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A STUDY OF MODULES RELATED WITH T-SEMISIMPLE MODULES

A Thesis

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By

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

إِنَّمَا وَلِيُّكُمُ اللَّهُ وَرَسُولُهُ وَالَّذِينَ آمَنُوا
الَّذِينَ يُقِيمُونَ الصَّلَاةَ وَيُؤْتُونَ الزَّكَاةَ
وَهُمْ رَاكِعُونَ

صدق الله العلي العظيم

سورة المائدة اية ٥٥

Dedication

To the man who has become a fixed mark of salvation.

To the man of humanity.

To the man who has awakened people from swoon and ignorance, and kindled the spark of faith often it went out.

To the man who was assassinated by the hands of injustice and tyranny because he always condemned and exposed them.

To the man whose lips never ceased to speak out and raise Supremes word of " God is Greatest"

Farhan

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Farhan

CERTIFICATION

I certify that this thesis was prepared under my supervision at the Department of Mathematics, College of Education for pure science Ibn Al-Haitham, University of Baghdad, as a partial fulfillment of the Requirements for the degree of Doctor in Mathematics.

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EXAMINATION COMMITTEE CERTIFICATION

We certify that we read this thesis entitled **"A Study of Modules Related with T-semisimple Modules "** and as examining committee examined the student **(Farhan Dakhil Shyaa)** in its contents and that in our opinion it is adequate for the partial fulfillment of the requirements for the degree of Doctor of Science in Mathematics

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List of Publications

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Contents

List of Symbols

Symbols	Title I would like
$N \leq M$	N is a submodule of M
$N < M$	N is a proper submodule of M
$N \ll M$	N is a small submodule of M
$N \leq_{ess} M$	N is an essential submodule of M
$N \leq_{tes} M$	N is t-essential submodule of M
$N \leq^{\oplus} M$	N is a direct summand of M
PIP	The pure intersection property
$M \simeq M'$	M isomorphic to M'
$\oplus$	Direct sum
Π	Direct product
$\circ$	Composition
$\mathrm{ann}_R(M)$	The annihilator of M in R
$\ker f$	The kernel of f
$\mathrm{Im} f$	The image of f
$\mathrm{Hom}_R(M, M')$	The set of all R-homomorphism from M into M'
$\bar{M}$	The quasi-injective hull (envelope) of M
$\mathrm{End}_R(M)$	The ring of endomorphism of M
$(N:_R M)$	The residual of N by M
$Rad(M)$	Radical of M
$N \leq_c M$	N is a closed submodule of M
$N \leq_{tc} M$	N is a t-closed submodule of M

Abstract

In 2013, the authors Asgari and Haghany introduced the concept of t-semisimple modules as a generalization of semisimple modules. Where ″an R-module M is called t-semisimple if for each $A \leq M$, there exists a direct summand B of M such that B is t-essential in A″. In fact the concept of t-essential is introduced by Asgari and Haghany in 2011 they said that a submodule A of ″an R-module M is t-essential in M(written $A \leq_{tes} M$) if whenever $A \cap C \leq Z_2(M), C \leq M$ implies $C \leq Z_2(M)$″ where $Z_2(M)$ is the second singular submodule of M. This dissertation is devoted for investigations the following:

- Extending the notions of t-semisimple modules to strongly t-semisimple modules.
- Generalizing the concepts t-semisimple modules, strongly t-semisimple modules in to FI-t-semisimple modules, purely t-semisimple modules, strongly FI-t-semisimple modules, strongly purely t-semisimple modules.
- Introducing various classes of modules related to types of t-semisimple modules and strongly t-semisimple modules, such as module satisfy strongly C_{11}-condition, strongly T_{11}-type modules, modules satisfy FI-C_{11}(strongly FI-C_{11})condition, FI-T_{11}(strongly FI-T_{11})-type modules, modules satisfy purely-C_{11}(strongly purely -C_{11})-condition and purely-T_{11}(strongly purely -T_{11}-type) modules.

Introduction

It is known that a submodule A of an R-module M is said to be essential in M (denoted by $A \leq_{ess} M$), if $A \cap W \neq (0)$ for every non-zero submodule W of M. Equivalently $A \leq_{ess} M$ if whenever $A \cap W=0$, then $W = 0$[23], [25], [26]. The concept of extending (also known as CS-module or module with C_1-condition) had been studied and generalized by several authors, (see [17],[34]). ″ A module M is called extending if for every submodule N of M there exists a direct summand $W(W \leq^{\oplus} M)$ such that $N \leq_{ess} W$ ″[17]. Equivalently ″ M is extending module if every closed submodule is a direct summand″, where a submodule C of M is called closed if $C \leq_{ess} C' \leq M$ implies that $C = C'$[23]. In 2011, Asgari and Haghany [6] introduced the notion of (t-essential) where ″ A submodule A of M is said to be t-essential in M (written $A \leq_{tes} M$)if for every submodule B of M ,$A \cap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$″[6] and ″ $Z_2(M)$ is the second singular (or Goldi torsion) defined by $Z(\frac{M}{Z(M)}) = \frac{Z_2(M)}{Z(M)}$ where $Z(M) = \{x \in M : xI = (0)$ for some essential ideal of $R\}$. In fact $Z(M) = \{x \in M : ann(x) \leq_{ess} R\}$ where $ann(x) = \{r \in R : xr = 0\}$ [23]. M is called singular (nonsingular) if $Z(M) = M(Z(M) = 0)$[23]. Note that $Z_2(M) = \{x \in M : xI = (0)$ for some t-essential ideal I of $R\}$.M is called Z_2-torsion if $Z_2(M) = M$ and a ring R is called right Z_2-torsion if $Z_2(R_R) = R_R$ [23].

Asgari and Haghany in [6] used the concept of t-essential submodule, to give the following: ″ A submodule C of an R-module M is called t-closed (denoted by $C \leq_{tc} M$) if whenever $C \leq_{tes} C' \leq M$ implies that $C = C'$[6]. The concepts of extending module, t-essential submodule, and t-closed submodule, led Asgari and Haghany in [6] to say that ″ a module M is t-extending if every t-closed submodule is a direct summand. Equivalently, M is t-extending if every submodule of M is t-essential in a direct summand ″[6]. It is known that a module is semisimple if every submodule is a direct summand [23].[25]. It is clear that every semisimple module is extending. The following observation: (A module M is semisimple if every submodule N of M contains a direct summand K such that $K \leq_{ess} N$). Motivated Asgari and Haghany in 2013[7] to introduced the notion of t-semisimple modules as a generalization of semisimple modules. They said that"A module M is t-semisimple if for every submodule N of M, there exists a direct summand K such that K $\leq_{\text{tes}}$ N" A ring R is right t-semisimple when the module R_R is t-semisimple [7]. Notice that for module:

Semisimple $\Longrightarrow$ t-semisimple $\Longrightarrow$ t-extending, but none of these implications is reversible (see [7, Examples 2.18]).

A comprehensive study of these modules and rings has been carried out by [7].

Our aims in this dissertation are to extend the notion of t-semisimple modules. So we introduce and study the concept: strongly t-semisimple modules. Also, we introduce many generalizations of t-semisimple modules and strongly t-semisimple modules. FI-t-semisimple modules, strongly FI-t-semisimple modules, purely t-semisimple modules and strongly purely t-semisimple modules. Beside these we investigate some types of modules which are related with above type of t-semisimple modules.

This thesis consists of four chapters. Chapter one is divided into five sections. In section one, some known concepts, propositions, Theorems and Examples which are useful in our work are recalled. Also, some new results are added (see, Theorem 1.1.51, Propositions 1.1.52, 1.1.53, 1.1.54, 1.1.55, 1.1.56, Corollaries 1.1.57, 1.1.58 and Proposition 1.1.59). In section two, the concept of strongly t-semisimple modules is introduced. An R-module is called strongly t-semisimple if for each submodule N of M there exists a fully invariant direct summand K such that K $\leq_{\text{tes}}$ N. It is clear that the class of t-semisimple modules contains the class of strongly t-semisimple; that is we have the following implication for modules.

Strongly t-semisimple $\Longrightarrow$ t-semisimple.

The reverse implication is not true in general (see, Remarks and Examples 1.2.2(8)). We investigate conditions which allow the converse to be hold (see, Proposition 1.2.5, Corollaries 1.2.13, 1.2.15 and Proposition 1.2.17). We provide several characterizations of strongly t-semisimple modules (see, Theorem 1.2.3 and Proposition 1.2.10). We show that the property of strongly t-semisimple is inherited by submodule (see, Proposition 1.2.7). However the direct sums do not inherit this property (see, Examples 1.2.4). But we explore condition which let a direct sum of strongly t-semisimple modules to be strongly t-semisimple (see, Theorem 1.2.9). Many other properties of strongly t-semisimple are presented.

In section three we focus on strongly t-extending module. In fact, as Asgari and Haghany in [7] proved that every t-semisimple module is t-extending.

. We verify analogous result that ″ every strongly t-semisimple module is strongly t-extending″ and the converse is not true in general (see, Theorem 1.3.5). Where ″ An R-module M is called strongly t-extending if every submodule is t-essential in a stable direct summand ″[20]. Some characterizations of strongly t-extending are given (see, Theorem 1.3.11). Beside these we have proved that every strongly extending is strongly t-extending, but not conversely (see, Proposition 1.3.7, Example 1.3.8), where ″ an R-module M is strongly extending if every submodule is essential in a stable direct summand ″[35]. The two concepts are equivalent under certain conditions (see, Propositions 1.3.9, 1.3.13). Also, under certain condition the direct sum of two strongly t-extending is strongly t-extending (see, Theorem 1.3.16) and we give a different proof of the property (strongly t-extending is inherited by a direct summand) which is given in [20] (see, Proposition 1.3.14).

Section four concerns with strongly t-semisimple rings. Several characterizations of commutative strongly t-semisimple ring are given. For examples (see, Propositions 1.4.8 and 1.4.9).

Section five deals with strongly t-Baer. The following implications for a modules hold:

t-semisimple $\Longrightarrow$ t-extending $\Longrightarrow$ t-Baer [7]. This motivate us to look for connections between strongly t-semisimple modules, strongly t-extending modules and strongly t-Baer modules, where ″An R-module is called t-Baer if $\mathrm{t_M(I) = \{m \in M : Im \leq Z_2(M)\}}$ is a direct summand of M for each left ideal I of End (M)″[6]. ″ An R-module is called strongly t-Baer if $\mathrm{t_M(I)}$ is a direct summand and fully invariant, for every left ideal I of S, where $S = End(M)$″[20]. :″A module M is called Baer if $r_M(I) \leq^{\oplus} M$ for every left ideal I of S where $S = End(M_R)$.″[33] and ″ A module M is called abelian Baer (or strongly Baer by some authors) if $r_M(I) \leq^{\oplus} M$ and fully invariant for every left ideal I of S where, $S = End(M_R)$"[34]. Many connections between these types of Baer modules are given (see, Remarks 1.5.5) several characterizations of strongly t-Baer module are presented (see, Theorem 1.5.8). Theorem 1.5.10, express the connections between the concepts strongly t-semisimple, strongly t-extending and strongly t-Baer modules. Beside these relationships between strongly t-extending, strongly t-Baer modules and strongly extending are given by (Theorem 1.5.12, Corollary 1.5.13). Then we introduce the concept strongly $\sum$-t-extending ring, where a ring R is called right strongly $\Sigma - t - extending$ if every free R- module is strongly t-extending.

Many equivalent statements for this concept is given, (see Theorem 1.5.16), Corollaries 1.5.18, 1.5.19). Finally (Theorem 1.5.20 and Corollary 1.5.21) present characterizations for strongly t-extending rings.

Chapter two consists of three sections. In section one FI-semisimple modules is introduced where an R-module M is called FI-semisimple if for each fully invariant submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{ess} N$. Clearly every semisimple module is FI-semisimple, but not conversely (see, Remarks and

Examples 2.1.3(1)). However they are equivalent under the class of duo modules (or multiplication modules). The homomorphic image of FI-semisimple need not be FI-semisimple (see, Remarks and Examples 2.1.3(7)). However it is true under certain condition (see, Corollary 2.1.5). Moreover, we prove the direct sum of two FI-semisimple modules is a FI-semisimple and the converse hold if each summand is a fully invariant submodules of M.

In section two, we provide a generalization of t-semisimple module, namely FI-t-semisimple, where An R-module M is called FI-t-semisimple if for each fully invariant submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. We observe the following every t-semisimple module is FI-t-semisimple module, and every FI-semisimple is FI-t-semisimple but the converses are not true in general (see, Remarks and Examples 2.2.2,(1),(2),(3)).

The property of FI-t-semisimple is inherited by fully invariant submodules (see, Proposition 2.2.3). We prove that every FI-t-semisimple is FI-t-extending if condition $(*)$ hold, where$(*)$: for an R-module M, a complement of $Z_2(M)$ is stable and ″an R-module M is called FI-t-extending module if every fully invariant t-closed submodule of M is a direct summand of M″[9].. Moreover, condition $(*)$ allows several statements to be equivalent with FI-t-semisimple module (see, Theorem 2.2.5(1⟷3⟷4), Proposition 2.2.10 and Theorem 2.2.12). Moreover other statements are equivalent to FI-t-semisimple module under certain condition are given (see, Proposition 2.2.11).

In section three, the notion of FI-t-semisimple module has been extended where an R-module M is called strongly FI-t-semisimple if for each fully invariant submodule N of M, there exists a fully invariant direct summand K such that $K \leq_{tes} N$. We have the following implications for a module

FI-semisimple $\Longrightarrow$ strongly FI-t-semisimple and strongly t-semisimple

⇒ strongly FI-t-semisimple

The reverse of each implication is not hold in general (see, Remarks and Examples 2.3.2).

We explore condition: any complement of any submodule of a module is stable which make FI-t-semisimple modules coincide with strongly FI-t-semisimple modules (see, Proposition 2.3.3). The property of strongly FI-t-semisimple is inherited by fully invariant direct summand (or nonsingular fully invariant submodule) (see, Proposition 2.3.6, Corollary 2.3.7). Then a direct sum of any two strongly FI-t-semisimple modules is strongly FI-t-semisimple module (see, Proposition 2.3.11).

Chapter three is divided into five sections. In section one another generalization of semisimple modules, which we called it purely semisimple is introduced and studied, where an R-module M is purely semisimple if for every pure submodule N of M there exists a direct summand K of M such that $K \leq_{ess} N$. Equivalently an R-module is purely semisimple if every pure submodule is a direct summand (see, Proposition 3.1.2). It is clear that every semisimple is purely semisimple, but the converse may be not true (see, Remarks and Examples 3.1.3(1)). Every pure simple module (or Noetherain projective or divisible module over a PID or prime injective) is purely semisimple module (see, Remarks and Examples 3.1.3(5), (6), (7), and (8)).If N is a pure submodule of purely semisimple module then N and $\frac{M}{N}$ are purely semisimple module (see, Remarks and Examples 3.1.3(3) and Proposition 3.1.4). Under certain conditions, we have that the direct sum of two purely semisimple modules is purely semisimple modules (see, Propositions 3.1.7, 3.1.8). Then we introduce the concept (M is N-purely projective) where M, N be any two R-modules (see, Definition 3.1.9). By using this concept, we get two equivalent statements for purely t-semisimple module (see, Theorem 3.1.10).

In section two, the notion of purely t-semisimple modules, which is a generalization of t-semisimple modules is given where an R- module M is called purely t-semisimple, if for each pure submodule N of M there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. We notice the following implications

Purely semisimple $\Longrightarrow$ t-semisimple $\Longrightarrow$ purely t-semisimple.

However purely t-semisimple module need not t-semisimple (see, Remarks and Examples 3.2.2 (1), (2) and the concept purely semisimple modules and t-semisimple modules are coincide in the class of nonsingular module (see Remarks and Examples 3.2.2(5)). Among many results in this section we have: The property of purely t-semisimple is inherited by pure submodule (see Proposition 3.2.3). We investigate conditions, under which the direct sum of two purely t-semisimple modules is purely t-semisimple (see, Propositions 3.2.5, 3.2.6). We get five equivalent statements for purely t-semisimple module if a complement of $Z_2(M)$ is a direct summand stable and M has PIP (pure intersection property), (see, Theorem 3.2.8). Another equivalent statement of purely t-semisimple module is given by Proposition 3.2.12.

In section three, as every t-semisimple is t-Baer, we hope to give an analogues statement for purely t-semisimple and so we investigate a concept (purely t-Baer module), where an R-module M is called purely t-Baer if for each ideal I of $End(M) = S$, $t_M(I)$ is a pure submodule of M. We study this type of modules; we have by (Theorem 3.3.4) a characterization of purely t-Baer module. We show that every purely t-extending module is purely t-Baer, (see, Proposition 3.3.5) and every purely t-semisimple module M with a complement of $Z_2(M)$ is pure is purely t-Baer. More properties related with purely t-Baer module are given by (Propositions 3.3.8, 3.3.9 and Corollary 3.3.10).

In section four, the notion of strongly purely t-semisimple module is introduced. An R-module M is called strongly purely t-semisimple if for each pure submodule

N of M, there exists a fully invariant direct summand K of M such that $K \leq_{tes} N$. Examples are provided to illustrate that the concept of purely t-semisimple doesn't imply strongly purely t-semisimple (see, Remark and Examples 3.4.2(1)). It is shown that every pure submodule of strongly purely t-semisimple module inherits the property. We obtain some characterizations of strongly purely t-semisimple module, under certain conditions (see, Theorem 3.4.6, Corollaries 3.4.7, 3.4.8). Then we focus on the direct sum of two strongly purely t-semisimple modules (see, Theorem 3.4.9 and Proposition 3.4.10).

In section five we introduce the notion of strongly purely t-Baer modules and looks for some connections between it and strongly purely t-semisimple modules. An R-module M is called strongly purely t-Baer if $\mathrm{t_M}(\mathrm{I})$ is a fully invariant pure submodule of M, for each left ideal I of S=EndM. We give a characterization of strongly purely t-Baer modules (see, Theorem 3.5.2). We put conditions on a module M to be strongly purely t-Baer (see, Propositions 3.5.3, 3.5.5). Next we prove that: For an R-module M such that a complement of $\mathrm{Z_2(M)}$ is a pure submodule in M. If M is strongly purely t-semisimple, then M is strongly t-Baer (see Theorem 3.5.6).

Chapter four is specified for introducing and studying certain types of modules which are related with the types of t-semisimple, t-extending modules, strongly extending, FI-t-extending modules. This chapter has six sections. In section one relevant concepts (modules satisfy C_{11}-condition and T_{11}-type modules) and results are recalled from [10], [38] where ″ An R-module M is said to be satisfy C_{11}-condition if every submodule of M has a complement which is a direct summand″ An R-module M said to be T_{11}-type module (or M satisfy T_{11}-condition) if every t-closed submodule has a complement which is a direct summand. A ring is said to be right T_{11}-type ring if R_R is a T_{11}-type module.″ Clearly every module satisfying C_{11}-condition and every t-extending module is T_{11}-type, but not conversely, see

[10]. Hence every t-semisimple modules is T_{11}-type module, but the converse may be not true (see, Remarks and Examples 4.1.1(4)). In section two, we introduce the notions of modules that satisfy strongly C_{11}-condition and strongly T_{11}-type modules, where an R-module M is said to be satisfy strongly C_{11}-condition if every submodule has a complement which is a fully invariant direct summand. An R-module is said to be strongly T_{11}(or strongly T_{11}-type module) if for each t-closed submodule, there exists a complement which is a fully invariant direct summand. We notice the following: module satisfies strongly-C_{11}-condition implies strongly T_{11}-type module which implies T_{11}-type, also module satisfies C_{11}-condition implies T_{11}-type module, but none of these implications is reversible, (see, Remarks 4.2.6 (1),(2),(3)) . Characterizations of both concepts: modules satisfy strongly C_{11}-condition and strongly T_{11}-type module are given (see, Proposition 4.2.4 and Theorem 4.2.9). Note that under the class nonsingular modules the two concepts are equivalent (see, Proposition 4.2.7). However, under the class of multiplication (or duo) modules, the T_{11}-type module equivalent to strongly T_{11}-type and module satisfies C_{11}-condition equivalent to module satisfies strongly C_{11}-condition. We prove that every strongly t-semisimple module is strongly T_{11}-type module (see, Proposition 4.2.11) and every strongly extending module is strongly T_{11}-type module (see, Theorem 4.2.12). Also, we have the property of strongly T_{11}-type module is inherited by a fully invariant direct summand.

In section three, the concepts of modules satisfy FI-C_{11}-condition and FI-T_{11}-type modules as generalizations of modules satisfy C_{11}-condition and T_{11}-type modules are presented where an R-module M is said to be satisfy FI-C_{11}-condition if every fully invariant submodule of M has a complement which is a direct summand. An R-module M is called FI-T_{11}-type module if every fully invariant t-closed submodule has a complement which is a direct summand. Module satisfies FI-C_{11}-condition implies FI-T_{11}-type module, and the converse may be not true. Many characterizations of modules satisfy C_{11}-condition and T_{11}-type modules are generalized for modules satisfy FI-C_{11}-condition and FI-T_{11}-type modules (see, Proposition 4.3.3 and Theorems 4.3.6, 4.3.10, 4.3.12). We prove that every FI-t-extending is FI-T_{11}-type (see, Proposition 4.3.7) and every FI-t-semisimple with condition $(*)$ imply FI-T_{11}-type (see, Corollary 4.3.9).

In section four, we extend the notions of modules satisfy FI-C_{11}-condition and FI-T_{11}-type modules. We say that an R-module M satisfies strongly FI-C_{11}-condition module if for each fully invariant submodule N there exists a fully invariant direct summand W which is a complement of N. An R-module M is called strongly FI-T_{11}-type module if for each fully invariant t-closed submodule N of M, there is a complement of N which is fully invariant direct summand. We noticed that module satisfies strongly C_{11}-condition module imply module satisfies strongly FI-C_{11}-condition which implies module satisfies FI-C_{11}-condition but none of these implications is reversible (see Remarks 4.4.3(1), (2)). Also, we have module satisfies strongly FI-C_{11}-condition implies strongly FI-T_{11}-type module which implies FI-T_{11}-type module and strongly T_{11}-type module implies strongly FI-T_{11}-type and each of these implications is not reversible (see, Remarks 4.4.5 (1),(2),(3)). Some characterizations of modules satisfy strongly FI-C_{11}-condition and strongly FI-T_{11}-type are given (see, Theorems 4.4.5 and 4.4.6).The property of strongly FI-T_{11}-type inherited by a fully invariant direct summand (see, Proposition 4.4.7). Also, we have if M is FI-t-extending module and every closed submodule is fully invariant, then M strongly FI-T_{11}-type module.

Section five deals with modules satisfy purely C_{11}-condition and purely T_{11}-type where, an R-module M is said to be satisfy purely C_{11}-condition if every pure submodule of M has a complement which is a direct summand. An R-module M is called purely T_{11}-type if every pure t-closed submodule of M has a complement which is a direct summand. Clearly every module satisfies C_{11}-condition is a module satisfies purely C_{11}-condition, but not conversely (see, Remarks and

Example 4.5.2(4)). Every pure simple module satisfies C_{11}-condition and it is purely T_{11}-type module but not conversely, for example $M = Z_8 \oplus Z_2$ as Z-module is purely T_{11}-type module and satisfies purely C_{11}-condition, but it is not pure simple module. Characterizations of such modules are given (see, Proposition 4.5.3). Under conditions we give some equivalent statements for purely T_{11}-type module (see, Theorem 4.4.12). Also, we prove that every purely t-semisimple and nonsingular module satisfies purely C_{11}-condition (see, Proposition 4.5.4). If M is a distributive module, the every pure submodule inherits the property of modules satisfy purely C_{11}-condition (see, Proposition 4.5.5). Every purely T_{11}-type module which is purely t-extending is T_{11}-type (see, Proposition 4.5.9).

Section six is devoted for modules satisfy strongly purely C_{11}-condition and strongly purely T_{11}-type module. An R-module M has strongly purely C_{11}-condition if every pure submodule has a complement which is a fully invariant direct summand. An R-module M is called strongly purely T_{11}-type module if every pure t-closed submodule has a complement which is a direct summand and fully invariant. Obviously, modules satisfy strongly purely C_{11}-condition implies modules satisfy purely C_{11}-condition and strongly purely T_{11}-type module implies purely T_{11}-type module. But each of these implications is not reversible. Many analogues properties of modules satisfy strongly C_{11}-condition and strongly T_{11}-type modules are given.

Finally, all modules are right unitary modules. Note that R need not be commutative except in some special cases and it will be mentioned. Thy symbol □ stands for the end of the proof.

Chapter One

Strongly T-semisimple Modules

and

Strongly T-semisimple Rings

Introduction

Asgari and Haghany in [6] introduced the concept of t-semisimple as follows" A module M is t-semisimple if for each $N \leq M$ there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$".

This chapter consists of five sections.

In section one, we recall some basic definitions. And list some important theorems and propositions that are relevant to our work. Also, we add several new results concerned with t-semisimple modules.

In section two, the notion of strongly t-semisimple is presented, where an R-module M is called strongly t-semisimple if for each submodule N of M there exists a fully invariant direct summand K such that $K \leq_{tes} N$.It is clear that every strongly t-semisimple module is t-semisimple. An example is given to show that the converse is not hold in general. In fact a comprehensive study of this class of modules is investigated.

In section three, we look for connections between strongly t-semisimple, strongly t-extending, and strongly extending modules. We proved that every strongly t-semisimple module is strongly t-extending, and every strongly extending module is strongly t-extending. Also, many characterizations and properties of strongly t-extending modules are given.

In section four, some properties of strongly t-semisimple rings are given.

In section five, we give connections between strongly t-semisimple, strongly t-Baer and strongly t-extending modules. Also, we investigate some new properties and characritzations of strongly t-Baer modules.

1.1 Preliminaries

In this section, we introduce some relevant concepts with some basic known results, which will be needed later; also we present some new results.

Definition (1.1.1)[25]**:** ″A submodule A of an R-module M is said to be essential in M (denoted by $A \leq_{ess} M$), if $A \cap W \neq (0)$ for every non-zero submodule W of M.Equivalently $A \leq_{ess} M$ if whenever $A \cap W = 0, W \leq M$ then $W = 0.$″

Definition (1.1.2) [**23**]: ″$\mathrm{Z}_2(\mathrm{M})$ is the Goldie torsion (or second singular) of an R-module M is defined by $\frac{\mathrm{Z}_2(\mathrm{M})}{\mathrm{Z}(\mathrm{M})} = Z(\frac{\mathrm{M}}{\mathrm{Z}(\mathrm{M})})$ **[**23]″,where ″ $Z(M) = \{x \in M: xI = 0$ for some $I \leq_{ess} R\}.$"

Definition (1.1.3) [17]**:″** A module M is called Z_2-torsion (or Goldie-Torsion) if $\mathrm{Z}_2(M) = M$″

″Examples (1.1.4):

(1) Consider Q as Z-module. one can easily show that $Z(Q) = 0$ and hence $\frac{\mathrm{Z}_2(\mathrm{Q})}{\mathrm{Z}(\mathrm{Q})} = Z\left(\frac{\mathrm{Q}}{\mathrm{Z}(\mathrm{Q})}\right) = Z(\frac{\mathrm{Q}}{0}) = Z(Q) = 0$. Thus $\mathrm{Z}_2(\mathrm{Q}) = \mathrm{Z}(\mathrm{Q}) = 0$

(2) For each, $n \in Z_+$ the module Z_n as Z-module is Z_2-torsion . We know that Z_n is singular as Z-module. So $\frac{\mathrm{Z}_2(Z_n)}{\mathrm{Z}(Z_n)} = Z\left(\frac{Z_n}{\mathrm{Z}(Z_n)}\right) = Z\left(\frac{Z_n}{Z_n}\right) = Z(0) = 0$. Thus $\mathrm{Z}_2(Z_n) = Z(Z_n) = Z_n$.

Z_4 as Z_4– module is Z_2-torsion since $\frac{Z_2(Z_4)}{Z(Z_4)} = \frac{Z_2(Z_4)}{\{0,2\}}$,$Z(\frac{Z_4}{Z(Z_4)}) = Z(\frac{Z_4}{\{0,2\}}) \cong Z(Z_2) \cong Z_2$. Hence $Z_2(Z_4) = Z_4$″.[27]

(1) **Proposition (1.1.5)** [27, Proposition 2.2.4]**: ″**Let M be an R-module and let A be a submodule of M. Then

(2) $\mathrm{Z}_2(A) \leq \mathrm{Z}_2(M)$;

(3) $\mathrm{Z}_2(A) = A \cap \mathrm{Z}_2(M)$″.

(4) **Corollary (1.1.6)[** 27, Corollary 2.2.5]**:″** Let M be an R-module and let A be an essential submodule of M such that $\mathrm{Z}_2(A) = 0, then\ \mathrm{Z}_2(M) = 0$.″

Remark (1.1.7)[6]**:** Every singular R-module is Goldie torsion.

Remarks (1.1.8):

(1) **″**Let M be an R-module, then $\mathrm{Z}_2(M) = 0$ if and only if M is nonsingular.″[27, Remarks 2.2.7]

(2) $\mathrm{Z}_2\left({}^{M}/_{\mathrm{Z}_2(M)}\right) = 0$ for every module M and this implies if $A \leq M$ such that A and ${}^{M}/_{A}$ are both Z_2-torsion, then M is Z_2-torsion.[6]

Proposition (1.1.9)[27, Proposition 2.2.8]**:** ″Let R be an integral domain and let A be a submodule of a nonsingular R-module M. If $\frac{M}{A}$ is Goldie torsion, then $A \leq_{ess} M$.″

Proposition (1.1.10) [27, Proposition 2.2.9] **: ″**Let M and N be two R-modules and let $f: M \to N$ be an R-homomorphism, then $f\big(\mathrm{Z}_2(M)\big) \leq \mathrm{Z}_2(N)$.″

Corollary (1.1.11) [27, Corollary 2.2.10]**:″** Let M and N be two R-modules and let $f: M \to N$ be an R-epimorphism. If M is Goldie torsion, then N is Goldie torsion″.

Proposition (1.1.12)[21]: ″Let C be an R-module, then C is nonsingular if and only if Hom $(Z_2(M), C) = 0$, for every R-module M''.

Proposition (1.1.13) [6], [27, proposition 2.2.13]**:** ″Let $M = \bigoplus_{\propto\in\Lambda} M_\propto$ be an R-module where $M_\propto$ is a submodule of M, for all $\propto\in\Lambda$. Then $\mathrm{Z}_2(\bigoplus_{\propto\in\Lambda} M_\propto) = \bigoplus_{\propto\in\Lambda} Z_2(M_\propto)$"

Definition (1.1.14)[8]: ″A submodule A of M is said to be t-essential in M (denoted by ($A \leq_{tes} M$) if for every submodule B of $M, A \cap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$″.

Remark (1.1.15)[6]:″ $Z_2(M)$ ={x ∈ M : $ann_R(x) \leq_{tes}$ R}, where $ann_R(x) = \{r \in R: xr = 0\}$.″

Example (1.1.16): Consider Z_{12} as Z-module. It is clear that Z_{12} is singular module. Hence Z_{12} is Z_2-torsion, that is $Z_2(Z_{12}) = Z_{12}$.

Let $A = (\bar{4}) \leq Z_{12}$. Then for all $B \leq Z_{12}$ and (4) $\cap B \leq Z_2(Z_{12})$=Z_{12} then $B \leq Z_2(Z_{12}) = Z_{12}$. Hence $(\bar{4}) \leq_{tes} M$, but $(\bar{4})$ is not essential of Z_{12}.

Proposition (1.1.17)[6, Proposition 2.2]**:** ″The following statements are equivalent for a submodule A of an R-module M

(1) A is t-essential in M ;
(2) $(A+Z_2(M))/Z_2(M)$ is essential in $M/Z_2(M)$;
(3) $A+Z_2(M)$ is essential in M;
(4) M/A is $Z_2 - torsion$.″

Corollary (1.1.18):″ If $A \leq_{ess} M$, then $A \leq_{tes} M$, but not conversely ″[6].

Proof: If $A \leq_{ess} M$, then $A+Z_2(M) \leq_{ess} M$ and hence by Proposition (1.1.17), $A \leq_{tes} M$. □

The converse is not true in general, see Example 1.1.16.

Corollary (1.1.19): Let M be a nonsingular module, $let\, N \leq M, N \neq$ (0) $N \leq_{tes} M$ if and only if $N \leq_{ess} M$.

Proof: As M is nonsingular, $Z_2(M) = 0$, then $N \leq_{tes} M$ if and only if $N \leq_{ess} M$. □

Proposition (1.1.20) [10,Corollary 1.2]**:**

"(1) Let M be an R-module $A \leq B \leq M$. Then $A \leq_{tes} M$ if and only if $A \leq_{tes} B$ and $B \leq_{tes} M$.

(2)*Let* $f: M \to N$ be a homomorphism of modules, and $A \leq_{tes} N$, then $f^{-1}(A) \leq_{tes} M$".

Corollary (1.1.21): Let M be an R-module and $A \leq B \leq N \leq M$, if $A \leq_{tes} M$ then $B \leq_{tes} N$.

Proof: A$\leq B \leq N \leq M$, if $A \leq_{tes} M$, then by Proposition (1.1.20) if $A \leq_{tes} N$ and $N \leq_{tes} M$. Again, $A \leq_{tes} N$, so by Proposition (1.1.20), $A \leq_{tes} B$ and $B \leq_{tes} N$. □

(1) **Proposition (1.1.22)[** 10,Corollary 1.3]**:"** Let A_λ be a submodule of M_λ for each λ in a set Λ. Then

(2) If Λ is a finite set and $A_\lambda \leq_{tes} M_{\lambda,}$then $\cap_\lambda A_\lambda \leq_{tes} \cap_\Lambda M_\lambda$ for all $\lambda \in \Lambda$.

(3) $\oplus_\Lambda A_\lambda \leq_{tes} \oplus_\Lambda M_\lambda$ If and only if $A_\lambda \leq_{tes} M_\lambda$ for all $\lambda \in \Lambda$."

We prove the following

Proposition (1.1.23): Let M and N be $R-$ modules and let $f: M \to N$ be a monomorphism if $A \leq_{tes} M$,then $f(A) \leq_{tes} f(M)$.

Proof: As $A \leq_{tes} M$, then $A+Z_2(M) \leq_{ess} M$ by (proposition 1.1.17(2)) .Since f is monomorphism, $f(A+Z_2(M)) \leq_{ess} f(M)$. Hence $f(A)+f(Z_2(M)) \leq_{ess} f(M)$, but $f(Z_2(M)) \leq Z_2(f(M)$by proposition (1.1.10) hence $f(A)+f(Z_2(M)) \leq f(A) + Z_2(f(M)) \leq f(M)$.It follows that $f(A)+Z_2(f(M)) \leq_{ess} f(M)$, thus $f(A) \leq_{tes} f(M)$ by Proposition 1.1.17. □

Recall that an "R-module M is called multiplication if for each submodule N of M, there exists an ideal I of R such that $N = MI$. Equivalently M is a multiplication R-module if for each submodule N of M, $N = M(N:_R M)$, where $(N: M) = \{r \in R: Mr \leq N\}$". [19]

The following Lemma will be needed in the next Proposition.

Lemma (1.1.24): Let M be a finitely generated faithful multiplication module over commutative ring R and let I, J be ideals of R. Then

(1) If $I \leq_{ess} J$,then $MI \leq_{ess} MJ$.

(2) $If\ MI \leq_{ess} MJ$, then $I \leq_{ess} J$, provided that R is regular ring (in sense of Von Neumann).

Proof: (1) Let $W \leq MJ$ and $MI \cap W = (0)$. It follows that $W \leq M$ and since M is multiplication, $W = MK$ for some ideal K of R, and by[19, Theorem 3.1], $K \leq J$. Hence $MI \cap MK = 0$; and since M is faithful, we get $M(I \cap K) = (0)$ by [19, Theorem 1.6]. As M is faithful, we get $I \cap K = (0)$ which implies $K = (0)$ because $I \leq_{ess} J$. It follows that $W = (0)$ and $MI \leq_{ess} MJ$.

(2)Let K be an ideal of J with $I \cap K = (0)$. K is an ideal of J and J is an ideal of R implies K is an ideal of R since R is a regular ring. It follows that $M(I \cap K) = (0)$ and by [19, Theorem 1.6] $MI \cap MK = (0)$. But K is an ideal of J, so MK is a submodule of MJ and as $MI \leq_{ess} MJ$, we conclude that $MK = (0)$. Hence $K = (0)$ since M is faithful. Thus $I \leq_{ess} J$. □

Proposition (1.1.25): Let M be finitely generated a faithful multiplication module over commutative ring R, and I, J be ideals of R. Then

(1) $MZ_2(R) = Z_2(M)$.

(2) If $I \leq_{tes} R$; then $MI \leq_{tes} M$.

(3) If $N \leq_{tes} M$, and N = MI then $I \leq_{tes} R$.

(4) If $I \leq_{tes} J$, then $MI \leq_{tes} MJ$, and the converse hold if R is regular.

Proof: (1) First $MZ_2(R) \leq Z_2(M)$ hold for any module M as follows. For each $a \in Z_2(R)$, then $ann(a) \leq_{tes} R$. Now for any $m \in M, ann(ma) \supseteq ann(a)$.This implies $ann(ma) \leq_{tes} R$; that is $ma \in Z_2(M)$ for each $m \in M$ and so

$MZ_2(R) \subseteq Z_2(M)$-----------(I)

But $Z_2(M) = M(Z_2(M){:}M)$ since M is multiplication, which implies $MZ_2(R)\ ) \leq M(Z_2(M){:}M)$. Let $a \in (Z_2(M){:}M)$, then $ann(ma) \leq_{tes} R$ for all $m \in M$. Since

M is finitely generated R-module, $M = < m_1, \dots\dots.. m_n >$ for some $m_1, \dots\dots.. m_n \in M$. Now $ann(Ma) = ann(\sum_{i=1}^{n} m_i\, a) = \bigcap_{i=1}^{n} ann(m_i\, a)$. But $ann(m_i a) \leq_{tes} R$ for all $i = 1, \dots, n$). So $\bigcap_{i=1}^{n} ann(m_i\, a) \leq_{tes} R$. Thus $ann(Ma) \leq_{tes} R$; . On the other hand, $ann_R(Ma) = ann(a)_R$ (since M is a faithful multiplication R-module), thus $ann(a) \leq_{tes} R$ and $a \in Z_2(R)$. Therefore $(Z_2(M){:}M) \leq Z_2(R)$ and hence $M(Z_2(M){:}M) \leq MZ_2(R)$. But $Z_2(M) = M(Z_2(M){:}M)$, since M is a multiplication module. Therefore

$Z_2(M) \subseteq MZ_2(R)$--(II).

Then $Z_2(M) = MZ_2(R)$ by (I) and (II).

(2) Since $I \leq_{tes} R$, $I{+}Z_2(R) \leq_{ess} R$ by Proposition (1.1.17). As M is faithful multiplication, then $MI{+}MZ_2(R) \leq_{ess} M$ by [19, Theorem 2.13]. But $MZ_2(R) \leq Z_2(M)$,so $MI{+}MZ_2(R) \leq MI + Z_2(M)$.Hence $MI + Z_2(M) \leq_{ess} M$, Thus $MI \leq_{tes} M,$ by proposition (1.1.17).

(3) Let $N \leq_{tes} M$. Since M is a multiplication R-module, then $N = MI$ for some ideal I of R. To prove $I \leq_{tes} R$.Assume $I \cap J \leq Z_2(R)$ for some $J \leq R$. As M is faithful multiplication,$M(I \cap J)$= $MI \cap MJ$ by [19, Theorem.1.5] so $MI \cap MJ \leq MZ_2(R)$. So that $N \cap MJ \leq M\ Z_2(R)$. But M is a faithful finitely generated multiplication module implies, $MZ_2(R) = Z_2(M)$ by (1). Thus $N \cap MJ \leq MZ_2(R) = Z_2(M)$ and hence $MJ \leq MZ_2(R)$, which implies $J \leq Z_2(R)$ by [19, Theorem 3.1]. Thus I $\leq_{tes} R$

(4) Since $I \leq_{tes} J$,then $I + Z_2(J) \leq_{ess} J$ by Proposition (1.1.17). Hence by Lemma 1.1.24(1) $M(I + Z_2(J)) \leq_{ess} MJ$. It follows that $+MZ_2(J) \leq_{ess} MJ$. But we can show that $MZ_2(J) = Z_2(MJ)$ as follows: $Z_2(MJ) = Z_2(M) \cap MJ$. But $Z_2(M) = MZ_2(R)$ so that $Z_2(MJ) = MZ_2(R) \cap MJ$. But by [19, Theorem 1.6] $MZ_2(R) \cap MJ = M(Z_2(R) \cap J) = MZ_2(J)$.Thus $Z_2(MJ) = MZ_2(J)$.Then

$MI + Z_2(MJ) \leq_{ess} MJ$ which implies $MI \leq_{tes} MJ$ by Proposition 1.1.17.

Now , if $MI \leq_{tes} MJ$, then $MI + Z_2(MJ) \leq_{ess} MJ$ by Proposition 1.1.17. But $Z_2(MJ) = MZ_2(J)$. So that $MI{+}MZ_2(J) \leq_{ess} MJ$; that is $M(I + Z_2(J)) \leq_{ess} MJ$.

Hence by Lemma 1.1.24(2), $I + Z_2(J) \leq_{ess} J$ and by Proposition 1.1.17, we have $I \leq_{tes} J$. □

Recall that ″a submodule N of an R-module M is called closed ($N \leq_c M$) if whenever $N \leq_{ess} W \leq M$, then $N = W$ in case a submodule $N \leq_{ess} W$ and W is closed in M, the submodule W is closure of N. ″[24].In other word,″ $N \leq_c M$, if N has no proper essential extension in M ″[25]. As a generalization of closed submodule, the concept t-closed, was introduced by Asgari [6].

Definition (1.1.26)[6]: ″A submodule C of M is t-closed in M (written $C \leq_{tc} M$ if $C \leq_{tes} C'$ whenever $C' \leq M$ implies $C = C'$. In other words, $C \leq_{tc} M$, if C has no proper t-essential extension in M″.

Lemma (1.1.27)[6, Lemma 2.5]**:″** Let M be an R- module. Then

(1) If $C \leq_{tc}$ M, then $Z_2(M) \leq C$.

(2) $0 \leq_{tc} M$ If and only if M is nonsingular.

(3) If $A \leq C$, then $C \leq_{tc}$ M if and only if $\frac{C}{A} \leq_{tc} \frac{M}{A}$.″

Proposition (1.1.28)[6, Proposition 2.6]**:″** Let C be a submodule of an R- module M. The following statements are equivalent:

(1) There exists a submodule S such that C is maximal with respect to the property that $C \cap S$ is Z_2-torsion,

(2) C is t-closed in M;

(3) C contains $Z_2(M)$ and $\frac{C}{Z_2(M)}$ is closed submodule of $\frac{M}{Z_2(M)}$;

(4) C contains $Z_2(M)$ and C is a closed submodule of M;

(5) C is complement to a nonsingular submodule of M;

(6) $\frac{M}{C}$ is nonsingular.″

By proposition 1.1.28(4), it follows directly that every t-closed submodule is closed. However the convers is not true in general. For example in any singular module M, we have (0) is a closed submodule and it is not t-closed.

Goodearl in [23], gave the following ″a submodule $N\ of\ M$ is called Y-closed if $\frac{M}{N}$ is nonsingular″. Hence by Proposition 1.1.28(6) the two concepts t-closed and Y-closed are coincide.

Corollary (1.1.29)[6, Corollary 2.7]**:″** Let M be a module .Then

(1) $Z_2(M)$ is t-closed in M;

(2) If φ is an endomorphism of M and C is t-closed submodule of M then $\varphi^{-1}\ (C)$ is t-closed in M.″

Corollary (1.1.30)[6, Corollary 2.8]: ″Let C be a submodule of a module M

(1) If $C \leq_{tc} M$, then $C = Z_2(M)$ iff C is Z_2-torsion and there exists a t-essential submodule S of M for which $C \cap S \leq Z_2(M)$.

(2) Let $C \leq N \leq M$. If $C \leq_{tc} M$, then $C \leq_{tc} N$.

(3) If $C \leq_{tc} N$ and $N \leq_{tc} M$, then $C \leq_{tc} M$.″

Proposition (1.1.31)[6, Proposition 2.9]:″ Let M be an R-mmodule. Then

(1) $C \leq M, C' \leq_{tc} M$ then $C \cap C' \leq_{tc} C$;

(2) $C \leq_{tc} M, C' \leq_{tc} M$ then $C \cap C' \leq_{tc} M$.″

Proposition (1.1.32)[6]: ″Let M be a nonsingular module and let A be a submodule of M. Then A is t-closed if and only if A is closed. ″

Proposition (1.1.33)[6]: ″Let M be a singular R-module. Then M is the only t-closed submodule of M.″

Examples (1.1.34):

(1) 0,Z are the only t-closed submodules of the Z-module Z.

(2) In the Z-module Q, the submodule Z is not t-closed in Q.

Next, we present the following

Definition (1.1.35):″ A submodule N of R-module M is fully invariant if $f(N) \leq N$ for each R-endomorphism f of M. ″[41]. ″A submodule N of an R-module M is called stable, if $f(N) \leq N$ for each R-homomorphism $f: N \to M$.″ [1]. It is clear every stable submodule is fully invariant but not conversely. For instance $2Z$ in Z-module Z is fully invariant and it isn't stable.

Remark (1.1.36): Let $K \leq N \leq^{\oplus} M$ such that K is a stable submodule in M, then K is stable in N.

Proof: Let $\theta: K \longrightarrow N$. Then $i \circ \theta: K \longrightarrow M$, where $(i: N \longrightarrow M)$ is inclusion mapping, so $(i \circ \theta)(K) \leq K$ (since K is stable in M). But $(i \circ \theta)(K) = \theta(K)$. Thus $\theta(K) \leq K$ and K is stable in N.

Definition (1.1.37): ″An R-module M is fully stable if every submodule of M is stable[1] and M is called duo if every submodule of M is fully invariant.″[31]

Proposition (1.1.38)[32]: ″Let R be a ring and let $L \leq K$ be submodules of an R-module M such that L is a fully invariant submodule of K and K is a fully invariant submodule of M. Then L is a fully invariant submodule of M.″

Lemma (1.1.39):″ Let M be a module.Then

(i) Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M.

