Republic of Iraq

Ministry of Higher Education and Scientific Research

University of Baghdad

College of Education for Pure Science/ Ibn Al-Haitham

Invariant Best Approximation in Modular spaces

A Thesis

Submitted to the College of Education for Pure Science/ Ibn Al- Haitham, University of Baghdad in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

Karrar Emad AbdulSada Allheby

Supervised by

Assist .Prof. Dr. Salwa Salman Abed

2018 AC **1439 A**

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

اللَّهُ نُورُ السَّمَاوَاتِ وَالأَرْضِ مَثَلُ نُورِهِ كَمِشْكَاةٍ فِيهَا
مِصْبَاحٌ الْمِصْبَاحُ فِي زُجَاجَةٍ الزُّجَاجَةُ كَأَنَّهَا كَوْكَبٌ
دُرِّيٌّ يُوقَدُ مِن شَجَرَةٍ مُّبَارَكَةٍ زَيْتُونَةٍ لَّا شَرْقِيَّةٍ وَلا
غَرْبِيَّةٍ يَكَادُ زَيْتُهَا يُضِيءُ وَلَوْ لَمْ تَمْسَسْهُ نَارٌ نُّورٌ
عَلَى نُورٍ يَهْدِي اللَّهُ لِنُورِهِ مَن يَشَاء وَيَضْرِبُ اللَّهُ
الأَمْثَالَ لِلنَّاسِ وَاللَّهُ بِكُلِّ شَيْءٍ عَلِيمٌ ﴿٣٥﴾

صدق الله العلي العظيم

إهداء

الى من بلغ الرسالة ... وادى الامانة ... ونصح الامة ... الى نبي الرحمة ونور العالمين سيدنا محمد صلى الله عليه واله وسلم وعلى ابن عمه امير المؤمنين علي ابن ابي طالب (ع) والتسعة المعصومين من ذرية الحسين (ع)

الى من أوصى الله سبحانه وتعالى بهم

﴿وَوَصَّيْنَا الْإِنسَانَ بِوَالِدَيْهِ إِحْسَانًا حَمَلَتْهُ أُمُّهُ كُرْهًا وَوَضَعَتْهُ كُرْهًا وَحَمْلُهُ

وَفِصَالُهُ ثَلَاثُونَ شَهْرًا حَتَّىٰ إِذَا بَلَغَ أَشُدَّهُ وَبَلَغَ أَرْبَعِينَ سَنَةً قَالَ رَبِّ أَوْزِعْنِي أَنْ

أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَىٰ وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَصْلِحْ لِي

فِي ذُرِّيَّتِي إِنِّي تُبْتُ إِلَيْكَ وَإِنِّي مِنَ الْمُسْلِمِينَ (15)

الاحقاف (15)

الى من علمني وارشدني في حياتي وشجعني على تحقيق حلمي ابي العزيز

الى معنى الحنان والتضحية والى سر نجاحي وسعادتي امي الغالية

الى من اعتمد عليهم في حياتي واكتسب منهم قوتي زوجتي واخواتي

الى اخوتي الذين لم تلدهم امي مشتاق – مازن- فراس- مؤنس- والاساتذه: السيد- احمد- حيدر- ايمن- خليل- محمد- محمود- سنان وخالي محمد.

*Acknowledgment*

Praise is to Allah and peace and blessings be upon his prophets and messengers of prophet Mohammed and his family and his extend.

I would like to thank who supervised on my thesis

Assist .Prof. Dr. Salwa Salman Abed

Doctor of pure mathematics - department of mathematics - college of education for pure science / Ibn Al-Haitham – University of Baghdad.

I would like to thank the

All staff of the department of mathematics-college of education for pure science / Ibn Al-Haitham – University of Baghdad.

Karrar Emad AbdulSada Allheby

Author's Publications

[1] S.S. Abed, K.A. Abdul Sada," *An Extension of Brosowski-Meinaraus Theorem in Modular Spaces*", Inter. J. of Math. Anal., Hikari Ltd., Vol. 11, No. 18, pp. 877 – 882(2017).

[2] S.S. Abed, K.A. Abdul Sada *"Approximatively Compactness and Best Approximation in Modular Spaces*" accepted in Conf. of Scie. Coll., Nahrain University.(2017).

[3] S.S. Abed, K.A. Abdul Sada, " *Fixed-Point and Best Approximation Theorems in Modular Spaces"*, to appear in Inter. J. os Pur. And Appli. Math.(2017).

[4] S.S. Abed, K.A. Abdul Sada, "common fixed point in modular space ", acceptable for publication. Ibn Al-Haitham 1^{st}. International Scientific Conference – 2017.

ABSTRACT

The purpose of this thesis is to study the properties of best approximations set and to apply some fixed\ coincidence point theorems to obtain invariant best approximations in modular spaces. The idea of obtaining these results was included in four pivots. The first one is to reform some concepts in the setting of modular spaces, such as, strong\ weak convergence, compactness, duality of a modular space, … and then prove some needed relative statements. The second is to prove some Brosowski-Minardus type theorems on an invariant best approximation. On the other hand, the third pivot is to apply a common fixed\ coincidence point theorems and using property of w −convex structure to get other results. Finally, the forth is to prove the existence of such results with respect to mappings of single\ set-valued non-expansive mappings, (P, Q) −nonexpansive mappings and generalized (P, Q)-non-expansive mappings.

CONTENTS

Introduction

The concept of modular spaces, as a generalization of metric spaces, was given by Nakano [31] in 1950. Musielak and Ortiz [30] in 1959 introduced a generalization of the classical function space L^P . Khamsi et al [23] proved the fixed point results in modular function space. There are literature on the fixed point theory in modular spaces, such as [1], [5], [8], [12], [14], [18], [23], [27], [29], [44] and the paper referenced there. Pata [7] proved banach $'$s contraction principle in modular spaces.

Paknazar et al. [7] used pata idea to prove another fixed point theorem and prepared an application of their results to existence of solution of megral equations in some of these spaces. Recently, S.S. Abed [2] introduced the concept of best approximation In modular spaces

The classical approximation problem is the best approximation to (a,b), along the straight line passing through the origin can be found by droping a perpendicular from (a,b) to the line.

Significant questions concerning y includes:

- How may y be found?
- Can be characterized?
- Is it unique?
- Does $A = M$?

The early problems of best approximation theory like Kyfan's theorem and Prolla's theorem depend on convexity properties which involve introducing a mapping with some hypothesis. This thesis deals with Brosowski-Meinardus type [38] which guarantees the existence of the invariant best approximation.

Fixed point theorems have been used at many places in approximation theory[15]. One of them is while existence of best approximation is proved. Later on, number of results were developed using fixed point theorem to prove the existence of best approximation. However, the result given by singh [36] was the fundamental result in this direction. An excellent reference can be seen in [39]. Another celebrated result was due to Jungch [20] also in fact extended the result of Hicks and Humpheries [17], Jungch and Sessa [21]. Latif [28], Khan [24], Singh [38] were some other authors who worked in this direction under different conditions following the line made by Singh [38].

In [17], Singh relaxed the condition of linearity of mapping and convexity of set but later, he observed that only the non-expansiveness is necessary to prove best approximation while applying fixed point theorem. Similary, Hicks and Humpheries said in their paper [17] that the element for the set of best approximation is not necessarily in the interior of the set.

In other papers, Jungch and Sessa [21] further weakend the hypothesis of carbon [10] and Singh [38] by replacing the condition of linearity by some properties to prove the existence of best approximation in a normed linear space. However, they used weak continuity of the mapping for such purpose in the second result. Recently, Latif [28] has removed the weak continuity from the hypothesis of Jungch and Sessa [21] and obtained the result in normed space.

Throughout this thesis, we seek about an invariant best approximation in the setting of normed spaces [35].

The existence of invariant best approximation in the setting of modular spaces. This thesis contains five chapters. In chapter zero we present some basic definitions and facts about vector spaces and topological vector spaces. In chapter one, we recall the notion of modular spaces and some related definitions, facts and examples. In chapter two, we prove the existence of invariant best approximation of ky-fan type with respect to set valued mappings. Also, prove some other results for non-expansive mappings in complete modular spaces. On the other hand, chapter three, is devoted to study common best approximation for non-commuting mappings depending on star-shaped and affineness conditions and finally, chapter four is devoted to present conclusions and future work.

CHAPTER 1

MODULAR SPACES

1-0 Introduction

This chapter contains four sections. Section one is devoted to recall the definition of a modular function on a linear spaces and some known definitions and facts.

In Section two there are some concepts of convergence sequences (strong and weak), compactness, approximative compactness, … . Also, includes the proof of some important results, such as, uniqueness of limit for weak convergent sequences, relation between strong and weak convergence and other results. Section three includes new considerations about the dual of modular spaces and linear functionals. In section four, there are some types of set-valued mappings and some related concepts.

1-1Basic Definitions and Examples of Modular Spaces

We start with the following:

Definition (1.1.1): [11]

Let M be a linear space over $F(=R)$. A function $\gamma: M \to [0,\infty]$ is called Modular if:

i. $\gamma(v) = 0 \Leftrightarrow v = 0; \forall\, v \in M.$

ii. $\gamma(\alpha v) = \alpha\gamma(v)$ for $\alpha \in F$ with $|\alpha| = 1$, $\forall\, v \in M$;

iii. $\gamma(\alpha v + \beta u) \leq \gamma(v) + \gamma(u) \Leftrightarrow \alpha, \beta \geq 0, \forall v, u \in M.$

Definition (1.1.3): [11]

A modular γ defines a corresponding modular space, i.e, the space M_γ given by

$$M_\gamma = \{v \in M : \gamma(\alpha v) \to 0 \; whenever \; \alpha \to 0\}.$$

Definition (1.1.2): [11]

If (iii) in definition modular space M_γ replaced by $\gamma(\alpha v + \beta u) \leq \alpha\gamma(v) + \beta\gamma(u)$, for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all v, u $\in M$, then M modular γ is called con$v$ex modular.

Remark (1.1.1): [11]

By condition (iii) above, if u$= 0$ then $\gamma(\alpha v) = \gamma\left(\frac{\alpha}{\beta}\beta v\right) \leq \gamma(\beta v)$, for all $\alpha, \beta \; in \; F, \; 0 < \alpha < \beta$. this shows that γ is increasing function.

Remark (1.1.2): [2]

i. A Modular space M_γ is a metric space with

$\gamma(v-u) = D_\gamma(v,A)$, for all , $\in M$.

ii. Any Modular space is a topological linear space, moreover, it is Hausdorff space. For the definition of topological linear space.

Definition (1.1.4): [11]

The γ-open ball, $B_r(u)$centered at $u \in M_\gamma$with radius $r > 0$ as

$$B_r(u) = \{\boldsymbol{v} \in M_\gamma; \gamma(v-u) < r\}.$$

The class of all γ-balls in a modular space M_γ generates a topology which makes M_γ Hausdorff topological linear space. Every γ-ball is convex set, therefore every modular space locally convex Hausdorff topological linear space [2].

Definition (1.1.5): [11]

$B \subset M_\gamma$ is said to be $\gamma-$ bounded if $daim_\gamma(B) < \infty$, where $daim_\gamma(B) = \text{Sup}\,\{\gamma(v-u);\ v,u \in B\}$ is the $\gamma-$ diameter of B.

Example (1.1.1):

Let $M_\gamma = R^2\ with\ \gamma(v\,,u) = |v| + |u|$ (| | is absolute value), for

Any pair $(v - u)$ in M_γ, then M_γ is modular space since it satisfies the conditions:

(i) $\gamma(v-u) = 0 \Leftrightarrow |v| + |u| = 0 \Leftrightarrow v = 0\,, u = 0$

(ii) $\gamma\big(\alpha(v-u)\big) = \gamma(\alpha v - \alpha u) = |\alpha v| + |\alpha u|$

$= |\alpha|(|v| + |u|) \quad \ldots |\alpha| = 1$

$= |v| + |u|$

$= \gamma(v-u)$

(iii) $\gamma\big(\alpha(v-u) + \beta(b-e)\big) = \gamma(\alpha v + \beta b - \alpha u - \beta e)$

$= |\alpha v + \beta b| + |\alpha u + \beta e|$

$\leq |v| + |u| + |\mathrm{b}| + |e|$

$= \gamma(v-u) + \gamma(b-e).$

Then $M_\gamma = M$ the modular space with respect to γ.

Example: [1.1.7]

As a classical example we mention to the Orlicz' modular defined for every measurable real function f by the formula

$$\gamma(f) = \int \phi\,(|f(t)|)\, d\mu(t),$$

where μ denotes the Lebesgue's measure in $\mathbb{R}$ and $\phi : \mathbb{R} \to [0,\infty)$ is continuous. We also assume that $\phi(u) = 0$ if and only if $u = 0$ and

$\phi(t) \to \infty$ as $t \to \infty$.

