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On Reliability Estimation of Stress-Strength Model

A Thesis

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BY

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بسم اللّٰه الرحمن الرحيم

إِن كُلُّ مَن فِي السَّمَاوَاتِ

وَالأَرْضِ إِلاَّ آتِي الرَّحْمَنِ

عَبْدًا۝

لَقَدْ أَحْصَاهُمْ وَعَدَّهُمْ عَدًّا۝

صدق اللّٰه العظيم

مريم 93 / 94

SUPERVISOR CERTIFICATION

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DEDICATION

To

Holy Prophet Muhammad

(peace be upon him)…

To my husband who helped me

through my scientific career, and

my family who rejoice for joy,

My Father, First Tutor,

My loving Mother, Heart Beat,

My Dear Brothers,

Each of the characters taught me

Everyone who appreciates science

and respects humanity…..

Fatima

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ABSTRACT

The thesis consists of three pivots,

The first pivot discusses and drives the mathematical formula of the reliability system of a studying three cases for the stress-strength model.

The first one is the system which contains one component for the strength random variable X subject to the stress random variable Y; (R=P(Y<X)) when the two random variables X and Y follow the Exponentiated Weibull Distribution (EWD).

The second case for the system which contains K^{th} parallel components for the strength X_1, X_2, …, X_k subject to a common stress Y (R_k=P(Y< Max X_1, X_2, …, X_k)), when they follow EWD.

The third one, for the multicomponent system (S-out of-K) the strength X works, when at least S of K are work; (1≤ S≤ K) subject to a common stress Y (R(s,k)=P(at least s of the X_1, X_2,…, X_k exceed Y), when X and Y follow EWD.

The second pivot dependent to estimate the reliability system in stress-strength of the three cases above using different estimation methods like:- (Maximum Likelihood Estimator (MLE), Moment Method (MOM), the Shrinkage Methods (Sh1, Sh2, and Th), Least Squared Method (LS), and the Ranked Set Sampling (RSS)).

Finally, the third pivot includes a comparison of the proposed estimation methods for three cases mentioned already using Monte-Carlo simulation depend on the Mean Square Error indicator (MSE).

As well as, Monte Carlo simulation results show that Shrinkage methods are the best in most cases since it checks the Mean Square Error is less compared to other methods.

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LIST OF SYMBOLS AND ABBREVIATIONS

The symbol	The meaning of Symbol
α	The shape parameter
Θ	The shape parameter
EWD	Exponentiated Weibull Distribution
E(X^r)	The r^{th} Moment about the origin
µ	Moment of population
X_{median}	The Median
Var(x)	The Variance
C.S	Coefficient of skewness
C.K	Coefficient of kurtosis
C.V	Coefficient of variation
$M_x(t)$	The moment generating function
PDF	Probability Density Function
CDF	Cumulative Distribution Function
R(t)	Reliability function
h(t)	Hazard function
r. v.	Random variable

$\hat{\alpha}$	Estimated value of shape parameter
n	Sample size
MLE	Maximum Likelihood Estimator
MOM	Moments Method
LS	Least Square Method
Th	Thompson-type Estimator
Sh	Shrinkage Estimator
RSS	Ranked Set Sampling
(S-S)	Stress-Strength model
R	Reliability system of stress-strength model consist one component
$\mathbf{R_k}$	Reliability system for K parallel components in stress-strength model
$\mathbf{R_{(s,k)}}$	Reliability system of the multi-components s-out of-k in the stress-strength model
$\hat{\alpha}_{MLE}$	Estimated value of shape parameter α by maximum likelihood estimator
$\hat{\alpha}_{MOM}$	Estimated value of shape parameter α by Moment Method
$\hat{\alpha}_{Sh1}$	Estimated value of shape parameter α by shrinkage weight function estimator
$\hat{\alpha}_{Sh2}$	Estimated value of shape parameter α by constant shrinkage weight factor estimator
$\hat{\alpha}_{Th}$	Estimated value of shape parameter α by modified Thompson type shrinkage weight factor method
$\hat{\alpha}_{LS}$	Estimated value of shape parameter α by least squares estimator
$\hat{\alpha}_{RSS}$	Estimated value of shape parameter α by ranked set sampling method

$\hat{R}_{MLE}$	Estimated value of reliability R by maximum likelihood estimator
$\hat{R}_{MOM}$	Estimated value of reliability R by Moment Method
$\hat{R}_{Sh1}$	Estimated value of reliability R by shrinkage weight function estimator
$\hat{R}_{Sh2}$	Estimated value of reliability R by constant shrinkage weight factor estimator
$\hat{R}_{Th}$	Estimated value of reliability R by modified Thompson type shrinkage weight factor method
$\hat{R}_{LS}$	Estimated value of reliability R by least squares estimator
$\hat{R}_{RSS}$	Estimated value of reliability R by ranked set sampling method
$\hat{R}_{k_{MLE}}$	Estimated value of reliability R_k by maximum likelihood estimator
$\hat{R}_{k_{MOM}}$	Estimated value of reliability R_k by Moment Method
$\hat{R}_{k_{Sh1}}$	Estimated value of reliability R_k by shrinkage weight function estimator
$\hat{R}_{k_{Sh2}}$	Estimated value of reliability R_k by constant shrinkage weight factor estimator
$\hat{R}_{k_{Th}}$	Estimated value of reliability R_k by modified Thompson type shrinkage weight factor method
$\hat{R}_{k_{LS}}$	Estimated value of reliability R_k by least squares estimator
$\hat{R}_{k_{RSS}}$	Estimated value of reliability R_k by ranked set sampling method
$\hat{R}_{(s,k)_{MLE}}$	Estimated value of reliability $R_{(s,k)}$ by maximum likelihood estimator
$\hat{R}_{(s,k)_{MOM}}$	Estimated value of reliability $R_{(s,k)}$ by Moment Method
$\hat{R}_{(s,k)_{Sh1}}$	Estimated value of reliability $R_{(s,k)}$ by shrinkage weight function

	estimator
$\widehat{\boldsymbol{R}}_{(\boldsymbol{s},\boldsymbol{k})_{\boldsymbol{Sh2}}}$	Estimated value of reliability $R_{(s,k)}$ by constant shrinkage weight factor estimator
$\widehat{\boldsymbol{R}}_{(\boldsymbol{s},\boldsymbol{k})_{\boldsymbol{Th}}}$	Estimated value of reliability $R_{(s,k)}$ by modified Thompson type shrinkage weight factor method
$\widehat{\boldsymbol{R}}_{(\boldsymbol{s},\boldsymbol{k})_{\boldsymbol{LS}}}$	Estimated value of reliability $R_{(s,k)}$ by least squares estimator
$\widehat{\boldsymbol{R}}_{(\boldsymbol{s},\boldsymbol{k})_{\boldsymbol{RSS}}}$	Estimated value of reliability $R_{(s,k)}$ by ranked set sampling method
MSE	Mean Square Error

Chapter One

Introduction, Review of literature and The aim

CHAPTER ONE

INTRODUCTION, REVIEW OF LITERATURE AND THE AIM

1-1 Introduction:-

Reliability is the probability of a unit to achieve a required task, under given environmental and operational circumstances during a specific time. "Most of the workers in the field of reliability have considered reliability as a function of time. It is well known that the constituent components of an industrial or defense equipment, while in operation, are generally subjected to a certain stress level and also that the components are capable of withstanding these stresses (environmental or due to conditions prevailing within the equipment) up to a certain limit only, i.e. the components possess some rated strength to withstand these stresses. If the stress level exceeds the rated value of the components' strength, the component will fail to work.

This fact was practically realized during the Second World War when it was found that defense equipment like radar and electronic communication systems which were quite reliable otherwise, failed to perform their functions adequately and efficiently when made to function under environmental conditions adverse to those for which these were designed. It, therefore, became necessary to consider the effects of environmental conditions while evaluating the reliability of an equipment";[66]

The widest practical for reliability estimation is the renowned as stress–strength (S-S) model. This model is using in several applications like physics and engineering similar to strength failure and the system breakdown. Therefore, its application has spread out into many fields, and one of the most developed usual applications focuses on engineering-oriented problems, as well as the (S-S) model used in engineering devices, it has been usually determining, how long range time of the system will be live.

Many of engineering systems, consist of more than one component which they fail individually or together;[26][28]

The stress-strength (S-S) reliability may be studied for some three models, the first model is the stress-strength (S-S) reliability of the system which is contained one component denoted by R=P(Y<X), where Y arise the stress random variable and X arise the strength random variable where X and Y are independent random variable, so the stress-strength (S-S) reliability formula can be obtained as $R = \int_0^\infty f_{\text{Strength}}(x).F_{\text{Stress}}(x)dx$;[44]. The second model is the system reliability which is content k^{th} parallel component say R_k and the formula of (S-S) reliability of this model obtained as R_k=P(Y< Max X_1, X_2, …, X_k) and in addition, the third model for the reliability system of multi-component system (s-out of-k) obtained as $R_{(s,k)}$ =P(at least s of the X_1, X_2,…, X_k exceed Y).

One of very common problem in the statistical literature was reliability estimation. Therefore, the objective of this thesis is to estimate the reliability of the three stress-strength models mentioned above via different methods suggested when X and Y are independent and follow Exponentiated Weibull Distribution (EWD) using a simulation and make a comparison between the different suggested estimation methods using Mean Squared Error (MSE) criteria.

“In the statistical approach (the stress-strength model), most of the considerations depend on the assumption that the component strengths are independently and identically distributed (iid). Unlike the practical situations, the components of a system are of the different structure so that the assumption of identical strength distributions may not be quite realistic";[26][28]

This thesis is organized as follows: **Chapter one** includes the introduction, literature review and the aim of this thesis. **Chapter two** will address some properties the Exponentiated Weibull distribution (EWD) and derive the theoretical formulas for three types of (S-S) reliability. **Chapter three** concern with estimation reliability of the three model of (S-S) using different estimation methods like Maximum Likelihood (ML), Moment (MOM), Shrinkage methods (Sh), Least Square (LS) and The Ranked Set Sampling (RSS) methods. **Chapter four** consists of the numerical results and including the comparison among the estimation methods made during simulation upon statistical indicator Mean Squared Error (MSE) based on a Monte-Carlo simulation. Finally, **chapter five** will cover the conclusions that the results of this thesis as well as introduce some suggestions for future works.

1-2 Review of literature:-

The issue of finding the reliability P(Y<X) in a stress-strength model has been discussed in the literature widely when X and Y have some specified distribution. In what follows, we show some available references and researchers have studied and estimated the reliability of three models in the stress-strength to contributions towards these models.

- In 1970, Church and Harris estimated the reliability from stress-strength relations and discussed the supply only by assuming that both X and Y are normal independent random variables with the derivation formula and estimated R using ML estimator;[16]

- In 1973, Yadav found reliability system formula in the stress-strength model by assuming the stress distributed as Normal and the strength distributed as Gamma distributions. An estimation of the reliability system introduced under different values of the distribution parameters;[66]
- In the same year, Bhattacharyya and Johnson extended the system of stress-strength to s-out of-k, which is having identical component and estimated the reliability by UMVU.[13]
- In 1974, Bhattacharyya and Johnson introduced and developed the reliability multi-components stress-strength system model $R_{(s,k)}$ which includes k of components and identical strength component put up with common stress function if s ($1 \leq s \leq k$) or more of the components simultaneously operate;[14]
- In 1980, Dhillon established (S-S) reliability system formula for four cases of an identical unit parallel system. The mentioned models were discussed for different cases: when system stress and unit strength are exponentially distributed, when system stress follows Maxwellian distribution and the strength follows an Exponential distribution, when both system stress and strength follow power series hazard rate distribution, finally, when stress and strength follow an Exponential distribution;[18]
- In 1982 Anne Chao compared the estimations of $Pr(Y<X)$ when X and Y are s-independent Exponential distribution random variables with mean β and α respectively. He made calculations for s-bias and mean square error (MSE) of the ML estimation of $Pr(Y<X)$ for the case when two-parameter β, α are unknown and when the only β is unknown;[15].
- In 1985, Pandey and Borhan Uddin developed the reliability in multi-components of stress- strength and they estimated the reliability in multi-

components stress-strength model when both of stress and strength follow Burr distribution[53]

- In 1988, Jaisingh studied the possibility of finding the lower bound for the reliability of a system such that the strength variable follows the Gamma distribution and the stress variable follows Chi-square distribution.[30]
- In 1991, Pandey and Borhan Uddin estimated the reliability system of multi-components in the stress-strength model for Burr distribution by using the Bayes estimators and the non-Bayes estimators and they conclude the Bayes estimators are the best;[52]
- Hanagal in 1996, studied estimation the reliability parallel system which contains two components strength (X_1, X_2) follow Bivariate Pareto (BVP) distribution subject to stresses (Y_1, Y_2) was R_1, or common stress (Y) was R_2, the stress which follows Pareto distribution;[23]
- In 1998, Hanagal estimated the reliability system in the stress-strength model with k^{th} components when will be parallel or series, supposed that the K^{th} components strength subject to common stress and they are independent and follows the two parameters Exponential distribution;[25].
- In 1999, the researcher Hanagal found reliability theoretically in the case of stress and strength for a system contain K independent components and assuming that the system remains work when at least s of the components are working ;$1<s<k$ where the vector of this components $X=(X_1,X_2,\dots,X_k)$ follows absolute continuous multivariate distribution (ACMVE) while the common stress random variable Y follows the Exponential distribution with mean μ, and the system reliability which does not depend on the time is $R=P[Y<X_{(k-s+1)}]$;[22]
- In 2001, Gupta & Brown estimated the reliability (R) of stress-strength on the supposition that each of stress and strength is independent and follows

the normal distribution (Skew-Normal Distribution) and the researchers enhanced their results by applying via a complete data set;[21]

- In 2003, Karam studied efficient estimations of stress-strength system reliability with four cases upon the parameters. She compared between the proposed estimators of the reliability in the stress-strength model via the MLE estimator with UMVUE, when both stress and strength are independent and identical and follow a Weibull distribution. She concludes the MLE is the best in the sense of MSE through simulation;[3]
- As well as, in 2003, Hanagal succeeded in finding and estimating the reliability of the series system contains k^{th} components strength put on common component stress (X_{k+1}), the stress and strength follow some of the failure distributions like Pareto distribution, Weibull distribution, and Gamma distribution when the shape parameter is known integer number, also estimate the parameters using MLE;[24]
- Also in 2003, Nadarajah & Kotz, succeeded in finding reliability in the case of stress-strength $R=P(Y<X)$ when each of X and Y is independent random variables and they belong to a Pareto distribution model;[44]
- In 2005, Ali and Woo considered conclusion the reliability $R=P(Y<X)$ via Pareto distribution with a known scale parameter, including point and interval estimation and a test of hypothesis;[7]
- also in 2005, Nadarajah calculated the reliability in the case of stress and strength model, in the case of the stress X and strength Y are dependent random variables follow Bivariate Beta distribution;[45]
- In 2006, Ng considered some test and confidence bounds for the system reliability $R=P(Y<X)$, where X and Y are independent follow an Exponential distribution with two parameters. The results are based on missing data and are applicable to some related distributions.;[48].

- In 2006, Mokhlis assessed reliability in the case of stress and strength of two components in parallel under the influence of four different stress models and on the assumption that the random variables of strength and stress are independent and distributed (Friday and Patial) Bivariate Exponential distribution and the common stress models follows Exponential distribution for two components and he studies other cases;[38]
- Also, in the same year Abed-El Fattah, Mondouh and Ashour estimated the system reliability R=P(Y<X) in the case of stress-strength using Bayesian estimation method and maximum likelihood method, when the stress and strength are identical and independent random variables and follow Lomax distribution;[12]
- In 2007, the researcher Alani evaluated the reliability of R = P [Y <X] using different estimation methods and on the assumption that the random variables of stress X and strength Y are independent and have the same distribution; So the methods of estimation were the maximum likelihood (ML), the moment method (MOM), the least squared method (LS); the shrinkage method (Sh), when the stress and strength follow Weibull model and the Pareto type 1 models. Comparison between the proposed estimators was made through simulation and she concludes that the MLE is the best for reliability R in the case of a Pareto distribution and shrinkage estimator is the best in the case of a Weibull distribution in the sense of MSE criteria;[1]
- Also, in 2009, Saraçoğlu, Kaya, and Abd-Elfattah estimated the reliability in the stress-strength model, when the stress and the strength are independent random variables and follow Gompertz distribution via three methods, MLE, UMVUE and Bayes estimator. A comparison between the

proposed estimators was made through simulation in the sense of MSE and they conclude that the Bayes estimator is the best and then the second was MLE and finally UMVUE;[61]

- In 2010, Panahi and Asadi considered the estimation of R= P(Y<X) when strength X and stress Y are independent r. v.^{s}, but non-identically follow two parameters Burr type XII distribution and the estimation of reliability R is obtained using MLE, UMVUE, and Bayes estimators. They conclude that the MLE and UMVUE are identical and they are better than Bayes estimator based on MSE criteria for sample size n=10,20,30;[51]
- In 2010, Rao & Kantam studied a system of k multi-components which have independently and identically strength X_1, X_2, …, X_k random variables distributed experiencing the random stress Y when the stress and the strength from the Log–Logistic population;[57]
- In 2011, Mokhlis and Khames estimated the system reliability for parallel and series multi-components in the stress-strength model, such that the components strengths and stresses follow Multivariate Marshall-Olkin Exponential distribution with (k+1) and (n+1) parameters, respectively;[39]
- In 2011, Panhi and Asadi estimated reliability system R=P(Y<X) in stress-strength (S-S) model when the stress and the strength are independent random variables follow Lomax distribution with independent with common scale parameter (λ) and different shape parameters α_1 and α_2, respectively via MLE and Bayes estimators. They conclude that, the MLE is the best for R in the sense of MSE criteria for sample size n=5,10,15,20;[50]
- in 2012, Ali, Pal, and Woo estimated the reliability R = P (Y <X), when X and Y are independent random variables follow Generalized Gamma

distribution including four parameters via a modified maximum likelihood method and Bayesian technique to estimated R. A simulation study has also been carried out to compare the two methods;[5]

- In 2012, Srinivasa Rao estimated reliability system in the multi-component, when the stress and the strength variables distributed as Log-Logistic for some shape parameters. The system reliability estimated via MO, ML, Modified ML, and BLUE methods;[54].
- Maha Abdullah in 2012, computed reliability system in stress-strength which contains K series components strength $X=(X_1, X_2, \ldots, X_k)$ subject to a common stress (X_{k+1}) when the stress and the strength follow an Exponential distribution. She estimated the reliability by different methods like (ML, UMVUE, MSE, LS, and Sh) methods and used Monte Carlo simulation. She found that the shrinkage method is the best comparing to other methods by via MSE and MAPE;[2]
- In the same year, Srinivasa Rao studied estimation for the reliability system for multi-components in case of stress-strength (S-S) model, when the stress and strength independent and identical distribution random variable considered as Generalized Exponential distribution via several shape parameters and estimated the mentioned reliability by ML estimation;[55]
- Also in the same year, Hassan & Basheikh studied the Bayes and non-Bayes estimation of a system reliability in S-out of-K model with non-identical component strengths was subject to a common stress, in the case when the stress and strength follow the Exponentiated Pareto distribution with a common and known shape parameter. Five classical estimation methods; ML, MO, LS, WLS, and percentile; while the Bayesian estimation considered using symmetric and asymmetric loss function;[26].

- Hussian in 2013, considered estimation R=Pr(Y<X), where X and Y follow GIED with different parameters and assuming common scale parameter is known. He obtained the maximum likelihood estimator MLE and Bayes estimators depend on informative and non-informative prior at the parameters. He also found C.I. of R. Comparisons between the proposed estimators were made based on Monte Carlo simulation.;[29].
- In the same year 2013, Amiri et al. considered the reliability estimation of R=P(Y<X), when the stress and strength are two independent random variables follow Weibull distribution with two parameters including different scale parameters and the known fixed shape parameter and they obtained ML, UMVUE, and Bayes estimator of R;[11]
- Also in the same year 2013, Salem estimated the reliability R=P(Y<X), where X and Y are independent follow Weighted Weibull distribution via several methods for estimating the reliability such as ML, LS, and Bayesian estimators based on non-informative and informative prior distribution;[59]
- And in the same year, Ghanim derived and estimated a formula for the (S-S) reliability P(X<Y) when the stress X follows the Gamma distribution with two parameters α_1 and β_1, and the strength Y follows Weibull distributions with two parameters λ_1 and μ_1. He assumed that the stress X follows Weibull distributions with two parameters α_2 and β_2 and the strength Y follows the Gamma distribution with two parameter λ_2 and μ_2. Also, estimated the reliability by using non-Bayesian method MLE and MOM. In addition, Ghanim used Bayesian method via prior distribution loss were Jeffery prior information, Extension of Jeffery and Inverse Gamma distribution. He compared these estimators upon MSE and MAPE.;[19]

- In 2013, Karam studied estimation of system reliability for one, two and s-out of-k components in stress-strength models and the system consists non-identical strength component subject to a common stress, using Exponentiated Exponential distribution with common scale parameter and made comparison between the ML, PC, and LS estimators of these system reliabilities when scale parameter is known;[33]
- Ali in 2013 estimated the reliability of the parallel stress-strength system with non-identical component subject to a common stress using Lomax distribution. She used many methods are Maximum Likelihood, Percentile, Moment and Least Squares of the system reliability function, also made a comparison between these methods by Mean Square Error and Mean Absolute Error;[6].
- Sezer and Kinaci in 2013, estimated the stress-strength reliability (R) for a system consists of two parallel components based on masked data by using Exponential distribution. Also used a simulation study to a comparison between Bayes and MLE;[62].
- In 2014, Karam and Ali considered estimation of system Reliability in stress-strength model contain non-identical parallel component subjected to a common stress using Lomax distribution. They made a comparison between the ML, PC and MO estimators of the reliability system;[32].
- Ghitany et al. in 2015, studied the estimation of the reliability system in the stress-strength model from Power Lindly distribution. The point and interval estimation of R;[20].
- Jani in 2015, discussed the reliability of the mathematical formula of multi-components reliability system in stress-strength for two models $R_{1(s,k)}$ and $R_{2(s,k)}$ for each of the Burr type III, Exponentiated Lomax and Generalized Inverse Rayleigh distributions. She estimated the reliability of

two models by using the ML, LS, WLS, Rg, MOM and RSS methods and made a comparison among them by MSE and MAPE.[31]

- In 2016, Srinivasa Rao et al. estimated a multi-components stress-strength reliability of system when stress and strength variables follow EWD for different shape parameters α and β, and common other shape and scale parameters γ and λ;[56].
- Najarzadegan et al. in 2016, studied estimation of R= P(Y<X); X and Y are independent r. v.[s] follow the L´evy distribution. They have considered three different point estimators for R: MLE, UMVUE and Bayes estimator. He considered two different interval estimators for R: asymptotic maximum likelihood estimator and bootstrap based percentile estimator. Comparisons between these estimators were made by simulation studies also by a real data application.;[46].
- In 2016, Karam considered the reliability function R in stress-strength model for a system has strength component put on independently of three stresses $R_{1,}$ the second case used two parallel components strength which is subjected to a common stress R_2, and the third case used multi-components $R_{(s,k)}$ via Gompertz model when the location parameter is unknown and the shape parameter was known and estimated the three type reliability models by three estimation methods ML, LS, and WLSE and using numerical simulation. A comparison between the three estimations was made based on MSE.;[34].
- In 2016, Mohamed et al. estimated the system reliability R=P(Y<X) in the stress-strength (S-S) model, where X and Y have the Lomax population with common scale parameter. They derived the MLE of R in the case with unknown scale parameter, and the MLE of the three unknown parameters. They proposed percentile bootstrap confidence intervals of R,

a Bayes estimator of R, and the corresponding credible interval using the MCMC sampling technique. Secondly, assuming the case when the common scale parameter is known, the MLE of R is obtained. In this case, Bayes estimators have been derived depend on Lindley's approximations;[37].

- In 2017, Hassan studied the estimation reliability in multi-components of (S-S) system model when the system consists of K components have strength each component experiencing a random stress. The stress and strength follow Lindley distribution. The ML, UMVUE and Bayes estimators of $R_{(s,k)}$ are obtained;[27].

1-3 Some Important Concepts::[49]

1-3-1 Shape parameter:-

A shape parameter controls the distribution and it is called indicator parameter or stop parameter. Some distributions have one shape parameter like Weibull and Gamma distributions and other distributions have two shape parameters like Burr and Beta distributions …etc.

1-3-2 The Cumulative Function (CDF)

The cumulative distribution function (CDF) is

$$F(x;\alpha) = P(X \leq x) = \int_{-\infty}^{x} f(t;\alpha)dt. \tag{1-1}$$

1-3-3 Reliability Function R:

Hence, the reliability function of an item is defined by

$$\mathrm{R(t)} = 1 - F(t) = \mathrm{P(T \geq t)} \qquad \text{for t} > 0 \tag{1-2}$$

$$\mathrm{R}(t) = \int_{t}^{\infty} f(u;\alpha)du.$$

1-3-4 Failure Rate Function h(t):

As a result, the Failure rate (hazard) function is defined as:

$$h(t:\alpha,\theta) = \frac{f(t:\alpha,\theta)}{R(t:\alpha,\theta)}$$

$$= \frac{f(t:\alpha,\theta)}{1-F(t)} \qquad (1\text{-}3)$$

And, the cumulative hazard function is defined by:

$$H(t:\alpha,\theta) = \int_0^t h(u:\alpha,\theta)du$$

$$H(t:\alpha,\theta) = \int_0^t \frac{f(u:\alpha,\theta)}{R(u:\alpha,\theta)}du = -\ln(1-F(t:\alpha,\theta)). \qquad (1\text{-}4)$$

1- 4 The aim of thesis:

The aim of this thesis organizes as follows:

1. Estimate the reliability of stress–strength (S-S) model for the system that contains one component ($R=P(Y<X)$) when the strength X and stress Y follow the two parameters Exponentiated Weibull distribution.
2. Estimate the reliability of K^{th} parallel components system in the stress-strength model $R_k = P(\ Y < \text{Max. } x_1, x_2, \ldots x_k\ )$ when each of X_i ;i=1, 2, …, k and Y follows the two parameters Exponentiated Weibull Distribution (EWD).
3. Estimate the multi-components system reliability in stress-strength model $R_{(s,k)}$ based on two parameters Exponentiated Weibull distribution.
4. Comparisons between the proposed estimators for three reliability models (Maximum Likelihood (ML), Shrinkage Methods (Sh), Least Square(LS), and Rank Set Sampling (RSS)) methods through Monte Carto simulation depends on Mean Squared Error (MSE) criteria for the Exponentiated Weibull distribution with known shape parameter θ and other unknown shape parameter α.

Chapter Two

Theoretical Formulas

CHAPTER TWO

THEORETICAL FORMULAS

2-1 Introduction:

This chapter involves some applications and properties of the two parameters Exponentiated Weibull Distribution (EWD) and also consists of derivation for the reliability system formula in stress-strength (S-S) model when the stress Y and the strength X are independent random variables follow Exponentiated Weibull distribution for the three cases below:-

1- When the system consists of one component strength X subject on a stress Y.
2- When the system consists of K^{th} parallel components strength (X_1, X_2, …, X_k) subject on a common stress Y.
3- When the system of multi-components s-out of-k (s,k) subject to a common stress Y.

2-2 Exponentiated Weibull Distribution:

2-2-1 The Original and Applications of Exponentiated Weibull Distribution:

"The Exponentiated Weibull family, a Weibull extension obtained by adding a second shape parameter, consists of regular distributions with bathtub shaped, unimodal and a broad variety of monotone hazard rates. It can be used for modeling lifetime data from reliability, survival and population studies, various extreme value data, and for constructing isotopes of the tests of the composite hypothesis of Exponentially";[43]. Mudholkar and Srivastava (1993) proposed a modification to the standard Weibull model through the introduction of an additional parameter v, such as G(x) is CDF of Weibull Distribution and F(x) is CDF of Exponentiated Weibull Distribution:

$$F(x) = (G(x))^{v} \quad ; \quad (0<v<\infty)$$

That is mean, when $v = 1$, the model reduces to the standard two-parameter Weibull model.

Also, in the same year, Mudholkar and Srivastava introduced the Exponentiated Weibull family distribution (EW) as an extension of Weibull family;[41]. In 1995, the above researchers introduced the applications of Exponentiated Weibull (EW) distribution;[42]. Mudholkar and Hutson in 1996, studied the reliability and survival of (EW) distribution;[40]. Nassar and Eissa in 2003, studied in more detail the properties of (EW) distribution;[47]. Pal and Woo in 2006, compared the Exponentiated Weibull with two-parameter Weibull and gamma distributions with respect to a failure rate, and also they showed that the EWD has an application in many fields such as health costs, civil engineering, economics, and others;[49]. Several authors have been using the (EWD) as a model of many fields like flood data, biological studies also in physical explanation…etc.