(ii) If $M = \oplus_{i \in I} X_i$ and S is a fully invariant submodule of M, then

$S = \oplus_{i \in I} \pi_i(S) = \oplus(X_i \cap S)$ where π_i is the $i-th$ projection homomorphism of M and $X_i \cap S$ is fully invariant in M_i, for all $i \in I$.″[11]

(iii) ″Let M be an R-module and let $M = K \oplus K'$, $K, K' \leq M$.Then K is a fully invariant submodule of M if and only if $\mathrm{Hom}(K, K') = 0$″[32,Lemma 2.6].

Lemma (1.1.40): (1) ″Let M be an R-module,let $K \leq L \leq M$. If $\frac{L}{K}$ is a fully invariant submodule of $\frac{M}{K}$ and K is a fully invariant submodule of M, then L is a fully invariant in M.″[9, Proposition 1.3]

(2)″Let M be an R- module. If $K \leq N$ is a fully invariant submodule of M and $N \leq^{\oplus} M$, then K is a fully invariant in N″[20] . However, we give a different proof.

Proof: Let $\theta \in End(N)$. Define $g: M \longrightarrow M$ by $g(x) = \begin{cases} \theta(x) & if\ x \in N \\ 0 & otherwise \end{cases}$.

As K is a fully invariant submodule of M, $g(K) \leq K$. But $g(K) = \theta(K)$ since $K \leq N$, hence $\theta(K) \leq K$ and K is a fully invariant submodule of N. □

″Recall that a module M is called SS-module if every direct summand is stable ″[35]. However Özcan et al in [31], gave the following. ″An R-module M is called weak duo if every direct summand of M is fully invariant. But every direct summand and fully invariant is stable [35, Lemma 2.1.6] hence the two concepts SS-module and weak duo are coincide″.

Lemma (1.1.41): ″Let M and N be R-modules, and let $f \in Hom(M, N)$ be an epimorphism. Then

(1) If $Ker\ f$ is a fully invariant in M and L is a fully invariant submodule of N then $f^{-1}(L)$ is a fully invariant submodule of M.

(2) If M is self-projective (quasi –projective) and U is a fully invariant submodule of M, then $f(U)$ is a fully invariant submodule of N″.[24]

In fact for R-modules N and A. N is said to be A- projective, if every submodule X of A, any homomorphism $\emptyset: N \longmapsto \frac{A}{X}$ can be lifted to a homorphism, $\psi: N \longmapsto A$, that is if $\pi: A \longmapsto \frac{A}{X}$, be the-natural epiomorphism, then there exists a homorphism $\psi: N \longmapsto A$ such that $\pi \circ \psi = \emptyset$.

$$\begin{tikzcd} & N \arrow[dl, dashed, "\psi"'] \arrow[dr, "\emptyset"] & \\ A \arrow[rr, "\pi"'] & & A/X \end{tikzcd}$$

M is called projective if M is N-projective for every R-module N. If M is M-projective, M is called self-projective″. [28] For examples:

(1) Z as Z-module is projective.
(2) Z_2 as Z-module is self-projective.
(3) Z_{P^∞} as Z- module is Z-projective.

Corollary (1.1.42): If M is self-projective and duo R-module and $K \leq M$, then $\frac{M}{K}$ is duo R-module.

Proof: Let $\pi: M \to \frac{M}{K}$, where π is the natural epimorphism. For any $\frac{N}{K} \leq \frac{M}{K}$, $\frac{N}{K} = \pi(N)$. Hence $\frac{N}{K}$ is fully invariant submodule of $\frac{M}{K}$ by Lemma 1.1.41(2). □

Recall that ″an R-module M is semisimple if every submodule is a direct summand of M ″[23]. Equivalently an R-module is semisimple if for each submodule N of M, there exists a direct summand K of M such that $K \leq_{ess} N$[7]. For more properties of semisimple modules see [23],[25].

Corollary (1.1.43): If M is self-projective and duo, $K \leq M$ such that $\frac{M}{K}$ is semisimple. Then $\frac{M}{K}$ is fully stable.

Proof: By Corollary (1.1.42), $\frac{M}{K}$ is duo. Hence every submodule of $\frac{M}{K}$ is fully invariant. But $\frac{M}{K}$ is semisimple, so every submodule is fully invariant and direct summand. Thus every submodule of $\frac{M}{K}$ is stable. □

In fact the above equivalent statement of semisimple module led Asgari and Haghany in [6], to introduce and study t-semisimple modules.

Definition (1.1.44)[7]: ″A module M is t-semisimple if for every submodule N of M there exists a direct summand K of M such that $K \leq_{tes} N$.″

Remarks (1.1.45):

(1) It is clear that every semisimple (hence every simple) module is t-semisimple but not conversely, see part (2).

(2) Z_n as Z-module is t-semisimple ,but not semisimple, where $n = 4,8,12$

Proof: $Z_2(Z_n) = Z_n$. For each $N \leq Z_n, Z_2(N) = Z_2(Z_n) \cap N = N$ so that $(\bar{0})$+ $Z_2(N) = N \leq_{ess} N$, hence by Proposition 1.1.17(3) $(\bar{0}) \leq_{tes} N$ and $(\bar{0}) \leq^{\oplus} M$. Thus Z_n is t-semisimple. □

(3) Let M be a non-singular R-module. Then M is t-semisimple if and only if M is semisimple.

Proof:⇒ Let $N \leq M$, so there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. But M is non-singular (so N is non-singular), hence $K \leq_{ess} N$, (by Lemma (1.1.19)), then M is semisimple.

⇐ It is clear. □

In particular the Z-module Z is nonsingular and it is not semisimple. So that it is not t-semisimple . Also Qas Z-module is nonsingular, Q is not semisimple so Q is not t-semisimple.

The following Theorem gave characterizations of t-semisimple modules.

Theorem (1.1.46)[7]:″ The following statements are equivalent for a module M:

(1) M is t-semisimple;

(2) $\frac{M}{Z_2\ (M)}$ is semisimple;

(3) M = Z_2 (M)⊕M' where $M'$is a non-singular semisimple module;

(4) Every nonsingular submodule of M is a direct summand;

(5) Every submodule of M which contains Z_2 (M) is a direct summand.″

By applying Theorem 1.1.46 we can give the following examples

(1) Consider the Z-module $M = Q \oplus Z_2$, $Z_2(M) = Z_2$. Hence $\frac{M}{Z_2(M)} = \frac{Q \oplus Z_2}{Z_2} \cong Q$ which is not semisimple. Hence M is not t-semisimple.

(2) Consider the Z-module $M = Z_8 \oplus Z_2$. $Z_2(M) = M$. Hence $\frac{M}{M} \cong (0)$ is semisimple. Thus M is t-semisimple.

(3) Let $M = Z_n \oplus Z$ as Z-module where $n \in Z_+$, $\frac{M}{Z_{2(M)}} \simeq Z$ which is not semisimple. Hence M is not t-semisimple by Theorem (1.1.46).

(4) Let $M = Z \oplus Z$ as Z-module. $\frac{M}{Z_2(M)} \simeq Z \oplus Z$ is not semisimple, so M is not t-semisimple.

Corollary (1.1.47)[7, Corollary 2.4]: ″Let M be a t-semisimple module.

(1) Every submodule of M is t-semisimple.

(2) Every homomorphic image of M is t-semisimple.″

Corollary (1.1.48)[7, Corollary 2.5]:″ Every direct sum of t-semisimple modules is t-semisimple.″

Corollary (1.1.49)[7, Corollary 2.7]: ″A module M is t-semisimple if and only if M has no proper t-essential submodule which contains $Z_2(M)$.″

Corollary (1.1.50)[7, Corollary 2.8]: ″A module M is t-semisimple if and only if N + $Z_2(M)$is closed in M, for every submodule N of M.″

We add the following results

Theorem (1.1.51): The following statements are equivalent:

(1) Every R-module M is t-semisimple and $Z_2(M)$ is projective.

(2) R is semisimple.

Proof: (1) $\Rightarrow$(2) Let M be an R-module. Then M is t-semisimple by hypothesis. Hence $M = Z_2(M) \oplus M'$, where M' is a nonsingular semisimple by Theorem 1.1.46. It follows that M' is projective, but by hypothesis $Z_2(M)$ is projective. Thus M is projective, that is every R-module is projective and so by [25, Corollary 8.2.2(e)] R is semisimple.

(2)$\Rightarrow$ (1) Since R is semisimple, every R-module is semisimple. Hence every R-module is t-semisimple. Also R is semisimple, then every R-module is projective [25, Corollary 8.2.2(e)]. Thus $Z_2(M)$ is projective. □

Proposition (1.1.52): If M is an indecomposable t-semisimple, then M either semisimple or Z_2-torsion.

Proof: Since M is t-semisimple, then by Proposition 1.1.46(3), $M = Z_2(M) \oplus M'$, where M' is nonsingular semisimple. But M is indecomposable, so either $Z_2(M) = 0$ or M'= (0). If $Z_2(M) = (0)$, then $M = M'$, but M' is semisimple, so that M is semisimple. If $M' = (0)$, then $M = Z_2(M)$ and hence M is Z_2-torsion. □

Recall that ″ if I is an ideal of a ring R, then the ring R is called I-semiperfect if $\frac{R}{I}$ is semisimple and I is strongly lifting (or that is idempotent lift strongly module I)(that is whenever $a^2 - a \in I, a \in R$, there exists $e^2 = e \in aR$ such that $e - a \in I$)"[30]. Note that "every nil right ideal is strongly lifting. ″[30]

By using in [30, Theorem 49] and Theorem (1.1.46). We get the following.

Proposition (1.1.53): The following assertions are equivalent:

(1) R_R is $Z_2(R)$-semiperfect ring.

(2) R_R is t-semisimple.

(3) Any module M_R, M is t-semisimple.

(4) Every nonsingular right R-module is injective.

Proof: (1) $\Leftrightarrow$ (3) For any module M_R, $\frac{M}{Z_2(M)}$ is semisimple by [30, Theorem 49(6)] that is M is t-semisimple by Theorem (1.1.46). Hence (1) $\Leftrightarrow$ (3)

(3) ⇔ (4) By Theorem 1.1.46, $\frac{M}{Z_2(M)}$ is semisimple. Hence the result follows by [30, Theorem 49].

(1)⇔(2) It follows by Theorem (1.1.46). □

By combining Lemma (48) in [30] and Theorem (1.1.46), we get another characterization, for t-semisimple modules.

Proposition (1.1.54): Let M be an R-module. Then M is a t-semisimple if and only if for each $N \leq M$,$N = A \oplus B$ for some $A \leq^{\oplus} M$ and $B \leq Z_2(M)$.

Proof: M is t-semisimple if and only if $M = Z_2(M) \oplus M'$ where $M'$is semisimple by Theorem (1.1.46). Hence the result follows by [30, Lemma 48 (1)⇔(2)]. □

Burcu, et al in [12] introduced the following″ Let F be a fully invariant submodule of a module M. Then M is called F-inverse split if $f^{-1}(F)$ is a direct summand of M for every $f \in S = End(M)$ [12]. Obviously, every module M is M-inverse split and every semisimple module M is F-inverse split, and so every module M over semisimple ring is F-inverse split. Recall that an R-module M is Rickart if $kerf$ is a direct summand of M [22]" For a module M, since ker $f = f^{-1}(0), M$ is Rickart if and only if it is (0)-inverse split ″[12]. It is clear that every semisimple module is Rickart.

We prove the following

Proposition (1.1.55): Every t-semisimple module M is $Z_2(M)$-inverse split.

Proof: Since M is t-semisimple, $M = Z_2(M) \oplus M'$, where M' is nonsingular, semisimple. But M' is semisimple implies, M' is a Rickart module. Hence by [12, Theorem 2.3] M is $Z_2(M)$-inverse split. □

The converse of proposition (1.1.55) is not true in general, for example: Consider the Z-module Z, Z is not t-semisimple. However $Z_2(Z) = (0)$ and for each $f \in End(Z)$, $f^{-1}(0) = (0)$, which is a direct summand, that is Z is $Z_2(Z) = (0)$-inverse split.

Proposition (1.1.56): If M is $Z_2(M)$-inverse split and M is an Artinian over commutative ring, then M is t-semisimple.

Proof: Since M is $Z_2(M)$-inverese split, then $M = Z_2(M) \oplus M'$, where M' is a Rickart module by [12, Theorem 2.3]. Since $M' \simeq \frac{M}{Z_2(M)}$, $M'$is nonsingular. But M is Artinain implies M' is an Artinian module. Hence by [22, Proposition 2.25] M' is semisimple. Hence by Theorem (1.1.46), M is t-semisimple. □

Corollary (1.1.5**7**): Let R be a Rickart Artinian commutative ring. Then R is t-semisimple.

Proof: Since R is a Rickart ring, R is nonsingular [22, Proposition 2.12], hence $Z_2(R) = (0)$. It follows that $R = Z_2(R) \oplus R = (0) \oplus R$. Hence by [12, Theorem 2.3] R is $Z_2(R)$-inverse. Then by proposition (1.1.56), R is t-semisimple. □

Corollary (1.1.58): Let M be an Artinain module over commutative ring. If M is Rickart and nonsingular, then M is t-semisimple.

Proof: Since M is nonsingular, $Z_2(M) = (0)$. Hence $M = (0) \oplus M = Z_2(M) \oplus M$, and since M is Rickart, so that M is $Z_2(M)$-inverse split by [12,Theorem 2.3]. Hence by Proposition (1.1.56), M is t-semisimple. □

Recall that "an R-module M is called F-regular (simply regular) if every submodule is pure, where a submodule N of M is pure if for every ideal I of R $MI \cap N = NI$.[4]

Next we have the following

Proposition (1.1.59): Let M be a F-regular R-module where R is a commutative ring. Then M is t-semisimple if and only if M is semisimple.

Proof:$\Rightarrow$ Since M is t-semisimple, then $M = Z_2(M) \oplus M'$, where $M'$is nonsingular semisimple submodule of M. As M is F-regular, $\frac{R}{ann(x)}$ is regular ring for all $x \in M$ [37, Theorem 1.10]. Let $x \in Z(M)$. Then $ann(x) \leq_{ess} R$ and hence $\frac{R}{ann(x)}$ is singular. That is $Z\left(\frac{R}{ann(x)}\right) = \frac{R}{ann(x)}$. But $\frac{R}{ann(x)}$ is a regular ring implies $Z\left(\frac{R}{ann(x)}\right) = (0)$.Thus $\frac{R}{ann(x)}$.=0 and so $R = ann(x)$, which implies $x = 0$ and so $Z(M) = 0$. It follows that $Z_2(M) = (0)$ and $M = M'$. Therefor M is semisimple.

$\Leftarrow$ It is clear. □

Proposition (1.1.60): Let M be a finitely generated faithful multiplication over a commutative regular ring (in sense of Von Neumann). If M is t-semisimple, then R is t-semisimple.

Proof: Let I be an ideal of R. Then $N = MI$ is a submodule of M. As M is t-semisimple, there exists a submodule U of M such that $U \leq^{\oplus} M$ and $U \leq_{tes} N = MI$. As M is a multiplication module $U = MJ$ for some $J \leq R$. Hence $MJ \leq_{tes} MI$. Hence by Proposition 1.1.25(4), $J \leq_{tes} I$. Also, since $U = MJ \leq^{\oplus} M$, then $J \leq^{\oplus} R$. Thus R is t-semisimple. □

Note that we see by Proposition 1.1.53 if R is t-semisimple module, then every R-module is t-semisimple.

1.2 Strongly t-semisimple Modules

We introduce the concept of strongly t-semisimple modules and give many characterizations and properties of this class of modules.

Definition (1.2.1): An R-module M is called strongly t-semisimple if for each submodule N of M there exists a fully invariant direct summand K such that $K \leq_{tes} N$.

Remarks and Examples (1.2.2):

(1) It is clear that every strongly t-semisimple module is t-semisimple, but the convers is not true.

(2) t-semisimple module need not be strongly t-semisimple, for example.

Let $T =$ M$\oplus$M where M is a non-singular semisimple R-module, M $\neq$ (0).

Hence T is semisimple, and so T is t-semisimple. Let N= M $\oplus(0)$, so there exists K $\leq^{\oplus}$ M such that K $\leq_{\text{tes}}$ N.

Hence K =$\mathrm{K}_1 \oplus(0)$ for some $\mathrm{K}_1 \leq$ M. If K_1= (0), then K =< (0, 0)> and K $\leq_{tes}$ M $\oplus$(0).

So < (0, 0)>+ Z_2(M)$\oplus(0))$ $\leq_{\text{ess}}$ $M\oplus$(0) (by Proposition 1.1.17(3))

Thus $\mathrm{Z}_2(M) \leq_{\text{ess}}$ M. But $\mathrm{Z}_2(\mathrm{M}) = (0)$, hence (0) $\leq_{\text{ess}}$ M and so M = (0), which is a contradiction. It follows that $\mathrm{K}_1 \neq (0)$, so K $\neq< (0,0 >$.But in this case K is not fully invariant submodule of T.

To see this: Let f:T $\longmapsto$ T defined by $f(\mathrm{x,y}) = (\mathrm{y,x})$, for all (x,y)$\in$ T, Then $f(\mathrm{K}_1 \oplus(0)) = (0) \oplus \mathrm{K}_1 \nsubseteq \mathrm{K}_1 \oplus(0)$. Thus K=$\mathrm{K}_1 \oplus(0)$ is not fully invariant submodule of T, such that K $\leq_{\text{tes}}$ N.Therefore T is not strongly t-semisimple. □

In particular, $\mathbb{R}$ as $\mathbb{R}$–module is simple non-singular $\mathbb{R}$ -module, so $\mathbb{R} \oplus \mathbb{R}$ as $\mathbb{R}$ -module is semisimple and so it is t-semisimple .But $\mathbb{R} \oplus \mathbb{R}$ is not strongly t-semisimple: To see this

Let N= $\mathbb{R} \oplus (0)$. As < (0, 0)> is the only direct summand fully invariant of $\mathbb{R} \oplus \mathbb{R}$, such that <(0,0)>≤N= $\mathbb{R} \oplus (0)$.But <(0,0)>$\nleq_{\text{tes}}$ N because if we assume that <(0,0)> $\leq_{tes}$ N then <(0,0)> +Z_2(N) $\leq_{ess}$ N, so that <(0,0)> +<(0,0)>= <(0,0)> $\leq_{ess}$N which is a contradiction.

(3) If M is Z_2-torsion, then M is strongly t-semisimple.

Proof: Since M is Z_2-torsion, Z_2(M) =M. So that for all A $\leq$ M,

$Z_2(A) = Z_2(M) \cap A = M \cap A = A$, then (0) + $Z_2(A) = A \leq_{ess} A$.

Thus (0) $\leq_{tes}$ A for all A $\leq$ M by Proposition (1.1.17(3)).But (0) is a direct summand of M, and (0) is fully invariant. Hence M is strongly t-semisimple. □

(4) Every singular module is strongly t-semisimple.

Proof: Let M be a singular R-module. Then Z (M) $=$ M, it follows that $Z_2(M) =$ Z (M) $=$ M. Thus M is Z_2 –torsion, hence M is strongly t-semisimple by part (2). □

Thus, in particular Z_n as Z-module is strongly t-semisimple for all n$\in Z_+$, n$>$ 1 .

(5) The converse of (4) is not true in general, for example

Z_4 as Z_4 -module is not singular, but it is Z_2-torsion, so it is strongly t-semisimple.

(6) If M is t-semisimple module and weak duo (SS-module). Then M is strongly t-semisimple.

Proof: Let N $\leq$ M. Since M is t-semisimple, there exists K$\leq^{\oplus}$ M such that K $\leq_{tes}$ N. But M is SS-module so K is stable; hence K is fully invariant direct summand. Thus M is strongly t-semisimple. □

(7) If M is a t-semisimple module and duo (or fully stable) then M is strongly t-semisimple. Hence every t-semisimple multiplication R-module is strongly t-semisimple.

(8) If M is cyclic t-semisimple module over commutative ring R then M is strongly t-semisimple.

Proof: Since M is cyclic module over commutative ring, then M is multiplication module. Thus M is duo. Therefore the result follows by part (7).

By the following Theorem we shall give several characterizations of strongly t-semisimple module.

Theorem (1.2.3): The following statements are equivalent for an R -module M:

(1) M is strongly t-semisimple,

(2) $\frac{M}{Z_2\,(M)}$ is a fully stable semisimple and isomorphic to a stable submodule of M,

(3) $M = Z_2\,(M) \oplus M'$ where $M'$is a nonsingular semisimple fully stable module and M' is stable in M,

(4) Every nonsingular submodule is stable direct summand,

(5) Every submodule of M which contains $Z_2\,(M)$ is a direct summand of M and $\frac{M}{Z_2\ (M)}$ is fully stable and isomorphic to a stable submodule of M.

Proof: (1) ⇒(4) Let N be a nonsingular submodule of M. Since M is strongly t-semisimple, there exists a fully invariant direct summand K of M such that $K \leq_{tes} N$. Assume that M =$K \oplus K'$ for some $K' \leq M$.Hence N=$(K \oplus K') \cap N$ and so N=$K \oplus (K' \cap N)$ by modular law. Thus $K \leq^{\oplus} N$ and $\frac{N}{K} \cong (N \cap\ K')$. But $K \leq_{tes} N$ implies $\frac{N}{K}$ is Z_2 -torsion, that is $Z_2\left(\frac{N}{K}\right) = \frac{N}{K}$ (by Proposition (1.1.17)). On the other hand $(N \cap\ K')$≤ N and N is nonsingular, so $(N \cap\ K')$ is nonsingular submodule, and hence $\frac{N}{K}$ is nonsingular, which implies that $Z_2\ \left(\frac{N}{K}\right) = 0$.Thus $\frac{N}{K} = 0$ and hence $N = K$. Therefore N is a fully invarent direct summand, and hence N is a stable direct summand.

(4)⇒(3) Let M' be a complement of $Z_2\ (M)$.Hence $M' \oplus Z_2\ (M) \leq_{ess} M$

And so $M' \leq_{tes} M$ by Proposition (1.1.17(3)).Thus $\frac{M}{M'}$ is Z_2 -torsion, by proposition (1.1.17. (4)).We claim that $M'$is nonsingular. To explain our assertion, suppose x∈ $Z(M')$, so $x \in M' \leq M$ and ann(x)$\leq_{ess} R$.Hence ann(x) $\leq_{tes}$R and this implies $x \in Z_2$(M). Thus $x \in Z_2(M)$∩ M'=(0), thus x=0 and M' is a nonsingular. So that by hypothesis, M' is a stable direct summand of M and so that M =$M' \oplus L$ for some $L \leq M$.Thus L≅ $\frac{M}{M'}$ which is Z_2 −torsion, hence L is Z_2-torsion .On other hand, $Z_2(M) = Z_2(M')$+ $Z_2(L)$=0+$L = L$.

It follows that $M = Z_2(M) \oplus M'$ and M' is a nonsingular hence M' is stable in M by condition (4). Now let $N \leq M'$, so N is a nonsingular and hence N is stable direct summand in M by hypothesis. It follows that $M = N \oplus W$ for some $W \leq M$ and hence $M' = (N \oplus W) \cap M'$ and so $M' = N \oplus (W \cap M')$ by modular law. Thus $N \leq^{\oplus} M'$ and hence M' is semisimple . On the other hand every submodule N of $M'$is fully invariant, by Lemma 1.1.40(2) but $N \leq^{\oplus} M$, so that N is stable in M' and hence M' is fully stable.

(3)⇒(1) Let $M = Z_2(M) \oplus M'$, where $M'$is a nonsingular semisimple fully stable module, M' is stable in M. Let $N \leq M$, then$(N \cap M') \leq M'$, so$(N \cap M') \leq^{\oplus} M'$ (since M' is semisimple). It follows that $M' = (N \cap M') \oplus W$ for some $W \leq M'$and hence $M = Z_2(M) \oplus (N \cap M') \oplus W$. Hence$(N \cap M') \leq^{\oplus} M$. On other hand,$\frac{N}{N \cap M'} \cong \frac{N+M'}{M'} \leq \frac{M}{M'} \cong Z_2(M)$. But $Z_2(M)$ is Z_2 −torision. Hence,$\frac{N}{N \cap M'}$ is Z_2 −torision and then by (Proposition 1.1.17(4)) $(N \cap M') \leq_{tes} N$. But $(N \cap M')$ is stable in M'(since M' is fully stable) and since M' is stable in M, then by Lemma (1.1.38) $N \cap M'$ is fully invariant in M. But $N \cap M'$is direct summand of M. Thus $N \cap M' \leq^{\oplus} M$ and $N \cap M' \leq_{tes} N$, hence M is strongly t-semisimple.

(3)⇒(5) Let $N \leq M, N \supseteq Z_2(M)$. Since $M = Z_2(M) \oplus M'$, where M' is a nonsingualr semisimple fully stable, M' is stable in M. Then $N = (Z_2(M) \oplus M') \cap N = Z_2(M) \oplus (N \cap M')$ by modular law. But $(N \cap M') \leq M'$ and $M'$is semisimple implies$(N \cap M') \leq^{\oplus} M'$. It follows that $(N \cap M') \oplus W = M'$for some $W \leq M'$ Hence $M = Z_2(M) \oplus (N \cap M') \oplus W = N \oplus W$.

Thus $N \leq^{\oplus} M$, also $\frac{M}{Z_2(M)} \cong M'$ and M' is a fully stable module and M' is stable in M, so that $\frac{M}{Z_2(M)}$ is fully stable semisimple and isomorphic to stable submodule of M.

(2)⇒(3) Since $Z_2(M)$ is t-closed, $\frac{M}{Z_2(M)}$ is nonsingular. By condition (2), $\frac{M}{Z_2(M)}$ is semisimple, hence $\frac{M}{Z_2(M)}$ is projective by [23, Corollary 1.25, P.35]. Now let $\pi: M \to \frac{M}{Z_2(M)}$ be the natural epiomorphism and as $\frac{M}{Z_2(M)}$ is projective, we get $ker\,\pi = Z_2(M) is\ a$ direct summand of M. Hence $M = Z_2(M) \oplus M'$.Thus $M' \simeq \frac{M}{Z_2(M)}$ which is a nonsingular semisimple fully stable module. Then M' is nonsingular semisimple fully stable .Also $M'$is stable submodule of M by condition (2).

(3)⇒(2) By condition (3), $M = Z_2(M) \oplus M'$, where M', is a nonsingular semisimple fully stable module and $M'$is stable in M. It follows that $\frac{M}{Z_2(M)} \simeq M'$. Thus $\frac{M}{Z_2(M)}$ is semisimple fully stable and isomorphic to stable submodule $M'\ of\ M$.

(2)⇒(5) It follows directly (since (2)⇔(3)⇒(5) then(2)⇒(5)).

(5)⇒(2) Let $\frac{N}{Z_2(M)} \leq \frac{M}{Z_2(M)}$. Then $N \supseteq Z_2(M)$, so by condition (5), N is stable direct summand of M, so that $N \oplus W = M$ for some $W \leq M$.Thus $\frac{N}{Z_2(M)} + \frac{W+Z_2(M)}{Z_2(M)} = \frac{M}{Z_2(M)}$. But we can show that $\frac{N}{Z_2(M)} \cap \frac{W+Z_2(M)}{Z_2(M)} = 0$, as follows.

Let $\bar{x} \in \frac{N}{Z_2(M)} \cap \frac{W+Z_2(M)}{Z_2(M)}$. Then $x = n + Z_2(M) = w + Z_2(M)$ for some $n \in N, w \in W$, and so $n - w \in Z_2(M) \subseteq N$. It follow that $n - w = n_1$ for some $n_1 \in N$ and hence $n - n_1 = w \in N \cap W = 0$. Thus $x = 0_{\frac{M}{Z_2(M)}}$ and so $\frac{N}{Z_2(M)} \oplus \frac{w+Z_2(M)}{Z_2(M)} = \frac{M}{Z_2(M)}$. This implies $\frac{M}{Z_2(M)}$ is semisimple. By condition (5), $\frac{M}{Z_2(M)}$fully stable and isomorphic to stable submodule of M. But $\frac{M}{Z_2(M)}$ is nonsingular, so $\frac{M}{Z_2(M)}$ is projective and hence $M = Z_2(M) \oplus M'$. Thus $M'$is nonsingular semisimple (since $M' \simeq \frac{M}{Z_2(M)}$). It follows that M' is a fully stable module and $M'$is stable in M. □

Examples (1.2.4):

(1) Let $M = Q \oplus Z_n$ as Z-module. $Z_2(M) = Z_n$, $\frac{M}{Z_2(M)} \cong \frac{M}{Z_n} \cong Q$ is not semisimple. Hence M is not t-semisimple, so it is not strongly t-semisimple.

(2) Let $\mathrm{M} = \mathrm{Z}_6 \oplus \mathrm{Z}_6$ as Z_6-module. M is t-semisimple since $\frac{M}{Z_2(M)} = \frac{M}{(0)} \cong M$ is semisimple. But $\frac{M}{Z_2(M)} \cong M$ is not fully stable, hence by Theorem (1.2.3) M is not strongly t-semisimple.

Recall that "an R-module M is called quasi-Dedekind if Hom $(\frac{M}{N}, M) = 0$ for all nonzero submodule N of M. Equivantally, M is quasi-Dadekind if for each $f \in End(M), f \neq 0$, then $Ker\ f = 0$.[29]

Proposition (1.2.5): If M is a quasi-Dedekind, then M is t-semisimple if and only if M is strongly t-semisimple.

Proof:$\Rightarrow$ since M is quasi-Dedekind, then for each $f \in End\ (M)\ \ if\ f \neq 0$, then f is monomorphism, and hence $\ker f = (0)$ which is stable and then by [36, Proposition 1.16], M is SS-module and so that M is strongly t-semisimple by Remarks and Examples 1.2.2(5).

$\Leftarrow$ It is clear. □

To prove the next result, we state and prove the following Lemma.

Lemma (1.2.6): Let $K \leq N \leq M$ such that K is a fully invariant direct summand of M. Then K is a fully invariant submodule in N.

Proof: To prove K is a fully invariant submodule of N. Let φ: $N \rightarrow N$ be an R-homomorphism, to show $\varphi\ (K) \leq K$.

Consider the sequence $M \xrightarrow{\rho} K \xrightarrow{i} N \xrightarrow{\varphi} N \xrightarrow{j} M$. Where ρ is the natural projection and i, j are the inclusion mappings. Then $(j \circ \varphi \circ i \circ \rho) \in End(M)$, and since K is a fully invariant in M,so $(j \circ \varphi \circ i \circ \rho)(K) \subseteq K$. But $j \circ \varphi \circ i \circ \rho(K) = \varphi(K)$. Thus K is a fully invariant submodule of N. □

Proposition (1.2.7): Every submodule of strongly t-semisimple module is strongly t-semisimple.

Proof: Let M be a strongly t-semisimple R-module and $N \leq M$. Assume $W \leq N$, so $W \leq M$. Since M is strongly t-semisimple, there exists fully invariant direct summand K of M such that $K \leq_{tes} W \leq N$. Hence by Lemma (1.2.6) K is fully invariant -submodule of N. As $K \leq^{\oplus} M$, $M = K \oplus K'$ for some $K' \leq M$ then, N $=N \cap (K \oplus K')$=$K \oplus (K' \cap N)$. So that $K \leq^{\oplus} N$. Therefore, K is fully invariant direct summand of N such that $K \leq_{tes} W \leq N$.Thus N is a strongly t-semisimple module. □

Now we consider the direct sum of strongly t-semisimple. First we notice that direct sum of strongly t-semisimple modules need not be strongly t-semisimple for example:

Consider $\mathbb{R}$ as $\mathbb{R}$ -module. $\mathbb{R}$ is strongly t-semisimple. But M= $\mathbb{R} \oplus \mathbb{R}$ is not strongly t-semisimple by Remarks and Examples 1.2.2(8). However, the direct sum of strongly t-semisimple is strongly t-semisimple under certain conditions. Before giving our next result, we present the following lemma.

Lemma (1.2.8): Let M_1 *and* M_2 be R-modules such that $annM_1 + annM_2 = R$.Then Hom (M_1, M_2) =0 and Hom$(M_2, M_1) = 0$.

Proof: Since $R = annM_1 + annM_2$, then $M_1 = M_1(annM_1) + M_1(annM_2)$.

Put $annM_1 = A_1$, $annM_2 = A_2$, therefore $M_1 = M_1A_1 + M_1A_2 = M_1A_2$, then for

each $\varphi \in Hom\ (M_1, M_2)$,$\varphi(M_1) = \varphi(M_1)A_2 \leq M_2A_2 = 0$,hence $\varphi = 0$.Thus Hom$(M_1, M_2) = 0$ .Similarly, $Hom(M_2, M_1) = 0$. □

Theorem (1.2.9): Let $M = M_1 \oplus M_2$ such that $annM_1 + annM_2 = R$. Then M_1, M_2 are strongly t-semisimple if and only if $M = M_1 \oplus M_2$ is strongly t-semisimple.

Proof: $\Leftarrow$ It follows by Proposition (1.2.7).

$\Rightarrow$ Let $N \le M$. Since $annM_1 + annM_2 = R$, $N = N_1 \oplus N_2$ for some N_1 *and* N_2 submodules of M_1 and M_2 respectively by [1, Theorem 4.2]. As M_1 *and* M_2 are strongly t-semisimple, then there exist $K_1 \le M_1$ and $K_2 \le M_2$ such that K_1 is a fully invariant direct summand of M_1, and K_1 is t-essential in N_1, K_2 is a fully invariant direct summand of M_2 , and K_2 is t-essential in N_2.But $K_1 \le^{\oplus} M_1$ and $K_2 \le^{\oplus} M_2$ imply $K_1 \oplus K_2 \le^{\oplus} M_1 \oplus M_2$ and $K_1 \le_{tes} N_1$, $K_2 \le_{tess} N_2$ imply $K_1 \oplus K_2 \le_{tess} N_1 \oplus N_2$ by Proposition 1.1.22 and $End(M) \cong \begin{pmatrix} EndM_1 & Hom(M_2, M_1) \\ Hom(M_1, M_2) & EndM_2 \end{pmatrix} = \begin{pmatrix} EndM_1 & 0 \\ 0 & EndM_2 \end{pmatrix}$ by Lemma 1.2.8.

Let $\varphi = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$ for some $\varphi_1 \in EndM_1$, $\varphi_2 \in End(M_2)$. Then $\varphi(K_1 \oplus K_2) = \varphi_1(K_1) \oplus \varphi_2(K_2) \le K_1 \oplus K_2$ since K_1 is fully invariant in M_1 and K_2 is fully invariant in M_2. Hence M is strongly t-semisimple. □

Now we shall give other characterizations of strongly t-semisimple module.

Proposition (1.2.10): The following statements are equivalent for a module M, such that any direct summand has a unique complement:

(1) M is strongly t-semisimple,

(2) For each submodule N of M, there exists a decomposition $\mathrm{M} = \mathrm{K} \oplus L$ such that $\mathrm{K} \le \mathrm{N}$ and L is stable in M and $N \cap L \le Z_2(L)$,

(3) For each submodule N of M , $N = K \oplus K'$ such that K is a direct summand stable in M and K' is Z_2-torsion.

Proof: (1) $\Rightarrow$ (2) Let $N \leq M$ and let K be a complement of $Z_2(N)$ in N. Then $K \oplus Z_2(N) \leq_{ess} N$ and let C be a complement of $K \oplus Z_2(M)$ in M.So $K \oplus Z_2(M) \oplus C \leq_{ess} M$ and hence $K \oplus Z_2(M) \oplus C \leq_{tes} M$. But M is strongly t-semisimple implies M is t-semisimple, hence $K \oplus Z_2(M) \oplus C = M$(by Corollary(1.1.49) .Put $Z_2(M) \oplus C = L$. Then $M = K \oplus L$ and hence $N = (K \oplus L) \cap N = K \oplus (N \cap L)$ (by modular law). But $K + Z_2(N) \leq_{ess} N$ implies $\frac{N}{K}$ is Z_2-torsion (by Proposition (1.1.17)). On the other hand, $\frac{N}{K} \cong N \cap L$, so that $N \cap L$ is Z_2-torsion. Thus $N \cap L = Z_2\,(L \cap N) \leq Z_2(L)$. Now, C is a complement of $K \oplus Z_2(M)$ which is a direct summand of M, so by hypothesis ,C is the unique complement and hence by[1,Theorem 4.8, p.31] C is stable in M and hence $L = Z_2(M) \oplus C$ is a stable submodule in M. Thus $M = K \oplus L$ is the desired decomposition.

(2)$\Rightarrow$(3) By condition (2) $M = K \oplus L$ such that $K \leq N$, L is stable in M and $N \cap L \leq Z_2(L)$. Hence $N = (K \oplus L) \cap N = K \oplus (L \cap N). Put$ $K' = L \cap N$, so $N = K \oplus K'$, $\frac{N}{K} \simeq K' = L \cap N$ is Z_2-torsion , K is stable in M (since K is complement of L which is direct summand of M).

(3)$\Rightarrow$(1) By condition (3), $N = K \oplus K'$, $K \leq^{\oplus} M$ and K is stable in M and K' is Z_2-torsion. Then $K \leq^{\oplus} M$, $K \leq N$ and $\frac{N}{K} \simeq K'$ is Z_2-torsion. Hence $K \leq_{tes} N$ and so that M is strongly t-semisimple. □

Definition (1.2.11)[5]: "An R-module M is called comultiplication if $ann_M\, ann_R N = N$ for every submodule N of M."

To prove the next result. We need the following Lemma.

Lemma (1.2.12): Every multiplication module is fully stable.

Proof: Let M be a comultiplication R-module. Then $ann_M ann_R N = N$ for all $N \leq M$.Hence $ann_M ann_R(xR) = xR$ for all cyclic submodule xR in M.Thus M is fully stable by [1, Corollary 3.5, p.22]. □

Proposition (1.2.13): Let M be a comultiplication $R-$module. Then M is t-semisimple if and only if M is strongly t-semisimple.

Proof: ⇐ It is clear

⇒ It follows directly by Lemma 1.2.12 and Remarks and Examples 1.2.2(6). □

Recall that" an R-module M is called a principally injective if for any $a \in R$, any R -homomorphism $f: aR \to M$ extends to an R-homomorphism from R_R to M''Equivalently M is principally injective if and only if $ann_M ann_R(x) = (x)''$ [26].

Corollary (1.2.14): Let M be a principally injective. Then M is t-semisimple if and only if M strongly t-semisimple.

Proof: ⇐ It is clear.

⇒ M is principally injective implies M is fully stable by [1,Corollary 3.5,P.22] and so by Remark and Examples1.2.2(6), M is strongly t-semisimple. □

Recall the following:

"For R-modules M and N. M is called N-injective if for each monomorphism $h: A \longmapsto N$ where A is any submodule of N and any homomorphism $\psi: A \longmapsto M$, there is a homomorphism $\phi: N \longmapsto M$ such that $\phi \circ h = \psi''$.[28].[17]." M is called injective module if M is N-injective, for any R-module M. M is called self-injective (quasi-injective) if M is M-injective "[28] ,[17].

Corollary (1.2.15): Let M be an injective R- module. Then M is t-semisimple $R-$ module if and only if M is strongly t-semisimple.

Recall the following definition.

Definition (1.2.16)[26]: "An R-module M is called scalar if for all $\varphi \in End\ (M)$, there exists $r \in R$ such that $\varphi(x) = xr$ for all $x \in M$, where R is a commutative ring."

Proposition (1.2.17): Let M be a scalar R-module, where R is commutative.

Then M is t-semisimple if and only if M is strongly t-semisimple

Proof: $\Leftarrow$ It is clear.

$\Rightarrow$ Let $N \leq M$ and, let $\varphi \in End\ (M)$. Since M is scalar, there exists $r \in R$ such that $\varphi(x) = xr$, for all $x \in M$. Hence $\varphi(N) = Nr \leq N$ and so that N is fully invariant submodule. Thus M is duo. But M is duo and t-semisimple implies M is strongly t-semisimple by Remarks and Examples 1.2.2(6). □

Proposition (1.2.18): If R is semisimple then every duo R-module is strongly t-semisimple.

Proof: Since R is semisimple, then every R-module is semisimple and so every R-module is t-semisimple. Then by Remarks and Examples 1.2.2(6), every duo R-module is strongly t-semisimple.□

Now we introduce the following:

Definition (1.2.19): An R-module M is called t-uniform if every submodule of M is t-essential.

Remark (1.2.20): A uniform modules and t-uniform are independent concepts. The following two examples show that.

(1) Z as Z –module is uniform, but $(0) \nleq_{tes}$ Z (since (0) + $Z_2(Z) = 0 \nleq_{ess}$Z).

(2) Let $M = Z_6$ as Z-module, $Z_2(M) = Z_6 = M$,$(\bar{0}) \leq_{tes} M$ since $(\bar{0}) + Z_2(M) = M \leq_{ess} M$. $N_1 = <\bar{2}> \leq_{tes} M$ since $<\bar{2}> + Z_2(M) = M \leq_{ess} M$, and similarly $N_2 = <\bar{3}> \leq_{tes} M, N_3 = M \leq_{tes} M$.Thus M is t-uniform, but M is not uniform.

Proposition (1.2.21): If M is a t-uniform module, then $\frac{M}{N}$ is strongly t-semisimple for all $N \leq M$.

Proof: For each $N \leq M$, $N \leq_{tes} M$. Then $\frac{M}{N}$ is Z_2-torsion (by proposition 1.1.17(4)).Hence $\frac{M}{N}$is strongly t-semisimple by Remarks and Examples 1.2.2(2). □

1.3 Strongly t-extending and strongly t-semisimple modules

Recall that″ an R-module M is called t-extending if every submodule is t-essential in a direct summand″ [6]. Equivalently″ M is t-extending if every t-closed is direct summand″ [6].

Some authors said that M is CLS-extending if every Y-closed submodule is a direct summand [40]. Thus the concepts t-extending modules and CLS-extending modules are coincide.

Recall that, "M is called strongly extending if every submodule of M is essential in a stable direct summand. Equivalently M is strongly extending if and only if every closed submodule is stable direct summand″ [35]. Also, this concept is studied in [18].

Asgari in [7] proved that every t-semisimple module is t-extending. We shall see later that strongly t-semisimple module implies strongly t-extending module which is introduced in [20]:

Definition (1.3.1)[20]: ″An R-module M is called strongly t-extending if every submodule is t-essential in a stable direct summand″.

We study this class of modules, so many characterizations and properties of this class of modules are given. Also some connections between strongly t-extending and other classes of modules such as strongly extending, strongly t-semisimple are introduced.

The following Proposition gives a characterization of strongly t-extending which is appeared in [20]. However we present a different proof.

Proposition (1.3.2):″ An R-module M is strongly t-extending if and only if every t-closed submodule is stable direct summand.″

Proof: $\Rightarrow$ Let N be a t-closed submodule of M.Since M is strongly t- extending, $N \leq_{tes} K$ for some $K \leq M$, K is stable direct summand .As N is t-closed, $N = K$, so N is stable direct summand.

$\Leftarrow$ Let $A \leq M$. By [20, Lemma 2.3] there exists a t-closed submodule K of M such that $A \leq_{tes} K$, . By hypothesis, K is a stable direct summand, .Thus M is strongly t-extending. □

Remarks (1.3.3):

(1) Every singular R-module is a strongly t-extending.

Proof: Let M be a singular R-module, then M is the only t-closed submodule of M by Proposition (1.1.33) and M is stable direct summand; hence M is strongly t-extending by Proposition 1.3.2. □

(2)″Every strongly t-extending module is t-extending, but the convers doesn't hold in general as the following example shows″**:″** Let F be a field, $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and M be an arbitrary R-module. Then $Z_2(M) \oplus R$ is a t-extending module which is not strongly t-extending since R_R is not strongly extending.″[20, Example 3.4]

(3) If M is a t-extending R-module and duo module then M is strongly t-extending.

Proof: Let $N \leq_{tc} M$. Since M is t-extending, then N is a direct summand. But M is duo, so N is fully invariant submodule of M. Hence N is a stable by [35, Lemma 2.1.6, P.21].Thus M is strongly t-extending. □

(4) If M is a multiplication t-extending module, then M is strongly t-extending.

Proof: As M is a multiplication module, then M is a duo module, hence the result follows by part (3). □

(5) If M is cyclic module over commutative ring and M is t-extending, then M is strongly t-extending.

Proof: As M is cyclic module over commutative ring, M is a multiplication module .Hence the result follows by part (4). □

(6) Let R be a commutative self-injective ring ″that is R as R-module is injective, then M is strongly t-extending.

Proof: R is self-injective implies R is extending, so R is t-extending but R is cyclic, hence R is strongly t-extending by part (5). □

(7) If M is a multiplication t-semisimple, then M is strongly t-extending.

Proof: Since M is t-semisimple, M is t-extending [7, Proposition 2.16].But M is multiplication t-extending hence M is strongly t-extending by part (4). □

(8) Every SS -t-extending -module is strongly t-extending.

Proof: It is easy. □

Examples (1.3.4):

(1) For all n$\in Z_+$, n>1, Z_n is t-semisimple multiplication Z-module. So Z_n is strongly t-extending by Remarks1.3.3 (7).

(2)Z_{P^∞} as Z-module is strongly t-extending, since (0), Z_{P^∞} are the only t-closed which are stable direct summands.

Theorem (1.3.5): If M is a strongly t-semisimple module, then it is strongly t-extending, hence every multiplication t-semisimple is strongly t-extending.

Proof: Let N be a t-closed submodule of M .As M is strongly t-semisimple, then M is t-semisimple and hence M is t-extending thus $N \leq^{\oplus} M$.To prove N is fully invariant in M. Since M is strongly t-semisimple, there exists a fully invariant direct summand K of M such that $K \leq_{tes} N$ and hence $K + Z_2(N) \leq_{ess} N$. Also M is strongly t-semisimple implies N is strongly t-semisimple, and so N is t-semisimple .Then by Corollary 1.1.49, N has no proper t-essential submodule containing Z_2(N).But $K + Z_2$(N) $\leq_{tes} N$, hence $K + Z_2(N) = N$. As N is t-closed, $N \supseteq Z_2(M)$ and so $Z_2(N) = Z_2(M) \cap N = Z_2(M)$.Thus $K + Z_2(N) = K + Z_2(M) = N$. Since $K\ and\ Z_2(M)$ are fully invariant submodules of M, then N is a fully invariant by Lemma 1.1.39(1).Hence M is strongly t-extending. □

The converse of Theorem 1.3.5 is not true in general. Consider Q as Z-module. (0) and Q are the only t-closed submodules of the Z-module Q and they are stable direct summands, hence Q is strongly t-extending .But Q is not strongly t-semisimple by Remarks and Examples 1.2.2(7).