Here, we omit the details about this space because it is not within the thesis objectives.

1.2 Convergences in Modular Spaces

In the following we recall some concept, facts of convergence in a modular space M_γ:

Definition (1.2.1): [11]

A sequence $(v_n) \subset M_\gamma$ is said to be γ-convergent (or strongly $\gamma-$ convergent) to $v \in M_\gamma$ and write $v_n \xrightarrow{\gamma} v$ if $\gamma(v_n - v) \to 0$ as n$\to \infty$.

Definition (1.2.2): [11]

A sequence (v_n) is called $\gamma-$ Cauchy whenever $\gamma(v_n - v_m) \to 0$ as, $m, n \to \infty$.

Definition (1.2.3): [11]

M_γ is called $\gamma-$ complete if any $\gamma-$ Cauchy sequence in M_γ is $\gamma-$ convergent**.**

Definition (1.2.4): [11]

A subset B of M_γ is called $\gamma-$ closed if for any sequence (v_n) subset of B $\gamma-$convergent to $v \in M_\gamma$, implies that $\in B$.

Definition (1.2.5): [11]

A $\gamma-$ closed subset B of M_γ is called $\gamma-$ compact if any sequence (v_n) a subset of B has a $\gamma-$ convergent subsequence**.**

Definition (1.2.6)

Let be M_γ a modular space. Then a mapping $S: M_\gamma \to M_\gamma$ is compact if the closure of A is compact whenever A is bunded subset of M_γ

Definition (1.2.7):

Let M_γ be a modular linear space, and A a subset of M_γ. We say that A is an approximatively compact if for every $v \in M_\gamma$. and every sequence (v_n) in A with $\lim_{n\to\infty} \gamma(v - v_n) = D_\gamma(v, A)$, there exists a subsequence $\left(v_{n_i}\right)$ converges to an element of A.

Since a modular space is metric space then we have:

Proposition (1.2.1):

Every convergent sequence in modular space has a unique limit.

Proof: It is clear.

1.3 Dual of a modular space

Definition (1.3.1):

let P be a linear functional with domain in a modular space M_γ and range in the scalar field K $P{:}D\left(M_\gamma\right) \to K,$ P is bounded linear functional such that for all $v \in D(P)$, $\gamma(Pv) \leq c\gamma(v)$. The set of all bounded linear functional on M_γ, $\boldsymbol{M'_\gamma}$ is linear space with point-wise operations. In the following, we reform some concepts about dual space in the setting of modular spaces, we being with following:

Proposition (1.3.1):

Let $P \in \boldsymbol{M'_\gamma}$, define $\boldsymbol{\gamma} : \boldsymbol{M'_\gamma} \to R^+ \quad \ni \boldsymbol{\gamma}(P) = \sup\{\gamma(Pv) : \gamma(v) = 1\}$ then

i. $\boldsymbol{\gamma}(\alpha P) = \boldsymbol{\gamma}(P)$, for $\alpha \in$ K with $|\alpha| = 1$

ii. $\boldsymbol{\gamma}(\alpha P + \beta Q) \le \boldsymbol{\gamma}(P) + \boldsymbol{\gamma}(Q)$,

iii. $\boldsymbol{\gamma}(P) = 0$ iff $P = 0$.

Proof:

For (i) $\boldsymbol{\gamma}(\alpha P) = \sup\{\gamma(\alpha P v)\} = \sup\{\gamma(Pv)\} = \boldsymbol{\gamma}(P)$.

For (ii) $\boldsymbol{\gamma}(\alpha P + \beta Q) = \sup\{\gamma(\alpha P v + \beta Q v)\}$

$$\le \sup\{\gamma(Pv) + \gamma(Qv)\}$$

$$= \sup\{\gamma(Pv)\} + \sup\{\gamma(Qv)\}$$

$$= \boldsymbol{\gamma}(P) + \boldsymbol{\gamma}(Q)$$

For(iii),

$\boldsymbol{\gamma}(P) = 0$ iff $\sup\{\gamma(Pv) : \gamma(v) = 1\}$ iff $\gamma(Pv) = 0$ for all$v$ iff $P = 0$.

A modular $\boldsymbol{\gamma}$ defines a corresponding modular spac, i.e the space$\boldsymbol{M}'_{\boldsymbol{\gamma}}$ given by

$$\boldsymbol{M}'_{\boldsymbol{\gamma}} = \{v \in M : \boldsymbol{\gamma}(\alpha P) \to 0 \text{ whenever } \alpha \to 0\}$$

Theorem (1.3.1):

$\boldsymbol{M}'_{\boldsymbol{\gamma}}$ is complete modular space.

Proof:

We consider an arbitrary Cauchy sequence (S_n) in $\boldsymbol{M}'_{\boldsymbol{\gamma}}$ and show that (S_n) converges to a $S \in \boldsymbol{M}'_{\boldsymbol{\gamma}}$ Since (S_n) is Cauchy, for every $\epsilon > 0$ there is an L such that

$$\boldsymbol{\gamma}(S_n - S_m) < \in , \qquad (n > L),$$

For any $v \in M_\gamma$ and $n > L$, this implies that

$$|S_n v - S_m v| = |(S_n - S_m)v| \leq \gamma(S_n - S_m)\gamma(v) \leq \in \gamma(v). \quad \ldots (2.1)$$

Now, for any fixed point v and given $\in'$ we may choose $\in = \in_v$ so that $\in_v \ \gamma(v) < \in'$.

Then from (2.1), we have $|S_n v - S_m v| <\in'$ and $(S_n v)$ is Cauchy in K. By completeness of K, $(S_n v)$ converges, say, $S_n v \to r$. Clearly, the limit $r \in K$ depends on the choice of $v \in M_\gamma$.

This defines a functional S: $M_\gamma \to K$ where $r = Sv$ .The functional S is linear since $\lim_{n\to\infty} S_n(\alpha v + \beta z) = \lim_{n\to\infty}(\alpha S_n v + \beta S_n z) = \alpha \lim_{n\to\infty} S_n v + \beta \lim_{n\to\infty} S_n z$. We prove that S is bounded and $S_n \to S$, that is $\boldsymbol{\gamma}(S_n - S) \to 0$.

Since (2.1) holds for every $m > L$ and $S_m v \to S$, we may let $m \to \infty$. Using the continuity of the modular, then for every $n > L$ and all $v \in M_\gamma$.

$$|S_n v - Sv| = \left|S_n v - \lim_{m\to\infty} S_m v\right|$$

$$= \lim_{m\to\infty} |S_n v - S_m v|$$

$$\leq \epsilon\gamma(v) \qquad \ldots(2.2)$$

This shows that $(S_n - S)$ with $n > L$ is a bounded linear functional. Since S_n is bounded, $S = S_n - (S_n - S)$ is bounded, that is, $S \in \boldsymbol{M'_\gamma}$. Furthermore, if in (2.2) we take the supremum over all v of modular 1, we obtain

$$\gamma(S_n - S) \leq \epsilon,\ n > L.$$

Hence $\boldsymbol{\gamma}(S_n - S) \to 0$. This completes proof.

Definition (1.3.2):

A sequence (v_n) in a modular space M_γ is said to be weakly convergent if there is an $v \in M_\gamma$ such that for every $P \in \boldsymbol{M}'_{\boldsymbol{\gamma}}$

$\lim_{n\to\infty} \gamma(Pv_n - Pv) = 0$ This denoted by $v_n \overset{\mathrm{w}}{\to} v$.

Proposition (1.3.2):

In a modular space M_γ, every convergent sequence is weakly convergent.

Proof:

By definition, $v_n \to v$ means $\gamma(v_n - v) \to 0$ and implies that for every P$\in M'_{\gamma,}$ $|P(v_n) - P(v)| = |P(v_n - v)| \le \gamma(P)\gamma(v_n - v) \to 0$. This shows that $v_n \overset{w}{\to} v$.

Note that, the converse of proposition (1.3.2) is not necessary true. To showing this recall the usual case in a normed space.In the following some other needed properties of weak convergence are given:

Proposition (1.3.3):

Let (v_n) be weakly convergent sequence in a modular space M_γ, say $v_n \overset{w}{\rightarrow} v$ Then:

i. The weak limit v of (v_n) is unique.

ii. Every subsequence of (v_n) converges weakly to v.

Proof:

For (i), suppose that $v_n \overset{w}{\rightarrow} v$ as well as $v_n \overset{w}{\rightarrow} u$. Then $\mathrm{P}(v_n) \rightarrow P(v)$ as well as $\mathrm{P}(v_n) \rightarrow P(u)$. Since $(\mathrm{P}(v_n))$ is a sequence of numbers, its limit is unique. Hence $P(v) = P(u)$, that is, for every P∈ $\boldsymbol{M}'_{\gamma}$. We have $P(v) - P(u) = P(v-u) = 0$.This implies $v-u = 0$ and shows that the weak limit is unique. Part (ii) follows from the fact that $(\mathrm{P}(v_n))$ is a convergent sequence of numbers. So that every subsequence of $(\mathrm{P}(v_n))$ converges and has same limit as the sequence.

Definition (1.3.3):

A a subset of a modular space M_γ is said to be weakly compact if every sequence in M_γ has a weak convergent subsequence

1.4 SomeTypes of Mappings of Modular Spaces

Let M_γ and N_ρ be two modular space, we state the following:

Definition (1.4.1):

Let M_γ be a modular space and 2^{M_γ} is the class of all subset of M_γ.

Then $S: M_\gamma \rightarrow 2^{M_\gamma}$ is called set-valued mapping if $\forall\, v \in M_\gamma, Sv \subset M_\gamma$.

Definition (1.4.2):

A set-valued mapping S is upper semi continuous (shortly, $u.s.c.$) if and only if the set $\{v \in M_\gamma: S(v) \cap B \neq \emptyset\}$ is closed for each closed subset B of N_ρ. Sv is a closed subset $M_\gamma \times N_\rho$.

Definition (1.4.3):

Let S be a set- valued mapping on M_γ and $v \in M_\gamma$, v is called a fixed point of S if $v \in Sv$.

(When S is single valued, v is fixed point of S if $v = Sv$, we denote to the of all fixed point of S by $F(S)$.

Definition (1.4.4)

A subset A of the modular spaceM_γ is an invariant under the mapping $S: M_\gamma \rightarrow M_\gamma$ under the mapping when $u \in A \Rightarrow Su \in A$.

Definition (1.4.5): [34]

Let S be a set-valued mapping on M_γ. A sequence (v_n) of points of M_γ is said to be an iteration of S at v if $v_n \in Sv_{n-1}$, for each

$n = 1, 2 \dots \dots$ (when S is single valued the iterative sequence of S at v is $v_n = Sv_{n-1}$, for each $n = 1, 2 \dots \dots ..$).

Definition (1.4.6): [26]

Let M_γ be a modular space and A subset of M_γ, S: $A \rightarrow A$, S is called contraction mapping if there is a fixed h $\in (0, 1)$ for all v, u in M

$$\gamma(Sv - Su) \le h(v - u)$$

And if h $= 1$ then S is called a non–expansive mapping.

Proved Banach's contriction principle in modular metric space, here we reform it in modular spaces [25].

Theorem (1.4.1): [26]

Let M_γ be a complete modular space and S: $M_\gamma \rightarrow M_\gamma$ such that $\gamma(Sv - Su) \le h(v - u)$, for all $v, u \in M_\gamma$, where $h \in (0,1)$. Suppose that $\exists\ v_\circ \in M_\gamma$ and there is some $v \in A \ni \gamma(Sv_\circ) < \infty$. Then, S has unique fixed point z $\in M_\gamma$ and the sequence$\left(S_{v_\circ}^n\right)$ converges to z.

CHAPTER 2

BEST APPROXIMATIONS IN MODULAR SPACES

2-0 Introduction

In this chapter there are three sections, in section one the concept of best approximation of a point v by a non-empty subset A of a modular space M_γ is introduced . And study it's existence. The existence of such element or not characterize three sets: proximinal, semi-Chebysev and Chebysev. Examples for these types and some conditions for existence of proximinal and Chebysev sets are given. Section two includes a studying the relation between best approximation and fixed point theorems, and proving a version of using Himmelberg's fixed point theorem of set-valued mappings, and then use it to prove that

Ky Fan's theorem in best approximation for set-valued mappings, we present Schauder's fixed point theorem for continuous mapping defined on a compact subset of a moduler space as a corollary. We illustrate an example for utility of compactness in Ky Fan's theorem. In section three, the definition of an approximatively compact is reformed in modular spaces and some it's properties are given. This concept has an efficacious in many results about best approximation.