The scheme below, refers to the relationship between the EWD and other distributions:

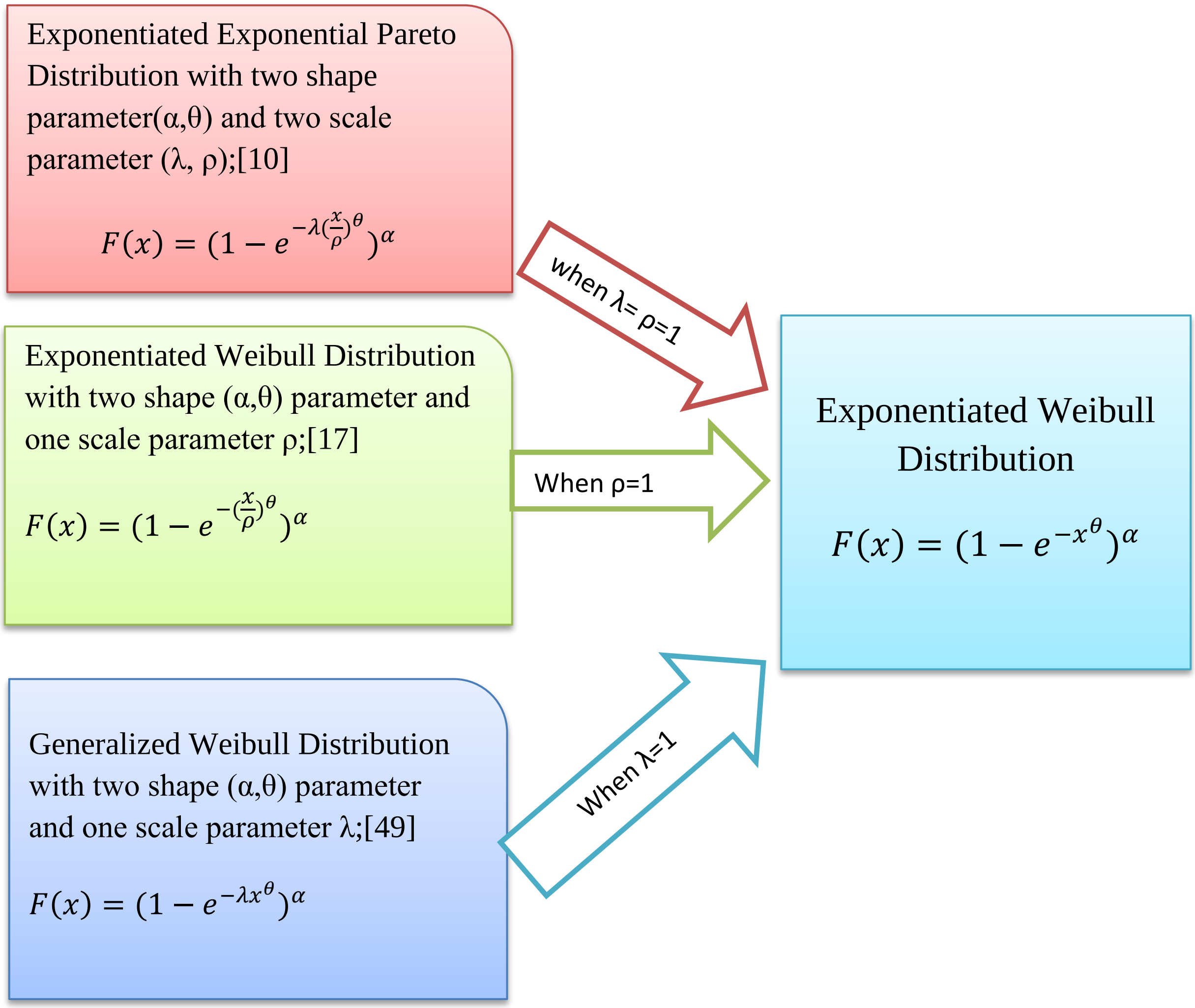

Figure (1): Relationship between the EWD and other distributions

2-2-2 Some Important Properties of Exponentiated Weibull Distribution; [10],[17],[49]

The probability density function (PDF) of a random variable X follow the two parameters Exponentiated Weibull Distribution given as below:

$$f(x;\alpha,\theta) = \begin{cases} \alpha\theta x^{\theta-1} e^{-x^{\theta}} (1 - e^{-x^{\theta}})^{\alpha-1} & ; x > 0, \alpha, \theta > 0 \\ 0 & otherwies \end{cases} . \qquad (2\text{-}1)$$

While, the Cumulative Distribution Function (CDF) is:

$$F(x;\alpha,\theta) = (1 - e^{-x^{\theta}})^{\alpha} \qquad ; x > 0,\ \alpha,\theta > 0\,. \tag{2-2}$$

Hence, the reliability function R(x) will be:

$$R(x) = 1 - \left(1 - e^{-x^{\theta}}\right)^{\alpha} \qquad x>0\ ;\alpha,\ \theta> 0. \tag{2-3}$$

And, the hazard function h(x) is:

$$h(\mathrm{x}) = \frac{f(\mathrm{x})}{R(\mathrm{x})} = \frac{\alpha\theta x^{\theta-1}e^{-x^{\theta}}(1-e^{-x^{\theta}})^{\alpha-1}}{1-\left(1-e^{-x^{\theta}}\right)^{\alpha}}\ . \tag{2-4}$$

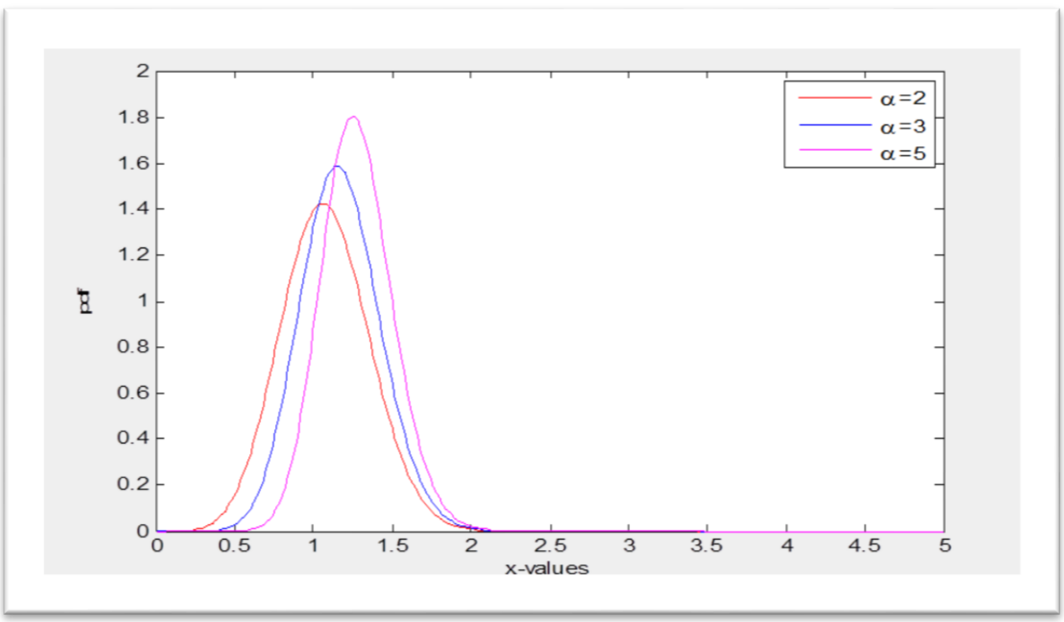

Figure (2): the PDF curve of EWD with different values of the parameters when α=2,3,5 and θ=3.

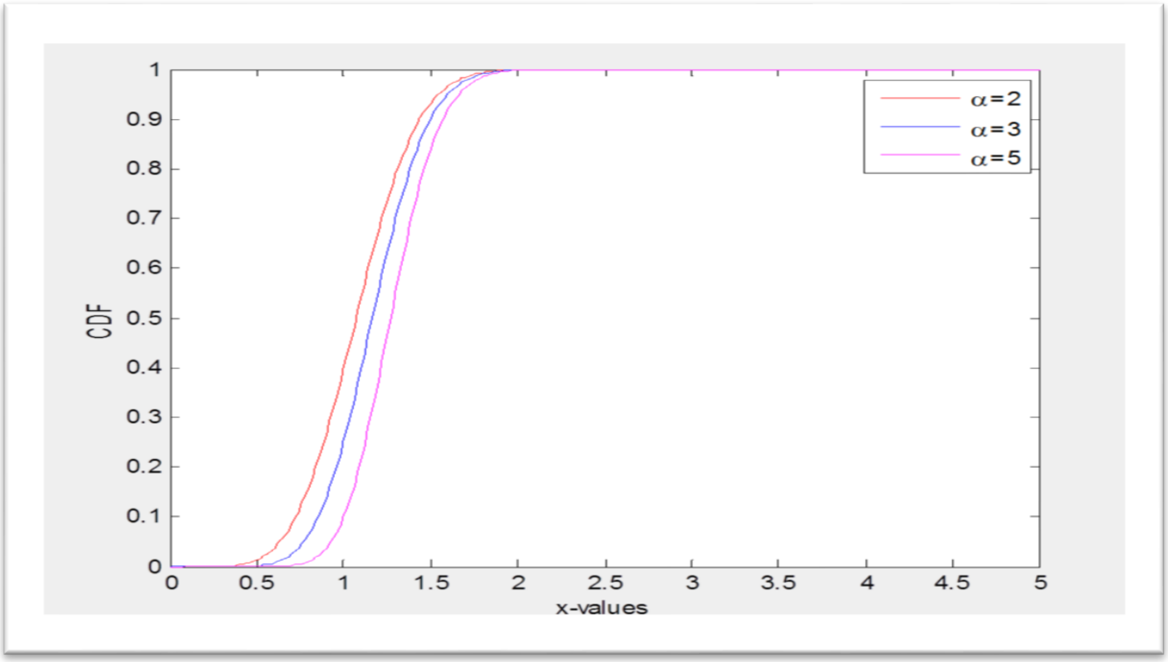

Figure (3): the CDF curve of EWD with different values of the parameters when α=2,3,5 and θ=3.

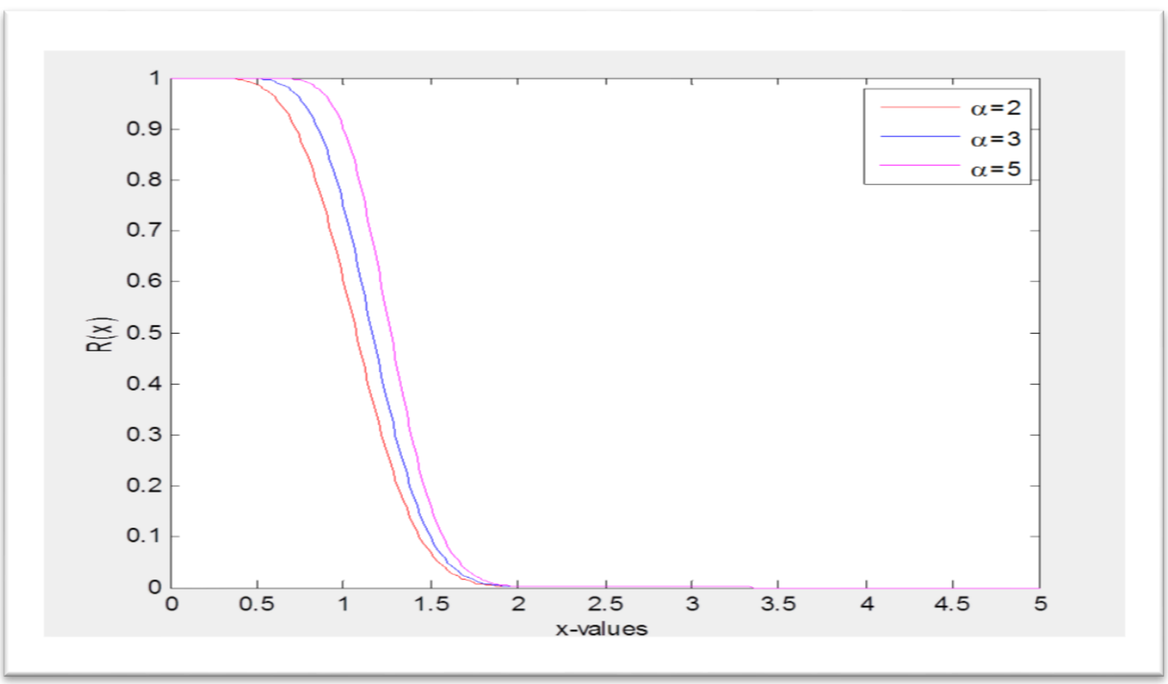

Figure (4): Reliability curve of EWD with different values of the parameters when α=2,3,5 and θ=3.

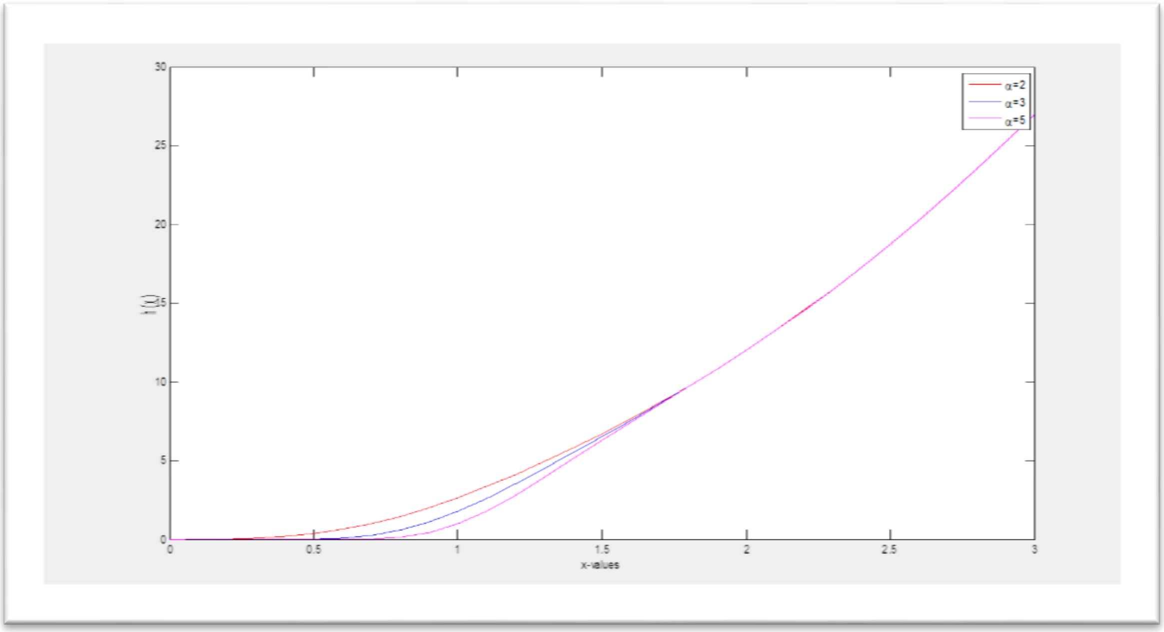

Figure (5): the hazard curve of EWD with different values of the parameters when α=0.5,1,3 and θ=3.

Now, some properties of the Exponentiated Weibull Distribution are listed below:-[10],[17],[49].

The rth moment about the origin:

$$E(X^r) = \alpha \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} (-1)^i \Gamma(\frac{r}{\theta} + 1)(i + 1)^{-(\frac{r}{\theta}+1)}. \qquad (2\text{-}3)$$

This implies that:

The mean:

$$E(X) = \mu = \alpha \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} (-1)^i \Gamma(\frac{1}{\theta} + 1)(i + 1)^{-(\frac{1}{\theta}+1)}. \qquad (2\text{-}4\text{-}a)$$

The Median:

$$X_{median} = \sqrt[\theta]{-\ln(1-\frac{1}{\sqrt[\alpha]{2}})} \qquad \text{(2-4-b)}$$

The variance:

$$Var(x) = \sigma^2 = \alpha \sum_{r=0}^{\infty} \sum_{j=0}^{2} (-1)^{i-j+2} \binom{\alpha-1}{i} \binom{2}{j} \mu^{2-j} \Gamma\left(\frac{1}{\theta}+1\right) (i+1)^{-\left(\frac{1}{\theta}+1\right)}.$$

…(2-5)

The coefficient of variation:

$$\mathrm{C.V.} = \frac{\sqrt{Var(x)}}{E(X)} = \frac{\sqrt{E(X-\mu)^2}}{E(X)}$$

$$= \frac{\sqrt{\alpha \sum_{r=0}^{\infty} \sum_{j=0}^{2} (-1)^{i-j+2} \binom{\alpha-1}{i} \binom{2}{j} \mu^{2-j} \Gamma(\frac{1}{\theta}+1)(i+1)^{-(\frac{1}{\theta}+1)}}}{\alpha \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} (-1)^{i} \Gamma(\frac{1}{\theta}+1)(i+1)^{-(\frac{1}{\theta}+1)}}. \qquad \text{(2-6)}$$

The coefficient of skewness:

$$\mathrm{C.S.} = \frac{E(X-\mu)^3}{(Var(x))^{\frac{3}{2}}}$$

$$= \frac{\alpha \sum_{r=0}^{\infty} \sum_{j=0}^{3} (-1)^{i-j+3} \binom{\alpha-1}{i} \binom{3}{j} \mu^{3-j} \Gamma(\frac{1}{\theta}+1)(i+1)^{-(\frac{1}{\theta}+1)}}{(\alpha \sum_{r=0}^{\infty} \sum_{j=0}^{2} (-1)^{i-j+2} \binom{\alpha-1}{i} \binom{2}{j} \mu^{2-j} \Gamma\left(\frac{1}{\theta}+1\right)(i+1)^{-\left(\frac{1}{\theta}+1\right)})^{\frac{3}{2}}}. \qquad \text{(2-7)}$$

The coefficient of kurtosis:

$$C.K = \frac{E(X-\mu)^4}{(Var(x))^2} = \frac{\alpha \sum_{r=0}^{\infty} \sum_{j=0}^{4} (-1)^{i-j+4} \binom{\alpha-1}{i} \binom{4}{j} \mu^{4-j} \Gamma(\frac{1}{\theta}+1)(i+1)^{-(\frac{1}{\theta}+1)}}{(\alpha \sum_{r=0}^{\infty} \sum_{j=0}^{2} (-1)^{i-j+2} \binom{\alpha-1}{i} \binom{2}{j} \mu^{2-j} \Gamma(\frac{1}{\theta}+1)(i+1)^{-(\frac{1}{\theta}+1)})^2}.$$

…(2-8)

The moment generating function:

$$M_X(t) = E(e^{tx}) = \alpha \sum_{r=0}^{\infty} \sum_{i=0}^{\alpha-1} \frac{t^r}{r!} (-1)^i \binom{\alpha-1}{i} \Gamma(\frac{r}{\theta}+1)(i+1)^{-(\frac{r}{\theta}+1)}. \qquad (2\text{-}9)$$

2-3 Theoretical formulas for the reliability system in (S-S) models:-

2-3-1 The Reliability System (R) of Stress–Strength Model Consist One Component;[11],[34],[35].

This section concerns with estimating the reliability system R in (S-S) model which contains one component via different estimation methods and make a comparison between the proposed methods using simulation depends on the statistical indicator MSE. Our hypothesis in (S-S) model. The stress (Y) and the strength (X) are independent variables follow two parameters Exponentiated Weibull Distribution (EWD).

Now, assume the two random variables X and Y follow the EWD with parameters (α_1,θ) and (α_2,θ) as strength and stress respectively.

Hence, the probability density functions (PDF) from (2-1) for each random variable X and Y is given as below:-

$$f(x;\alpha_1,\theta) = \begin{cases} \alpha_1 \theta x^{\theta-1} e^{-x^\theta} (1-e^{-x^\theta})^{\alpha_1-1} & x>0 \,;\, \alpha_1,\theta>0 \\ 0 & \text{otherwise} \end{cases}. \qquad (2\text{-}12)$$

$$f(y;\alpha_2,\theta) = \begin{cases} \alpha_2\, \theta y^{\theta-1} e^{-y^\theta} (1-e^{-y^\theta})^{\alpha_2-1} & y>0 \,;\, \alpha_2,\theta>0 \\ 0 & \text{otherwise} \end{cases}. \qquad (2\text{-}13)$$

And, from equation (2-2), the CDF of X and Y are, respectively, given as below:-

$$F(x;\alpha_1,\theta) = (1-e^{-x^\theta})^{\alpha_1} \qquad x>0 \,;\, \alpha_1,\theta>0. \qquad (2\text{-}14)$$

$F(y;\alpha_2,\theta)=(1-e^{-y^{\theta}})^{\alpha_2}$ y>0 ; α_2,θ>0. (2-15)

The (S-S) reliability system R for this model is defined as follows:

R = P(Y<X)

$=\iint_{y<x} f(x)f(y)dydx.$

Therefore,

R $=\int_0^{\infty} \alpha_1\theta x^{\theta-1}e^{-x^{\theta}}(1-e^{-x^{\theta}})^{\alpha_1-1}dx\int_0^{x} \alpha_2\,\theta y^{\theta-1}e^{-y^{\theta}}(1-e^{-y^{\theta}})^{\alpha_2-1}\,dy$

$=\int_0^{\infty} \alpha_1\theta x^{\theta-1}e^{-x^{\theta}}\,(1-e^{-x^{\theta}})^{\alpha_2+\alpha_1-1}dx$

$R=\frac{\alpha_1}{\alpha_1+\alpha_2}$ (2-16)

2-3-2 The Reliability System (R_K) for Kth Parallel Components in The Stress-Strength Model;[32],[25]and [32]

This section concerns with finding formula of the reliability system for K^{th} parallel components system in the stress-strength model when (X_1, X_2, …, X_k) are strength subjected to a common stress Y, when the stress and strength follow the Exponentiated Weibull Distribution (EWD) with unknown shape parameter α and the other shape parameter θ to be common and known.

Assume x_1, x_2, …, x_k arises strength have (EWD) with parameter α_i, i=1,2,…k and common parameter θ. All components are subjected to the stress y have (EWD) with two parameters α_{k+1} , θ and assumed to be independent.

Then, the reliability system R_k of the K^{th} parallel components in the stress-strength model is computed as below:

R_k=P(Y<max $x_1, x_2, \ldots, x_k$).

Suppose Z=max($x_1, x_2, \ldots, x_k$)

$$R_k = \int_0^{\infty} \bar{F}_z(y) f(y)\, dy \tag{2-17}$$

Where f(y) and F(y) are defined in equations (2-13) and (2-15) above.

As well known,

$F_z(z)$=P(Z<z)

$=P(x_1<z)P(x_2<z)\ldots P(x_k<z)$

$=(1-e^{-z^{\theta}})^{\alpha_1}(1-e^{-z^{\theta}})^{\alpha_2}\ldots(1-e^{-z^{\theta}})^{\alpha_k}$

$=(1-e^{-z^{\theta}})^{\sum_{i=1}^{k}\alpha_i}$.

This implies that a random variable Z follows EWD with the parameters $\sum_{i=1}^{k}\alpha_i$ and θ.

Therefore,

$$\bar{F}_z(y) = 1-\left(1-e^{-y^{\theta}}\right)^{\sum_{i=1}^{k}\alpha_i}. \tag{2-18}$$

Substitute equation (2-13) and equation (2-18) in equation (2-17), we obtain:

$$R_k = \int_0^{\infty}\left(1-(1-e^{-y^{\theta}})^{\sum_{i=1}^{k}\alpha_i}\right)\alpha_{k+1}\theta y^{\theta-1}e^{-y^{\theta}}(1-e^{-y^{\theta}})^{\alpha_{k+1}-1}dy$$

$$= \int_0^{\infty} \alpha_{k+1}\theta y^{\theta-1} e^{-y^{\theta}}(1-e^{-y^{\theta}})^{\alpha_{k+1}-1}\mathrm{dy} - \int_0^{\infty} \alpha_{k+1}\theta y^{\theta-1} e^{-y^{\theta}}(1-e^{-y^{\theta}})^{\sum_{i=1}^{k+1}\alpha_i-1}\mathrm{dy}$$

$$= 1 - \frac{\alpha_{k+1}}{\sum_{i=1}^{k+1}\alpha_i}$$

$$\mathrm{R_k} = \frac{\sum_{i=1}^{k}\alpha_i}{\sum_{i=1}^{k+1}\alpha_i}. \qquad (2\text{-}19)$$

2-3-3 The Reliability System $R_{(s,k)}$ of The Multi-Components s-out of-k in The Stress-Strength Model:- ;[14[, [22], and [34]

This section concerns with finding the reliability system of the multi-components in stress-strength model $R_{(s,k)}$, when the stress and strength are independent random variables and follow the Exponentiated Weibull Distribution (EWD) with known first shape parameter θ and unknown second shape parameter α.

The reliability of system model, s out of k (s-k) denoted by $R_{(s,k)}$ which means that the system functioning when at least s (1≤s≤k) of components survive was introduced by Bhattacharya and Johnson (1974). "They developed the reliability multi-components stress-strength system model $R_{(s,k)}$ which includes k components and identical strength component put up with a common stress function if s (1≤s≤k) or more of the components simultaneously operate";[5]. In other word, this system works successfully if at least s out of k components resist the stress. Noted that, if s=1 and s=k it corresponded, respectively, to parallel and series systems.

Now, we have to find a system reliability consisting of k^{th} identical components (when at least s out of k function) for the strength X_1, X_2, …, X_k which are random variables with EWD (α_1, θ) subjected to a stress Y which is a random variable follows EWD (α_2, θ) depend on Bhattacharya and Johnson (1974).

Hence the reliability system of a multi-components stress-strength model $R_{(s,k)}$ will be;

$R_{(s,k)}$ =P(at least s of the X_1, X_2,…, X_k exceed Y)

$$=\sum_{i=s}^{k}\binom{k}{i}\int_{o}^{\infty}(1-F_x(y))^i(F_x(y))^{k-i}dF(y). \quad (2\text{-}20)$$

Substituted equation (2-14) and equation (2-13) in equation (2-20) becomes:

$$R_{(s,k)} = \sum_{i=s}^{k}\binom{k}{i}\int_{o}^{\infty}(1-(1-e^{-y^\theta})^{\alpha_1})^i((1-e^{-y^\theta})^{\alpha_1})^{k-i}\alpha_2\theta y^{\theta-1}e^{-y^\theta}(1-e^{-y^\theta})^{\alpha_2-1}dy$$

$$=\sum_{i=s}^{k}\binom{k}{i}\int_{o}^{\infty}(1-(1-e^{-y^\theta})^{\alpha_1})^i\alpha_2\theta y^{\theta-1}e^{-y^\theta}(1-e^{-y^\theta})^{\alpha_1 k-i\alpha_1+\alpha_2-1}dy.$$

Suppose $z = 1-e^{-y^\theta}$, and $dz = \theta y^{\theta-1}e^{-y^\theta}dy$

When 0<y<∞ implies 0<z<1,

and by some simplification, we get

$$R_{(s,k)}= \alpha_2\sum_{i=s}^{k}\binom{k}{i}\int_{o}^{1}(1-z^{\alpha_1})^i z^{\alpha_1 k-i\alpha_1+\alpha_2-1}dz. \quad (2\text{-}21)$$

Assume $t = z^{\alpha_1} \Rightarrow z = (t)^{\frac{1}{\alpha_1}}$; this results $dz = \frac{1}{\alpha_1}(t)^{\frac{1}{\alpha_1}-1}dt$,

and when 0<z<1 implies 0<t<1.

The equation (2-21) become as:

$$R_{(s,k)}= \frac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\binom{k}{i}\int_{o}^{1}(1-t)^i t^{k-i+\frac{\alpha_2}{\alpha_1}-\frac{1}{\alpha_1}}(t)^{\frac{1}{\alpha_1}-1}dt \quad (2\text{-}22)$$

$$R_{(s,k)}= \frac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\binom{k}{i}\int_{o}^{1}(1-t)^i t^{k-i+\frac{\alpha_2}{\alpha_1}-1}dt. \qquad (2\text{-}23)$$

The integration in equation (2-23) above looks like the integration of a function of a random variable T which follows the Beta distribution with parameters (k-i+$\frac{\alpha_2}{\alpha_1}$) and (i+1)

$$R_{(s,k)}= \frac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\frac{k!}{i!(k-i)!}\ \mathrm{B}\,(\mathrm{k}-\mathrm{i}+\frac{\alpha_2}{\alpha_1},\mathrm{i}+1).$$

Using the definition of Beta distribution, we get:

$$R_{(s,k)}= \frac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\frac{k!}{i!(k-i)!}\frac{\Gamma(\mathrm{k-i}+\frac{\alpha_2}{\alpha_1})\Gamma(\mathrm{i}+1)}{\Gamma(\mathrm{k}+\frac{\alpha_2}{\alpha_1}+1)}. \qquad (2\text{-}24)$$

$$R_{(s,k)}= \frac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\frac{k!}{i!(k-i)!}\frac{\Gamma\left(\mathrm{k}+\frac{\alpha_2}{\alpha_1}-\mathrm{i}\right)\mathrm{i}!}{(\mathrm{k}+\frac{\alpha_2}{\alpha_1})\Gamma(\mathrm{k}+\frac{\alpha_2}{\alpha_1})}. \qquad (2\text{-}25)$$

$$R_{(s,k)}= \frac{\alpha_2}{\alpha_1}\sum_{i=s}^{k}\frac{k!}{(k-i)!}\left[\prod_{j=0}^{i}(k+\frac{\alpha_2}{\alpha_1}-j)\right]^{-1} \quad ;\ \text{k, i, j are integers.} \qquad (2\text{-}26)$$

Chapter Three

Some Estimation Methods for Reliability System in Three Proposed Stress-Strength Models

CHAPTER THREE

SOME ESTIMATION METHODS FOR RELIABILITY SYSTEM IN THREE PROPOSED STRESS-STRENGTH MODELS

3-1 Introduction:

This chapter concerns with the estimation of the reliability system in three proposed cases in the stress–strength model when the stress (Y) and the strength (X) are independent variables follow the Exponentiated Weibull distribution (EWD) with two parameters using some estimation methods like Maximum Likelihood (ML), Moment (MO), Shrinkage methods(Sh), Least Squares Estimator (LS), and Rank Set Sampling (RSS).