The following Theorem is a consequence of Theorem 1.3.5 and Proposition 1.3.2

Theorem (1.3.6): If M is a strongly t-semisimple module, then $N + Z_2(M)$ is t-closed stable direct summand for every submodule N of M.

Proof: As M is strongly t-semisimple, implies M is t-semisimple, hence $N + Z_2(M)$ is a closed submodule of M,for every submodul N of M by Corollary 1.1.50. But $N + Z_2(M)$ is closed and $N + Z_2(M) \geq Z_2(M)$ imply $N + Z_2(M)$ is t-closed, by proposition 1.1.28 (4⇔ 2). On the other hand, M is strongly t-semisimple implies M is strongly t-extending by Theorem (1.3.5) and hence by Proposition (1.3.2) $N + Z_2(M)$ is stable direct summand. □

The following observation mention in [20].

Remark (1.3.7): "If M is strongly extending then M is strongly t-extending."

Proof: Let N be a submodule of M. Since M is strongly extending, then N is essential in a stable direct summand K of M. Hence N is t-essential in K, and K is stable direct summand .Thus M is strongly t-extending. □

Example (1.3.8): Consider M $= Z_n \oplus Z$ as a Z-module where n is a positive integer. We shall see by Theorem 1.3.11(4), M is strongly t-extending. However M is not strongly extending, of M.

The following Theorem was given in [20]. A different proof is introduced

Proposition (1.3.9): Let M be a nonsingular module. Then M is strongly extending if and only if M is strongly t-extending.

Proof: $\Rightarrow$ It follows by Remark (1.3.7).

$\Leftarrow$ Let $N \leq M$. Since M is strongly t-extending,$N \leq_{tes} K$, for some stable direct summand K of M. But M is nonsingular, then K is nonsingular hence $N \leq_{ess} K$. Thus M is strongly extending. □

Proposition (1.3.10): Let M be a multiplication t-semisimple R-module.Then $\frac{M}{C}$ is semisimple fully stable for every t-closed submodule C of M, and the converse hold .

Proof: Let C be a t-closed submodule of M. Then $\frac{M}{C}$ is nonsingular by Proposition 1.1.28(6) and semisimple by [7, Corollary 2.17]. But M is a multiplication R-module implies $\frac{M}{C}$ is multiplication t-semisimple module so $\frac{M}{C}$ is duo and hence $\frac{M}{C}$ is strongly t-semisimple. Then by Theorem 1.3.5, $\frac{M}{C}$ is strongly t-extending. As $\frac{M}{C}$ is nonsingular, we conclude that $\frac{M}{C}$ is strongly extending by Proposition 1.3.9. It follows that $\frac{M}{C}$ is fully stable by [35, Remarks and Examples 2.2.2(11)].

The converse holds by Theorem 1.2.3. □

The following Theorem gives characterizations of strongly t-extending is appeared in [20]. However we present a different proof.

Theorem (1.3.11): The following statements are equivalent for an R-module M.

(1) M is strongly t-extending;

(2) Each t-closed submodule of M is a fully invariant direct summand.

(3) M is t-extending and each direct summand of M which contains $Z_2(M)$ is fully invariant.

(4) $M = Z_2(M) \oplus M'$ Where M' is a strongly extending module.

(5) Every submodule of M which contains $Z_2(M)$ is essential in a fully invariant direct summand.

(6) Every submodule of M which contains $Z_2(M)$ is t-essential in a fully invariant direct summand.

(7) For every submodule A of M, N is a fully invariant direct summand of M, where $N \supseteq A$ and $\frac{N}{A} = Z_2(\frac{M}{A})$.

(8) For each submodule A, of M, there exists a decomposition $\frac{M}{A} = \frac{N}{A} \oplus \frac{N'}{A}$ such that N is fully invariant direct summand of M and $N' \leq_{tes} M, N \supseteq A$.

Proof: (1)⇒(7) $\frac{M}{N} = \frac{M/A}{N/A} \simeq \frac{M/A}{Z_2(M/A)}$ is nonsingular. Hence N is t-closed in M so N is stable direct summand.

(7)⇒ (4) $\frac{M}{Z_{2}(M)}$ is nonsingular (since $Z_2(M)$ is t-closed) so $Z_2\left(\frac{M}{Z_2(M)}\right) = (0) = \frac{Z_2(M)}{Z_2(M)}$ .By condition (7), $Z_2(M)$ is a fully invariant direct summand, so it is stable. Thus $M = Z_2(M) \oplus M'$, hence $M' \simeq \frac{M}{Z_2(M)}$ is nonsingular. Let C be a closed submodule in M'. Since M' is nonsingular, C is t-closed, hence $\frac{M'}{C}$ is nonsingular. This implies $\frac{M}{Z_2(M)+C}$ is nonsingular, thus $Z_2(M) + C$ is t-closed in M by Proposition (1.1.28). Therefore $Z_2(M) + C$ is stable direct summand in M by condition (7). To prove C is a stable direct summand in M'. Since $Z_2(M) + C$ is stable direct summand of M, then $(Z_2(M) + C) \oplus W = M$. But C is a submodule of M'.

Hence $M' = ([Z_2(M) + C) \oplus W)] \cap \quad M' = C \oplus [(Z_2(M) \oplus W) \cap M']$ by modular law, so $C \leq^{\oplus} M'$. To prove C is stable in M', it is enough to prove C is fully invariant in M'. Let $f: M' \longmapsto M'$. Consider the sequnces $M \xrightarrow{\rho} M' \xrightarrow{f} M' \xrightarrow{i} M$ where ρ is the natural projection and i is the inclusion mapping. Hence $i \circ f \circ \rho \in$ End (M).Therefore $(i \circ f \circ \rho)(Z_2(M) \oplus C) \leq Z_2(M) \oplus C$.But $i \circ f \circ \rho)(Z_2(M) \oplus C) =$ $i \circ f(\rho(Z_2(M) \oplus C)) = i \circ f(C) = f(C)$.Thus $f(C) \leq Z_2(M) \oplus C$, so for any $x \in C$, $f(x) = y + t$ for some $y \in Z_2(M)$ and $t \in C$. But $f: M' \mapsto M'$, so $f(x) \in M'$, hence $f(x) - t = y \in Z_2(M) \cap M'$,so $f(x) - t = y$ =0 , then y=0 and $f(x) = t$ $\in C$. Thus C is a fully invariant in M', but $C \leq^{\oplus} M'$.Hence C is a stable direct summand M'.

(4)⇒(5) Let K be a submodule of M which contains $Z_2(M)$. Since $M = Z_2(M) \oplus M'$, where M' is nonsingular strongly extending. Hence $K = (Z_2(M) \oplus M') \cap K =$ $Z_2(M) \oplus (K \cap M')$ by modular law. Hence $K \cap M' \leq M'$, and $M'$is strongly extending, so $(K \cap M') \leq_{ess} L \leq M'$, for some stable direct summand L of M'.Therefore $Z_2(M) \oplus (K \cap M') \leq_{ess} Z_2(M) \oplus L$, by [25, Corollary 5.1.7, P.110] , hence $K \leq_{ess} Z_2(M) \oplus L$. But we can prove that $Z_2(M) \oplus L$ is stable direct summand of M. Since L $\leq^{\oplus} M'$,so $L \oplus W = M'$, for some $W \leq M'$, but $M =$ $Z_2(M) \oplus \ M'$.Hence $M = Z_2(M) \oplus (L \oplus W)$, thus $Z_2(M) \oplus L \ \leq^{\oplus} M$ .To prove $Z_2(M) \oplus L$ is fully invariant in M . Let f:$M \mapsto M$ so $(f(Z_2(M) \oplus L) = f(Z_2(M)) +$ $f(L) \leq Z_2(M) + f(L)$.

Consider the sequence $M' \xrightarrow{i} M \xrightarrow{f} M \xrightarrow{\rho} M'$ where ρ is the natural projection , and i is the inclusion mapping therefore $(\rho \circ f \circ i)(L) = \rho \circ f(L) = \rho\big(f(L)\big)$. But L is a fully invariant in M', so $(\rho \circ f \circ i)(L) \leq L$.

Hence $\rho\big(f(L)\big) \leq L \leq M'$.Now for any $x \in L$, f$(x) \in M$, so that

$f(x) = y + c$ for some $y \in Z_2(M), c \in M'$. On the other hand, $\rho\big(f(x)\big) =$ $\rho(y + c) = c$, and hence $c \in L$. This implies $f(x) = y + c \in Z_2(M) \oplus L$ that is $f(L) \leq Z_2(M)\oplus L$. Now $f(Z_2(M) \oplus L) = f\big(Z_2(M)\big) + f(L)) \leq Z_2(M) + f(L) \leq Z_2(M) + (Z_2(M)\oplus L)$. Hence $f(Z_2(M)\oplus L)\ ) \leq Z_2(M)\oplus L$. Thus $Z_2(M)\oplus L$ is a fully invariant direct summand of M, so it is stable direct summand. Thus M is strongly t-extending.

(5)⇒(1) Let $A \leq M$. Then $A + Z_2(M) \supseteq Z_2(M)$ so by condition (5), there exists a stable direct summand L of M such that A+ $Z_2(M) \leq_{ess}$ L. But $Z_2(M) \leq A + Z_2(M) \leq L$, so $Z_2(M) \leq L$, also $Z_2(L) = Z_2(M) \cap L = Z_2(M)$. So that $A + Z_2(L) \leq_{ess} L$. Hence $A \leq_{tes} L$ and L is stable direct summand. Thus condition (1) hold.

(1)⇒ (8) Let $A \leq M$. By condition (1) there exists a stable direct summand N of M such that $A \leq_{tes} N$. Since $N \leq^{\oplus} M$, $N\oplus L = M$ for some $L \leq N$. Hence $\frac{M}{A} = \frac{N}{A}\oplus\frac{L+A}{A}$, but $\frac{M}{L+A} \simeq \frac{M/_{A}}{(L+A)/_{A}} \simeq \frac{N}{A}$ .Which is Z_2-torsion by Proposition (1.1.17).Thus $\frac{M}{L+A}$ is Z_2-torsion, hence we conclude that $L + A \leq_{tes} M$ by Proposition (1.1.17).

(8)⇒(1) Let C be a t-closed submodule of M, then $\frac{M}{C}$ is nonsingular, that is $Z(\frac{M}{C})$=0. By condition (8), there exists a decomposition $\frac{M}{C} = \frac{N}{C}\oplus \frac{N'}{C}$ where N is stable direct summand and $N' \leq_{tes} M$. But $\frac{M}{C}$ is nonsingular then $\frac{N}{C}$ is nonsingular, thus C is t-closed in N. Also we have $N' \leq_{tes} M$, hence $\frac{M}{N'}$ is Z_2-torsion and so $\frac{N+N'}{N'} \simeq \frac{N}{C}$ is Z_2-torsion, hence $C \leq_{tes} N$, but C is t-closed in N, so $C = N$. Thus C is stable direct summand of M.

(5) ⇒(6) It is clear.

(6) ⇒(5) Let $A \leq M$ and $A \supseteq Z_2(M)$. Since $A \leq_{tes} K$, where K is fully invariant direct summand, then $A + Z_2(K) \leq_{ess} K$ by Proposition (1.1.17).But $A + Z_2(K) \leq A + Z_2(M) = A$, so $A + Z_2(K) = A$ and $A \leq_{ess} K$.

(1) $\Leftrightarrow$ (2) It follows by Proposition (1.3.2) and Definition 1.3.1.

(3) $\Rightarrow$ (2) Let N be a t-closed of M. Since M is t-extending, N is a direct summand of M. But N is a t-closed, so $N \supseteq Z_2(M)$. Thus by hypothesis N is a fully invariant, and so N is a fully invariant direct summand.

(1) $\Rightarrow$ (3) since M is strongly t-extending, and then M is t-extending. Let $N \leq^{\oplus} M$ and $N \supseteq Z_2(M)$. Since $N \leq^{\oplus} M, N$ is closed. Hence N is t-closed by Proposition (1.1.28). Then by Proposition (1.3.2) N is a stable direct summand, so N is fully invariant. □

Now we give the following result which is another characterization of strongly t-extending

Theorem (1.3.12): Let M be an R –module. Then M is strongly t-extending if and only if for each submodule A of M there exists direct decomposition $M = M_1 \oplus M_2$,such that $A \leq M_1$, where M_1 is stable submodule of M and $A + M_2 \leq_{tes} M$.

Proof:$\Rightarrow$ Suppose M is strongly t-extending. Let A$\leq M$, then there exists a stable direct summand M_1 of M such that $A \leq_{tes} M_1$.Since $M_1 \leq^{\oplus} M$, $M_1 \oplus M_2 = M$ for some $M_2 \leq M$, and as $A \leq_{tes} M_1$ and $M_2 \leq_{tes} M_2$ then $A \oplus M_2 \leq_{tes} M_1 \oplus M_2 = M$ by Proposition 1.1.22(2).

$\Leftarrow$Let $A \leq M$. By hypothesis $M = M_1 \oplus M_2$ with $A \leq M_1, M_1$ is a stable direct summand of M and $A \oplus M_2 \leq_{tes} M$.Let A $\cap B \leq Z_2(M_1)$, B$\leq M_1$. Since $A + M_2 \leq_{tes} M$, then A+ $M_2 + Z_2(M) \leq_{ess} M$ by Proposition. 1.1.17. Hence A+M_2+ $(Z_2 (M_1) \oplus Z_2(M_2)) \leq_{ess} M$. So that A+$Z_2(M_1) \oplus M_2 \leq_{ess} M$ (since $Z_2(M_2) \leq M_2$). Thus $A + Z_2(M_1) \oplus M_2 \leq_{ess} M_1 \oplus M_2$ which implies A+$Z_2(M_1) \leq_{ess} M_1$,hence A$\leq_{tes} M_1$ by Proposition 1.1.17(3). Thus M is strongly t-extending by Definition 1.3.1. □

Proposition (1.3.13): Let M be an R- module such that for every submodule N of M, there exists a t-closed submodule A with $N \leq_{ess} A$ .Then M is strongly t-extending if and only if M is strongly extending.

Proof: $\Rightarrow$ Let $N \leq M$. By our assumption there exists t-closed submodule A of M such that $N \leq_{ess} A$. Since M is strongly t-extending, then A is stable direct summand of M by Theorem 1.3.11(2) and hence M strongly extending.

$\Leftarrow$ It is clear by Remark (1.3.7). □

The following Proposition was given in [20]. A different proof is introduced

Proposition (1.3.14): Any direct summand of a strongly t-extending module is strongly t-extending module.

Proof: If $A \leq^{\oplus} M$.Let $M = A \oplus B$ be a strongly t-extending module, let K be a t-closed submodule of A, we have $\frac{M}{K \oplus B} = \frac{A \oplus B}{K \oplus B} \simeq \frac{A}{K}$. which is nonsingular since K is a t-closed submodule of A.Thus K⊕B is a t-closed of M, but M is strongly t-extending module, therefore K⊕B is a stable direct summand in M and so M = (K⊕B)⊕D for some submodule D of M. Hence $A = [K \oplus (B \oplus D)] \cap A = K \oplus [(B \oplus D) \cap A]$. So that $K \leq^{\oplus} A$. Also $K \oplus B$ is stable in $M = A \oplus B$, which implies that K is stable in A. Let $f: A \to A$ be any R-homorphism. Define $h: M \to M$ by $h(x) = \begin{cases} f(x) & if \ x \in A \\ 0 & otherwise \end{cases}$, hence $h(K) = f(K) \leq A$ (since $f \in End\,(A)$). Moreover $h(K \oplus B) = h(K) + h(B) = f(K) + 0 = f(K)$. But $h(K \oplus B) \leq K \oplus B$(since $K \oplus B$ is stable in M). So that $h(K) \leq (K \oplus B) \cap A = K$. Thus K is a fully invariant submodule of A, so it is stable..Therefore A is strongly t-extending. □

Corollary (1.3.15): Every t-closed submodule of a strongly t-extending is a strongly t-extending.

Proof: It follows directly by Proposition 1.3.2 and Proposition 1.3.14. □

Theorem (1.3.16): Let $\mathrm{M} = \mathrm{M}_1 \oplus \mathrm{M}_2$ where M_1 and M_2 are R-modules with $\mathrm{annM}_1 + \mathrm{annM}_2 = \mathrm{R}$. Then M is strongly t-extending module if and only if M_1 and M_2 are strongly t-extending modules.

Proof:$\Rightarrow$ It is clear by Proposition (1.3.14).

$\Leftarrow$ Let $\mathrm{N} \leq_{\mathrm{tc}} \mathrm{M}$. Since $\mathrm{annM}_1 + \mathrm{annM}_2 = \mathrm{R}$, then $\mathrm{N} = \mathrm{N}_1 \oplus \mathrm{N}_2$ for some $\mathrm{N}_1 \leq \mathrm{M}_1$, $\mathrm{N}_2 \leq \mathrm{M}_2$ by [1,Proposition 4.2,P.28]. As $\mathrm{N} \leq_{\mathrm{tc}} \mathrm{M}$, N_1 is a t-closed in M_1 and N_2 is a t-closed in M_2 by[27,Proposition 2.1.20,P.29]. But M_1 and M_2 are strongly t-extending. N_1 is a fully invariant direct summand of M_1 and N_2 is a fully invariant direct summand of M_2 by Theorem 1.3.11. Since $\mathrm{N}_1 \leq^{\oplus} \mathrm{M}_1$ and $\mathrm{N}_2 \leq^{\oplus} \mathrm{M}_2$ imply $\mathrm{N}_1 \oplus \mathrm{N}_2 \leq^{\oplus} M$, so it is enough to verify that $\mathrm{N}_1 \oplus \mathrm{N}_2$ is a fully invariant in M

$\mathrm{End(M)} \simeq \begin{pmatrix} \mathrm{End(M_1)} & \mathrm{Hom(M_2, M_1)} \\ \mathrm{Hom(M_1, M_2} & \mathrm{End(M_2)} \end{pmatrix}$. But $\mathrm{Hom(M_1, M_2)} = 0$, $\mathrm{Hom(M_2, M_1)} = 0$ by Lemma 1.2.8. Hence $\mathrm{End(M)} \simeq \begin{pmatrix} \mathrm{End(M_1)} & 0 \\ 0 & \mathrm{End(M_2)} \end{pmatrix}$ and so for each $\mathrm{f} \in \mathrm{End(M)}$ $\mathrm{f} = \begin{pmatrix} \mathrm{f}_1 & 0 \\ 0 & \mathrm{f}_2 \end{pmatrix}$ where $\mathrm{f}_1 \in \mathrm{End(M_1)}$, $\mathrm{f}_2 \in \mathrm{End(M_2)}$. Hence $\mathrm{f(N_1 \oplus N_2)} = \mathrm{f_1(N_1) \oplus f_2(N_2)} \subseteq \mathrm{N_1 \oplus N_2}$. Thus $\mathrm{N_1 \oplus N_2}$ is fully invariant submodule of M. □

Proposition (1.3.17): Let A and B be submodules of a module M. If B is a strongly t-extending module and $A \leq_{tc} M$, then $A \cap B$ is a stable direct summand of B.

Proof: As $A \leq_{tc} M$ and $B \leq M$, then $A \cap B \leq_{tc} B$ by Proposition 1.1.31(1). But B is strongly t-extending, so that $A \cap B$ is a stable direct summand in B. □

Proposition (1.3.18): Let M be a semisimple module. Consider the following:

(1) M is strongly t-extending;

(2) M is SS-module (weak duo);

(3) M is duo;

(4) M is fully stable;

(5) M is strongly extending. Then (1)⇔(2)⇔(3)⇔(4)⇔(5).

Proof: (2)⇔(3)⇔(4)⇔(5). [35, Remarks and Examples 2.2.2(11)p.40]

(2)⇒ (1) Let N be a t-closed submodule of M. Since M is semisimple, $N \leq^{\oplus} M$. But M is SS-module, so that every direct summand is stable. Thus N is stable direct summand. Therefor M is strongly t-extending by Proposition (1.3.2)

(1) ⇒ (4) Let A ≤ M. Since M is semisimple, $M = A \oplus B$ for some submodule B of M. On the other hand, M is strongly t-extending, hence by Theorem (1.3.11) there exists a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and M_1 is stable in M and $A \oplus M_2 \leq_{tes} M$. It follows that $M = A \oplus B \leq M_1 \oplus B$ and hence $M = A \oplus B = M_1 \oplus B$, and since M_1 is a stable submodule of M, $A = M_1$ by [1, Theorem 4.8,p.30]. Thus A is a stable submodule of M. Thus every submodule A of M is stable that is , M is fully stable. □

Recall that" an R −module M satisfies SIP if the intersection of two summands of M is a summand in M "[42].

Proposition (1.3.19): If M is a nonsingular strongly t-extending module then M has SIP.

Proof: Since M is strongly t-extending and nonsingular, then, M is strongly extending. Thus M has SIP by [35, Corollary 2.2.8]. □

Remark (1.3.20): If M is t-uniform, then $\frac{M}{N}$ is strongly t-extending for each $N \leq M$.

Proof: It follows by Proposition 1.2.21 and Theorem 1.3.5. □

Proposition (1.3.21): If M is strongly t-extending and indecomposable then $N \leq_{tes} M$ for each $0 \neq N \leq M$

Proof: Let $(0) \neq N \leq M$. Since M is strongly t-extending, there exists a stable direct summand submodule K of M such that $N \leq_{tes} K$.But M is indecomposable, so $K = M$. Thus $N \leq_{tes} M$. □

1.4 Strongly t-semisimple rings

This section concerns with strongly t-semisimple rings. Several characterizations of commutative strongly t-semisimple rings are introduced. Also some characterizations of nonsingular strongly t-semisimple ring, are given.

Proposition (1.4.1): Every commutative t-semisimple ring R is strongly t-semisimple ring R.

Proof: Since R is commutative ring, then R is duo R-module. This implies R is strongly t-semisimple, by Examples and Remarks 1.2.2(6). □

Proposition (1.4.2): Let R be a commutative Artinian ring with $Rad\, R \leq_{tes} R$. Then R is strongly t-semisimple. In particular every commutative local Artinain ring is strongly t-semisimple.

Proof: By [7, Proposition 3.1], R is t-semisimple ring. Hence by Proposition (1.4.1), R is strongly t-semisimple. □

Examples (1.4.3):

(1) The ring Z_{P^∞} is Artnian and $Rad\, Z_{P^\infty} = Z_{P^\infty} \leq_{ess} Z_{P^\infty}$. Thus Z_{P^∞} is t-semisimple. Hence by Proposition (1.4.2), Z_{P^∞} is strongly t-semisimple.

(2) Let R be the ring Z_{P^n}, R is an Artnian local ring , so by(Proposition(1.4.2)) R is strongly t-semisimple

Proposition (1.4.4): The following statements are equivalent for a commutative ring

(1) R is strongly t-semisimple;

(2) R is t-semisimple;

(3) Every R-module is t-semisimple;

(4) Every nonsingular R −module is semisimple;

(5) Every nonsingular R −module is injective;

(6) Fore every R −module M there is an injective submodule M' such that $M = Z_2(M) \oplus M'$;

(7) $\frac{R}{Z_2(R)}$ is a semisimple ring.

(8) Every maximal ideal which contains $Z_2(R)$ is a direct summand;

(9) R is a direct product of two ring, one is Z_2- torsion and the other is semisimple ring.

Proof: (1)⇒(2) it is clear

(2) ⇒ (1) It is follows by (Proposition 1.4.1).

(2) ⇔ (3) ⇔ (4) ⇔ (5) ⇔ (6) ⇔ (7) (see [7, Theorem (3.2)]).

(2) ⇔ (8) ⇔(9) It follows by [7, Theorem 3.8]. □

Corollary (1.4.5):″Let R be a t-semisimple ring(and hence if R is strongly t-semisimple).

(1) A maximal right ideal I of R is a direct summand if and only if it contains $Z_2(R)$.

(2) A minimal right ideal J of R is a direct summand if and only if it is nonsingular ″[7,Corollary 3.9].

Recall that a ring R ″is called quasi-Frobenius if R is self-injective and Noetherain″. Equivalently ″R is quasi-Frobenius if R is self-injective and Artinian.″[21]

Corollary (1.4.6)[7, Corollary 4.6]: ″Let R be a right nonsingular. Then R is quasi-Frobenius if and only if R is semisimple″.

Proposition (1.4.7): Let R be a nonsingular commutative ring. Then the following statements are equivalent:

(1) R is quasi-Frobenius;

(2) R is semisimple ;

(3) R is t-semisimple (R is strongly t-semisimple);

(4) Every R −module is t-semisimple;

(5) Every nonsingular R −module is semisimple;

(6) Every nonsingular R −module is injective;

(7) For every R −module M **,** there exists an injective submodule M' such that $M = Z_2(M) \oplus M'$;

(8) $\frac{R}{Z_2(R)}$ is a semisimple ring.

Proof: (3) ⇔ (4) ⇔ (5) ⇔ (6) ⇔ (7) ⇔ (8) by Proposition (1.4.4).

(1)⇔ (2) It follows by Corollary (1.4.6)

(2) ⇔(3) It follows by [7,Theorem 3.2] and Proposition (1.4.4). □

Proposition (1.4.8): The following statements are equivalent for a commutative ring R

(1) $R\ is\ t$-semisimple (R is strongly t-semisimple);

(2) Every weak duo module (SS-module) is strongly t-semisimple;

(3) Every R −module is t-semisimple.

Proof: (1) ⇔ (3) by Proposition (1.4.4)

(3)⇒(2) It follows by Remarks Examples 1.2.2(5).

(2) ⇒ (1) R is duo (because R is commutative ring with unity), so R is strongly t-semisimple. □

Proposition (1.4.9): The following statements are equivalent for a commutative ring R:

(1) R is t-semisimple(R is strongly t-semisimple);

(2) Every nonsingular R-module is strongly t-semisimple;

(3) For every R-module M, there exists a strongly t-semisimple R-module M' such that $M = Z_2(M) \oplus M'$.

Proof: (1) $\Rightarrow$ (2) Let M be a nonsingular R-module. Hence M is semisimple by Proposition (1.4.7)(3⇒5) and so M is t-semisimple. Also M is injective by (Proposition(1.4.7) ((3) $\Rightarrow$ (6)). It follows that M is strongly t-semisimple by (Corollary (1.2.15))

(2)⇒(1) By condition (2) every nonsingular module M is strongly t-semisimple, hence every nonsingular module M is t-semisimple. Thus every nonsingular is semisimple by (Remarks 1.1.45(3)). It follows that R is t-semisimple by (Proposition (1.4.4) (4)⇒(2)).

(1) $\Rightarrow$ (3) By (Proposition (1.4.4) (2)⇒ (6)), $M = Z_2(M) \oplus M'$ for some injective R-module M'. But $M' \simeq \frac{M}{Z_2(M)}$ which is nonsingular module. Hence M' is semisimple by (proposition (1.4.4) (2)⇒(4)). Thus M' is t-semisimple and injective, so M' is strongly t-semisimple by Corollary (1.2.15).

(3) ⇒(1) $M = Z_2(M) \oplus M'$ where M' is strongly t-semisimple. Hence M' is t-semisimple. But $M' \simeq \frac{M}{Z_2(M)}$ which is nonsingular, so $M'$is nonsingular t-semisimple. Thus $M'$is semisimple by Remarks 1.1.45(3). So that $M'$is injective. Thus R is t-semisimple by (Proposition (1.4.4) (6) ⇒(2)). □

1.5 Strongly t-Baer modules and strongly t- semisimple modules

"For a left ideal I of End (M), set the right annihilator in M of I by $r_M(I) = \{m \in M : Im = 0\}$ and $t_M(I) = \{m \in M : Im \leq Z_2(M)\}$"[33]. Recall that "a module M is (quasi)-Baer if the right annihilator in M of any left (two sided) ideal I of $End(M)$ $r_M(I)$ is a direct summand of M " [33]. A close connection was established between (quasi-)Baer modules and FI-extending modules which is introduced in [33]. As a generalization of t-extending, hence extending module and of a nonsingular Baer, the notion of t-Baer is introduced in [6]. Connections between t-extending and t-Baer were established; see [6, Theorem 3.9].

In this section we introduce the notions of strongly Baer module and strongly t-Baer module. Many connections between these concepts and other related concepts such as Baer module, t-Baer, strongly t-semisimple modules, strongly extending strongly t-extending and noncosingular modules are presented.

Definition (1.5.1):"A module M is called Baer if $r_M(I) \leq^{\oplus} M$ for every left ideal I of S where $S = End(M_R)$."[33].

Definition (1.5.2):" A module M is called abelian Baer (or strongly Baer by some authors) if $r_M(I) \leq^{\oplus} M$ and fully invariant for every left ideal I of S where, $S = End(M_R)$"[34].

Definition (1.5.3):"A module M is called t-Baer if $t_M(I) \leq^{\oplus} M$ for every left ideal I of S where $S = End(M_R)$."[6]

Definition (1.5.4): "A module M is called strongly t-Baer if $t_M(I) \leq^{\oplus} M$ and fully invariant, for every left ideal I of S where, $S = End(M_R)$"[20].

Remarks and Examples (1.5.5):

(1) It is clear every strongly Baer module is Baer module, and every strongly t-Baer module is t-Baer module.

(2) Every Z_2-torsion module is strongly t-Baer.

Proof: Let M be a Z_2-torsion module, so $Z_2(M) = M$. For any left ideal I of $S = End(M_R)$, $t_M(I) = \{m \in M: Im \leq Z_2(M) = M\} = M$.But $M \leq^{\oplus} M$ and it is fully invariant .Thus M is a strongly t-Baer. □

(3) If M is a nonsingular module, then $r_M(I) = t_M(I)$ for every left ideal I of S.

Proof: Since M is nonsingular, $Z_2(M) = 0$. Hence $r_M(I) = \{m \in M: Im = 0\} = \{m \in M: Im \leq Z_2(M) = 0\} = t_M(I)$. □

(4) Let M be a nonsingular R −module. Then M is strongly Baer if and only if M strongly t-Baer.

Proof: It follows directly by (3). □

(5) Let M be a nonsingular R-module. Then M is Baer if and only if M is t-Baer.

(6) The Z-module $Z \oplus Z_2$ is t-Baer which is neither Baer nor Z_2-torsion [6, Example 3.4(1)]

(7) The Z-module Z is strongly t-Baer, since End(Z) $\cong Z$ and so for any ideal I of End(Z),$I = nZ$.Thus $t_Z(I) = \{m \in Z: mnZ \leq Z_2(Z) = 0\} = \{m \in Z: mn = 0\} = \{0\}$ is a fully invariant direct summand of Z.

(8) Let $M = Z \oplus Z$ as Z-module, End $M \simeq \begin{pmatrix} End\ (Z) & End\ (Z) \\ End\ (Z) & End\ (Z) \end{pmatrix} \simeq \begin{pmatrix} Z & Z \\ Z & Z \end{pmatrix}$ and $Z_2(M) = (0) \oplus (0)$

Let $I = \left\{ \begin{pmatrix} 0 & nz \\ 0 & nz \end{pmatrix} : n\ is\ a\ fixed\ postive\ integer \right\}$,then for any $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in End(M)$

$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & n \\ 0 & n \end{pmatrix} = \begin{pmatrix} 0 & xn + yn \\ 0 & zn + wn \end{pmatrix}$. Thus I is a left ideal of $End(M)$.$t_M(I) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in Z \oplus Z: \begin{pmatrix} 0 & n \\ 0 & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} ny \\ ny \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$,hence $t_M(I) = Z \oplus (0) \leq^{\oplus} M$, but $t_M(M)$ is not fully invariant.

For $\varphi: Z\oplus Z \to Z\oplus Z$ which is defined by $\varphi(x,y) = (y,x)$ for all $(x,y) \in Z\oplus Z$. Then, $\varphi(x,0) = (0,x)$ for all $x \in Z$ and $\varphi(t_M(I)) = (0)\oplus Z \nleq t_M(I)$. Thus M is not strongly t-Baer.

Lemma (1.5.6): Let $M = Z_2(M)\oplus M'$, let I be a left ideal of $S = End(M)$, let $\pi: M \to M'$ be the natural epimorphism, and

$A' = \{\pi \circ \theta\left|_{M'}\right.: \theta \in I\}$, let $I' = S'A'$, where $S' = EndM'$. Then $t_M(I) = Z_2(M)\oplus r_{M'}(I')$.

Proof: Let $m_1 + m_2 \in t_M(I)$ so $m_1 \in Z_2(M), m_2 \in M'$ Then $I(m_1 + m_2) \subseteq Z_2(M)$, hence for any $\theta \in I$, $\theta(m_1 + m_2) \in Z_2(M)$, that is $\theta(m_1)$+ $\theta(m_2) \in Z_2(M)$, so $\theta(m_2) \in Z_2(M)$.Now $\pi \circ \theta\left|_{M'}\right.(m_2) = \pi\big(\theta(m_2)\big) = 0$.Thus $m_2 \in r_{M'}(I')$,so $t_M(I) \subseteq Z_2(M)\oplus r_{M'}(I')$.Let $m_1 + m_2 \in Z_2(M)\oplus r_{M'}(I')$, so $m_1 \in Z_2(M)$ and $I'm_2 = 0$. Hence $S'A'm_2 = S'\pi\,\theta\left|_{M'}\right.(m_2) = 0$. This implies $\theta(m_2) \in Z_2(M)$ and so $\theta(m_1 + m_2) = \theta(m_1)$+ $\theta(m_2) \in Z_2(M)$. Thus $m_1 + m_2 \in t_M(I)$, hence $Z_2(M)\oplus r_{M'}(I') \subseteq t_M(I)$.Thus $t_M(I) = Z_2(M)\oplus r_{M'}(I')$. □

Lemma (1.5.7): Let $M = Z_2(M)\oplus M'$, let I' be an ideal of $S' = End(M')$ let $A = \{Id_{Z_2(M)} + \psi: \psi \in I'\}$ and $I = SA$. Then $t_M(I) = Z_2(M)\oplus r_{M'}(I')$.

Proof: Let $m_1 + m_2 \in t_M(I)$, so $\emptyset(m_1 + m_2) \in Z_2(M)$ for any $\emptyset \in I$, then $SA(m_1 + m_2) \subseteq Z_2(M)$, so $S(I + \psi)(m_1 + m_2) = S(m_1 + \psi m_2)$, hence $(m_1 + \psi m_2)\,) \subseteq Z_2(M)$. As $m_1 \in Z_2(M)$, so that $\psi(m_2) \in Z_2(M)$. Hence $\psi(m_2) \in Z_2(M) \cap M'$=0, so $m_2 \in r_{M'}(I')$. Thus $m_1 + m_2 \in Z_2(M)\oplus r_{M'}(I')$. Conversely, let $m_1 + m_2 \in Z_2(M)\oplus r_{M'}(I')$. Then $m_1 \in Z_2(M)$ and $m_2 \in r_{M'}(I')$. So $m_1 \in Z_2(M)$ and $\psi(m_2) = 0$, $I(m_1 + m_2) = SA(m_1 + m_2) = S(Id_{Z_2(M)} + \psi)(\,m_1, m_2) = S(m_1 + \psi(m_2) = Sm_1 \subseteq Z_2(M)$,then $m_1 + m_2 \in t_M(I)$.Thus $Z_2(M)\oplus r_{M'}(I') \subseteq t_M(I)$. Hence $t_M(I) = Z_2(M)\oplus r_{M'}(I')$. □

The following result gives some characterizations of strongly t-Baer module which are analogous generalization of Theorem 3.2 in [6].

Theorem (1.5.8): The following statements are equivalent for a module M:

(1) M is strongly t-Baer;

(2) $M = Z_2(M) \oplus M'$ where M' is a (nonsingular) strongly Baer module;"[20]

(3) M has strongly summand intersection property for direct summands which contain $Z_2(M)$ and $\varphi^{-1}(Z_2(M)) \leq^{\oplus} M$ and fully invariant for all $\varphi \in S = End(M)$;

(4) $\cap\, \varphi^{-1}(Z_2(M)) \leq^{\oplus} M$ and fully invariant , for every, $\varphi \in S = End(M)$.

Note that (1)⇔(2) is given in[20, Theorem 4.2]. But our proof is different.

Proof: (1) $\Rightarrow$ (2) Since M is strongly t-Baer, $Z_2(M) = t_M(S) \leq^{\oplus} M$ where $S = End(M)$ Hence $M = Z_2(M) \oplus M'$ for some $M' \leq M$; M' is nonsingular. To prove M' is strongly Baer. Let I' be a left ideal of $S' = End(M')$, let $A = \{I + \psi : \psi \in I'\}$ and $I = SA$. Then $t_M(I) = Z_2(M) \oplus r_{M'}(I')$by Lemma (1.5.7). Since M is strongly t-Baer,$t_M(I) \leq^{\oplus} M$ and fully invariant, so $M = t_M(I) \oplus K$ for some $K \leq M$; that is $M = Z_2(M) \oplus r_{M'}(I') \oplus K$.But M'=$(Z_2(M) \oplus r_{M'}(I') \oplus K) \cap M'$= $r_{M'}(I') \oplus [(Z_2(M)) \oplus K) \cap M']$ by modular law. Hence $r_{M'}(I') \leq^{\oplus} M'$. Let $f \in End(M')$,then $I_{Z_2(M)} \oplus f \in S$.Hence $(I \oplus f)(t_M(I) \leq t_M(I)$ since $t_M(I)$ is fully invariant in M so $(I \oplus f)\big(t_M(I)\big) = Z_2(M) \oplus f(r_{M'}(I') \leq Z_2(M) \oplus r_{M'}(I') = t_M(I)$. Thus $f(r_{M'}(I') \leq r_{M'}(I')$, which implies $r_{M'}(I')$ is fully invariant in M'. Thus M' is strongly Baer.

(2) $\Rightarrow$ (1) Assume $M = Z_2(M) \oplus M'$, where M' is strongly Baer. Let $S' = End(M')$, let I be a left ideal of S . Let $\varphi \in I$, hence $\varphi : M \to M$. Consider the sequence $M \overset{\varphi}{\to} M \overset{\pi'}{\to} M'$,where π' the natural projection. Put $A' = \{\pi' \circ \varphi|_{M'} : \varphi \in I\}$, $A' \leq S'$, put $I' = S'A'$. Then $t_M(I) = Z_2(M) \oplus r_{M'}(I')$ by Lemma (1.5.6). Since M' is strongly Baer, $r_{M'}(I') \leq^{\oplus} M'$ and fully invariant in M' . Hence $M' = r_{M'}(I')$

$\oplus K$ for some $K \leq M'$, thus $M = Z_2(M) \oplus M' = Z_2(M) \oplus r_{M'}(I') \oplus K = t_M(I) \oplus K$. Hence $t_M(I) \leq^{\oplus} M$.

To prove $t_M(I)$ is fully invariant in M, let $\psi: M \to M$. Since $t_M(I) = Z_2(M) \oplus r_{M'}(I')$, then for all $m \in t_M(I)$, $m = m_1 + m_2$ such that $m_1 \in Z_2(M)$, $m_2 \in r_{M'}(I')$. $\psi(m) = \psi(m_1) + \psi(m_2)$. Assume $\psi(m_2) = c + d \in M$ such that $c \in Z_2(M)$, $d \in M'$. Now $\pi' \circ \psi|_{r_{M'}(I')}: r_{M'}(I') \to M'$, where π' the natural projection, then $\pi' \circ \psi|_{r_{M'}(I')}(r_{M'}(I')) \subseteq r_{M'}(I')$, hence $\pi' \circ \psi(m_2) \in r_{M'}(I')$, that is $\pi'(c+d) \in r_{M'}(I')$, hence $d \in r_{M'}(I')$.Thus $\psi(m_2) = c + d \in t_M(I)$ where $c \in Z_2(M), d \in r_{M'}(I')$ and so $\psi(m) = \psi(m_1) + \psi(m_2) \in t_M(I)$. Therefore $t_M(I)$ is fully invariant in M.

(1)$\Rightarrow$ (3) Let $\varphi \in S$. Since $\varphi^{-1}\left(Z_2(M)\right) = t_M(S\varphi)$ and M is strongly t-Baer, so $\varphi^{-1}\left(Z_2(M)\right) \leq^{\oplus} M$ and fully invariant submodule of M. To prove M has a strongly summand intersection property for direct summand which contain $Z_2(M)$.Let $N_\lambda \leq^{\oplus} M$,$N_\lambda \supseteq Z_2(M)$, $\lambda \in \wedge$, then for each $\lambda \in \wedge$, $N_\lambda = e_\lambda(M)$ where, e_λ is an idempotent of S.

Let $\mathrm{I} = \Sigma_{\lambda\in\wedge} S(1 - e_\lambda)$ where 1 is the identity mapping on M and let $m \in t_M(I)$. Then $Im = \Sigma_{\lambda\in\wedge}$ S(1-e_λ)m $\leq Z_2(M)$, hence $(1-e_\lambda)m \in Z_2(M)$, for all $\lambda \in \wedge$ and hence , $m \in (1-e_\lambda)^{-1} Z_2(M)$. *Hence* $t_M(I) \subseteq (1-e_\lambda)^{-1}(Z_2(M)) \leq e_\lambda M$, for all $\lambda \in A$. To show this, let $m \in (1-e_\lambda)^{-1}(Z_2(M))$, then $(1-e_\lambda)(m) \in Z_2(M)$ and so $(m - e_\lambda(m) \in Z_2(M)$. This implies $m \in Z_2(M) + e_\lambda(M) = e_\lambda(M)$ (since $e_\lambda(M) \supseteq Z_2(M)$. Thus $(1-e_\lambda)^{-1} Z_2(M) \subseteq e_\lambda(M)$. So that $t_M(I) \subseteq \cap_{\lambda\in\wedge} e_\lambda M = \cap_{\lambda\in\wedge} N_\lambda$.

Now assume that $m \notin t_M(I)$, then $Im \nleq Z_2(M)$. Thus there exists $\lambda_0 \in \wedge$ such that $\left(1 - e_{\lambda_0}\right)m \notin Z_2(M)$; hence $m \notin (1 - e_{\lambda_0})^{-1}(Z_2(M))$, but $(1 - e_{\lambda_0})^{-1}(Z_2(M)) = e_{\lambda_0}(M)$. To show this. Let $x \in e_{\lambda_0}(M)$. Then $x = e_{\lambda_0}(y)$ for some $y \in M$ and so $x = e^2{}_{\lambda_0}(y) = e_{\lambda_0}\left(e_{\lambda_0}(y_0)\right) = e_{\lambda_0}(x)$. Hence $x - e_{\lambda_0}(x) = \left(1 - e_{\lambda_0}\right)(x) = 0 \in$

$Z_2(M)$ and this implies $x \in (1-e_{\lambda_0})^{-1}(Z_2(M))$. Thus $e_{\lambda_0}(M) \subseteq (1-e_{\lambda_0})^{-1}(Z_2(M))$.But $(1-e_{\lambda_0})^{-1}Z_2(M) \subseteq e_{\lambda_0}(M)$. Therefore $(1-e_{\lambda_0})^{-1}(Z_2(M)) = e_{\lambda_0}(M)$. Thus $m \notin \cap_{\lambda\in\Lambda} e_\lambda M$ and so $\cap_{\lambda\in\Lambda} e_\lambda M = t_M(I)$ But (by condition (1) $t_M(I) \leq^{\oplus} M$ and fully invariant). So $\cap_{\lambda\in\Lambda} e_\lambda M$ is a fully invariant direct summand.

(3) $\Rightarrow$ (4) Since $\varphi^{-1}(Z_2(M)) \leq^{\oplus} M$ and a fully invariant submodule of M for each $\varphi \in S$ by condition (3). Then $\cap_{\varphi\in\mathcal{F}} \varphi^{-1}(Z_2(M)) \leq^{\oplus} M$ and fully invariant.

(4)$\Rightarrow$(1) Let I be a left ideal of S. Cleary $t_M(I) = \cap_{\varphi\in I} \varphi^{-1}(Z_2(M))$. By condition(4), $\cap_{\varphi\in I} \varphi^{-1}(Z_2(M))$ is a direct summand and fully invariant of M.Thus $t_M(I)$ is a direct summand and fully invariant of M. □

To prove the next Theorem, first the following Proposition is presented.

Proposition (1.5.9): Let M be a nonsingular strongly extending R-module .Then M is strongly Baer.

Proof: To prove $r_M(I) \leq^{\oplus} M$ and fully invariant for each left ideal I of S. Since every strongly extending module is extending. So M is nonsingular and extending, which implies M is Baer by [33]. Thus $r_M(I) \leq^{\oplus} M$, hence $r_M(I)$is closed submodule and so $r_M(I)$ is stable, since M is strongly extending. Thus M is strongly Baer. □

Recall that "for a submodule N of M. The set $\{\varphi \in S: \varphi(N) \leq Z_2(M)\}$ is denoted by $t_S(N)$" [7]

The following Theorem explains connections between strongly t-semisimple modules with strongly t-extending modules and strongly t-Baer modules.

Theorem (1.5.10): For an R-module M. Consider the following assertions

(1) M is strongly t-semisimple;

(2) M is strongly t-extending and $N = t_M t_S(N)$ for every submodule N of M, which contains $Z_2(M)$.

(3) M is strongly t-Baer and $N = t_M t_S(N)$ for every submodule N of M which contains $Z_2(M)$.

Then (1)→(2)→(3) and((3)→(1) if a complement of $Z_2(M)$ is unique).

Proof: (1) ⇒ (2) By Theorem (1.3.5), M is strongly t-extending Also M is strongly t-semisimple, implies M is t-semisimple. Hence by [7, Proposition 2.19], $N = t_M t_S(N)$ for every submodule N of M which contains $Z_2(M)$.

(2) ⇒(3) Since M is strongly t-extending, $M = Z_2(M) \oplus M'$, M' is nonsingular strongly extending by Theorem(1.3.7). It follows that M' is strongly Baer by Proposition (1.5.9).Hence by Theorem ((1.5.8) (2) ⇒(1)), M is strongly t-Baer.