2.1 Properties of the Best Approximations Set of Modular Spaces

Definition (2.1.1): [2]

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$, an element u$\in A$ is called the best approximation for v in M_γ if

$$\gamma(v-u) = D_\gamma(\,(v,A)\,) = inf\,\{\gamma(v-u): u \in A\}$$

We shall denote by $P_A(v)$ or P_A the set of all elements of best approximation of v by $P(v)$, that is $P_A\ (v) = \left\{u \in A: \gamma(v-u) = D_\gamma(\,(v,A)\,)\right\}$.

Proposition (2.1.1):

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$ and $\gamma: M \to [0,\infty]$, then, $P_A(v)$ is closed and bounded set.

Proof:

Suppose that u is an accumulation point of $P_A(v)$ and $D_\gamma\left(\mathrm{u}, \mathrm{P}(v)\right) = 0$

$$\gamma(v-u) \leq D_\gamma\left(v, \mathrm{P}(v)\right) + D_\gamma\left(\mathrm{u}, \mathrm{P}(v)\right)$$

$$= D_\gamma\left(v, \mathrm{P}(v)\right)$$

$$= \inf\{\gamma(v-z): z \epsilon A\}$$

$$= D_\gamma(v, A)$$

Since $D_\gamma(v, A) \le \gamma(v-u)$, $thus$ $\gamma(v-u) = D_\gamma(v, A)$ and $u \in P_A(v)$, which means $P_A(v)$ is closed.

$P_A(v)$ is bounded since $P_A(v) < \infty$ and, $P_A(v)$ containing in $B_r(u)$,

where $r = d(v, r) + 1$.

Proposition (2.1.2):

Let M_γ be a modular space and $\emptyset \ne A \subset M_\gamma$. If $\gamma : M \to [0, \infty]$ is convex Then $P_A(v)$ is convex set.

Proof:

Let $0 \le \lambda \le 1$ and $u_1, u_2 \in P_A(v)$ then

$\gamma(u_1 - v) = D_\gamma(v, A)$ and $\gamma(u_2 - v) = D_\gamma(v, A)$

$\lambda\, \gamma(u_1 - v) = \lambda\, D_\gamma(v, A)$

$(1-\lambda)\, \gamma(u_2 - v) = (1-\lambda) D_\gamma(v, A)$

$\gamma(\lambda u_1 - \lambda v) = \lambda\, D_\gamma(v, A)$ and $\gamma((1-\lambda)u_2 - (1-\lambda)v = (1-\lambda)\, D_\gamma$ (v, A)

$\gamma(\lambda u_1 - \lambda v) + \gamma((1-\lambda)\, u_2 - (1-\lambda)v) = D_\gamma(v, A)$

But $\gamma(\lambda u_1 - \lambda v + (1-\lambda)\, u_2 - (1-\lambda)) \le \gamma(\lambda u_1 - \lambda v) + \gamma((1-\lambda)u_2 - (1-\lambda)v) = D_\gamma(v, A)$ …(2.1)

Now, since $u_1, u_2 \in P_A(v) \subset A$, then $u_1, u_2 \in A$ and A is convex set

So $\lambda u_1 + (1-\lambda)u_2 \in A$ therefore

$$D_\gamma(v, A) \le \gamma(\lambda u_1 + (1-\lambda)u_2 - v) \quad \ldots(2.2)$$

By (2.1), (2.2), we have $\gamma\,(\lambda u_1 + (1-\lambda)u_2\,) = D_\gamma(v, A)$

Hence $\lambda u_{\,1} + (1-\lambda)u_2 \in P_A(v)$. then $P_A(v)$ is convex set.

Definition (2.1.2): [2]

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$.. A is called proximinal if for all $v \in M_\gamma$, there exist a u$\in A$ such that

$$\gamma(v-u) = D_\gamma(v, A).$$

Definition (2.1.3): [2]

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$.. A is called semi-chebysev if there most one u$\in A$ satisfying

$$\gamma(v-u) = D_\gamma(v, A), \forall\ v \in M_\gamma$$

Definition (2.1.4): [2]

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$. A is called Chebyse$v$ if $\forall\ v \in M_\gamma$, there is an unique element u$\in A$ such that

$$\gamma\,(v-u) = D_\gamma(v, A)$$

Example (2.1.1):

Consider $M_\gamma = R^2$, where v= $(v_1\ ,\ v_2\ )$. Setting= $(1,1)$, and $u = (1,0)$, we have

$$\gamma(v-\alpha u) = D_\gamma(1-\alpha, 1)$$

The value, will be the minimum if and only if $\alpha = \mathbf{1}$. Thus the unique best approximation of v by A: the closed linear subspace spanned by u. And is Chebysev set.

Example (2.1.2):

Consider $M_\gamma = R^2$ with $\gamma(v) = \max\{|v_1|, |v_2|\}$ where $v = (v_1, v_2)$. Setting $v = (1,1)$, $u = (1,0)$, we have $\gamma(v - \alpha u) = D_\gamma(1-\alpha, 1) = \max\{|1-\alpha, 1|\}$.

There exists infinitely many best approximation of v by A: The closed linear subspace spanned by , that is $P(v) = \{\alpha u = 0 \leq \alpha \leq 2\}$. And A is proximinal set.

Proposition (2.1.2): [32]

A Hausdorff topological vector space is locally compact if and only if A is finite dimensional.

Proposition (2.1.3):

If M_γ is modular space and A is a finite dimensional subspace of M_γ, then is A proximinal subspace.

Proof:

Let A be a finite dimensional subspace of a modular linear space M_γ, and $\in M_\gamma$. The space Q= $\{v\} \cup A$ is finite dimensional. By proposition (2.1.2) Q is locally compact.

Clearly, $D_\gamma(v, A) \leq \gamma(v)$. If $e \in A$ and $\gamma(v - e) \leq \gamma(v)$

$\Longrightarrow |\gamma(v) - \gamma(e)| \leq \gamma(v - e) \leq \gamma(v)$

$\Longrightarrow |\gamma(v) - \gamma(e)| \leq \gamma(v)$

$\Longrightarrow -\gamma(v) \leq \gamma(v) - \gamma(e) \leq \gamma(v)$

$\Longrightarrow -\gamma(v) \leq \gamma(v) - \gamma(e)$

$\Longrightarrow \gamma(e) \leq 2\gamma(v)$

Hance, to find $e \in A$ such that $(v - e) = D_\gamma(v, A)$, let

$K = \{u \in M : \gamma(u) \leq 2\gamma(v)\}$. Since, by the previous observation, K is compact set, then there exist $e \in K$ such that, therefore A is proximinal set.

Definition (2.1.5):

Let M_γ be a modular space. M_γ is said to be strictly moduler space when $\gamma(v + u) = \gamma(v) + \gamma(u) \Leftrightarrow$ u= αv $(\alpha \geq 0)$.

Proposition (2.1.4):

If M_γ is a strictly moduler space and A is a finite dimensional subspace of M_γ, then A is Chebysev set.

Proof:

Since M_γ is modular space, and A is finite dimensional subspace of M, then by proposition (2.1.3) A is proximinal set, so there is a linear m $\in A$ such that

$$\gamma(v-m) = D_\gamma(v, A)$$

If $v \in A \Longrightarrow 0 = D_\gamma(v, A) = \gamma(v-m)$

$\Longrightarrow 0 = \gamma(v-m)$

$\Longrightarrow v = m$

$\Longrightarrow m$ is unique and then A is Chebyse$v$.

We consider if $v \notin A$

If $\{v_1, \ldots\ldots, v_n\}$ is a base for A, suppose that, and with $\gamma(v-m) = \gamma(v-z)$

Since M_γ is strictly moduler space, then for some $v \geq 0$

Since $v \notin A \Longrightarrow v = 1$

Since $v_{1,\ldots\ldots,} v_n$ are linearly independent, then $v_i = u_i$ for $i = 1, 2, \ldots\ldots, n$,

and thus $m = z \Longrightarrow A$ is Chebysev.

2.2 Ky Fan Type of Invariant Approximation

Now we give the following concept in modular space:

Definition (2.2.1):

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$. A is called almost convex if for any $\epsilon > 0$, $B_\in(0)$ and any finite set of points of A u_1, u_2,…….., $u_n \in A$ there exist $v_1, v_2 \dots v_n$ M_γ such that $v_i - u_i \in B_\in(0)$for all i, and $co\{v_1, v_2 \dots v_n\} \subset A$.

Theorem (2.2.1):

Let $\emptyset \neq A$ be a compact subset of modular space M_γ with modular function γ and $S : A \to CB(A)$ be an $(u.s.c.)$ mapping ($CB(A)$is the set of all non –empty closed and bounded subsets of A) with (v) is convex for all v in some dense almost convex of A. Then S has a fixed point.

Proof:

For each $\epsilon > 0$, let $F_\in = \{v \in A: \in S(v) + \bar{B}_\epsilon(0)\}$

To prove the existence fixed point of S it is sufficient to show $\cap F_\in \neq \emptyset$. Since for any$\epsilon > \delta$,$F_\in \supset F_\delta$, it is sufficient, by the compactness of A, to show that each $F_\in$, is closed and nonempty. So let $\epsilon > 0$. Define the set-valued mappings

$S_\epsilon : A \to 2^A$, $S_\epsilon(v) = (S + \bar{B}_\epsilon(0)) \cap A$

and $R_\epsilon: A \to 2^A$, $R_\epsilon(v) = (v + \bar{B}_\epsilon(0)) \cap A$, for $v \in A$

Then $S_\epsilon = R_\epsilon \circ S$, $R\epsilon$ is a closed subset of $A \times A$ since

R_ϵ = $\{(v,u)$ ∈A× A $|u$-v ∈ $\bar{B}_\epsilon(0)\}$ and since $\bar{B}_\epsilon(0)$ is closed subset of A× A and A is compact is follows that both $R\epsilon$ and S are $(u.s.c.)$. Hence $S\epsilon$ is $(u.s.c.)$.and S_ϵ is closed subset of A× A.

Let Δ be the diagonal in A× A. Then

$F_\in$ is the projection of the compact set $\Delta \cap S\epsilon$ onto the domain of S_ϵ. It follows that $F_\in$ is closed. Now choose $z_1, \dots, z_m \in K$ such that $K \subset \{$ $\bar{B}_\epsilon(0)$: $1 \le i \le m\}$, and C= $co\{z_1, \dots, z_m\} \subset K$. Define$H_\epsilon \subset C$× C byH_ϵ= S_ϵ $\cap (C \times C)$. For each $v \in C$, $H_\epsilon(v)$ is closed, convex (since $C \subset A$) and nonempty (since S_ϵ+ $\bar{B}\epsilon$ contain some z_i). Moreover, $H_\epsilon$is a closed subset of C× C (since S_ϵ is closed). Thus H_ϵ has a fixed point by Kakutani′s fixed point theorem [18], say, u. And u belongs to $F_\in$, which is not empty.

Theorem (2.2.2):

Let $\emptyset \neq A$ convex subset of complete modular space M_γ with modular function. Let S:$A \to CB(A)$ an $(u.s.c.)$ such that $S(v)$ is convex for all v∈ A and $S(A)$ is contained in some compact subset C of A. Then S has fixed point.

Proof:

Let B =coC and K= $\bar{B}$. Then K is compact, $B \subset A$ and $S(B) \subset C \subset B$. Let $H = S \cap B \times B$. Then H is relatively closed subset of $B \times B$. Consider $\overline{H} \subset$ K × K with closure relative to $K \times K$. $H$is a set-valued mapping from $K$to K, i.e., $\overline{H}^{-1}(K) = K$ since$\overline{H}^{-1}(K)$ closed and contains$B$. Moreover $\overline{H}(K) \subset C \subset B$ and $= \overline{H} \cap (B \times B)$; so $\overline{H}(v)$ =H(v) = $S(v)$ for all v∈ B. Thus by Theorem (2.2.1) $\overline{H}$ has fixed point say v in K. But $v \in \overline{H}(v) \subset$ C ⊂ B. So $v \in S(v)$. Hence S has fixed point.

Definition (2.2.2):

Let M_γ be a modular space with modular function γ and $\emptyset \neq A \subset M_\gamma$ for P_A $(v)=$ $\{u \in A: \gamma(v-u) = D_\gamma(v, A)\}$ is the set of all best approximation of v by A and the set-valued mapping P: $M_\gamma \to 2^A$ is said to the metric projection on M_γ.

Theorem (2.2.3):

Let A be a compact convex subset of a convex modular M_γ and P: $A \to M_\gamma$ be a continuous function, then there exist a $u \in A$ such that

$$\gamma\big(u - P(u)\big) = D_\gamma(P(u), A) \tag{1.4}$$

Proof:

Let i: $A \to R^+$ be defined $i(v) =$ in f$\{\gamma(u-v), u \in A\}$. Since P is continuous on A for each $v \in A$, then there exist a $u \in A$ such that $i(v) = \gamma\big(u - P(v)\big)$(because$A$ is compact)**.** Define a set valued mapping S:$A \to 2^A$ by: $S(v) = \{u \in A: i(v) = \gamma\big(u - P(v)\big)\} \subseteq A \neq \emptyset$

(as above). We will prove that

i. $S(v)$ is closed set;
ii. $S(v)$ is convex set;
iii. S is $(u.s.c.)$.