3-2 Estimation Methods for The Reliability System in (S-S) Models:

3-2-1 Maximum Likelihood Estimator (MLE):[34]

The Maximum Likelihood is one of the important methods of estimation, because it has a good characteristics distinguish from the rest of the methods which is Invariance Property.

3-2-1-1 The Maximum Likelihood Estimator for Reliability System (R) of Stress–Strength Model Consist One Component:

Let $x_1, x_2, \ldots, x_n$ be a random sample from $EW(\alpha_1,\theta)$ and $y_1, y_2, \ldots, y_m$ be random samples from $EW(\alpha_2,\theta)$ then, the likelihood function of the observed sample is given as:

$$l \equiv L(x, y; \alpha_1, \alpha_2, \theta) = \prod_{i=1}^{n} f(x_i) \prod_{j=1}^{m} g(y_j). \tag{3-1}$$

From equation (2-12) and equation (2-13), the equation (3-1) become:

$$= \prod_{i=1}^{n} \alpha_1 \theta x_i^{\theta-1} e^{-x_i^{\theta}} (1 - e^{-x_i^{\theta}})^{\alpha_1 - 1} \prod_{j=1}^{m} \alpha_2 \theta y_j^{\theta-1} e^{-y_j^{\theta}} (1 - e^{-y_j^{\theta}})^{\alpha_2 - 1}.$$

$$= \alpha_1^n \alpha_2^m \theta^{n+m} e^{-\sum_{i=1}^{n} x_i^{\theta} - \sum_{j=1}^{m} y_j^{\theta}} \prod_{i=1}^{n} x_i^{\theta-1} \prod_{j=1}^{m} y_j^{\theta-1} \prod_{i=1}^{n} (1 - e^{-x_i^{\theta}})^{\alpha_1 - 1} \prod_{j=1}^{m} (1 - e^{-y_j^{\theta}})^{\alpha_2 - 1}.$$

Take Ln to both sides, we get:-

$$\mathrm{Ln}(l) = \mathrm{nLn}\alpha_1 + \mathrm{mLn}\alpha_2 + (\mathrm{n} + \mathrm{m})\mathrm{Ln}\theta + \sum_{i=1}^{n} Lnx_i^{\theta-1} + \sum_{j=1}^{m} Lny_j^{\theta-1} - \sum_{i=1}^{n} x_i^{\theta} - \sum_{j=1}^{m} y_j^{\theta} + \sum_{i=1}^{n} Ln(1 - e^{-x_i^{\theta}})^{\alpha_1 - 1} + \sum_{j=1}^{m} Ln(1 - e^{-y_j^{\theta}})^{\alpha_2 - 1}. \quad (3\text{-}2)$$

By taking the partial derivatives of Ln(l) with respect to α_1 and α_2 respectively, we conclude:

$$\frac{\partial Ln(l)}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} Ln\left(1 - e^{-x_i^{\theta}}\right), \quad (3\text{-}3)$$

$$\frac{\partial Ln(l)}{\partial \alpha_2} = \frac{m}{\alpha_2} + \sum_{j=1}^{m} Ln\left(1 - e^{-y_j^{\theta}}\right). \quad (3\text{-}4)$$

Equating the equations (3-3) and (3-4) to zero will be:

$$\frac{n}{\alpha_1} + \sum_{i=1}^{n} Ln\left(1 - e^{-x_i^{\theta}}\right) = 0, \quad (3\text{-}5)$$

$$\frac{m}{\alpha_2} + \sum_{j=1}^{m} Ln\left(1 - e^{-y_j^{\theta}}\right) = 0. \quad (3\text{-}6)$$

Then, the ML's estimator for the unknown shape parameters α_i (i=1, 2) are respectively given as below:

$$\hat{\alpha}_{1_{MLE}}=\frac{-n}{\sum_{i=1}^{n} Ln\left(1-e^{-x_i^{\theta}}\right)}\ , \tag{3-7}$$

$$\hat{\alpha}_{2_{MLE}}=\frac{-m}{\sum_{j=1}^{m} Ln\left(1-e^{-y_j^{\theta}}\right)}. \tag{3-8}$$

Noted that, $\hat{\alpha}_{i_{MLE}}$ is biased since E($\hat{\alpha}_{i_{MLE}}$)=$\frac{\mathfrak{z}\alpha_i}{\mathfrak{z}-1} \neq \alpha_i$, $\mathfrak{z}$ may refers to n or m depend on (i) respectively.

Hence, $\hat{\alpha}_{i_{ub}}=\frac{\mathfrak{z}-1}{\mathfrak{z}}\hat{\alpha}_{i_{MLE}}$ will be unbiased estimator of α_i, i=1,2.

Therefore,

$$E(\hat{\alpha}_{i_{ub}}) = \alpha_i \text{ and, } Var(\hat{\alpha}_{i_{ub}})=\frac{(\alpha_i)^2}{(\mathfrak{z}-2)}\ ;i=1,2. \tag{3-9}$$

Then, $$\hat{\alpha}_{1_{ub}}=\frac{n-1}{-\sum_{i=1}^{n} Ln\left(1-e^{-x_i^{\theta}}\right)}\ . \tag{3-10}$$

And, $$\hat{\alpha}_{2_{ub}}=\frac{m-1}{-\sum_{j=1}^{m} Ln\left(1-e^{-y_j^{\theta}}\right)}\ . \tag{3-11}$$

By substituting equation (3-7) and equation (3-8) in equation (2-16), we get the reliability estimation in (S-S) model which is contain one component as below:-

$$\hat{\boldsymbol{R}}_{\boldsymbol{MLE}} = \frac{\hat{\boldsymbol{\alpha}}_{\boldsymbol{1_{MLE}}}}{\hat{\boldsymbol{\alpha}}_{\boldsymbol{1_{MLE}}}+\hat{\boldsymbol{\alpha}}_{\boldsymbol{2_{MLE}}}}\ . \tag{3-12}$$

3-2-1-2 The Maximum Likelihood Estimator for Reliability System (R_K) for K^{th} Parallel Components in The Stress-Strength Model:-

In order to obtain the maximum likelihood estimator of R_k, suppose that k components of the system are put on the life–testing experiment. Let x_{11}, x_{12}, …, x_{1n1}; x_{21}, x_{22}, …,x_{2n2}; x_{k1}, x_{k2}, …, x_{knk} form EW(α_i, θ), respectively, for i=1,2,…,k; and y_t ;t=1,2,…,m form EW(α_{k+1}, θ). Then the likelihood function of the mentioned system will be:

$$l \equiv \mathrm{L}(\alpha_i, \theta; x_i, y) = \prod_{j=1}^{n_i}\left(\prod_{i=1}^{k} f(x_{ij})\right)\prod_{t=1}^{m} g(y_t). \tag{3-13}$$

From equation (2-12) and equation (2-13) in equation (3-13) become:

$$l = \prod_{i=1}^{k} {\alpha_i}^{n_i}\, \theta^{n_i k} \prod_{j=1}^{n_i}\prod_{i=1}^{k} {x_{ij}}^{\theta-1} e^{-\sum_{j=1}^{n_i}\sum_{i=1}^{k} {x_{ij}}^{\theta}} \prod_{j=1}^{n_i}\prod_{i=1}^{k}(1-e^{-{x_{ij}}^{\theta}})^{\alpha_i-1}$$

$$(\alpha_{k+1})^m \theta^m \prod_{t=1}^{m} {y_t}^{\theta-1}\, e^{-\sum_{t=1}^{m} {y_t}^{\theta}} \prod_{t=1}^{m}(1-e^{-{y_t}^{\theta}})^{\alpha_{k+1}-1}. \tag{3-14}$$

Take Ln to both sides in equation (3-14) we get:-

$$Ln(l) =$$

$$n_i \sum_{i=1}^{k} ln\alpha_i + (kn_i + m)ln\theta + (\theta-1)\sum_{j=1}^{n_i}\sum_{i=1}^{k} lnx_{ij} - \sum_{j=1}^{n_i}\sum_{i=1}^{k} {x_{ij}}^{\theta} +$$
$$(\alpha_i - 1)\sum_{j=1}^{n_i}\sum_{i=1}^{k} ln\left(1-e^{-{x_{ij}}^{\theta}}\right) + m\ ln\alpha_{k+1} + (\theta-1)\sum_{t=1}^{m} lny_t -$$
$$\sum_{t=1}^{m} {y_t}^{\theta} + (\alpha_{k+1}-1)\sum_{t=1}^{m} \ln\left(1-e^{-{y_t}^{\theta}}\right). \qquad \ldots(3\text{-}15)$$

The partial derivatives for the equation (3-15) with respect to the unknown shape parameters α_i (i=1,2,…,k+1) we get:

$$\frac{\partial \mathrm{Ln}(l)}{\partial \alpha_i} = \frac{n_i}{\alpha_i} + \sum_{j=1}^{n_i} ln\left(1-e^{-{x_{ij}}^{\theta}}\right) \qquad ;i=1,2,\ldots,k \tag{3-16}$$

$$\frac{\partial \mathrm{Ln}(l)}{\partial \alpha_{k+1}} = \frac{m}{\alpha_{k+1}} + \sum_{t=1}^{m} ln\left(1 - e^{-y_t{}^{\theta}}\right). \tag{3-17}$$

Equating the equations (3-16) and (3-17) to zero will be:

$$\frac{n_i}{\alpha_i} + \sum_{j=1}^{n_i} ln\left(1 - e^{-x_{ij}{}^{\theta}}\right) = 0 \qquad ;i=1,2,\ldots,k \tag{3-18}$$

$$\frac{m}{\alpha_{k+1}} + \sum_{t=1}^{m} ln\left(1 - e^{-y_t{}^{\theta}}\right) = 0. \tag{3-19}$$

Thus, the maximum likelihood estimator of the parameter α_i (i=1,2,…,k+1) will be as follows:

$$\hat{\alpha}_{i_{MLE}} = \frac{-n_i}{\sum_{j=1}^{n_i} ln\left(1-e^{-x_{ij}{}^{\theta}}\right)} \;;i=1,2,\ldots,k \tag{3-20}$$

$$\hat{\alpha}_{k+1_{MLE}} = \frac{-m}{\sum_{t=1}^{m} ln\left(1-e^{-y_t{}^{\theta}}\right)} . \tag{3-21}$$

By substituting $\hat{\alpha}_{i_{MLE}}$ and $\hat{\alpha}_{k+1_{MLE}}$in equation (2-19), we get the reliability estimation for K component parallel system in (S-S) model using Maximum Likelihood method as below:

$$\hat{\mathbf{R}}_{\boldsymbol{k}_{MLE}} = \frac{\sum_{\mathrm{i=1}}^{\mathrm{k}} \hat{\boldsymbol{\alpha}}_{\boldsymbol{i}_{MLE}}}{\sum_{\mathrm{i=1}}^{\mathrm{k+1}} \hat{\boldsymbol{\alpha}}_{\boldsymbol{i}_{MLE}}} . \tag{3-22}$$

3-2-1-3 The Maximum Likelihood Estimator for Reliability System $R_{(s,k)}$ of The Multi-Components s-out of-k in The Stress-Strength Model:-

Suppose that k of components are put on life-testing experiment. In this case, let $x_1, x_2, \ldots, x_n$ be a random sample of size n follow EWD (α_1, θ), and $y_1, y_2, \ldots, y_m$ be a random sample of size m follows EWD (α_2, θ).

Then, the maximum likelihood estimator $\hat{\alpha}_{i_{mle}}$ (i=1,2) for the mentioned parameters α_i will be the same as the equations (3-7) and (3-8) in subsection 3-2-1-1.

By substituting $\hat{\alpha}_{i_{mle}}$ (i=1,2) in equation (2-26), we get the reliability estimation for $R_{(s,k)}$ model via Maximum Likelihood method as below:

$$\widehat{\boldsymbol{R}}_{(s,k)_{MLE}} = \frac{\hat{\alpha}_{2_{MLE}}}{\hat{\alpha}_{1_{MLE}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} \left(k + \frac{\hat{\alpha}_{2_{MLE}}}{\hat{\alpha}_{1_{MLE}}} - j\right) \right]^{-1}. \tag{3-23}$$

3-2-2 Moment Method (MOM):-

A moment method was one of the first methods used to estimate the population parameter, it was introduced by Person (1894).

3-2-2-1 The Moment Method for Reliability System (R) in Stress–Strength Model Consist One Component:[6],[49]

We need the population moments for X and Y of EWD, which is given below:-

$$E(X^r)=\begin{cases} \alpha \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} (-1)^i (i+1)^{-\frac{r}{\theta}-1} \Gamma\left(\frac{r}{\theta}+1\right), if \alpha \in N \\ \alpha \sum_{i=0}^{\infty} \frac{\Gamma(\alpha)}{i!\Gamma(\alpha-i)} (-1)^i (i+1)^{-\frac{r}{\theta}-1} \Gamma\left(\frac{r}{\theta}+1\right), if\ \alpha \notin N \end{cases} \text{for r=1,2,3...} \tag{3-24}$$

Then, the population means of X and Y are, respectively, as below;[49].

$$E(X) = \begin{cases} \alpha_1 \sum_{i=0}^{\alpha_1-1} \binom{\alpha_1-1}{i} (-1)^i (i+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if \alpha_1 \in N \\ \alpha_1 \sum_{i=0}^{\infty} \frac{\Gamma(\alpha_1)}{i!\Gamma(\alpha_1-i)} (-1)^i (i+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if\ \alpha_1 \notin N \end{cases} \quad (3\text{-}25)$$

$$E(Y) = \begin{cases} \alpha_2 \sum_{j=0}^{\alpha_2-1} \binom{\alpha_2-1}{j} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if \alpha_2 \in N \\ \alpha_2 \sum_{j=0}^{\infty} \frac{\Gamma(\alpha_2)}{j!\Gamma(\alpha_2-j)} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if\ \alpha_2 \notin N \end{cases} \quad (3\text{-}26)$$

Equating the sample mean with corresponding population mean, we get:

$$\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n} = E(X)\ . \quad (3\text{-}27)$$

And,

$$\bar{Y} = \frac{\sum_{j=1}^{m} y_j}{m} = E(Y). \quad (3\text{-}28)$$

Then, the Moment estimation method of α_i (i=1,2), say $\hat{\alpha}_{i_{MOM}}$ can be obtained by solving the above equations (3-27) and (3-28) with respect to α_i (i=1,2), when θ is known based on median procedure which is explained in an appendix;[67].

Substitution $\hat{\alpha}_{i_{MOM}}$ (i=1,2) in the equation (2-16), we get the estimation of (S-S) reliability by moment method as below:

$$\hat{\boldsymbol{R}}_{MOM} = \frac{\hat{\alpha}_{1_{MOM}}}{\hat{\alpha}_{1_{MOM}} + \hat{\alpha}_{2_{MOM}}}\ . \quad (3\text{-}29)$$

3-2-2-2 The Moment Method for Reliability System (R_K) for K^{th} Parallel Components in The Stress-Strength Model:-[32].[34]

Let x_{ij} ;j=1,2,.., n_i be the strength random samples from EWD(α_i, θ) with size n_i where α_i are unknown parameters, i=1,2,…,k, and let y_t ; t=1,2,…,m be the stress random samples from EWD (α_{k+1}, θ) with size m, where α_i (i= 1,2,…,k+1) are unknown parameters and θ is known parameter.

The population mean of a random variables X_i which is follows the above distribution is given by equation (3-25).

Therefore,

$$E(X_i) = \begin{cases} \alpha_i \sum_{j=0}^{\alpha_i-1} \binom{\alpha_i-1}{j} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if\, \alpha_i \in N \\ \alpha_i \sum_{j=0}^{\infty} \frac{\Gamma(\alpha_i)}{j!\Gamma(\alpha_i-j)} (-1)^j (j+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) & , if\ \alpha_i \notin N \end{cases}$$

for i=1,2,…,k. …(3-30)

And by the same way, the population of the random variable Y will be:

$$\mathrm{E(Y)} = \begin{cases} \alpha_{k+1} \sum_{t=0}^{\alpha_{k+1}-1} \binom{\alpha_{k+1}-1}{t} (-1)^t (t+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right), if\, \alpha_{k+1} \in N \\ \alpha_{k+1} \sum_{t=0}^{\infty} \frac{\Gamma(\alpha_{k+1})}{t!\Gamma(\alpha_{k+1}-t)} (-1)^t (t+1)^{-\frac{1}{\theta}-1} \Gamma\left(\frac{1}{\theta}+1\right) \quad , if\ \alpha_{k+1} \notin N \end{cases}$$

…(3-31)

Equating the sample mean with the corresponding population mean as follows:-

$$\overline{\mathrm{X}}_i = \frac{\sum_{j=1}^{n_i} x_{ij}}{n_i} = E(X_i) \quad ; \text{for i=1,2,…,k.} \qquad (3\text{-}32)$$

And,

$$\overline{\mathrm{Y}} = \frac{\sum_{t=1}^{m} y_t}{m} = E(Y). \tag{3-33}$$

Then, the Moment estimation method of α_i (i=1, 2, …, k+1), say $\hat{\alpha}_{i_{MOM}}$ can be obtained by solving the above equations (3-32) and (3-33) with respect to α_i (i=1, 2, …, k+1) when θ is known based on median procedure which is explained in the appendix;[67].

Substitution $\hat{\alpha}_{i_{MOM}}$ (i=1, 2, …, k+1) in equation (2-19), we obtain the reliability estimation of K components parallel system in (S-S) model using moment method as below:

$$\hat{\mathbf{R}}_{k_{MOM}} = \frac{\sum_{\mathrm{i=1}}^{\mathrm{k}} \hat{\alpha}_{i_{MOM}}}{\sum_{\mathrm{i=1}}^{\mathrm{k+1}} \hat{\alpha}_{i_{MOM}}} . \tag{3-34}$$

3-2-2-3 The Moment Method for Reliability System $R_{(s,k)}$ of The Multi-Components s-out of-k in The Stress-Strength Model:- ;[31]

Let x_1, x_2, …, x_n be a random sample of size n for strength X follows EW(α_1, θ), and y_1 , y_2, …, y_m be a random sample of size m for stress Y follows EW(α_2, θ). Let $\bar{X}$ and $\bar{Y}$ are the means of samples of strength and stress respectively, and the population moments of X, Y are respectively given in equations (3-25) and (3-26). Then the estimation of unknown shape parameters α_i (i=1,2) will be the same as equations (3-27) and (3-28), respectively, in subsection 3-2-2-1.

Substitution $\hat{\alpha}_{i_{mom}}$ (i=1,2) in equation (2-26), we conclude the reliability estimation for $R_{(s,k)}$ model via moment method as below:

$$\widehat{\boldsymbol{R}}_{(s,k)_{MOM}} = \frac{\hat{\alpha}_{2_{MOM}}}{\hat{\alpha}_{1_{MOM}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} \left(k + \frac{\hat{\alpha}_{2_{MOM}}}{\hat{\alpha}_{1_{MOM}}} - j\right) \right]^{-1}. \quad (3\text{-}35)$$

3-2-3 Shrinkage Estimation Method (Sh)::[4], [8], [9], [60] and [65]

Thompson in 1968 has suggested the problem of shrink a usual estimator $\hat{\alpha}$ of the parameter α to prior information α_0 using shrinkage weight factor $\emptyset(\hat{\alpha})$, such that $0 \le \emptyset(\hat{\alpha}) \le 1$. Thompson say that "We are estimating α and we believe α_0 is closed to the true value of α or We fear that α_0 may be near the true value of α, that is mean something bad happens if $\alpha_0 \approx \alpha$ and we do not use α_0". Thus, the form of shrinkage estimator of α denoted by $\hat{\alpha}_{Sh}$ is defined as below:

$$\hat{\alpha}_{Sh} = \emptyset(\hat{\alpha})\hat{\alpha} + (1 - \emptyset(\hat{\alpha}))\hat{\alpha}_0. \quad (3\text{-}36)$$

By using the unbiased estimator $\hat{\alpha}_{ub}$ as a usual estimator and the moment estimator as a prior information of α in equation (3-36) above.

Where $\emptyset(\hat{\alpha})$ denote the shrinkage weight factor as we mentioned above such that $0 \le \emptyset(\hat{\alpha}) \le 1$, which may be a function of $\hat{\alpha}_{ub}$, a function of sample size (n_i,m) or may be constant or can be found by minimizing the mean square error of $\hat{\alpha}_{Sh}$.

Thus, the shrinkage estimator of the shape parameter α of EWD will be as follows:

$$\hat{\alpha}_{Sh} = \emptyset(\hat{\alpha})\hat{\alpha}_{ub} + \left(1 - \emptyset(\hat{\alpha})\right)\hat{\alpha}_{MOM}. \quad (3\text{-}37)$$

3-2-3-1 The Shrinkage Weight Function (Sh1):

3-2-3-1-1 Shrinkage Weight Function for Reliability System (R) of Stress–Strength Model Consist One Component:

In this subsection, consider the shrinkage weight factor as a function of n and m, respectively, in equation (3-37)

$\emptyset(\hat{\alpha}_1) = e^{-n}$, and $\emptyset(\hat{\alpha}_2) = e^{-m}$.

Therefore,

$$\hat{\alpha}_{i_{Sh1}} = \emptyset(\hat{\alpha}_i)\hat{\alpha}_{i_{ub}} + (1 - \emptyset(\hat{\alpha}_i))\hat{\alpha}_{i_{MOM}} \qquad \text{for i=1, 2} \qquad (3\text{-}38)$$

Where $\hat{\alpha}_{i_{ub}}$ i=1,2 are defined as in equation (3-10) and (3-11), respectively.

The corresponding (S-S) reliability $\hat{R}_{Sh1}$ using shrinkage method in equation (3-38) will be obtained by substituting equation (3-38) in equation (2-16) as follows:

$$\hat{\mathbf{R}}_{\mathbf{Sh1}} = \frac{\hat{\alpha}_{1_{Sh1}}}{\hat{\alpha}_{1_{Sh1}} + \hat{\alpha}_{2_{Sh1}}} \,. \qquad (3\text{-}39)$$

3-2-3-1-2 Shrinkage Weight Function for Reliability System (R_K) for K^{th} Parallel Components in The Stress-Strength Model:-

In this subsection, consider the shrinkage weight factor as a function of n_i and m, respectively, as below.

$\emptyset(\hat{\alpha}_i) = e^{-n_i}$, and $\emptyset(\hat{\alpha}_{k+1}) = e^{-m}$;i=1,2,…,k.

Therefore, the shrinkage estimator of α which is defined in equation (3-37) using above shrinkage weight function will be:

$$\hat{\alpha}_{i_{Sh1}} = \emptyset(\hat{\alpha}_i)\hat{\alpha}_{i_{ub}} + (1 - \emptyset(\hat{\alpha}_i))\hat{\alpha}_{i_{MOM}} \qquad \text{for } i=1,2,\ldots,k+1. \qquad (3\text{-}40)$$

By substituting equation (3-40) in equation (2-19), we obtain the shrinkage reliability estimation of K components parallel system in (S-S) model using shrinkage weight function above as follows:

$$\hat{\mathbf{R}}_{k_{Sh1}} = \frac{\sum_{i=1}^{k} \hat{\alpha}_{i_{Sh1}}}{\sum_{i=1}^{k+1} \hat{\alpha}_{i_{Sh1}}}\,. \qquad (3\text{-}41)$$

3-2-3-1-3 Shrinkage Weight Function for Reliability System $R_{(s,k)}$ of The Multi-Components s-out of-k in The Stress-Strength Model:-

In this subsection, we consider the shrinkage weight factor as a function of sizes n and m respectively. Such that,

$\emptyset(\hat{\alpha}_1) = e^{-n}$, and $\emptyset(\hat{\alpha}_2) = e^{-m}$.

Therefore, the shrinkage estimator of α_i is defined in equation (3-38) upon above shrinkage weight function.

Substitute $\hat{\alpha}_{i_{Sh1}}$ i=1,2 in equation (2-26), we obtain the shrinkage reliability estimation of $R_{(s,k)}$ model as follows:

$$\hat{\boldsymbol{R}}_{(s,k)_{Sh1}} = \frac{\hat{\alpha}_{2_{Sh1}}}{\hat{\alpha}_{1_{Sh1}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} \left(k + \frac{\hat{\alpha}_{2_{Sh1}}}{\hat{\alpha}_{1_{Sh1}}} - j\right) \right]^{-1}. \qquad (3\text{-}42)$$

3-2-3-2 Constant Shrinkage Weight Factor (Sh2):

3-2-3-2-1 Estimate The Reliability System (R) of Stress–Strength Model Consist One Component by Constant Shrinkage Weight Factor (Sh2) :

This subsection concerns with the shrinkage estimator in equation (3-37) when the constant shrinkage weight factor will be assumed as:

$\varphi(\hat{\alpha}_i) = 0.3$;i=1,2. ; $0 \leq \varphi(\hat{\alpha}_i) \leq 1$

Therefore, the shrinkage estimator for α_i using constant shrinkage weight factor will be:

$$\hat{\alpha}_{i_{Sh2}} = \varphi(\hat{\alpha}_i)\hat{\alpha}_{i_{ub}} + (1 - \varphi(\hat{\alpha}_i))\hat{\alpha}_{i_{MOM}} \quad ;i=1,2. \qquad (3\text{-}43)$$

Substituting the equation (3-43) in the equation (2-16), we get the shrinkage estimation of (S-S) reliability $\hat{R}_{Sh2}$ by using constant shrinkage weight factor as below:

$$\hat{\boldsymbol{R}}_{\boldsymbol{Sh2}} = \frac{\hat{\alpha}_{1_{Sh2}}}{\hat{\alpha}_{1_{Sh2}} + \hat{\alpha}_{2_{Sh2}}} . \qquad (3\text{-}44)$$

3-2-3-2-2 Estimate The Reliability System (R_K) for K^{th} Parallel Components in The Stress-Strength Model by Constant Shrinkage Weight Factor (Sh2):-

In this subsection, constant shrinkage weight factor $\varphi(\hat{\alpha}_i) = 0.3$;i=1,2,…,k+1 will be suggested. Therefore, the shrinkage estimator using specific constant weight factor will be as follows:

$\hat{\alpha}_{i_{Sh2}} = \varphi(\hat{\alpha}_i)\hat{\alpha}_{i_{ub}} + (1 - \varphi(\hat{\alpha}_i))\hat{\alpha}_{i_{MOM}}$ for i=1,2,…,k+1. (3-45)

Substituting equation (3-45) in equation (2-19) to find reliability shrinkage estimation of K^{th} components parallel system in (S-S) model using the constant shrinkage weight factor as follows:

$$\widehat{\mathbf{R}}_{k_{Sh2}} = \frac{\sum_{i=1}^{k} \hat{\alpha}_{i_{Sh2}}}{\sum_{i=1}^{k+1} \hat{\alpha}_{i_{Sh2}}} . \tag{3-46}$$

3-2-3-2-3 Estimate The Reliability system $R_{(s,k)}$ of The Multi-Components s-out of-k in The Stress-Strength Model by Constant Shrinkage Weight Factor (Sh2):-

As same as in subsection 3-2-3-2, we suggest in this subsection a constant shrinkage weight factor $\varphi(\hat{\alpha}_i) = 0.3$;(i=1,2).

And the shrinkage estimator for α_i ($\hat{\alpha}_{i_{Sh2}}$) which is defined in equation (3-43).

Substitute $\hat{\alpha}_{i_{Sh2}}$;i=1,2 in equation (2-26) to find shrinkage estimation of $R_{(s,k)}$ using the above constant shrinkage weight factor as follows:

$$\hat{R}_{(s,k)_{Sh2}} = \frac{\hat{\alpha}_{2_{Sh2}}}{\hat{\alpha}_{1_{Sh2}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} \left(k + \frac{\hat{\alpha}_{2_{Sh2}}}{\hat{\alpha}_{1_{Sh2}}} - j\right) \right]^{-1} . \tag{3-47}$$

3-2-3-3 Modified Thompson Type Shrinkage Weight Function (Th):

This subsection, introduces a modifying the shrinkage weight factor consider by Thompson in 1968 as follow:

3-2-3-3-1 Estimate The Reliability System (R) of Stress–Strength Model Consist One Component by Modified Thompson Type Shrinkage Weight Function (Th):

Assume a modified thompson type shrinkage weight factor as follows:

$$\gamma(\hat{\alpha}_i) = \frac{(\hat{\alpha}_{i_{ub}} - \hat{\alpha}_{i_{MOM}})^2}{(\hat{\alpha}_{i_{ub}} - \hat{\alpha}_{MOM})^2 + var(\hat{\alpha}_{i_{ub}})}(0.01) \qquad \text{for } i = 1, 2.$$

Where $Var(\hat{\alpha}_{i_{ub}})$ is defined in equation (3-9).