(3) ⇒(1) Since M is strongly t-Baer, $Z_2(M) = t_M(S)$ is a direct summand of M and fully invariant submodule of M, say $M = Z_2(M) \oplus M'$, hence M' is nonsingular.Let

$N \leq M$ and $N \supseteq Z_2(M)$.Then $N = M \cap N = (Z_2(M) \oplus M') \cap N = Z_2(M) \oplus (M' \cap N)$. Since M is strongly t-Baer, $t_M(t_S(N))$ is a fully invariant and direct summand of M.But by condition (3), $N = t_M t_S(N)$, so N is fully invariant direct summand of M. Thus every submodule which contains $Z_2(M)$ is stable direct summand. Hence $\frac{M}{Z_2(M)}$ is semisimple, but $M' \simeq \frac{M}{Z_2(M)}$, so that M' is semisimple. To prove M' is a fully stable, let $N' \leq M'$. Since $N' \leq^{\oplus} M'$, it is enough to show that N' is fully invariant submodule of M'. Assume $\varphi \in End(M')$. Since $M = Z_2(M) \oplus M'$, φ can be extended to $g: M \to M$ which is defined by $g:(m) = \begin{cases} \varphi(m) & if \quad m \in M' \\ 0 & otherwise \end{cases}$.

Hence $g(Z_2(M) \oplus N') \leq Z_2(M) \oplus N'$, since $N = Z_2(M) \oplus N'$ is a fully invariant submodule of M. But $g(Z_2(M) \oplus N') = g(Z_2(M)) \oplus g(N')$=$g(N') = \varphi(N')$. Thus $g(N') \leq Z_2(M) \oplus N'$ and since $\varphi(N') \leq M'$, then $\varphi(N') \leq (Z_2(M) \oplus N') \cap M' =$

N', that is N' is a fully invariant submodule of M'. Thus M' is fully stable, and so $\frac{M}{Z_{2(M)}}$ is fully stable. To prove M' is a fully invariant in M. Since $M'$is a complement of $Z_2(M)$ and by hypothesis, M' is a unique complement of $Z_2(M)$ so M' is stable in M by [1, Theorem 4.8, P.30] and hence $\frac{M}{Z_{2(M)}} \simeq$ stable submodule. Therefore by Proposition 1.2.3(5→1), M is strongly t-semisimple. □

Asgari proved that ″if M is t-Baer, then so is every direct summand″ [7, Theorem 3.6]. However ″every direct summand of strongly t-Baer is strongly t-Baer” [20, Theorem 4.4]: we give a different a proof, for this fact.

Theorem (1.5.11): If M is strongly t-Baer module, then so is every direct summand of M.

Proof: First, we show that if $\mathrm{M} = M_1 \oplus M_2$ and M_1 is Z_2-torsion, then M_2 is strongly t-Baer. Let I_2 be a left ideal of $S_2 = \mathrm{End}(M_2)$. Since M is strongly t-Baer, M is t-Baer. Hence by the same proof of the 1st paragraph of proof of Theorem 3.6 in [6], $t_{M_2}(I_2) \leq^{\oplus} M_2$. To prove $t_{M_2}(I_2)$ is a fully invariant submodule of M_2.put $A = \{1_{\boldsymbol{M_1}} \oplus \varphi : \varphi \in \mathrm{I}_2\}$, and $\mathrm{I} = \mathrm{SA}$. Since M is strongly t-Baer, $t_M(\mathrm{I}) \leq^{\oplus} M$ and fully invariant. But $t_M(\mathrm{I}) = M_1 \oplus t_{M_2}(I_2)$. Assume $f \in End(M_2)$. Define $h: M \to M$ by $h(x) = \begin{cases} f(x) & if\ x \in M_2 \\ 0 & otherwise \end{cases}$. Hence $h(t_M(\mathrm{I})) \leq t_M(\mathrm{I})$, so that $h(M_1) + h(t_{M_2}(I_2)) \leq t_M(\mathrm{I})$, which implies $h\left(t_{M_2}(I_2)\right) \leq t_M(\mathrm{I}) = M_1 \oplus t_{M_2}(I_2)$. Thus $h(t_{M_2}(I_2)) \leq [M_1 \oplus t_{M_2}(I_2)] \cap M_2 = t_{M_2}(I_2)$. Therefore $t_{M_2}(I_2)$ is fully invariant submodule of M_2 and so M_2 is strongly t-Baer.

Let N be a direct summand of M,say $M = K \oplus \mathrm{N}$, hence $Z_2(M) = Z_2(K) \oplus Z_2(N)$ and as M is strongly t-Baer, $Z_2(M) \leq^{\oplus} M$. It follows that. $Z_2(M) \oplus \mathrm{W} = \mathrm{M}$. Since

$\mathrm{K} = \mathrm{M} \cap \mathrm{K} = (Z_2(M) \oplus \mathrm{W}) \cap \mathrm{K} = (Z_2(K) \oplus Z_2(N) \oplus \mathrm{W}) \cap \mathrm{K}$, so that $K = (Z_2(K) \oplus [Z_2(N) \oplus \mathrm{W}) \cap \mathrm{K}]$, let $U = (Z_2(\mathrm{N}) \oplus \mathrm{W}) \cap \mathrm{K}$, hence $Z_2(K) \leq^{\oplus} K$, and $K = Z_2(K) \oplus U$.Set $L = Z_2(K) \oplus N$. As $M = K \oplus N$, so $M = (Z_2(K) \oplus U) \oplus N$, $(Z_2(K) \oplus N) \oplus U = L \oplus U$,then $L \leq^{\oplus} M$. Also, $L \geq Z_2(K)$, $L \geq N \geq Z_2(N)$. So $L \geq Z_2(K) \oplus Z_2(N) = Z_2(M)$.

But $\mathrm{Z}_2(\mathrm{L}) = \mathrm{Z}_2(\mathrm{Z}_2(\mathrm{K})) \oplus \mathrm{Z}_2(\mathrm{N}) = \mathrm{Z}_2(\mathrm{K}) \oplus \mathrm{Z}_2(\mathrm{N}) = \mathrm{Z}_2(\mathrm{M})$ By using Theorem 1.5.8, we can show that L has strongly summand intersection property for submodules which contains $Z_2(L) = Z_2(M)$ .Now, let $\psi \in \mathrm{End}\ (\mathrm{L})$, $\varphi = \psi\pi_{\mathrm{L}}$, where π_{L} is canonical projection onto L. $\varphi^{-1}\ (\mathrm{Z}_2(\mathrm{M})) = {\pi_{\mathrm{L}}}^{-1}\ \psi^{-1}(\mathrm{Z}_2(\mathrm{M}))$. Since M is strongly t-Baer, then , $\varphi^{-1}\ (\mathrm{Z}_2(\mathrm{M})) \leq^{\oplus} \mathrm{M}$ and fully invariant in M. Say , $\mathrm{M} = \varphi^{-1}\ (\mathrm{Z}_2(\mathrm{M})) \oplus \mathrm{M}'$, hence $\mathrm{L} = \psi^{-1}\ (\mathrm{Z}_2(\mathrm{M})) \oplus \pi_{\mathrm{L}}(\mathrm{M}')$.Thus $\psi^{-1}\ (\mathrm{Z}_2(\mathrm{M})) \leq^{\oplus} \mathrm{L}$. Also, since $\psi^{-1}\ (\mathrm{Z}_2(\mathrm{M}))$ is fully invariant in M, hence by Lemma(1.2.6) $\psi^{-1}\ (Z_2(M))$ is fully invariant in L. Thus L is strongly t-Baer and so (by first paragraph), we have N is strongly t-Baer. □

Recall that "a module M is called t-cononsingular if every submodule N of M with $t_S(N) = t_S(M)$ implies $N \leq_{tes} M$"[6]." M is strongly t-cononsingular if every submodule N of M,$t_S(N) = t_S(M)$ implies $N \leq_{ess} M$"[6]. Asgari establish a close connection between t-extending modules and t-Baer modules; in fact, a module is t-extending if and only if it is t-Baer and t-cononsingular.[6,Proposition 3.9]

In the following Theorem, we establish connection between the strongly t-extending and strongly t-Baer modules. Also this Theorem is a generalization of Theorem 3.9 in [6]

Theorem (1.5.12): The following statements are equivalent for a module M.

(1) M is strongly t-extending;

(2) M is strongly t-Baer and t-cononsingular;

(3) M strongly t-Baer and $C = t_M t_S(C)$ for every t-closed submodule C.

(4) M is strongly t-Baer and for any t-closed submodule C of M if $t_S(M) = t_S(C)$, then $C = M$.

Proof: (1)⇒(2) Since M is strongly t-extending, $M = Z_2(M) \oplus M'$ where M' is strongly extending by Theorem1.3.11(4). But M' is nonsingular and strongly extending implies M' is strongly Baer by Proposition (1.5.9). Thus by Theorem (1.5.10), M is strongly t-Baer. Also M is t-cononsingular follows by the same proof of [6, Theorem (3.9) (1→2))].

(1)⇒(3) By the same proof of (1)⇒(2) M is strongly t-Baer. As M is t-extending, then $C = t_M t_S(C)$ for every t-closed submodule C of M follows [6, Theorem (3.9) (1→3)]

(2)⇒(4) For any t-closed submodule C of M if $t_S(C) = t_S(M)$,then $C \leq_{tes} M$ (by definition of t-cononsingular). But C is t-closed, so $C = M$.

(3)⇒(4) Let C be a t-closed submodule of M , such that $t_S(C) = t_S(M)$.

Hence $t_M t_S(C) = t_M t_S(M)$, thus $C = M$.

(4)⇒(1) By Theorem 1.3.11 to prove M is strongly t-extending, it suffices to show that any submodule which contains $Z_2(M)$ is essential in stable direct summand of M.Let N be a such a submodule . Since M strongly t-Baer,$t_M(t_S(N)) \leq^{\oplus} M$ and fully invariant in M, so $t_M\big(t_S(N)\big) = eM$ for some idempotent $e \in S$. But $N \leq t_M\big(t_S(N)\big) = eM \leq^{\oplus} M$ and $t_M\big(t_S(N)\big)$ fully invariant in M. Moreover $N \leq_{ess} eM$ by the same proof of [6, Theorem (3.9) (4→1)] .Thus N is essential in stable direct summand. □

Corollary (1.5.13): The following statements are equivalent for a module M:

(1) M is nonsingular strongly extending;

(2) M is strongly t-Baer and strongly t-cononsingular;

(3) M is strongly t-Baer and $C = t_M t_S(C)$ for every closed submodule C of M;

(4) M is strongly t-Baer and for any closed submodule C of M,$t_S(C) = t_S(M)$, then $C = M$.

Proof: (1) ⇒ (2) is obvious by Remark 1.3.7 and Theorem 1.5.12.

(1)⇒ (3) Since condition (1) implies M is strongly t-extending. Then M is strongly t-Baer and $C = t_M t_S(C)$ for every t-closed submodule C by (Theorem (1.5.12)(1)→(3)). As M is nonsingular, every closed is t-closed. Hence the result is obtained.

(2)⇒ (4) Let C be any closed submodule. If $t_S(C) = t_S(M)$ then $C \leq_{ess} M$. Hence $C = M$.

(3)⇒ (4) It follows by the same proof (Theorem (1.5.12) (3)→(4)).

(4)⇒ (1) By Theorem 1.5.8, $M = Z_2(M) \oplus M'$ for some $M' \leq M$. As $M' \leq^{\oplus} M$, so M' is closed in M. But $t_S(M') = t_S(M)$ since if $\varphi \in t_S(M')$, then $\varphi(M') \leq Z_2(M)$, so $\varphi(M) = \varphi(Z_2(M) \oplus M') = \varphi\big(Z_2(M)\big) + \varphi(M') \leq Z_2(M)$. Hence $t_S(M') \leq t_S(M)$. Now let $\varphi \in t_S(M)$, $\varphi(M) \leq Z_2(M)$ so that $\varphi(Z_2(M) \oplus M') \leq Z_2(M)$. And this implies $\varphi\big(Z_2(M)\big) + \varphi(M') \leq Z_2(M)$. Thus $\varphi(M') \leq Z_2(M)$, hence $\varphi \in t_S(M')$. Then $t_S(M) \leq t_S(M')$ and so $t_S(M') = t_S(M)$. It follows that $M = M'$,by condition (4) and, hence M is nonsingular which implies every closed submodule is t-closed. Thus M is strongly t-Baer and for any t-closed submodule C of, if $t_S(C) = t_S(M)$, then $C = M$. Hence by (Theorem (1.5.12) (4)→(1)), M is strongly t-extending. Thus it is strongly extending. □

Now we introduce the following

Definition (1.5.14): A ring R is called right strongly $\Sigma - t - extending$ if every free R- module is strongly t-extending.

Example (1.5.15): Let R be a right Z_2-torsion ring, that is $Z_2(R_R) = R$. For any module M, $Z_2(M) \geq MZ_2(R_R)$. Hence $Z_2(M) = M$, that is M is Z_2-torsion, hence by Remarks and Examples 1.2.2(2), M is strongly t-semisimple and by Theorem(1.3.5), M is strongly t- extending and so every free R-module is strongly t-extending. Thus R is a right strongly t-extending.

Theorem (1.5.16): The following statements are equivalent for a ring R.

(1) R is a right strongly $\Sigma - t - extending$ **;**

(2) Every nonsingular R-module is projective and strongly t-extending;

(3) For every R-module M, there exists a projective submodule M' of M, with $M = Z_2(M) \oplus M'$, $M'$is strongly t-extending.

(4) Every R-module is strongly t-Baer;

(5) Every R-module is strongly t-extending;

(6) Every projective R-module is strongly t-extending;

(7) Every nonsingular R-module is strongly t-Baer and $Z_2(R_R) \leq^{\oplus} R$;

(8) Every nonsingular R-module is strongly extending and $Z_2(R_R) \leq^{\oplus} R$.

Proof: (1) ⇒ (2) Let M be a nonsingular R-module.There exists a free R-module F and $K \leq F$, such that $M \simeq \frac{F}{K}$. Hence $\frac{F}{K}$ is nonsingular and so K is t-closed. Then by (Theorem (1.3.11) (2)). K is a fully invariant direct summand of F. Hence $F = K \oplus W$. But F is projective, implies W is projective. As $W \simeq \frac{F}{K} \simeq M$, so M is projective and by Proposition (1.3.14), W is strongly t-extending. Thus M is strongly t-extending.

(2) ⇒ (3) Let M be an R-module. Then $\frac{M}{Z_2(M)}$ is nonsingular, and hence by hypothesis it is projective and strongly t-extending. Since $\frac{M}{Z_2(M)}$ is projective, $M = Z_2(M) \oplus M'$and hence $M' \simeq \frac{M}{Z_2(M)}$, is projective . Also, M' is nonsingular (since $M' \simeq \frac{M}{Z_2(M)}$ is nonsingular) . So M' is strongly t-extending by hypothesis.

(3) $\Rightarrow$ (2) Let M be a nonsingular R-module. By condition (3), $M = Z_2(M) \oplus M'$, M' is projective and strongly t-extending. As M is nonsingular, $Z_2(M) = 0$ and so $M = M'$ is projective, strongly t-extending.

(3) $\Rightarrow$ (4) Let M be an R-module. Then $M = Z_2(M) \oplus M'$, where M' is strongly t-extending and M' is projective, so M' is strongly extending (since M' *is* nonsingular). But M' is nonsingular and strongly extending, implies M' is strongly Baer by Proposition (1.5.9).Thus M is strongly t-Baer by Theorem (1.5.8) (2).

(4) $\Rightarrow$ (5) Let M be an R-module, let $K \leq M$. Define $\varphi : M \oplus \frac{M}{K} \to M \oplus \frac{M}{K}$ by $\varphi(m, m' + K) = (0, m + K)$. Since $M \oplus \frac{M}{K}$ is strongly t-Baer, $\varphi^{-1}(Z_2\left(M \oplus \frac{M}{K}\right))$ is a fully invariant direct summand in $M \oplus \frac{M}{K}$ by Theorem 1.5.8(3). But $Z_2\left(M \oplus \frac{M}{K}\right) = Z_2(M) \oplus Z_2\left(\frac{M}{K}\right)$. Put $Z_2\left(\frac{M}{K}\right) = \frac{K_1}{K}$. We can show that $\varphi^{-1}[(Z_2(M) \oplus Z_2\left(\frac{M}{K}\right)] = K_1 \oplus \frac{M}{K}$,as follows: let $(m, m_1 + K) \in \varphi^{-1}[(Z_2(M) \oplus \left(\frac{K_1}{K}\right)]$, where $m, m_1 \in M$ $\varphi(m, m_1$+k)= $(0, m + k) \in Z_2(M) \oplus \frac{K_1}{K}$ and hence $m \in K_1$. Thus $(m, m_1$+k)$\in$ $K_1 \oplus \frac{M}{K}$, that is $\varphi^{-1}[(Z_2(M) \oplus Z_2\left(\frac{M}{K}\right)] \leq K_1 \oplus \frac{M}{K}$ …………….(I)

Conversely, let $(m, m_1$+k)$\in$ $K_1 \oplus \frac{M}{K}$ where $m \in K_1$, $m_1 \in M$, $\varphi(m, m_1$+k)=$(0, m + K) \in Z_2(M) \oplus \frac{K_1}{K} = Z_2(M) \oplus Z_2\left(\frac{M}{K}\right)$. Thus $(m, m_1$+k) $\in \varphi^{-1}[(Z_2(M) \oplus Z_2\left(\frac{M}{K}\right)]$ .Then $K_1 \oplus \frac{M}{K} \leq \varphi^{-1}[(Z_2(M) \oplus Z_2\left(\frac{M}{K}\right)]$ …………(II) . By(I),(II), we get $\varphi^{-1}[\left(Z_2(M) \oplus Z_2\left(\frac{M}{K}\right)\right] = K_1 \oplus \frac{M}{K}$. It follows that $K_1 \oplus \frac{M}{K}$ is a fully invariant direct summand of $M \oplus \frac{M}{K}$. So that $\left(K_1 \oplus \frac{M}{K}\right) \oplus W = M \oplus \frac{M}{K}$ for some $W \leq M \oplus \frac{M}{K}$, hence $K_1 \leq^{\oplus} M$. To prove K_1 is fully invariant in M. Let $f : M \to M$. Define $h : M \oplus \frac{M}{K} \to M \oplus \frac{M}{K}$ by $h(x) = \begin{cases} f(x) & if\ x \in M \\ 0 & otherwise \end{cases}$

$h(K_1 \oplus \frac{M}{K}) \subseteq K_1 \oplus \frac{M}{K}$,since $K_1 \oplus \frac{M}{K}$ is fully invariant in $M \oplus \frac{M}{K}$. But $h(K_1 \oplus \frac{M}{K}) = f(K_1) \leq M$. Thus $f(K_1) \subseteq K_1$, that is K_1 is a fully invariant submodule of M and hence by Theorem(1.3.11)(2), M is strongly t-extending.

(5) $\Rightarrow$ (6) It is clear.

(7) $\Rightarrow$ (8) Let K be a closed submodule of M where, M is nonsingular, so K is a t-closed and $\frac{M}{K}$ is nonsingular. Since $M, \frac{M}{K}$ are nonsingular, then $M \oplus \frac{M}{K}$ is nonsingular, so by hypothesis, $M \oplus \frac{M}{K}$ is strongly t-Baer. Then by similar proof (4) $\rightarrow$ (5), M is strongly t-extending.

(8) $\Rightarrow$ (1) Let F be a free R-module. Then F $\simeq \oplus_{i \in \wedge} R_i$,$i \in \wedge, R_i = R$ for each $i \in \wedge$, so that $Z_2(F) = Z_2(\oplus_{i \in \wedge} R_i = \oplus_{i \in \wedge} Z_2(R_i)$ by Proposition 1.1.13. Since $Z_2(R_R) \leq^{\oplus} R$, so $Z_2(F) \leq^{\oplus} F$. Hence $F = Z_2(F) \oplus W$, hence W is nonsingular since $W \simeq \frac{F}{Z_2(F)}$. By condition (8), W is strongly extending. Thus F is strongly t-extending by Theorem 1.3.11(4$\rightarrow$1)

(6) $\Rightarrow$ (1) Let F be a free R-module. Then F is projective, hence by condition (6), F is strongly t-extending. Thus R is a right strongly $\Sigma - t - extending$.

(4) $\Rightarrow$ (7) By condition (4), every R-module is strongly t-Baer, hence every R-module is t-Baer by Remarks and Examples 1.5.5(2). Then by [6, Theorem (3.12) (4) $\rightarrow$ (7)], every nonsingular R-module is Baer and $Z_2(R_R)$ is a direct summand of R. But by condition, every R-module is strongly t-Baer. Thus every R-module is strongly t-Baer and $Z_2(R_R)$ is a direct summand of R. □

Corollary (1.5.17): The following statements are equivalent for a nonsingular ring R.

(1) R is a right strongly Σ-t-extending ring,

(2) Every nonsingular R-module is projective and strongly t-extending;

(3) For every R-module M,there a projective submodule $M'$of M with $M = Z_2(M) \oplus M'$ and M' is strongly t-extending;
(4) Every R-module is strongly t-Baer;
(5) Every R-module is strongly t-extending;
(6) Every projective -module is strongly t-extending;
(7) Every nonsingular R-module is strongly Baer;
(8) Every nonsingular R-module is strongly extending.

Corollary (1.5.18): Let R be a ring consider the following statements.

(1) R is right strongly Σ-t-extending and all Z_2-torsion modules are projective and strongly t-extending
(2) R is semisimple;
(3) Every R-module is t-semisimple.

Then (1) ⇒ (2)⇒(3), and (3)⇒(2) if R is nonsingular.

Proof: (1)⇒ (2) Let M be an R-module , if M is nonsingular , then M is projective by Theorem 1.5.16(1→2). If M is Z_2-torsion, then M is projective by hypothesis. Now if M is neither nonsingular nor Z_2-torsion, then

$M = Z_2(M) \oplus M'$, where M' is projective and strongly t-extending by Theorem 1.5.16(1→3). But $Z_2(M) = Z_2(Z_2(M))$, so $Z_2(M)$ is Z_2-torsion, hence $Z_2(M)$ is projective. Then $M = Z_2(M) \oplus M'$ is projective. Thus all R-modules are projective, and so R is semisimple by [25, Corollary 8.2.2(e), P.196].

(2)⇒(3) It follows by Proposition 1.4.4.

(3)⇒(2) It follows by Remarks 1.1.45(3). □

Proposition (1.5.19): The following statements are equivalent for a ring R.
(1) R_R is strongly t-extending;
(2) Every nonsingular cyclic R-module is projective, strongly t-extending;

(3) For every cyclic R-module, there is a projective strongly t-extending M', with $M = Z_2(M) \oplus M'$;

(4) Every cyclic R-module is strongly t-extending;

(5) Every cyclic projective R-module is strongly t-extending.

Proof: (1) $\Rightarrow$ (2) Let $M = xR$ be a cyclic nonsingular R-module. Then $\frac{R}{ann_R(x)} \cong M$ (is nonsingular). But $\frac{R}{ann(x)}$ is nonsingular, implies $ann_R(x)$ is t-closed. As R_R is strongly t-extending, $ann_R(x)$ is a stable direct summand of R. Hence $R = ann_R(x) \oplus J$, so J is projective. Now $M \cong \frac{R}{ann_R(x)} \simeq J$ hence M is projective. Also J is a direct summand of strongly t-extending, so it is strongly t-extending. Thus M is strongly t-extending.

(2)$\Rightarrow$ (3) Let M be a cyclic R-module. Then, $\frac{M}{Z_2(M)}$ is a nonsingular cyclic module. By condition (2), $\frac{M}{Z_2(M)}$ is projective and strongly t-extending, hence $M = Z_2(M) \oplus M'$, for some $M' \leq M$. Thus M' is nonsingular projective. Also M' is strongly t-extending.

(3)$\Rightarrow$(2) Let M be a nonsingular cyclic, so $M = Z_2(M) \oplus M'$, M' is nonsingular , projective, strongly t-extending by condition (3). Hence $M = M'$(since $Z_2(M)$ =0).Thus M is projective, strongly t-extending.

(3) $\Rightarrow$ (4) Let M be cyclic R-module. Then $M = Z_2(M) \oplus M'$, M' is projective, and strongly t-extending. Since M' is nonsingular, M' is strongly extending. Hence M is strongly t-extending by Theorem (1.3.11).

(4) $\Rightarrow$ (5) $\Rightarrow$ (1) are clear. □

Corollary (1.5.20): The following statements are equivalent for a nonsingular ring R.

(1) R_R is strongly extending (R is strongly t-extending);

(2) Every nonsingular cyclic R-module is projective strongly extending;

(3) For every cyclic R-module M, there is a projective strongly extending M' with $M = Z_2(M) \oplus M'$;

(4) Every cyclic R-module is strongly t- extending;

(5) Every cyclic projective R-module is strongly t- extending.

Chapter Two

FI-semisimple Modules,

FI- t-semisimple Modules

and

Strongly FI-t-semisimple Modules

Introduction

In this chapter we introduce the notions of FI-semisimple and FI-t-semisimple modules as generalizations of semisimple modules; also we extend the notion of FI-t-semisimple in to strongly FI-t-semisimple. This chapter have three sections.

In section one, properties of FI-semisimple module are studied, we investigate connection between FI-semisimple and FI-extending modules. We also show that the direct sum of two FI-semisimple modules is a FI-semisimple module.

Section two is devoted for studying FI-t-semisimple modules. We obtain characterizations of FI-t-semisimple when a module satisfies condition$(*)$, where $(*)$ means: For an R-module M, a complement of $Z_2(M)$ is stable. We also, provide a connection between FI-t-semisimple M and FI-t-Baer modules, when M satisfies condition$(*)$. We generalize the property every t-semisimple module is t-extending. We get every FI-t-semisimple module is FI-t-extending.

In section three we introduce the notion of strongly FI-t-semisimple. The two concepts FI-t-semisimple and strongly FI-t-semisimple modules are coincide when condition $(*)$ hold. It is shown that every fully invariant submodule of strongly FI-t-semisimple inherits the property. A direct sum of two strongly FI-t-semisimple R-modules M_1 and M_2 is strongly FI-t-semisimple, if $annM_1 + annM_2 = R$.

2.1 FI-semisimple modules

In this section, we present the concept namely FI-semisimple modules as a generalization of semisimple modules. Many properties about this concept, and connections between it and other related concepts are introduced.

Definition (2.1.1): An R-module M is called FI-semisimple if for each fully invariant submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{ess} N$.

The following is a characterization of FI-semisimple modules.

Proposition (2.1.2): An R-module M is FI-semisimple if and only if every fully invariant submodule of M is a direct summand.

Proof: $\Rightarrow$ Let N be a fully invariant submodule of M, so there exists $K \leq^{\oplus} M$ such that $K \leq_{ess} N$. But $K \leq^{\oplus} M$ implies K is closed in M, so it has no proper essential extension in M. Thus $K = N$ and so $N \leq^{\oplus} M$.

$\Leftarrow$ Let N be a fully invariant submodule of M. By hypothesis $N \leq^{\oplus} M$. But $N \leq_{ess} N$ and $N \leq^{\oplus} M$. Thus M is FI-semisimple. □

Remarks and Examples (2.1.3):

(1) It is clear that every semisimple module is FI-semisimple, but the converse is not true in general, for example: The Z-module Q has only two fully invariant submodules which are (0), Q. Hence Q is FI-semisimple, but it is not semisimple.

(2) t-semisimple module does not imply FI-semisimple in general for example Z_{12} as Z-module is t-semisimple but it is not FI-semisimple. Also FI-semisimple module does not imply t-semisimple, for example Q as Z-module is FI-semisimple and it is not t-semisimple.

(3) If M is a duo module (hence if M is a multiplication module), then M is a semisimple module if and only if M is FI-semisimple. In particular the Z-

modules Z, Z_4, Z_{12} are not FI-semisimple. Also, every commutative ring R is semisimple if and only if R is FI-semisimple.

(4) A fully invariant submodule of FI-semisimple module is FI-semisimple.

Proof: Let N be a fully invariant submodule of M and M is a FI-semisimple module. Let U be a fully invariant submodule of N, hence U is a fully invariant in M by Proposition (1.1.38). It follows that $U \leq^{\oplus} M$. Thus $U \oplus U' = M$ for some $U' \leq M$ and so $N = (U \oplus U') \cap N = U \oplus (U' \cap N)$ by modular law. Then $U \leq^{\oplus} N$. Thus N is FI-semisimple by Proposition (2.1.2). □

(5) Every FI-semisimple module M is FI-extending. Where" an R-module M is called FI-extending if every fully invariant submodule is essential in a direct summand "[9].

Proof: Let N be a fully invariant submodule of M. As M is FI-semisimple, $N \leq^{\oplus} M$. But $N \leq_{ess} N$. So that M is FI-extending. □

(6) If M and N are isomorphic R-modules, then M is FI-semisimple if and only if N is FI-semisimple.

(7) If $f: M \rightarrow M'$ be an epimorophism and M' is FI-semisimple, then it is not necessary that M is FI-semisimple. For example $\pi: Z \rightarrow \frac{Z}{(6)} \cong Z_6$, Z_6 is FI-semisimple, but Z is not.

Proposition (2.1.4): Let M be a FI-semisimple and N is fully invariant in M then $\frac{M}{N}$ is a FI-semisimple.

Proof: Let $\frac{W}{N}$ be a fully invariant submodule of $\frac{M}{N}$. Since N is a fully invariant submodule of M. Then W is a fully invariant submodule of M by Lemma (1.1.40). But M is FI-semisimple, so $W \leq^{\oplus} M$. Then $W \oplus K = M$ for some $K \leq M$. This implies $\frac{W}{N} \oplus \frac{K+N}{N} = \frac{M}{N}$. Thus $\frac{W}{N} \leq^{\oplus} \frac{M}{N}$ and $\frac{M}{N}$ is FI-semisimple. □

Corollary (2.1.5): Let $f: M \to M'$ be an R- epimorphism and $Kerf$ is a fully invariant submodule of M. If M is a FI-semisimple R-module, then M' is a FI-semisimple.

Proof: Since $f: M \to M'$ epimorphism, $\frac{M}{Kerf} \cong M'$. But $\frac{M}{Kerf}$ is a FI-semisimple by proposition (2.1.4), hence M' is a FI-semisimple by Remarks and Examples 2.1.3(6). □

Corollary (2.1.6): Let M be a FI-semisimple R-module. Then $\frac{M}{Z_2(M)}$ is FI-semisimple and $M = Z_2(M) \oplus M'$ where M' is a nonsingular FI-semisimple.

Proof: As $Z_2(M)$ is a fully invariant submodule of M , then $\frac{M}{Z_2(M)}$ is FI-semisimple module by Proposition (2.1.4). Also, $Z_2(M)$ is a fully invariant in M implies $Z_2(M) \leq^{\oplus} M$, by Proposition (2.1.2). Thus $M = Z_2(M) \oplus M'$ for some $M' \leq M$. But $M' \simeq \frac{M}{Z_2(M)}$, so M' is nonsingular FI-semisimple. □

Next the following proposition concerned with the direct sum of FI-semisimple modules

Proposition (2.1.7): Let $M = M_1 \oplus M_2$, where $M_1, M_2 \leq M$. If M_1 and M_2 are FI-semisimple, then M is FI-semisimple and the converse hold if M_1 and M_2 are fully invariant submodules of M.

Proof: $\Rightarrow$ Let N be a fully invariant submodule of M. Then $N = (N \cap M_1) \oplus (N \cap M_2)$ and, $(N \cap M_1)$,$(N \cap M_2)$ are fully invariant submodules of M_1 and M_2 respectively by Proposition 1.1.39(ii). Put $N_1 = N \cap M_1$, $N_2 = N \cap M_2$. Hence $N_1 \leq^{\oplus} M_1$, $N_2 \leq^{\oplus} M_2$, since M_1 and M_2 are FI-semisimple. It follows that $N = N_1 \oplus N_2 \leq^{\oplus} M$ and so M is FI-semisimple.

$\Leftarrow$ Since M_1 is a fully invariant submodule of M, $\frac{M}{M_1}$ is FI-semisimple by Proposition (2.1.4). But $\frac{M}{M_1} \simeq M_2$, hence M_2 is FI-semisimple. Similarly, M_1 is FI-semisimple. □

2.2 FI-t-semisimple Modules

In this section a generalization of t-semisimple modules namely, FI-t-semisimple which is also a generalization of semisimple modules is introduced and studied. Several properties concerned with this concept are given.

Definition (2.2.1): An R-module M is called FI-t-semisimple if for each fully invariant submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$.

Remarks and Examples (2.2.2):

(1) It is clear that every t-semisimple module is FI-t-semisimple, but the converse is not true, for example Q as Z-module is not t-semisimple and clearly it is FI-t-semisimple.

(2) It is clear that every FI-semisimple module is FI-t-semisimple, hence each of the Z-module $Q, Q \oplus Z_2, Z_2 \oplus Z_6$ is FI-t-semisimple, since each of them is FI-semisimple module.

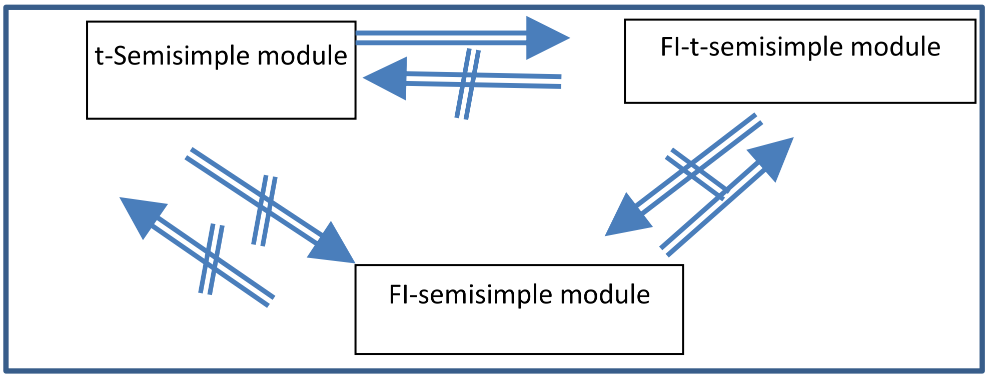
(3) The converse of part (2) is not true in general, for example, Z_{12} as a Z-module is a FI-t-semisimple (since it is t-semisimple) but it is not FI-semisimple.

(4) Let M be a nonsingular R-module. Then M is FI-semisimple if and only if M is FI-t-semisimple. In particular, Z as Z-module is not FI-t-semisimple, also if $R = Z[x]$, then R_R is not FI-t-semisimple.

Proof: $\Rightarrow$ It is clear by part (2)

$\Leftarrow$ Let M be a FI-t-semisimple module and N be a fully invariant submodule of M, there exists $K \leq^{\oplus} M$ and $K \leq_{tes} N$. But M is nonsingular implies N is nonsingular and hence $K \leq_{ess} N$. But $K \leq^{\oplus} M$ implies K is a closed submodule of M and so that $K = N$. It follows that M is FI-semisimple by Proposition (2.1.2). □

Thus Remarks 2.2.2 can be illustrated by the following diagrams.



Proposition (2.2.3): Every fully invariant submodule of FI-t-semisimple module is FI-t-semisimple.

Proof: Let N be a fully invariant submodule of a FI-t-semisimple R-module M. To prove N is FI-t-semisimple, let W be a fully invariant submodule of N. Hence W is a fully invariant submodule of M by Proposition 1.1.38. It follows that there exists $K \leq^{\oplus} M$ and $K \leq_{tes} W$, since M is FI-t-semisimple. Hence $M = K \oplus C$ for some $C \leq M$ and so that $N = K \oplus (C \cap N)$, thus $K \leq^{\oplus} N$ and so that N is FI-t-semisimple. □

Proposition (2.2.4): Let $M = M_1 \oplus M_2$ where $M_1 \leq M, M_2 \leq M$. If M_1 and M_2 are FI-t-semisimple, then M is a FI-t-semisimple. The converses hold if $ann(M_1) + ann(M_2) = R$.

Proof:$\Rightarrow$ Let N be a fully invariant submodule of M. Then $N = N_1 \oplus N_2$, where N_1 is fully invariant in M_1 and N_2 is fully invariant in M_2 by Lemma (1.1.39)(ii). Hence, there exists $K_1 \leq^{\oplus} M_1$ and $K_2 \leq^{\oplus} M_2$ such that $K_1 \leq_{tes} N_1$, $K_2 \leq_{tes} N_2$. Hence $K = K_1 \oplus K_2 \leq^{\oplus} M$ and $K = K_1 \oplus K_2 \leq_{tes} N_1 \oplus N_2 = N$ by Proposition 1.1.22(2).

$\Leftarrow$ Since $M = M_1 \oplus M_2$ and $ann(M_1) + ann(M_2) = R$, then $M_1 = M_1 ann(M_2), M_2 = M_2 ann(M_1)$ and so for each $f \in End(M)$, $f(M_1) = f\big(M_1\ ann(M_2)\big) = f(M_1)ann(M_2) \leq (M_1 \oplus M_2)ann(M_2)$ that is $f(M_1) \leq M_1 ann(M_2) = M_1$. Similarly $f(M_2) \leq M_2$. Thus M_1 and M_2 are fully invariant, and hence by Proposition (2.2.3), M_1 and M_2 are FI-t-semisimple. □

Let (⁎) means the following: For an $\boldsymbol{R}$-module $\boldsymbol{M}$, a complement of $\boldsymbol{Z_2(M)}$ is stable in $\boldsymbol{M}$.

Theorem (2.2.5): For an R-module M consider the following statements

(1) M is an FI-t-semisimple module;

(2) $\frac{M}{Z_{2(M)}}$ is a FI-semisimple module;

(3) $M = Z_2(M) \oplus M'$, where $M'$is nonsingular, FI-semisimple, M' is stable in M;

(4) Every nonsingular fully invariant submodule of M is a direct summand.

(5) Every fully invariant submodule of M which contains $Z_2(M)$ is direct summand.

Then (3) ⇒ (5) ⇒ (2) and (3) ⇒ (1) ⇒ (4). (4)⇒ (3) if condition (⁎) hold and so that (3)⇔(1)⇔(4) (if condition (⁎) hold).

Proof: (3) ⇒ (5) Let N be a fully invariant submodule of M, $N \supseteq Z_2(M)$. Since $M = Z_2(M) \oplus M'$ where M' is FI-semisimple nonsingular and stable in M. Then $N = N \cap (Z_2(M) \oplus M') = Z_2(M) \oplus (N \cap M')$. As N and M' are fully invariant in M, so $(N \cap M')$ is fully invariant in M. Since $(N \cap M') \leq M' \leq^{\oplus} M$ and $N \cap M'$ is fully invariant in M, then $N \cap M'$ is a fully invariant in M' by Lemma 1.1.40(2).But M' is FI-semisimple, so $N \cap M' \leq^{\oplus} M'$. It follows that $M' = (N \cap M') \oplus W$, for some $W \leq M'$ and so that $M = Z_2(M) \oplus [(N \cap M') \oplus W = [Z_2(M) \oplus (N \cap M')] \oplus W$

$= N \oplus W$.Therefore $N \leq^{\oplus} M$.

(5)⇒(2) Let $\frac{N}{Z_2(M)}$ be a fully invariant submodule of $\frac{M}{Z_2(M)}$. Since $Z_2(M)$ is fully invariant in M, then N is fully invariant in M by Lemma 1.1.40(1).Also $N \supseteq Z_2(M)$, so by condition (5), $N \leq^{\oplus} M$. Thus $N \oplus K = M$ for some $K \leq M.$ it follows that $\frac{M}{Z_2(M)} = \frac{N}{Z_2(M)} \oplus \frac{K+Z_2(M)}{Z_2(M)}$ So that $\frac{N}{Z_2(M)} \leq^{\oplus} \frac{M}{Z_2(M)}$ and so $\frac{M}{Z_2(M)}$ is FI-semisimple.

(3) ⇒ (1) By hypothesis, $M = Z_2(M) \oplus M'$, where M' is nonsingular FI-semisimple and M' is stable in M. Let N be a fully invariant submodule of M, so $N \cap M'$ is a fully invariant submodule in M. Hence by Lemma 1.1.40(2) $N \cap M'$ is a fully invariant submodule in M' and so $(N \cap M') \leq^{\oplus} M'$. It follows that $(N \cap M') \leq^{\oplus} M$. On the other hand, $\frac{N}{(N \cap M')} \cong \frac{(N+M')}{M'} \leq \frac{M}{M'}$ which is Z_2-torsion, hence, $\frac{N}{(N \cap M')}$ is Z_2-torsion and so that $(N \cap M') \leq_{tes} N$ by Proposition (1.1.17). Thus $(N \cap M') \leq^{\oplus} M$ and $(N \cap M') \leq_{tes} N$ which implies that M is FI-t-semisimple.

(1) ⇒ (4) Let N be a nonsingular fully invariant submodule of M. By condition (1) there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. As N is nonsingular, $K \leq_{ess} N$. But $K \leq^{\oplus} M$, implies K is closed in M, hence $K = N$. Thus $N \leq^{\oplus} M$.

(4) ⇒ (3) Let M' be a complement of $Z_2(M)$. Hence $Z_2(M) \oplus M' \leq_{ess} M$, implies $M' \leq_{tes} M$ (by proposition 1.1.17). Hence $\frac{M}{M'}$ is Z_2-torsion. This implies M' is nonsingular by the same argument of proof of Theorem 1.2.3(4)→(3). By condition (⁎), M' is stable, hence $M' \leq^{\oplus} M$ by condition (4). Thus $M = M' \oplus L$, for some $L \leq M$ and so $Z_2(M) = Z_2(M') + Z_2(L)$. But $Z_2(M') = 0$ and $L \simeq \frac{M}{M'}$ is Z_2-torsion, so $Z_2(L) = L$. Hence $Z_2(M) = L$. Thus $M = M' \oplus Z_2(M)$ such that M' is nonsingular and stable.

To prove M' is FI-semisimple, let N be a fully invariant submodule of M'. As M' is fully invariant in M, so N is fully invariant in M by Proposition 1.1.38. Also, M' is nonsingular, implies N is nonsingular. Thus N is nonsingular fully invariant in M. Hence by condition (4), $N \leq^{\oplus} M$, and so $N \oplus W = M$, for some $W \leq M$. Then

$M' = (N \oplus W) \cap M' = N \oplus (W \oplus M')$ (since N $\leq M'$, and so $N \leq^{\oplus} M'$. Thus M' is a FI-semisimple module. □

By applying Theorem 2.2.5(1⇒4). If M is a duo (or multiplication) module, then M is a FI-t-semisimple implies every nonsingular submodule is a direct summand, and hence by Theorem 1.1.46, M is t-semisimple. Thus the concepts t-semisimple and FI-t-semisimple under the class of duo(or multiplication) modules are equivalent.

Recall that ″an R-module M is called FI-t-extending if every fully invariant t-closed submodule of M is a direct summand ″[9] ″an R-module M is called FI-t-Baer if $t_M(I)$ is a direct summand of M for any two-sided ideal I of $End(M)$ ″[9].By Theorem 3.9 in [9] every FI-t-extending module is FI-t-Baer. we have the following:

Proposition (2.2.6): Let M be an R-module which satisfies condition (⁎) .If M is FI-t-semisimple, then M is FI-t-extending.

Proof: By Theorem (2.2.5) (1→5) for each fully invariant submodule N with $N \supseteq Z_2(M)$, $N \leq^{\oplus} M$. As every t-closed submodule contains $Z_2(M)$,so for each fully invariant t-closed submodule N of M is a direct summand. Thus M is FI-t-extending.□

Theorem (2.2.7): Let M be an R-module such that a complement of fully invariant submodule is fully invariant. If M is an FI-t-semisimple implies $\frac{M}{C}$ is an FI-t-semisimple, for each fully invariant t-closed submodule C of M.

Proof: By Proposition (2.2.6), M is FI-t-extending. Hence, any fully invariant t-closed submodule C is a direct summand. Thus $C \oplus C' = M$ for some $C' \leq M$. C' is a complement of C and by hypothesis C' is a fully invariant submodule of M. Hence C' is a FI-t-semisimple by Proposition (2.2.3). But $C' \simeq \frac{M}{C}$ is a FI-t-semisimple. □

Proposition (2.2.8): Let M be an R-module such that condition($*$) hold. If M is a FI-t-semisimple, then $N + Z_2(M)$ is closed, for each fully invariant submodule N of

M. The converse holds if a complement of a fully invariant submodule is fully invariant.

Proof: $\Rightarrow$ For each fully invariant submodule N of M, $N + Z_2(M) \supseteq Z_2(M)$ and it is fully invariant submodule of M by Lemma 1.1.39(i), so that $N + Z_2(M)$ is a direct summand by Theorem 2.2.5(1$\Rightarrow$5) and hence $N + Z_2(M)$ is a closed submodule of M.

$\Leftarrow$ To prove M is a FI-t-semisimple. Let K be a nonsingular fully invariant submodule of M. Assume L is a complement of K, then by hypothesis, L is a fully invariant submodule of M. Thus $K \oplus L \leq_{ess} M$, and $K \oplus L$ is a fully invariant submodule of M. It follows that $(K \oplus L)+Z_2(M) \leq_{ess} M$. But $(K \oplus L) + Z_2(M)$ is a fully invariant submodule containing $Z_2(M)$, so that $(K \oplus L) + Z_2(M)$ is a direct summand, so it is closed. Hence $(K \oplus L) + Z_2(M = M$. But $K \cap L = 0$ and $K \cap Z_2(M) = Z_2(K) = (0)$, since K is nonsingular. Moreover, we can show that $K \cap (L + Z_2(M)) = 0$. Suppose there exists $0 \neq x \in K \cap (L + Z_2(M))$, then $x = l + y$, $l \in L, y \in Z_2(M)$. Since K is nonsingular $Z_2(K) = 0$ and hence, $ann(x) \not\leq_{tes} R$. But $x - l = y$,so $ann(x - l) = ann(y) \leq_{tes} R$.But it is known that $ann(x - l) = ann(x) \cap ann(l)$. It follows that $ann(x) \cap ann(l) \leq_{tes} R$, which implies $ann(x) \leq_{tes} R$ which is a controduction. Thus $\big(K \oplus (L + Z_2(M)\big) = M$, that is $K \leq^{\oplus} M$. Then M is a FI-t-semisimple by Theorem (2.2.5)4$\rightarrow$1. □

Proposition (2.2.9): Let M be an R-module such that condition (⁎) hold. Then M is FI-t-semisimple if and only if M has no proper nonzero fully invariant submodule N containing $Z_2(M)$ and $N \leq_{ess} M$.

Proof: $\Rightarrow$ By Theorem 2.2.5(1$\rightarrow$5) M is FI-t-semisimple, implies that for each fully invariant submodule N of M containing $Z_2(M)$, $N \leq^{\oplus} M$. Hence for each proper nonzero fully invariant submodule $N \supseteq Z_2(M)$, $N \not\leq_{ess} M$.