For (i), suppose that z is an accumulation point of $S(v)$, then there exists a sequence $(z_n) \subseteq S(v)$such that $z_n \to z$. And we have

$$\gamma\big(z - P(v)\big) = \gamma\big(\lim_{n\to\infty} z_n - P(v)\big) = \lim_{n\to\infty} \gamma\big(z_n - P(v)\big) = i(v).$$

Thus $\in S(v)$, and then $S(v)$ is closed set.

For (ii), suppose that $0\leq \lambda \leq 1$ and $u_1, u_2 \in S(v) \subset A$. Since A is convex, then

$\lambda u_1+(1-\lambda)u_2 \in A$ and $D_\gamma(v, A) \leq \gamma(u_1+(1-\lambda)u_2-v)$

Now,

$$\gamma(\lambda u_1 + (1-\lambda)u_2 - v) \leq \lambda\gamma(u_1 - v) + (1-\lambda)\gamma(u_2 - v)$$

$$= i(v)$$

$$= \gamma(\lambda u_1 + (1-\lambda)u_2 - v)$$

And this prove that $S(v)$ is convex set.

For (iii), let C be a closed subset of A, we will prove that $S^{-1}(C) = \{u \in A: S(u) \cap C \neq \emptyset\}$ is closed subset C of M_γ . Suppose that $v_0 \in A$ be an accumulation point$S^{-1}(C)$, then there exists a net $(v_a) \subseteq S^{-1}(C)$ converge to v_0. This implies that there is a net $u_a \in S(v_a) \cap C$. That is, $u_a \in C$ and $u_a \in S(v_a)$ so, $\gamma\big(u_a - P(v_a)\big) = i(v_a)$ for each a. Since A is compact and C is closed subset of, then C is compact, so there is a $u_0 \in C$ and a subnet (u_β) of(u_a). Hence, $u_\beta \in S(v_a)$

$\Rightarrow\gamma\left(u_\beta - P(v_a)\right) = i(v)$, for each β.

$\Rightarrow\gamma\big(u_0 - P(v_0)\big) = i(v_0)$, which means that $u_0 \in Sv_0 \cap C$. This implies that $v_0 \in S^{-1}(C)$. Thus S is $(u.s.c.)$ set-valued mapping. Since A is compact and $S(A) \subset A$, then $S(A)$ is contained in compact set. Therefor by theorem (2.2.2) there is a $u_0 \in A$ such that $u_0 \in Su_0$ that is

$\gamma\big(u_0 - P(u_0)\big) = d(P(u_0), A)$.

To illustrate the utility of compactness condition in Theorem (2.2.3), we have the following:

Example (2.2.1):

Consider the unit ball $B_1(0)$in modular space l^2 with convex modular function $\gamma(x) = \sqrt{\sum_1^\infty |x_i|^2}$, where $x = (x_1, x_2, \dots)$ and $|\ \ |$is absolute valued. $B_r(0)$ is closed and bounded but non-compact with topology induced by γ. For each $x\ in\ B_r(0)$, define the continuous function f by

$$f(x) = (\sqrt{1 - \left(\gamma(x)\right)^2}, x_1, x_2, .., x_n, ..)$$

Clearly, $\gamma(f(x)) = 1$

Suppose that f has a fixed point z, so, $\gamma\left(f(z)\right) = \gamma(z) = 1$ this implies that $z = 0$, i.e., $\gamma(z) = 0$. Which is a contradiction.

2.3 Approximately Compactness and Best Approximation

We begin with the following results:

Proposition (2.3.1):

If A is compact subset of a modular space M_γ, then A is an approximately compact.

Proof:

Let $v \in M_\gamma$ and (v_n) be a sequence in A with

$$\lim_{n\to\infty} \gamma(v - v_n) = D_\gamma(v, A)$$

Since A is compact set, then by Definition (1.2.5) there is a subsequence (v_{n_i}) of converging to (v_n) an element in A, which proved the proposition.

The converse of the above proposition is not true. To explain this fact, we need the following definition and then show a general statement.

Definition (2.3.5):

A modular space M_γ is called uniformly convex if for any ε> 0, there exists a δ(ε) > 0, such that if

$\gamma(v)$ = $\gamma(u)$ = 1 and $\gamma(v-u)$≥ε, then $\gamma\left(\frac{1}{2}(v+u)\right) \leq 1-\delta$.

Example (2.3.1) Consider the unit closed ball $A = \{\boldsymbol{v} \in M_\gamma; \gamma(v) \leq r\}$.in uniformly convex complete modular space $l^2(R)$ with convex modular function $\gamma(x) = \sqrt{\sum_1^\infty |x_i|^2}$, where $x = (x_1, x_2, \dots)$ and $|\ \ |$is absolute valued on real numbers R. $A$is not compact with topology induced by γ butapproximativley compact

Proposition (2.3.2):

A closed convex set A in an uniformly convex modular space M_γ is an approximativley compact.

Proof:

Let $u \in M_\gamma$ and $(u_n) \subseteq A$ such that $\gamma(u_n - u) \rightarrow D_\gamma(u, A)$. Then sup $\gamma(u_n)<\infty$. Since A is closed and convex, then there exists a $u_\circ \in A$ and a sequence $(u_n) \subseteq A$ such that $u_h \rightarrow u_\circ$. Since $\lim_{n\to\infty} \gamma(u_h - u) = u_\circ - u$, then $u_h - u \to u_\circ - u$.So

$$\gamma(u_\circ - u) \leq \liminf_{i\to\infty} \gamma(u_h - u) = D_\gamma(u, A) \leq \gamma(u_\circ - u)$$

that is $\gamma(u_\circ - u)|$ = $D_\gamma(u,\ A)$. By definition of <u_n>, we get $u_h - u \to D_\gamma(u, A) = \gamma(u_\circ - u)$. Since M_γ is an uniformly convex modular space, then we get $u_h - u \to u_\circ - u$, Then $u_h \to u \in A$, then A is an approximately compact.

Example (2.3.2):

Consider the unit closed ball $A = \{\boldsymbol{v} \in M_\gamma; \gamma(v) \leq r\}$. In uniformly convex complete convex modular space $l^2(R)$ with convex modular function $\gamma(x) = \sqrt{\sum_1^\infty |x_i|^2}$, where $x = (x_1, x_2, \dots)$ and $|\quad|$is absolute valued on real numbers R. A is not compact with topology induced by γ but approximately compact.

Theorem (2.3.1):

If A is an approximatively compact subset of modular space M_γ, then A is a proximal and closed.

Proof:

Let $v \in M_\gamma$. By definition of $D_\gamma(v, A)$, from the set of the numbers

$$\{\gamma(v - u) : u \epsilon A\}$$

we can extract a sequence $(\gamma(v - u_n))$ such that

$\lim_{n\to\infty} \gamma(v - u_n) = D_\gamma(v, A)$, Since A is an approximatively compact, then we can extract from (u_n) a subsequence converging to a point $u_\circ \in A$. We then have by the continuity of γ

$$\gamma(v-u_\circ)=\gamma\left(v-\lim_{i\to\infty} u_{n_i}\right)=\lim_{i\to\infty}\gamma\left(v-u_{n_i}\right)=D_\gamma(v,A)$$

When $u_\circ \in P_A(v)$, which complete the proof of proximinality. Finally, let v is an accumulation point of A, then there exist a $u\in A$ such that

$$\gamma(v-u) = D_\gamma(v,A) = 0,$$

So $v \in A,$ and A is closed set.•

Conversely, if A is proximal set, then it is not necessary that A is an approximatively compact. To illustrate this we give the following example.

Example (2.3.3):

Consider M_γ as in Example (2.3.1) and let A be the sequence defined by

u_1 = **0** and $u_n = \left(1,\frac{1}{n},0,\dots,0,1,0,\dots\right)$

A is a proximal set (since for every $v\in M_\gamma$, the sequence of non-negative numbers $\langle\gamma(v-u_n)\rangle$ is convergent, whence $\inf\gamma(v-u_n)=D_\gamma(v,A)$),

but it is not approximately compact (since for $v=(1,0,0,\dots)\in M_\gamma$, we have

$$\lim_{n\to\infty}\gamma(v-u_n)=D_\gamma(v,A),$$

But $<u_n>$ has no convergent subsequence, by virtue of the relation $\gamma\,(u_i-u_j)\neq 0$ (for i ≠□j).

Theorem (2.3.2):

Let$\emptyset \neq A$ be approximatively compact subset modular linear space M_γ. If $\gamma(u) < \infty$, for each u. Then P_A maps M_γ into $\mathrm{C}B(A)$, is $u.s.c$.

Proof:

By virtue of Theorem (2.3.1), A is proximal set, hence $P_A(v)$ is non – empty for each v in M_γ. By [proposition 2.1.1], $P_A(v)$ is closed and bounded thus $P_A(v)$ maps M into $CB(A)$.

Now, let K be an arbitrary closed subset of A. We show that the set

$B = \{v \in M_\gamma .: \ P_A(v) \cap K \neq \emptyset\}$

Is closed set, which will complete the proof ;

Let (v_n) be a sequence in B, converging to an element $v \in M_\gamma$.

Since $\langle v_n \rangle \subseteq B$, then there exists a sequence $(u_n) \subseteq A$ such that $u_n \in P_A(v_n) \cap K$, $(n = 1,2, \dots .)$

By $u_n \in P_A(v_n)$, $(n = 1,2, \dots .)$, we have

$$D_\gamma(v_n, A)\gamma(v_n - u_n) \mapsto \lim_{n\to\infty} D_\gamma(v_n, A) = \lim_{n\to\infty} \gamma(v_n - u_n)$$

$$\mapsto D_\gamma(v, A) = \lim_{n\to\infty} \gamma(v - u_n)$$

$$= \lim_{n\to\infty} \gamma(v - v_n + v_n - u_n)$$

$$\leq \lim_{n\to\infty} \gamma(v - v_n) + \lim_{n\to\infty} \gamma(v_n - u_n)$$

$$= 0 + \lim_{n\to\infty} \gamma(v - u_n)$$

$$= \quad D_\gamma(v, A)$$

Thus $\lim_{n\to\infty} \gamma(v - u_n) = \quad D_\gamma(v, A)$.Consequently, being an approximatively compact, then there exists a subsequence (u_{n_h}) of (u_n) converging to an element $u_\circ \in A$, which implies that there exists a subsequence (v_{n_h}) of(v_n).

Now, since $u_\circ \in A$, then

$$D_\gamma(v, A) \leq \gamma(v - u_\circ)$$

$$\leq \gamma\left(v - u_{n_h} + u_{n_h} - u_\circ\right)$$

$$\leq \gamma\left(v - u_{n_h}\right) + \gamma\left(u_{n_h} - u_\circ\right)$$

$$\leq \gamma\left(v - v_{n_h} + v_{n_h} - u_{n_h}\right) + \quad \gamma\left(u_{n_h} - u_\circ\right)$$

$$\leq \gamma\left(v - v_{n_h}\right) + \gamma\left(v_{n_h} - u_{n_h}\right) + \gamma\left(u_{n_h} - u_\circ\right)$$

$$= \gamma\left(v - v_{n_h}\right) + D_\gamma\left(v_{n_h}, A\right) + \gamma\left(u_{n_h} - u_\circ\right)$$

$$= D_\gamma(v, A) \leq \gamma(v - u_\circ)$$

For $h \to \infty$, $\gamma(v - u_\circ) = D_\gamma(v, A)$, that is$u_\circ \in P_A(v)$. On the other hand, since K is a closed and $\langle u_{n_h} \rangle \subseteq$ M, $\lim_{h\to\infty} u_{n_h} = u_\circ$ we have $u_\circ \in P_A(v) \cap K$, whence x $\in$B, which complete the proof. ■

Theorem (2.3.3):

Let $\emptyset \neq A$ be approximatively compact subset of a modular space M_γ, and $P_A : M_\gamma \to 2^A$ be the metric projection of M_γ onto A. Then $P_A(C) = \cup \{P_A(v) : v \in C\}$ is compact for any compact subset C of M.

Proof:

Let $\langle u_n \rangle$ be a sequence in $P_A(C)$. Then there is a sequenes $\langle v_n \rangle \subseteq C$ such that for each n

$u_n \in P_A(v_n)$, that is $\gamma(v_n - u_n) = D_\gamma(v_n, A)$.