And, the shrinkage estimator for α_i (i=1,2) using modified thompson type shrinkage weight factor will be:

$$\hat{\alpha}_{i_{Th}} = \gamma(\hat{\alpha}_i)\hat{\alpha}_{i_{ub}} + (1 - \gamma(\hat{\alpha}_i))\alpha_{i_{MOM}} \qquad \text{for } i=1,2. \qquad (3\text{-}48)$$

By substituting equation (3-48) in equation (2-16), we get the modified Thompson type shrinkage estimation of the (S-S) reliability as below:

. (3-49)

3-2-3-3-2 Estimate The Reliability System ((R_K) for K^{th} Parallel Components in The Stress-Strength Model by Modified Thompson Type Shrinkage Weight Function (Th):

Take a modified thompson type shrinkage weight function as below:

$$\gamma(\hat{\alpha}_i) = \frac{\left(\hat{\alpha}_{i_{ub}} - \hat{\alpha}_{i_{MOM}}\right)^2}{\left(\hat{\alpha}_{i_{ub}} - \hat{\alpha}_{i_{MOM}}\right)^2 + Var\left(\hat{\alpha}_{i_{ub}}\right)}(0.01) \quad ; \quad \text{for } i=1,2,\ldots,k+1.$$

Therefore, the shrinkage estimator using above modified shrinkage weight factor will be:

$$\hat{\alpha}_{i_{Th}} = \gamma(\hat{\alpha}_i)\hat{\alpha}_{i_{ub}} + (1 - \gamma(\hat{\alpha}_i))\hat{\alpha}_{i_{MOM}} \quad ; \quad \text{for } i=1,2,\ldots,k+1. \qquad (3\text{-}50)$$

Substituting equation (3-50) in equation (2-19), we find reliability estimation of K^{th} components parallel system in (S-S) model $\hat{R}_{k_{Th}}$ using modified Thompson type shrinkage as follows:

. (3-51)

3-2-3-3-3 Estimate The Reliability System $R_{(s,k)}$ of The Multi-Components s-out of-k in The Stress-Strength Model by Modified Thompson Type Shrinkage Weight Function (Th):

Same as in subsection 3-2-3-3-1, we suggest in this subsection the shrinkage estimator for$\hat{\alpha}_i$ (i=1,2) which is defined in equation (3-48).

Substitute equation (3-48) in equation (2-26), we conclude the reliability estimation of $R_{(s,k)}$ based on modified Thompson type shrinkage weight factor as follows:

(3-52)

3-2-4 Least Squares Estimator Method (LS):- ;[11], [31], and [35]

In this subsection, we discuss the estimator of Least Squares method. This method used for many mathematical and engineering application;[6]

The main idea for this method is to minimize the sum of square error between the values and the expected value.

3-2-4-1 Estimate The Reliability System (R) of Stress–Strength Model Consist One Component by Least Squares Estimator Method (LS):

To find the parameters α_i (i=1,2) via mentioned least squares method, let:

$S=\sum_{i=1}^{n}[F(x_i) - E(F(x_i))]^{2}$ i=1,2,3……n. (3-53)

Where, $F(x_i)$ refers to the CDF of two parameters EWD which is defined in equation (2-14) as follows:

$F(x_i)=(1-e^{-x_i^{\theta}})^{\alpha_1}$.

And, $E(F(x_i))=P_i$.

Such as; $P_i=\frac{i}{n+1}$, i=1,2,…,n.

$F(x_i)=E(F(x_i))$

$(1-e^{-x_i^{\theta}})^{\alpha_1}=\frac{i}{n+1}$. (3-54)

Now, taking the natural logarithm for both sides in equation (3-54) as below:

$$\alpha_1 \text{Ln}(1 - e^{-x_i{}^{\theta}}) = \text{Ln}P_i \ . \tag{3-55}$$

Now, by putting equation (3-55) in equation (3-53) we obtain:

$$S = \sum_{i=1}^{n} [\alpha_1 \text{Ln}(1 - e^{-x_i{}^{\theta}}) - \text{Ln}P_i\,]^2 \ . \tag{3-56}$$

And be finding the partial derivatives for the equation (3-56) with respect to α_1.

$$\frac{\partial S}{\partial \alpha_1} = 2 \sum_{i=1}^{n} [\alpha_1 \text{Ln}(1 - e^{-x_i{}^{\theta}}) - \text{Ln}P_i\,] \text{Ln}(1 - e^{-x_i{}^{\theta}}) \ . \tag{3-57}$$

Then, equate the equation (3-57) to zero to get:

$$\sum_{i=1}^{n} \left[\alpha_1 \text{Ln}\left(1 - e^{-x_i{}^{\theta}}\right) - \text{Ln}P_i\right] \text{Ln}\left(1 - e^{-x_i{}^{\theta}}\right) = 0 \ .$$

$$\sum_{i=1}^{n} \alpha_1 \text{Ln}\left(1 - e^{-x_i{}^{\theta}}\right) \text{Ln}\left(1 - e^{-x_i{}^{\theta}}\right) = \sum_{i=1}^{n} \text{Ln}P_i \, \text{Ln}\left(1 - e^{-x_i{}^{\theta}}\right).$$

$$\hat{\alpha}_{1_{LS}} = \frac{\sum_{i=1}^{n} \text{Ln}P_i \text{Ln}\left(1 - e^{-x_i{}^{\theta}}\right)}{\sum_{i=1}^{n} (\text{Ln}\left(1 - e^{-x_i{}^{\theta}}\right))^2} \ . \tag{3-58}$$

By same way, one can estimate the parameter α_2 for Y which represents stress random variable and follows EWD (α_2,θ) with size m as follow:

$$\hat{\alpha}_{2_{LS}} = \frac{\sum_{j=1}^{m} \text{Ln}P_j \text{Ln}\left(1 - e^{-y_j{}^{\theta}}\right)}{\sum_{j=1}^{m} (\text{Ln}\left(1 - e^{-y_j{}^{\theta}}\right))^2} \ . \tag{3-59}$$

$P_j = \frac{j}{m+1}$, j=1,2,…,m.

We put $\hat{\alpha}_{1_{LS}}$ and $\hat{\alpha}_{2_{LS}}$ in equation (2-16) to obtain the reliability estimation of R based on LS as follows:

$$\hat{\boldsymbol{R}}_{LS} = \frac{\hat{\alpha}_{1_{LS}}}{\hat{\alpha}_{1_{LS}}+\hat{\alpha}_{2_{LS}}} . \tag{3-60}$$

3-2-4-2 Estimate The Reliability System ((R$_K$) for Kth Parallel Components in The Stress-Strength Model by Least Squares Estimator Method (LS):

Suppose the strength random samples x_{ij} follow EWD with two parameters α_i and θ of size n_i ;i=1,2,…,k, and j=1,2,…,n_i, and the stress random samples y_t ;t=1,2,…,m follow EWD with two parameter α_{k+1} and θ of size m.

$$S = \sum_{j=1}^{n_i}[F(x_{ij}) - E(F(x_{ij}))]^{2} \qquad j=1,2,\dots,n_i \quad \text{and } i=1,2,\dots,k. \tag{3-61}$$

Where $F(x_{ij})$ refers to the CDF of x_{ij} for the two parameters EWD which is defined in equation (2-14) as follows:

$$F(x_{ij}) = (1 - e^{-x_{ij}{}^{\theta}})^{\alpha_i}.$$

And, $E(F(x_{ij})) = P_{ij}$,

such that; $P_{ij} = \frac{j}{n_i+1}$; j=1,2,…,n_i and i=1,2,…,k.

$$F(x_{ij}) = E(F(x_{ij}))$$

$$(1 - e^{-x_{ij}{}^{\theta}})^{\alpha_i} = \frac{j}{n_i+1} . \tag{3-62}$$

Now, taking the natural logarithm for equation (3-62) to both sides we get:

$$\alpha_i Ln\,(1-e^{-x_{ij}{}^{\theta}}) = LnP_{ij}\ . \tag{3-63}$$

Now by putting equation (3-63) in equation (3-61) we obtain:

$$S=\sum_{j=1}^{n_i}[\alpha_i \text{Ln}(1-e^{-x_{ij}{}^{\theta}})\ -\ \text{Ln}P_{ij}\ ]^{\,2}. \tag{3-64}$$

And, find the partial derivatives of the equation (3-64) with respect to α_i as below:

$$\frac{\partial S}{\partial \alpha_i}=2\sum_{j=1}^{n_i}\left[\alpha_i \text{Ln}\left(1-e^{-x_{ij}{}^{\theta}}\right)-\ \text{Ln}P_{ij}\ \right]\text{Ln}\left(1-e^{-x_{ij}{}^{\theta}}\right). \tag{3-65}$$

And equating the result to zero, we get:

$$\sum_{j=1}^{n_i}[\alpha_i Ln(1-e^{-x_{ij}{}^{\theta}})\ -\ LnP_{ij}\ ]\,Ln(1-e^{-x_{ij}{}^{\theta}})\ =0$$

$$\sum_{j=1}^{n_i}\alpha_i \text{Ln}\left(1-e^{-x_{ij}{}^{\theta}}\right)\text{Ln}\left(1-e^{-x_{ij}{}^{\theta}}\right)=\sum_{j=1}^{n_i}\ \text{Ln}P_{ij}\ \text{Ln}\left(1-e^{-x_{ij}{}^{\theta}}\right)$$

$$\hat{\alpha}_{i_{LS}}=\frac{\sum_{j=1}^{n_i}\ \text{Ln}P_{ij}\text{Ln}\left(1-e^{-x_{ij}{}^{\theta}}\right)}{\sum_{j=1}^{n_i}(\text{Ln}\left(1-e^{-x_{ij}{}^{\theta}}\right))^2}\quad ;i=1,2,\ldots,k. \tag{3-66}$$

By the same way, one can estimate the parameter α_{k+1} for Y which is the stress random variable follows EWD(α_{k+1},θ) with size m as follows:

$$\hat{\alpha}_{k+1_{LS}}=\frac{\sum_{t=1}^{m}\ \text{Ln}P_t\text{Ln}\left(1-e^{-y_t{}^{\theta}}\right)}{\sum_{t=1}^{m}(\text{Ln}\left(1-e^{-y_t{}^{\theta}}\right))^2}\ . \tag{3-67}$$

Where, $P_t=\frac{t}{m+1}$, t=1,2,…,m.

By putting $\hat{\alpha}_{1_{ls}}$ and $\hat{\alpha}_{2_{ls}}$ in equation (2-19), we obtain the reliability estimation of R_k based on LS as follows:

$$\widehat{\mathbf{R}}_{k_{LS}} = \frac{\sum_{i=1}^{k} \hat{\alpha}_{i_{LS}}}{\sum_{i=1}^{k+1} \hat{\alpha}_{i_{LS}}} . \qquad (3\text{-}68)$$

3-2-4-3 Estimate The Reliability System $R_{(s,k)}$ of The Multi-Components s-out of-k in The Stress-Strength Model by Least Squares Estimator Method(LS):

Suppose the strength $x_1, x_2, \ldots, x_n$ is a random sample of size n follow EWD and assume $x_{(i)}$;i=1, 2, .., n result the order sample.

Now, to find the parameters α_i by the least squares estimator, we can use the same procedure in subsection 3-2-4-1 as in equations (3-58) and (3-59).

Put $\hat{\alpha}_{1_{LS}}$ and $\hat{\alpha}_{2_{LS}}$ in equation (2-26), we obtain the reliability estimation of $R_{(s,k)}$ based on LS as follows:

$$\widehat{\boldsymbol{R}}_{(s,k)_{LS}} = \frac{\hat{\alpha}_{2_{LS}}}{\hat{\alpha}_{1_{LS}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} \left(\boldsymbol{k} + \frac{\hat{\alpha}_{2_{LS}}}{\hat{\alpha}_{1_{LS}}} - \boldsymbol{j}\right) \right]^{-1} \qquad \text{i, j, k are integer.} \qquad (3\text{-}69)$$

3-2-5 Ranked Set Sampling Method (RSS) ;[31], [63], and [64]

“Ranked set sampling was introduced and applied to the problem of estimating mean pasture yields by Mclntyre in 1952 this function was to improve the efficiency of the sample mean as an estimator of the population mean in situations in which the characteristic of interest was difficult or expensive to measure, but using ranked, it become cheaper”. ;[63], [22]

“The concept of rank set sampling is a recent development that enables one to provide more structure to the collected sample items”[64]

3-2-5-1 Estimate The Reliability System (R) of Stress–Strength Model Consist One Component by Ranked Set Sampling Method (RSS)

Let $x_{1}, x_{2}, \ldots, x_{n}$ be a random sample EWD, let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be order statistics increasing order. The PDF of $x_{(q)}$ is

$$f(x_{(q)}) = \frac{n!}{(q-1)!(n-q)!}\left[F\left(x_{(q)}\right)\right]^{q-1}\left[1-F\left(x_{(q)}\right)\right]^{n-q} f(x_{(q)}). \tag{3-70}$$

By substituting the PDF from equation (2-12) and the CDF from equation (2-14) in equation (3-92) will be get:

$$f(x_{(q)}) = \frac{n!}{(q-1)!\,(n-q)!}\left[\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]^{q-1}\left[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]^{n-q}$$

$$\alpha_1\theta x_{(q)}{}^{\theta-1}e^{-x_{(q)}{}^{\theta}}(1-e^{-x_{(q)}{}^{\theta}})^{\alpha_1-1}\,. \tag{3-71}$$

$$f\left(x_{(q)}\right) = \mathrm{Q}\alpha_1\theta x_{(q)}{}^{\theta-1}e^{-x_{(q)}{}^{\theta}}\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{q\alpha_1-1}\left[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]^{n-q}.$$

Such that; $\mathrm{Q} = \frac{n!}{(q-1)!(n-q)!}$.

The likelihood function of order sample $x_{(1)},x_{(2)},\ldots,x_{(n)}$ is:

$$\mathrm{L}\left(x_{(1)}, x_{(2)}, \ldots, x_{(n)};\ \alpha_1, \theta\right) = Q^n{\alpha_1}^n\theta^n\prod_{q=1}^{n}x_{(q)}{}^{\theta-1}\prod_{q=1}^{n}e^{-x_{(q)}{}^{\theta}}\prod_{q=1}^{n}(1-$$

$$e^{-x_{(k)}{}^{\theta}})^{q\alpha_1-1}\prod_{q=1}^{n}[1-\left(1-e^{-x_{(k)}{}^{\theta}}\right)^{\alpha_1}]^{n-q} \qquad \ldots(3\text{-}72)$$

Take (Ln) for both side of the equation (3-72) as below:

$$\mathrm{Ln}(l) = \mathrm{nLnQ} + \mathrm{nLn}\alpha_1 + \mathrm{nLn}\theta + (\theta-1)\sum_{q=1}^{n}Lnx_{(q)} - \sum_{q=1}^{n}x_{(q)}{}^{\theta} +$$

$$(q\alpha_1-1)\sum_{q=1}^{n}Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right) + (n-q)\sum_{q=1}^{n}Ln[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}].$$

Take the partial derivative with respect to the parameter α_1 get the following:

$$\frac{\partial \mathrm{Ln}(l)}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{q=1}^{n} q \, Ln\left(1 - e^{-x_{(q)}{}^{\theta}}\right) + \sum_{q=1}^{n} (n-q) \frac{-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1} Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right)}{\left[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]}$$

…(3-73)

Now, equate the equation (3-73) to zero, we get:

$$\frac{n}{\alpha_1} + \sum_{q=1}^{n} q \, Ln\left(1 - e^{-x_{(q)}{}^{\theta}}\right) - \sum_{q=1}^{n} (n-q) \frac{\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1} Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right)}{\left[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]} = 0,$$

…(3-74)

$$\frac{n}{\alpha_1} = \sum_{q=1}^{n} (n-q) \frac{\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1} Ln\left(1-e^{-x_{(q)}{}^{\theta}}\right)}{\left[1-\left(1-e^{-x_{(q)}{}^{\theta}}\right)^{\alpha_1}\right]} - \sum_{q=1}^{n} q \, Ln\left(1 - e^{-x_{(q)}{}^{\theta}}\right).$$

…(3-75)

Then, the Ranked Set Sampling estimation method of α_i (i=1, 2), say $\hat{\alpha}_{i_{RSS}}$ can be obtained by solving the above equation with respect to α_i (i=1, 2) when θ is known upon the median procedure which is explained in the appendix;[67].

By putting $\hat{\alpha}_{1_{RSS}}$ and $\hat{\alpha}_{2_{RSS}}$ in equation (2-16), we obtain the reliability estimation of R based on RSS method as follows:

$$\widehat{\boldsymbol{R}}_{\boldsymbol{RSS}} = \frac{\hat{\alpha}_{1_{RSS}}}{\hat{\alpha}_{1_{RSS}} + \hat{\alpha}_{2_{RSS}}} . \tag{3-76}$$

3-2-5-2 Estimate The Reliability System (R_K) for K^{th} Parallel Components in The Stress-Strength Model by by Ranked Set Sampling Method (RSS):

Firstly find the order statistic of strength $x_{i(1)}, x_{i(2)}, \ldots, x_{i(ni)}$ with size n_i and let $y_{(1)}, y_{(2)}, .., y_{(t)}$ be stress order statistic with size m.

The PDF of $x_{i_{(j)}}$ is:

$$f(x_{i_{(j)}}) = \frac{n_i!}{(j-1)!(n_i-j)!}\left[F\left(x_{i_{(j)}}\right)\right]^{j-1}\left[1-F\left(x_{i_{(j)}}\right)\right]^{n_i-j} f(x_{i_{(j)}}) \qquad (3\text{-}77)$$

for i= 1,2,…,k.

By substituting the PDF from equation (2-12) and the CDF from equation (2-14) in equation (3-77), we get:

$$f(x_{i_{(j)}}) = \frac{n_i!}{(j-1)!\,(n_i-j)!}\left[\left(1-e^{-x_{i_{(j)}}{}^{\theta}}\right)^{\alpha_i}\right]^{j-1}\left[1-\left(1-e^{-x_{i_{(j)}}{}^{\theta}}\right)^{\alpha_i}\right]^{n_i-j}$$

$$\alpha_i \theta x_{i_{(j)}}{}^{\theta-1} e^{-x_{i_{(j)}}{}^{\theta}}(1-e^{-x_{i_{(j)}}{}^{\theta}})^{\alpha_i-1}. \qquad (3\text{-}78)$$

$$f(x_{i_{(j)}}) = B\alpha_i\theta x_{i_{(j)}}{}^{\theta-1} e^{-x_{i_{(j)}}{}^{\theta}}(1-e^{-x_{i_{(j)}}{}^{\theta}})^{j\alpha_i-1}[1-\left(1-e^{-x_{i_{(j)}}{}^{\theta}}\right)^{\alpha_1}]^{n_i-j}$$

Such that; $B = \frac{n_i!}{(j-1)!(n_i-j)!}$.

The likelihood function of order sample $x_{i(1)}, x_{i(2)}, \ldots, x_{i(ni)}$;i=1,2,…,k is as below:

$$L(x_{i_{(j)}};\, \alpha_i, \theta) = B^{n_i}\alpha_i{}^{n_i}\theta^{n_i}\prod_{j=1}^{n_i} x_{i_{(j)}}{}^{\theta-1}\prod_{j=1}^{n_i} e^{-(x_{i_{(j)}})^{\theta}}\prod_{j=1}^{n_i}(1-$$

$$e^{-x_{i_{(j)}}{}^{\theta}})^{j\alpha_i-1}\prod_{j=1}^{n_i}[1-\left(1-e^{-(x_{i_{(j)}})^{\theta}}\right)^{\alpha_i}]^{n_i-j}. \qquad (3\text{-}79)$$

Take (Ln) for both sides of the equation (3-79), we get:

$$Ln(l) = n_i LnB + n_i Ln\alpha_i + n_i Ln\theta + (\theta - 1)\sum_{j=1}^{n_i} Ln\, x_{i_{(j)}} - \sum_{j=1}^{n_i}(x_{i_{(j)}})^{\theta} + (j\alpha_i - 1)\sum_{j=1}^{n_i} Ln\left(1 - e^{-x_{i_{(j)}}{}^{\theta}}\right) + (n_i - j)\sum_{j=1}^{n_i} Ln[1 - \left(1 - e^{-x_{i_{(j)}}{}^{\theta}}\right)^{\alpha_i}] .$$

…(3-80)

Now, the partial derivation for Ln(l) and equating the result to zero we get:

$$\frac{dLn(l)}{d\alpha_i} = \frac{n_i}{\alpha_i} + \sum_{j=1}^{n_i} j\, Ln\left(1 - e^{-(x_{i_{(j)}})^{\theta}}\right) + \sum_{j=1}^{n_i}(n_i - j)\frac{-\left(1-e^{-(x_{i_{(j)}})^{\theta}}\right)^{\alpha_i} Ln\left(1-e^{-(x_{i_{(j)}})^{\theta}}\right)}{[1-\left(1-e^{-(x_{i_{(j)}})^{\theta}}\right)^{\alpha_i}]} ,$$

…(3-81)

$$\frac{n_i}{\alpha_i} + \sum_{j=1}^{n_i} j\, Ln\left(1 - e^{-(x_{i_{(j)}})^{\theta}}\right) - \sum_{j=1}^{n_i}(n_i - j)\frac{\left(1-e^{-\left(x_{i_{(j)}}\right)^{\theta}}\right)^{\alpha_1} Ln\left(1-e^{-\left(x_{i_{(j)}}\right)^{\theta}}\right)}{\left[1-\left(1-e^{-\left(x_{i_{(j)}}\right)^{\theta}}\right)^{\alpha_1}\right]} = 0.$$

…(3-82)

Then, the Ranked Set Sampling estimation method of α_i (i=1, 2, …, k+1), say $\hat{\alpha}_{i_{RSS}}$ can be obtained by solving the above equation with respect to α_i (i=1, 2, …, k+1) when θ is known, upon the median procedure which is explained in the appendix;[67].

By putting $\hat{\alpha}_{i_{RSS}}$;(i=1,2,…,k+1) in equation (2-19), we obtain the reliability estimation of R_k based on RSS as follows:

$$\hat{\mathbf{R}}_{k_{RSS}} = \frac{\sum_{i=1}^{k}\hat{\alpha}_{i_{RSS}}}{\sum_{i=1}^{k+1}\hat{\alpha}_{i_{RSS}}} . \qquad (3\text{-}83)$$

3-2-5-3 Estimate The Reliability System $R_{(s,k)}$ of The Multi-Components s-out of-k in The Stress-Strength Model by Ranked Set Sampling Method (RSS):

In this subsection, we have to find the parameters α_i (i=1,2) using the RSS method based on the same procedure and steps in subsection 3-2-5-1 as we obtained in equation (3-75).

Substitution the estimation parameter $\hat{\alpha}_{i_{RSS}}$ (i=1,2) in equation (2-26), will obtain the reliability estimation of $R_{(s,k)}$ based on RSS as follows:

$$\hat{\boldsymbol{R}}_{(s,k)_{RSS}} = \frac{\hat{\alpha}_{2_{RSS}}}{\hat{\alpha}_{1_{RSS}}} \sum_{i=s}^{k} \frac{k!}{(k-i)!} \left[\prod_{j=0}^{i} \left(k + \frac{\hat{\alpha}_{2_{RSS}}}{\hat{\alpha}_{1_{RSS}}} - j\right) \right]^{-1} . \qquad (3\text{-}84)$$

Chapter Four

Experimental Aspect

CHAPTER FOUR

EXPERIMENTAL ASPECT

4-1 Introduction:

The concept of Monte Carlo simulation involves generating random numbers and describing the stages of experiments simulation in terms of samples sizes generated, as well as different values of default parameters.

In this chapter, the Monte Carlo simulation technique is used to make a comparison between some estimation methods for the three proposed cases system reliability in stress-strength models.

The process of the experiment simulation was repeated 1000 times. The Mean Squared Errors (MSE) were used to compare between the estimation methods as a statistical indicator. The results were obtained using programs written in Matlab version 2010b.

4-2 Simulation;[58], [36]

Many simulation methods were developed, especially after the rapid development of using the electronic calculator, which provided to the researchers a lot of effort, money and time and satisfied for them numerical solutions.

The simulation method provides an experimental basis which is an evidence with the theoretical basis for choosing the appropriate method or the appropriate algorithms to analyze and study the phenomena data by matching their characteristics with the types to which the simulation was applied.

Simulations are defined as representations or imitations of actual reality using specific models. In real fact, there are complicated processes of understanding and analysis, therefore, it is better to describe these processes with specific models similar to the real ones. It is natural that the degree of similarity between any experiments simulation and real reality depends on how conformity or similar the simulation model with the real system.

There are different methods of simulation, such as the Analog Method, the Mixed Method, and the Monte Carlo Method. The Monte Carlo Method is one of the most widely used methods, and it is used to generate observations for most of the probability distribution function which is used in many phenomena since this method depends on sampling methods taken from a theoretical population experiments on that simulates real population.

The random numbers are formulated and the flexibility of simulation gives the ability to experiment and test by repeating the process many times by interpreting the special input for the particular process of estimation at all times, as well as the importance of the simulation in random, since the series of random numbers used in the first experiment is independent of the series of random numbers used in the second experiment and so on.

We use Monte Carlo simulation to observe the behavior of different methods of estimation for R, $R_{k,}$ and $R_{(s,k)}$ for different sample size and use different values for shape parameters.

And as mentioned, the experiment simulation based on repeated 1000 times to obtain the independent samples of different sizes.

4-3 The Simulation Manner:

In this section, numerical results were studied to compare the performance of the different estimators of reliability which is obtained in chapter three, using different sample size =(10, 30, 50 and 100), based on 1000 replication via MSE criteria. It concerns with simulation manner for estimating proposed three cases of reliability system.

A. The simulation manner is written using the Matlab program to estimate the system reliability R which contains one component depends on the following steps:

Step1: Generate the random sample which follows the continuous uniform distribution over (0,1) as $u_1, u_2, \dots, u_n$.

Step2: Generate the random sample which follows the continuous uniform distribution over (0, 1) as $v_1, v_2, \dots, v_m$.

Step3: Transform the uniform random samples in step1 to random samples follows EWD, applying the theorem that using the inverse cumulative probability distribution function (c. d. f.) as below:

$$F(x) = (1 - e^{-x_i^{\theta}})^{\alpha_1}$$

$$U_i = (1 - e^{-x_i^{\theta}})^{\alpha_1}$$

$$x_i = [-\ln(1 - U_i^{\frac{1}{\alpha_1}})]^{\frac{1}{\theta}}.$$

And, calculate V_j from step2 by the same method to obtain the random variable y_j:

$y_j = [-\ln(1 - V_j^{\frac{1}{\alpha_2}})]^{\frac{1}{\theta}}$.

Step4: Recall the R as in equation (2-16).

Step5: Compute the Maximum Likelihood Estimator of R using equation (3-12).

Step6: Apply the Moment method on R using equation (3-29).

Step7: Calculate the three Shrinkage estimators of R using equations (3-39), (3-44) and (3-49).

Step8: Compute the Least Squares estimator of R using equation (3-60).

Step9: Compute the Ranked Set Sampling method of R using equation (3-76).

Step10: Based on (L=1000) replication, the MSE for all proposed estimation methods of R is utilized as follows:

$$\mathbf{MSE} = \frac{1}{L}\sum_{i=1}^{L}(\hat{\mathbf{R}}_i - \mathbf{R})^2.$$

Where $\hat{\mathbf{R}}$ refers the proposed estimators of real value of reliability R.

By using a random samples for x_i and y_j of size (n,m)= (10,10), (30,10), (50,10), (100,10), (10,30), (30,30), (50,30), (100,30), (10,50), (30,50), , (50,50), (100,50), (10,100), (30,100), (50,100), and (100.100).

B. The simulation manner is written using the Matlab program to estimate the system reliability R_k which contains k^{th} parallel component depend on the following steps:

step1: We generate the random sample which follows the continuous uniform distribution defined on the interval (0,1) as $u_{i1}, u_{i2}, \dots, u_{in_i}$;i=1,2,…,k.

step2: We generate the random sample which follows the continuous uniform distribution defined on the interval (0, 1) as $w_1, w_2, \ldots, w_m$.

step3: Transform the above uniform random samples to random samples follows EWD using the cumulative distribution function (c. d. f.) as follows:

$$F(x_{ij}) = (1 - e^{-x_{ij}^{\theta}})^{\alpha_i}$$

$$Uij = (1 - e^{-x_{ij}^{\theta}})^{\alpha_i}$$

$$x_{ij} = [-\ln(1 - U_{ij}^{\frac{1}{\alpha_i}})]^{\frac{1}{\theta}} \qquad ;i=1,2\ldots,k \ .$$

And, by the same method, we get:

$$y_t = [-\ln(1 - W_t^{\frac{1}{\alpha_{k+1}}})]^{\frac{1}{\theta}}.$$

Step4: Recall the R as in equation (2-19).

Step5: We compute the Maximum Likelihood estimator of R_k using equation (3-22).

Step6: We compute the Moment method of R_k using equation (3-34).

Step7: We compute the three shrinkage estimators of R_k using equations (3-41), (46) and (51).

Step8: Compute the Least Squares estimator of R_k using equation (3-68).

Step9: Compute the Ranked Set Sampling method of R_k using equation (3-83).

Step10: Based on (L=1000) Replication, we calculate the MSE as follows:

$$\mathbf{MSE} = \frac{1}{L}\sum_{i=1}^{L}(\hat{R}_{ki} - R_k)^2.$$

Where $\widehat{\mathbf{R}}_{\mathbf{ki}}$ refer the proposed estimators of real value of Reliability R_k.

By using a random samples for xij and yt of size n_i and m respectively (10,10,10,10), (30,50,100,10), (50,100,30,10), (100,30,50,10), (30,50,100,30), (50,100,30.30), (100,30,50,30), (30,50,100,50), (50,100,30,50), (100,30,50,50), (30,30,30,30), (50,50,50,50), and (100,100,100,100).

C. The simulation manner is written using the Matlab program to estimate the system reliability $R_{(s,k)}$ for this purpose, Monte Carlo simulation was used as the following steps:

Step1: We generate the random sample which follows the uniform distribution over (0,1) as same as in subsection (A).

Step2: We generate the random sample which follows the uniform distribution over (0,1) as same as in subsection (A).

Step3: By the same way, transform the above uniform random samples to random samples follows EWD using the cumulative distribution function (c. d. f.) for x_i and y_t .

Step4: Recall the $R_{(s,k)}$ as in equation (2-29).

Step5: Compute the Maximum Likelihood estimator of $R_{(s,k)}$ using equation (3-23).

Step6: Compute the Moment method of $R_{(s,k)}$ using equation (3-35).