$\Leftarrow$ Let M' be a complement of $Z_2(M)$, so that $M' \oplus Z_2(M) \leq_{ess} M$. But by hypothesis, $M'$is a fully invariant submodule of M and so $M' \oplus Z_2(M)$ is a fully invariant submodule of M .Thus $M' \oplus Z_2(M) = M$. Hence, $M' \simeq \frac{M}{Z_2(M)}$ is nonsingular and stable. Let N be a fully invariant submodule of M'. Since M' is a fully invariant in M, then N is a fully invariant submodule in M by Lemma 1.1.38. Hence $N+Z_2(M)$ is fully invariant in M. Let K be a complement of $N+Z_2(M)$. Hence $(N+Z_2(M))\oplus K \leq_{ess} M$. But by hypothesis $(N+Z_2(M))\oplus K = M$. We can show that $N\oplus(Z_2(M)+K)=M$, as follows. Let $x \in N \cap (Z_2(M)+K)$. Then $x=a+b$ for some $a \in Z_2(M), b \in K$. Then $x-a=b \in (N+Z_2(M)) \cap K = 0$, hence $x-a=b=0$, and so that $x=a \in N \cap Z_2(M)$= $Z_2(N)=0$. Thus $x=0$ and $N \cap (Z_2(M)+K)=0$, hence $N\oplus(Z_2(M)+K)=M$, that is

$N \leq^{\oplus} M$. Now $M'=[N\oplus(Z_2(M)+K)]\cap M' = N\oplus[(Z_2(M)+K)\cap M']$. That is $N \leq^{\oplus} M'$. Hence M' is FI-semisimple. Thus by Theorem 2.2.5(3→1), M is FI-t-semisimple. □

Recall that " if N and K are submodules in an R-module M. K is called a weak supplement of N if M = K + N and K ∩ N ≪ M (the notation ≪ denotes a small submodule). "[13], where a submodule W of M is called a small submodule of M if whenever M = W + U , U is a submodule of M implies U = M.

Proposition (2.2.10): Let M be an R-module such that condition (⁎) hold. If $Rad(M)$ is Z_2-torsion and every nonsingular fully invariant submodule of M has a weak supplement, then M is FI-t-semisimple

Proof: Let N be a nonsingular fully invariant submodule of M. As M has weak supplement there exists a submodule K of M such that $M=K+N$ and $K\cap N \ll M$. Cleary $M=\left(K+Rad(M)\right)+N$. Now we will show that $\left(K+Rad(M)\right)\cap N=0$.Assume that $x \in \left(K+Rad(M)\right)\cap N$. Then $x=y+z$ where $y \in K$ and $z \in Rad(M)$. Since $Rad(M)$ is Z_2-torsion, there exists a t-essential right ideal I of R such that $(x-y)I=0$. Thus $xI=yI \subseteq K\cap N \leq Rad(M) \leq Z_2(M)$ since $K \cap N \ll M$ implies $K\cap N \subseteq RadM$, also $RadM = Z_2(RadM)$ since $RadM$ is Z_2-torsion hence $RadM \subseteq Z_2(M)$. So $x+Z_2(M) \in Z_2\left(\frac{M}{Z_2(M)}\right)=0$. It follows that $x \in Z_2(M)\cap N = Z_2(N)=0$ and this implies that N is a direct summand of M. Hence by Theorem 2.2.5(4) M is FI-t-semisimple. □

Proposition (2.2.11): The following assertions are equivalent for a module M, such that for any $B \leq M$, a complement of $Z_2(B)$ is stable in B.

(1) M is FI-t-semisimple

(2) For each fully invariant submodule N of M, there exists a decomposition $M = K \oplus L$ such that $K \leq N$ and L is stable in M and $N \cap L \leq Z_2(L)$.

(3) For each fully invariant submodule N of M, $N = K \oplus K'$ such that K is a direct summand stable and K' is Z_2-torsion.

Proof: (1) ⇒ (2) Let N be a fully invariant submodule of M. Let K be a complement of $\mathrm{Z}_2(\mathrm{N})$ in N. Then $\mathrm{K} \oplus \mathrm{Z}_2(\mathrm{N}) \leq_{\mathrm{ess}} \mathrm{N}$ and K is a fully invariant of N by hypothesis . By proposition (2.2.3) and proposition (2.2.9), $\mathrm{K} \oplus \mathrm{Z}_2(\mathrm{N}) = \mathrm{N}$. Let C be a complement of $\mathrm{K} \oplus \mathrm{Z}_2(\mathrm{M})$, so C is a stable submodule of M and $(\mathrm{K} \oplus \mathrm{Z}_2(\mathrm{M})) \oplus \mathrm{C} \leq_{\mathrm{ess}} \mathrm{M}$. But M is FI-t-semisimple, hence by proposition (2.2.9), $(\mathrm{K} \oplus \mathrm{Z}_2(\mathrm{M})) \oplus \mathrm{C} = \mathrm{M}$. Put $\mathrm{Z}_2(\mathrm{M}) \oplus \mathrm{C} = \mathrm{L}$, hence L is a stable in M. Moreover, $\mathrm{N} = (\mathrm{K} \oplus \mathrm{L}) \cap \mathrm{N} = \mathrm{K} \oplus (\mathrm{N} \cap \mathrm{L})$. But $K \oplus Z_2(N) = N$ implies $\frac{N}{K} \simeq Z_2(N)$ which is Z_2-torsion. On other hand, $\frac{N}{K} \simeq N \cap L$, so that $N \cap L$ is Z_2-torsion. Then $N \cap L = Z_2(N \cap L) \leq Z_2(L)$. Thus $M = K \oplus L$ is a desired decompodition.

(2) ⇒ (3) Let N be a fully invariant submodule of M. By condition (2) , $M = K \oplus L$ where $K \leq N$ and L is stable in M and $N \cap L \leq Z_2(L)$. Hence $N = (K \oplus L) \cap N = K \oplus (L \cap N)$. Put $K' = N \cap L$, so that $N = K \oplus K'$, and $\frac{N}{K} \simeq K' = N \cap L$ which is Z_2-torsion. Also K stable in M, since K is a complement of L in M.

(3) ⇒ (1) Let N be a fully invariant submodule of M. By condition (3), $N = K \oplus K'$, where K is stable direct summand in M and $K'$is Z_2-torsion. Now $K \leq N$and $\frac{N}{K} \simeq K'$ which is Z_2-torsion. Hence $K \leq_{tes} N$ and so that M is FI-t-semisimple. □

Proposition (2.2.12): Let M be an R-module such that condition $(*)$ hold. Then the following statements are equivalent.

(1) M is FI-t-semisimple.

(2) M is FI-t-extending and $N = t_M t_S(N)$ for every fully invariant submodule N of M contain $Z_2(M)$.

(3) M is FI-t-Baer and $N = t_M t_S(N)$ for every fully invariant submodule N of M contain $Z_2(M)$.

Proof: (1) $\Rightarrow$ (2) M is FI-t-semisimple implies M is FI-t-extending by Proposition (2.2.6). Now, let N be a fully invariant submodule of M and $N \supseteq Z_2(M)$. Hence $N \leq^{\oplus} M$ by Proposition 2.2.5(1⇒5). This implies, $M = N \oplus N'$ for some $N' \leq M$. It is obvious, that $N \leq t_M t_S(N)$. Let π' be the canonical projection on N', that is $\pi': M \longrightarrow N' \leq N \oplus N'$, so $\pi' \in S$, $\pi'(N) = 0 \leq Z_2(M)$, so $\pi' \in t_S(N)$. If $m \in t_M t_S(N)$, then $\pi'(m) \in Z_2(M) \leq N$. Hence $\pi'(m) = 0$ and so $m \in N$. Thus $N = t_M t_S(N)$

(2) $\Rightarrow$ (3) It is obvious, since every FI-t-extending is FI-t- Baer, see [9, Theorem 3.9].

(3) $\Rightarrow$(1) Since M is FI-t-Baer, $Z_2(M) = t_M(S)$ is a direct summand and then $M = Z_2(M) \oplus M'$, where M' is nonsingular [9, Theorem 3.2]. Hence M' is a complement of $Z_2(M)$, so it is stable. Now, let N' be a fully invariant submodule of M', so that N' is a fully invariant submodule of M. Put $N = Z_2(M) \oplus N'$. Then N is a fully invariant submodule of M containing $Z_2(M)$. So $N = t_M t_S(N)$ by hypothesis. On the other hand, M is FI-t-Baer and $t_S(N)$ is a two sided ideal of S, hence $t_M t_S(N) \leq^{\oplus} M$. Thus $N \leq^{\oplus} M$. It follows that $M = N \oplus W$ for some $W \leq M$. Then $M = Z_2(M) \oplus N' \oplus W$. By hypothesis complement of $Z_2(M)$ is stable so by [1, Theorem 4.8, p31], $N' \oplus W = M'$ and hence $N' \leq^{\oplus} M'$, and this implies M' is FI-semisimple. Therefore M is FI-t-t-semisimple by Proposition (2.2.5). □

2.3 Strongly FI-t-semisimple Modules

In this section, we extend the notion of FI-t-semisimple into strongly FI-t-semisimple. Many properties about this concept, and many connections between it and other related concepts are presented.

Definition (2.3.1): An R-module M is called strongly FI-t-semisimple if for each fully invariant submodule N of M, there exists a fully invariant direct summand K such that $K \leq_{tes} N$.

Remarks and Examples (2.3.2):

(1) Every strongly FI-t-semisimple is FI-t-semisimple. We claim that the converse is not true but we have no example to ensure this.

(2) Every strongly t-semisimple is strongly FI-t-semisimple but the converses is not true in general as following example shows:

Q as Z-module is strongly FI-t-semisimple, since Q has only two fully invariant submodules (0), Q. But Q is not strongly t-semisimple.

(3) Every FI-semisimple module is strongly FI-t-semisimple.

Proof: Let N be a fully invariant submodule of N. Then $N \leq^{\oplus} M$, since M is a FI-semisimple . But $\mathrm{N} \leq_{\mathrm{tes}} \mathrm{N}$, hence M is strongly FI-t-semisimple. □

(4) Every Z_2-torsion M is strongly FI-t-semisimple.

Proof: Let N be a FI-submodule of M, $(0) \leq^{\oplus} M$, ($(0) \leq_{tes} N$ since $(0) + Z_2(N) = N \leq_{ess} N)$.

(5) Let M be a duo (or multiplication) module then M is FI-t-semisimple if and only if M is strongly FI-t-semisimple

Proposition (2.3.3): Let M be an R-module with the property, a complement of any submodule of M is fully invariant. The following statements are equivalent.

(1) M is strongly FI-t-semisimple;

(2) M is FI-t-semisimple;

Proof: (1) $\Rightarrow$ (2) It is clear.

(2)$\Rightarrow$ (1) Let N be a fully invariant submodule of M. Since M is FI-t-semisimple, there exists $K \leq^{\oplus} M$ and $K \leq_{tes} N$. Hence $M = K \oplus W$ for some $W \leq M$. One can check easily that K is a complement of W. But by hypothesis K is fully invariant. Thus M is strongly FI-t-semisimple. □

The following result follows by combining Proposition (2.3.3) and Proposition (2.2.9).

Proposition (2.3.4): Let M be an R-module such that a complement of any submodule is fully invariant. Then the following are equivalent:

(1) M is strongly FI-t-semisimple;
(2) M is FI-t-semisimple;
(3) M has no proper nonzero fully invariant submodule N containg $Z_2(M)$ and $N \leq_{ess} M$.

Proposition (2.3.5): A fully invariant direct summand N of a strongly FI-t-semisimple is strongly FI-t-semisimple.

Proof: Let W be a fully invariant submodule of N. Then $W$is a fully invariant submodule of M by Proposition (1.1.38). Since M is strongly FI-t-semisimple, there exists $K \leq^{\oplus} M$, K is a fully invariant submodule of M and $K \leq_{tes} W \leq N$. But $K \leq^{\oplus} M$ implies $M = K \oplus A$ for some $A \leq M$ and this implies $N = K \oplus (A \cap N)$; that is $K \leq^{\oplus} N$. Beside this by Lemma (1.2.6), K is a fully invariant submodule of N. Thus N is strongly FI-t-semisimple.

Remark (2.3.6): Q as Z-module is strongly FI-t-semisimple, $Z < Q$. But Z is not strongly FI-t-semisimple. However and $Z \nleq^{\oplus} Q$ and Z is not fully invariant submodule of Q.

The following three results follow by Proposition 2.3.5.

Corollary (2.3.7): Every nonsingular fully invariant submodule of strongly FI-t-semisimple is strongly FI-t-semisimple.

Proof: Let N be a nonsingular fully invariant submodule of M, where M is strongly FI-t-semisimple. Hence M is FI-t-semisimple by Remarks and Examples 2.3.2(1) and then by Theorem 2.2.5(1→4) N is a direct summand of M. Thus N is strongly FI-t-semisimple by Proposition (2.3.5). □

Corollary (2.3.8): For an R-module M which satisfies (⁎), if M is strongly FI-t-semisimple then every fully invariant submodule N of M such that $N \supseteq Z_2(M)$, is strongly FI-t-semisimple,.

Proof: Since M is strongly FI-t-semisimple, M is a FI-t-semisimple. Hence by Proposition 2.2.5(1⇒5), N is a direct summand and then by Proposition (2.3.5), N is strongly FI-t-semisimple. □

Corollary (2.3.9): For any strongly FI-t-semisimple module M which satisfies condition(⁎), $Z_2(M)$ is strongly FI-t-semisimple.

Proof: It follows directly by Theorem 2.2.5(1→5) and Proposition (2.3.5). □

Proposition (2.3.10): Let M be an R-module which satisfies (⁎). If M is strongly FI-t-semisimple, then $\frac{M}{Z_2(M)}$ is FI-semisimple, and hence it is strongly FI-t-semisimple.

Proof: Let $\frac{L}{Z_2(M)}$ be a fully invariant submodule of $\frac{M}{Z_2(M)}$. Then L is a fully invariant submodule in M by Lemma 1.1.40.As M is strongly FI-t-semisimple, M is FI-t-semisimple and hence by Theorem 2.2.5(1→2), $\frac{M}{Z_2(M)}$ is FI-semisimple. □

Theorem (2.3.11): Let $M = M_1 \oplus M_2$ where M_1 and M_2 are submodules of M such that $annM_1 \oplus annM_2 = R$. Then M is strongly FI-t-semisimple if and only if M_1 and M_2 are strongly FI-t-semisimple.

Proof: $\Rightarrow$ Let N be a fully invariant submodule of M. Then $N = (N \cap M_1) \oplus (N \cap M_2)$ and $N \cap M_1$, $N \cap M_2$ are fully invariant in M_1, M_2 respectively by Lemma 1.1.39(iii) Put $N_1 = N \cap M_1$, $N_2 = N \cap M_2$. Hence there exist K_1, K_2 are fully invariant direct summands in M_1 and M_2 respectively and $K_1 \leq_{tes} N_1$, $K_2 \leq_{tes} N_2$since $M_1$and M_2 are strongly FI-t-semisimple. It follows easily that $K_1 \oplus K_2 \leq^{\oplus} M$ and by Corollary 1.1.22(2) $K_1 \oplus K_2 \leq_{tes} N_1 \oplus N_2 = N$. To show that $K_1 \oplus K_2$ is a fully invariant in M. Let $\theta \epsilon EndM = \begin{pmatrix} EndM_1 & Hom(M_2, M_1) \\ Hom(M_1, M_2) & EndM_2 \end{pmatrix}$. But by Lemma (1.2.8), $Hom\,(M_1, M_2) = 0$, $Hom(M_2, M_1) = 0$. It follows that $\theta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ for some $\alpha_1 \in EndM_1$ and $\alpha_2 \in EndM_2$. Thus $\theta(K_1 \oplus K_2) = \alpha_1(K_1) \oplus \alpha_2(K_2)$. But $\alpha_1(K_1) \leq K_1$, $\alpha_2(K_2) \leq K_2$ since K_1 and K_2 are fully invariant in M_1 and M_2 respectively. Thus $K_1 \oplus K_2$ is fully invariant in M and so M is strongly FI-t-semisimple.

$\Leftarrow$ Since $M = M_1 \oplus M_2$ and $ann(M_1) \oplus ann(M_2) = R$, then $M_1 = M_1 ann(M_2)$ and $M_2 = M_2 ann(M_1)$. Hence for any $f \in End(M)$, $f(M_1) \leq M_1, f(M_2) \leq M_2$, that is M_1 and M_2 are fully invariant in M. Then by Proposition 2.3.5, M_1 and M_2 are strongly FI-t-semisimple. □

Chapter Three

Purely Semisimple Modules,

Purely t-semisimple Modules

and

Strongly Purely t-semisimple Modules

Introduction

Our goals in this chapter are generalizing semisimple modules into purely semisimple, purely t-semisimple modules and extending the concept t-semisimple modules into strongly purely t-semisimple.

In section one many basic properties and examples of purely semisimple modules are introduced.

In section two purely t-semisimple modules is presented. It is clear that t-semisimple module implies purely t-semisimple but not conversely. We generalize many properties of t-semisimple modules into purely t-semisimple modules. Also, we have every purely t-semisimple module is purely t-extending if M satisfies that a complement of $Z_2(M)$ is a pure submodule.

In section three, the property which is mentioned in chapter one: Every t-semisimple module is t-Baer, led us to introduce the concept of purely t-Baer module. So we study this class of modules and we prove that every purely t-extending is purely t-Baer. Also, every purely t-semisimple module M such that a complement $Z_2(M)$ is pure is purely t-Baer.

In section four, we present and study the concept of strongly purely t-semisimple modules as an extension of purely t-semisimple and as a generalization of strongly t-semisimple modules.

In section five, the result in chapter one every strongly t-semisimple is strongly t-Baer, led us to define and study the concept of strongly purely t-Baer modules. We prove that: For an R-module M such that a complement of $Z_2(M)$ is pure submodule. If M is strongly purely t-semisimple, then M is strongly purely t-Baer.

3.1purely semisimple modules

In this section we define and study purely semisimple as a generalization of semisimple modules. As well as, we study many properties related with this concept.

Definition (3.1.1): An R-module M is called purely semisimple if for each pure submodule N of M, there exists $K \leq^{\oplus} M$ such that $K \leq_{ess} N$.

Proposition (3.1.2): An R-module M is purely semisimple if and only if every pure submodule is a direct summand.

Proof:$\Rightarrow$ Let N be a pure submodule of M, so there exists $K \leq^{\oplus} M$ such that $K \leq_{ess} N$. But $K \leq^{\oplus} M$ implies K is closed in M. Hence $K = N$ and so K is a direct summand.

$\Leftarrow$ Let N be a pure submodule. Since $N \leq^{\oplus} M$ and $N \leq_{ess} N$, then M is purely semisimple. □

Remarks and Examples (3.1.3):

(1) It is clear that every semisimple module is purely semisimple, but the converse is not hold in general for example : The Z-module $M = Z_8 \oplus Z_2$ is not semisimple, however it is purely semisimple since the only pure submodules of M are $N_1 = Z_8 \oplus 0$, $N_2 = (0) \oplus Z_2$, $N_3 =< (\bar{1}, \bar{1})>$, $N_4 =< (\bar{4}, \bar{1})>$, $N_5 =< (\bar{0}, \bar{0})>$, $N_6 = Z_8 \oplus Z_2$ and each of them is a direct summand of M , since $N_1 \oplus N_2 = M$, $N_3 \oplus N_2 = M$, $N_4 \oplus N_1 = M$, $N_5 \oplus M = M$ and $N_6 \oplus < (\bar{0}, \bar{0})>= M$.

Also, each of Z-modules Z,Q,Z_{P^∞} are purely semisimple but they aren't semisimple.

(2) Let M be a regular module (every submodule of M is pure). Then M is purely semisimple if and only if M is semisimple.

(3) A pure submodule of purely semisimple module M is purely semisimple.

Proof: Let N be a pure submodule of M. Let K be a pure submodule of N, hence K is pure in M . Since M is purely semisimple $K \leq^{\oplus} M$ so that $K \oplus K' = M$ for some $K' \leq M$, and hence $N = K \oplus (K' \cap N)$. Thus $K \leq^{\oplus} N$ and N is purely semisimple. □

(4) Let M be a purely semisimple. Then M is purely extending if and only if M is

Extending, where″an R-module is called purely extending if for each $N \leq M$, there exists pure submodule K of M such that $N \leq_{ess} K$″ [14]. Equivalently ″M is purely extending if every closed is pure″.

Proof: ⇐ It is clear

⇒ Let $N \leq M$. As M is purely extending, there exists a pure submodule K in M such that $N \leq_{ess} K$. But M is purely semisimple, $K \leq^{\oplus} M$. Then $N \leq_{ess} K \leq^{\oplus} M$ and so M is extending. □

(5) Every pure simple module is purely semisimple, where ″ an R-module is pure simple if it has only two pure submodules $(0), M$ ″[2].

(6) Every Noetherain projective R-module is purely semisimple.

Proof: By [2, Proposition 2.11, p.63], every pure submodule is a direct summand, of M. Hence M is purely semisimple. □

In particular $M = Z \oplus Z$ as Z-module is Notherain projective, so M is purely semisimple.

(7) If M is divisible over a PID, then M is purely semisimple.

Proof: By [2, Proposition 2.7, P.61], every pure submodule of M is a direct summand that is M is purely semisimple. □

As examples Q and $\frac{Q}{Z}$ as Z-module are divisible module over a PID Z, so that both of them are purely semisimple.

(8) If M is prime injective, then M is purely semisimple.

Proof: It follows by applying [2, Proposition 2.7, p.61].

(9) Let M and M' be two isomorphic R-modules.Then M is purely semisimple if and only if M' is purely semisimple.

Proof: $\Rightarrow$ Since $M \simeq M'$, there exists $f: M \mapsto M'$ such that f is an isomorphism. Let W be a pure submodule of M'. Then $W = f(N)$ for some $N \leq M$. It follows that N is a pure in M and hence $N \leq^{\oplus} M$, that is $N \oplus N_1 = M$ for some $N_1 \leq M$. Then $f(N \oplus N_1) = f(M) = M'$. This implies $W \oplus f(N_1) = M'$, $W \leq^{\oplus} M'$ and M' is purely semisimple.

$\Leftarrow$ The converse is similar.□

Proposition (3.1.4): Let N be a pure submodule of a purely semisimple module M. Then $\frac{M}{N}$ is purely semisimple.

Proof: Let $\frac{L}{N}$ be a pure submodule of $\frac{M}{N}$. As N is pure in M, L is pure in M. Hence $L \leq^{\oplus} M$ .That is $L \oplus W = M$ for some $W \leq M$. It follows that $\frac{L}{N} \oplus \frac{W+N}{N} = \frac{M}{N}$. Thus $\frac{M}{N}$ is purely semisimple. □

Corollary (3.1.5): Let $f: M \rightarrow M'$ be an epiomorphism and $Ker(f)$ is pure in M. If M is purely semisimple, then M' is purely semisimple.

Proof: $\frac{M}{Ker(f)} \simeq M'$ by the 1st fundamental theorem. But $\frac{M}{Ker(f)}$ is purely semisimple by Proposition (3.1.4). Hence $M'$is purely semisimple by Remarks and Examples 3.1.3(9). □

Proposition (3.1.6): If every pure submodule of M which contains $Z_2(M)$ is a direct summand, and $Z_2(M)$ is pure then $\frac{M}{Z_2(M)}$ is purely semisimple.

Proof: Let $\frac{N}{Z_2(M)}$ be a pure submodule of $\frac{M}{Z_2(M)}$. Hence N is a pure submodule in M with $N \supseteq Z_2(M)$. By hypothesis, N is a direct summand of M, and so $N \oplus W = M$ for some $W \leq M$. It follows that $\frac{N}{Z_2(M)} \oplus \frac{W + Z_2(M)}{Z_2(M)} = \frac{M}{Z_2(M)}$, that is $\frac{N}{Z_2(M)} \leq^{\oplus} \frac{M}{Z_2(M)}$ and $\frac{M}{Z_2(M)}$ is purely semisimple. □

Proposition (3.1.7): Let $M = M_1 \oplus M_2$, where $M_1, M_2 \leq M$ and $annM_1 + annM_2 = R$. Then M is purely semisimple if and only if M_1 and M_2 are purely semisimple.

Proof: $\Leftarrow$ let N be a pure submodule of M. Then $N = N_1 \oplus N_2$, $N_1 \leq M_1$, $N_2 \leq M_2$[1,Proposition 4.2,P.28]. Hence N_1 is pure in M_1, N_2 is pure in M_2. Which implies $N_1 \leq^{\oplus} M_1$, $N_2 \leq^{\oplus} M_2$. Thus $N = N_1 \oplus N_2 \leq^{\oplus} M$.

$\Rightarrow$Since $M_1 \leq^{\oplus} M$ and $M_2 \leq^{\oplus} M$, then M_1 and M_2 are pure in M. Hence M_1 and M_2 are purely semisimple by Remarks and Examples 3.1.3(3). □

Proposition (3.1.8): Let $M = \oplus_{i \in I} M_i$ where $M_i \leq M$ for each $i \in I$. If every pure submodule of M is fully invariant. Then M is purely semisimple if and only if M_i is purely semisimple, for all $i \in I$.

Proof:$\Rightarrow$It follows by Remarks and Examples 3.1.3(3).

$\Leftarrow$ Let N be a pure submodule of M, by hypothesis N is a fully invariant submodule of M. Then $N = \oplus_{i \in I} N_i$ where $N_i = N \cap M_i$ for each $i \in I$ by Lemma 1.1.39(ii). Hence N_i is pure in M_i for each $i \in I$ .As M_i is purely semisimple, $N_i \leq^{\oplus} M_i$ for each $i \in I$. It follows that $N = \oplus_{i \in I} N_i \leq^{\oplus} \oplus_{i \in I} M_i = M$. Thus M is purely semisimple. □

Now we introduce the following

Definition (3.1.9): Let M and N be R-modules. M is called N-purely projective if every homomorphism $f: M \rightarrow \frac{N}{K}$, where K is a pure submodule of N,there exists

$g \in Hom(M, N)$ such that $\pi \circ g = f$, where π is the natural epiomorphism from N in to $\frac{N}{K}$, that is the digram is commutative.

$$\begin{tikzcd} & M \arrow[dl, dashed, "g"'] \arrow[dr, "f"] & \\ N \arrow[rr, "\pi"'] & & N/K \end{tikzcd}$$

M is called purely projective if M is N-purely projective for each R-module. M is called self purely projective if M is M-purely projective.

Theorem (3.1.10): The following statements are equivalent for an R-module M.

(1) M is purely semisimple.

(2) Every R-module is M-purely projective.

(3) For each pure submodule K of M , $\frac{M}{K}$ is M-purely projective.

Proof: (1) $\Rightarrow$ (2) Let M' be an R-module, let N be a pure submodule of M, and let $f \in Hom(M', \frac{M}{N})$ and let $\pi: M \to \frac{M}{N}$ be the natural epiomorphism . By condition (1) M is purely semisimple, so $N \leq^{\oplus} M$; that is $N \oplus W = M$ for some $W \leq M$. Hence $\frac{M}{N} \simeq W$ which implies that there exists an isomorphism $\psi: \frac{M}{N} \to W$. Set $\vartheta = j\psi f$ where $j: W \to M$ be the inclusion mapping. Then $\pi \circ \vartheta = f$ since for each $m' \in M'$, let $f(m') = n + w + N$ then $\pi\vartheta(m') = \pi j\psi\big(f(m')\big) = \pi j\psi(n + w + N) = \pi j\psi(w + N) = \pi j(w) = \pi(w) = w + N = n + w + N = f(m')$.

(2) $\Rightarrow$ (3) It is obvious

(3)$\Rightarrow$ (1) Consider the following diagram

$$\begin{tikzcd} & & M/K \arrow[dl, dotted, "g"'] \arrow[d, "I"] \\ K \arrow[r, "i"'] & M \arrow[r, "\pi"'] & M/K \end{tikzcd}$$

Since $\frac{M}{K}$ is M-purely projective, then (identity mapping) I on $\frac{M}{K}$ can be lifted to g: $\frac{M}{K} \rightarrow M$ with $\pi g = I$. Hence π has a right inverse. Therefore the short exact sequence (in the bottom) splits and so $K \leq^{\oplus} M$. Thus M is purely semisimple. □

The following Corollary is a direct consequence of Theorem 3.1.10(1⇒2).

Corollary (3.1.11): Every purely semisimple is self purely projective.

3.2 Purely t-semisimple modules

A generalization of t-semisimple module namely purely t-semisimple module is introduced and studied in this section.

Definition (3.2.1): An R- module M is called purely t-semisimple, if for each pure submodule N of M there exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$.

Remarks and examples (3.2.2):

(1) It is clear that every t-semisimple is purely t-semisimple, but not conversely,(see part (2)).

(2) Every pure simple module M, is purely t-semisimple, since (0) and M are the only pure submodules in M, but they are direct summands in M and $(0) \leq_{tes} (0)$, $M \leq_{tes} M$.

In particular, each of the Z-module Z, Q and Z_{P^∞} (where P is a prime number) is pure simple, so each of them is purely t-semisimple. However, Z, Q and Z_{P^∞} are not t-semisimple.

(3) Let M be a regular R- module (that is every submodule is pure). Then M is purely t-semisimple if and only if M is t-semisimple.

(4) It follows easily that every purely semismple is purely t-semisimple. In particular each of the Z-modules $Z_8 \oplus Z_2$, $Z_4 \oplus Z_2$ is purely semisimple, so they are purely t-semisimple. Also, they aren't pure simple.

(5) Let M be a nonsingular R-module. Then M is purely semisimple if and only if M is purely t-semisimple.

Proof: $\Rightarrow$ It follows by (4).

$\Leftarrow$ Let N be a pure submodule, there exists $K \leq^{\oplus} M$, $K \leq_{tes} N$. Since M is nonsingular, N is nonsingular so that $K \leq_{tes} N$, implies $K \leq_{ess} N$. But $K \leq^{\oplus} M$, implies K is closed in M. Hence $K \leq_{ess} N$ implies $K = N$. Thus $N \leq^{\oplus} M$, and therefore M is purely semisimple. □

(6) Let R be a regular ring. Then the following statements are equivalent:

(i) M is purely t-semisimple;

(ii) M is t-semisimple;

(iii) M is semisimple.

Proof: Since R is a regular, M is a regular module. Hence (ii)⇔(iii) by Proposition 1.1.59. Also since M is regular, it is clear that (i)⇔(ii) by part (3). □

Proposition (3.2.3): A pure submodule N of purely t-semisimple R-module M is purely t-semisimple.

Proof: Let N be a pure submodule of M, and assume that W is a pure submodule of N. Hence W is a pure submodule of M and so there exists $K \leq^{\oplus} M$ such that

$K \leq_{tes} W$. It follows that $M = K \oplus L$ for some $L \leq M$ and hence $N = (K \oplus L) \cap N = K \oplus (L \cap N)$ by modular law. Thus $K \leq^{\oplus} N$ and so N is purely t-semisimple. □

Since every direct summand is pure, we get the following result directly.

Corollary (3.2.4): A direct summand of purely t-semisimple module is purely t-semisimple.

Proposition (3.2.5): Let $M = \oplus_{i \in I} M_i$ where $M_i \leq M$ for all $i \in I$, and every pure submodule of M is fully invariant submodule. Then M is purely t-semisimple if and only if M_i is purely t-semisimple for each $i \in I$.

Proof: $\Rightarrow$ It is clear by Proposition 3.2.3.

$\Leftarrow$ Let N be a pure submodule of M. By hypothesis N is a fully invariant submodule of M and hence $N = \oplus_{i \in I}(N \cap M_i)$ by Lemma 1.1.39(ii). Then $N \cap M_i$ is pure in N,

but N is pure in M, so $N \cap M_i$ is pure in M. Since $N \cap M_i \leq M_i \leq M$,then $N \cap M_i$ is pure in M_i. But M_i is purely t-semisimple, there exists $K_i \leq^{\oplus} M_i$ such that $K_i \leq_{tes} N \cap M_i$. It follows that $\oplus_{i \in I} K_i \leq^{\oplus} M$ and $\oplus_{i \in I} K_i \leq_{tes} \oplus_{i \in I} (N \cap M_i) = N$ by Corollary 1.1.22(ii). □

Proposition (3.2.6): Let M_1, M_2 be R-module,$M = M_1 \oplus M_2$ with $annM_1 \oplus annM_2 = R$. Then M is purely t-semisimple if and only if M_1 and M_2 are purely t-semisimple.

Proof: Let N be a pure submodule of M. Then by the proof of [1, Proposition 4.2], $N = N_1 \oplus N_2$ for some $N_1 \leq M_1$ and $N_2 \leq M_2$. Then by the same argument of Proposition (3.2.5) the result holds. □

Proposition (3.2.7): Let $M = M_1 \oplus M_2$ with $M_1 \leq M$ and $M_2 \leq M$ and M is a distributive module. Then M is purely t-semisimple if and only if M_1 and M_2 are purely t-semisimple.

Proof: Let N be a pure submodule of M. As M is distributive, $N = (N \cap M_1) \oplus (N \cap M_2)$. Hence by the same procedure of Proposition 3.2.6 the result follows. □

Recall that "an R-module M has PIP property if the intersection of two pure submodules is pure" [2].

Theorem (3.2.8): For an R- module M. Consider the following assertions.

(1) M is purely t-semisimple;
(2) $\frac{M}{Z_2(M)}$ is purely semisimple;
(3) $M = Z_2(M) \oplus M'$ where M' is a nonsingular purely semisimple;
(4) Every nonsingular pure submodule of M is a direct summand;
(5) Every pure submodule of M which contains $Z_2(M)$ is a direct summand.

Then (1)⇒(4), (3)⇒(5) and (3)⇒(2), [(4)⇒(3), if complement of $Z_2(M)$ is pure] and [(2)⇒(3), if a complement of $Z_2(M)$ is direct summand], (3)⇒(1) if M has PIP, [(5)⇒(3) if complement of $Z_2(M)$ is a direct summand stable in M]. Thus all statements (1) through (5) are equivalent if a complement of $Z_2(M)$ is direct summand stable and M has PIP and ((1)⇔(4)⇔(3) if complement of $Z_2(M)$ is pure and M has PIP).

Proof: (1)⇒(4) Let N be a nonsingular pure submodule of M. There exists $K \leq^{\oplus} M$ such that $K \leq_{tes} N$. Assume that $M = K \oplus K'$ for some $K' \leq M$. By modular law, $N = K \oplus (N \cap K')$. Thus $(N \cap K') \simeq \frac{N}{K}$ which is Z_2-torsion by Proposition 1.1.17(4). But $N \cap K'$ is nonsingular, hence $\frac{N}{K} = (0)$, that is $N = K$ and so $N \leq^{\oplus} M$.

(3)⇒(5) Let N be a pure submodule of M and $N \supseteq Z_2(M)$. Since $M = Z_2(M) \oplus M'$, where M' is a nonsingular purely semisimple. Then by modular law, $N = (Z_2(M) \oplus M') \cap N = Z_2(M) \oplus (N \cap M')$. Hence $N \cap M' \leq^{\oplus} N$, so $N \cap M'$ is pure in N, but N is pure in M, so that $(N \cap M')$ is pure in M. On other hand $N \cap M' \leq$

M',hence $N \cap M'$ is pure in M'. As M' is purely semisimple, $N \cap M' \leq^{\oplus} M'$. Thus $M' = (N \cap M') \oplus W$ for some $W \leq M'$, and this implies that $M = Z_2(M) \oplus (N \cap M') + W = N \oplus W$. Thus $N \leq^{\oplus} M$.

(3)⇒(2) By condition (3), $M = Z_2(M) \oplus M'$, where M' is a nonsingular purely semisimple. As $\frac{M}{Z_2(M)} \simeq M'$, so $\frac{M}{Z_2(M)}$ is purely semisimple.

(4)⇒(3) Assume a complement of $Z_2(M)$ is pure in M. Let M' be a complement of $Z_2(M)$. Then $Z_2(M) \oplus M' \leq_{ess} M$ and, hence $M' \leq_{tes} M$ by Proposition 1.1.17(3). Thus $\frac{M}{M'}$ is Z_2-torsion. But $M'$is nonsingular, thus $M' \leq^{\oplus} M$ by condition (4) and so that $M = L \oplus M'$ for some $L \leq M$. Thus $L \simeq \frac{M}{M'}$ which is Z_2-torsion. Beside this $Z_2(M) = Z_2(L) \oplus Z_2(M') = L \oplus (0) = L$. Thus $M = Z_2(M) \oplus M'$. Now $M' \leq^{\oplus} M$ and M is purely t-semisimple. Then M' is purely t-semisimple, by proposition (3.2.5). On other hand M' is nonsingular, so that M' is purely semisimple.

(2)⇒(3) Assume a complement of $Z_2(M)$ is a direct summand. Let M' be a complement of $Z_2(M)$. So that, $M' \leq^{\oplus} M$. Also, M' is nonsingular. It follows by the same of proof of part (4)→(3), $M = Z_2(M) \oplus M'$. Then by condition (2),$\frac{M}{Z_2(M)}$ is purely semisimple and so that M' is purely semisimple.

(3)⇒(1) Let N be a pure submodule of M. As $M' \leq^{\oplus} M$, M' is a pure in M. Since M has PIP, $N \cap M'$ is a pure submodule of M. But $N \cap M' \leq M'$, so $N \cap M'$ is pure in M', and as M' is purely semisimple, then $N \cap M' \leq^{\oplus} M'$. It follows that $M' = (N \cap M') \oplus W$ for some $W \leq M'$. Hence $M = Z_2(M) \oplus (N \cap M') \oplus W$, that is $(N \cap M') \leq^{\oplus} M$. On the other hand, we have $\frac{N}{N \cap M'} \simeq \frac{N+M'}{M'} \leq \frac{M}{M'} \simeq Z_2(M)$. Hence $\frac{N}{N \cap M'}$ is Z_2-torsion; that is $(N \cap M') \leq_{tes} N$. Thus M is purely t-semisimple.

(5)⇒(3) (If a complement of $Z_2(M)$ is direct summand stable). By the same way of proof ((4)⇒(3)), $M = Z_2(M) \oplus M'$ where M' is a nonsingular. To prove M' is purely semisimple. Let N be a pure submodule of M'. Then $Z_2(M) \oplus N$ is pure in $Z_2(M) \oplus M' = M$. But $Z_2(M) \oplus N \supseteq Z_2(M)$, so that by condition (5), $Z_2(M) \oplus N \leq^{\oplus} M$. Thus $(Z_2(M) \oplus N) \oplus L = M$ for some $L \leq M$ and so $M = Z_2(M) \oplus (N \oplus L)$. But $M = Z_2(M) \oplus M'$, this implies $N \oplus L = M'$ by [1, Lemma 4.8]. □

By [2, Proposition 2.3, P.33], every multiplication module satisfies PIP. Hence by using this fact and Theorem 3.2.8, the following result is obtained.

Corollary (3.2.9): For multiplication R-module M with a complement of $Z_2(M)$ is a pure submodule of M. The following statements are equivalent:

(1) M is purely t-semisimple.

(2) Every nonsingular pure submodule of M is a direct summand.

(3) $M = Z_2(M) \oplus M'$ Where $M'$is a nonsingular module and purely semisimple.

Also by applying Theorem 3.2.8, we get the following

Corollary (3.2.10): If M is purely t-semisimple module and a complement of $Z_2(M)$ is pure. Then every pure submodule N of M, $N \supseteq Z_2(M)$ is closed.

Proposition (3.2.11): For an R-module M such that a complement of $Z_2(M)$ is pure and M satisfies PIP. If $Rad(M)$ is Z_2-torsion and every nonsingular pure submodule of M has a weak supplement. Then M is purely t-semisimple.

Proof: Let N be a nonsingular pure submodule of M. By hypothesis N has a weak supplement submodule K of M such that $M = K + N$ and $K \cap N \ll M$. Cleary $M = \left(K + Rad(M)\right) + N$. Now we claim that $\left(K + Rad(M)\right) \cap N = 0$. To prove our assertion, assume that $x \in \left(K + Rad(M)\right) \cap N$. Then $x = y + z$ where $y \in K$ and $z \in Rad(M)$ and, since $Rad(M)$ is Z_2-torsion there exist a t-essential right ideal I of R such that $(x - y)I = 0$. Thus $xI = yI \leq K \cap N \leq Rad(M) \leq Z_2(M)$ and so

$x + Z_2(M) \in Z_2\left(\frac{M}{Z_2(M)}\right) = 0$. Hence $x \in Z_2(M)$. Thus $x \in Z_2(M) \cap N = Z_2(N) = 0$ and this implies that N is direct summand of M. Hence by Theorem 3.2.8[(4)⇒(3) and (3)⇒(1)] M is purely-t-semisimple. □

The following Proposition is a characterization of purely t-semisimple module.

Proposition (3.2.12): An R-module M is purely t-semisimple if and only if for each pure submodule N of M, there is a decomposition $N = K \oplus K'$ such that K is a direct summand of M and K' is Z_2-torsion.

Proof: ⇒ If M is purely t-semisimple. Let N be a pure submodule of M. Then there exists $K \leq^{\oplus} M$ and $K \leq_{tes} N$. As $K \leq^{\oplus} M$, $M = K \oplus W$ for some $W \leq M$. Then by modular law, $N = K \oplus (\mathrm{W} \cap N)$. Put $\mathrm{W} \cap N = K'$ so $N = K \oplus K'$. Also, since $K \leq_{tes} N$, then $\frac{N}{K} \cong K'$ is Z_2-torsion by Proposition 1.1.17(4).

⇐ Let N be a pure submodule of M. By hypothesis, $N = K \oplus K'$ and $K \leq^{\oplus} M$, K' is Z_2-torsion so $\frac{N}{K} \cong K'$ is Z_2-torsion, hence $K \leq_{tes} N$ by Proposition 1.1.17(4). Thus M is purely t-semisimple. □

Proposition (3.2.14): For an R- module M which satisfies the condition, a complement of $Z_2(M)$ is pure. If M is purely t-semisimple, then M has no proper t-essential pure submodule which contains $Z_2(M)$.

Proof: Assume L is a proper t-essential pure submodule of M with $L \supseteq Z_2(M)$. By Proposition 1.1.17(2) $,\frac{L}{Z_2(M)} \leq_{ess} \frac{M}{Z_2(M)}$. But by Theorem 3.2.8((1)⇒(4)⇒(3)⇒(2)),$\frac{M}{Z_2(M)}$ is purely semisimple, and since $\frac{L}{Z_2(M)}$ is pure in $\frac{M}{Z_2(M)}$, so that $\frac{L}{Z_2(M)} \leq^{\oplus} \frac{M}{Z_2(M)}$. Thus $\frac{L}{Z_2(M)} = \frac{M}{Z_2(M)}$, hence $L = M$ which is a contradiction. Therefore M has no proper t-essential pure submodule of M containing $Z_2(M)$. □

Theorem (3.2.15): Let M be a finitely generated faithful multiplication R-module. Then M is purely t-semisimple, if R is purely t-semisimple.

Proof: Let N be a pure in M. Then $N = MI$ for some ideal I of R. We claim that I is pure in R. Assume J be any ideal of R. $M(I \cap J) = MI \cap MJ$ by [19, Corollary 1.7]. As MI is pure in M, $MJ \cap MI = MIJ$. Thus $M(I \cap J) = M(IJ)$ and so $I \cap J = IJ$ by [19, Theorem 3.1] since M is finitely generated faithful multiplication. Thus I is pure in R, and hence there exists a direct summand L of R, such that $L \leq_{tes} I$. As $L \leq^{\oplus} R$, so $ML \leq^{\oplus} M$. Also, $L \leq_{tes} I$ implies that $ML \leq_{tes} MI = N$ by Lemma 1.1.24(2). Thus M is purely t-semisimple. □

Note (3.2.16): If M is a finitely generated faithful multiplication module over regular commutative ring. Then M is purely t-semisimple (t-semisimple) implies R is semisimple.

Proof: Since R is a regular ring, M is regular. Cleary M is purely t-semisimple equivalently to M is t-semisimple. Then by Proposition 1.1.59, M is semisimple and this implies R is semisimple. □

Remark (3.2.17): A purely t-semisimple module need not be purely extending module, for example: The Z-module $Z_8 \oplus Z_2$ is purely t-semisimple but it is not purely extending since, it is easy to see that, $N =< (\bar{2}, \bar{1}) >$ is closed and it is not pure.

We introduce the following

Definition (3.2.18): Let M be an R-module. M is called purely t-extending if for each submodule N of M, N is t-essential in pure submodule of M.

Lemma (3.2.19): An R –module M is purely t-extending if and only if every t-closed submodule of M is pure in M.

Proof: ⇒Let $N \leq M$, N is t-closed. Then $N \leq_{tes} K$ and K is pure. Hence $N = K$ since every t-closed has no proper t-essential extension. Thus N is pure.

$\Leftarrow$ Let $N \leq M$. By [10, Lemma 2.3], there exists a t-closed submodule H of M such that $N \leq_{tes} H$. As H is t-closed, hence by hypothesis H is pure. Thus M is purely t-extending. □

Al-Bahraany in [3] said that " an R-module M is purely Y-extending if every Y-closed submodule of M is pure in M ". But as we mention in chapter one, Y-closed and t-closed are coincide hence purely t-extending and purely Y-extending are coincide.

We know that t-semisimple module implies t-extending module but we cannot generalize this for purely t-semisimple. However, we have

A purely t-semisimple module implies purely t-extending, however we have the following result.

Proposition (3.2.20): Let M be an R-module such that $M = Z_2(M) \oplus M'$, M' is purely t-extending. Then M is purely t-extending.

Proof: Let N be a t-closed in M. Then $N \supseteq Z_2(M)$, so by modular law, $N = Z_2(M) \oplus (M' \cap N)$. As N is t-closed in M, then $(M' \cap N)$ is t-closed in M' by Proposition 1.1.31(i). But M' is purely t-extending implies that $M' \cap N$ is pure in M'. Hence $Z_2(M) \oplus (M' \cap N)$ is pure in $Z_2(M) \oplus M' = M$, that is N is pure in M and so M is purely t-extending. □

Proposition (3.2.21): For a ring R. The following statements are equivalent:

(1) $\oplus_I R$ is purely t-semisimple.

(2) Every projective R-module is purely t-semisimple.

Proof: (1) $\Rightarrow$ (2) Assume that M is a projective R-module. Then there exist a free R-module F and an epiomorphism $f: F \longmapsto M$. Since F is free, then by [24, Lemma 4.4.1, p.88] $F \cong \oplus_I R$ for some index I. Consider the following short exact sequence: $0 \longrightarrow Ker f \overset{i}{\rightarrow} \oplus_I R \longrightarrow M \longrightarrow 0$ where i is the inclusion mapping. Since M

is projective, the sequence is split. Thus $\oplus_I R = kerf \oplus M$. Then M is purely t-semisimple, by Corollary 3.2.4.