Since C is compact, then we may assume that there is a $v \in C$ with $v_n \to v$. Now, $D_\gamma(v_n, A) = inf\{\gamma(v_n - u): u \in A\}$

$$= inf\{\gamma(v - u): u \in A\} = D_\gamma(v, A)$$

thus$(v_n, A) \to D_\gamma(v, A)$, and

$$D_\gamma(v, A) \leq \gamma(v - u_n) = \gamma(v - v_n + v_n - u_n)$$

$$\leq \gamma(v - v_n) + \gamma(v_n - u_n)$$

$$= \gamma(v - v_n) + D_\gamma(v_n, A)$$

therefore$D_\gamma(v, A) \leq \gamma(v - u_n) \leq \gamma(v - v_n) + D_\gamma(v_n, A)$

$$\lim_{n\to\infty} D_\gamma(v, A) \leq \lim_{n\to\infty} \gamma(v - u_n) \leq \lim_{n\to\infty} \gamma(v - v_n) \leq \lim_{n\to\infty} D_\gamma(v_n, A)$$

$$D_\gamma(v, A) \leq \lim_{n\to\infty} \gamma(v - u_n) \leq 0 + D_\gamma(v, A)$$

$$D_\gamma(v, A) = \lim_{n\to\infty} \gamma(v - u_n)$$

Since $\langle u_n \rangle \subseteq P_A(C) \subseteq A$ and A is an approximatively compact set, then the above relation implies the existence of $u \in A$ and subsequence$\langle u_{n_i} \rangle$ of $\langle u_n \rangle$ with $u_{n_i} \to u$. This prove that $P_A(C)$ is compact subset of M_γ. ■

CHAPTER 3

FIXED POINTS, COMMON FIXED POINTS AND BEST APPROXIMATIONS

3-0 Introduction

The purpose of this chapter is to study the existence of an invariant best approximation in the setting of a modular space for single valued or set-valued mappings by weakening the hypothesis in some known results or form new cases which guarantee the existence of an invariant best approximation. These results hold by applying some fixed point theorems and common point theorems. This chapter contains three sections where in section one, there is a generalization of fixed point theorem for non-expansive mappings and the use it to extend and unified the above results [43], [14] and [2]. In section two, two common fixed point theorems for P-non-expansive mapping defined on a star-shaped weakly compact subset are proved, Here the conditions of affineness and demi-closedness and Opial's property play an active role in the proving our results will be general case for the other results. The object of section three is to prove the existence of best approximations by applying a common fixed point theorem without any one of star-shapedness, affineness and commuting conditions by using property of non-convexity which is given by Dotson [13], say (w)-convex structure. Therefore the results of this section will be the extension of Nashine's results [33]

3.1 An Extension of Brosowski $-$ Meinaraus Theorem in Modular Spaces

Mongkolkeha, Sintunavarat and Kumam [26] showed that existence of $v \in M_\gamma$ with $\gamma(Sv) < \infty$ is necessary to guarantee fixed point. The result in Proposition (2.1.1) also hold, if we replace this condition by boundeness of modular function γ.

Definition (3.1.1):

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$. $S: A \to A$, S is called Banach operator of modular space if

$$\gamma(Sv - S^2v) \leq h\,\gamma(v - Sv)$$

for all $v \in A$ where h is constant with $0 < h < 1$.

Proposition (3.1.1):

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$. S: $A \to A$ a continuous Banach operator. $\exists v \in A \ni (Sv) < \infty$. Then S has fixed point in A.

Proof:

Since $(Sv - S^2v) \leq h\,\gamma(v - Sv)$, by adding $\gamma(v - Sv)$ to both sides, we get

$$\gamma(Sv - S^2v) + \gamma(v - Sv) \leq h\,\gamma(v - Sv) + \gamma(v - Sv)$$

which can be rewritten as

$$\gamma(v - Sv) - h\,\gamma(v - Sv) \le \gamma(v - Sv) - \gamma(Sv - S^2v)$$

$$\gamma(v - Sv)\,[1 - h] \le \gamma(v - Sv) - \gamma(Sv - S^2v)$$

$$\gamma(v - Sv) \le [1 - h]^{-1}[\gamma(v - Sv) - \gamma(Sv - S^2v)]$$

Now define the function $Q: M_\gamma \to R^+$by setting

$Q(v) = (1 - h)^{-1}\,\gamma(v - Sv), v \in M_\gamma$

Thus, $\gamma(v - Sv) \le Q(v) - Q(Sv)$. Therefore if $v \in M_\gamma$ and, $n \in N$ with $n < m$

$\gamma\,(S^{n+1}v - S^{m+1}v) \le \sum_{i=n}^{m} \gamma\left(S^i v - S^{i+1}v\right) \le Q(S^n v) - Q(S^{n+1}v)$

In particular, by taking n= 1 and letting m→ ∞ we conclude that

$\sum_{i=1}^{\infty} \gamma\left(S^i v - S^{i+1}v\right) \le Q(Sv) < \infty$

This implies that $\{S^n v\}$ is Cauchy sequence, since $\overline{S(A)}$is complete there exist $v_0 \in M_\gamma$ such that $\lim_{n\to\infty} S^n v = v_0$ and since S is continuous

$$v_0 = \lim_{n\to\infty} S^n v = \lim_{n\to\infty} S^{n+1} v = S\,v_0$$

Thus v_0 is fixed point of S.

The above theorem Remains true when A is closed subset of modular space M_γ and $\overline{S(A)}$is compact this fact with Proposition (3.1.1) we get the following extending of Dotson's theorem ([13], Theorem 2) for non-expansive mappings in modular spaces.

Theorem (3.1.1):

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$ and A a closed and star -shaped is non – expansive mapping with $\overline{S(A)}$ is compact and exist $v \in M_\gamma \ni \gamma(Sv) < \infty$, then S has a fixed point in A.

Proof:

Let u be a star –center of A, for each $n \geq 1$ define S_n by

$$S_n(v) = (1 - h_n)\, u + h_n\, S(v) \text{for all } v \in A,$$

when $\{h_n\}$ is a sequences of real numbers with $0 \leq h_n < 1$and $\lim_{n\to\infty} h_n = 1$. Clearly, $S_n : A \to A$, for each n.

Now, since $S$is non-expansive, for any $n \geq 1$ and $v \in A$, we get

$$\gamma(S_n\, v - S_n^2\, v)$$

$$= \gamma[\big((1 - h_n)u + h_n\, Sv\big) - S_n((1 - h_n)u + h_n\, Sv)]$$

$$= \gamma[\big((1 - h_n)u + h_n\, Sv\big) - (h_n\, S\, (h_n\, Sv + (1 - h_n)u) - (1 - h_n)u\,)]$$

$$= h_n\gamma\big(Sv - S(h_nSv + (1 - h_n)u)\big)$$

$$\leq h_n\gamma(v - h_nSv + (1 - h_n)u)$$

$$= h_n\gamma(v - S_nv)$$

Since S is continuous mapping then $S_n$is continuous, since $\overline{S(A)}$ is compact then $(1 - h_n)\ u + h_n\, Sv$ is compact. Therefore, by Proposition (3.1.1) there exist $v_n \in A$ such that$S_nv_n = v_n\,, n \geq 1$. By compactness of $\overline{S(A)}$, $\{Sv_n\}$has a convergent subsequence $\{Sv_{n_i}: i \geq 1\}$ with $\lim_{i\to\infty} Sv_{n_i} = v$ in A. Since

$$v_{n_i} = S_{n_i}v_{n_i} = \big(1 - h_{n_i}\big)u + h_{n_i}Sv_{n_i}$$

and $\lim_{i\to\infty} h_{n_i} = 1$, we have $v_{n_i} \to v$. Consequently $\lim_{i\to\infty} Sv_{n_i} = v$■

In the following example, we say that theorem(3.1.1) need not true if either A is not closed, star-shaped or $\overline{S(A)}$ is not compact. Consider

$M_\gamma = R^2$ and $\gamma(v-u) = |v| + |u|$, for all $v,u \in R^2$.

Example (3.1.1):

Let $A = \{(v{-}u) \in M_\gamma\colon 0 < v < 1,\ 0 < u < 1\}$ and $S\colon A \longrightarrow A$ defined by $S(v{-}u) = (v/3, u/4)$, $(v{-}u) \in A$.

It is clear that A is not closed and S is non-expansive mapping and has no fixed point.

Let (v, u), $(z, u) \in A$

$$\gamma(S(v,u) - S(z,y)) = \gamma\left(\left(\frac{v}{3},\frac{u}{4}\right) - \left(\frac{z}{3},\frac{y}{4}\right)\right)$$

$$\leq \left|\frac{1}{2}\right| \gamma((v-z),(u-y))$$

$$= \left|\frac{1}{2}\right| \gamma((v,u)-(z,y))$$

$$\leq h((v,u)-(z,y))$$

and (0,0) is fixed point of S. But $(0,0) \notin A$. ■

Example (3.1.2):

Let $A = E \cup F$, where $E = \{(v{-}u) \in M_\gamma\colon 0 \le v \le 1,\ 0 \le u \le 6\}$ and $F = \{(v{-}u) \in M_\gamma\colon 3 \le v \le 4,\ 0 \le u \le 6\}$, and $S : A \longrightarrow A$ defined by

$$S(v-u)=\begin{cases}(2,u) & \text{if } (v,u)\in E\\ (1,u) & \text{if } (v,u)\in F\end{cases}$$

It is clear that S is non-expansive mapping and has no fixed point.

A has no star-shaped since $\forall$ $u\in A$, $(v,u) \in A$, then h_n $S(v-u)+(1-h_n)u\notin A$, where $h_n \in (0,1)$ and $\lim_{n\to\infty} h_n = 1$.

Example (3.1.3):

Let $A = \{(v-u) \in M_\gamma: 0 \leq v \leq \infty, 0 \leq u \leq 1\}$ and $S : A \longrightarrow A$ defined by $S(v-u) = (v + 1,u)$, $(v-u) \in S$. Then $\overline{S(A)} = \{(v-u) \in M_\gamma: 1 \leq v < \infty, \ \ 0 \leq u \leq 1\}$.

It is clear that S is non-expansive mapping, has no fixed point, and $\overline{S(A)}$ is not compact.

Theorem (3.1.2):

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$ and $S: M_\gamma \to M_\gamma$ a non – expansive mapping with a fixed point $v \in M_\gamma$ and exist $v \in M_\gamma$ such that $\gamma(Sv) < \infty$. If A is closed S- invariant of M_γ and the restriction $S|A$ is compact , then the set $P_A(v) \neq \emptyset$.

Proof:

Let $\delta = D_\gamma(v, A)$. Then there exists sequence $\langle u_n \rangle$in A such that $\lim_{n\to\infty} D_\gamma(v, u_n) = \delta$. Which implies that $\langle u_n \rangle$ is bounded sequence. By hypothesis, $\overline{\{Su_n\}}$ is a compact subset of $A$and so $\{Su_n\}$ has a convergent subsequence $\{Su_{n_i}: i \geq 1\}$ with $\lim_{i\to\infty} Su_{n_i} = u$, say, in A.

Therefore,

$\delta \leq D_\gamma(v, u)$

$$= \lim_{i\to\infty} D_\gamma\left(Sv, Su_{n_i}\right) \leq \lim_{i\to\infty} D_\gamma\left(v, u_{n_i}\right) = \lim_{n\to\infty} D_\gamma(v, u_n) = \delta$$

Hence $D_\gamma(v, u) = \delta$ and then $u \in P_A(v)$. This complete the proof•

Using Theorem (3.1.1) and Theorem (3.1.2) to prove the following:

Theorem (3.1.3):

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$, and $S : M_\gamma \to M_\gamma$, A is non – expansive mapping with a fixed point $v \in M_\gamma$ and exist $v \in M_\gamma$ such that $\gamma(Sv) < \infty$. If A is a closed with $S|A$ compact, $S(A) = A$ and $P_A(v)$ is star-shaped, then there exist an element in $P_A(v)$ which also a fixed point of S.

Proof:

Let $Z = P_A(v)$, by proposition (3.1.2) then $Z \neq \emptyset$, let $\in Z$.

Set $\delta = D_\gamma(v, A)$. Then

$D_\gamma(u, v) = D_\gamma(v, A)$

since $u \in Z$ then $u \in A$ and $S(A) = A$ therefore $Su \in A$.

Now, since $D_\gamma(v, Su) \geq \delta$ … (3.1)

and $D_\gamma(v, Su) = D_\gamma(Sv, Su)$, also

$D_\gamma(Sv, Su) \leq D_\gamma(v, u)$ … (3.2)

Therefore by (3.1) and (3.2), we have

$\delta \leq D_\gamma(v, Su) = D_\gamma(Sv, Su) \leq D_\gamma(v, u) = \delta$

hence, $\delta \leq D_\gamma (v, Su) \leq \delta$

Thus $D_\gamma (v, Su) = \delta r = D_\gamma (v, A)$, therefore $Su \in Z$

If Z is singleton, i.e., $Z = \{u\}$ and $Su \in Z$ then Su $= u$.