Step7: Compute the three Shrinkage estimators of $R_{(s,k)}$ using equations (3-42), (3-47) and (3-52).

Step8: Compute the Least Squares estimator of $R_{(s,k)}$ using equation (3-69).

Step9: Compute the Ranked Set Sampling method of $R_{(s,k)}$using equation (3-84).

Step10: Based on (L=1000) Replication, we calculate the MSE for all proposed estimation methods of $R_{(s,k)}$ as follows:

$$\mathbf{MSE} = \frac{1}{L}\sum_{i=1}^{L}(\hat{\mathbf{R}}_{(s,k)_i} - \mathbf{R}_{(s,k)})^2.$$

Where $\hat{\mathbf{R}}_{(s,k)}$ refers to the proposed estimators of real value of Reliability $R_{(s,k)}$.

By using a random samples for xi and yj of size (n,m)= (10,10), (30,10), (50,10), (100,10), (10,30), (30,30), (50,30), (100,30), (10,50), (30,50), , (50,50), (100,50), (10,100), (30,100), (50,100), and (100.100).

4-4 Simulation Results:

After applying the simulation steps in section 4-3 for each three model R, R_k, and $R_{(s,k)}$, we obtained:

A. The model R:

This model consists of estimation of R for EWD (α,θ) when parameters

(α_1, α_2) = (2,2), (2, 4), (5, 3), (2.3, 2.3), (2.5, 5.2), and (5.1, 1.5), with common shape parameter θ=3 for each cases.

Table (4-1): Shown estimation value of R, when α_1=2, α_2=2, and θ =3

(n,m)	R	$\widehat{R}_{MLE}$	$\widehat{R}_{Sh1}$	$\widehat{R}_{Sh2}$	$\widehat{R}_{Th}$	$\widehat{R}_{LS}$	$\widehat{R}_{RSS}$
(10,10)	**0.50000**	**0.504977**	**0.492579**	**0.500000**	**0.499988**	**0.513860**	**0.506525**
(10,30)	**0.50000**	**0.512359**	**0.496640**	**0.500004**	**0.500081**	**0.249262**	**0.724399**
(10,50)	**0.50000**	**0.507472**	**0.495096**	**0.500004**	**0.500094**	**0.157255**	**0.805732**
(10,100)	**0.50000**	**0.513591**	**0.498067**	**0.500006**	**0.500061**	**0.085890**	**0.892115**
(30,10)	**0.50000**	**0.487309**	**0.483697**	**0.499995**	**0.499925**	**0.746869**	**0.274708**
(30,30)	**0.50000**	**0.501731**	**0.495814**	**0.500000**	**0.500007**	**0.510170**	**0.499090**
(30,50)	**0.50000**	**0.499757**	**0.495795**	**0.500001**	**0.499993**	**0.380737**	**0.614035**
(30,100)	**0.50000**	**0.503232**	**0.499276**	**0.500002**	**0.500022**	**0.216342**	**0.760393**
(50,10)	**0.50000**	**0.493075**	**0.488151**	**0.499995**	**0.499889**	**0.835964**	**0.199089**
(50,30)	**0.50000**	**0.498143**	**0.495650**	**0.499998**	**0.500015**	**0.633772**	**0.381098**
(50,50)	**0.50000**	**0.498827**	**0.496894**	**0.499999**	**0.499975**	**0.495030**	**0.498247**
(50,100)	**0.50000**	**0.505054**	**0.500262**	**0.500001**	**0.500008**	**0.341256**	**0.666527**
(100,10)	**0.50000**	**0.483275**	**0.480895**	**0.499993**	**0.499893**	**0.911308**	**0.106848**
(100,30)	**0.50000**	**0.497566**	**0.497137**	**0.499997**	**0.500004**	**0.765676**	**0.243579**
(100,50)	**0.50000**	**0.497829**	**0.497246**	**0.499998**	**0.499965**	**0.667934**	**0.337112**
(100,100)	**0.50000**	**0.500945**	**0.499658**	**0.500000**	**0.499975**	**0.492771**	**0.501881**

Table (4-2): shown MSE values when α_1=2, α_2=2, and θ =3 and R= 0.500000

(n,m)	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$	Best
(10,10)	**0.012302**	**0.006431**	**0.2 E-9**	**0.7E-6**	**0.061263**	**0.044716**	$\hat{R}_{Sh2}$
(10,30)	**0.008648**	**0.002128**	**0.1 E-9**	**0.8E-6**	**0.100461**	**0.069589**	$\hat{R}_{Sh2}$
(10,50)	**0.007613**	**0.001230**	**0.1 E-9**	**0.7E-6**	**0.142304**	**0.103825**	$\hat{R}_{Sh2}$
(10,100)	**0.006871**	**0.000589**	**0.1 E-9**	**0.7E-6**	**0.184731**	**0.157139**	$\hat{R}_{Sh2}$
(30,10)	**0.008720**	**0.006664**	**0.1 E-9**	**0.8E-6**	**0.102618**	**0.071310**	$\hat{R}_{Sh2}$
(30,30)	**0.004167**	**0.002067**	**0.8E-10**	**0.8E-6**	**0.049958**	**0.017960**	$\hat{R}_{Sh2}$
(30,50)	**0.003373**	**0.001211**	**0.7E-10**	**0.8E-6**	**0.056870**	**0.025622**	$\hat{R}_{Sh2}$
(30,100)	**0.002756**	**0.000640**	**0.6E-10**	**0.8E-6**	**0.106516**	**0.073194**	$\hat{R}_{Sh2}$
(50,10)	**0.007930**	**0.006568**	**0.1E-9**	**0.8E-6**	**0.140643**	**0.101459**	$\hat{R}_{Sh2}$
(50,30)	**0.003408**	**0.002140**	**0.7E-10**	**0.8E-6**	**0.058558**	**0.026927**	$\hat{R}_{Sh2}$
(50,50)	**0.002543**	**0.001267**	**0.6E-10**	**0.8E-6**	**0.045029**	**0.010829**	$\hat{R}_{Sh2}$
(50,100)	**0.001849**	**0.000633**	**0.5E-10**	**0.8E-6**	**0.062954**	**0.033772**	$\hat{R}_{Sh2}$
(100,10)	**0.007031**	**0.006433**	**0.1E-9**	**0.8E-6**	**0.184444**	**0.157806**	$\hat{R}_{Sh2}$
(100,30)	**0.002768**	**0.002089**	**0.7E-10**	**0.8E-6**	**0.101308**	**0.071825**	$\hat{R}_{Sh2}$
(100,50)	**0.001945**	**0.001192**	**0.5E-10**	**0.8E-6**	**0.065026**	**0.033086**	$\hat{R}_{Sh2}$
(100,100)	**0.001237**	**0.000609**	**0.4E-10**	**0.9E-6**	**0.036399**	**0.005327**	$\hat{R}_{Sh2}$

Table (4-3): Shown estimation value of R, when α_1=2, α_2=4, and θ =3

(n,m)	R	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$
(10,10)	**0.333333**	**0.3369308**	**0.325408**	**0.333330**	**0.333336**	**0.357300**	**0.386252**
(10,30)	**0.333333**	**0.342812**	**0.328934**	**0.333333**	**0.333446**	**0.153133**	**0.616869**
(10,50)	**0.333333**	**0.346275**	**0.332061**	**0.333334**	**0.333367**	**0.093859**	**0.723831**
(10,100)	**0.333333**	**0.346563**	**0.331608**	**0.333334**	**0.333422**	**0.046686**	**0.832055**
(30,10)	**0.333333**	**0.330952**	**0.326924**	**0.333326**	**0.333262**	**0.638930**	**0.188643**
(30,30)	**0.333333**	**0.337453**	**0.331115**	**0.333330**	**0.333362**	**0.368993**	**0.381909**
(30,50)	**0.333333**	**0.335446**	**0.331172**	**0.333330**	**0.3333655**	**0.253132**	**0.490475**
(30,100)	**0.333333**	**0.339675**	**0.332952**	**0.333331**	**0.333360**	**0.333360**	**0.658445**
(50,10)	**0.333333**	**0.327242**	**0.324791**	**0.333325**	**0.333264**	**0.752072**	**0.122067**
(50,30)	**0.333333**	**0.334048**	**0.329821**	**0.333329**	**0.333303**	**0.503847**	**0.272456**
(50,50)	**0.333333**	**0.333963**	**0.332298**	**0.333329**	**0.333366**	**0.361054**	**0.373814**
(50,100)	**0.333333**	**0.332757**	**0.332395**	**0.333329**	**0.333366**	**0.218912**	**0.535929**
(100,10)	**0.333333**	**0.328809**	**0.327350**	**0.333324**	**0.333279**	**0.861869**	**0.066589**
(100,30)	**0.333333**	**0.331887**	**0.330566**	**0.333327**	**0.333361**	**0.654581**	**0.157285**
(100,50)	**0.333333**	**0.331502**	**0.330320**	**0.333328**	**0.333319**	**0.532097**	**0.232091**
(100,100)	**0.333333**	**0.335739**	**0.333710**	**0.333329**	**0.333315**	**0.358024**	**0.374920**

Table (4-4): Shown MSE values when α_1=2, α_2=4, and θ =3 and R=0.333333

(n,m)	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$	Best
(10,10)	**0.010148**	**0.004877**	**0.1E-9**	**0.5E-6**	**0.054059**	**0.058840**	$\hat{R}_{Sh2}$
(10,30)	**0.006782**	**0.001704**	**0.1E-9**	**0.6E-6**	**0.057368**	**0.103391**	$\hat{R}_{Sh2}$
(10,50)	**0.006557**	**0.000931**	**0.9E-10**	**0.5E-6**	**0.073211**	**0.167399**	$\hat{R}_{Sh2}$
(10,100)	**0.006033**	**0.000533**	**0.9E-10**	**0.6E-6**	**0.089957**	**0.255515**	$\hat{R}_{Sh2}$
(30,10)	**0.006377**	**0.004726**	**0.1E-9**	**0.6E-6**	**0.143137**	**0.029677**	$\hat{R}_{Sh2}$
(30,30)	**0.003239**	**0.001704**	**0.6E-10**	**0.6E-6**	**0.045379**	**0.015258**	$\hat{R}_{Sh2}$
(30,50)	**0.002800**	**0.001001**	**0.5E-10**	**0.6E-6**	**0.039485**	**0.036743**	$\hat{R}_{Sh2}$
(30,100)	**0.002033**	**0.000487**	**0.4E-10**	**0.6E-6**	**0.057808**	**0.113648**	$\hat{R}_{Sh2}$
(50,10)	**0.005579**	**0.004893**	**0.1E-9**	**0.5E-6**	**0.212391**	**0.049034**	$\hat{R}_{Sh2}$
(50,30)	**0.002731**	**0.001626**	**0.6E-10**	**0.6E-6**	**0.073472**	**0.011450**	$\hat{R}_{Sh2}$
(50,50)	**0.001873**	**0.001017**	**0.5E-10**	**0.7E-6**	**0.041455**	**0.009221**	$\hat{R}_{Sh2}$
(50,100)	**0.001486**	**0.000527**	**0.4E-10**	**0.7E-6**	**0.041391**	**0.048244**	$\hat{R}_{Sh2}$
(100,10)	**0.005242**	**0.004781**	**0.1E-10**	**0.5E-6**	**0.299677**	**0.072280**	$\hat{R}_{Sh2}$
(100,30)	**0.002130**	**0.001644**	**0.6E-10**	**0.6E-6**	**0.145371**	**0.033551**	$\hat{R}_{Sh2}$
(100,50)	**0.001473**	**0.000942**	**0.5E-10**	**0.7E-6**	**0.081891**	**0.013556**	$\hat{R}_{Sh2}$
(100,100)	**0.000949**	**0.000474**	**0.4E-10**	**0.6E-6**	**0.036944**	**0.005431**	$\hat{R}_{Sh2}$

Table (4-5): Shown estimation value of R, when α_1=5, α_2=3, and θ =3

(n,m)	R	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$
(10,10)	**0.625000**	**0.616384**	**0.606075**	**0.625001**	**0.625005**	**0.591166**	**0.587248**
(10,30)	**0.625000**	**0.627356**	**0.620916**	**0.625004**	**0.625069**	**0.321262**	**0.796290**
(10,50)	**0.625000**	**0.628725**	**0.620036**	**0.625005**	**0.625098**	**0.219678**	**0.864874**
(10,100)	**0.625000**	**0.632184**	**0.624488**	**0.625006**	**0.625073**	**0.127857**	**0.928666**
(30,10)	**0.625000**	**0.612533**	**0.609240**	**0.624998**	**0.624904**	**0.813103**	**0.362485**
(30,30)	**0.625000**	**0.622077**	**0.619245**	**0.625001**	**0.625031**	**0.611309**	**0.594260**
(30,50)	**0.625000**	**0.620802**	**0.620535**	**0.625001**	**0.625006**	**0.468170**	**0.704002**
(30,100)	**0.625000**	**0.628351**	**0.624372**	**0.625003**	**0.625057**	**0.311408**	**0.828833**
(50,10)	**0.625000**	**0.613511**	**0.611111**	**0.624997**	**0.624956**	**0.885567**	**0.259642**
(50,30)	**0.625000**	**0.622551**	**0.621086**	**0.625000**	**0.625016**	**0.728090**	**0.480440**
(50,50)	**0.625000**	**0.623836**	**0.622109**	**0.625001**	**0.624998**	**0.594427**	**0.601250**
(50,100)	**0.625000**	**0.626848**	**0.624946**	**0.625002**	**0.625002**	**0.425603**	**0.751918**
(100,10)	**0.625000**	**0.609041**	**0.607747**	**0.624997**	**0.6249629**	**0.940218**	**0.150321**
(100,30)	**0.625000**	**0.623730**	**0.621965**	**0.625000**	**0.624990**	**0.841843**	**0.324782**
(100,50)	**0.625000**	**0.622757**	**0.621321**	**0.625001**	**0.625007**	**0.755405**	**0.435292**
(100,100)	**0.625000**	**0.622571**	**0.622437**	**0.625002**	**0.625006**	**0.606221**	**0.598149**

Table (4-6): Shown MSE values when α_1=5, α_2=3, and θ =3 and R= 0.625000

(n,m)	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$	Best
(10,10)	**0.010786**	**0.006006**	**0.1E-9**	**0.6E-6**	**0.057024**	**0.032338**	$\hat{R}_{Sh2}$
(10,30)	**0.007657**	**0.001908**	**0.1E-9**	**0.6E-6**	**0.1393876**	**0.039126**	$\hat{R}_{Sh2}$
(10,50)	**0.006376**	**0.001152**	**0.1E-9**	**0.6E-6**	**0.199150**	**0.061884**	$\hat{R}_{Sh2}$
(10,100)	**0.005950**	**0.000533**	**0.1E-9**	**0.6E-6**	**0.270419**	**0.093374**	$\hat{R}_{Sh2}$
(30,10)	**0.008073**	**0.005943**	**0.1E-9**	**0.7E-6**	**0.069357**	**0.089356**	$\hat{R}_{Sh2}$
(30,30)	**0.003532**	**0.001928**	**0.4E-10**	**0.7E-6**	**0.046314**	**0.012024**	$\hat{R}_{Sh2}$
(30,50)	**0.003182**	**0.001179**	**0.4E-10**	**0.7E-6**	**0.069744**	**0.013765**	$\hat{R}_{Sh2}$
(30,100)	**0.002376**	**0.000612**	**0.4E-10**	**0.7E-6**	**0.134251**	**0.043955**	$\hat{R}_{Sh2}$
(50,10)	**0.007015**	**0.006106**	**0.1E-9**	**0.6E-6**	**0.084751**	**0.145253**	$\hat{R}_{Sh2}$
(50,30)	**0.002816**	**0.001822**	**0.3E-10**	**0.6E-6**	**0.044841**	**0.030620**	$\hat{R}_{Sh2}$
(50,50)	**0.002076**	**0.001080**	**0.2E-10**	**0.7E-6**	**0.045579**	**0.007357**	$\hat{R}_{Sh2}$
(50,100)	**0.001670**	**0.000575**	**0.2E-10**	**0.7E-6**	**0.079788**	**0.019512**	$\hat{R}_{Sh2}$
(100,10)	**0.006637**	**0.006141**	**0.1E-9**	**0.6E-6**	**0.109270**	**0.230131**	$\hat{R}_{Sh2}$
(100,30)	**0.002382**	**0.001928**	**0.3E-10**	**0.7E-6**	**0.068154**	**0.096432**	$\hat{R}_{Sh2}$
(100,50)	**0.001728**	**0.001168**	**0.2E-10**	**0.7E-6**	**0.047962**	**0.042001**	$\hat{R}_{Sh2}$
(100,100)	**0.001109**	**0.000534**	**0.1E-10**	**0.7E-6**	**0.034779**	**0.004312**	$\hat{R}_{Sh2}$

Table (4-7): Shown estimation value of R, when α_1=2.3, α_2=2.3, and θ =3

(n,m)	R	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$
(10,10)	**0.50000**	**0.500159**	**0.489304**	**0.500000**	**0.500005**	**0.498990**	**0.500519**
(10,30)	**0.50000**	**0.507491**	**0.494305**	**0.500003**	**0.500104**	**0.243499**	**0.718200**
(10,50)	**0.50000**	**0.511232**	**0.496680**	**0.500003**	**0.500059**	**0.147513**	**0.810613**
(10,100)	**0.50000**	**0.511206**	**0.498694**	**0.500003**	**0.500070**	**0.083116**	**0.892288**
(30,10)	**0.50000**	**0.487063**	**0.483719**	**0.499996**	**0.499918**	**0.761057**	**0.268267**
(30,30)	**0.50000**	**0.500005**	**0.495442**	**0.499999**	**0.499945**	**0.497206**	**0.504331**
(30,50)	**0.50000**	**0.498551**	**0.496666**	**0.499999**	**0.500017**	**0.360171**	**0.614426**
(30,100)	**0.50000**	**0.499685**	**0.498779**	**0.499999**	**0.500035**	**0.223686**	**0.755960**
(50,10)	**0.50000**	**0.488326**	**0.486873**	**0.499996**	**0.499917**	**0.844047**	**0.188704**
(50,30)	**0.50000**	**0.498437**	**0.495508**	**0.499999**	**0.499979**	**0.633696**	**0.380758**
(50,50)	**0.50000**	**0.498428**	**0.497856**	**0.499999**	**0.500062**	**0.500062**	**0.497920**
(50,100)	**0.50000**	**0.499085**	**0.498089**	**0.500000**	**0.500050**	**0.326961**	**0.659023**
(100,10)	**0.50000**	**0.488472**	**0.488537**	**0.499996**	**0.499906**	**0.919352**	**0.106278**
(100,30)	**0.50000**	**0.497377**	**0.494731**	**0.500000**	**0.500022**	**0.774109**	**0.241266**
(100,50)	**0.50000**	**0.498725**	**0.497394**	**0.499999**	**0.499986**	**0.671408**	**0.338404**
(100,100)	**0.50000**	**0.501799**	**0.498560**	**0.500000**	**0.500023**	**0.502432**	**0.503820**

Table (4-8): Shown MSE values when α_1=2.3, α_2=2.3, and θ =3 and R=0.50000

(n,m)	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$	Best
(10,10)	**0.011455**	**0.005861**	**0.1E-9**	**0.6E-6**	**0.056851**	**0.042707**	$\hat{R}_{Sh2}$
(10,30)	**0.008490**	**0.002260**	**0.1E-9**	**0.7E-6**	**0.1053895**	**0.065755**	$\hat{R}_{Sh2}$
(10,50)	**0.007944**	**0.001289**	**0.1E-9**	**0.7E-6**	**0.145362**	**0.105781**	$\hat{R}_{Sh2}$
(10,100)	**0.006871**	**0.000640**	**0.1E-9**	**0.8E-6**	**0.187367**	**0.157170**	$\hat{R}_{Sh2}$
(30,10)	**0.008492**	**0.006577**	**0.1E-9**	**0.7E-6**	**0.105749**	**0.070279**	$\hat{R}_{Sh2}$
(30,30)	**0.004133**	**0.002006**	**0.7E-10**	**0.8E-6**	**0.047096**	**0.015965**	$\hat{R}_{Sh2}$
(30,50)	**0.003127**	**0.001267**	**0.5E-10**	**0.8E-6**	**0.061595**	**0.024469**	$\hat{R}_{Sh2}$
(30,100)	**0.002589**	**0.000658**	**0.5E-10**	**0.8E-6**	**0.104028**	**0.071028**	$\hat{R}_{Sh2}$
(50,10)	**0.007317**	**0.006210**	**0.1E-9**	**0.7E-6**	**0.141773**	**0.105475**	$\hat{R}_{Sh2}$
(50,30)	**0.003147**	**0.001981**	**0.6E-10**	**0.8E-6**	**0.057325**	**0.024790**	$\hat{R}_{Sh2}$
(50,50)	**0.002552**	**0.001357**	**0.5E-10**	**0.8E-6**	**0.044359**	**0.009815**	$\hat{R}_{Sh2}$
(50,100)	**0.001899**	**0.000682**	**0.5E-10**	**0.8E-6**	**0.065083**	**0.031087**	$\hat{R}_{Sh2}$
(100,10)	**0.006489**	**0.006248**	**0.1E-9**	**0.8E-6**	**0.187354**	**0.157849**	$\hat{R}_{Sh2}$
(100,30)	**0.002784**	**0.002120**	**0.6E-10**	**0.8E-6**	**0.104744**	**0.072282**	$\hat{R}_{Sh2}$
(100,50)	**0.001890**	**0.001232**	**0.5E-10**	**0.8E-6**	**0.067223**	**0.031888**	$\hat{R}_{Sh2}$
(100,100)	**0.001318**	**0.000661**	**0.5E-10**	**0.9E-6**	**0.041233**	**0.005013**	$\hat{R}_{Sh2}$

Table (4-9): Shown estimation value of R, when α_1=2.5, α_2=5.2, and θ =3

(n,m)	R	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$
(10,10)	**0.324675**	**0.335265**	**0.318016**	**0.324673**	**0.324652**	**0.365276**	**0.381690**
(10,30)	**0.324675**	**0.337064**	**0.324764**	**0.324676**	**0.324762**	**0.154082**	**0.602644**
(10,50)	**0.324675**	**0.339457**	**0.324921**	**0.324676**	**0.324745**	**0.091961**	**0.711788**
(10,100)	**0.324675**	**0.339754**	**0.324214**	**0.324677**	**0.324758**	**0.046662**	**0.824506**
(30,10)	**0.324675**	**0.321325**	**0.317341**	**0.324669**	**0.324642**	**0.630029**	**0.177061**
(30,30)	**0.324675**	**0.328150**	**0.322718**	**0.324673**	**0.324690**	**0.351585**	**0.365937**
(30,50)	**0.324675**	**0.327222**	**0.322514**	**0.324673**	**0.324709**	**0.233742**	**0.474942**
(30,100)	**0.324675**	**0.329381**	**0.325260**	**0.324674**	**0.324697**	**0.139966**	**0.642862**
(50,10)	**0.324675**	**0.320830**	**0.316931**	**0.324669**	**0.324632**	**0.742162**	**0.116220**
(50,30)	**0.324675**	**0.323519**	**0.322063**	**0.324672**	**0.324662**	**0.489146**	**0.254191**
(50,50)	**0.324675**	**0.325705**	**0.324420**	**0.324673**	**0.324663**	**0.351922**	**0.359057**
(50,100)	**0.324675**	**0.327710**	**0.324507**	**0.324673**	**0.324720**	**0.216059**	**0.524649**
(100,10)	**0.324675**	**0.314289**	**0.312751**	**0.324669**	**0.324611**	**0.853105**	**0.060479**
(100,30)	**0.324675**	**0.321751**	**0.320650**	**0.324672**	**0.324690**	**0.639732**	**0.148076**
(100,50)	**0.324675**	**0.324001**	**0.322530**	**0.324673**	**0.324681**	**0.528353**	**0.221326**
(100,100)	**0.324675**	**0.325285**	**0.324134**	**0.324674**	**0.324683**	**0.351857**	**0.356567**

Table (4-10): Shown MSE values when α_1=2.5, α_2=5.2, and θ =3 and R=0.324675.

(n,m)	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$	Best
(10,10)	**0.009533**	**0.004523**	**0.1E-9**	**0.5E-6**	**0.055852**	**0.034551**	$\hat{R}_{Sh2}$
(10,30)	**0.006390**	**0.001575**	**0.8E-10**	**0.5E-6**	**0.053982**	**0.097955**	$\hat{R}_{Sh2}$
(10,50)	**0.005911**	**0.000940**	**0.8E-10**	**0.6E-6**	**0.067788**	**0.163791**	$\hat{R}_{Sh2}$
(10,100)	**0.005704**	**0.000477**	**0.8E-10**	**0.5E-6**	**0.083391**	**0.256335**	$\hat{R}_{Sh2}$
(30,10)	**0.005735**	**0.004567**	**0.1E-9**	**0.5E-6**	**0.143671**	**0.029112**	$\hat{R}_{Sh2}$
(30,30)	**0.003134**	**0.001605**	**0.4E-10**	**0.6E-6**	**0.042710**	**0.012712**	$\hat{R}_{Sh2}$
(30,50)	**0.002521**	**0.000962**	**0.3E-10**	**0.6E-6**	**0.036326**	**0.033072**	$\hat{R}_{Sh2}$
(30,100)	**0.002050**	**0.000481**	**0.2E-10**	**0.6E-6**	**0.054718**	**0.108518**	$\hat{R}_{Sh2}$
(50,10)	**0.005753**	**0.004861**	**0.1E-9**	**0.6E-6**	**0.214244**	**0.046845**	$\hat{R}_{Sh2}$
(50,30)	**0.002514**	**0.001584**	**0.3E-10**	**0.6E-6**	**0.073027**	**0.010516**	$\hat{R}_{Sh2}$
(50,50)	**0.001995**	**0.000964**	**0.2E-10**	**0.6E-6**	**0.038922**	**0.008443**	$\hat{R}_{Sh2}$
(50,100)	**0.001510**	**0.000519**	**0.2E-10**	**0.6E-6**	**0.040171**	**0.04616**	$\hat{R}_{Sh2}$
(100,10)	**0.005211**	**0.004775**	**0.1E-9**	**0.5E-6**	**0.300140**	**0.070702**	$\hat{R}_{Sh2}$
(100,30)	**0.002136**	**0.001584**	**0.3E-10**	**0.66E-6**	**0.140514**	**0.033305**	$\hat{R}_{Sh2}$
(100,50)	**0.001345**	**0.000961**	**0.1E-10**	**0.6E-6**	**0.082894**	**0.013372**	$\hat{R}_{Sh2}$
(100,100)	**0.000986**	**0.000474**	**0.1E-10**	**0.6E-6**	**0.034374**	**0.004651**	$\hat{R}_{Sh2}$

Table (4-11): Shown estimation value of R, when α_1=5.1, α_2=1.5, and θ =3

(n,m)	R	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$
(10,10)	**0.772727**	**0.758051**	**0.755666**	**0.772730**	**0.7727693**	**0.706620**	**0.667186**
(10,30)	**0.772727**	**0.772766**	**0.769158**	**0.772733**	**0.772748**	**0.445102**	**0.858615**
(10,50)	**0.772727**	**0.773750**	**0.769535**	**0.772732**	**0.772788**	**0.327827**	**0.909868**
(10,100)	**0.772727**	**0.770002**	**0.771047**	**0.772732**	**0.772804**	**0.197734**	**0.952169**
(30,10)	**0.772727**	**0.763945**	**0.763054**	**0.772729**	**0.772710**	**0.888515**	**0.474449**
(30,30)	**0.772727**	**0.767415**	**0.767453**	**0.772730**	**0.772757**	**0.725429**	**0.693388**
(30,50)	**0.772727**	**0.771107**	**0.770077**	**0.772730**	**0.772752**	**0.621960**	**0.793703**
(30,100)	**0.772727**	**0.771361**	**0.771435**	**0.772730**	**0.772746**	**0.457024**	**0.884522**
(50,10)	**0.772727**	**0.756907**	**0.756924**	**0.772728**	**0.772679**	**0.933719**	**0.350891**
(50,30)	**0.772727**	**0.769518**	**0.769359**	**0.772729**	**0.772708**	**0.813793**	**0.595662**
(50,50)	**0.772727**	**0.770079**	**0.769251**	**0.772730**	**0.772718**	**0.727951**	**0.703224**
(50,100)	**0.772727**	**0.772537**	**0.770890**	**0.772730**	**0.772722**	**0.584219**	**0.826194**
(100,10)	**0.772727**	**0.761520**	**0.761484**	**0.772728**	**0.772661**	**0.967524**	**0.232377**
(100,30)	**0.772727**	**0.767254**	**0.767747**	**0.772729**	**0.772676**	**0.909501**	**0.428812**
(100,50)	**0.772727**	**0.771863**	**0.770758**	**0.772730**	**0.772657**	**0.854326**	**0.554514**
(100,100)	**0.772727**	**0.771976**	**0.772080**	**0.772729**	**0.772762**	**0.739401**	**0.710936**

Table (4-12): Shown MSE values when α_1=5.1, α_2=1.5, and θ =3 and

R= 0.772727

(n,m)	$\hat{R}_{MLE}$	$\hat{R}_{Sh1}$	$\hat{R}_{Sh2}$	$\hat{R}_{Th}$	$\hat{R}_{LS}$	$\hat{R}_{RSS}$	Best
(10,10)	**0.007232**	**0.004020**	**0.1E-9**	**0.3E-6**	**0.049227**	**0.067986**	$\hat{R}_{Sh2}$
(10,30)	**0.004138**	**0.001061**	**0.8E-10**	**0,3E-6**	**0.158545**	**0.015968**	$\hat{R}_{Sh2}$
(10,50)	**0.003268**	**0.000610**	**0.7E-10**	**0.3E-6**	**0.241198**	**0.021412**	$\hat{R}_{Sh2}$
(10,100)	**0.003397**	**0.000332**	**0.7E-10**	**0.4E-6**	**0.357931**	**0.032816**	$\hat{R}_{Sh2}$
(30,10)	**0.004505**	**0.003517**	**0.6E-10**	**0.3E-6**	**0.032535**	**0.120549**	$\hat{R}_{Sh2}$
(30,30)	**0.001975**	**0.001096**	**0.2E-10**	**0.3E-6**	**0.036668**	**0.019880**	$\hat{R}_{Sh2}$
(30,50)	**0.001567**	**0.000644**	**0.2E-10**	**0.4E-6**	**0.064908**	**0.006185**	$\hat{R}_{Sh2}$
(30,100)	**0.001267**	**0.000293**	**0.2E-10**	**0.3E-6**	**0.144762**	**0.013974**	$\hat{R}_{Sh2}$
(50,10)	**0.004524**	**0.003684**	**0.6E-10**	**0.3E-6**	**0.036148**	**0.201669**	$\hat{R}_{Sh2}$
(50,30)	**0.001689**	**0.001091**	**0.2E-10**	**0.3E-6**	**0.028718**	**0.044980**	$\hat{R}_{Sh2}$
(50,50)	**0.001344**	**0.000671**	**0.1E-10**	**0.4E-6**	**0.036706**	**0.012960**	$\hat{R}_{Sh2}$
(50,100)	**0.000960**	**0.000321**	**0.1E-10**	**0.4E-6**	**0.080245**	**0.005394**	$\hat{R}_{Sh2}$
(100,10)	**0.004139**	**0.003737**	**0.1E-10**	**0.4E-6**	**0.042351**	**0.306010**	$\hat{R}_{Sh2}$
(100,30)	**0.001392**	**0.001064**	**0.1E-10**	**0.4E-6**	**0.030320**	**0.130512**	$\hat{R}_{Sh2}$
(100,50)	**0.000896**	**0.000549**	**0.1E-10**	**0.4E-6**	**0.025804**	**0.055663**	$\hat{R}_{Sh2}$
(100,100)	**0.000636**	**0.000324**	**0.1E-10**	**0.4E-6**	**0.029863**	**0.007479**	$\hat{R}_{Sh2}$

B. The model R_k:

This model consists estimation of R_k for EWD(α_i,θ) i= 1,2,…, k+1 when known parameters ($\alpha_1\alpha_2\alpha_3$, α_4)=(2,2,2,2), (2,4,3,5) ,(5, 3,2,1), (2.5,3.3,1.2,4.2), (2.2,5.1,3.4,2) and (5, 1, 2, 4.2) as well as unknown parameter θ=3 for all cases.