(2)⇒(1) It is clear. □

3.3 Purely t-semisimple and purely t-Baer modules

It known that every t-semisimple is t-Baer [7]. In this section we introduce purely t-Baer and prove that every purely t-semisimple modules with certain conditions is purely t- Baer. Moreover we prove that every purely t-extending is purely t-Baer.

Definition (3.3.1): An R-module M is called purely t-Baer if for each left ideal $I$of $End(M) = S$, $t_M(I)$ is a pure submodule of M.

As we mention in chapter one Remarks and Examples 1.5.5(3), for a nonsingular R-module M, $t_M(I) = r_M(I)$ for left ideal I of S. Hence we get the following remark.

Remark (3.3.2): Let M be a nonsingular R-module. Then M is purely t-Baer if and only if M is purely Baer.

The following Theorem is a characterization of purely t-Baer module. Before giving this Theorem we need the following Lemma.

Lemma (3.3.3): Let $A \leq B \leq M$, where M is an R-module. If $A \leq_{ess} B$, then for each $b \in$ B**,** there exists an essential ideal J of R such that bJ ≤ A**.**

Proof: Let $b \in B$. As $A \leq_{ess} B$, there exists $r \in R$ such that $0 \neq br \in A$. Put $J = \{r \in R: br \in A\}$. J is a right ideal of R. We claim that $J \leq_{ess} R$ and $bJ \leq A$. Suppose $J \cap C = (0)$ for some ideal C of $R, C \neq 0$. Hence for each $c \in C, c \notin J$. It follows $bc \notin A$. As $b \in B$, $bc \in B$, hence there exists $r_1 \in R$ such that $0 \neq (bc)r_1 \in A$. Thus $0 \neq b(cr_1) \in A$ and so $0 \neq cr_1 \in J$. On the other hand $cr_1 \in C$, and hence $0 \neq cr_1 \in J \cap C$ and this is a contradiction. Thus $J \leq_{ess} R$. Also it is clear that $bJ \leq A$. □

Theorem (3.3.4): An R-module M is purely t-Baer if and only if for each left ideal I of $End(M)$, $t_M(I)$ is t-essential in pure submodule of M.

Proof: $\Rightarrow$ Since M is purely t-Baer, $t_M(I)$ is pure in M. But $t_M(I) \leq_{tes} t_M(I)$.

$\Leftarrow$ $t_M(I) \leq_{tes} K$ for some pure submodule K of M. As $Z_2(K) \leq Z_2(M) \leq t_M(I)$, then $t_M(I) + Z_2(K) \leq_{ess} K$ by Proposition 1.1.17 and so $t_M(I) \leq_{ess} K$. Now for each $k \in$ K, there exists an essential ideal J of R such that $kJ \leq t_M(I)$ by Lemma (3.3.3). Hence for each $f \in I$,$f(k.J) = f(k)J \leq Z_2(M)$. Thus $\big(f(k) + Z_2(M)\big)J = Z_2(M) = 0_{\frac{M}{Z_2(M)}}$. But $J \leq_{ess} R$, hence $(f(k) + Z_2(M) \in Z\left(\frac{M}{Z_2(M)}\right) = 0$. This implies $f(k) \in Z_2(M)$ and hence $k \in t_M(I)$. Thus $K = t_M(I)$ and so $t_M(I)$ is pure and M is purely t-Baer. □

Proposition (3.3.5): Every purely t-extending is purely t-Baer.

Proof: Let M be a purely t-extending. As $t_M(I) \leq M$, then by Definition 3.2.18 $t_M(I) \leq_{tes} K$ and K is pure. Thus M is purely t-Baer by Theorem (3.3.4). □

To give the next result, we need the following Lemma.

Lemma (3.3.6): Let M be an R-module such that $\mathrm{M} = \mathrm{Z}_2(\mathrm{M}) \oplus \mathrm{M}'$, M' is stable in M. Then $\mathrm{t}_\mathrm{M}(\mathrm{I}) \cap \mathrm{M}' = \mathrm{r}_{\mathrm{M}'}(\mathrm{I})$ for each left ideal I of S=End(M).

Proof: Let $\mathrm{m} \in \mathrm{r}_{\mathrm{M}'}(\mathrm{I})$. Then $\mathrm{m} \in \mathrm{M}'$ and $\mathrm{Im} = 0 \leq \mathrm{Z}_2(\mathrm{M})$. Hence $\mathrm{m} \in \mathrm{t}_\mathrm{M}(\mathrm{I}) \cap \mathrm{M}'$.Now if $\mathrm{m} \in \mathrm{t}_\mathrm{M}(\mathrm{I}) \cap \mathrm{M}'$, then $\mathrm{m} \in \mathrm{M}'$ and $\mathrm{Im} \leq \mathrm{Z}_2(\mathrm{M})$. So that for any $f \in \mathrm{I}$, $f(\mathrm{m}) \in \mathrm{Z}_2(\mathrm{M})$. But M' is stable in M implies $f(\mathrm{m}) \in \mathrm{M}'$ and so $f(\mathrm{m}) \in \mathrm{Z}_2(\mathrm{M}) \cap \mathrm{M}' = (0)$; that is $Im = 0$ and so $\mathrm{m} \in \mathrm{r}_{\mathrm{M}'}(\mathrm{I})$. Thus $\mathrm{t}_\mathrm{M}(\mathrm{I}) \cap \mathrm{M}' = \mathrm{r}_{\mathrm{M}'}(\mathrm{I})$. □

Proposition (3.3.7): Let M be an abelian Baer (strongly Baer) module such that a complement of $Z_2(M)$ is pure stable in M and M satisfies PIP. If M is purely t-semisimple, then M is purely t-Baer.

Proof: By Theorem 3.2.8(1→4) and (4→3 if a complement of $Z_2(M)$ is pure). Hence $M = Z_2(M) \oplus M'$where M' is nonsingular purely semisimple. Then for each $I \leq End(M)$, $Z_2(M) \leq t_M(I)$, so $t_M(I) = Z_2(M) \oplus (t_M(I) \cap M')$. Thus $t_M(I) = Z_2(M) \oplus r_{M'}(I)$ by Lemma 3.3.6.As M is an abelian Baer (strongly Baer module), $r_M(I)$ is fully invariant direct summand, hence it is pure in M. But it is clear that $r_{M'}(I) = r_M(I) \cap M'$ and since M has PIP, $r_{M'}(I)$ is pure in M'. It follows that $t_M(I)$=$Z_2(M) \oplus r_{M'}(I)$ is pure in $Z_2(M) \oplus M' = M$. Thus $t_M(I)$ is pure in M and M is purely t-Baer. □

Proposition (3.3.8): If $M = Z_2(M) \oplus M'$ for some $M' \leq M$ such that $t_M(I) \cap M'$ is pure in M', then M is purely t-Baer.

Proof: Since $Z_2(M) \leq t_M(I)$,for each $I \leq End(M)$ then $t_M(I) = Z_2(M) \oplus (t_M(I) \cap M')$ by modular law. But $Z_2(M)$ is pure in $Z_2(M)$ and $(t_M(I) \cap M')$ is pure in $M'$by hypothesis, the $Z_2(M) \oplus (t_M(I) \cap M')$ is pure in $Z_2(M) \oplus M' = M$. Thus $t_M(I)$ is pure in M and M is purely t-Baer. □

Corollary (3.3.9): If $M = Z_2(M) \oplus M'$ for some $M' \leq M$. If M has PIP and M is purely t-Baer then $(t_M(I) \cap M')$ is pure in M'.

Proof: By the same proof of Proposition (3.3.8) $t_M(I) = Z_2(M) \oplus (t_M(I) \cap M')$ for each $I \leq End(M)$. But $t_M(I)$ is pure in M and M' is pure in M. So that $t_M(I) \cap M'$ is pure in M by PIP. Hence $(t_M(I) \cap M')$ is pure in M' since $t_M(I) \cap M' \leq M'$. □

3.4 Strongly purely t-semisimple Modules

In this section we extend the notion of purely t-semisimple module into strongly purely t-semisimple module. Also this concept is a generalization of strongly t-semisimple modules. A comprehensive study of this concept and its connections with some related modules are introduced.

Definition (3.4.1): An R-module M is called strongly purely t-semisimple if for each pure submodule N of M, there exists a fully invariant direct summand K of M such that $K \leq_{tes} N$.

Remarks and Examples (3.4.2):

(1) Every strongly purely t-semisimple is purely t-semisimple. But the converse is not true as the following example, shows.

Let $M = \mathbb{R}\oplus\mathbb{R}$ as $\mathbb{R}$-module, let $N = \mathbb{R}\oplus(0)$, N is a pure submodule of M. But $K =< (0,0) >$ is the only fully invariant direct summand such that $K \leq N$. However $K \nleq_{tes} N$, see (Remarks and Example 1.2.2(8)). Thus M is not strongly purely t-semisimple. On other hand M is purely t-semisimple since it is semisimple.

(2) Every singular module is strongly purely t-semisimple, for example $M = Z_8 \oplus Z_2$ as Z-module is strongly purely t-semisimple.

Proof: Let N be a pure submodule of M, there exists $(0) \leq^{\oplus} M$, and $(0) + Z(N) = N \leq_{ess} N$, hence $(0) \leq_{tes} N$ and (0) is fully invariant. □

(3) Every pure simple is strongly purely t-semisimple, for example Z as Z-module is strongly purely t-semisimple, but it is not t-semisimple (hence it is not strongly t-semisimple).

(4) Purely t-semisimple and strongly purely t-semisimple are coinciding in the class of multiplication modules.

(5) A pure submodule of strongly purely t-semisimple is strongly purely t-semisimple.

Proof: Let N be a pure submodule of M and M is strongly purely t-semisimple, let W be a pure submodule of N. Hence W is a pure submodule of M and so there exists $K \leq^{\oplus} M$ and K is fully invariant in M, $K \leq_{tes} W$. By Lemma (1.2.6), K is fully invariant in N. But $K \leq^{\oplus} M$, then $K \oplus K' = M$, for some $K' \leq M$ and by modular law $N = K \oplus (N \cap K')$, so $K \leq^{\oplus} N$. Hence K is a fully invariant direct summand of N and $K \leq_{tes} W$. Thus N is strongly purely t-semisimple. □

(6) A direct summand of strongly purely t-semisimple is strongly purely t-semisimple.

Proof: Since every direct summand is pure , the result follows directly by part(5). □

We introduce the following and will be used in our work.

Definition (3.4.3): An R-module M is called purely fully stable if every pure submodule of M is stable.

Remark (3.4.4): Every fully stable module is purely fully stable .

Example (3.4.5): Consider the Z-module Q. Q is purely fully stable. But Q is not fully stable since $Z \leq Q$ and $f: Z \rightarrow Q$ defined by $f(n) = \frac{n}{2}$, implies that $f(Z) \nleq Z$.

Theorem (3.4.6): For an R-module M. Consider the following conditions.

(1) M is strongly purely t-semisimple.

(2) $M = Z_2(M) \oplus M'$, where M' is a nonsingular fully invariant submodule of M and purely fully stable, purely semisimple.

(3) For each nonsingular pure submodule N of M, N is fully invariant direct summand .

(4) $\frac{M}{Z_2(M)}$ is purely semisimple and isomorphic to purely fully stable submodule of M.

(5) For each pure submodule N in M with $N \supseteq Z_2(M)$, N is fully invariant direct summand of M.

Then (1) $\Rightarrow$ (3), (2) $\Rightarrow$ (4) and (2) $\Rightarrow$(5). ((3)$\Rightarrow$ (2) if (a complement of $Z_2(M)$ is pure). (2) $\Rightarrow$ (1) if (M satisfies PIP). (5) $\Rightarrow$ (2) (if complement of $Z_2(M)$ is direct summand stable).Thus (1)$\Leftrightarrow$(3)$\Leftrightarrow$(2) (if a complement of $Z_2(M)$ is pure and M has

PIP). (1)⇔(3)⇔(2)⇔(5) if (a complement of $Z_2(M)$ is a stable direct summand and M has PIP).

Proof: (1) ⇒(3**)** Assume N is a pure nonsingular submodule of M. By condition (1), there exists a fully invariant direct summand K of M with $K \leq_{tes} N$. So $\frac{N}{K}$ is Z_2-torsion. But K is a direct summand of M implies $M = K \oplus K'$ for some $K' \leq M$ and hence by modular law, $N = K \oplus (N \cap K')$ and so $N \cap K' \simeq \frac{N}{K}$ which is Z_2-trosion. On other hand $N \cap K'$ is a nonsingular submodule of N, so that $\frac{N}{K}$=(0), that is $N = K$ and so N is a fully invariant direct summand .

(2)⇒(4) By condition (2), $M = Z_2(M) \oplus M'$, M' is a nonsingular purely fully stable , M' is fully invariant in M, and purely semisimple . Hence $\frac{M}{Z_2(M)} \simeq M'$, that is $\frac{M}{Z_2(M)}$ is purely semisimple and isomorphic to a purely fully stable submodule.

(2)⇒(5) Let N be a pure submodule of M and $N \supseteq Z_2(M)$. Since $M = Z_2(M) \oplus M'$, $N = Z_2(M) \oplus (N \cap M')$. Hence $(N \cap M')$ is pure in M', so $(N \cap M') \leq^{\oplus} M'$ and stable since M' is purely semisimple and purely fully stable. So that $M' = (N \cap M') \oplus W$ for some $W \leq M'$. Hence, $M = Z_2(M) \oplus (N \cap M') \oplus W = N \oplus W$. Thus $N \leq^{\oplus} M$. To prove N is a fully invariant submodule in M. Since $N = Z_2(M) \oplus (N \cap M')$ and $(N \cap M')$ is fully invariant in M' and M' is fully invariant in M, so $(N \cap M')$ is fully invariant in M, hence $N = Z_2(M) \oplus (N \cap M')$ is fully invariant in M.

(3)⇒(2) (If a complement of $Z_2(M)$ is pure) . Suppose M' be a complement of $Z_2(M)$. Then $M' \oplus Z_2(M) \leq_{ess} M$, hence $M' \leq_{tes} M$ and $\frac{M}{M'}$ is Z_2-torsion. But M' is nonsingular, to show our assertion, suppose $x \in Z(M')$, so $x \in M' \leq M$ and $ann(x) \leq_{ess} R$.Hence $ann(x) \leq_{tes} R$ and this implies $x \in Z_2(M)$. Thus $x \in Z_2(M) \cap M'$=(0), thus x=0 and M' is a nonsingular. Thus M' is pure nonsingular, so that $M' \leq^{\oplus} M$ and M' is fully invariant submodule of M. Thus $M = L \oplus M'$ for some

$L \leq M$. It follows that $\frac{M}{M'} \simeq L$ and hence L is a Z_2-torsion. Hence $Z_2(M) = Z_2(L) \oplus Z_2(M') = L \oplus (0)$ and so $M = Z_2(M) \oplus M'$. Let N be a pure submodule of M', then N is pure in M since M' is pure in M. Also, N is a nonsingular submodule of M.Then by condition (3), N is a fully invariant direct summand of M and so $M = N \oplus W$ for some $W \leq M$ which implies that $M' = N \oplus (W \cap M')$. Thus $N \leq^{\oplus} M'$ and M' is purely semisimple. On other hand, N is fully invariant submodule of M and $N \leq M$ imply N is a fully invariant submodule of $M'$by Lemma (1.2.6) and so N is a stable submodule of M' since $N \leq^{\oplus} M'$. Thus M' is purely fully stable.

(2)⇒ (1) (If M has PIP). Let N be a pure submodule of M. As $M' \leq^{\oplus} M$, M' is pure submodule of M and by PIP, $N \cap M'$ is pure in M. But $N \cap M' \leq M'$, hence $N \cap M'$ is pure in M'. It follows that $(N \cap M') \leq^{\oplus} M'$ since M' is purely semisimple. Thus $M' = (N \cap M') \oplus W$ for some $W \leq M'$ and so $M = Z_2(M) \oplus (N \cap M') \oplus W$, that is $(N \cap M')$ is a direct summand of M. On other hand ,$\frac{N}{N \cap M'} \simeq \frac{N+M'}{M'} \leq \frac{M}{M'} \simeq Z_2(M)$. Hence $\frac{N}{N \cap M'}$ is Z_2-torsion; that is $(N \cap M') \leq_{tes} N$. Beside this $(N \cap M')$ is fully invariant in M', since M' is fully stable. But M' is fully invariant in M, so $(N \cap M')$ is fully invariant in M. Therefor $(N \cap M')$ is fully invariant direct summand of M and $N \cap M' \leq_{tes} N$ and so M is strongly purely t-semisimple.

(5)⇒(2) (If a complement $Z_2(M)$ is direct summand stable). Assume M' is a complement of $Z_2(M)$ is pure, then by a similar proof of part (3)⇒(2) $M = Z_2(M) \oplus M'$, where M' is a nonsingular. To prove M' is purely semisimple, let N be a pure submodule of M'. Hence $Z_2(M) \oplus N$ is pure in $Z_2(M) \oplus M' = M$ and as $Z_2(M) \oplus N \supseteq Z_2(M)$, so $Z_2(M) \oplus N$ is fully invariant direct summand of M by condition (5). Thus $(Z_2(M) \oplus N) \oplus L = M$ for some $L \leq M$. Thus $M = Z_2(M) \oplus (N \oplus L)$. But $M = Z_2(M) \oplus M'$. So that $N \oplus L = M'$[1, Theorem 4.8, p.30]. Therefore M' is purely semisimple. Now to prove M' is purely fully stable. Let W be a pure submodule of M'. Then $Z_2(M) \oplus W$ is pure in $Z_2(M) \oplus M' =$

M and so $Z_2(M) \oplus W$ is fully invariant direct summand in $M = Z_2(M) \oplus M'$. But $W = (Z_2(M) \oplus W) \cap M'$ to see this: $W \leq M'$ and $W \leq Z_2(M) \oplus W$ implies $W \leq (Z_2(M) \oplus W) \cap M'$. Let $x \in (Z_2(M) \oplus W) \cap M')$, then $x = a + b \in M'$ such that $a \in Z_2(M), b \in W \leq M'$, $a = x - b \in Z_2(M) \cap M' = 0$, implies $a = 0$, then $x = b \in W$. Hence $W = (Z_2(M) \oplus W) \cap M'$ .Also $Z_2(M) \oplus W$ is fully invariant submodule of M and M' is a fully invariant in M, so that W is a fully invariant in M. Beside this $(Z_2(M) \oplus W) \leq^{\oplus} M$ implies $W \leq^{\oplus} M$ it follows that $W$is a fully invariant in M' by Lemma 1.2.6. Also, we have $W \leq^{\oplus} M'$(since $M'$is purely semisimple), so W is fully invariant direct summand of M' . Thus W is pure stable submodule of M' and hence M' is purely fully stable. □

As every multiplication satisfies PIP [2] we get the following

Corollary (3.4.7): For a multiplication R-module M with the condition complement of $Z_2(M)$ is pure. The following assertions are equivalent:

(1) M is strongly purely t-semisimple.
(2) $M = Z_2(M) \oplus M'$, M' is a nonsingular fully invariant submodule of M, M' is purely fully stable and purely semisimple.
(3) Every nonsingular pure submodule of M is fully invariant direct summand.

Corollary (3.4.8): Let M be an R-module such that complement of $Z_2(M)$ is direct summand stable and M is a multiplication module. The following statements are equivalent.

(1) M is strongly purely t-semisimple.
(2) $M = Z_2(M) \oplus M'$, where M' is a nonsingular fully invariant submodule of M and purely fully stable, purely semisimple.
(3) For each nonsingular pure submodule N of M, N is fully invariant direct summand.
(4) For each pure in M, $N \supseteq Z_2(M)$, N is fully invariant direct summand of M.

Now we will consider the direct sum of strongly purely t-semisimple. First we have

Theorem (3.4.9): Let $M = M_1 \oplus M_2$, where M_1 and M_2 are R-module and every pure submodule of M is fully invariant. Then M is strongly purely t-semisimple if and only if M_1 and M_2 are strongly purely t-semisimple.

Proof:$\Rightarrow$ It is clear by Remarks and Examples 3.4.2(6).

$\Leftarrow$ Let N be a pure submodule of M. Then N is fully invariant in M and so $N = N_1 \oplus N_2$, N_1 is fully invariant in M_1, N_2 is fully invariant in M_2 where $N_1 = N \cap M_1$, $N_2 = N \cap M_2$ by Lemma 1.1.39(ii). Also, N is pure in M implies N_1 is pure in M_1 and N_2 is pure in M_2. Since M_1 and M_2 are strongly purely t-semisimple there exist fully invariant direct summands K_1, K_2 of M_1 , M_2 respectively where $K_1 \leq_{tes} N_1$ and $K_2 \leq_{tes} N_2$. It follows $K_1 \oplus K_2$ is a direct summand of M and $K_1 \oplus K_2 \leq_{tes} N_1 \oplus N_2 = N$ by Proposition 1.1.22(2). To show that $K_1 \oplus K_2$ is fully invariant in M. $End(M) \simeq \begin{pmatrix} End(M_1) & Hom(M_2, M_1) \\ Hom(M_1, M_2) & End(M_2) \end{pmatrix}$. But M_1 and M_2 are pure in M, so they are fully invariant by hypothesis. Hence $Hom(M_1, M_2) = 0, Hom(M_2, M_1) = 0$ by Lemma 1.1.39(iii). Thus for any $f \in End(M), f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ where $f_1 \in End(M_1)$,$f_2 \in End(M_2)$. So $f(K_1 \oplus K_2) = f_1(K_1) \oplus f_2(K_2) \leq K_1 \oplus K_2$. Thus $K_1 \oplus K_2$ is fully invariant in M. Therefor M is a strongly purely t-semisimple. □

Note that $\mathbb{R}$ as $\mathbb{R}$-module is strongly purely t-semisimple, but $M = \mathbb{R} \oplus \mathbb{R}$ as $\mathbb{R}$-module is not strongly purely t-semisimple by Remarks and Examples 3.4.2(1). Also $\mathbb{R} \oplus (0)$ is pure submodule of M but it is not fully invariant.

Proposition (3.4.10): Let $M = M_1 \oplus M_2$, where M_1 and M_2 are R-modules. $annM_1 + annM_2 = R$. Then M is strongly purely t-semisimple if and only if M_1 and M_2 are strongly purely t-semisimple.

Proof: $\Rightarrow$ It is clear by Remarks and Examples 3.4.2(6).

$\Leftarrow$ Let N be a pure submodule of M. Since $annM_1 + annM_2 = R$, then $N = N_1 \oplus N_2$ for some $N_1 \leq M_1$, $N_2 \leq M_2$ by[1,Proposition 4.2]. Since N is pure in M, we get N_1 is pure in M_1 and N_2 is pure in M_2. As M_1 and M_2 are strongly purely t-semisimple there exist fully invariant direct summands K_1, K_2 of M_1 , M_2 respectively and $K_1 \leq_{tes} N_1$ and $K_2 \leq_{tes} N_2$. It follows $K_1 \oplus K_2$ is a direct summand of M and $K_1 \oplus K_2 \leq_{tes} N_1 \oplus N_2 = N$.But $annM_1 + annM_2 = R$, implies Hom(M_1 , M_2)=0, Hom(M_2, M_1) by Lemma(1.2.8). Thus for any $f \in End(M)$ $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ where $f_1 \in End(M_1)$ and $f_2 \in End(M_2)$ and hence $f(K_1 \oplus K_2) \leq f_1(K_1) \oplus f_2(K_2) \leq K_1 \oplus K_2$, that is $K_1 \oplus K_2$ is a fully invariant submodule of M. Therefore M is strongly purely t-semisimple. □

3.5 Strongly purely t-semisimple and strongly purely t-Baer Modules

In this section we define and study strongly purely t-Baer modules. We present characterization of strongly purely t-Baer module; we have a necessary condition for a module to be strongly purely t-Baer. Also, we give a connection between strongly purely t-semisimple and strongly purely t-Baer.

Definition (3.5.1): An R-module M is called strongly purely t-Baer if $t_M(I)$ is fully invariant pure submodule of M, for each left ideal I of S=End(M).

The following Theorem is a characterization of strongly t-Baer modules

Theorem (3.5.2): An R-module M is strongly purely t-Baer if and only if for each left ideal I of S=End(M), $t_M(I)$ is t-essential in fully invariant pure submodule.

Proof: $\Rightarrow$ it is clear (since $t_M(I) \leq_{tes} t_M(I)$)

$\Leftarrow$ It follows by a similar proof of Theorem (3.3.4). □

Proposition (3.5.3): Let M be an R-module such that $M = Z_2(M) \oplus M'$. If $t_M(I) \cap M'$is pure and fully invariant in M'.Then M is strongly purely t-Baer.

Proof: As $Z_2(M \leq \mathrm{t_M(I)}, \mathrm{t_M(I)} = Z_2(M) \oplus (\mathrm{t_M(I)} \cap M')$, by modular law. Since $Z_2(M)$ is pure in $Z_2(M), \mathrm{t_M(I)} \cap M')$ is pure in M'.Then $\mathrm{t_M(I)}$ is pure in M.To prove $t_M(I)$ is fully invariant in M.

$End(M) \simeq \begin{pmatrix} End\ (Z_2(M)) & Hom(M', Z_2(M)) \\ Hom(Z_2(M), M') & End(M') \end{pmatrix}$.Note that as $Z_2(M)$ is a fully invariant in M implies $Hom(Z_2(M), M') = 0$ by Lemma 1.1.39(iii). Let $f \in End(M)$ then $f = \begin{pmatrix} \mathrm{f_1} & \mathrm{f_2} \\ 0 & \mathrm{f_3} \end{pmatrix}$,where $f_1 \in \mathrm{End(Z_2(}M)), f_2 \in \mathrm{Hom(M', Z_2(M)}$ and $f_3 \in \mathrm{End(M')}$,$f(\mathrm{t_M(I)} = \begin{pmatrix} f_1 & f_2 \\ 0 & f_3 \end{pmatrix} \begin{pmatrix} \mathrm{Z_2(M)} \\ \mathrm{t_M(I) \cap M'} \end{pmatrix} = \begin{pmatrix} f_1\mathrm{(Z_2(M)) + f_2(t_M(I) \cap M')} \\ \mathrm{f_3(t_M(I) \cap M')} \end{pmatrix} \leq$ $\begin{pmatrix} \mathrm{Z_2(M)} \\ \mathrm{t_M(I) \cap M'} \end{pmatrix} = \mathrm{t_M(I)}$. Thus $\mathrm{t_M(I)}$ is fully invariant in M and hence M is strongly purely t-Baer. □

To prove the next result, we state and prove the following Lemma.

Lemma (3.5.4): Let $\mathrm{M = M_1 \oplus M_2}$. Then

(1) If $\mathrm{M_1}$is a fully invariant submodule in M and B is a fully invariant in $\mathrm{M_2}$,then $M_1 \oplus B$ is fully invariant in M.

(2)If A is a fully invariant submodule of $\mathrm{M_1}$ and $\mathrm{M_2}$ is a fully invariant in M, then $A \oplus \mathrm{M_2}$ is a fully invariant submodule of M.

Proof: (1) $\mathrm{End(M)} \simeq \begin{pmatrix} End(M_1) & Hom(M_2, M_1) \\ Hom(M_1, M_2) & End(M_2) \end{pmatrix}$ But $Hom(M_1, M_2) = 0$ by Lemma 1.1.39(iiii).Let $f \in \mathrm{End(M)}$ then $f = \begin{pmatrix} f_1 & f_2 \\ 0 & f_3 \end{pmatrix}$. Where $f_1 \in End(M_1), f_2 \in Hom(M_2, M_1), f_3 \in End(M_2)$. Hence $f(\mathrm{M_1 \oplus B}) = \begin{pmatrix} f_1 & f_2 \\ 0 & f_3 \end{pmatrix} \begin{pmatrix} \mathrm{M_1} \\ \mathrm{B} \end{pmatrix} =$

$\begin{pmatrix} f_1(\mathrm{M}_1)+f_2(\mathrm{B}) \\ f_3(\mathrm{B}) \end{pmatrix}$. But $f_1(\mathrm{M}_1) \leq \mathrm{M}_1, f_2(\mathrm{B}) \leq M_1$ and $f_3(\mathrm{B}) \leq \mathrm{B}$, so that $f(\mathrm{M}_1 \oplus \mathrm{B}) \leq \mathrm{M}_1 \oplus \mathrm{B}$, that is $\mathrm{M}_1 \oplus \mathrm{B}$ is fully invariant.

The proof of (2) is similarly. □

Proposition (3.5.5): Let $\mathrm{M} = \mathrm{Z}_2(\mathrm{M}) \oplus \mathrm{M}'$, where M' is a fully invariant submodule of M, If M' is strongly purely Baer module. Then M is strongly purely t-Baer.

Proof: Since $M = Z_2(M) \oplus M'. Then\ t_M(I) = Z_2(M) \oplus (t_M(I) \cap M') = Z_2(M) \oplus \mathrm{r}_{\mathrm{M}'}(\mathrm{I})$ by Lemma 3.3.6. On the other hand, let $\mathrm{I}' = \{f|_{\mathrm{M}'}: f \in \mathrm{I}\}$. Note $f|_{\mathrm{M}'}: \mathrm{M}' \to \mathrm{M}$, but M' is fully invariant in M (hence stable) so $\mathrm{f}|_{\mathrm{M}'} \in \mathrm{End}(\mathrm{M}')$. We claim that I' is an ideal of $End(M')$. To show this. Let $\mathrm{g}_1, \mathrm{g}_2 \in I'$, then $\mathrm{g}_1 = f_1|_{M'}, \mathrm{g}_2 = f_2|_{M'}$ where $f_1, f_2 \in I$,so $\mathrm{g}_1 - \mathrm{g}_2 = (f_1 - f_2)|_{M'} \in I'$ Let $h \in End(M')$. Then there exists $h_1: M \mapsto M$ defined by

$$h_1(x) = \begin{cases} h(x) & if\ x \in M' \\ 0 & otherwise \end{cases}$$

$h \circ \mathrm{g} = (h_1 \circ f_1)|_{M'}$since $(h \circ \mathrm{g}_1)(m') = h(\mathrm{g}_1(m')) = h(f_1(m') \in M' = (h_1 \circ f_1)(m')$ .Also $h_1 \circ f_1 \in I$ since $f_1 \in I$ and I left ideal of S. We claim that $r_{M'}(I') = r_{M'}(I)$ .Let $\mathrm{m}' \in \mathrm{r}_{\mathrm{M}'}(\mathrm{I})$, Then $\mathrm{Im}' = 0,$ so $\mathrm{f}(\mathrm{m}') = 0$, for each $\mathrm{f} \in \mathrm{I} \leq \mathrm{End}(\mathrm{M})$. Then $f|_{M'}(m') = 0$, but $f|_{M'} \in I'$, then $m' \in r_{M'}(I')$, hence $r_{M'}(I) \leq r_{M'}(I')$.

Conversely Let $m' \in r_{M'}(I')$, then $I'm' = 0$ .So for each $\mathrm{g} \in \mathrm{I}'$, $\mathrm{g}(\mathrm{m}') = 0$, but $\mathrm{g} \in \mathrm{I}'$so there exists $f \in \mathrm{I}$ and $f|_{\mathrm{M}'} = \mathrm{g}$,then $\mathrm{g}(\mathrm{m}') = f(\mathrm{m}') = 0$. hence $\mathrm{m}' \in \mathrm{r}_{\mathrm{M}'}(\mathrm{I})$ and we get $\mathrm{r}_{\mathrm{M}'}(\mathrm{I}') \leq \mathrm{r}_{\mathrm{M}}(\mathrm{I})$. Thus $r_{M'}(I') = r_{M'}(I)$ and hence $t_M(I) = Z_2(M) \oplus r_{M'}(I')$. But $Z_2(M)$ is pure in $Z_2(M)$, $r_{M'}(I')$ is pure in M' so that $t_M(I)$ is pure in M. Since M' is strongly purely Baer module implies $r_{M'}(I')$ is fully invariant in M'. Hence by Lemma 3.5.4(1) $t_M(I)$ is fully invariant in M and so $t_M(I)$ is fully invariant pure in M. Hence M is strongly purely t-Baer . □

Next we have the following Theorem

Theorem (3.5.6): Let M be an R-module such that a complement of $Z_2(M)$ is a pure submodule in M. If M is strongly purely t-semisimple, then M is strongly purely t-Baer.

Proof: By Theorem 3.4.6(1→3→2), $M = Z_2(M) \oplus M'$, where M' is nonsingular, fully invariant submodule, M' is purely fully stable and purely semisimple. By the proof of Proposition (3.5.5), $M = Z_2(M) \oplus r_{M'}(I'), I' \leq End(M')$. So $r_{M'}(I')$ is pure in M. But $r_{M'}(I') \leq M'$, so $r_{M'}(I')$ is pure in M'. But M' is purely fully stable, hence $r_{M'}(I')$ is fully invariant in M', and so M' is strongly purely Baer. Then by Proposition (3.5.5) M is a strongly purely t-Baer module. □

Chapter Four

Certain types of Modules

Related with types of

T-semisimple Modules

Introduction

In this chapter, we investigate certain types of module which have a close connection with the types of t-semisimple modules which are introduced in previous chapters. This chapter consists of six sections.

In section one we give a review about modules that satisfy C_{11}-condition and T_{11}-type modules.

In section two we introduce modules that satisfy strongly C_{11}-condition and strongly T_{11}-type modules. We notice that every module satisfies strongly C_{11}-condition module is strongly T_{11}-type module and strongly T_{11}-type implies T_{11}-type. Examples to show that the converses may be not hold are given. Also, every strongly t-semisimple module is strongly T_{11}-type module. Many other properties for these classes of modules are presented.

In section three, modules that satisfy FI-C_{11}-condition and FI-T_{11}-type modules as generalizations of modules that satisfy C_{11}-condition and T_{11}-type modules are introduced. A module satisfies FI-C_{11}-condition is FI-T_{11}-type module, but the converse may be not true. Beside other results in this section, we have if M is FI-t-semisimple modules such that M satisfies condition $(*)$ then M is FI-T_{11}-type module.

In section four, the concepts modules satisfy strongly FI-C_{11}-condition and strongly FI-T_{11}-type modules are investigated. Many properties related with these concepts and many connections between these concepts and other related concepts such as modules satisfy FI-C_{11}-condition, FI-T_{11}-type modules and strongly T_{11}-type modules. Also, we have if M is a FI-extending and every closed submodule is fully invariant, then M is strongly FI-T_{11}-type module.

In section five we introduce modules that satisfy purely C_{11}-conditions and purely T_{11}-type modules. We study these concepts and their connections with purely t-semisimple modules.

Section six is devoted for presenting and studying the concepts modules satisfy strongly purely C_{11}-condition and strongly purely T_{11}-type modules. Also, we study their connections with strongly purely t-semisimple modules.

4.1 Modules satisfy C_{11}-condition and T_{11}-type Modules

Recall that**:** ″ An R-module M is said to be satisfy C_{11}-condition if every submodule of M has a complement which is a direct summand″ [38]. Asgari [10], restricted C_{11}- condition to t-closed condition of M. She defined the following. ″An R-module M said to be T_{11}-type module (or M satisfy T_{11}-type) if every t-closed submodule has a complement which is a direct summand. A ring is said to be right T_{11}-type ring if R_R is a T_{11}-type module.″[10]

Proposition (4.1.1)[38, Proposition 2.3]:″ The following statements are equivalent for a module M.

(1) M has C_{11}-condition

(2) For any complement submodule L in M, there exists a direct summand K of M such that K is a complement of L in M.

(3) For any submodule N of M, there exists a direct summand K of M such that $N \cap K = O$ and $N \oplus K$ is an essential submodule of M.

(4) For any complement submodule L in M, there exists a direct summand K of M such that $L \cap K = O$ and $L \oplus K$ is an essential submodule of M″.

Theorem (4.1.2)[38]: ″Any direct sum of modules with C_{11}-condition satisfies C_{11}-condition.″

Remarks and Examples (4.1.3):

(1) It is clear that ″every module satisfying C_{11} is T_{11}-type-module″, but the convers is not true [10].For example. The Z-module $\prod_{i=0}^{\infty} Z_i$ does not satisfy C_{11} by [38, Proposition 3.6]. But it is T_{11}-type module, since it is Z_2-torsion.

(2) ″Every t-extending module (hence every extending module) is a T_{11}-type module ″[10].But the convers is not true. For example Let $R = Z[X]$, R_R is uniform, nonsingular. By [36, Theorem 2.4] $R\oplus R$ satisfies C_{11}-condition. Hence $R\oplus R$ is T_{11}-type module. But $R\oplus R$ is not t-semisimple ,because if it is so, then $R\oplus R$ is t-extending, which is a contradiction since by [15,Example 2.4] $R\oplus R$ is not extending, hence not t-extending, since $R\oplus R$ is nonsingular.

(3) ″Every Z_2-torsion is T_{11}-type module and every finitely generated Abelian group is a T_{11}-type module″ [10].

(4) The Z-module Z and Q are not t-semisimple. But Z and Q are indecomposable and nonsingular uniform, so Z and Q are T_{11}-type module by[10, Corollary 2.8].

(5) Any direct sum of uniform modules has C_{11}-condition module by [38, Corollary 2.6], so is T_{11}-type module. In particular each of $Q\oplus Z$, $Z_4\oplus Z_8$, $Z_8\oplus Z_2$ is T_{11}-type module . Also notice that $Q\oplus Z$ is not t-semisimple.

Proposition (4.1.4): Every t-semisimple module is T_{11}-type module.

Proof: By [7, Proposition 2.16], every t-semisimple is t-extending, hence by Remarks and Examples 4.1.3(2), it is T_{11}-type module. □

Theorem (4.1.5)[10,Theorem 2.4]**:** ″ The following statements are equivalent for a module M:

(1) M *is* T_{11}-type;

(2) $M = Z_2(M)\oplus M'$,where M' satisfies C_{11}-condition;

(3) For every submodule A of M, there exists a direct summand D of M such that $A\oplus D \leq_{tes} M$;

(4) For every t-closed submodule C of M, there exists a direct summand D of M such that $C \oplus D \leq_{tes} M$;

(5) For every t-closed submodule C of M, there exists a direct summand D of M such that $C \oplus D \leq_{ess} M''$.

Corollary (4.1.6)[10, Corollary 2.5]: ″ Let M be a nonsingular module. Then M is T_{11}-type if and only if M satisfies C_{11} condition″.

Corollary (4.1.7)[10, Corollary 2.6]:″ A module M satisfies C_{11} condition if and only if M is T_{11}-type and $Z_2(M)$ satisfies C_{11} condition″.

Corollary (4.1.8)[10, Corollary 2.7]:″ Every direct sum of T_{11}-type modules satisfies T_{11} condition″.

Corollary (4.1.9)[10, Corollary 2.8]:″ An indecomposable module M is T_{11}-type if and only if M is either a nonsingular uniform module or a Z_2-torsion module ″.

Corollary (4.1.10)[10, Corollary 2.10]: ″If M is a T_{11}-type module and L is a fully invariant submodule of M, then L and $\frac{M}{L}$ are T_{11}-type modules. ″

Proposition (4.1.11)[10, Proposition 2.11]:″ Every T_{11}-type module M is FI-t-extending.″

4.2 Modules satisfy strongly C_{11}-condition and strongly T_{11}-type modules.

In this section, we extend the notions of modules satisfy C_{11}-condition and T_{11}-type modules into modules satisfy strongly C_{11}-conditions and strongly T_{11}-type modules. We study these concepts and their connections with strongly t-semisimple modules

Definition (4.2.1): An R-module M said to be satisfy strongly C_{11}-condition if every submodule has a complement which is a fully invariant direct summand.

The following Lemmas are needed in the next Proposition.

Lemma (4.2.2)[38, Lemma 2.2]: ″ Let $N \leq M$, let K be a direct summand of M . K is a complement of N if and only if $K \cap N = 0$ and $K \oplus N \leq_{ess} M$″.

The following Lemma is clear.

Lemma (4.2.3): If $N \leq M$ and K is a fully invariant direct summand of M then K is a fully invariant complement of N if and only if $K \cap N = 0$ and $K \oplus N \leq_{ess} M$.

The following Proposition gives characterizations for modules which have strongly C_{11}-condition.

Proposition (4.2.4): The following statements are equivalent for a module M

(1) M satisfies strongly C_{11}-condition;

(2) For any complement submodule L in M, there exists a fully invariant direct summand K of M such that K is a complement of L in M;

(3) For any submodule N of M, there exists a fully invariant direct summand K of M such that $N \cap K = 0$ and $N \oplus K$ is an essential submodule of M;

(4) For any complement submodule L in M, there exists a fully invariant direct summand K of M such that $L \cap K = 0$ and $L \oplus K \leq_{ess} M$.

Proof: (1) ⇒ (2) For any complement submodule L in M. By strongly C_{11}-condition, L has a complement in M which is a fully invariant direct summand K of M.

(3) ⇒ (4) and (2)⇒(4) are obvious.

(1) ⇒ (3) It is clear by Lemma (4.2.3).

(4) ⇒ (1) Let A be any submodule of M. Then there exists a complement so (closed submodule B in M) such that $A \leq_{ess} B$ by[23, Exercise 13,P.20]. By hypothesis, there exists a fully invariant direct summand K of M such that $B \cap K = 0$ and $B \oplus K \leq_{ess} M$. Hence by Lemma (4.2.2) K is a complement of B in M. Now $B \cap K = 0$, which implies $K \cap A = 0$. Suppose that $K' \leq M$ and $K' > K$.

Therefore $K'\cap B \neq 0$ and hence $(K'\cap B)\cap A \neq 0$ (since $A \leq_{ess} B$), so that $K'\cap A \neq 0$. Thus K is a complement of A in M. □

As we have seen t-semisimple is T_{11}-type module. We claim that strongly t-semisimple modules imply modules which are strongly than T_{11}-type module. Hence this leads us to define the following:

Definition (4.2.5): An R-module is said to be strongly T_{11}(or strongly T_{11}-type module) if for each t-closed submodule, there exists a complement which is a fully invariant direct summand.

Remarks (4.2.6):

(1) It is clear that every module, which satisfies strongly C_{11}-condition, is a strongly T_{11}-type module, but the converse is not true in general, as the following example shows:

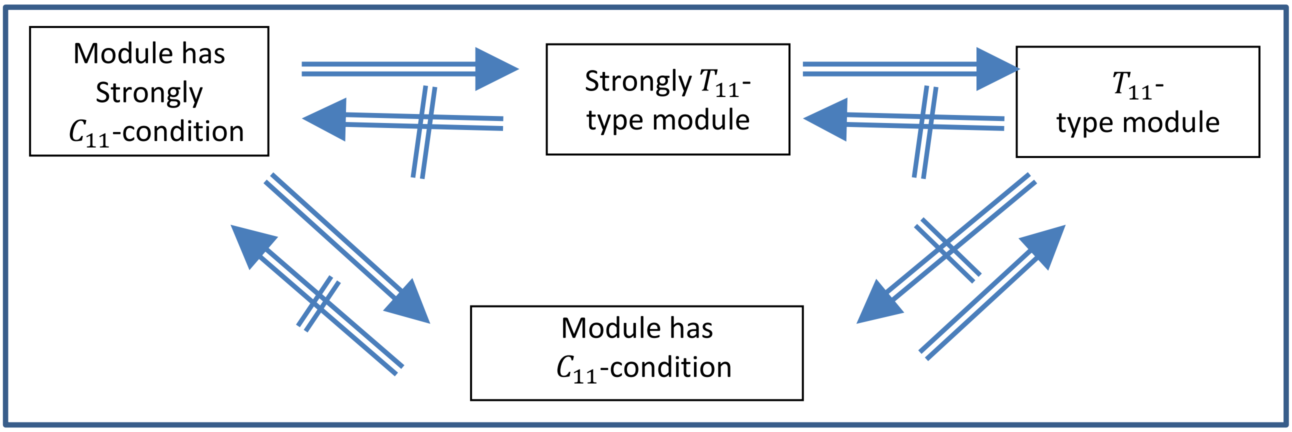
Let $M = Z_8 \oplus Z_2$ as Z-module. M is strongly T_{11}-type module, since M is the only t-closed submodule in M and (0) is a complement of M, which is a fully invariant direct summand. To show that M has not strongly C_{11}-condition. Let $N =< (\bar{2}, \bar{0}) >$. The only submodules of M which have zero intersections with N are: $W = (\bar{0}) \oplus Z_2$ and $K =<(\bar{4}, \bar{1}) >= \{(\bar{4}, \bar{1}), \bar{0}, \bar{0})\}$,$K_1 =< (\bar{0}, \bar{0})>$, $W \leq^{\oplus} M$ and $N \oplus W = (\bar{2}) \oplus Z_2 \leq_{ess} M$, also $K \leq^{\oplus} M$ and $N \oplus K$= $(\bar{4}, \bar{1}), (\bar{0}, \bar{0}), (\bar{6}, \bar{1}), (\bar{2}, \bar{0}), (\bar{0}, \bar{1}), (\bar{4}, \bar{0})$,$(\bar{2}, \bar{1}), (\bar{6}, \bar{0})\}$ $\leq_{ess} M$, $K_1 \leq^{\oplus} M$

But $N \oplus K_1 = N \nleq_{ess} M$. As $W \leq^{\oplus} M$ and $K \leq^{\oplus} M$, then by Lemma 4.2.2, W and K are complement of N. But W is not a fully invariant in M (not stable in M) since there exists$f: W \longmapsto M$ defined by:$f(\bar{0}, \bar{1}) = (\bar{4}, \bar{1}), f(\bar{0}, \bar{0}) = (\bar{0}, \bar{0})$. Hence $f(W) \nleq W$. Also K is not a fully invariant in M (not stable in M) since there exists $g: K \longmapsto M$ defined by $g(\bar{0}, \bar{0}) = (\bar{0}, \bar{0})$, $g(\bar{4}, \bar{1}) = (\bar{0}, \bar{1})$. Hence $g(K) \nleq K$. Thus M does not satisfy strongly C_{11}-condition.