Now, by definition $S|A$ then $\overline{S(A)}$ is compact. Since A is closed and have all conditions in theorem (3.1.1) then exist $v \in Z$ such that $Z = v$.

3.2 Common Fixed Point for Commuting Mappings

Definition (3.2.1):

Let M_γ be a modular space and P, S: $M_\gamma \rightarrow M_\gamma$ be a mappings then S is said to be P – contraction if there exists h$\in (0, 1)$ such that

$\gamma(Sv - Su) \leq h\, \gamma(Pv - Pu)\ \forall\ v, u$ in M_γ If h= 1 in then S is called P– non– expansive mapping.

Definition (3.2.2):

A two mappings S and P on M_γ are said to be commute if $SPv = PSv\ \forall\ v \in M_\gamma$

Proposition (3.2.1):

Let P be a continuous self-mapping of Banach operator of M_γ , if S: $M_\gamma \rightarrow M_\gamma$is P- contraction mapping which commutes with P and $S(M_\gamma) \subseteq P(M_\gamma)$ and $\exists\ v \in M_\gamma$ such that $\gamma(P(v)) < \infty$ then

$F(P) \cap F(S) =$ singleton.

Proof:

Suppose $\mathrm{P}(a) = a$ for some $a \in M_\gamma$, define S: $M_\gamma \to M_\gamma$ by $S(v) = a \ \forall\ v \in M_\gamma$ then $S(P(v)) = a$ and $P(S(v)) = P(a)$ for all $v \in M_\gamma$ so $S(P(v)) = P(S(v)) \ \forall\ v \in M_\gamma$ and S commutes with P moreover $S(v) = a = P(a) \ \forall\ v \in M_\gamma$ so that

$S(M_\gamma) \subseteq P(M_\gamma)$ finally for any $a \in (0,1)$ we have $\forall$ v,u in M_γ:

$\gamma(S(v) - S(u)) = \gamma(a - a) = 0 \le a\,\gamma(P(v) - P(u))$. Thus holds this proof.

The following lemma is needed.

Lemma (3.2.1):

Let M_γ be a modular space, S: $M_\gamma \to M_\gamma$ be mapping, and $u \in M_\gamma$. If $S(hu + (1-h)v) = hSu + (1-h)v$, $\forall v \in M_\gamma$ and $h \in (0,1)$, then u is a fixed point.

Definition (3.2.4):

Let M_γ be a modular space and $\emptyset \neq A \subset M_\gamma$ and S: $A \to M_\gamma$ be a mapping, S is called demi-closed of $v \in A$, if for every sequence (v_n) in A such that $v_n \xrightarrow{w} v$ and $v_n \to u \in M_\gamma$ then $u = Sv$ and S is demi-closed on A if it is demi-closed of each v in A.

Theorem (3.2.1):

Let $\emptyset \neq A$ weakly compact subset of Banach operator . Let P be a continuous and affine mapping on M_γ with $\mathrm{p}(A) = A$, S: $A \to A$ be an P-non – expansive mapping commutes with P. If A is star-shaped with respect to S,and there is some $v \in A$ $\gamma\big(S(v)\big) < \infty$ and $(P - S)$ is demi-closed on M_γ, then $F(S) \cap F(P) \neq \emptyset$.

Proof:

Since A is star-shaped with respect to $u \in A$, then S: $A \to A$, we define S_n on A for any v in A by, $S_n(v) = h_n Sv + (1 - h_n)u$ and there is $u \in A$, and the sequence $h_n \to 1$ as $n \to \infty$, $0 < h_n < 1$ such that $(1 - h_n)u + h_n Sv \in A \;\forall\; v,u \in A$. It is clear that $S_n : A \to A$.

Note that $S(A) \subseteq A$ and $S_n(A) \subseteq P(A)$. Since S commutes with P and P is affine mapping, for each $v \in A$.

$$S_n Pv = h_n Spv + (1 - h_n)Pu$$

$$= h_n PSv + (1 - h_n)Pu$$

$$= P\big(h_n Sv + (1 - h_n)\big)$$

$$= PS_n v$$

$\ni$ $S_n$commutes with P. Further, we observe that for each $n \geq 1$, S is P- non-expansive mapping,

$$\gamma(S_n v - S_n u) = \gamma(h_n Sv + (1 - h_n)u - h_n Su - (1 - h_n)u)$$

$$= h_n \gamma(Sv - Su)$$

$$\leq h_n \gamma(Pv - Pu)$$

$\forall\; v, u \in A$ hence $S_n$is P- contraction. Thus by proposition (3.2.1),

there is a unique $v_n \in A$ such that $v_n = S_n = Pv_n$ for all $n \geq 1$.

Since A is weakly compact, there is a subsequence $\big(v_{n_i}\big)$ of sequence (v_n) which converges weakly to some $v_0 \in A$.

Since P is a continuous affine mapping then P is weakly continuous and so, since $Sv_{ni} = \frac{S_{ni}\, v_{ni}\ +(1-h_{ni})u}{h_{ni}}$ and $Pv_{ni} = v_{ni}$.

Now, $(P-S)v_{ni} = Pv_{ni} - Sv_{ni}$

$$= v_{ni} - \left(\frac{S_{ni}v_{ni} - (1-h_{ni})u}{h_{ni}}\right)$$

$$= \frac{h_{ni}v_{ni} - S_{ni}v_{ni} + (1-h_{ni})u}{h_{ni}}$$

$$= \frac{-v_{ni}(1-h_{ni}) + (1-h_{ni})u}{h_{ni}}$$

$$= \frac{(1-h_{ni})(u-v_{ni})}{h_{ni}}$$

$$= \frac{(1-h_{ni})}{h_{ni}}(u-v_{ni})$$

$$= \left(\frac{1}{h_{ni}} - 1\right)(u-v_{ni})$$

Therefore $(P-S)v_{ni} = \left(\frac{1}{h_{ni}} - 1\right)(u-v_{ni})$

Thus $(P-S)v_{ni} = \left|\frac{1}{h_{ni}} - 1\right| \gamma(u-v_{ni}) \leq \left|\frac{1}{h_{ni}} - 1\right| [\gamma(v_{ni}) + \gamma(u)]$.

Since A is bounded, $v_{ni} \in A$ implies $(\gamma(v_{ni}))$ is bounded and so by the fact that $h_{ni} \to 1$,

We have $\gamma(P-S)v_{ni} \to 0$

Now, since $P{-}S$ is demi-closed then $(P-S)v_0 = 0$ and thus $Pv_0 = v_0 = Sv_0$. Hence, $F(S) \cap F(P) \neq \emptyset$.

Another common fixed point theorem will be given for opial space.

Definition (3.2.5):

A modular space M_γ is said to be Opial of modular space if for every sequence (v_n) in M_γ weakly convergent to $v \in M_\gamma$ the inequality

$\lim_{n\to\infty} \inf \gamma(v_n - v) < \lim_{n\to\infty} inf \gamma(v_n - u)$ Holds for all $u \neq v$.

Theorem (3.2.2):

Let$\emptyset \neq A$ weakly compact subset of Banach operator . Let P be a continuous and affine mapping on M_γ with $P(A) = A$, S: $A \to A$ be P-non-expansive mapping commutes with P.∃v $\in A \ni \gamma\big(S(v)\big) < \infty$ and the modular space M_γ is Opial. If A is star-shaped with respect to S, then $F(S) \cap F(P) \neq \emptyset$.

Proof:

Since A has star-shaped then S:$A\to A$ and there is $u \in A$ and the sequence $h_n \to 1$, as $n\to \infty$, $(0 < h_n < 1) \ni (1 - h_n)u + h_n Sv \in A$ for all $v\in A$. Now, define $S_n$on A for any v in A by, $S_n(v) = h_n Sv + (1 - h_n)u$ and there is $u\in A$, it is clear that $S_n: A \to A$. Note that $S(A) \subseteq A$ and $S_n(A) \subseteq P(A)$. Since S commutes with P and P is affine mapping, for each $v \in A$.

$$
\begin{aligned}
S_n P v &= h_n SPv + (1 - h_n)Pu \\
&= h_n PSv + (1 - h_n)Pu \\
&= P(h_n Sv + (1 - h_n)u) \\
&= PS_n v
\end{aligned}
$$

Thus each h_n commute with P. Further observe that for each $n \geq 1$, S is P – non-expansive mapping.

$$\gamma(S_n v - S_n u) = \gamma(h_n Sv + (1-h_n)u - h_n Su - (1-h_n)u)$$

$$= h_n \gamma(Sv - Su)$$

$$\leq h_n \gamma(Pv - Pu)$$

$\forall\ u \in\ A$, hence S_n is P- contraction.

Thus by proposition (3.2.1), there is a unique $v_n \in A$ such that $v_n = S_n\, v_n = Pv_n$ for all $n\ \geq 1$. Since A is weakly compact, there is a subsequence (v_{ni}) of sequence (v_n) which converges weakly to some $v_0 \in A$. Since P is a continuous affine mapping then P is weakly continuous and so we have :

$$Pv_0 = \lim \quad Pv_{ni} = \lim \quad v_{ni} = v_0$$

Since $Sv_{ni} = \frac{S_{ni} v_{ni} + (1-h_{ni})u}{h_{ni}}$ and $Pv_{ni} = v_{ni}$, we have:

$$(P-S)v_{ni} = Pv_{ni} - Sv_{ni}$$

$$= v_{ni} - \left(\frac{S_{ni} v_{ni} + (1-h_{ni})u}{h_{ni}}\right)$$

$$= \frac{h_{ni} v_{ni} \ - v_{ni} + (1-h_{ni})u}{h_{ni}} \quad = \frac{-v_{ni}(1-h_{ni}) + (1-h_{ni})u}{h_{ni}}$$

$$= \frac{(1-h_{ni})(u - v_{ni})}{h_{ni}}$$

$$= \frac{(1-h_{ni})}{h_{ni}}(u - v_{ni})$$

$$(P-S)v_{ni} = \left(\frac{1}{h_{ni}} - 1\right)(u - v_{ni})$$

Therefore $(P-S)v_{ni} = \left(\frac{1}{h_{ni}} - 1\right)(u - v_{ni})$.

Thus $\gamma(P-S)v_{ni} = \left|\frac{1}{h_{ni}} - 1\right| \gamma(u - v_{ni}) \leq \left|\frac{1}{h_{ni}} - 1\right| [\gamma(v_{ni}) + \gamma(u)]$.

Since A is bounded by A is weakly compact, $v_{ni} \in A$ implies $\left(\gamma(v_{ni})\right)$ is bounded and so by the fact that $h_{ni} \to 1$, we have

$$\gamma(P-S)v_{ni} \to 0$$

Now, since M_γ is Opial space and suppose that, $Sv_0 \neq v_0$ we have:

$$\lim_{i\to\infty} inf \gamma(v_{ni}-v_0) < \lim_{i\to\infty} \inf \gamma(v_{ni}-Sv_0)$$

$$= \lim_{i\to\infty} \inf \gamma(Sv_{ni}+(P-S)v_{ni}-Sv_0)$$

$$\leq \lim_{i\to\infty} \inf \gamma(Sv_{ni}-Sv_0) + \lim_{i\to\infty} inf \gamma(P-S)v_{ni},$$ since

$v_{ni} = (P-S)v_{ni} + Sv_{ni}$

And thus

$$\lim_{i\to\infty} inf \gamma(v_{ni}-v_0) < \lim_{i\to\infty} \inf \gamma(Sv_{ni}-Sv_0)$$

But on the other hand we have

$$\lim_{i\to\infty} \inf \gamma(Sv_{ni}-Sv_0) \leq$$
$$\lim_{i\to\infty} \inf \gamma(Pv_{ni}-Pv_0) = \lim_{i\to\infty} \inf \gamma(v_{ni}-v_0)$$

Which is a contradiction. Hence $v_0 \in F(S) \cap F(P) \Rightarrow F(S) \cap F(P) \neq \emptyset$.

Lemma (3.2.2):

Let A be a subset of modular space M_γ. Then for any $v \in M_\gamma$, $P_A(v) \subseteq \partial A$.

Proof:

Let $u \in P_A(v)$, then every neighborhood of u contains a point strictly between u and v on $\gamma(v-u)$.Since u is best approximation to v then is closer to v than u, so, it cannot be in A. Thus u is not interior of A. Then $u \in \partial A$.

Corollary:

Let M_γ complete opial space, let S and $P: M_\gamma \to M_\gamma$ and $A \subseteq M_\gamma \ni S(\partial A) \subseteq A$ and $v \in F(S) \cap F(P). \emptyset \neq P_A(v)$ weakly compact, is star-shaped to S and $q \in F(P)$ and let S be a P-non-expansive mapping on $P_A(v) \cup \{v\}$, $\exists v \in A \ni \gamma\big(S(v)\big) < \infty$, where P is affine, continuous on $P_A(v)$, $P\big(P_A(v)\big) = P_A(v)$ and commute with S on $P_A(v)$ then $P_A(v) \cap F(S) \cap F(P) \neq \emptyset$.

