Education, Babylon University, 2000.
22. Sinan, H. A., "Variational Approach for Solving Two-Dimensional Electrochemical Machining Problems", M.Sc. Thesis, College of Education (Ibn Al-Haitham), University of Baghdad, 2002.
23. Vallentine, H.R., “Applied Hydrodynamics”, S.I. Eddition London Butter Worths, 1969.
24. Zainab, Z. H., " Variational Formulation for Solving Three Dimensional Moving Boundary Value Problems and its Application in Laser Ablation Problem", M.Sc. Thesis, College of Science of Al-Mustansiriya, 200.

Appendix

Using MATLAB R2016a.

```
clc
clear
syms x y A B C D b2;
x0=0.5;
x1=1;
x2=1.5;
L=2;
H0=1;
H1=0.5;
m0=H0/x0;
m1=H1/(L-x2);
b0=H0;
b1=H0-(1/m0);
B=H1/(x0-x1);
b3=-(m1+1/m0+2*B*(x1-x0))/(3*(x1-x0)^2);
b2=B;
E1(x)=m0*x;
E2(x)=m1-m1*x;
H(x)=b0+b1*(x-x0)+b2*(x-x0)^2+b3*(x-x0)^3
u1=y^2*(y-E1(x))*(A+C*x+D*y)+H0;
u2=y^2*(y-H(x))*(A+C*x+D*y)+H(x);
u3=y^2*(y-E2(x))*(A+C*x+D*y)+E1(x);
u4=y^2*(y-E2(x))*(A+C*x+D*y)+H1;
u=u1+u2+u3+u4;
U1=diff(u1,A);
U2=diff(u2,A);
U3=diff(u3,A);
```

```
U4=diff(u4,A);
a1=int(int(U1,y,0,E1(x)),x,0,x0);
a2=int(int(U2,y,0,H(x)),x,x0,x1);
a3=int(int(U3,y,0,E2(x)),x,x1,x2);
a4=int(int(U4,y,0,E2(x)),x,x2,L);
A=a1+a2+a3+a4;
C1=diff(u1,C);
C2=diff(u2,C);
C3=diff(u3,C);
C4=diff(u4,C);
c1=int(int(C1,y,0,E1(x)),x,0,x0);
c2=int(int(C2,y,0,H(x)),x,x0,x1);
c3=int(int(C3,y,0,E2(x)),x,x1,x2);
c4=int(int(C4,y,0,E2(x)),x,x2,L);
C=c1+c2+c3+c4;
D1=diff(u1,D);
D2=diff(u2,D);
D3=diff(u3,D);
D4=-diff(u4,D);
d1=int(int(D1,y,0,E1(x)),x,0,x0);
d2=int(int(D2,y,0,H(x)),x,x0,x1);
d3=int(int(D3,y,0,E2(x)),x,x1,x2);
d4=int(int(D4,y,0,E2(x)),x,x2,L);
D=d1+d2+d3+d4;
u(x)=diff(u,x);
u(y)=diff(u,y);
u2=y^2*(y-H(x))*(A+C*x+D*y)+H(x);
```

```
R=diff(u2,x);

N=diff(u2,y);

g=diff(R,y);

G=int(g,y,0,H(x));

j=diff(N,y);

J=int(j,y,0,H(x));

t1=2*G*((y^2*(-(x-x0)^2)-(2*(x-x0)^3/(3*x1-3*x0)))*C+y^2*((-
2*x+2*x0)-2*(x-x0)^2/(x1-x0))*(A+C*x+D*y))+2*J*(-2*y*((x-
x0)^2+2*(x-x0)^3/(3*x1-3*x0)))*(A+C*x+D*y)-y^2*D*((x-
x0)^2+2*(x-x0)^3/(3*x1-3*x0));

B1=int(int(t1, y,0,H(x)),x,x0,x1);

t2=((x-x0)^2+2*(x-x0)^3/(3*x1-3*x0))*(G^2+J^2);

B2=int(t2,x,x0,x1);

B=B1+B2

[A,B,C,D];

[b0,b1,b2,b3]

x=linspace(0,2);

e=H(x)-2*x

e1=(H(x))-3*x

e2=H(x)-4*x

v=plot(x,e,'k',x,e1,'^',x,e2,'<')
```