(2) It is clear that every strongly T_{11}-type module is T_{11}-type module. But the converse may be not hold for example: Let $M = Z \oplus Z$ as Z-module. Then M is T_{11}-type module by Corollary 4.1.8. Let $N = Z \oplus (0)$. Then $\frac{M}{N} = \frac{Z \oplus Z}{Z \oplus (0)} \simeq (0) \oplus Z$ which is nonsingular. Hence N is t-closed, but any fully invariant submodule W of M has the form $W = (W \cap Z) \oplus (W \cap Z)$, so that if $W \cap Z \neq 0$, then $W \cap N \neq 0$ and W is not a complement of N. If $W \cap Z = 0$, then $W = 0$ and $W \oplus N = N \not\leq_{ess} M$ and hence W is not a complement of N. Thus N has no fully invariant complement which is a direct summand and hence M is not strongly T_{11}-type module.

(3) The same example in part (2) M satisfies C_{11}-condition by Remarks and Examples 4.1.3(5), but M does not satisfy strongly C_{11}-condition.

We can summarize these relations by the following diagram



Proposition (4.2.7): Let M be a nonsingular R-module. Then M satisfies strongly C_{11}-condition module if and only if M is strongly T_{11}-type module.

Proof:$\Rightarrow$ It is clear.

$\Leftarrow$ Let $A \leq M$. By [23, Exercise 13, P.20], there exists a closed submodule W of M such that $A \leq_{ess} W$. Since M is nonsingular, W is a t-closed of M. Hence there exists a fully invariant direct summand D of W in M such that $W \oplus D \leq_{ess} M$ since M is strongly T_{11}-type module. It follows that $A \oplus D \leq_{ess} W \oplus D \leq_{ess} M$. Thus

$A \oplus D \leq_{ess} M$ such that D is a fully invariant. Thus D is a complement of A by Lemma 4.2.2. So that M satisfies strongly C_{11}-condition. □

Proposition (4.2.8): Let M be a multiplication (hence if M is duo or fully stable). Then

(1) M is T_{11}-type module if and only if M is strongly T_{11}-type module.

(2) M satisfies C_{11}-condition if and only if M satisfies strongly C_{11}-condition module.

We will give some properties of strongly T_{11}-type modules.

Theorem (4.2.9): Consider the following statements for a module M

(1) M is strongly T_{11}-type module;

(2) $M = Z_2(M) \oplus M'$, where M' is a fully invariant submodule in M and satisfies strongly C_{11}-condition;

(3) For every submodule A of M , there exists a fully invariant direct summand D of M such that $A \oplus D \leq_{tes} M$.

(4) For every t-closed submodule C of M , there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{tes} M$.

(5) For every t-closed submodule C of M, there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{ess} M$.

Then (1),(3),(4) and (5) are equivalent, (2)⇒(5) and [(1) ⇒(2) if $\frac{L}{Z_{2(M)}}$ is fully invariant of $\frac{M}{Z_{2(M)}}$ for each fully invariant submodule L of M containg $Z_2(M)$,.

Proof: (1)⇒(5) Let C be a t-closed submodule of M. By condition (1) there exists a complement D to C such that $D \leq^{\oplus} M$, D is fully invariant. Thus $C \oplus D \leq_{ess} M$.

(3)⇒ (1) Let C be a t-closed submodule of M. By hypothesis there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{tes} M$. Let E be a complement of C, then $C \cap E = 0$ and $C \oplus E \leq_{ess} M$. We claim that $C \oplus D \leq_{ess} C \oplus E$. Let $(C \oplus D) \cap X = (0)$, where $X \leq C \oplus E$. $(C \oplus D) \cap X = (0) \leq Z_2(M)$. Thus implies $X \leq Z_2(M)$ since $C \oplus D \leq_{tes} M$. But $Z_2(M) \leq C$ (since C is t-closed) hence $X \leq C$. It follows that $(C \oplus D) \cap X = X = (0)$. Thus $(C \oplus D) \leq_{ess} C \oplus E$. It follows that $D \leq_{ess} E$. However,$D \leq^{\oplus} M$ so D is closed in M, which implies $D = E$,that is E a complement of C, which is a fully invariant direct summand. Thus M is a strongly T_{11}-type module.

(5) $\Rightarrow$(4) The implication is clear since every essential submodule is t-essential submodule.

(4) $\Rightarrow$ (3) Let $A \leq M$. By [10, Lemma 2.3], there exists a t-closed C of A such that $A \leq_{tes} C$. By hypothesis, there exists a fully invariant direct summand D such that $C \oplus D \leq_{tes} M$. But $A \leq_{tes} C$, we conclude that $A \oplus D \leq_{tes} C \oplus D$ and hence $A \oplus D \leq_{tes} M$.

(2) $\Rightarrow$(5) Let C be a t-closed submodule of M. Hence by Lemma (1.1.27), $Z_2(M) \leq C$ and so $C = Z_2(M) \oplus (C \cap M')$. Moreover, $C \cap M'$is a t-closed submodule of $M'$by Proposition 1.1.31(1). Since M' satisfies strongly C_{11} condition, there exists a fully invariant direct summand D of M' such that $(\cap M') \oplus D \leq_{ess} M'$ by Proposition 4.2.4(3). But $D \leq^{\oplus} M'$ and $M' \leq^{\oplus} M$, then $D \leq^{\oplus} M$ and $C \oplus D = [Z_2(M) \oplus (C \cap M')] \oplus D = Z_2(M) \oplus [(C \cap M')) \oplus D] \leq_{ess} Z_2(M) \oplus M' = M$. Hence $C \oplus D \leq_{ess} M$, but D is fully invariant in M' and M' is fully invariant in M. Hence D is fully invariant in M.

(1) $\Rightarrow$(2) Since M is strongly T_{11}-type module and $Z_2(M)$ is a t-closed submodule of M, there exists a complement $M'$to $Z_2(M)$ which is a fully invariant direct summand, say $M = L \oplus M'$. Since M' is nonsingular, we have $Z_2(M) = Z_2(L)$. But $Z_2(M) \oplus M' \leq_{ess} M$ since $M'$is complement to $Z_2(M)$, so by Proposition (1.1.17) $\frac{M}{M'}$ is Z_2-torsion, thus L is Z_2-torsion (since $L \simeq \frac{M}{M'}$). So $L = Z_2(L) = Z_2(M)$ and hence $L = Z_2(M)$. Therefore, $M = Z_2(M) \oplus M'$. Now to show that $M' \simeq \frac{M}{Z_2(M)} \simeq \bar{M}$ satisfies strongly C_{11} condition. Let $\bar{C}$=$\frac{C}{Z_2(M)}$ be a closed submodule of $\bar{M}$ so $\bar{C}$ is t-closed in $\bar{M}$ and hence C is t-closed submodule of M by Lemma 1.1.27(3).But M is a strongly T_{11}-type, so there exists a complement D of C in M which is a fully invariant direct summand of M. Say $M = D \oplus D'$ for some $D' \leq M$. Since $Z_2(M) = Z_2(D) \oplus Z_2(D')$ we get $\bar{M} = \frac{M}{Z_2(M)} = \frac{D \oplus D'}{Z_2(D) \oplus Z_2(D')} \simeq \frac{D}{Z_2(D)} \oplus \frac{D'}{Z_2(D')} = \bar{D} \oplus \overline{D'}$. Cleary $\bar{D} \cap \overline{D'}$=0 and $\bar{C} \oplus \bar{D} \leq_{ess} \bar{M}$. But $D, Z_2(M)$ are fully invariant in M and by hypothesis $\frac{D+Z_2(M)}{Z_2(M)} = \bar{D}$ is fully invariant in $\bar{M}$. □

Remark (4.2.10): If an R-module M is fully stable and semisimple, then M satisfies strongly C_{11}-condition module.

Proof: Let $N \leq M$, then $N \leq^{\oplus} M$, and so there exists $W \leq M$ such that $N \oplus W = M$, hence W is a complement of N. But M is fully stable, so W is a fully invariant, moreover $W \leq^{\oplus} M$. Thus M is strongly C_{11}-condition module. □

In Particular, Z_6 as Z-module satisfies strongly C_{11}-condition.

Proposition (4.2.11): If an R-module M is strongly t-semisimple, then M is a strongly T_{11}-type module.

Proof: By Theorem 1.2.3, $M = Z_2(M) \oplus M'$, where M' is nonsingular semisimple fully stable and M' is stable in M'. But M' is fully stable semisimple then $M'$is strongly C_{11}-condition module by Remark 4.2.10. Hence M satisfies condition (2) which implies condition(1) of Theorem 4.2.9. Thus M is a strongly T_{11}-type module. □

Theorem (4.2.12: Every strongly extending module is strongly T_{11}-type module.

Proof: Let N be a t-closed submodule of M. Hence N is a closed submodule. As M is strongly extending, N is a fully invariant direct summand. Then $M = N \oplus W$ for some $W \leq M$ and so W is a complement of N. To see this let $W' \leq M$ and $W' \geq W$ and $N \cap W' = (0)$, then $M = N \oplus W \subseteq N \oplus W'$, so $M = N \oplus W' = N \oplus W$. Assume $x \in W'$ then $x = n + y, n \in N, y \in W \leq W'$, then $x - y = n \in N \cap W' = 0$, hence $x - y = 0$ implies $x = y \in W$. Hence $W' = W$, moreover $W \leq^{\oplus} M$, so W is closed submodule and hence W is a fully invariant direct summand. Thus M is strongly T_{11}-type module. □

Proposition (4.2.13): If M is a strongly t-extending R-module, then M is strongly T_{11}-type module and every complement to a nonsingular direct summand is fully invariant direct summand.

.**Proof:** Since M is strongly t-extending, then $M = Z_2(M) \oplus M'$, M' is strongly extending module by Theorem 1.3.11. Hence, M' is strongly T_{11}-type module by Proposition (4.2.12). But M' is nonsingular, so M' satisfies strongly C_{11}-condition module by Proposition (4.2.7). Thus M satisfies condition (2) of Theorem 4.2.9 which implies M is a strongly T_{11}-type module. Now let C be a complement of a nonsingular submodule of M, so by Proposition 1.1.28(5⇔2) C is a t-closed submodule of M. Hence C is a fully invariant direct summand of M by definition of strongly t-extending. □

Note that if every complement of nonsingular submodule of an R-module M is fully invariant direct summand then M is strongly t-extending, since by Proposition 1.1.28(5⇔2) every t-closed is a complement of nonsingular submodule and so that every t-closed submodule is fully invariant direct summand. Thus M is strongly t-extending.

Proposition (4.2.14): Let $M = M_1 \oplus M_2$, M_2 is a fully invariant submodule in M. The following conditions are equivalent:

(1) M_1 is a strongly T_{11}-type module;

(2) For every submodule A of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $A \oplus D \leq_{tes} M$.

(3) For every t-closed submodule C of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{tes} M$;

(4) For every t-closed submodule C of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{ess} M$.

Proof: (1) $\Rightarrow$ (2) Since M_1 is strongly T_{11}-type module, then by condition (3) of Theorem 4.2.9 for each $A \leq M_1$, there exists a fully invariant direct summand D_1 of M_1 such that $A \oplus D_1 \leq_{tes} M_1$. But $D_1 \leq^{\oplus} M_1$ implies that $D = D_1 \oplus M_2 \leq^{\oplus} M$. Also, $D = D_1 \oplus M_2$ is fully invariant in M by Lemma 3.5.4(2). Moreover, $A \oplus D_1 \leq_{tes} M_1$ implies that $(A \oplus D_1) \oplus M_2 \leq_{tes} M_1 \oplus M_2 = M$.Thus $A \oplus (D_1 \oplus M_2) \leq_{tes} M$;that is $A \oplus D \leq_{tes} M$.

(2) $\Rightarrow$(3) It is obvious.

(3) $\Rightarrow$(4) For every t-closed submodule C of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{tes} M$. Then $C \oplus D + Z_2(M) \leq_{ess} M$ by Proposition 1.1.17. But $Z_2(M) = Z_2(M_1) \oplus Z_2(M_2)$. As C is t-closed in M_1, $C \supseteq Z_2(M_1)$ by Lemma 1.1.27(1). Also as $M_2 \leq D$,then $Z_2(M_2) \leq Z_2(D) \leq D$. It follows that $C \oplus D + Z_2(M) = C \oplus D + Z_2(M_1) \oplus Z_2(M_2) = C \oplus D \leq_{ess} M$.

(4) $\Rightarrow$(1) Let C be t-closed of M_1. By condition (4) there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{ess} M$. But D is a fully a invariant submodule in M implies, $D = (D \cap M_1) \oplus (D \cap M_2)$, such that $D \cap M_1$ is fully invariant in M_1 by Lemma 1.1.39(ii) and $D \cap M_2 = M_2$ since $M_2 \leq D$, hence $D = (D \cap M_1) \oplus M_2$ and $D \cap M_1 \leq^{\oplus} M_1$.Now $C \oplus D = C \oplus [(D \cap M_1) \oplus M_2] \leq_{ess} M$

$= M_1 \oplus M_2$ Hence $[C \oplus [(D \cap M_1)] \leq_{ess} M_1$. Thus M_1 satisfies condition (5) of Theorem (4.2.9), which implies M_1 is a strongly type -T_{11} module.□

Proposition (4.2.15): If L is a fully invariant direct summand of strongly T_{11}-type module, then

(1) L is a strongly T_{11}-type module.

(2) $\frac{M}{L}$ is strongly T_{11}-type if M is self-projective.

Proof: (1) To prove L is strongly T_{11}-type module. Let A be a submodule of L, hence A is a submodule of M, and so by condition (3) of Theorem 4.2.9, there exists a fully invariant direct summand D of M, such that $A\oplus D \leq_{tes} M$. Hence $(A\oplus D) \cap L \leq_{tes} L$ and so $A\oplus(D \cap L) \leq_{tes} L$. On the other hand, $D \leq^{\oplus} M$ implies $M = D\oplus D'$ for some $D' \leq M$. As L is a fully invariant submodule in M, $L = (D \cap L)\oplus(D' \cap L)$, where $D \cap L$ is fully invariant in $D, D'\cap L$ is fully invariant submodule in D' by Lemma 1.1.39(ii). Now $D \cap L \leq^{\oplus} L$ and $L \leq^{\oplus} M$, imply $D\cap L \leq^{\oplus} M$. But $D\cap L$ is fully invariant in M since D and L are fully invariant in M. Hence by Lemma 1.2.6, $D\cap L$ is fully invariant in L. Thus L is strongly T_{11}-type.

(2)Let $\frac{C}{L}$ be a t-closed submodule in $\frac{M}{L}$. Then C is a t-closed in M. As M is strongly T_{11}-type module there exists a fully invariant direct summand D of M such that $C\oplus D \leq_{ess} M$ by Theorem 4.2.9. Let $M = D\oplus D'$ for some $D' \leq M$ and since L is fully invariant in M, $L = (D\cap L)\oplus(D'\cap L)$ such that $D\cap L$ is fully invariant in D, $D'\cap L$ is fully invariant in D'. Then $\frac{M}{L} = \frac{D\oplus D'}{(D\cap L)\oplus(D'\cap L)} \simeq \frac{D}{D\cap L}\oplus\frac{D'}{D'\cap L} \simeq \frac{D+L}{L}\oplus\frac{D'+L}{L}$. But it is easy to see that $\frac{C}{L}\oplus\frac{D+L}{L} \leq_{ess} \frac{M}{L}$. As $L \leq^{\oplus} M$, L is closed and this implies that $\frac{C\oplus D}{L} \leq_{ess} \frac{M}{L}$ by [23, Proposition 1.4, P.18]. Thus $\frac{C}{L}\oplus\frac{D+L}{L} \leq_{ess} \frac{M}{L}$. On the other hand, since D is a fully invariant submodule in M and, L is fully invariant in M, then $D\oplus L$ is fully invariant in M. Hence $\frac{D+L}{L}$ is fully invariant in $\frac{M}{L}$, by Lemma 1.1.41(2). Thus $\frac{D+L}{L}$ is a fully invariant direct summand of $\frac{M}{L}$ and $\frac{C}{L}\oplus \frac{D+L}{L} \leq_{ess} \frac{M}{L}$. Therefore $\frac{M}{L}$ is strongly T_{11}-type module by Theorem 4.2.9(1⇔3). □

Corollary (4.2.16): If R is a commutative strongly T_{11}-type module and $L \leq^{\oplus} R$, then $\frac{R}{L}$ is a strongly T_{11}-type module.

Corollary (4.2.17): Let M be a multiplication strongly T_{11}-type module and $L \leq^{\oplus} M$. Then $\frac{M}{L}$ is strongly T_{11}-type module, provided M is self-projective.

4.3. Modules satisfy FI-C_{11}-condition and FI-T_{11}-type Modules

In this section we generalize the concepts of modules satisfy C_{11}-condition and T_{11}-type modules by restricted the C_{11}-condition on fully invariant submodule and the condition of T_{11}-type modules to fully invariant t-closed. We give some properties of these concepts. Also we study their relationships between them and with FI-t-semisimple modules.

Definition (4.3.1): An R-module M is said to be satisfies FI-C_{11}-condition if every fully invariant submodule of M has a complement which is a direct summand.

Remarks and Examples (4.3.2):

(1) It is clear that every module satisfies C_{11}-condition also satisfies FI-C_{11}-condition, but the converse is not true in general, for example. Let $M = \prod^{\infty} Z$ as Z-module, M is not C_{11}-condition [38,Lemma 3.4]. But M has only two fully invariant submodules namely, M and (0). So that M satisfies FI-C_{11}-condition.

(2) Let M be a multiplication (or duo) module. Then M satisfies C_{11}-condition if and only if M satisfies FI-C_{11}-condition.

In particular Every submodule of Z_{12} as Z-module is fully invariant and every submodule of Z_{12} has a complement which is a direct summand. Thus Z_{12} satisfies C_{11} condition and so satisfies FI-C_{11}-condition.

(3) Every uniform module satisfies FI-C_{11}-condition. In particular, each of the Z-module Q, Z,Z_{P^∞} satisfies FI-C_{11}-condition.

(4) $M = Z_8 \oplus Z_2$ satisfies C_{11}-condition so it satisfies FI-C_{11}-condition.

Proposition (4.3.3): Consider the following statements for an R-module M.

(1) M has FI-C_{11} -condition

(2) For any fully invariant complement submodule L in M, there exist a direct summand K of M such that K is a complement of L in M.

(3) For any fully invariant submodule N of M, there exists a direct summand K of M such that $N \cap K = (0)$ and $N \oplus K \leq_{ess} M$.

(4) For any fully invariant complement submodule L in M, there exists a direct summand K of M such that $L \cap K = (0)$ and $L \oplus K \leq_{ess} M$.

Then (1)⇒(2)⇔(4), (1)⇔(3)⇒(4), (4)⇒(1) if every fully invariant submodule has a fully invariant closure.

Proof: (1) ⇒(2) , (3)⇒(4) are obvious.

(1)⇔(3) ,(2)⇔(4) clear by Lemma 4.2.2.

(4)⇒(1) (if every fully invariant submodule has a fully invariant closure).

Let A be a fully invariant submodule of M. Then there exists a fully invariant closed submodule B in M such that $A \leq_{ess} B$. By hypothesis, there exists a direct summand K of M such that $B \cap K = (0)$ and $B \oplus K \leq_{ess} M$.Hence by Lemma 4.2.2 K is a complement of B in M. We claim that K is a complement of A. Assume U is a submodule of M contain K. Then $U \cap B \neq 0$ and so $U \cap B \cap A \neq 0$, since $A \leq_{ess} B$, that is $U \cap A \neq 0$. Thus K is complement A. □

Definition (4.3.4): An R-module M is called a FI-T_{11}-type module if every fully invariant t-closed submodule has a complement which is a direct summand.

Remark (4.3.5):

(1) It is clear every module satisfies FI-C_{11}-condition implies FI-T_{11}-type module, but the converse is not true for examples.

(i) If $R = Q[u, v]$ with the relation $u^2 = v^2 = uv = 0$. Then R_R is T_{11}-type and R_R does not satisfy C_{11}-condition [10, Example 2.2] and hence it is FI-T_{11}-type. As R is duo, so R_R does not satisfy FI-C_{11}-condition.

(ii) Let $M = M_1 \oplus M_2$ be a singular R-module. If M_2 is a fully invariant in M and M_1 doesn't satisfy FI-C_{11}-condition, then M doesn't satisfy FI-C_{11}-condition.

Proof: As M is singular, M is FI-T_{11}-type. Since M_1 doesn't satisfy FI-C_{11}-condition, there exists a fully invariant submodule N_1 of M_1 such that N_1 has no complement which is a direct summand of M_1. Assume M has FI-C_{11}-condition let $N = N_1 \oplus M_2$ then N is a fully invariant in M by Lemma 3.5.4(2), hence N has a complement W which is a direct summand. Thus $W \cap (N_1 \oplus M_2) = 0$ and so $W \cap N_1 = 0$, $W \cap M_2 = (0)$, hence $W \leq M_1$. It follows $W \leq^{\oplus} M_1$. Moreover $W \oplus (N_1 \oplus M_2) \leq_{ess} M$ implies $(W \oplus N_1) \oplus M_2 \leq_{ess} M$ and so $W \oplus N_1 \leq_{ess} M_1$. Thus W is a complement of N_1, by Lemma(4.2.2) and hence M_1 has FI-C_{11}-condition which is a contradiction.

(2) Every singular module is T_{11}-type, so it is FI-T_{11}type.

Recall that C is a t-closure, let M be a module. Then every submodule N of M is contained in a t-closed submodule C of M, where $N \leq_{tes} C$, we call C is a t-closure[10].

Theorem (4.3.6): Consider the following statements for an R-module M.

(1) M is FI-T_{11}-type.

(2) $M = Z_2(M) \oplus M'$ where M' satisfies FI-C_{11}-condition.

(3) For every fully invariant submodule C of M there exists a direct summand D of M such that $C \oplus D \leq_{tes} M$.

(4) For every fully invariant t-closed submodule C of M, there exists $D \leq^{\oplus} M$ such that $C \oplus D \leq_{tes} M$.

(5) For every fully invariant t-closed submodule C of M , there exists $D \leq^{\oplus} M$ such that $C \oplus D \leq_{ess} M$.

Then (3) ⇒(1) ⇔(2) ⇔ (5) ⇔(4).[(4) ⇒(3) (if every fully invariant submodule has fully invariant t-closure)]. Thus (1),(2),(3),(4) and (5) are equivalent if every fully invariant submodule has a fully invariant t-closure.

Proof: (1) ⇒(2) As $Z_2(M)$ is fully invariant t-closed, there exists M' a complement of $Z_2(M)$ which is a direct summand of M, say $M = M' \oplus L$. Since M' is nonsingular, $Z_2(M) = Z_2(L) \leq L$. But $M' \oplus Z_2(M) \leq_{ess} M$, so $M' \leq_{tes} M$ by Proposition (1.1.17). Thus $\frac{M}{M'} \simeq L$ is Z_2-torsion and so $Z_2(L) = L$, hence $Z_2(M) = L$. Thus $M = Z_2(M) \oplus M'$. Now to show that $\bar{M} \simeq M'$ satisfies FI-C_{11}-condition where

$\bar{M}=\frac{M}{Z_2(M)}$. Let $\bar{C}$ be a fully invariant closed submodule of $\bar{M}$. Let $\bar{C}=\frac{C}{Z_2(M)}$ and as $\bar{M}$ is nonsingular, $\bar{C}$ is t-closed, hence C is t-closed in M by Lemma 1.1.27(3). But $\frac{C}{Z_2(M)}$ is fully invariant in $\bar{M}$ and $Z_2(M)$ is fully invariant in M, hence C is fully invariant in M by Lemma 1.1.40. Thus C is fully invariant t-closed submodule of M. But M is FI-T_{11}-type module, there exists a complement D to C in M, which is a direct summand of M, say $M=D\oplus D'$. Since $Z_2(M)=Z_2(D)\oplus Z_2(D')$, we conclude that $\bar{M}=\frac{M}{Z_2(M)}=\frac{D}{Z_2(D)}\oplus\frac{D'}{Z_2(D')}=\bar{D}\oplus\overline{D'}$. It is clear that $\bar{D}\cap\bar{C}=(0)$. Also $C\oplus D\leq_{ess} M$ implies that $C\oplus D\leq_{tes} M$ and so by Proposition 1.1.17 $\bar{C}\oplus\bar{D}\leq_{ess} M$. Thus $\bar{M}$ satisfies FI-C_{11} –condition by Proposition 4.3.3(4).

(2)⇒(5) Let C be a fully invariant t-closed submodule of M. Hence $Z_2(M)\leq C$ and so $C=(Z_2(M)\oplus M')\cap C=Z_2(M)\oplus(C\cap M')$. Hence $C\cap M'$ is a t-closed submodule of C and by Corollary 1.1.30(2) $C\cap M'$ is closed submodule of M'. But $C\cap M'$ is fully invariant in M', to see this. Let $f: M'\longmapsto M'$, define $g: M\longmapsto M$ by $g(x)=\begin{cases} f(x) & x\in M' \\ 0 & otherwise\end{cases}$ .Now for all $a\in C\cap M', a\in C \; and \; a\in M'$, hence $g(a)=f(a)$ but $g(a)\in C$ since C is fully invariant in M, then $f(a)\in C$. Also, $a\in M'$ then $f(a)\in M'$. Thus $f(a)\in C\cap M'$. By Proposition 4.3.3(4), there exists a direct summand D of M' such that $(C\cap M')\oplus D\leq_{ess} M'$. Hence D is a direct summand of M, and $C\oplus D=Z_2(M)\oplus(C\cap M')\oplus D\leq_{ess} Z_2(M)\oplus M'=M$.

(5)⇒(4) It follows directly since every essential submodule is t-essential submodule.

(4)⇒(3) Let A be a fully invariant submodule of M. By hypothesis, A has a fully invariant t-closure say B. Hence by condition (4), there exists a direct summand D of M such that $B\oplus D\leq_{tes} M$. Since $A\leq_{tes} B$, then $A\oplus D\leq_{tes} B\oplus D$ by Proposition 1.1.22(2). Hence $A\oplus D\leq_{tes} M$ by Proposition 1.1.20(1).

(3) ⇒(1) Let C be a fully invariant t-closed submodule of M. By hypothesis there exists $D \leq^{\oplus} M$ such that $C \oplus D \leq_{tes} M$. Since $C \oplus D \leq_{tes} M$ then $C \oplus D \oplus Z_2(M) \leq_{ess} M$. But $Z_2(M) \leq C$, so $C \oplus D \leq_{ess} M$. As $D \leq^{\oplus} M$ and $C \oplus D \leq_{ess} M$. Thus D is a complement of C which is a direct summand. □

Recall that t-extending module implies T_{11}-type module, we claim that M is FI-t-extending implies M is FI-T_{11}-type module. So we have.

Proposition (4.3.7): If M is a FI-t-extending, then M is a FI-T_{11}-type module.

Proof: Let N be a fully invariant t-closed submodule of M. Since M is a FI-t-extending, N is a direct summand of M; that is $N \oplus W = M$ for some $W \leq M$. Hence W is a complement of N and $W \leq^{\oplus} M$. Thus M is FI-T_{11}-type module. □

Example (4.3.8): Let $R = \begin{pmatrix} Z & Z \\ 0 & Z \end{pmatrix}$ and M be an arbitrary R-module. Then $R \oplus Z_2(M)$ is FI-t-extending R-module which is not t-extending [9, Example 2.10].Hence $R \oplus Z_2(M)$ is FI- T_{11}-type module.

Proposition (4.3.9): Let M be a multiplication (hence if M is cyclic) over a commutative ring R-module. Then M is FI-T_{11}-type module if and only if M is t-extending module.

Proof: Since M is a multiplication (or M is a cyclic) R -module, M is duo. Hence M is FI-T_{11}-type if and only if M is T_{11}-type module. Then the result follows by [10, Proposition 2.14]. □

Recall that every t-semisimple module implies is t-extending which implies T_{11}-type module. However every FI-t-semisimple module is FI-t-extending if condition $(*)$ hold by Proposition 2.2.6, where condition $(*)$: For an R-module, a complement of $Z_2(M)$ is stable in M. The following is an analogous result.

Corollary (4.3.10): If M is FI-t-semismple module and satisfies condition$(*)$, then M is FI-T_{11}-type module.

Proof: by Proposition 2.2.6 M is FI-t-extending. Hence the result follows directly by Proposition (4.3.7). □

Theorem (4.3.11): Let M_1 and M_2 be two R-modules that satisfy FI-C_{11}-condition. Then $M = M_1 \oplus M_2$ satisfies FI-C_{11}-condition.

Proof: Let N be a fully invariant submodule of M. Then $N = N_1 \oplus N_2$ where N_1 and N_2 are fully invariant in M_1, M_2 respectively by Lemma 1.1.39(ii). As M_1 and M_2 satisfy FI-C_{11}-condition, there exist $W_1 \leq M_1, W_2 \leq M_2$ such that W_1 a complement of N_1 and it is a direct summand of M_1, W_2 is a complement of N_2 and it is a direct summand of M_2. As $W_1 \leq^{\oplus} M_1, W_2 \leq^{\oplus} M_2$.Then $W_1 \oplus W_2 \leq^{\oplus} M$. Moreover $(W_1 \oplus W_2) \cap (N_1 \oplus N_2) = (0)$ and $(W_1 \oplus W_2) \oplus (N_1 \oplus N_2) = (W_1 \oplus N_1) \oplus (W_2 \oplus N_2)$, but $W_1 \oplus N_1 \leq_{ess} M_1$ and $W_2 \oplus N_2 \leq_{ess} M_2$,so that $(W_1 \oplus W_2) \oplus (N_1 \oplus N_2) \leq_{ess} M$. Then by Lemma 4.2.2 ,$W_1 \oplus W_2$ is a complement of $N_1 \oplus N_2$. Thus M satisfies FI-C_{11}-condition. □

Theorem (4.3.12): Let M_1 and M_2 be FI-T_{11}-type modules. Then $M = M_1 \oplus M_2$ is FI-T_{11}-type module.

Proof: Let N be a fully invariant t-closed submodule of M. As N is a fully invariant in M.Then $N = N_1 \oplus N_2$ where N_1 is a fully invariant in M_1 and N_2 is a fully invariant in M_2 by Lemma 1.1.39. As N is t-closed in M, then N_1 is t-closed in M_1 and N_2 is t-closed in M_2. But M_1 and M_2 are FI-T_{11}-type modules, so there exist $W_1 \leq^{\oplus} M_1, W_2 \leq^{\oplus} M_2$ with W_1 is a complement of N_1 and W_2 is a complement of N_2. But $W_1 \leq^{\oplus} M_1$ and $W_2 \leq^{\oplus} M_2$ imply $W \leq^{\oplus} M$. Also, $W_1 \cap N_1 = (0)$, $W_2 \cap N_2 = (0)$ imply $W \cap N = (0)$ and since $W_1 \oplus N_1 \leq_{ess} M_1$ and $W_2 \oplus N_2 \leq_{ess} M_2$, we conclude that $(W_1 \oplus W_2) \oplus (N_1 \oplus N_2) = (W_1 \oplus N_1) \oplus W_2 \oplus N_2 \leq_{ess} M$. Thus W is a complement of N by Lemma 4.2.2 which is a direct summand. Thus M is a FI-T_{11}-type module. □

4.4 Modules satisfy strongly FI-C_{11} condition and strongly FI-T_{11}-type Modules

In this section, we extend the concept of module that satisfy FI- C_{11}-condition and FI-T_{11}-type modules into modules with strongly FI-C_{11} conditions and strongly FI-T_{11}-type modules. We establish many properties related with these concepts. Also, a relationship between strongly FI-T_{11}-type modules and FI-extending is given.

Definition (4.4.1): An R-module M has (or satisfies) strongly FI-C_{11}-condition module if for each fully invariant submodule N there exists a fully invariant direct summand W which is a complement of N.

Theorem (4.4.2): Consider the following statements for a module M:

(1) M satisfies strongly FI-C_{11}-condition module;

(2) For any fully invariant complement submodule L of M, there exists a fully invariant direct summand K of M such that K is a complement of L in M;

(3) For any fully invariant submodule N of M, there exists a fully invariant direct summand K of M such that $N \cap K = (0)$ and $N \oplus K \leq_{ess} M$;

(4) For any fully invariant complement submodule L in M, there exists a fully invariant direct summand K of M such that $L \cap K = (0)$ and $L \oplus K \leq_{ess} M$.

Then (1)⇔(3)⇒(4), (1)⇒(2) ⇒(4).((4)⇒(1) if every fully invariant submodule has a fully invariant closure).

Proof: (1)⇒(2) and (3) ⇒(4) are clear .

(1) ⇔(3) and (2) ⇒(4) are clear by Lemma 4.2.2.

(4)⇒(1) Let A be a fully invariant submodule , there exists B fully invariant closed submodule such that $A \leq_{ess} B$. By condition (4) there exists a fully invariant direct summand C such that $B \oplus C \leq_{ess} M$. But $A \leq_{ess} B$, then $A \oplus C \leq_{ess} B \oplus C$. Thus $A \oplus C \leq_{ess} M$ and by Lemma 4.2.3, C is a fully invariant complement. Therefore M has strongly FI-C_{11}-condition. □

Remarks (4.4.3):

(1) Every module satisfies strongly FI-C_{11}-condition is a module satisfying FI-C_{11}-condition. But not conversely as shown by the following example.

Let $M = M_1 \oplus M_2$ be a singular R-module where M_1 and M_2 satisfy FI- C_{11}-condition. If M_2 is a fully invariant submodule of M, then M doesn't satisfy strongly FI-C_{11}-condition.

Proof: M satisfies FI-C_{11}-condition by Theorem 4.3.11. Let N be a fully invariant submodule of M, then $N = (N \cap M_1) \oplus (N \cap M_2)$, where $N \cap M_1$ is a fully invariant in M_1 and $N \cap M_2$ is a fully invariant in M_2. Set $K = (N \cap M_1) \oplus M_2$. Then by Lemma 3.5.4, K is a fully invariant submodule of M. Assume that K has a complement K' such that K' is a fully invariant direct summand. Hence $K' = (K \cap M_1) \oplus (K \cap M_2)$ where $K \cap M_1$ is fully invariant in M_1 and $K \cap M_2$ is a fully invariant in M_2 by Lemma 1.1.39 . Since $K \cap K' = [(N \cap M_1) \cap (K \cap M_1)] \oplus [(M_2 \cap (K \cap M_2)] = (0)$. Hence $K \cap M_2 = (0)$, so $K' = (K \cap M_1) \oplus (0)$. Beside this $End(M) \simeq \begin{pmatrix} End(M_1) & 0 \\ Hom(M_1, M_2) & End(M_2) \end{pmatrix}$.

Let $f \in End(M)$, then $f = \begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}, f_1 \in End(M_1), f_2 \in Hom(M_1, M_2), f_3 \in End(M_2)$ $f(K') = f\begin{pmatrix} K \cap M_1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}\begin{pmatrix} K \cap M_1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1(K \cap M_1) \\ f_2(K \cap M_1) + f_3(0) \end{pmatrix} \leq \begin{pmatrix} K \cap M_1 \\ M_2 + 0 \end{pmatrix} = \begin{pmatrix} K \cap M_1 \\ M_2 \end{pmatrix} \nleq K'$, which is a contradiction. Thus K has no complement which is a fully invariant direct summand and so M doesn't satisfies strongly FI-C_{11}-condition.

(2)Every module satisfies strongly C_{11}-condition implies module satisfies strongly FI-C_{11}-condition. But not conversely for example:

Let $M = Z_8 \oplus Z_2$ as Zmodule. The only fully invariant submodules of M are $N_1 = (\bar{2}) \oplus Z_2$, $N_2 = (\bar{4}) \oplus Z_2$, $N_3 =< (\bar{0}, \bar{0})$, $N_4 = M$, $N_5 = Z_8 \oplus (\bar{0}), N_6 =$

$(\bar{0})\oplus Z_2$, $N_7 = (\bar{2})\oplus(\bar{0})$, $N_8 = (\bar{4}\oplus(\bar{0})$. But, N_1, N_2, and N_4 are essential in M, so that $< (\bar{0}, \bar{0})$ is a complement of each N_1, N_2 and N_4 and so it is fully invariant direct summand. Also, N_4 is a complement of N_3. But N_5 is a complement of N_6 and N_6 is a complement of N_5 and each of them are fully invariant direct summand. N_6 is a complement of N_7 which is a fully invariant direct summand also N_6 is a complement of N_8 which is a fully invariant direct summand. Thus M satisfies strongly FI-C_{11}-condition. But it doesn't satisfy strongly C_{11}-condition by Remarks 4.2.6(1).

By restricting the definition of modules have strongly FI-C_{11}-condition to fully invariant t-closed submodule, we introduce the following.

Definition (4.4.4): An R-module M is to be strongly FI-T_{11}-type module if for each fully invariant t-closed submodule N of M, there is a complement of N which is fully invariant direct summand.

Remarks (4.4.5):

(1) Every module that satisfies strongly FI-C_{11}- condition implies strongly FI-T_{11}-type module. But the converse is not true for example:

Let $M = M_1 \oplus M_2$ be a singular R-module such that M_1 does not satisfies strongly FI- C_{11}- condition and M_2 is a fully invariant submodule of M is strongly FI-T_{11}-type and does not satisfy strongly FI-C_{11}-condition.

Proof: M is strongly FI-T_{11}-type since it is singular. As M_1 does not satisfy strongly FI-C_{11}-condition, there exists a fully invariant submodule N_1 of M_1 such that N_1 has no complemented which is a fully invariant direct summand. Assume M satisfy strongly FI-C_{11}-condition. Let $N = N_1 \oplus M_2$. Then by Lemma 3.5.4, N is a fully invariant submodule of M and hence N has a complement W such that W is a fully invariant direct summand of M. As W is a fully invariant submodule of M, $W = W_1 \oplus W_2$, where W_1 is a fully invariant in M_1 and W_2 is a fully invariant in M_2 by Lemma 1.1.39(ii). It follow that $W \cap (N_1 \oplus M_2) = (0)$ and so $W_1 \cap N_1 = 0$ and

$W_2 \cap M_2 = 0 = W_2$.As $W \oplus (N_1 \oplus M_2) \leq_{ess} M$, then $W_1 \oplus N_1 \leq_{ess} M_1$. Also, as $W = W_1 \oplus (0) \leq^{\oplus} M$, then $W_1 \leq^{\oplus} M_1$. Thus W_1 is a complement of N_1 by Lemma 4.2.3 which is a fully invariant direct summand of M_1. But this is a contradiction since M_1 hasn't strongly FI-C_{11}-condition.

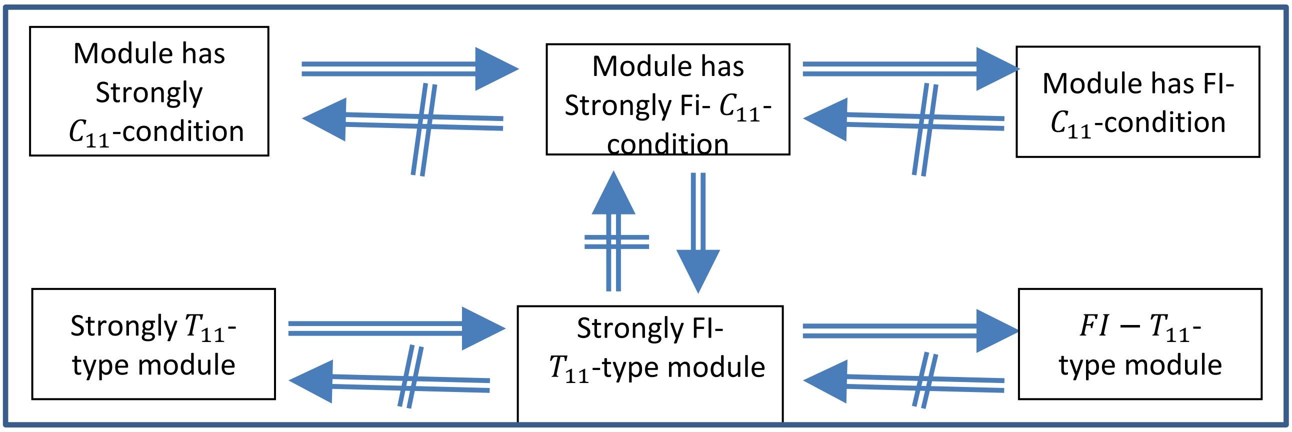
(2) Every strongly FI-T_{11}-type module is FI-T_{11}-type. But not conversely for example:

Let $M = M_1 \oplus M_2$ where M_1 and M_2 are FI-T_{11}-type modules and M_2 is a fully invariant submodule of M. Then M_1 is not strongly FI-T_{11}-type module.

Proof: Since M_1 and M_2 are FI-T_{11}-type module, M is a FI-T_{11}-type module by Theorem 4.3.12. Let N be a fully invariant t-closed submodule of M. As N is fully invariant in M, then $N = (N \cap M_1) \oplus (N \cap M_2)$ where $N \cap M_1$ is fully invariant in M_1 , $N \cap M_2$ is fully invariant in M_2 by Lemma 1.1.39(ii). Also, since N is t-closed in $M, N \cap M_1$ is t-closed in M_1 and $N \cap M_2$ is t-closed in M_2 . Set $K = (N \cap M_1) \oplus M_2$. Then by Lemma 3.5.4, K is a fully invariant in M and $\frac{M}{K} \simeq \frac{M_1 \oplus M_2}{(N \cap M_1) \oplus M_2} \simeq \frac{M_1}{N \cap M_1}$ which is a nonsingular so by Proposition 1.1.28, K is a t-closed in M. Assume K has a complement say W in M such that W is a fully invariant direct summand of M. It follows that $W = (W \cap M_1) \oplus (W \cap M_2)$. Since $K \cap W = (0)$ we conclude that $W \cap M_2 = 0$ and so $W = (W \cap M_1) \oplus (0)$. But $End(M) \simeq \begin{pmatrix} End(M_1) & 0 \\ Hom(M_1, M_2) & End(M_2) \end{pmatrix}$. Let $f \in EndM$, then $= \begin{pmatrix} f_1 & 0 \\ f_2 & f_3 \end{pmatrix}$, $f_1 \in End(M_1), f_2 \in Hom(M_1, M_2), f_3 \in End(M_2). f(W) = f\begin{pmatrix} W \cap M_1 \\ 0 \end{pmatrix} =$ $\begin{pmatrix} f_1(W \cap M_1) \\ f_2(W \cap M_1) \end{pmatrix} \leq \begin{pmatrix} W \cap M_1 \\ M_2 \end{pmatrix} \nleq W$, hence W is not fully invariant in M. Which, is a contradiction. Thus K has no complement which is a fully invariant direct summand and so M doesn't satisfies strongly FI- T_{11}-type.

(3) Clearly every strongly T_{11}-type is strongly FI-T_{11}-type. But not conversely for example: Let $M = Z \oplus Z$ as a Z-module. M is not strongly T_{11}-type module by Remarks 4.2.6(2). If N is a fully invariant t-closed of M. As N is a fully invariant in M, $N = (N \cap Z) \oplus (N \cap Z)$ by Lemma 1.1.39(ii). Also, since N is t-closed then $N \cap Z$ is a t-closed in Z, but Z has only t-closed namely $Z, 0$. Thus either $N \cap Z$=0 or $N \cap Z = Z$ and so $N = (0)$ or $N = M$. If $N = (0)$, then M is a complement of N which is a fully invariant direct summand. If $N = M$, then (0) is a complement of N which is a fully invariant direct summand. Thus M is strongly FI-T_{11}-type module.

We can summarize these remarks by the following diagram



Theorem (4.4.6): Consider the following statements for an R-module M.

(1) M is a strongly FI-T_{11}-type module;

(2) $M = Z_2(M) \oplus M'$, where M' is a fully invariant in M and satisfies strongly FI-C_{11}-condition;

(3) For every fully invariant submodule A of M, there exists a fully invariant direct summand D of M such that $A \oplus D \leq_{tes} M$;

(4) For every fully invariant t-closed submodule C of M, there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{tes} M$;

(5) For every fully invariant t-closed submodule C of M, there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{ess} M$.

Then (2) $\Rightarrow$(5) $\Rightarrow$(4), (3) $\Rightarrow$(1)$\Leftrightarrow$(5) and [(4) $\Rightarrow$(3) if every fully invariant submodule has a fully invariant t-closure] that is (1),(3),(4) and (5) are equivalent if every fully invariant submodule has a fully invariant t-closure.

(1)$\Rightarrow$(2) if $\frac{L}{Z_2(M)}$ is fully invariant submodule in M for each fully invariant submodule L of M, $L \supseteq Z_2(M)$ and every fully invariant submodule has a fully invariant t-closure.

Proof: (1)$\Rightarrow$(2) Since $Z_2(M)$ is a fully invariant t-closed, there exists a fully invariant direct summand M' of M such that M' is a complement of $Z_2(M)$ in M, say $M = L \oplus M'$ for some $L \leq M$. Since M' is nonsingular, so $Z_2(M) = Z_2(L) \leq L$. But $Z_2(M) \oplus M' \leq_{ess} M$, hence $M' \leq_{tes} M$ and so $\frac{M}{M'}$ is Z_2-torsion. Thus L is a Z_2-torsion which implies that $Z_2(L) = L$ and $Z_2(M) = L$. Thus $M = Z_2(M) \oplus M'$. Now to prove $M' \simeq \frac{M}{Z_2(M)}$ satisfies strongly FI-C_{11}-condition. Let $\bar{C} = \frac{C}{Z_2(M)}$ be a fully invariant t-closed and so C is a t-closed in M and as $\frac{C}{Z_2(M)}$ is fully invariant and $Z_2(M)$ is fully invariant, we have C is a fully invariant t-closed in M. But M is strongly FI-T_{11}-type, there exists a fully invariant direct summand D of M, which is a complement of C say $M = D \oplus D'$ for some $D' \leq M$. since $Z_2(M) = Z_2(D) \oplus Z_2(D')$, we get $\bar{M} = \frac{M}{Z_2(M)} = \frac{D \oplus D'}{Z_2(D) \oplus Z_2(D')} \simeq \frac{D}{Z_2(D)} \oplus \frac{D'}{Z_2(D')} = \overline{D} \oplus \overline{D'}$. It is clear that $\overline{D} \cap \overline{D'} = \bar{0}$ and $\bar{C} \oplus \overline{D'} \leq_{ess} \bar{M}$. But $D, Z_2(M)$ are fully invariant in N, so $D + Z_2(M)$ is fully invariant in M and by hypothesis $\frac{D+Z_2(M)}{Z_2(M)} \simeq \overline{D}$ is fully invariant in $\bar{M}$. Thus $\bar{M}$ is strongly FI-C_{11}condition by Theorem 4.2.2((4)$\Rightarrow$(1)).