Table (4-13): Shown estimation value of R_k, when α_1= α_2= α_3= α_4=2 and θ=3

(n_1,n_2,n_3,m)	R_k	$\hat{R}_{k_{MLE}}$	$\hat{R}_{k_{Sh1}}$	$\hat{R}_{k_{Sh2}}$	$\hat{R}_{k_{Th}}$	$\hat{R}_{k_{LS}}$	$\hat{R}_{k_{RSS}}$
(10,10,10,10)	**0.750000**	**0.744182**	**0.748582**	**0.747160**	**0.748552**	**0.739348**	**0.750741**
(30,50,100,10)	**0.750000**	**0.741109**	**0.750955**	**0.752054**	**0.750917**	**0.506018**	**0.947560**
(50,100,30,10)	**0.750000**	**0.740331**	**0.750516**	**0.751548**	**0.750491**	**0.511926**	**0.947491**
(100,30,50,10)	**0.750000**	**0.737277**	**0.750010**	**0.750284**	**0.749967**	**0.486345**	**0.947428**
(10,10,10,30)	**0.750000**	**0.759601**	**0.749834**	**0.749329**	**0.749866**	**0.868083**	**0.502255**
(30,50,100,30)	**0.750000**	**0.749912**	**0.750410**	**0.750882**	**0.750413**	**0.680309**	**0.857636**
(50,100,30,30)	**0.750000**	**0.749218**	**0.750224**	**0.750539**	**0.750221**	**0.690331**	**0.857581**
(100,30,50,30)	**0.750000**	**0.747700**	**0.750142**	**0.750016**	**0.750133**	**0.677476**	**0.857352**
(10,10,10,50)	**0.750000**	**0.764313**	**0.750623**	**0.750665**	**0.750673**	**0.898505**	**0.377363**
(30,50,100,50)	**0.750000**	**0.747163**	**0.749238**	**0.748564**	**0.749225**	**0.760340**	**0.783392**
(50,100,30,50)	**0.750000**	**0.750352**	**0.750088**	**0.750113**	**0.750088**	**0.761263**	**0.783202**
(100,30,50,50)	**0.750000**	**0.751512**	**0.750246**	**0.750565**	**0.750249**	**0.739352**	**0.782914**
(10,10,10,100)	**0.750000**	**0.764071**	**0.750039**	**0.749684**	**0.750084**	**0.939668**	**0.233004**
(30,50,100,100)	**0.750000**	**0.750802**	**0.749552**	**0.749356**	**0.749554**	**0.823460**	**0.644161**
(50,100,30,100)	**0.750000**	**0.750603**	**0.749862**	**0.749517**	**0.749860**	**0.832874**	**0.644155**
(100,30,50,100)	**0.750000**	**0.751674**	**0.749995**	**0.749936**	**0.750000**	**0.835356**	**0.643917**
(30,30,30,30)	**0.750000**	**0.750412**	**0.750021**	**0.750117**	**0.750024**	**0.758260**	**0.750910**
(50,50,50,50)	**0.750000**	**0.749615**	**0.750015**	**0.749890**	**0.750012**	**0.746758**	**0.750732**
(100,100,100,100)	**0.750000**	**0.749066**	**0.749799**	**0.749570**	**0.749795**	**0.756429**	**0.750860**

Table (4-14) :Shown MSE values when α_1=2, α_2=2, α_3=2, α_4=2 and θ=3 and R_k=0.750000

(n_1,n_2,n_3,m)	$\hat{R}_{k_{MLE}}$	$\hat{R}_{k_{Sh1}}$	$\hat{R}_{k_{Sh2}}$	$\hat{R}_{k_{Th}}$	$\hat{R}_{k_{LS}}$	$\hat{R}_{k_{RSS}}$	Best
(10,10,10,10)	**0.004868**	**0.000300**	**0.001068**	**0.000309**	**0.059318**	**0.000020**	$\hat{R}_{k_{RSS}}$
(30,50,100,10)	**0.004537**	**0.000243**	**0.000916**	**0.000251**	**0.135220**	**0.039032**	$\hat{R}_{k_{Sh1}}$
(50,100,30,10)	**0.004193**	**0.000240**	**0.000845**	**0.000247**	**0.133861**	**0.039004**	$\hat{R}_{k_{Sh1}}$
(100,30,50,10)	**0.004465**	**0.000261**	**0.000905**	**0.000268**	**0.144731**	**0.038978**	$\hat{R}_{k_{Sh1}}$
(10,10,10,30)	**0.002572**	**0.000157**	**0.000558**	**0.000162**	**0.045587**	**0.061771**	$\hat{R}_{k_{Sh1}}$
(30,50,100,30)	**0.001523**	**0.000094**	**0.000327**	**0.000096**	**0.071428**	**0.011604**	$\hat{R}_{k_{Sh1}}$
(50,100,30,30)	**0.001452**	**0.000093**	**0.000313**	**0.000095**	**0.069220**	**0.011591**	$\hat{R}_{k_{Sh1}}$
(100,30,50,30)	**0.001475**	**0.000092**	**0.000316**	**0.000095**	**0.072806**	**0.011526**	$\hat{R}_{k_{Sh1}}$
(10,10,10,50)	**0.002048**	**0.000125**	**0.000426**	**0.000128**	**0.045753**	**0.139339**	$\hat{R}_{k_{Sh1}}$
(30,50,100,50)	**0.001020**	**0.000064**	**0.000220**	**0.000066**	**0.054211**	**0.001168**	$\hat{R}_{k_{Sh1}}$
(50,100,30,50)	**0.000960**	**0.000062**	**0.000210**	**0.000064**	**0.050526**	**0.001139**	$\hat{R}_{k_{Sh1}}$
(100,30,50,50)	**0.000955**	**0.000063**	**0.000209**	**0.000065**	**0.058852**	**0.001086**	$\hat{R}_{k_{Sh1}}$
(10,10,10,100)	**0.001845**	**0.000107**	**0.000378**	**0.000110**	**0.046739**	**0.267929**	$\hat{R}_{k_{Sh1}}$
(30,50,100,100)	**0.000583**	**0.000039**	**0.000129**	**0.000040**	**0.047961**	**0.011332**	$\hat{R}_{k_{Sh1}}$
(50,100,30,100)	**0.000578**	**0.000036**	**0.000122**	**0.000037**	**0.044259**	**0.011318**	$\hat{R}_{k_{Sh1}}$
(100,30,50,100)	**0.000599**	**0.000036**	**0.000127**	**0.000037**	**0.046471**	**0.011303**	$\hat{R}_{k_{Sh1}}$
(30,30,30,30)	**0.001579**	**0.000101**	**0.000344**	**0.000104**	**0.051579**	**0.000074**	$\hat{R}_{k_{RSS}}$
(50,50,50,50)	**0.000973**	**0.000061**	**0.000211**	**0.000063**	**0.058605**	**0.000014**	$\hat{R}_{k_{RSS}}$
(100,100,100,100)	**0.000512**	**0.000031**	**0.000110**	**0.000032**	**0.053907**	**0.000066**	$\hat{R}_{k_{Sh1}}$

Table (4-15): Shown estimation value of R_k, when α_1=2, α_2=4, α_3=3, α_4=5 and θ=3

(n_1,n_2,n_3,m)	R_k	$\hat{R}_{k_{MLE}}$	$\hat{R}_{k_{Sh1}}$	$\hat{R}_{k_{Sh2}}$	$\hat{R}_{k_{Th}}$	$\hat{R}_{k_{LS}}$	$\hat{R}_{k_{RSS}}$
(10,10,10,10)	0.642857	0.645624	0.643229	0.643229	0.643224	0.681222	0.750631
(30,50,100,10)	0.642857	0.636430	0.643936	0.646205	0.643914	0.409251	0.947569
(50,100,30,10)	0.642857	0.632828	0.643146	0.644553	0.643111	0.408313	0.947485
(100,30,50,10)	0.642857	0.629079	0.642634	0.642937	0.642588	0.414180	0.947441
(10,10,10,30)	0.642857	0.654607	0.642760	0.641859	0.642787	0.815616	0.502013
(30,50,100,30)	0.642857	0.643216	0.643248	0.643909	0.643248	0.588887	0.857697
(50,100,30,30)	0.642857	0.639176	0.642292	0.642015	0.642277	0.586119	0.857522
(100,30,50,30)	0.642857	0.642442	0.642884	0.643218	0.642882	0.598339	0.857341
(10,10,10,50)	0.642857	0.656987	0.642574	0.641810	0.642618	0.847321	0.377691
(30,50,100,50)	0.642857	0.644024	0.643007	0.643219	0.643008	0.673383	0.783328
(50,100,30,50)	0.642857	0.641922	0.642261	0.642023	0.642257	0.657943	0.783312
(100,30,50,50)	0.642857	0.644403	0.642832	0.642964	0.642831	0.683978	0.783015
(10,10,10,100)	0.642857	0.659236	0.642772	0.641915	0.642818	0.914082	0.232808
(30,50,100,100)	0.642857	0.644341	0.642781	0.642595	0.642787	0.769032	0.644306
(50,100,30,100)	0.642857	0.646271	0.643099	0.643374	0.643113	0.774551	0.643993
(100,30,50,100)	0.642857	0.645499	0.642738	0.642701	0.642748	0.774267	0.643582
(30,30,30,30)	0.642857	0.642007	0.642579	0.642220	0.642569	0.688674	0.750699
(50,50,50,50)	0.642857	0.642407	0.642713	0.642530	0.642714	0.685892	0.750623
(100,100,100,100)	0.642857	0.643745	0.642942	0.643137	0.642945	0.669423	0.750811

Table (4-16) :Shown MSE values when α_1=2, α_2=4, α_3=3, α_4=5 and θ=3 and

R_k= 0.642857

(n_1,n_2,n_3,m)	$\hat{R}_{k_{MLE}}$	$\hat{R}_{k_{Sh1}}$	$\hat{R}_{k_{Sh2}}$	$\hat{R}_{k_{Th}}$	$\hat{R}_{k_{LS}}$	$\hat{R}_{k_{RSS}}$	Best
(10,10,10,10)	**0.007619**	**0.000246**	**0.001405**	**0.000259**	**0.065728**	**0.011657**	$\hat{R}_{k_{Sh1}}$
(30,50,100,10)	**0.005910**	**0.000184**	**0.001020**	**0.000193**	**0.124826**	**0.092852**	$\hat{R}_{k_{Sh1}}$
(50,100,30,10)	**0.006061**	**0.000173**	**0.001016**	**0.000182**	**0.125591**	**0.092799**	$\hat{R}_{k_{Sh1}}$
(100,30,50,10)	**0.006343**	**0.000190**	**0.001066**	**0.000199**	**0.129343**	**0.092772**	$\hat{R}_{k_{Sh1}}$
(10,10,10,30)	**0.003755**	**0.000138**	**0.000687**	**0.000144**	**0.075211**	**0.020057**	$\hat{R}_{k_{Sh1}}$
(30,50,100,30)	**0.002050**	**0.000067**	**0.000359**	**0.000070**	**0.081115**	**0.046210**	$\hat{R}_{k_{Sh1}}$
(50,100,30,30)	**0.002193**	**0.000070**	**0.000381**	**0.000073**	**0.078076**	**0.046092**	$\hat{R}_{k_{Sh1}}$
(100,30,50,30)	**0.002210**	**0.000072**	**0.000385**	**0.000075**	**0.074916**	**0.046004**	$\hat{R}_{k_{Sh1}}$
(10,10,10,50)	**0.003306**	**0.000123**	**0.000596**	**0.000128**	**0.082252**	**0.070842**	$\hat{R}_{k_{Sh1}}$
(30,50,100,50)	**0.001452**	**0.000048**	**0.000255**	**0.000050**	**0.068434**	**0.019772**	$\hat{R}_{k_{Sh1}}$
(50,100,30,50)	**0.001477**	**0.000050**	**0.000262**	**0.000053**	**0.070877**	**0.019777**	$\hat{R}_{k_{Sh1}}$
(100,30,50,50)	**0.001567**	**0.000052**	**0.000277**	**0.000055**	**0.067880**	**0.019649**	$\hat{R}_{k_{Sh1}}$
(10,10,10,100)	**0.002487**	**0.000091**	**0.000429**	**0.000095**	**0.092477**	**0.168778**	$\hat{R}_{k_{Sh1}}$
(30,50,100,100)	**0.000863**	**0.000031**	**0.000155**	**0.000032**	**0.069145**	**0.000154**	$\hat{R}_{k_{Sh1}}$
(50,100,30,100)	**0.000856**	**0.000031**	**0.000153**	**0.000033**	**0.066422**	**0.000058**	$\hat{R}_{k_{Sh1}}$
(100,30,50,100)	**0.001042**	**0.000036**	**0.000184**	**0.000038**	**0.069626**	**0.000022**	$\hat{R}_{k_{Sh1}}$
(30,30,30,30)	**0.002565**	**0.000088**	**0.000463**	**0.000092**	**0.065287**	**0.011655**	$\hat{R}_{k_{Sh1}}$
(50,50,50,50)	**0.001322**	**0.000045**	**0.000232**	**0.000047**	**0.069126**	**0.011650**	$\hat{R}_{k_{Sh1}}$
(100,100,100,100)	**0.000664**	**0.000023**	**0.000117**	**0.000024**	**0.068213**	**0.011695**	$\hat{R}_{k_{Sh1}}$

Table (4-17): Shown estimation value of R_k, when α_1=5, α_2=3, α_3=2, α_4=1and θ=3

(n_1,n_2,n_3,m)	R_k	$\widehat{R}_{k_{MLE}}$	$\widehat{R}_{k_{Sh1}}$	$\widehat{R}_{k_{Sh2}}$	$\widehat{R}_{k_{Th}}$	$\widehat{R}_{k_{LS}}$	$\widehat{R}_{k_{RSS}}$
(10,10,10,10)	**0.909091**	**0.908351**	**0.909383**	**0.909432**	**0.909388**	**0.872212**	**0.750813**
(30,50,100,10)	**0.909091**	**0.903633**	**0.909493**	**0.909775**	**0.909483**	**0.715735**	**0.947546**
(50,100,30,10)	**0.909091**	**0.902337**	**0.909359**	**0.909395**	**0.909342**	**0.700408**	**0.947484**
(100,30,50,10)	**0.909091**	**0.902960**	**0.909552**	**0.909736**	**0.909542**	**0.679197**	**0.947438**
(10,10,10,30)	**0.909091**	**0.912588**	**0.909255**	**0.908960**	**0.909268**	**0.936814**	**0.502128**
(30,50,100,30)	**0.909091**	**0.907606**	**0.908884**	**0.908814**	**0.908881**	**0.837772**	**0.857659**
(50,100,30,30)	**0.909091**	**0.907163**	**0.908864**	**0.908754**	**0.908861**	**0.833968**	**0.857474**
(100,30,50,30)	**0.909091**	**0.907534**	**0.909004**	**0.908963**	**0.909000**	**0.816446**	**0.857341**
(10,10,10,50)	**0.909091**	**0.913233**	**0.909126**	**0.908725**	**0.909143**	**0.958393**	**0.377108**
(30,50,100,50)	**0.909091**	**0.907921**	**0.908807**	**0.908527**	**0.908804**	**0.883764**	**0.783414**
(50,100,30,50)	**0.909091**	**0.908035**	**0.908808**	**0.908648**	**0.908806**	**0.860371**	**0.783251**
(100,30,50,50)	**0.909091**	**0.909025**	**0.909078**	**0.909144**	**0.909080**	**0.866642**	**0.783033**
(10,10,10,100)	**0.909091**	**0.913248**	**0.908930**	**0.908340**	**0.908943**	**0.972631**	**0.232962**
(30,50,100,100)	**0.909091**	**0.909149**	**0.908895**	**0.908709**	**0.908897**	**0.927702**	**0.644279**
(50,100,30,100)	**0.909091**	**0.909060**	**0.909003**	**0.908847**	**0.909003**	**0.927864**	**0.643941**
(100,30,50,100)	**0.909091**	**0.909506**	**0.909060**	**0.909023**	**0.909062**	**0.918627**	**0.643956**
(30,30,30,30)	**0.909091**	**0.909058**	**0.909167**	**0.909255**	**0.909171**	**0.875959**	**0.750767**
(50,50,50,50)	**0.909091**	**0.907783**	**0.908881**	**0.908610**	**0.908877**	**0.873285**	**0.750790**
(100,100,100,100)	**0.909091**	**0.909004**	**0.909105**	**0.909109**	**0.909106**	**0.882051**	**0.750852**

Table (4-18) :Shown MSE values when α_1=5, α_2=3, α_3=2, α_4=1and θ=3 and

R_k=0.909091

(n_1,n_2,n_3,m)	$\hat{R}_{k_{MLE}}$	$\hat{R}_{k_{Sh1}}$	$\hat{R}_{k_{Sh2}}$	$\hat{R}_{k_{Th}}$	$\hat{R}_{k_{LS}}$	$\hat{R}_{k_{RSS}}$	Best
(10,10,10,10)	**0.000999**	**0.000096**	**0.000245**	**0.000097**	**0.030381**	**0.025115**	$\hat{R}_{k_{Sh1}}$
(30,50,100,10)	**0.000960**	**0.000093**	**0.000222**	**0.000095**	**0.098788**	**0.001481**	$\hat{R}_{k_{Sh1}}$
(50,100,30,10)	**0.000953**	**0.000089**	**0.000215**	**0.000091**	**0.105920**	**0.001474**	$\hat{R}_{k_{Sh1}}$
(100,30,50,10)	**0.000851**	**0.000087**	**0.000199**	**0.000089**	**0.119214**	**0.001470**	$\hat{R}_{k_{Sh1}}$
(10,10,10,30)	**0.000510**	**0.000040**	**0.000120**	**0.000041**	**0.014412**	**0.165937**	$\hat{R}_{k_{Sh1}}$
(30,50,100,30)	**0.000330**	**0.000031**	**0.000081**	**0.000032**	**0.044984**	**0.002674**	$\hat{R}_{k_{Sh1}}$
(50,100,30,30)	**0.000278**	**0.000030**	**0.000070**	**0.000031**	**0.042574**	**0.002669**	$\hat{R}_{k_{Sh1}}$
(100,30,50,30)	**0.000291**	**0.000031**	**0.000075**	**0.000032**	**0.049948**	**0.002679**	$\hat{R}_{k_{Sh1}}$
(10,10,10,50)	**0.000399**	**0.000026**	**0.000089**	**0.000027**	**0.010801**	**0.283462**	$\hat{R}_{k_{Sh1}}$
(30,50,100,50)	**0.000204**	**0.000018**	**0.000048**	**0.000019**	**0.028256**	**0.015843**	$\hat{R}_{k_{Sh1}}$
(50,100,30,50)	**0.000174**	**0.000018**	**0.000044**	**0.000019**	**0.036000**	**0.015881**	$\hat{R}_{k_{Sh1}}$
(100,30,50,50)	**0.000184**	**0.000018**	**0.000046**	**0.000019**	**0.034885**	**0.015900**	$\hat{R}_{k_{Sh1}}$
(10,10,10,100)	**0.000322**	**0.000017**	**0.000067**	**0.000018**	**0.010280**	**0.457748**	$\hat{R}_{k_{Sh1}}$
(30,50,100,100)	**0.000134**	**0.000011**	**0.000031**	**0.000012**	**0.017052**	**0.070248**	$\hat{R}_{k_{Sh1}}$
(50,100,30,100)	**0.000122**	**0.000010**	**0.000030**	**0.000011**	**0.013387**	**0.070416**	$\hat{R}_{k_{Sh1}}$
(100,30,50,100)	**0.000106**	**0.000010**	**0.000026**	**0.000011**	**0.018133**	**0.070368**	$\hat{R}_{k_{Sh1}}$
(30,30,30,30)	**0.000318**	**0.000031**	**0.000079**	**0.000032**	**0.033071**	**0.025090**	$\hat{R}_{k_{Sh1}}$
(50,50,50,50)	**0.000181**	**0.000018**	**0.000046**	**0.000019**	**0.033329**	**0.025098**	$\hat{R}_{k_{Sh1}}$
(100,100,100,100)	**0.000098**	**0.000010**	**0.000025**	**0.000011**	**0.027187**	**0.025119**	$\hat{R}_{k_{Sh1}}$

Table (4-19): Shown estimation value of R_k, when α_1=2.5, α_2=3.3, α_3=1.2, α_4=4.2 and θ=3

(n_1,n_2,n_3,m)	R_k	$\widehat{R}_{k_{MLE}}$	$\widehat{R}_{k_{Sh1}}$	$\widehat{R}_{k_{Sh2}}$	$\widehat{R}_{k_{Th}}$	$\widehat{R}_{k_{LS}}$	$\widehat{R}_{k_{RSS}}$
(10,10,10,10)	**0.625000**	**0.628942**	**0.627499**	**0.627258**	**0.627452**	**0.652338**	**0.750936**
(30,50,100,10)	**0.625000**	**0.612315**	**0.627052**	**0.626942**	**0.626951**	**0.414401**	**0.947554**
(50,100,30,10)	**0.625000**	**0.614494**	**0.627520**	**0.628414**	**0.627432**	**0.382354**	**0.947459**
(100,30,50,10)	**0.625000**	**0.612192**	**0.627085**	**0.627065**	**0.626963**	**0.411508**	**0.947452**
(10,10,10,30)	**0.625000**	**0.635315**	**0.626603**	**0.624779**	**0.626588**	**0.803362**	**0.502646**
(30,50,100,30)	**0.625000**	**0.623997**	**0.626915**	**0.626474**	**0.626866**	**0.591583**	**0.857652**
(50,100,30,30)	**0.625000**	**0.621485**	**0.626683**	**0.625911**	**0.626608**	**0.561022**	**0.857539**
(100,30,50,30)	**0.625000**	**0.625224**	**0.626925**	**0.626931**	**0.626856**	**0.576556**	**0.857403**
(10,10,10,50)	**0.625000**	**0.641353**	**0.627063**	**0.626080**	**0.627074**	**0.852312**	**0.377130**
(30,50,100,50)	**0.625000**	**0.625413**	**0.626831**	**0.626119**	**0.626788**	**0.676039**	**0.783418**
(50,100,30,50)	**0.625000**	**0.625690**	**0.627315**	**0.626844**	**0.627257**	**0.640504**	**0.783167**
(100,30,50,50)	**0.625000**	**0.627822**	**0.627249**	**0.627183**	**0.627195**	**0.675143**	**0.782946**
(10,10,10,100)	**0.625000**	**0.640141**	**0.626621**	**0.624912**	**0.626626**	**0.912633**	**0.232757**
(30,50,100,100)	**0.625000**	**0.628563**	**0.627137**	**0.626706**	**0.627106**	**0.762840**	**0.644140**
(50,100,30,100)	**0.625000**	**0.627269**	**0.627006**	**0.626544**	**0.626960**	**0.747024**	**0.644015**
(100,30,50,100)	**0.625000**	**0.627173**	**0.626907**	**0.626153**	**0.626842**	**0.757850**	**0.643707**
(30,30,30,30)	**0.625000**	**0.627142**	**0.627035**	**0.626838**	**0.626988**	**0.664785**	**0.750712**
(50,50,50,50)	**0.625000**	**0.625260**	**0.626864**	**0.626287**	**0.626804**	**0.670057**	**0.750777**
(100,100,100,100)	**0.625000**	**0.626084**	**0.627249**	**0.626863**	**0.627189**	**0.655584**	**0.750823**

Table (4-20) :Shown MSE values when α_1=2.5, α_2=3.3, α_3=1.2, α_4=4.2 and θ=3 and R_k=0.625000

(n_1,n_2,n_3,m)	$\widehat{R}_{k_{MLE}}$	$\widehat{R}_{k_{Sh1}}$	$\widehat{R}_{k_{Sh2}}$	$\widehat{R}_{k_{Th}}$	$\widehat{R}_{k_{LS}}$	$\widehat{R}_{k_{RSS}}$	Best
(10,10,10,10)	**0.008058**	**0.000321**	**0.001536**	**0.000335**	**0.073804**	**0.015979**	$\widehat{R}_{k_{Sh1}}$
(30,50,100,10)	**0.006745**	**0.000224**	**0.001174**	**0.000234**	**0.119078**	**0.104043**	$\widehat{R}_{k_{Sh1}}$
(50,100,30,10)	**0.005815**	**0.000226**	**0.001031**	**0.000234**	**0.129442**	**0.103982**	$\widehat{R}_{k_{Sh1}}$
(100,30,50,10)	**0.006299**	**0.000221**	**0.001090**	**0.000230**	**0.122714**	**0.103976**	$\widehat{R}_{k_{Sh1}}$
(10,10,10,30)	**0.004334**	**0.000185**	**0.000808**	**0.000192**	**0.076429**	**0.015879**	$\widehat{R}_{k_{Sh1}}$
(30,50,100,30)	**0.002525**	**0.000099**	**0.000451**	**0.000103**	**0.075068**	**0.054147**	$\widehat{R}_{k_{Sh1}}$
(50,100,30,30)	**0.002177**	**0.000081**	**0.000381**	**0.000084**	**0.081309**	**0.054097**	$\widehat{R}_{k_{Sh1}}$
(100.30,50,30)	**0.002327**	**0.000091**	**0.000415**	**0.000094**	**0.080587**	**0.054014**	$\widehat{R}_{k_{Sh1}}$
(10,10,10,50)	**0.003387**	**0.000150**	**0.000605**	**0.000155**	**0.086799**	**0.061931**	$\widehat{R}_{k_{Sh1}}$
(30,50,100,50)	**0.001554**	**0.000065**	**0.000280**	**0.000068**	**0.074278**	**0.025147**	$\widehat{R}_{k_{Sh1}}$
(50,100,30,50)	**0.001402**	**0.000060**	**0.000253**	**0.000062**	**0.071252**	**0.025059**	$\widehat{R}_{k_{Sh1}}$
(100,30,50,50)	**0.001550**	**0.000060**	**0.000273**	**0.000063**	**0.066921**	**0.024951**	$\widehat{R}_{k_{Sh1}}$
(10,10,10,100)	**0.002965**	**0.000133**	**0.000540**	**0.000138**	**0.102514**	**0.154377**	$\widehat{R}_{k_{Sh1}}$
(30,50,100,100)	**0.001074**	**0.000052**	**0.000200**	**0.000054**	**0.071931**	**0.000471**	$\widehat{R}_{k_{Sh1}}$
(50,100,30,100)	**0.000868**	**0.000043**	**0.000163**	**0.000044**	**0.069691**	**0.000423**	$\widehat{R}_{k_{Sh1}}$
(100,30,50,100)	**0.001133**	**0.000046**	**0.000201**	**0.000048**	**0.073525**	**0.000515**	$\widehat{R}_{k_{Sh1}}$
(30,30,30,30)	**0.002707**	**0.000106**	**0.000487**	**0.000111**	**0.068972**	**0.015837**	$\widehat{R}_{k_{Sh1}}$
(50,50,50,50)	**0.001466**	**0.000058**	**0.000259**	**0.000061**	**0.070870**	**0.015838**	$\widehat{R}_{k_{Sh1}}$
(100,100,100,100)	**0.000714**	**0.000033**	**0.000129**	**0.000034**	**0.067533**	**0.015866**	$\widehat{R}_{k_{Sh1}}$