Using MATLAB R2016a

```
clc
clear
syms x y A B C D;
x0=-0.25;
x1=-.75;
x2=-1.25;
L=-1.75;
H0=0.5;
H1=0.25;
m0=H0/x0;
m1=H1/(L-x2);
r0=H0;
r1=H0-(1/m0);
B=H1/(x0-x1);
r2 =B;
r3=-(m1+1/m0+2*B*(x1-x0))/(3*(x1-x0)^2);
E1(x)=m0*x;
E2(x)=m1-m1*x;
I(x)=r0+r1*(x-x0)+r2*(x-x0)^2+r3*(x-x0)^3;
u1=y^2*(y-E1(x))*(A+C*x+D*y)+H0;
u2=y^2*(y-I(x))*(A+C*x+D*y)+I(x);
u3=y^2*(y-E2(x))*(A+C*x+D*y)+E1(x);
u4=y^2*(y-E2(x))*(A+C*x+D*y)+H1;
u=u1+u2+u3+u4;
U1=diff(u1,A);
U2=diff(u2,A);
U3=diff(u3,A);
```

```
U4=diff(u4,A);
a1=int(int(U1,y,0,E1(x)),x,0,x0);
a2=int(int(U2,y,0,I(x)),x,x0,x1);
a3=int(int(U3,y,0,E2(x)),x,x1,x2);
a4=int(int(U4,y,0,E2(x)),x,x2,L);
A=a1+a2+a3+a4;
C1=diff(u1,C);
C2=diff(u2,C);
C3=diff(u3,C);
C4=diff(u4,C);
c1=int(int(C1,y,0,E1(x)),x,0,x0);
c2=int(int(C2,y,0,I(x)),x,x0,x1);
c3=int(int(C3,y,0,E2(x)),x,x1,x2);
c4=int(int(C4,y,0,E2(x)),x,x2,L);
C=c1+c2+c3+c4;
D1=diff(u1,D);
D2=diff(u2,D);
D3=diff(u3,D);
D4=-diff(u4,D);
d1=int(int(D1,y,0,E1(x)),x,0,x0);
d2=int(int(D2,y,0,I(x)),x,x0,x1);
d3=int(int(D3,y,0,E2(x)),x,x1,x2);
d4=int(int(D4,y,0,E2(x)),x,x2,L);
D=d1+d2+d3+d4;
u(x)=diff(u,x);
u(y)=diff(u,y);
u2=y^2*(y-I(x))*(A+C*x+D*y)+I(x);
```

```
R=diff(u2,x);
N=diff(u2,y);
v=diff(R,y);
G=int(v,y,0,I(x));
j=diff(N,y);
J=int(j,y,0,I(x));
t1=2*G*((y^2*(-(x-x0)^2)-(2*(x-x0)^3/(3*x1-3*x0)))*C+y^2*((-
2*x+2*x0)-2*(x-x0)^2/(x1-x0))*(A+C*x+D*y))+2*J*(-2*y*((x-
x0)^2+2*(x-x0)^3/(3*x1-3*x0)))*(A+C*x+D*y)-y^2*D*((x-
x0)^2+2*(x-x0)^3/(3*x1-3*x0));
B1=int(int(t1, y,0,I(x)),x,x0,x1);
t2=((x-x0)^2+2*(x-x0)^3/(3*x1-3*x0))*(G^2+J^2);
B2=int(t2,x,x0,x1);
B=B1+B2;
[A,B,C,D]
[r0,r1,r2,r3]
x=linspace(0,-2)
e=I(x)-x
e1=(I(x))-x/2
e2=I(x)-x/5
v=plot(x,e,'k',x,e1,'^',x,e2,'<')
```

المستخلص

الهدف الرئيسي الاول لهذه الرسالة هو لدراسة الصياغة التغايرية لمسائل القيم الحدودية ذات السطح الحر والمتحرك. المسائل التي تناولناها كانت معادلتها المتحكمة من نوع معادلة لابلاس والشروط الحدودية من نوع السطح الحر.

الهدف الثاني لهذه الرسالة هو اشتقاق الصياغة الفيزيائية والرياضياتية قيد الدراسة ، وهي مسألة النضوح في البرك والتي يمكن اعتبارها مسألة ذات سطح حر. ثم صياغة وحل المسألة قيد الدراسة باستخدام اسلوب التغاير. كما واستخدمت طريقة رتز العددية لحل المسألة تقريبيا، حيث تم كتابة برامج حاسوبية باستخدام لغة ((MATLAB 2016a) لحل المسألة وايجاد النتائج العددية.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بغداد
كلية التربية للعلوم الصرفة /ابن الهيثم
قسم الرياضيات

الصياغة التغايرية لحل مسائل القيم الحدودية الخطية ذات السطح الحر والمتحرك

رسالة
مقدمة الى كلية التربية للعلوم الصرفة / ابن الهيثم – جامعة بغداد
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
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