(2)$\Rightarrow$(5) Let C be a fully invariant t-closed submodule of M. Hence $Z_2(M) \leq C$. As $M = Z_2(M) \oplus M'$, then $C = Z_2(M \oplus (C \cap M')$ and $C \cap M'$ is a fully invariant submodule of M, since C and M' are fully invariant in M. But $C \cap M' \leq M' \leq^{\oplus} M$, hence by Lemma 1.1.40(2), $C \cap M'$ is a fully invariant in M'. As M' has strongly

C_{11}-condition, there exists a fully invariant direct summand D of M' such that $(C \cap M') \oplus D \leq_{ess} M'$. On other hand, as $D \leq^{\oplus} M'$ and $M' \leq^{\oplus} M$, we get D is adirect summand of M and $C \oplus D = [Z_2(M) \oplus (C \cap M')] \oplus D = Z_2(M) \oplus [(C \cap M') \oplus D] \leq_{ess} Z_2(M) \oplus M' = M$; thus $C \oplus D \leq_{ess} M$. But D is a fully invariant submodule of M' and M' is a fully invariant in M, hence D is fully invariant in M by Proposition 1.1.38.

(5)⇒(4) It is clear(since every essential is t-essential).

(4)⇒(3) Let A be a fully invariant submodule of M. Then there exists a fully invariant t-closed C of M such that $A \leq_{tes} C$ by [20, Lemma 2.3].

By condition (4) there exists a fully invariant direct summand D such that $C \oplus D \leq_{tes} M$. But $A \leq_{tes} C$, so we concluded that $A \oplus D \leq_{tes} C \oplus D$ and hence $A \oplus D \leq_{tes} M$.

(3)⇒(1) Let C be a fully invariant t-closed. By condition (3) there exists a fully invariant direct summand D such that $C \oplus D \leq_{tes} M$. We claim that D is a complement of C. Assume E is a complement of C, so $C \cap E = (0)$ and $C \oplus E \leq_{ess} M$. Let $X \leq C \oplus E$ and $(C \oplus D) \cap X = (0) \leq Z_2(M)$.Hence $X \leq Z_2(M)$ (since$C \oplus D \leq_{tes} M$). But $Z_2(M) \leq C$, so $X \leq C$. It follows that $(C \oplus D) \cap X = (0)$. Thus $C \oplus D \leq_{ess} C \oplus E$ which implies $D \leq_{ess} E$. As D is a closed in M since $D \leq^{\oplus} M$. It follows that $D = E$. Thus D is a complement of C which is a fully invariant. Hence M is strongly FI-T_{11}-type module.

(1)⇒(5) Let C be a fully invariant t-closed submodule of M. By condition (1) there exists a fully invariant direct summand D of M such that D is a complement of C. Hence $C \oplus D \leq_{ess} M$ so $C \oplus D \leq_{tes} M$ and D is a fully invariant direct summand of M.

(5)⇒(1) It is clear by Lemma 4.2.2. □

Proposition (4.4.7): Let $M = M_1 \oplus M_2$, M_2 is fully invariant submodule in M and every fully invariant submodule of M_1 has a fully invariant t-closure. Then the following assertions are equivalent.

(1) M_1 is a strongly FI-T_{11}-type module;

(2) For every fully invariant submodule A of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $A \oplus D \leq_{tes} M$;

(3) For every fully invariant t-closed submodule C of M_1 there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{tes} M$;

(4) For every fully invariant t-closed C of M, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{ess} M$.

Proof: (1)⇒(2) Since M_1 is strongly FI-T_{11}-type, then by condition (3) of Theorem 4.4.6 for each fully invariant submodule A of M_1, there exists a fully invariant direct summand D_1 of M_1 such that $A \oplus D_1 \leq_{tes} M_1$. As $D_1 \leq^{\oplus} M_1$, then $D_1 \oplus M_2 \leq^{\oplus} M$. Also, $D = D_1 \oplus M_2$ is fully invariant in M by Lemma 3.5.4(2).But $A \oplus D_1 \leq_{tes} M_1$ implies $(A \oplus D_1) \oplus M_2 \leq_{tes} M_1 \oplus M_2 = M$. Thus $A \oplus D \leq_{tes} M$

(2)⇒(3) It is clear.

(3)⇒(4) For every fully invariant t-closed submodule C of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{tes} M$. Then $C \oplus D + Z_2(M) \leq_{ess} M$ by Proposition (1.1.17). But $Z_2(M) = Z_2(M_1) \oplus Z_2(M_2)$ and as C is t-closed in M_1,$C \supseteq Z_2(M_1)$. Also as $M_2 \leq D$, then $Z_2(M_2) \leq Z_2(D) \leq D$.It follows that $C \oplus D + Z_2(M) = C \oplus D + Z_2(M_1) \oplus Z_2(M_2) = C \oplus D \leq_{ess} M$.

(4)⇒(1) Let C be a fully invariant t-closed of M_1. By condition (4) there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{ess} M$. But D is a fully invariant submodule in M implies, $D = (D \cap M_1) \oplus (D \cap M_2)$, such that $D \cap M_1$ is fully invariant in M_1 and $D \cap M_2$ is fully invariant in M_2. But $D \cap M_2 = M_2$ since $M_2 \leq D$. Hence $D = (D \cap M_1) \oplus M_2$ and $D \cap M_1 \leq^{\oplus} M_1$.Now $C \oplus D =$

$C\oplus[(D\cap M_1)\oplus M_2] \leq_{ess} M = M_1\oplus M_2$. Hence,$[C\oplus[(D\cap M_1)] \leq_{ess} M_1$. Thus M_1 satisfies condition (5) of Theorem (4.4.6), which implies that M_1 is strongly type-T_{11} module. □

Proposition (4.4.8): If M is strongly FI-T_{11}-type and L is a fully invariant direct summand of . Then

(1) L is strongly FI-T_{11}-type; provided every fully invariant submodule has a fully invariant t-closure.

(2) $\frac{M}{L}$ is strongly FI-T_{11}-type (provided M is self-projective).

Proof: (1) To prove L is strongly FI-T_{11}-type module. Let A be a fully invariant submodule of L. As L is fully invariant in M, then A is a fully invariant submodule of M. Hence by Theorem 4.4.6(3) there exists fully invariant direct summand D of M such that $A\oplus D \leq_{tes} M$. Hence $(A\oplus D) \cap L \leq_{tes} L$ and so $A\oplus(D \cap L) \leq_{tes} L$. Let $M = D\oplus D'$ for some $D' \leq M$. $L = (D \cap L)\oplus(D' \cap L)$, where $D \cap L$ is fully invariant in D, $D'\cap L$ is fully invariant submodule in D' . But $D \cap L$ is fully invariant in D and D is fully invariant in M, so $D \cap L$ is fully invariant in M. But $\mathrm{L} \leq^{\oplus} M$, and $D \cap L \leq \mathrm{L}$ so $D \cap L$ is fully invariant in L by Lemma 1.1.40(2). Thus $D \cap L$ is fully invariant direct summand in L and $A\oplus((D \cap L) \leq_{tes} L$. Thus L is strongly FI-T_{11}-type by Theorem 4.4.6(3).

(2)Let $\frac{C}{L}$ be a fully invariant t-closed submodule in $\frac{M}{L}$. Then C is a fully invariant t-closed in M. As M is strongly FI-T_{11}-type module there exists a fully invariant direct summand D of M such that $C\oplus D \leq_{ess} M$ by Theorem 4.4.6(5). Let $M = D\oplus D'$ for some $D' \leq M$ and since L is fully invariant in M, $L = (D\cap L)\oplus(D'\cap L)$ such that $D\cap L$ is fully invariant in D, $D'\cap L$ is fully invariant in D'. Then $\frac{M}{L} = \frac{D\oplus D'}{(D\cap L)\oplus(D'\cap L)} \simeq \frac{D}{D\cap L}\oplus\frac{D'}{D'\cap L} \simeq \frac{D+L}{L}\oplus\frac{D'+L}{L}$. But it is easy to see that $\frac{C}{L}\oplus\frac{D+L}{L} \leq_{ess} \frac{M}{L}$. As $L \leq^{\oplus} M$, L is closed and this implies that $\frac{C\oplus D}{L} \leq_{ess} \frac{M}{L}$ by [23,Proposition1.4,P.18]. Thus

$\frac{C}{L} \oplus \frac{D+L}{L} \leq_{ess} \frac{M}{L}$. On the other hand, since D is a fully invariant submodule in M and, L is fully invariant in M, then D⊕L is fully invariant in M. Hence $\frac{D+L}{L}$ is fully invariant in $\frac{M}{L}$ (since M is self-projective) by Lemma 1.1.41(2). Thus $\frac{D+L}{L}$ is a fully invariant direct summand of $\frac{M}{L}$ and $\frac{C}{L} \oplus \frac{D+L}{L} \leq_{ess} \frac{M}{L}$. Therefore $\frac{M}{L}$ is strongly FI-T_{11}-type module by Theorem 4.4.6(1⇔5). □

Proposition (4.4.9): Let M be a FI-extending such that every closed submodule is fully invariant. Then M is strongly FI-T_{11}-type.

Proof: Let N be a fully invariant t-closed. Then N is a fully invariant closed submodule. Hence $N \leq^{\oplus} M$ since M is FI-extending. Say $N \oplus W = M$, hence W is a complement of N.Then by hypothesis, W is a fully invariant. Thus M is strongly FI-T_{11}-type. □

4.5 Modules satisfy purely C_{11}-condition and purely T_{11}-type modules

In this section, we generalize modules that satisfy C_{11}-condition and T_{11}-type modules into modules satisfy purely C_{11}-conditions and purely T_{11}-type modules. We study these concepts and their connections with purely t-semisimple modules

Definition (4.5.1): An R-module M is said to be satisfies purely C_{11}-condition if every pure submodule of M has a complement which is a direct summand.

Remarks and Examples (4.5.2):

(1) Every module satisfies C_{11}-condition has purely C_{11}-condition, but the converse is not true see (4)

(2) Every purely semisimple module (every pure is a direct summand) satisfies purely C_{11}-condition. In particular every Noetherain projective module (or every divisible module over principle ideal domain) is purely semisimple by Remarks and Examples 3.1.3(6, 7) hence satisfies purely C_{11}-condition. In particular it is clear that

$M = Z_3 \oplus Z_6$ as Z_6-module is Noetherain and it is projective by[25,Corollary 8.2.8(c)] so M satisfies purely C_{11}-condition. Not that M is not pure simple.

(3) Every pure simple module satisfies purely C_{11}-condition but not conversely.

(4) Every pure simple module and not uniform satisfies purely C_{11}-condition and doesn't satisfies C_{11}-condition.

Proof: Since M is a pure simple, then M is an indecomposable. Hence M is an indecomposable and not uniform and M doesn't satisfy C_{11}-condition by [38, Proposition 2.3 (iii)]. However M satisfies purely C_{11}-condition since M is pure simple. □

Proposition (4.5.3): Consider the following statements on M.

(1) M satisfies purely C_{11}-condition;

(2) For any pure submodule L of M, there exists $K \leq^{\oplus} M$ such that K is a complement of L;

(3) For any pure submodule N, there exists $D \leq^{\oplus} M$ such that $N \cap D = (0)$ and $N \oplus K \leq_{ess} M$;

(4) For any pure complement $L \leq M$, there exists $K \leq^{\oplus} M$ such that $K \cap L = (0)$, $K \oplus L \leq_{ess} M$.

Then (1)⇔(2)⇔(3) ,(2)⇒(4) and (4)⇒(1) if every pure submodule has pure closure.

Proof: (1) ⇔(3), (1) ⇔(2) and (2) ⇒(4) are clear.

(4)⇒(1) (If pure submodule has pure closure). Let N be a pure submodule, there exists W (closed pure) such that $N \leq_{ess} W$. By condition (4), there exists $K \leq^{\oplus} M$ such that $K \cap W = (0), K \oplus W \leq_{ess} M$, then $K \oplus N \leq_{ess} K \oplus W \leq_{ess} M$. Hence $K \leq^{\oplus} M$ and $K \oplus N \leq_{ess} M$ so K is a complement of N. By Lemma 4.2.2. □

Proposition (4.5.4): If M is purely t-semisimple and nonsingular module. Then M satisfies purely C_{11}-condition.

Proof: Let N be a pure submodule of M. Since M is nonsingular, N is nonsingular submodule of M. Hence by Theorem 3.2.8(1⇒4), $N \leq^{\oplus} M$, so that $M = N \oplus K$ for some $K \leq M$. It follows that K is a complement direct summand of N. Thus M is purely C_{11}-condition. □

Proposition (4.5.5): Let M be a distributive module which satisfies purely C_{11}-condition. Then a pure submodule N of M satisfies purely C_{11}-condition.

Proof: Let W be a pure submodule of N. As N is pure in M, so W is pure in M. But M satisfies purely C_{11}-condition, implies there exists $K \leq^{\oplus} M$ such that K is a complement of W. Then $K \oplus W \leq_{ess} M$, $(K \oplus W) \cap N \leq_{ess} M \cap N = N$, so $W \oplus (K \cap N) \leq_{ess} N$, to prove $K \cap N \leq^{\oplus} N$. Since $K \oplus K' = M$ for some $K' \leq M$, then $N = (K \oplus K') \cap N = (K \cap N) \oplus (K' \cap N)$ because M is a distributive module hence $K \cap N \leq^{\oplus} N$. Thus N has purely C_{11}-condition. □

Corollary (4.5.6): Let M be a distributive module and satisfies purely C_{11}-condition .Then a direct summand of M is purely C_{11}-condition.

Now we introduce the following.

Definition (4.5.7): An R-module M is called purely T_{11}-type if every pure t-closed submodule of M has a complement which is a direct summand.

Remarks (4.5.8):

(1) Every modules satisfies purely C_{11}-condition is purely T_{11}-type module.
(2) Every T_{11}-type module is purely T_{11}-type module.
(3) Every pure simple is purely T_{11}-type module, but not conversely for example: $Z_8 \oplus Z_2$ as Z-module is purely T_{11}-type module but it isn't pure simple.

Proposition (4.5.9): If M is a purely T_{11}-type module and M is purely t-extending, then M is T_{11}-type.

Proof: Let N be a t-closed submodule. Since M is purely t-extending, N is pure .Hence N is pure t-closed, but M is purely T_{11}-type, so N has a complement W which is a direct summand. Thus M is T_{11}-type. □

Proposition (4.5.10): Let M be a purely t-semisimple such that complement of $Z_2(M)$ is pure. Then M is purely T_{11}-type.

Proof: Let N be a pure t-closed submodule of M. Hence $N \supseteq Z_2(M)$ and so by Theorem 3.2.8(5),$N \leq^{\oplus} M$, say $N \oplus W = M$ for some $W \leq M$. Thus W is a complement of N, which is a direct summand and M is purely T_{11}-type module. □

Corollary (4.5.11): Let M be a purely t-semisimple such that complement of $Z_2(M)$ is direct summand. Then M is purely T_{11}-type.

Theorem (4.5.12): Consider the following assertions for an R-module M.

(1) M is purely T_{11}-type;

(2) $M = Z_2(M) \oplus M'$, where M' is nonsingular, satisfies purely C_{11}-condition;

(3) For each pure submodule A of M, there exists a direct summand D of M such that $A \oplus D \leq_{tes} M$;

(4) For any pure t-closed submodule C of M, there exists a direct summand D such that $C \oplus D \leq_{tes} M$;

(5) For any pure t-closed submodule C of M, there exists $D \leq^{\oplus} M$ such that $C \oplus D \leq_{ess} M$.

Then (2)⇒(5)⇒(4) , (3)⇒(1)⇔(5), (1)⇒(2) if $Z_2(M)$ is pure and every t-closure of pure is pure, (1)⇒(3) if every t-closure of pure is pure, and (1)⇒(4) if M is nonsingular.

Proof: (2)⇒(5) Let C be a pure t-closed submodule M , so that $C \supseteq Z_2(M)$. As $M = Z_2(M) \oplus M'$, then $C = Z_2(M) \oplus (C \cap M')$. Hence $(C \cap M')$ is pure in C and hence $(C \cap M')$ is pure in M (since C is pure in M) and as $(C \cap M') \leq M'$ then $(C \cap M')$ is pure in M'. Moreover, by Proposition 1.1.31(1) $(C \cap M')$ is t-closed in

M'. But M' is purely C_{11}-condition, so there exists $D \leq^{\oplus} M'$ such that $(C \cap M') \oplus D \leq_{ess} M'$. Then $C \oplus D = [Z_2(M) \oplus (C \cap M'] \oplus D = Z_2(M) \oplus [(C \oplus \cap M') \oplus D] \leq_{ess} Z_2(M \oplus M' = M$. Beside this $D \leq^{\oplus} M'$, then $D \leq^{\oplus} M$. Thus condition (5) hold.

(5)⇒(4) It is clear.

(3)⇒(1) Let C be a pure t-closed submodule of M. By hypothesis, there exists $D \leq^{\oplus} M$ such that $C \oplus D \leq_{tes} M$. We claim that D is a complement of C. Assume that, there exists $E \supseteq D$ and E is a complement of C. Let $X \leq C \oplus E$ and $(C \oplus D) \cap X = (0) \leq Z_2(M\ )$. Hence $X \leq Z_2(M)$ (since $C \oplus D \leq_{tes} M$). But $Z_2(M) \leq C$, so $X \leq C$ and hence $(C \oplus D) \cap X = X = 0$, thus $C \oplus D \leq_{ess} C \oplus E$ and so $D \leq_{ess} E$. But $D \leq^{\oplus} M$, so that $D = E$. Thus D is a complement of C.

(1)⇔(5) It follows directly.

(1)⇒(2) As $Z_2(M)$ is pure t-closed, there exists $M' \leq^{\oplus} M$, M' is a complement of $Z_2(M)$, then $M = L \oplus M'$. Since M' is nonsingular, $Z_2(M) = Z_2(L) \leq L$. But $Z_2(M) \oplus M' \leq_{ess} M$ implies $\frac{M}{M'} \simeq L$ is Z_2-torsion, hence $Z_2(L) = L$. Thus $M = Z_2(M) \oplus M', M' \simeq \frac{M}{Z_2(M)} = \bar{M}$. To prove M' satisfies purely C_{11}-condition. Let $\bar{C} = \frac{C}{Z_2(M)}$ be a pure closed (t-closed) in $\frac{M}{Z_2(M)}$. Hence C is pure t-closed in M. Since M is purely T_{11}-type, there exists a complement D of C, $D \leq^{\oplus} M$, say $M = D \oplus D'$. As $Z_2(M) = Z_2(D) \oplus Z_2(D')$. We get $\bar{M} = \bar{D} \oplus \overline{D'}$. It is $\bar{C} \cap \bar{D} = (0)$. Beside these $C \oplus D \leq_{ess} M$ implies $C \oplus D \leq_{tes} M$ and so $\bar{C} \oplus \bar{D} \leq_{ess} \bar{M}$. Thus $\bar{M}$ satisfies purely C_{11}-condition by Theorem 4.5.3((4)⇒(1)).

(1)⇒(3) Let A be a pure submodule. Then there exists a t-closure B of $A(A \leq_{tes} B$ and B is t-closed by [10, Lemma 2.3]. Also, B is pure by hypothesis. Thus B is pure t-closed. Since M is purely T_{11}-type, there exists $D \leq^{\oplus} M$ such that $D \oplus B \leq_{ess} M$. Hence $D \oplus B \leq_{tes} M$ and so $D \oplus A \leq_{tes} D \oplus B$ (since $A \leq_{tes} B$). Thus $D \oplus A \leq_{tes} M$.

(4)⇒(1) It is easy. □

Theorem (4.5.13): Let M be a finitely generated faithful multiplication over commutative ring R which is purely T_{11}-type. Then M is purely T_{11}-type.

Proof: Let $N = MI$ be a pure t-closed of M. Then I is a pure in R. To prove I is a t-closed in R. Let $I \leq_{tes} J$, then $MI \leq_{tes} MJ$ by Proposition 1.1.25(4). Hence $MI = MJ$, but M is finitely generated faithful multiplication so by [19, Theorem 3.1]

$I = J$ .But I is a pure t-closed in R and R is purely T_{11}-type ring imply that there exists a direct summand, J of R such that $I \oplus J \leq_{ess} R$. Then by [19, Theorem 2.13] $M(I \oplus J) \leq_{ess} M$, so $MI \oplus MJ \leq_{ess} M$ and as $J \leq^{\oplus} R$ implies $MJ \leq^{\oplus} M$, hence MJ is a complement of $N = MI$ which is a direct summand. Thus M is purely T_{11}-type. □

Proposition (4.5.14): Let M be a finitely generated faithful multiplication over a regular ring R. If M is T_{11}-type (purely T_{11}-type) then R is T_{11}-type (purely T_{11}-type).

Proof: Let I be a t-closed ideal of R. Let $N = MI$, then $MI \leq_{tc} M$, since if $MI \leq_{tes} MJ$, then $I \leq_{tes} J$ by Lemma 1.1.25(4), so $I = J$ and $MI = MJ$. Thus $MI \leq_{tc} M$. As M is T_{11}-type, there exists $W = MJ \leq^{\oplus} M$ and $N \oplus W \leq_{ess} M$. But $W = MJ \leq^{\oplus} M$, implies $J \leq^{\oplus} R$ and $N \oplus W = MI \oplus MJ = M(I \oplus J) \leq_{ess} M$ implies $I \oplus J \leq_{ess} R$ by[19,Theorem 2.13]. Thus R is T_{11}-type.

The second case is similarly. □

4.6 Modules satisfy Strongly purely C_{11}-condition and strongly purely T_{11}-type Modules

In this section, we generalize modules satisfy C_{11}-condition and T_{11}-type modules into modules satisfy strongly purely C_{11}- conditions and strongly purely T_{11}-type modules. We study these concepts and many properties related with these concepts.

Definition (4.6.1): An R-module M has strongly purely C_{11}-condition if every pure submodule has a complement which is a fully invariant direct summand.

Remarks and Examples (4.6.2):

(1) Every module satisfies strongly purely C_{11}-condition implies module satisfies purely C_{11}-condition, but the converse may be not hold, as the following example shows.

Let $M = Z_8 \oplus Z_2$ as Z-module. M is a direct sum of uniform modules, so M has C_{11}-condition by Remarks and Examples 4.1.3(5) and hence M has purely C_{11}-condition.

Let $N =< (\bar{1},\bar{1}) >= \{(\bar{1},\bar{1}),(\bar{2},\bar{0}),(\bar{3},\bar{1}),(\bar{4},\bar{0}),(\bar{5},\bar{1}),(\bar{6},\bar{0}),(\bar{7},\bar{1}),(\bar{0},\bar{0})\}$. N is a pure submodule of M. However there are only $W_1 =< \bar{4},\bar{1} >= \{(\bar{4},\bar{1}),(\bar{0},\bar{0})\}$ and $W_2 = (\bar{0}) \oplus Z_2 = \{(\bar{0},\bar{0}),(\bar{0},\bar{1})\}, W_3 = \{(\bar{0},\bar{0})\}$ such that $N \cap W_1 = N \cap W_2 = N \cap W_3 = 0$.But $N \oplus W_1 =$
$\{(\bar{1},\bar{1}),)(\bar{5},\bar{0}),(\bar{2},\bar{0}),(\bar{6},\bar{1}),(\bar{3},\bar{1}),(\bar{7},\bar{0}),(\bar{4},\bar{0}),(\bar{0},\bar{1}),(\bar{5},\bar{1}),$
$,(\bar{1},\bar{0}),(\bar{6},\bar{0}),(\bar{2},\bar{1}),(\bar{7},\bar{1}),(\bar{3},\bar{0}),(\bar{4},\bar{1}),(\bar{0},\bar{0})\}=M$. Hence W_1

is a complement direct summand of N, however W_1 is not fully invariant submodule of M, since if we define $f: W_1 \mapsto M$ by $f(\bar{4},\bar{1}) = (\bar{0},\bar{1}), f(\bar{0},\bar{0}) = (\bar{0},\bar{0})$, f is a Z-homomorphism and $f(W_1) \nleq W_1$.

Now,
$N \oplus W_2 =$
$\{(\bar{1},\bar{1}),(\bar{2},\bar{0}),(\bar{3},\bar{1}),(\bar{4},\bar{0}),(\bar{5},\bar{1}),(\bar{6},\bar{0}),(\bar{7},\bar{1}),(\bar{0},\bar{0}),(\bar{1},\bar{0}),(\bar{2},\bar{1}),(\bar{3},\bar{0}),(\bar{4},\bar{1}),$

$(\bar{5},\bar{0}),(\bar{6},\bar{1}),(\bar{7},\bar{0}),(\bar{0},\bar{1})\} = M \leq_{ess} M$. Hence W_2 is a complement direct summand in M of N but $W_2$is not fully invariant submodule of M since if we define $f: W_2 \mapsto M$ by $f(\bar{0},\bar{1}) = (\bar{4},\bar{1}), f(\bar{0},\bar{0})$=$(\bar{0},\bar{0})$, then f is a Z-homorphism. But $f(W_2) \nleq W_2$ hence W_2 is not fully invariant. Also, $N \oplus W_3 = N \leq^{\oplus} M$, but $N \nleq_{ess} M$,so W_3 is not a complement of N. Thus N has no complement which is fully invariant direct summand of M. Therefore M does not satisfy strongly purely C_{11}-condition.

(2) $M = Z_2 \oplus Z_2$ as Z-module is purely C_{11}-condition but doesn't satisfy strongly purely C_{11}-condition.

(3) If M is purely fully stable. M is purely C_{11}-condition if and only if M has strongly purely C_{11}-condition.

(4) If M is weak duo(hence if M is multiplication or duo). M has purely C_{11}-condition if and only if M has strongly purely C_{11}-condition.

(5) If M is an R-modules (every pure submodule is fully invariant) then M has purely C_{11}-condition if and only if M has strongly purely C_{11}-confition.

Theorem (4.6.3): Let M be a finitely generated faithful multiplication over commutative ring R. Then the following statements are equivalent:

(1) M is purely C_{11}-condition;

(2) R is purely C_{11}-condition;

(3) R is strongly purely C_{11}-condition.

(4) M is strongly purely C_{11}-condition.

Proof: (1)⇒(2) Let I be a pure ideal of R. Then MI is a pure submodule of M. Since M has purely C_{11}-condition, there exists $W \leq^{\oplus} M$ such that W is a complement of MI, $W = MJ$ for some ideal J of R, since M is a multiplication module. Thus $W \oplus K = M$ for some $K \leq M$, let $K = MT$ for some ideal T of R. Hence $MJ \oplus MT = M$, which implies $J \oplus T = R$; that is $J \leq^{\oplus} R$. Beside this $W = MJ$ is a complement of

MI, implies $MJ \oplus MT \leq_{ess} M$ and by [19,Theorem 2.13], $J \oplus T \leq_{ess} R$. Thus I has a complement J which is a direct summand.

(2) ⇒(1) Let N be a pure submodule of M. As M finitely generated faithful multiplication $N = MI$ for some pure ideal I of R. But R has purely C_{11}-condition, there exists J (a complement of I) and $J \leq^{\oplus} R$. Hence $J \oplus I \leq_{ess} R$. Hence by [19, Theorem 2.13] $MJ \oplus MI \leq_{ess} M$. As $J \leq^{\oplus} R$, then $J \oplus T = R$ for some $T \leq R$ and $M(J \oplus T) = M$, so $MJ \oplus MT = M$ that is $MJ \leq^{\oplus} M$. Thus MJ is a complement of $MI = N$ and $MJ \leq^{\oplus} M$. Thus M is purely C_{11}-condition.

(2) ⇒(3) and (1) ⇔ (4) are clear by Remarks and Examples 4.6.2(4). □

Next we have

Proposition (4.6.4): Consider the following statements for an R-module M:

(1) M satisfies strongly purely C_{11}-condition;

(2) For any pure complement submodule L in M, there exists a fully invariant direct summand K of M such that K is a complement of L in M;

(3) For any pure submodule A of M, there exists a fully invariant direct summand B of M such that $A \cap B = (0)$ and $A \oplus B \leq_{ess} M$;

(4) For any pure complement submodule L of M, there exists a fully invariant direct summand K of M such that $L \cap K = (0)$ and $L \oplus K \leq_{ess} M$.

Then (1) ⇔ (3) ,(2)⇔(4), (1)⇒(2), (4)⇒(1) if every closure of pure submodule is pure .

Proof: (1) ⇔ (3) and (2) ⇔(4) are clear.

(1)⇒(2) Let N be a pure complement submodule. By condition (1) N has a fully invariant complement W which is direct summand in M.

(4)⇒(1) Let A be a pure submodule of M, so there exists a closed pure submodule such that $A \leq_{ess} B$ by hypothesis. As B is pure complement, there exists $K \leq^{\oplus} M$

and K is fully invariant such that $B \cap K = (0), B \oplus K \leq_{ess} M$. Hence K is a complement fully invariant of B. As $A \leq B$, $A \cap K \leq B \cap K = (0)$. But $A \oplus K \leq_{ess} B \oplus K \leq_{ess} M$, so $A \oplus K \leq_{ess} M$ and $K \leq^{\oplus} M$. Hence K is a fully invariant complement of A. □

By restricting the condition of modules satisfy strongly pure-C_{11}-condition in to pure t-closed submodules, we give the following:

Definition (4.6.5): An R-module M is called strongly purely T_{11}-type module if every pure t-closed submodule has a complement which is a direct summand and fully invariant.

Remarks and Examples (4.6.6):

(1) $M = Z_8 \oplus Z_2$ as Z-module. M is the only pure t-closed submodule of M, there exists $(\bar{0}, \bar{0}) \leq^{\oplus} M$ such that $\{(\bar{0}, \bar{0})\}$ is fully invariant submodule and $\{(\bar{0}, \bar{0})\}$ is a complement of M . Thus M is strongly purely T_{11}-type.

(2) It is clear that every module satisfies strongly purely C_{11}-condition is strongly purely T_{11}-type but converses is not true for examples:

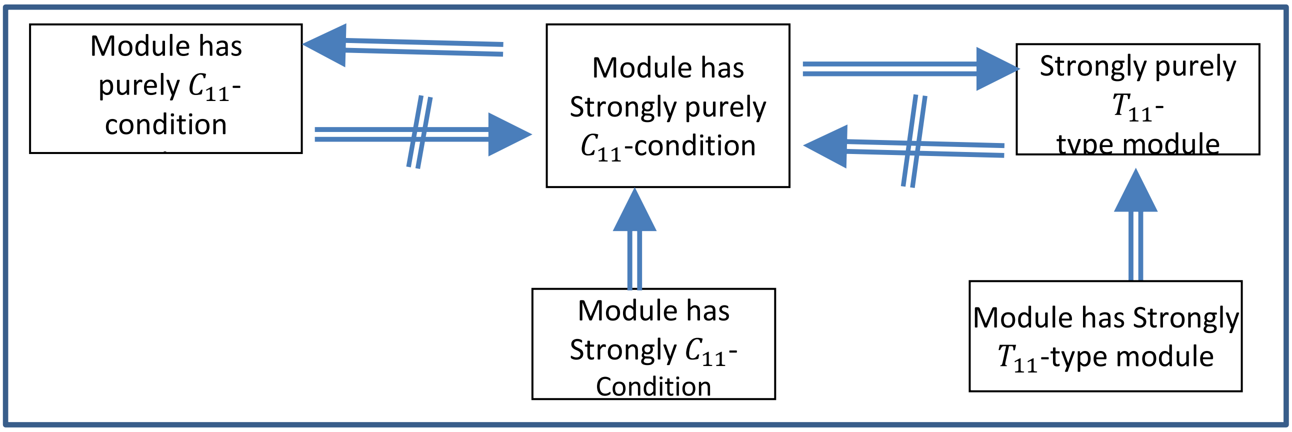
- (I) $M = Z_8 \oplus Z_2$ as Z-module is strongly purely T_{11} by part (1), and it does not satisfy strongly purely C_{11}-condition by Remarks and Examples 4.6.2(1).
- (II) $M = Z_2 \oplus Z_2$ as Z-module. M is singular, so M is the only pure t-closed. Thus M is strongly purely T_{11}-type. But $N = Z_2 \oplus (0)$, N has a complement $W = (\bar{0} \oplus Z_2) \leq^{\oplus} M$. But W is not fully invariant. Thus M is not strongly purely C_{11}-condition. Hence strongly purely T_{11}-type need not be strongly purely C_{11}-cnodition. □

(3) Let M be a multiplication (hence if M is duo or fully stable) M is purely T_{11}-type if and only if M is strongly purely T_{11}-type.

(4) M is pure simple then M is strongly purely T_{11}-type.

(5) M is strongly T_{11}-type implies strongly purely T_{11}-type.

We can summarize these relations by the following diagram



Proposition (4.6.7): Let M be a nonsingular. If M is strongly purely t-semisimple, then M is strongly purely C_{11}(hence strongly purely T_{11}-type).

Proof: Let N be a pure submodule of M. Since M is nonsingular, so N is nonsingular. Hence by Theorem 3.4.6(1→3), N is a fully invariant direct summand, say $M = N \oplus W$ for some $W \leq M$. As $W \leq^{\oplus} M$, W is a pure submodule. Also, W is nonsingular and so again by Theorem 3.4.6(1→3), W is a fully invariant direct summand. On the other hand, W is a complement of N. Thus M is purely C_{11}-condition. □

Remarks (4.6.8):

(1) Let M be a purely fully stable. Then M is purely T_{11}-type if and only if M is strongly purely T_{11}-type.

Proof: It is clear. □

(2) Let M be a regular R-module (every submodule is pure). Then

(I) M is T_{11}-type if and only if $M$is purely T_{11}-type.
(II) M is strongly T_{11}-type if and only if M is strongly purely T_{11}-type.
(III) M is strongly C_{11}-condition if and only if M is strongly purely C_{11}-condition.

Theorem (4.6.9): Consider the following statements for an R-module M:

(1) M is strongly purely T_{11}-type;

(2) $M = Z_2(M) \oplus M'$, where M' is nonsingular strongly purely C_{11}, M' is fully invariant submodule of M;

(3) For every pure submodule A of M, there exists a fully invariant direct summand B of M such that $A \oplus B \leq_{tes} M$;

(4) For every pure t-closed submodule C of M, there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{tes} M$;

(5) For every pure t-closed submodule C of M, there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{ess} M$.

Then (1) ⇔ (5) ⇒ (4), (2) ⇒ (1). (4) ⇒ (3) (if t-closure of pure is pure).

Thus (1), (3), (4), (5) are equivalent if (t-closure of pure is pure) and (1) ⇔ (2) if $Z_2(M)$ is pure.

Proof: (1)⇔(5) ⇒(4) It is clear.

(2) ⇒(1) Let C be a pure t-closed of M. Then $C \supseteq Z_2(M)$, hence $C = Z_2(M) \oplus (C \cap M')$. Since $C \leq_{tc} M$, then by Proposition 1.1.31(1) $C \cap M'$ is a t-closed in M'. Moreover, $C \cap M' \leq^{\oplus} C$ implies $C \cap M'$ is pure in C, but C is pure in M, so $C \cap M'$is pure in M'. Since $M'$is strongly purely C_{11}-condition, there exists a fully invariant direct summand D of M' such that $(C \cap M') \oplus D \leq_{ess} M'$. But $D \leq^{\oplus} M'$ implies $D \leq^{\oplus} M$. Also D is a fully invariant in M' and M' is fully invariant in M implies D is fully invariant in M. Now $Z_2(M) \oplus [C \cap M' \oplus D] = [Z_2(M) \oplus (C \cap M')] \oplus D = C \oplus D \leq_{ess} Z_2(M) \oplus M' = M$. Thus (1) hold.

(3)⇒(1) Let A be a pure t-closed submodule of M, there exists a fully invariant direct summand B of M such that $A \oplus B \leq_{tes} M$, then $(A \oplus B) + Z_2(M) \leq_{ess} M$ by Proposition 1.1.17. But $Z_2(M) \leq A$ since A is t-closed. So $(A \oplus B) + Z_2(M) = A \oplus B \leq_{ess} M$, and B is a fully invariant direct summand. Thus condition (1) hold.

(4)⇒(3) Let A be a pure submodule of M By[10, Lemma 2.3], there exists B (t-closed of M) such that $A \leq_{tes} B$ and by hypothesis B is pure. Thus B is pure t-closed in M, hence by condition (4) there exists a fully invariant direct summand D

of M such that $B \oplus D \leq_{ess} M$, so $B \oplus D \leq_{tes} M$. But $A \leq_{tes} B$, so $A \oplus D \leq_{tes} B \oplus D$. Thus $A \oplus D \leq_{tes} M$ by Proposition 1.1.20(1).

(1)⇒(2) If $Z_2(M)$ is pure , then $Z_2(M)$ is pure t-closed then by the same proof of (Theorem 4.2.9(1→2)), condition (2) hold. □

Proposition (4.6.10): Let M be a strongly purely t-semisimple such that complement of $Z_2(M)$ is pure. Then M is strongly purely T_{11}-type.

Proof: Since complement $Z_2(M)$ is pure, then by Theorem 3.4.6(1→3→2),

$M = Z_2(M) \oplus M'$, where M' is nonsingular fully invariant submodule of M and purely fully stable, purely semisimple. It is enough to show that M' is strongly purely C_{11}-condition. Let A be a pure submodule of M'. Since M' is purely semisimple, $A \leq^{\oplus} M'$ and hence $M' = A \oplus B$. So that $B$is a complement of A in M'. But M' is fully stable, so B is stable in M' , hence it is fully invariant. Thus A has a complement B in M' such that B is a fully invariant direct summand. Therefore M' is strongly purely C_{11}-condition and so by Theorem 4.6.9(2→1), M is strongly purely T_{11}-type. □

Theorem (4.6.11): Let $M = M_1 \oplus M_2$, where $M_1, M_2 \leq M$ and M_2 is fully invariant in M. The following statements are equivalent.

(1) M_1 is strongly purely T_{11}-type module;
(2) For every pure submodule A of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $A \oplus D \leq_{tes} M$;
(3) For every pure t-closed submodule C of M_1, there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{tes} M$;
(4) For every pure t-closed submodule C of M_1, there exists a fully invariant direct summand of M such that $M_2 \leq D$ and $C \oplus D \leq_{ess} M$.

Proof: (1)⇒(2) Let C be a pure submodule of M_1. By Theorem 4.6.9(1→4), there exists a fully invariant direct summand D_1 of M_1 such that $C \oplus D_1 \leq_{ess} M_1$. Hence $C \oplus D_1 \leq_{tes} M_1$ which implies $(C \oplus D_1) \oplus M_2 \leq_{tes} M_1 \oplus M_2 = M$, $C \oplus (D_1 \oplus M_2) \leq_{tes} M$, but $D_1 \leq^{\oplus} M_1$ implies $D = D_1 \oplus M_2 \leq^{\oplus} M$ and $M_2 \leq D$. Also $D = D_1 \oplus M_2$ is fully invariant in M by Lemma 3.5.4(2). Hence D is a fully invariant direct summand in M. Thus condition (2) hold.

(2) ⇒(3) It is obvious.

(3)⇒(4) Let C be a pure t-closed submodule of M_1. Then by condition (3), there exists a fully invariant direct summand of M such that $M_2 \leq D$ and $C \oplus D \leq_{tes} M$. Hence by Proposition 1.1.17 $C \oplus D + Z_2(M) \leq_{ess} M$. But $Z_2(M) = Z_2(M_1) \oplus Z_2(M_2)$. As C is t-closed in M_1, $C \supseteq Z_2(M_1)$ and as $M_2 \leq D$,$Z_2(M_2) \leq D$. Thus $C \oplus D + Z_2(M) = C \oplus D \leq_{ess} M$.

(4)⇒(1) Let C be a pure t-closed submodule of M_1. By condition (4), there exists a fully invariant direct summand D of M such that $M_2 \leq D$ and $C \oplus D \leq_{ess} M$. As D is a fully invariant submodule of M, $D = (D \cap M_1) \oplus (D \cap M_2)$, where $D \cap M_1$ is fully invariant in M_1, $D \cap M_2 = M_2$ which is fully invariant in M. Thus $D = (D \cap M_1) \oplus M_2$. As $D \leq^{\oplus} M$, hence $D \cap M_1 \leq^{\oplus} M_1$, $C \oplus D = C \oplus [(D \cap M_1) \oplus M_2] \leq_{ess} M = M_1 \oplus M_2$ so that $C \oplus (D \cap M_1) \leq_{ess} M_1$ by Theorem 4.6.9(4→1) M_1 is strongly purely T_{11}-type module. □

Theorem (4.6.12): Let M be a strongly purely T_{11}-type, L is a fully invariant submodule direct summand of M. Then (1) L is strongly purely T_{11}-type (provided that every t-closure of pure submodule is pure) and

(2)$\frac{M}{L}$ is strongly purely T_{11}(provided M is self-projective).

Proof: (1) Let W be a pure submodule of L. Since L is pure in M(because $L \leq^{\oplus} M$), then W is pure in M, so there exists W'(t-closure of M); $W \leq_{tes} W'$ and W' is t-closed by [20, Lemma 2.3]. By hypothesis W' is pure in M. Thus W' is pure t-closed. Since M is strongly purely T_{11}-type, there exists a fully invariant direct summand K of M such that $K \oplus W' \leq_{ess} M$, so $K \oplus W' \leq_{tes} M$. As $W \leq_{tes} W'$, then $K \oplus W \leq_{tes} K \oplus W'$.Hence $K \oplus W \leq_{tes} M$ and $(K \oplus W) \cap L \leq_{tes} L$. It follows that $W \oplus (K \cap L) \leq_{tes} L$. Beside this $K \leq^{\oplus} M$ implies $K \oplus K' = M$ for some $K' \leq M$ and since L is fully invariant submodule of M,$L = (K \cap L) \oplus (K' \cap L)$ where $K \cap L$ is fully invariant in K, $K' \cap L$ is fully invariant in K', and $K \cap L \leq^{\oplus} L$. Now $K \cap L$ is fully invariant in K and K is fully invariant in M. So $K \cap L$ is fully invariant in M. $K \cap L$ is a direct summand of L and $L \leq^{\oplus} M$ so $K \cap L \leq^{\oplus} M$ and so by Lemma 1.2.6($K \cap L$) is a fully invariant submodule in L.

(2)To prove $\frac{M}{L}$ is strongly purely T_{11}-type. Let $\frac{C}{L}$ be a pure t-closed of $\frac{M}{L}$. As $\frac{C}{L}$ is t-closed, then C is t-closed in M and as $\frac{C}{L}$ is pure in $\frac{M}{L}$ and L is pure in M,we have C is pure in M. Since M is strongly purely T_{11}-type, there exists a fully invariant direct summand D of M such that $C \oplus D \leq_{ess} M$. Let $M = D \oplus D'$ for some $D' \leq M$, and since L is fully invariant in M, $L = (D \cap L) \oplus (D' \cap L)$ wher $D \cap L$ is fully invariant in $D, D' \cap L$ is fully invariant in D' and $D \cap L \leq^{\oplus} L$. Now $\frac{M}{L} = \frac{D \oplus D'}{(D \cap L) \oplus (D' \cap L)} \simeq \frac{D}{D \cap L} \oplus \frac{D'}{D' \cap L} \simeq \frac{D+L}{L} \oplus \frac{D'+L}{L}$, then $\frac{D+L}{L} \leq^{\oplus} \frac{M}{L}$. As $D \; and \; L$ are fully invariant in M, then $D + L$ is fully invariant in M and so $\frac{D+L}{L}$ is fully invariant in $\frac{M}{L}$ (since M is self-projective). Moreover, we can show that $\frac{C}{L} \oplus \frac{D+L}{L} \leq_{ess} \frac{M}{L}$. Since $C \oplus D \leq_{ess} M$ and L is closed in M (because$L \leq^{\oplus} M$). Hence $\frac{C \oplus D}{L} \leq_{ess} \frac{M}{L}$ by [23, Proposition 1.4, P.18]. It follows that $\frac{C}{L} \oplus \frac{D+L}{L} \leq_{ess} \frac{M}{L}$. Thus $\frac{M}{L}$ is a strongly purely T_{11}-type module by Theorem 4.6.9(5→1). □

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المستخلص

في سنة 2013 , الباحثان اشكاري وهاكاري قدما مفهوم المقاسات شبه البسيطة من النمط-t كتعميم لمفهوم المقاسات شبه البسيطة , حيث يقال لموديول M على R شبه بسيط من النمط-t اذا كان لكل $A \leq M$ يوجد مركبة مباشرة B في M بحيث ان A (واسع من النمط-t) في B . في الحقيقة ان المفهوم (واسع من النمط-t) قُدم من قبل اشكاري وهاكاري سنة 2011 , وقد اطلقا على مقاس جزئي A من مقاس M على R واسعا من النمط-t في M (وتكتب $A \leq_{tes} M$) اذا كان لكل $C \leq M$, $A \cap C \leq Z_2(M)$ يؤدي الى $C \leq Z_2(M)$, حيث $Z_2(M)$ هو المقاس الجزئي المنفرد الثاني في M.

هذه الاطروحة مخصصة لتقديم مايلي:

- توسيع مفهوم المقاسات شبه البسيطة من النمط-t الى المقاسات شبه البسيطة من النمط-t القوية .
- تعميم المفهومان المقاسات شبه البسيطة من النمط-t , والمقاسات شبه البسيطة من النمط-t القوية الى المقاسات شبه البسيطة من النمط-FI-t , المقاسات شبه البسيطة من النمط-t النقية , المقاسات شبه البسيطة من النمط-t النقية القوية .
- تقديم اصناف مختلفة من المقاسات ذات علاقة مع انواع من المقاسات شبه البسيطة من لنمط-t ,فمثلا المقاسات التي تحقق الشرط C_{11} القوي , المقاسات من النوع T_{11} القوية , المقاسات التي تحقق الشرط $FI\text{-}C_{11}$ (التي تحقق الشرط $FI\text{-}C_{11}$ القوي) , المقاسات من النوع $FI\text{-}T_{11}$ ($FI\text{-}T_{11}$ القوي) , المقاسات التي تحقق الشرط C_{11} النقي (التي تحقق الشرط C_{11} النقي القوي) والمقاسات من النوع T_{11} النقي (T_{11} النقي القوي) .

جمهورية العراق

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(ابن الهيثم)

قسم الرياضيات

دراسة المقاسات المرتبطه بالمقاسات الشبه البسيطة من النمط T

اطروحة

مقدمة إلــى كلية التربية للعلوم الصرفة/ ابن الهيثم- جامعـة بغداد وهي جزء من متطلبات نيـل درجة الدكتوراه فلسفة فــي علوم الرياضيـــات

مــن قبل

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