Table (4-21): Shown estimation value of R_k, when α_1=2.2, α_2=5.1, α_3=3.4, α_4=2, and θ=3

(n_1,n_2,n_3,m)	R_k	$\widehat{R}_{k_{MLE}}$	$\widehat{R}_{k_{Sh1}}$	$\widehat{R}_{k_{Sh2}}$	$\widehat{R}_{k_{Th}}$	$\widehat{R}_{k_{LS}}$	$\widehat{R}_{k_{RSS}}$
(10,10,10,10)	**0.842519**	**0.836556**	**0.838908**	**0.838487**	**0.838899**	**0.808145**	**0.750745**
(30,50,100,10)	**0.842519**	**0.834567**	**0.840468**	**0.841904**	**0.840457**	**0.596738**	**0.947560**
(50,100,30,10)	**0.842519**	**0.834006**	**0.840156**	**0.841514**	**0.840150**	**0.596900**	**0.947489**
(100,30,50,10)	**0.842519**	**0.831817**	**0.839767**	**0.840450**	**0.839749**	**0.600966**	**0.947426**
(10,10,10,30)	**0.842519**	**0.848261**	**0.839758**	**0.840120**	**0.839795**	**0.908274**	**0.502341**
(30,50,100,30)	**0.842519**	**0.841311**	**0.840058**	**0.841019**	**0.840072**	**0.751741**	**0.857537**
(50,100,30,30)	**0.842519**	**0.841346**	**0.839973**	**0.840970**	**0.839985**	**0.760799**	**0.857581**
(100,30,50,30)	**0.842519**	**0.840151**	**0.839619**	**0.840226**	**0.839625**	**0.774512**	**0.857255**
(10,10,10,50)	**0.842519**	**0.849058**	**0.839643**	**0.839754**	**0.839679**	**0.941523**	**0.376949**
(30,50,100,50)	**0.842519**	**0.842474**	**0.840070**	**0.840881**	**0.840089**	**0.800879**	**0.783414**
(50,100,30,50)	**0.842519**	**0.842870**	**0.840039**	**0.840972**	**0.840056**	**0.815620**	**0.783191**
(100,30,50,50)	**0.842519**	**0.841603**	**0.839633**	**0.840168**	**0.839645**	**0.838734**	**0.783052**
(10,10,10,100)	**0.842519**	**0.850125**	**0.839686**	**0.839796**	**0.839731**	**0.963327**	**0.232966**
(30,50,100,100)	**0.842519**	**0.843519**	**0.839954**	**0.840741**	**0.839976**	**0.876934**	**0.644161**
(50,100,30,100)	**0.842519**	**0.843478**	**0.839945**	**0.840702**	**0.839963**	**0.884254**	**0.644024**
(100,30,50,100)	**0.842519**	**0.843667**	**0.839742**	**0.840513**	**0.839762**	**0.887382**	**0.643919**
(30,30,30,30)	**0.842519**	**0.841435**	**0.839650**	**0.840286**	**0.839661**	**0.815402**	**0.751016**
(50,50,50,50)	**0.842519**	**0.841364**	**0.839736**	**0.840313**	**0.839749**	**0.817763**	**0.750783**
(100,100,100,100)	**0.842519**	**0.841894**	**0.839745**	**0.840433**	**0.839759**	**0.815803**	**0.750911**

Table (4-22): Shown MSE values when α_1=2.2, α_2=5.1, α_3=3.4, α_4=2, and θ=3 and R_k=0.842519

(n_1,n_2,n_3,m)	$\hat{R}_{k_{MLE}}$	$\hat{R}_{k_{Sh1}}$	$\hat{R}_{k_{Sh2}}$	$\hat{R}_{k_{Th}}$	$\hat{R}_{k_{LS}}$	$\hat{R}_{k_{RSS}}$	Best
(10,10,10,10)	**0.002703**	**0.000152**	**0.000562**	**0.000156**	**0.047028**	**0.008441**	$\hat{R}_{k_{Sh1}}$
(30,50,100,10)	**0.002541**	**0.000125**	**0.000475**	**0.000129**	**0.132741**	**0.011035**	$\hat{R}_{k_{Sh1}}$
(50,100,30,10)	**0.002303**	**0.000125**	**0.000434**	**0.000128**	**0.134022**	**0.011020**	$\hat{R}_{k_{Sh1}}$
(100,30,50,10)	**0.002501**	**0.000139**	**0.000478**	**0.000143**	**0.131060**	**0.011005**	$\hat{R}_{k_{Sh1}}$
(10,10,10,30)	**0.001352**	**0.000073**	**0.000282**	**0.000075**	**0.026758**	**0.116197**	$\hat{R}_{k_{Sh1}}$
(30,50,100,30)	**0.000776**	**0.000050**	**0.000163**	**0.000051**	**0.063185**	**0.000244**	$\hat{R}_{k_{Sh1}}$
(50,100,30,30)	**0.000732**	**0.000050**	**0.000157**	**0.000052**	**0.061682**	**0.000244**	$\hat{R}_{k_{Sh1}}$
(100,30,50,30)	**0.000852**	**0.000054**	**0.000179**	**0.000056**	**0.051582**	**0.000217**	$\hat{R}_{k_{Sh1}}$
(10,10,10,50)	**0.001037**	**0.000057**	**0.000220**	**0.000058**	**0.019976**	**0.217612**	$\hat{R}_{k_{Sh1}}$
(30,50,100,50)	**0.000494**	**0.000036**	**0.000109**	**0.000037**	**0.051723**	**0.003535**	$\hat{R}_{k_{Sh1}}$
(50,100,30,50)	**0.000452**	**0.000033**	**0.000098**	**0.000034**	**0.044135**	**0.003552**	$\hat{R}_{k_{Sh1}}$
(100,30,50,50)	**0.000571**	**0.000040**	**0.000125**	**0.000041**	**0.037103**	**0.003546**	$\hat{R}_{k_{Sh1}}$
(10,10,10,100)	**0.000873**	**0.000044**	**0.000181**	**0.000045**	**0.021413**	**0.372134**	$\hat{R}_{k_{Sh1}}$
(30,50,100,100)	**0.000307**	**0.000023**	**0.000067**	**0.000024**	**0.031822**	**0.039488**	$\hat{R}_{k_{Sh1}}$
(50,100,30,100)	**0.000289**	**0.000023**	**0.000063**	**0.000024**	**0.029348**	**0.039501**	$\hat{R}_{k_{Sh1}}$
(100,30,50,100)	**0.000406**	**0.000027**	**0.000086**	**0.000028**	**0.0293538**	**0.039508**	$\hat{R}_{k_{Sh1}}$
(30,30,30,30)	**0.000823**	**0.000056**	**0.000178**	**0.000057**	**0.041665**	**0.008462**	$\hat{R}_{k_{Sh1}}$
(50,50,50,50)	**0.000507**	**0.000037**	**0.000111**	**0.000038**	**0.041106**	**0.008459**	$\hat{R}_{k_{Sh1}}$
(100,100,100,100)	**0.000257**	**0.000022**	**0.000058**	**0.000023**	**0.044359**	**0.008444**	$\hat{R}_{k_{Sh1}}$

Table (4-23): Shown estimation value of R_k, when α_1=5, α_2=1,α_3=2, α_4=4.2, and θ=3

(n_1,n_2,n_3,m)	R_k	$\hat{R}_{k_{MLE}}$	$\hat{R}_{k_{Sh1}}$	$\hat{R}_{k_{Sh2}}$	$\hat{R}_{k_{Th}}$	$\hat{R}_{k_{LS}}$	$\hat{R}_{k_{RSS}}$
(10,10,10,10)	**0.655737**	**0.649189**	**0.647616**	**0.647691**	**0.647607**	**0.661403**	**0.750741**
(30,50,100,10)	**0.655737**	**0.647032**	**0.649820**	**0.653161**	**0.649813**	**0.435153**	**0.947561**
(50,100,30,10)	**0.655737**	**0.646392**	**0.649495**	**0.652981**	**0.649505**	**0.433618**	**0.947494**
(100,30,50,10)	**0.655737**	**0.642345**	**0.649080**	**0.651824**	**0.649065**	**0.359318**	**0.947432**
(10,10,10,30)	**0.655737**	**0.665832**	**0.648777**	**0.649868**	**0.648854**	**0.815974**	**0.502088**
(30,50,100,30)	**0.655737**	**0.655980**	**0.649309**	**0.651720**	**0.649354**	**0.618619**	**0.857634**
(50,100,30,30)	**0.655737**	**0.655319**	**0.649210**	**0.651604**	**0.649248**	**0.614254**	**0.857584**
(100,30,50,30)	**0.655737**	**0.651853**	**0.648789**	**0.650676**	**0.648809**	**0.558026**	**0.857371**
(10,10,10,50)	**0.655737**	**0.664764**	**0.648194**	**0.648377**	**0.648260**	**0.865533**	**0.377280**
(30,50,100,50)	**0.655737**	**0.656879**	**0.648839**	**0.650913**	**0.648883**	**0.681659**	**0.783371**
(50,100,30,50)	**0.655737**	**0.654549**	**0.648689**	**0.650249**	**0.648718**	**0.680570**	**0.783213**
(100,30,50,50)	**0.655737**	**0.655404**	**0.649200**	**0.651244**	**0.649235**	**0.640374**	**0.783267**
(10,10,10,100)	**0.655737**	**0.672714**	**0.649187**	**0.650932**	**0.649289**	**0.907689**	**0.232780**
(30,50,100,100)	**0.655737**	**0.656303**	**0.648662**	**0.650037**	**0.648698**	**0.789044**	**0.644199**
(50,100,30,100)	**0.655737**	**0.657564**	**0.648998**	**0.650857**	**0.649043**	**0.782315**	**0.644043**
(100,30,50,100)	**0.655737**	**0.657066**	**0.648977**	**0.651042**	**0.649022**	**0.751132**	**0.644012**
(30,30,30,30)	**0.655737**	**0.656419**	**0.649083**	**0.651188**	**0.649119**	**0.689030**	**0.750881**
(50,50,50,50)	**0.655737**	**0.655148**	**0.648846**	**0.650661**	**0.648882**	**0.688914**	**0.750936**
(100,100,100,100)	**0.655737**	**0.656844**	**0.649253**	**0.651498**	**0.649297**	**0.675468**	**0.750869**

Table (4-24): Shown MSE values when α_1=5, α_2=1,α_3=2, α_4=4.2, and θ=3 and R_k=0.655737

(n_1,n_2,n_3,m)	$\hat{R}_{k_{MLE}}$	$\hat{R}_{k_{Sh1}}$	$\hat{R}_{k_{Sh2}}$	$\hat{R}_{k_{Th}}$	$\hat{R}_{k_{LS}}$	$\hat{R}_{k_{RSS}}$	Best
(10,10,10,10)	0.007150	0.000330	0.001387	0.000341	0.070673	0.009046	$\hat{R}_{k_{Sh1}}$
(30,50,100,10)	0.006353	0.000235	0.001117	0.000245	0.124754	0.085163	$\hat{R}_{k_{Sh1}}$
(50,100,30,10)	0.005783	0.000234	0.001006	0.000242	0.125978	0.085123	$\hat{R}_{k_{Sh1}}$
(100,30,50,10)	0.006060	0.000251	0.001063	0.000260	0.153729	0.085085	$\hat{R}_{k_{Sh1}}$
(10,10,10,30)	0.004230	0.000206	0.000825	0.000212	0.068391	0.023852	$\hat{R}_{k_{Sh1}}$
(30,50,100,30)	0.002463	0.000125	0.000450	0.000129	0.075665	0.040780	$\hat{R}_{k_{Sh1}}$
(50,100,30,30)	0.002218	0.000122	0.000409	0.000125	0.074765	0.040760	$\hat{R}_{k_{Sh1}}$
(100,30,50,30)	0.001972	0.000118	0.000373	0.000372	0.087270	0.040660	$\hat{R}_{k_{Sh1}}$
(10,10,10,50)	0.003619	0.000193	0.000739	0.000198	0.074638	0.078114	$\hat{R}_{k_{Sh1}}$
(30,50,100,50)	0.001753	0.000109	0.000334	0.000112	0.069858	0.016339	$\hat{R}_{k_{Sh1}}$
(50,100,30,50)	0.001593	0.000108	0.000316	0.000110	0.067410	0.016323	$\hat{R}_{k_{Sh1}}$
(100,30,50,50)	0.001306	0.000093	0.000256	0.000095	0.071525	0.016312	$\hat{R}_{k_{Sh1}}$
(10,10,10,100)	0.003231	0.000164	0.000606	0.000168	0.087735	0.179510	$\hat{R}_{k_{Sh1}}$
(30,50,100,100)	0.001194	0.000091	0.000243	0.000093	0.064550	0.000228	$\hat{R}_{k_{Sh1}}$
(50,100,30,100)	0.001098	0.000084	0.000219	0.000086	0.066490	0.000218	$\hat{R}_{k_{Sh1}}$
(100,30,50,100)	0.000777	0.000078	0.000165	0.000079	0.065766	0.000256	$\hat{R}_{k_{Sh1}}$
(30,30,30,30)	0.002472	0.000137	0.000469	0.000140	0.066428	0.009092	$\hat{R}_{k_{Sh1}}$
(50,50,50,50)	0.001589	0.000102	0.000306	0.000104	0.064434	0.009130	$\hat{R}_{k_{Sh1}}$
(100,100,100,100)	0.000726	0.000069	0.000148	0.000070	0.068265	0.009090	$\hat{R}_{k_{Sh1}}$

C. The model $R_{(s,k)}$:

This model includes estimation of for EWD with (s,k)=(2,3), (2,4) and (3,4)

When the known parameters are (α_1, α_2)=(2,4), and (4,2), as well as known parameter θ=3 for each case.

The results of this model have put it in the tables below:

Table (4-25): Shown estimation value of $R_{(s,k)}$, and $\widehat{R}_{(s,k)}$ when s=2 & k=3 , α_1=2, α_2=4, and θ=3

(n,m)	$R_{(s,k)}$	$\widehat{R}_{(s,k)MLE}$	$\widehat{R}_{(s,k)Sh1}$	$\widehat{R}_{(s,k)Sh2}$	$\widehat{R}_{(s,k)Th}$	$\widehat{R}_{(s,k)LS}$	$\widehat{R}_{(s,k)RSS}$
(10,10)	**0.300000**	**0.307234**	**0.299470**	**0.300568**	**0.299487**	**0.251414**	**0.294016**
(10,30)	**0.300000**	**0.615900**	**0.299873**	**0.362317**	**0.301685**	**0.074871**	**0.632114**
(10,50)	**0.300000**	**0.737024**	**0.299882**	**0.376514**	**0.302058**	**0.052877**	**0.753338**
(10,100)	**0.300000**	**0.852828**	**0.299497**	**0.387354**	**0.301930**	**0.016355**	**0.864906**
(30,10)	**0.300000**	**0.089494**	**0.301271**	**0.186780**	**0.296786**	**0.675384**	**0.070013**
(30,30)	**0.300000**	**0.303634**	**0.300208**	**0.300815**	**0.300223**	**0.144981**	**0.293619**
(30,50)	**0.300000**	**0.445046**	**0.299532**	**0.334611**	**0.300563**	**0.220380**	**0.445574**
(30,100)	**0.300000**	**0.641729**	**0.299936**	**0.366283**	**0.301845**	**0.044228**	**0.642825**
(50,10)	**0.300000**	**0.040140**	**0.299543**	**0.124472**	**0.289886**	**0.879577**	**0.031017**
(50,30)	**0.300000**	**0.180077**	**0.300171**	**0.252438**	**0.298656**	**0.335177**	**0.169282**
(50,50)	**0.300000**	**0.301163**	**0.299638**	**0.299724**	**0.299633**	**0.079645**	**0.293936**
(50,100)	**0.300000**	**0.502280**	**0.300196**	**0.345830**	**0.301549**	**0.108095**	**0.496693**
(100,10)	**0.300000**	**0.012960**	**0.300919**	**0.062278**	**0.279538**	**0.868436**	**0.009068**
(100,30)	**0.300000**	**0.072448**	**0.300009**	**0.172745**	**0.294050**	**0.653057**	**0.066214**
(100,50)	**0.300000**	**0.143031**	**0.299427**	**0.231925**	**0.296906**	**0.337029**	**0.136715**
(100,100)	**0.300000**	**0.301467**	**0.299805**	**0.300158**	**0.299812**	**0.225289**	**0.293595**

Table (4-26): Shown MSE values for $\hat{R}_{(s,k)}$ when s=2 & k=3 , α_1=2, α_2=4, θ=3, and $R_{(s,k)}$=0.30000

(n,m)	$\hat{R}_{(s,k)MLE}$	$\hat{R}_{(s,k)Sh1}$	$\hat{R}_{(s,k)Sh2}$	$\hat{R}_{(s,k)Th}$	$\hat{R}_{(s,k)LS}$	$\hat{R}_{(s,k)RSS}$	Best
(10,10)	**0.014510**	**0.000750**	**0.003065**	**0.000775**	**0.010566**	**0.000831**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,30)	**0.108198**	**0.000560**	**0.005847**	**0.000579**	**0.052690**	**0.110765**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,50)	**0.196101**	**0.000552**	**0.007945**	**0.000574**	**0.061851**	**0.205783**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,100)	**0.307526**	**0.000500**	**0.009582**	**0.000520**	**0.080730**	**0.319215**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,10)	**0.046512**	**0.000458**	**0.015064**	**0.000532**	**0.152318**	**0.052965**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,30)	**0.004997**	**0.000263**	**0.001030**	**0.000272**	**0.028850**	**0.000246**	$\mathbf{R}_{(s,k)RSS}$
(30,50)	**0.025702**	**0.000219**	**0.001979**	**0.000227**	**0.010266**	**0.021386**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,100)	**0.119754**	**0.000193**	**0.005112**	**0.000203**	**0.065790**	**0.117652**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,10)	**0.068001**	**0.000343**	**0.032242**	**0.000537**	**0.339584**	**0.072363**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,30)	**0.016572**	**0.000187**	**0.003088**	**0.000202**	**0.006448**	**0.017168**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,50)	**0.002865**	**0.000144**	**0.000573**	**0.000149**	**0.054989**	**0.000153**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,100)	**0.043334**	**0.000119**	**0.002514**	**0.000125**	**0.037914**	**0.038784**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,10)	**0.082453**	**0.000311**	**0.057119**	**0.000919**	**0.328643**	**0.084642**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,30)	**0.052248**	**0.000134**	**0.016840**	**0.000190**	**0.126220**	**0.054670**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,50)	**0.025503**	**0.000090**	**0.005079**	**0.000107**	**0.010334**	**0.026692**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,100)	**0.001336**	**0.000066**	**0.000261**	**0.000068**	**0.007150**	**0.000091**	$\hat{\mathbf{R}}_{(s,k)Sh1}$

Table (4-27): Shown estimation value of $R_{(s,k)}$, and $\widehat{R}_{(s,k)}$ when s=2 & k=3 , α_1=4, α_2=2, and θ=3

(n,m)	$R_{(s,k)}$	$\widehat{R}_{(s,k)MLE}$	$\widehat{R}_{(s,k)Sh1}$	$\widehat{R}_{(s,k)Sh2}$	$\widehat{R}_{(s,k)Th}$	$\widehat{R}_{(s,k)LS}$	$\widehat{R}_{(s,k)RSS}$
(10,10)	**0.685714**	**0.681971**	**0.687349**	**0.687573**	**0.687366**	**0.767945**	**0.690091**
(10,30)	**0.685714**	**0.871231**	**0.684807**	**0.733387**	**0.686357**	**0.184243**	**0.887485**
(10,50)	**0.685714**	**0.919958**	**0.684477**	**0.743024**	**0.686345**	**0.208016**	**0.931340**
(10,100)	**0.685714**	**0.959338**	**0.685073**	**0.751398**	**0.687162**	**0.037001**	**0.965135**
(30,10)	**0.685714**	**0.378822**	**0.686579**	**0.559832**	**0.682645**	**0.884442**	**0.359493**
(30,30)	**0.685714**	**0.681366**	**0.685649**	**0.684863**	**0.685631**	**0.636610**	**0.691262**
(30,50)	**0.685714**	**0.785444**	**0.685009**	**0.713136**	**0.685891**	**0.433780**	**0.798204**
(30,100)	**0.685714**	**0.885188**	**0.685658**	**0.737451**	**0.687301**	**0.366270**	**0.891471**
(50,10)	**0.685714**	**0.242645**	**0.686644**	**0.464720**	**0.678128**	**0.960079**	**0.220965**
(50,30)	**0.685714**	**0.546355**	**0.685078**	**0.638346**	**0.683724**	**0.543288**	**0.554361**
(50,50)	**0.685714**	**0.683347**	**0.685861**	**0.685451**	**0.685849**	**0.637006**	**0.691228**
(50,100)	**0.685714**	**0.819922**	**0.685751**	**0.722440**	**0.686920**	**0.481200**	**0.826016**
(100,10)	**0.685714**	**0.109142**	**0.686695**	**0.315483**	**0.667214**	**0.969280**	**0.093340**
(100,30)	**0.685714**	**0.347822**	**0.685362**	**0.543695**	**0.680113**	**0.885155**	**0.348498**
(100,50)	**0.685714**	**0.496933**	**0.685552**	**0.618960**	**0.683343**	**0.789847**	**0.503020**
(100,100)	**0.685714**	**0.684333**	**0.685509**	**0.685334**	**0.685508**	**0.664694**	**0.691194**

Table (4-28): Shown MSE values for $\hat{R}_{(s,k)}$ when s=2 & k=3 , α_1=4, α_2=2, θ=3, and $R_{(s,k)}$= 0.685714

(n,m)	$\hat{R}_{(s,k)MLE}$	$\hat{R}_{(s,k)Sh1}$	$\hat{R}_{(s,k)Sh2}$	$\hat{R}_{(s,k)Th}$	$\hat{R}_{(s,k)LS}$	$\hat{R}_{(s,k)RSS}$	Best
(10,10)	**0.010871**	**0.000577**	**0.002252**	**0.000595**	**0.007885**	**0.000623**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,30)	**0.036083**	**0.000302**	**0.003141**	**0.000309**	**0.268655**	**0.040771**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,50)	**0.055516**	**0.000272**	**0.004095**	**0.000278**	**0.231084**	**0.060355**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,100)	**0.075028**	**0.000235**	**0.005027**	**0.000243**	**0.420973**	**0.078081**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,10)	**0.104595**	**0.000457**	**0.020051**	**0.000534**	**0.039722**	**0.107038**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,30)	**0.003660**	**0.000181**	**0.000719**	**0.000187**	**0.006097**	**0.000174**	$\hat{\boldsymbol{R}}_{(s,k)RSS}$
(30,50)	**0.011611**	**0.000144**	**0.001191**	**0.000148**	**0.071215**	**0.012713**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,100)	**0.040255**	**0.000100**	**0.002973**	**0.000105**	**0.106612**	**0.042351**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,10)	**0.203144**	**0.000392**	**0.054171**	**0.000561**	**0.075601**	**0.216372**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,30)	**0.023660**	**0.000163**	**0.003175**	**0.000180**	**0.041891**	**0.017432**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,50)	**0.002322**	**0.000112**	**0.000454**	**0.000116**	**0.001041**	**0.000118**	$\mathbf{R}_{(s,k)Sh1}$
(50,100)	**0.018767**	**0.000075**	**0.001591**	**0.000079**	**0.044080**	**0.019710**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,10)	**0.334599**	**0.000377**	**0.142440**	**0.000960**	**0.080463**	**0.351006**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,30)	**0.117834**	**0.000159**	**0.021757**	**0.000219**	**0.040664**	**0.113870**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,50)	**0.038322**	**0.000098**	**0.005095**	**0.000113**	**0.014107**	**0.033491**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,100)	**0.001141**	**0.000056**	**0.000224**	**0.000058**	**0.005266**	**0.000071**	$\hat{\mathbf{R}}_{(s,k)Sh1}$

Table (4-29): Shown estimation value of $R_{(s,k)}$, and $\widehat{R}_{(s,k)}$ when s=2 & k=4 , α_1=2, α_2=4, and θ=3

(n,m)	$R_{(s,k)}$	$\widehat{R}_{(s,k)MLE}$	$\widehat{R}_{(s,k)Sh1}$	$\widehat{R}_{(s,k)Sh2}$	$\widehat{R}_{(s,k)Th}$	$\widehat{R}_{(s,k)LS}$	$\widehat{R}_{(s,k)RSS}$
(10,10)	**0.40000**	**0.406299**	**0.400560**	**0.401522**	**0.400571**	**0.414676**	**0.391909**
(10,30)	**0.40000**	**0.701578**	**0.399553**	**0.465218**	**0.401522**	**0.045353**	**0.717129**
(10,50)	**0.40000**	**0.800801**	**0.399086**	**0.478165**	**0.401422**	**0.136655**	**0.816852**
(10,100)	**0.40000**	**0.891835**	**0.398006**	**0.488272**	**0.400623**	**0.059801**	**0.902586**
(30,10)	**0.40000**	**0.140953**	**0.401170**	**0.268895**	**0.396296**	**0.632153**	**0.113632**
(30,30)	**0.40000**	**0.400099**	**0.399631**	**0.399552**	**0.399628**	**0.568324**	**0.393196**
(30,50)	**0.40000**	**0.546205**	**0.398734**	**0.435994**	**0.399853**	**0.100903**	**0.548431**
(30,100)	**0.40000**	**0.725815**	**0.400360**	**0.470192**	**0.402431**	**0.068933**	**0.725929**
(50,10)	**0.40000**	**0.068639**	**0.399573**	**0.191134**	**0.389261**	**0.805181**	**0.053683**
(50,30)	**0.40000**	**0.259107**	**0.399792**	**0.346286**	**0.398147**	**0.497372**	**0.248017**
(50,50)	**0.40000**	**0.401989**	**0.400166**	**0.400539**	**0.400168**	**0.308320**	**0.392952**
(50,100)	**0.40000**	**0.600130**	**0.399760**	**0.447918**	**0.401228**	**0.293959**	**0.597158**
(100,10)	**0.40000**	**0.022842**	**0.400313**	**0.100443**	**0.376551**	**0.925445**	**0.016836**
(100,30)	**0.40000**	**0.117258**	**0.399983**	**0.251818**	**0.393463**	**0.757374**	**0.108422**
(100,50)	**0.40000**	**0.215764**	**0.400163**	**0.324593**	**0.397430**	**0.541421**	**0.205766**
(100,100)	**0.40000**	**0.400018**	**0.400201**	**0.400085**	**0.400202**	**0.566140**	**0.393265**

Table (4-30): Shown MSE values for $\widehat{R}_{(s,k)}$ when s=2 & k=4 , α_1=2, α_2=4, θ=3, and $R_{(s,k)}$=0.40000

(n,m)	$\widehat{R}_{(s,k)MLE}$	$\widehat{R}_{(s,k)Sh1}$	$\widehat{R}_{(s,k)Sh2}$	$\widehat{R}_{(s,k)Th}$	$\widehat{R}_{(s,k)LS}$	$\widehat{R}_{(s,k)RSS}$	Best
(10,10)	**0.016877**	**0.000904**	**0.003708**	**0.000936**	**0.012371**	**0.001066**	$\widehat{R}_{(s,k)Sh1}$
(10,30)	**0.097576**	**0.000694**	**0.006445**	**0.000716**	**0.127157**	**0.100932**	$\widehat{R}_{(s,k)Sh1}$
(10,50)	**0.164065**	**0.000658**	**0.008203**	**0.000678**	**0.075736**	**0.173943**	$\widehat{R}_{(s,k)Sh1}$
(10,100)	**0.242941**	**0.000542**	**0.009636**	**0.000554**	**0.128930**	**0.252642**	$\widehat{R}_{(s,k)Sh1}$
(30,10)	**0.071111**	**0.000512**	**0.020448**	**0.000602**	**0.058749**	**0.082152**	$\widehat{R}_{(s,k)Sh1}$
(30,30)	**0.005400**	**0.000292**	**0.001107**	**0.000301**	**0.061584**	**0.000283**	$R_{(s,k)RSS}$
(30,50)	**0.025637**	**0.000261**	**0.002150**	**0.000268**	**0.092480**	**0.022212**	$\widehat{R}_{(s,k)Sh1}$
(30,100)	**0.108207**	**0.000226**	**0.005650**	**0.000238**	**0.111509**	**0.106317**	$\widehat{R}_{(s,k)Sh1}$
(50,10)	**0.110919**	**0.000403**	**0.046092**	**0.000632**	**0.166508**	**0.119970**	$\widehat{R}_{(s,k)Sh1}$
(50,30)	**0.023149**	**0.000213**	**0.003958**	**0.000233**	**0.015605**	**0.023223**	$\widehat{R}_{(s,k)Sh1}$
(50,50)	**0.003550**	**0.000177**	**0.000708**	**0.000183**	**0.009752**	**0.000192**	$\widehat{R}_{(s,k)Sh1}$
(50,100)	**0.042329**	**0.000153**	**0.002796**	**0.000159**	**0.020153**	**0.038961**	$\widehat{R}_{(s,k)Sh1}$
(100,10)	**0.142399**	**0.000353**	**0.090984**	**0.001158**	**0.278559**	**0.146817**	$\widehat{R}_{(s,k)Sh1}$
(100,30)	**0.080980**	**0.000164**	**0.023052**	**0.000237**	**0.130708**	**0.085053**	$\widehat{R}_{(s,k)Sh1}$
(100,50)	**0.035378**	**0.000111**	**0.006277**	**0.000127**	**0.024601**	**0.037777**	$\widehat{R}_{(s,k)Sh1}$
(100,100)	**0.001715**	**0.000085**	**0.000339**	**0.000088**	**0.044552**	**0.000110**	$\widehat{R}_{(s,k)Sh1}$