3.3 A Best Approximations for (w) Convex Set

Definition (3.3.1):

A family of maps $\{P_\alpha\}$ $\alpha \in M_\gamma$ is said to be (w)-convex structure on modular space M_γ, if it satisfies the following conditions:

i. P_α: $[0, 1] \to M_\gamma$, i.e. P_α is map from $[0, 1]$ into M_γ for each $\alpha \in M_\gamma$,

ii. P_α (1) = α for each $\alpha \in M_\gamma$,

iii. $P_\alpha(t)$ is a jointly continuous in (α, t), i.e., $P_\alpha(t) \to P\alpha_0(t_0)$ for $\alpha \to \alpha_0$ in M_γ and $t \to t_0$ in [0, 1],

iv. If P is a map from M_γ into itself, then for any $v \in M_\gamma$, $P_{sv}(t) \subseteq Sv \; \forall \; t \in [0, 1]$,

v. $\gamma(P_\alpha(t)- P_\beta(t)) \leq [\phi(t)]\, \gamma(\alpha - \beta)$, where ϕ is function from [0, 1] into itself.

Now, we recall the following definition.

Definition (3.3.2):

Let $\{P_\alpha\}$ be a sequence of (w) – convex structure on a modular space M_γ. A self-mapping S of M_γ is said to satisfy the property (I), if for any $t \in [0, 1]$, $\forall\, v \in M_\gamma$ and $\forall\, Pv$ we have $S\,(P_v\,(t)) = P_{sv}(t)$.

Remark (3.3.1):

It is clear that the commute pair (S, P) is Banach operator but the converse is not true.For convers, one can see the following simple example:

Example (3.3.1):

Consider P, S in modular space $M_\gamma = [0, 1]$ as
$P\,(v) = 1 - v$ and

$$S(v) = \begin{cases} 1 - v & 0 \le v \le \frac{1}{2} \\ \\ 1 - \frac{v}{2} & \frac{1}{2} < v \le 1 \end{cases}$$

It is clear that P and S are not commute and $(\mathrm{F}(P)) = \mathrm{F}(P) = \left\{ \frac{1}{2} \right\}$.

In the next work, we quote the condition of Banach operator of modular space and incorporate it with (w)- convexity condition to give two results in invariant best approximation.

Theorem (3.3.1):[40]

Suppose S and P are two self-mapping of a closed subset A of the metric space M such that (S, P) is Banach operator pair on A and S is Q-contraction on A, if $F(P) \neq \emptyset$ and $\overline{S(A)}$ is complete, then $F(S) \cap F(P) =$ singleton

Theorem (3.3.2):

Let M_γ be a modular space with (w)-convex structure. Let S, $P: M_\gamma \to M_\gamma$ be Banach operator and $A \subseteq M_\gamma$ such that $S(\partial A) \subseteq A$. let $v_0 \in$ F $(S) \cap$ F (P). Suppose that S is h-non-expansive mapping on $P_{\mathrm{A}}(vo) \cup \{v_0\}$,with $S\big(F(P)\big) \subset F(P)$ P is continuous and $S(\mathrm{F}\ (P)) \subseteq F(P)$ on $P_{\mathrm{A}}(v_0)$, $(\mathrm{P}_{\mathrm{A}}(v_0))$ is compact. If $P_{\mathrm{A}}\ (v_0) \neq \emptyset$, closed,$\exists v \in A \ni \gamma\big(S(v)\big) < \infty$ and $h\ (\mathrm{P}_{\mathrm{A}}(v_0)) \subseteq \mathrm{P}_{\mathrm{A}}(v_0)$ then $\mathrm{P}_{\mathrm{A}}(v_0) \cap$ F $(S) \cap$ F $(P) \neq \emptyset$.

Proof:

Let D = $P_A(v_0)$. First, we show that S:D → D. Let $u \in$ D then $u \in \partial A$ by Lemma (3.2.2). Also, since $S\ (\partial A) \subseteq A$ then $Su \in A$.

Now, since $Pu \in$ D by P (D) $\subseteq$ D and since $Sv_0 = v_0$ and S, P non-expansive mapping, we have

$$\gamma(Sv - vu) = \gamma(Su - Sv_0) \leq \gamma(P\ u - P\ v_0)$$

As $P\ v_0 = v_0$ we therefore have

$$\gamma(Sv - vu) \leq \gamma(P\ u - v_0) = D_\gamma(v_0,\ A)$$

Thus Su is also closest to v_0, so $Su \in$ D.

By (w) – convexity property (I) there is a family $\{P_v\}$ $v \in$ D satisfies condition of definition (3.3.1), choose $h_n \in (0,1)$ such that $<h_n> \to 1$, and define S_n as $S_n\ (v) = Ps_u\ (P\ _n)$, for all $v \in$ D.

It is clear that S_n is well-defined map from D into D for each n,

Now, we have S_n, S, P: D → D and S (F(P)) ⊆ F(P) on D ∀ v, $u \in$ D,

For each n, we have

$$\gamma(S_n v \text{ -} S_n u) = \gamma(Ps_v\ (P\ _n) \text{ - } Ps_u\ (P\ _n))$$
$$\leq [\phi\ (h_n)]\ \gamma(Sv - vu)$$
$$\leq [\phi\ (h_n)]\ \gamma(P\ v - P\ u)$$

i.e.,

$$\gamma\ (S_n v \text{ -} S_n u) \leq [\phi\ (h_n)]\ \gamma\ (P\ \ v - P\ u) \text{ for all } v, u \in \text{D}$$

Hence S_n is P -contraction on D.

Now, we have to show that S_n(F (P)) ⊆ F(P), if $v \in$ F(P) then $Sv \in$ F(P) by S F(P) ⊆ F(P), and $S_n\ (v) = Ps_v\ (P\ _n)$ then $Sv\ (P\ _n) \subseteq Sv$ and $Sv \in$ F(P), implies $S_n(w) \in$ F(P). Hence (S_n, P) is Banach operator on D.

Since $\overline{S\ (\text{D})}$ is compact, each $\overline{S_n(\text{D})}$ is compact, hence $\overline{S_n(\text{D})}$ is complete.

By theorem (3.3.1), there exists $v_n \in$ D and $S_n\ v_n = P\ v_n = v_n$ for all $n \in$. Since $\overline{S_n(\text{D})}$ is compact, there is a subsequence (Sv_{n_1}) of a sequence (Sv_n) which converges to $u \in$ A.

$$v_{n_i} = P\ v_{n_i} = S_n\ v_n = s_{v_{n_i}}(P\ _{n_i})$$

By the continuity of S, $\{\ v_{n_i}\}$ converges to Su. But Sv_{n_i} tends to u by the assumption,

$$S_{n_i}\ v_{n_i} = Ps_{v_{n_i}}\ (P\ _{n_i}) \to Ps_u\ (1) = Su, \text{ as } i \to \infty$$

Thus, $Su = u$. Also from the continuity of h, we have

$P\ u = P$ $(\lim v_{n_i}) = \lim P\ \ v_{n_i} = \lim v_{n_i} = u$, as $i \to \infty$, i.e. $P\ u = u$.

Hence D ∩ F (S) ∩ F $(P\ ) \neq$ ϕ. This complete the proof. ■

Also, we have another result on an invariant best approximation.

Theorem (3.3.3):

Let M_γ be a modular space with (w)-convex structure. Let S, P: $M_\gamma \to M_\gamma$ and $A \subseteq M_\gamma$ such that $S\ (\partial A) \subseteq A$

Let $v_0 \in$ F (S) ∩ F $(P\ )$. Suppose that S is P -non-expansive mapping on

$\mathrm{P_A}(\ (v_0)\ \cup\ \{v_0\})$, P is weakly continuous. If $\mathrm{P_A}(v_0) \neq \emptyset, \gamma\big(S(v)\big) < \infty$ weakly compact. If P $(\mathrm{P_A}\ (v_0)) \subseteq \mathrm{P_A}\ (v_0)$ and S (F $(P\ )) \subseteq$ F (P) on $\mathrm{P_A}\ (v_0)$, then $\mathrm{P_A}\ (v_0)$ ∩ F (S) ∩ F $(P\ ) \neq \emptyset$ provided $(P - S)$ is demi-closed.

Proof:

Let D = $\mathrm{P_A}\ (v_0)$. First, we show that S is a self-mapping on D. let $u \in$ D then $u \in \partial A$ Lemma (3.2.2). Also, since $S\ (\partial A) \subseteq A$ then $Su \in A$.

Now, since $P\ u \in$ D by P (D) ⊆ D and $S_{v_o} = v_o$ and S is P -non-expansive

Mapping, we have

$$\gamma(Su - v_o) = \gamma\ (Su - Sv_o) \leq \gamma\ (P\ u - P\ \ v_0)$$

As $P\ \ v_0 = \ v_0 \to$ note we therefore have

$$\gamma(Su - v_o) \leq \gamma(P\ u - \ v_0) = (v_o, A)$$

Thus Su is also closet to v_o, so $Su \in$ D. By (w)-convexity property(I) there is a family $\{P_v\}_{v \in D}$ satisfies condition of definition (3.3.1), choose $h_n \in (0, 1)$ such that $< h_n > \rightarrow 1$, and define S_n as $S(v) = P_{Sv}(P_n)$, $\forall\ v \in$ D. It is clear that S_n: D→ D is well defind $\forall n$. $\forall\ v, u \in$ D, for each n, we have

$$\gamma(S_n v - S_n u) = \gamma(P_{Sv}(P_n) - P_{Su}(P_n))$$

$$\leq [\phi\ (h_n)]\ \gamma(Sv - Su)$$

$$\gamma(S_n v - S_n u) \leq [\phi\ (h_n)]\ \gamma(P\ v - P\ u)$$

i.e.,

$$\gamma(S_n v - S_n u) \leq [\phi\ (h_n)]\ \gamma(P\ v - P\ u)\ \forall\ v, u \in \text{D}.$$

Hence S_n P -construction on D.

Now, we have to show that (F(P)) ⊆ F(h), if $s \in$ F(h) then $S_v \in S$(F(P))

By (F(P)) ⊆ F(P), $S_n\ (v) = f_{Sv}(P_n)$ then $f_{Sv}(P_n) \subseteq S_v$ and $S_v \in$ F(P),

Implies $S_n\ (v) \subseteq$ F(P), therefore S_n (F(P)) ⊆ F(P).

Now, we have S_n, S, P : D→D and hence (S_n, P) is Banach operator on D. Since (D) ⊆ D ⊆ M_γ then $\overline{S_n(\text{D})} \subseteq M_\gamma$ and M_γ is a complete then $\overline{S_n(\text{D})}$ is complete. By theorem (3.3.1), we conclude that, there exists $v_n \in$ D and $S_n v_n = P\ v_n = v_n$ for all $n \in N$. Since D weakly compact, there is a subsequence (v_{n_i}) of sequence (v_n) which converges to $u \in A$.

$$v_{n_i} = P\ \ v_{n_i} = S_{n_i} v_{n_i} = P_{S_{v_{n_i}}}\ (P_{n_i})$$

From the weakly continuity of P , we have

$P\ u = P$ (lim v_{n_i}) = lim $P\ v_{n_i} = v_{n_i} = u$, as $i \rightarrow \infty$, i.e. $P\ u = u$.

Now we have to show that lim $(P \ - S)\ v_{n_i} = 0$

$(P \ - S)\ v_{n_i} = P\ v_{n_i}$ - $Sv_{n_i} = v_{n_i}$ - $Sv_{n_i} = P_{S_{v_{n_i}}}(P\ {n_i})$ - Sv_{n_i}, thus

$\lim_{i\to\infty}(P \ - \mathrm{S})v_{n_i}$= lim $P_{S_{v_{n_i}}}\ (h_{n_i})$ – lim Sv_{n_i}

$$= S_u(1) - Su$$

$\lim_{i\to\infty}(P \ - \mathrm{S})v_{n_i} = Su - Su = 0$.Now, $(P \ - S)$ is demi-closed at 0 and sequence converges weakly to u.

$(P \ - \mathrm{S})\ u = 0$ implies that $u = Su$

Hence u is fixed point of S in D. Hence D ∩ F (S) ∩ F $(P\ ) \neq \emptyset$.