Table (4-31): Shown estimation value of $R_{(s,k)}$, and $\widehat{R}_{(s,k)}$ when s=2 & k=4 , α_1=4, α_2=2, and θ=3

(n,m)	$R_{(s,k)}$	$\widehat{R}_{(s,k)MLE}$	$\widehat{R}_{(s,k)Sh1}$	$\widehat{R}_{(s,k)Sh2}$	$\widehat{R}_{(s,k)Th}$	$\widehat{R}_{(s,k)LS}$	$\widehat{R}_{(s,k)RSS}$
(10,10)	**0.761904**	**0.747667**	**0.761519**	**0.759175**	**0.761487**	**0.673914**	**0.767082**
(10,30)	**0.761904**	**0.907778**	**0.761597**	**0.801242**	**0.762888**	**0.323210**	**0.918983**
(10,50)	**0.761904**	**0.943138**	**0.761141**	**0.808757**	**0.762680**	**0.514587**	**0.951098**
(10,100)	**0.761904**	**0.970843**	**0.761126**	**0.814357**	**0.762842**	**0.119738**	**0.975460**
(30,10)	**0.761904**	**0.475257**	**0.762012**	**0.651222**	**0.758698**	**0.941906**	**0.463461**
(30,30)	**0.761904**	**0.756802**	**0.761543**	**0.760725**	**0.761532**	**0.625135**	**0.766494**
(30,50)	**0.761904**	**0.844976**	**0.762375**	**0.786058**	**0.763121**	**0.598051**	**0.850925**
(30,100)	**0.761904**	**0.917282**	**0.761593**	**0.803719**	**0.762948**	**0.347763**	**0.922068**
(50,10)	**0.761904**	**0.332848**	**0.762635**	**0.567287**	**0.755723**	**0.978823**	**0.310347**
(50,30)	**0.761904**	**0.640858**	**0.761708**	**0.722452**	**0.760590**	**0.801723**	**0.649154**
(50,50)	**0.761904**	**0.757446**	**0.761455**	**0.760623**	**0.761438**	**0.823898**	**0.766767**
(50,100)	**0.761904**	**0.867499**	**0.761639**	**0.791272**	**0.762595**	**0.584677**	**0.873118**
(100,10)	**0.761904**	**0.169103**	**0.762474**	**0.415978**	**0.746432**	**0.944001**	**0.146969**
(100,30)	**0.761904**	**0.451273**	**0.762136**	**0.640760**	**0.757835**	**0.928136**	**0.451163**
(100,50)	**0.761904**	**0.595848**	**0.761885**	**0.705446**	**0.760045**	**0.901543**	**0.602799**
(100,100)	**0.761904**	**0.760103**	**0.761719**	**0.761406**	**0.761715**	**0.779137**	**0.766413**

Table (4-32): Shown MSE values for $\hat{R}_{(s,k)}$ when s=2 & k=4 , α_1=4, α_2=2, θ=3, and $R_{(s,k)}$=**0.761904**

(n,m)	$\hat{R}_{(s,k)MLE}$	$\hat{R}_{(s,k)Sh1}$	$\hat{R}_{(s,k)Sh2}$	$\hat{R}_{(s,k)Th}$	$\hat{R}_{(s,k)LS}$	$\hat{R}_{(s,k)RSS}$	Best
(10,10)	**0.008499**	**0.000405**	**0.001618**	**0.000418**	**0.016350**	**0.000427**	$\hat{R}_{(s,k)Sh1}$
(10,30)	**0.022280**	**0.000211**	**0.002147**	**0.000218**	**0.195409**	**0.024708**	$\hat{R}_{(s,k)Sh1}$
(10,50)	**0.033196**	**0.000183**	**0.002699**	**0.000187**	**0.075270**	**0.035805**	$\hat{R}_{(s,k)Sh1}$
(10,100)	**0.043743**	**0.000156**	**0.003208**	**0.000160**	**0.413675**	**0.045609**	$\hat{R}_{(s,k)Sh1}$
(30,10)	**0.092224**	**0.000265**	**0.015321**	**0.000318**	**0.032651**	**0.089734**	$\hat{R}_{(s,k)Sh1}$
(30,30)	**0.002599**	**0.000123**	**0.000503**	**0.000127**	**0.027046**	**0.000122**	$\hat{R}_{(s,k)RSS}$
(30,50)	**0.007894**	**0.000098**	**0.000874**	**0.000102**	**0.031084**	**0.007963**	$\hat{R}_{(s,k)Sh1}$
(30,100)	**0.024405**	**0.000068**	**0.001943**	**0.000071**	**0.174943**	**0.025661**	$\hat{R}_{(s,k)Sh1}$
(50,10)	**0.192445**	**0.000258**	**0.042502**	**0.000371**	**0.047248**	**0.204394**	$\hat{R}_{(s,k)Sh1}$
(50,30)	**0.018179**	**0.000113**	**0.002236**	**0.000124**	**0.002697**	**0.012866**	$\hat{R}_{(s,k)Sh1}$
(50,50)	**0.001525**	**0.000074**	**0.000293**	**0.000076**	**0.041739**	**0.000080**	$\hat{R}_{(s,k)Sh1}$
(50,100)	**0.011556**	**0.000046**	**0.001002**	**0.000048**	**0.034354**	**0.012382**	$\hat{R}_{(s,k)Sh1}$
(100,10)	**0.355610**	**0.000262**	**0.126153**	**0.000678**	**0.037734**	**0.378345**	$\hat{R}_{(s,k)Sh1}$
(100,30)	**0.100123**	**0.000095**	**0.015790**	**0.000129**	**0.028450**	**0.096729**	$\hat{R}_{(s,k)Sh1}$
(100,50)	**0.029942**	**0.000069**	**0.003684**	**0.000080**	**0.019944**	**0.025407**	$\hat{R}_{(s,k)Sh1}$
(100,100)	**0.000720**	**0.000035**	**0.000140**	**0.000037**	**0.011507**	**0.000045**	$\hat{R}_{(s,k)Sh1}$

Table (4-33): Shown estimation value of $R_{(s,k)}$, and $\widehat{R}_{(s,k)}$ when s=3& k=4 , α_1=2, α_2=4, and θ=3

(n,m)	$R_{(s,k)}$	$\widehat{R}_{(s,k)MLE}$	$\widehat{R}_{(s,k)Sh1}$	$\widehat{R}_{(s,k)Sh2}$	$\widehat{R}_{(s,k)Th}$	$\widehat{R}_{(s,k)LS}$	$\widehat{R}_{(s,k)RSS}$
(10,10)	**0.2000**	**0.219593**	**0.201034**	**0.204687**	**0.201086**	**0.192554**	**0.194045**
(10,30)	**0.2000**	**0.528642**	**0.19952**	**0.259132**	**0.201190**	**0.032576**	**0.547621**
(10,50)	**0.2000**	**0.669854**	**0.199587**	**0.272857**	**0.201580**	**0.035161**	**0.690376**
(10,100)	**0.2000**	**0.811693**	**0.199286**	**0.283069**	**0.201511**	**0.001478**	**0.827573**
(30,10)	**0.2000**	**0.038518**	**0.200518**	**0.104537**	**0.196465**	**0.373333**	**0.026225**
(30,30)	**0.2000**	**0.205556**	**0.200330**	**0.201063**	**0.200336**	**0.247670**	**0.194053**
(30,50)	**0.2000**	**0.344286**	**0.199426**	**0.232476**	**0.200383**	**0.047300**	**0.342197**
(30,100)	**0.2000**	**0.553778**	**0.198916**	**0.260474**	**0.200638**	**0.098344**	**0.560822**
(50,10)	**0.2000**	**0.013541**	**0.201464**	**0.062877**	**0.193006**	**0.737473**	**0.007946**
(50,30)	**0.2000**	**0.100570**	**0.200325**	**0.158308**	**0.198951**	**0.621764**	**0.090772**
(50,50)	**0.2000**	**0.204738**	**0.200536**	**0.201369**	**0.200546**	**0.170772**	**0.194091**
(50,100)	**0.2000**	**0.400020**	**0.199719**	**0.241796**	**0.200947**	**0.107772**	**0.397427**
(100,10)	**0.2000**	**0.002393**	**0.200149**	**0.022484**	**0.180718**	**0.844758**	**0.001349**
(100,30)	**0.2000**	**0.028321**	**0.200148**	**0.094151**	**0.194766**	**0.561053**	**0.023989**
(100,50)	**0.2000**	**0.072126**	**0.199586**	**0.140417**	**0.197278**	**0.374189**	**0.067256**
(100,100)	**0.2000**	**0.202708**	**0.200484**	**0.200908**	**0.200491**	**0.284482**	**0.194093**

Table (4-34): Shown MSE values for $\hat{R}_{(s,k)}$ when s=3 & k=4 , α_1=2, α_2=4, θ=3, and $R_{(s,k)}$=0.2000

(n,m)	$\hat{R}_{(s,k)MLE}$	$\hat{R}_{(s,k)Sh1}$	$\hat{R}_{(s,k)Sh2}$	$\hat{R}_{(s,k)Th}$	$\hat{R}_{(s,k)LS}$	$\hat{R}_{(s,k)RSS}$	Best
(10,10)	**0.013026**	**0.000680**	**0.002680**	**0.000701**	**0.006464**	**0.000739**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,30)	**0.119369**	**0.000478**	**0.005492**	**0.000497**	**0.028083**	**0.121481**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,50)	**0.228185**	**0.000464**	**0.007387**	**0.000484**	**0.031239**	**0.240859**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(10,100)	**0.377097**	**0.000419**	**0.008850**	**0.000438**	**0.039415**	**0.393984**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,10)	**0.026787**	**0.000359**	**0.010338**	**0.000416**	**0.056512**	**0.030215**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,30)	**0.003763**	**0.000194**	**0.000746**	**0.000200**	**0.003180**	**0.000190**	$\mathbf{R}_{(s,k)RSS}$
(30,50)	**0.025586**	**0.000179**	**0.001744**	**0.000186**	**0.031930**	**0.020420**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(30,100)	**0.129328**	**0.000169**	**0.004355**	**0.000175**	**0.016834**	**0.130379**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,10)	**0.034881**	**0.000299**	**0.019478**	**0.000416**	**0.293339**	**0.036886**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,30)	**0.011216**	**0.000149**	**0.002348**	**0.000160**	**0.246962**	**0.011976**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,50)	**0.002350**	**0.000120**	**0.000464**	**0.000124**	**0.007443**	**0.000129**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(50,100)	**0.043174**	**0.000110**	**0.002180**	**0.000115**	**0.009981**	**0.039112**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,10)	**0.039053**	**0.000265**	**0.031673**	**0.000786**	**0.416099**	**0.039461**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,30)	**0.029628**	**0.000118**	**0.011591**	**0.000162**	**0.132374**	**0.030983**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,50)	**0.016821**	**0.000090**	**0.003896**	**0.000105**	**0.038184**	**0.017637**	$\hat{\mathbf{R}}_{(s,k)Sh1}$
(100,100)	**0.001214**	**0.000060**	**0.000238**	**0.000063**	**0.010234**	**0.000077**	$\hat{\mathbf{R}}_{(s,k)Sh1}$

Table (4-35): Shown estimation value of $R_{(s,k)}$, and $\widehat{R}_{(s,k)}$ when s=3 & k=4 , α_1=4, α_2=2, and θ=3

(n,m)	$R_{(s,k)}$	$\widehat{R}_{(s,k)MLE}$	$\widehat{R}_{(s,k)Sh1}$	$\widehat{R}_{(s,k)Sh2}$	$\widehat{R}_{(s,k)Th}$	$\widehat{R}_{(s,k)LS}$	$\widehat{R}_{(s,k)RSS}$
(10,10)	**0.609523**	**0.602488**	**0.610189**	**0.608960**	**0.610166**	**0.686199**	**0.615164**
(10,30)	**0.609523**	**0.836269**	**0.608895**	**0.666578**	**0.610728**	**0.211974**	**0.855956**
(10,50)	**0.609523**	**0.899128**	**0.609417**	**0.680432**	**0.611630**	**0.078722**	**0.911161**
(10,100)	**0.609523**	**0.947606**	**0.608884**	**0.688751**	**0.611374**	**0.015579**	**0.954827**
(30,10)	**0.609523**	**0.271872**	**0.609264**	**0.460948**	**0.604454**	**0.8168664**	**0.257978**
(30,30)	**0.609523**	**0.603633**	**0.608888**	**0.607899**	**0.608872**	**0.564084**	**0.616216**
(30,50)	**0.609523**	**0.732511**	**0.608868**	**0.643298**	**0.609933**	**0.602711**	**0.744351**
(30,100)	**0.609523**	**0.853553**	**0.609318**	**0.671176**	**0.611264**	**0.194573**	**0.860944**
(50,10)	**0.609523**	**0.148650**	**0.608419**	**0.358499**	**0.598167**	**0.683180**	**0.132544**
(50,30)	**0.609523**	**0.452310**	**0.609535**	**0.555551**	**0.607966**	**0.760478**	**0.459225**
(50,50)	**0.609523**	**0.608029**	**0.609329**	**0.609319**	**0.609335**	**0.688661**	**0.615774**
(50,100)	**0.609523**	**0.770065**	**0.609186**	**0.652259**	**0.610556**	**0.303646**	**0.779349**
(100,10)	**0.609523**	**0.050410**	**0.609290**	**0.215274**	**0.586475**	**0.971170**	**0.039453**
(100,30)	**0.609523**	**0.246022**	**0.609381**	**0.447394**	**0.603163**	**0.796836**	**0.245682**
(100,50)	**0.609523**	**0.398014**	**0.609341**	**0.532118**	**0.606732**	**0.287269**	**0.403266**
(100,100)	**0.609523**	**0.606108**	**0.608988**	**0.608285**	**0.608974**	**0.686903**	**0.616414**

Table (4-36): Shown MSE values for $\hat{R}_{(s,k)}$ when s=3 & k=4 , α_1=4, α_2=2, θ=3, and $R_{(s,k)}$=0.609523

(n,m)	$\hat{R}_{(s,k)_{MLE}}$	$\hat{R}_{(s,k)_{Sh1}}$	$\hat{R}_{(s,k)_{Sh2}}$	$\hat{R}_{(s,k)_{Th}}$	$\hat{R}_{(s,k)LS}$	$\hat{R}_{(s,k)RSS}$	Best
(10,10)	**0.015156**	**0.000789**	**0.003181**	**0.000815**	**0.035769**	**0.000831**	$\hat{R}_{(s,k)_{Sh1}}$
(10,30)	**0.054141**	**0.000428**	**0.004567**	**0.000441**	**0.164102**	**0.060830**	$\hat{R}_{(s,k)_{Sh1}}$
(10,50)	**0.084965**	**0.000376**	**0.006202**	**0.000390**	**0.289069**	**0.091023**	$\hat{R}_{(s,k)_{Sh1}}$
(10,100)	**0.114598**	**0.000336**	**0.007434**	**0.000350**	**0.352798**	**0.119244**	$\hat{R}_{(s,k)_{Sh1}}$
(30,10)	**0.124218**	**0.000603**	**0.027290**	**0.000724**	**0.047863**	**0.124156**	$\hat{R}_{(s,k)_{Sh1}}$
(30,30)	**0.004878**	**0.000263**	**0.001002**	**0.000271**	**0.076640**	**0.000237**	$\hat{R}_{(s,k)RSS}$
(30,50)	**0.017703**	**0.000197**	**0.001796**	**0.000204**	**0.015200**	**0.018273**	$\hat{R}_{(s,k)_{Sh1}}$
(30,100)	**0.060249**	**0.000135**	**0.004217**	**0.000141**	**0.174607**	**0.063237**	$\hat{R}_{(s,k)_{Sh1}}$
(50,10)	**0.215797**	**0.000520**	**0.066992**	**0.000765**	**0.101759**	**0.227919**	$\hat{R}_{(s,k)_{Sh1}}$
(50,30)	**0.029717**	**0.000223**	**0.004109**	**0.000244**	**0.029624**	**0.022813**	$\hat{R}_{(s,k)_{Sh1}}$
(50,50)	**0.002960**	**0.000143**	**0.000581**	**0.000148**	**0.009312**	**0.000154**	$\hat{R}_{(s,k)_{Sh1}}$
(50,100)	**0.0268399**	**0.000102**	**0.002144**	**0.000106**	**0.094988**	**0.028878**	$\hat{R}_{(s,k)_{Sh1}}$
(100,10)	**0.313392**	**0.000529**	**0.159944**	**0.001391**	**0.102762**	**0.324946**	$\hat{R}_{(s,k)_{Sh1}}$
(100,30)	**0.135203**	**0.000193**	**0.027988**	**0.000267**	**0.037017**	**0.132511**	$\hat{R}_{(s,k)_{Sh1}}$
(100,50)	**0.047745**	**0.000130**	**0.006808**	**0.000151**	**0.124563**	**0.042668**	$\hat{R}_{(s,k)_{Sh1}}$
(100,100)	**0.001588**	**0.000077**	**0.000312**	**0.000080**	**0.030976**	**0.000095**	$\hat{R}_{(s,k)_{Sh1}}$

Chapter Five

The Conclusions

CHAPTER FIVE

THE CONCLUSIONS AND THE RESULTS

5-1 Introduction:-

This chapter brings the main summary and most important conclusions and recommendations according to simulation results which have been discussed in chapter four.

5-2 Conclusions with Respect to R.

From the numerical results in the previous chapter, one can find the proposed shrinkage estimation method using constant shrinkage weight function $(\hat{R}_{sh2})$ which depends on unbiased estimator and prior estimate (Moment Method) as a linear combination, performance good behavior and it is the best estimator than the others in the sense of MSE.

i. **When n=m, we show**:

Order	1	2	3
Estimation method	Sh2	Th	Sh1

ii. **When n<m, we note:**

Order	1	2	3
Estimation method	Sh2	Th	Sh1

iii. When n>m, one may show:

Order	1	2	3
Estimation method	Sh2	Th	Sh1

For any (n,m) some of the proposed estimator (mle, sh_1, and sh_2) are decreasing and the other methods are vibration.

5-3 Conclusions with Respect to R_k:

For estimation reliability for K^{th} components parallel system of the stress-strength model which are subjected to a common stress, when the stress and strength follow the Exponentiated Weibull Distribution (EWD).

From the tables (4-13), …, (4-24) which contain (k=3) component for strength and one stress component in chapter four, one can find the proposal shrinkage estimation method using shrinkage weight factor as function of n_i and m ($\hat{R}_{sh1}$), performance good behavior and it is the best estimator than the others in the sense of MSE except model (1) when n=m=10,30 and 50 as follows:

I. Model (1): when $\alpha_i \in N$; $\alpha_1=\alpha_2=\alpha_3=\alpha_4=2$

i. When n=m=10,30 and 50

Order	1	2	3
Estimation method	RSS	Sh1	Th

ii. When n=m=100

Order	1	2	3
Estimation method	Sh1	Th	RSS

iii. When n=30,50 100 for any m

Order	1	2	3
Estimation method	Sh1	Th	Sh2

II. Model (2): when $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_4$ for any $\alpha_i \in N$;i=1,2,3,4

i. For any n_i when m=10,30, and 50, and when n_i =m we note:

Order	1	2	3
Estimation method	Sh1	Th	Sh2

ii. When (n_1,n_2,n_3,m) =(50,100,30,100) and (100,30 50,100)

Order	1	2	3
Estimation method	Sh1	Th	RSS

iii. When (n_1,n_2,n_3,m) =(10,10,10,100) and (30,50,100,100), we show:

Order	1	2	3
Estimation method	Sh1	Th	Sh2

III. In model(3): when $\alpha_1, \alpha_2, \alpha_3 > \alpha_4$

For any n_i and m ;i=1,2,3, we conclude:

Order	1	2	3
Estimation method	Sh1	Th	Sh2

IV. Model(4): While in case $\alpha_i \notin N$, we note

Order	1	2	3
Estimation method	Sh1	Th	Sh2

V. Model(5): when $\alpha_1, \alpha_2, \alpha_3 \notin N$ and $\alpha_4 \in N$, one may show:

Order	1	2	3
Estimation method	Sh1	Th	Sh2

VI. Model (6): when $\alpha_1, \alpha_2, \alpha_3 \in N$ and $\alpha_4 \notin N$, we note:

i. When m=10,30 and 50 for any n_i , and (n_1,n_2,n_3,m)=(10,0,10,100), so (100,30,50,100), we note:

Order	1	2	3
Estimation method	Sh1	Th	Sh2

ii. Either in case $(n_1,n_2,n_3,m)=(50,100,30,100)$ and $(30,50,100,100)$, we show:

Order	1	2	3
Estimation method	Sh1	Th	RSS

5-4 Conclusions with Respect to $R_{(s,k)}$:

In this section, the Mean Square Error (MSE) for reliability estimation of $R_{(s,k)}$ model for the Exponentiated Weibull distribution is held using the shrinkage weight factor as a function of sizes n and m ($\hat{R}_{sh1}$) for three case When $\alpha_1 < \alpha_2$ and when $\alpha_1 > \alpha_2$.

5-4-4 Conclusions with Respect to $R_{(s,k)}$ when s=2 and k=3:

i. When n=m=10,50,100, we show:

Order	1	2	3
Estimation method	Sh1	Th	RSS

ii. When n=m=30, one may show:

Order	1	2	3
Estimation method	RSS	Sh1	Th

iii. When $n \neq m$, we note:

Order	1	2	3
Estimation method	Sh1	Th	Sh2

5-4-5 Conclusions with Respect to $R_{(s,k)}$ when s=2 and k=4:

i. When n=m=10,50,100 , we note:

Order	1	2	3
Estimation method	Sh1	Th	RSS

ii. When n=m=30, we show:

Order	1	2	3
Estimation method	RSS	Sh1	Th

iii. When (n,m)=(100 ,10) one may show:

Order	1	2	3
Estimation method	Sh1	Th	LS

iv. When n ≠ m, one may note:

Order	1	2	3
Estimation method	Sh1	Th	Sh2

5-4-6 Conclusions with Respect to $R_{(s,k)}$ when s=3 and k=4:

i. When n=m=10,50,100 , one may note:

Order	1	2	3
Estimation method	Sh1	Th	RSS

ii. When n=m=30, one may show:

Order	1	2	3
Estimation method	RSS	Sh1	Th

iii. When $n \neq m$, we show:

Order	1	2	3
Estimation method	Sh1	Th	Sh2

iv. When (n,m)=(100 ,10) when $\alpha_1 > \alpha_2$, we note:

Order	1	2	3
Estimation method	Sh1	Th	LS

5-5 Recommendations:-

Based on the conclusions reached, the main recommendations can be summarized as follows.

1. Recommends using the shrinkage method to estimate the reliability system in the case of stress and strength since it's at most performance good behavior and be the best when compared with the other methods.
2. Recommends one can find the proposed shrinkage estimation method using constant shrinkage weight function ($\hat{R}_{sh2}$) when the system contains one component.
3. Recommends when the system contains k^{th} components, one can find the proposed shrinkage estimation method using shrinkage weight factor as function of n_i and m ($\hat{R}_{sh1}$), performance good behavior and it is the best estimator than the others in the sense of MSE.
4. When system contains multi-components $R_{(s,k)}$, it is recommended using the shrinkage weight factor as a function of sizes n and m ($\hat{R}_{sh1}$) for three cases when (s,k)=(2,3), (2,4) and (3,4).
5. Recommends when n=m=30 using the Ranked Set Sampling method to estimate the reliability system $R_{(s,k)}$ contains multi-components.
6. Recommends when n=m=10,30 and 50, using the Ranked Set Sampling method to estimate the reliability system R_k contains K_{th} components.

5-6 Future Work:

1- As a future work, one can use the Bayesian methods as an estimation method in the stress-strength model for three cases.

2- One can estimate the reliability system which contains K^{th} series components in cases model.

3- It is possible to find the formula and estimate the reliability system for redundancy system.

4- It is possible to find the formula and estimate the reliability of two-part multicomponent stress-strength system model.

5- It is possible to discuss the case when the two shape parameters of Exponentiated Weibull Distribution (EWD).

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Appendix

APPENDIX

Median procedure:

We explain the steps of the median procedure as below: see;[67].

Step 1:-

The population median of the random variable X which follows EWD can be obtained as below:

$F(x;\alpha,\theta)=\frac{1}{2}$.

From the equation (2-2), we get:

$$(1-e^{-X^{\theta}})^{\alpha}=\frac{1}{2}$$

$$\left(1-e^{-X^{\theta}}\right)=\frac{1}{\sqrt[\alpha]{2}}$$

$$e^{-X^{\theta}}=1-\frac{1}{\sqrt[\alpha]{2}}$$

$$X^{\theta}=-\ln\left(1-\frac{1}{\sqrt[\alpha]{2}}\right)$$

$$X_{median}=\sqrt[\theta]{-\ln\left(1-\frac{1}{\sqrt[\alpha]{2}}\right)}$$

Step 2:-

In this stage, equating the population median (X_{median}) with the sample median (x_{median}) , we get:

$$X_{median}=x_{median}$$

$$\sqrt[\theta]{-\ln\left(1-\frac{1}{\sqrt[\alpha]{2}}\right)}=x_{median}$$

$$-(x_{median})^{\theta} = ln\left(1 - \frac{1}{\sqrt[\alpha]{2}}\right)$$

$$e^{-x_{median}{}^{\theta}} = 1 - \frac{1}{\sqrt[\alpha]{2}}$$

$$1 - e^{-x_{median}{}^{\theta}} = \frac{1}{\sqrt[\alpha]{2}}$$

$$(1 - e^{-x_{median}{}^{\theta}})^{\alpha} = \frac{1}{2}$$

$$\alpha \ln\left(1 - e^{-x_{median}{}^{\theta}}\right) = \ln\frac{1}{2}$$

$$\alpha_0 = \frac{ln\frac{1}{2}}{ln\left(1 - e^{-x_{median}{}^{\theta}}\right)}$$

Now, the estimation of α_i using Moment method will be as below:

$$\hat{\alpha}_{i_{mom}} = \frac{\bar{x}_i}{\sum_{i=0}^{\infty}\frac{\Gamma(\alpha_0)}{i!\Gamma(\alpha_o - i)}(-1)^i(i+1)^{-\frac{r}{\theta}-1}\Gamma\left(\frac{r}{\theta}+1\right)} \quad ;i=1,2,\ldots,k+1.$$

And, the estimation of α_i using Ranked Set Sampling method will be as below:

$$\hat{\alpha}_{i_{RSS}} = \frac{n_i}{\sum_{j=1}^{n_i}(n_i - j)\frac{\left(1-e^{-(x_{i_{(j)}})^{\theta}}\right)^{\alpha_{0i}} Ln\left(1-e^{-(x_{i_{(j)}})^{\theta}}\right)}{\left[1-\left(1-e^{-(x_{i_{(j)}})^{\theta}}\right)^{\alpha_{0i}}\right]} - \sum_{j=1}^{n_i} jLn\left(1-e^{-(x_{i_{(j)}})^{\theta}}\right)}$$

;i=1,2,…,k+1.

المستخلص

تتضمن الرسالة ثلاث محاور:

المحور الاول: يتضمن ايجاد الصيغ الرياضية لمعوليه نظام الاجهاد-المتانة في الحالات الثلاثة المقترحة التالية.

الحالة الاولى يكون النظام مكون من مركبة واحدة تمثل متغير المتانة (X) التي وضعت تحت متغير الاجهاد (Y) عندما يتبعان توزيع ويبل الاسي.

الحالة الثانية يكون فيها النظام مكون من مركبات(X_k) عددها (k) مربوطة على التوازي التي تمثل المتانة التي وضعت تحت اجهاد مشترك (Y) بحيث يتبع كل من متغيرا الاجهاد والمتانة توزيع ويبل الاسي.

الحالة الثالثة عندها يكون النظام متعدد المركبات (S-out of-K) حيث يعمل النظام اذا عملت s من k لمركبات النظام التي وضعت تحت اجهاد مشترك (Y) التي يتبعان توزيع ويبل الاسي.

المحور الثاني: فيتعلق بتقدير دالة المعولية لنظام الاجهاد- المتانة للحالات الثلاثة المذكورة اعلاه باستخدام طرائق التقدير مختلفة منها:-

طريقة الامكان الاعظم MLE، طريقة العزومMOM، طرائق التقلص (Sh1, Sh2, and Th)، طريقة المربعات الصغرى LS،و طريقة معاينة مجموعة الرتب RSS.

فيما يتعلق بالمحور الثالث: يتضمن مقارنة الطرائق المقترحة اعلاه و الحالات الثلاثة المذكورة انفا باستخدام محاكاة مونت – كارلو بالاعتماد على المؤشر الاحصائي متوسط مربعات الخطأ(MSE).

وتبين من خلال نتائج المحاكاة ان طرائق التقلص هي الافضل لأنها تحقق أقل متوسط مربعات خطأ (MSE) مقارنة بالطرائق الاخرى.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بغداد
كلية التربية /أبن الهيثم

حول تقدير المعولية لنظام الاجهاد-المتانة

رسالـــة

مقدمة إلى كلية التربية للعلوم الصرفة /أبن الهيثم
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