CHAPTER 4

INVARIANT BEST APPROXIMATION FOR NON-EXPANSIVE MAPPINGS

4-0 Introduction

Throughout this chapter, the definitions of a (P,Q) $-$contraction mappings and a generalized (P,Q)-contraction in the setting of modular spaces are presented and common fixed points and coincidence theorems for these mappings are applied to have many results on invariant best approximation. Here, the condition of P and Q are commuting is replaced with weakly compatible (in special case to C_u-subcommuting, R-subcommuting or R-subweakly commuting). In section one, theorems about common fixed point and coincidence point for (P,Q)-nonexpansive mapping and proved which are general cases for the results in [41], [36], [37] and [6] these theorems are employed to get invariant approximations. In section two, with the same above hypotheses, some results of previous section are extended for a generalized (P,Q)nonexpansive mapping. This results will be a general case for results in [41], [42] and other special case. Finally, in section three the conditions of a fineness' is also omitted in addition to non-commute non-convexity and replaced by the (w)-convexity property to have more general results in invariant best approximation for (P,Q) $-$nonexpansive mappings.

4.1 Coincidence Points for (P, Q) −Non-expansive Mappings and Best Approximations

Definition (4.1.1):

An element u of a modular space M_γ is called a coincidence point of the pair of mappings $S: M_\gamma \to M_\gamma$ and $P: M_\gamma \to M_\gamma$ if $SPu = PSu$.

Definition (4.1.1):

Let be M_γ a modular space and $A \subseteq M_\gamma$ and P, S: $A \longrightarrow M_\gamma$ be mappings, then

i. P and S are called compatible if $P\ v_n$, $Sv_n \in A$ $\forall$ n and $\lim_{n\to\infty} \gamma(P\ v_n - Sv_n) = 0$, for a sequence $(v_n) \ni \lim_{n\to\infty} Sv_n = \lim_{n\to\infty} p\ v_n = t$. $t \in A$.

ii. P and S are called weakly compatible if P, S commute at thier coincidence points (i.e.) $SP\ v = P\ Sv$ whenever $Pv = Sv$.

Remark (4.1.1):

1. If M_γ is compact and P, S are continuous mappings then P and S are compatible if P and S are weakly compatible.

2. $\forall$ compatible is weakly compatible, but the converse is not true.

To see this consider the following example.

Example (4.1.1):

Let $M_\gamma = [0,2]$, $\gamma(v) = |v|$(| | is the absolute value on R) $\forall$ v, u in M_γ, define S and P as follows

$$Sv = \begin{cases} 1 & \text{if } v \in [0,1) \\ 2 & \text{if } v = 1 \\ \dfrac{v+3}{5} & \text{if } v \in (1,2] \end{cases}$$

$$pv = \begin{cases} 2 & \text{if } v \in [0,1] \\ \dfrac{v}{2} & \text{if } v \in (1,2] \end{cases}$$

The coincidence points of S, P are $\{1, 2\}$, we have:

SP (1) = P (1) and SP (2) = P (2) = 2.

Therefore (S, P) is weakly compatible.

To show that (S, P) not compatible

Taking $v_n = 2 - \dfrac{1}{2n}$, for all n then $P(v_n) \longrightarrow 1$ and $S(v_n) \longrightarrow 1$. Hence $\lim_{n\to\infty} Pv_n = \lim_{n\to\infty} Sv_n$ but $\lim P\, Sv_n \neq \lim S\, Pv_n$.

Had been mentioned to some relation between some generalization of commuting mappings [3].

Definition (4.1.2):

Let (M_γ, γ) be a modular space and S, P, Q:$M_\gamma \to M_\gamma$ S is said to be (P, Q)-contraction if there is $0 < h < 1$, ∋

$$\gamma(Sv - Su) \leq h\, \gamma(Pv - Qu),\ \forall\ v, u \text{ in } M_\gamma \qquad \ldots(3.1.1)$$

If $h = 1$ the S is (P, Q)-non-expansive mapping.

If $P = Q =$ I (I is the identity mapping) then S is contraction (or non-expansive).

In [16], [37] define the concepts of R-subcomuting, R-subweakly commuting and mappings in the case of normed spaces, here we reform these definitions in modular spaces:

Definition (4.1.3):

Let $\emptyset \neq A \subset M_\gamma$ and P, S:$A \to A$ with $u \in$ F(P,S). A pair (P,) is called:

i. R-subcommuting on A if $\forall\ v \in A$, $\exists$ R > 0 $\ni$

$$\gamma(PSv - SPv) \leq \frac{\mathrm{R}}{h}\gamma(Pv - |\mathrm{S}v, u|)$$

where $|Sv,u|$ = $\{(1 - h)u + hSv: 0 < h \leq 1\}$, $u \in A$.

ii. R-subweakly commuting on A if $\forall\ v \in A$, $\exists$ R > 0 $\ni$

$$\gamma(PSv - SPv) \leq \mathrm{R}\,\gamma(Pv - |\mathrm{S}v, u|)$$

iii. C_u $-$commuting if $PSv = SPv$ $\forall$ $v \in C(P,S) = \cup \{ (P,S_h): 0 \leq h \leq 1\}$ and $S_h v = (1 - h)u + hSv$.

Remarks (4.1.2):

i. C_u $-$commuting mappings are weakly compatible but the converse is not true.

ii. R-subcommuting mappings and R-subweakly commuting mappings are C_u $-$commuting but the converse is not true.

For more details see the same reference.

Theorem (4.1.1):

Let $\emptyset \neq A \subset M_\gamma \ni M_\gamma$ complete modular space and A is star-shaped, and S, P, Q be three mappings on A and S be a (P, Q)-contraction which satisfies $\overline{S(A)} \subset (A) \cap (A)$. If either $\overline{S(A)}$ or (A) or (A) is complete, and there is some $v \in A \ni S(v) < \infty$ then

i. $\exists\, z, u, v \in A \ni Pu = Su = z = Sv = Qv$, that is $u \in C\,(S, P)$ and $v \in C\,(S, P)$;

If, in addition, (S, P) and (S, Q) are weakly compatible, and then

ii. $F(S) \cap F(P) \cap F(Q)$ is singleton.

Proof:

Take $v_0 \in A$. As $\overline{S(A)} \subset (A) \cap (A)$, choose a sequence $\{v_n\}$ in $A \ni$ $Sv_{2n} = Pv_{2n+1}$ and Sv_{2n+1} $\forall\, n \geq 0$. By

$$\gamma(Sv_{2n+1} - Sv_{2n}) \leq h\gamma(Pv_{2n+1} - Qv_{2n}) = h\gamma(Sv_{2n} - Sv_{2n-1}).$$

Similarly, we also have that

$$\gamma(Sv_{2n-1} - Sv_{2n}) \leq h\gamma(Pv_{2n-1} - Qv_{2n}) = h\gamma(Sv_{2n-1} - Sv_{2n-1}).$$

Therefore, $\forall\, n \geq 0$,

$$\gamma(Sv_{2n+1} - Sv_{2n}) \leq h\gamma(Sv_{n-1} - Sv_n)\, h^n\gamma(Sv_1 - Sv_0).$$

Thus,

$$\gamma\big(Sv_{n+p} - Sv_n\big) \leq \gamma(Sv_{n+i} - Sv_{n+i+1}) \leq h^{n+i}\gamma(Sv_1 - Sv_0).$$

Hence, $\{Sv_n\}$ is a Cauchy sequence. By the definition of $\{Sv_n\}$,
$\exists\, a$ sequence $\{Pv_{2n+1}\}$ and $\{Qv_{2n+2}\}$ are also Cauchy sequence.

Since either $\overline{S(A)}$ or (A) or (A) is complete, if (A) is complete.

Then $Pv_{2n+1} \to z \in A$, and by the definition of $\{Sv_n\}$, we obtain that.

$$Qv_{2n,}\, Pv_{2n+1,}\, Sv_n \to\ z\ \in\ \overline{S(A)} \subset\ P(A) \cap Q(A).$$

Hence $\exists\ d, e \in A \ni Pd = z = Qe$. Then as $n \to \infty$,

$$\gamma(Sv_{2n+1}Se) \leq\ h\gamma(Pv_{2n+1} -\ Qe) = h\gamma(Pv_{2n+1} - z) \to 0.$$

Thus $Sv_n \to Se = z = Qe$. Similarly, also $\ = z = Pd$. (i)

Finally we prove (ii). As (S, P) and (S, Q) are weakly compatible and $Qe = Se = z = Sd = Pd$, then

$$Qz = QSe = SQe = Sz = SPd = PSd = Pz.$$

We claim that z is common fixed point of S, P, d. Since

$\gamma(z -\ Sz) =\ \gamma(Sd -\ Sz) \leq h\gamma(Pd -\ Qz) = h\gamma(z - Sz),$

Then $z = Sz$, i.e., $z \in (S) \cap (P) \cap (Q)$. $\exists\ e \in A \ni e = Se = Qe = Pe$, then

$$\gamma(z - e) = \gamma(Sz -\ Sv) \leq A\gamma(Pz - Qe) = A\gamma(z -\ e).$$

Hence $z = v$. The proof is complete

For modular space we prove the following:

Theorem (4.1.2):

Let $\emptyset \neq A \subset M_\gamma \ni M_\gamma$ complete modular space and A is star-shaped at $u \in A$, P, Q: $A \longrightarrow A$ be affine mappings, and S: $M_\gamma \longrightarrow M_\gamma$ be (P, Q)–non-expansive mapping. If $\overline{S(A)} \subset P\ (A) \cap (A)$ and $\exists v \in A \ni \gamma(Sv) < \infty$. Assume that either $\overline{S(A)}$ or (A) or (A) is compact, then

i. $\exists$ z, u, $v \in A \ni Pu = Su = z = Sv = Qv$, that is $u \in C(S, P)$ and

$v \in C(S, P)$,

If, in addition, (S,P) and (S, P) are weakly compatible, and $PPv = Pv$ $\forall$ $v \in (S, P)$, then:

ii. $F(S) \cap F(P) \cap F(Q) \neq \emptyset$.

Proof:

Since A is star-shaped $\exists$ a sequence (h_n) $(0 < h_n < 1)$ converging to 1 $\ni (1 - h_n)\, u + h_n Sv \in A$, $\forall$ v in A. define the mapping $S_n: A \longrightarrow A$ as the following: $S_n v = (1 - h_n)\, u + h_n Sv$

Since $\overline{S(A)} \subset P(A) \cap (A)$ we can prove that $\overline{S_n(A)} \subset P(A) \cap (A)$ as follows:

$$\overline{S_n(A)} = \{\overline{(1-h_n)u + h_n Sv}\} = (1-h_n)u + h_n\{\overline{Sv : v \in A}\}$$

Since $(u) = u$ and $(u) = u$ then:

$P\,((1 - h_n)\, u + h_n\, v) = (1 - h_n)\, Pu + h_n\, Pv = (1 - h_n)\, u + h_n\, Pv$

Also $((1 - h_n)\, u + h_n\, v) = (1 - h_n)\, Qu + h_n\, Qv = (1 - h_n)\, u + h_n\, Qv$

$\forall$ $v \in A$. Thus $\overline{S_n(A)} \subset (A) \cap (A)$.$\forall$ $v, u \in A$

$$\gamma(S_n v - S_n u) = \gamma((1-h_n)u + h_n Sv - (1-h_n)u - h_n Su)$$

$$= |h_n|\ \gamma(Sv - Su)$$

$$\leq |h_n| \gamma(Pv - Qu)$$

So S_n is (P, Q)-contraction mappings $h_n \in (0.1)$.

Since either $\overline{S(A)}$ or P (A) or (A) is compact, then $\overline{S(A)}$ or (A) or (A) is complete. Also, if $\overline{S(A)}$ is compact then $\overline{S_n(A)}$ is compact $\Rightarrow$ $\overline{S(A)}$ is complete. By Theorem (4.1.1) that $\forall\, n$, $\exists v_{m(n)}$, $v_{t(n)}$, $u_n \in A \ni$

$Pv_{m(n)} = S_n\, v_{m(n)} = u_n = S_n\, v_{t(n)} = Q\, v_{t(n)}$.

by compactness of either $\overline{S_n(A)}$ or $P(A)$ or Q (A) $\exists$ $< u_{n_i} > \subset < u_n >$ and

$z \in A \ni u_{n_i} = Pv_{m(n_i)} = Qv_{t(n)} \longrightarrow z$, (i $\longrightarrow \infty$),

$$Sv_{m(n_i)} = Sv_{t(n_i)} = \frac{u_{n_i} - (1 - h_{n_i})u}{h_{n_i}} \rightarrow z \in \overline{S(A)}.$$

and $z \in P(A) \cap Q(A)$ by $\overline{S(A)} \subset P(A) \cap Q(A)$.

hence, $\exists\, u, v \in A \ni z = Pu = Qv$, as i $\longrightarrow \infty$.

$\gamma(Su - Sv_{t(n_i)}) \leq \gamma(Pu - Qv_{t(n_i)}) = \gamma(z - Qv_{t(n_i)}) \rightarrow 0$, therefore

$Sv_{t(n_i)} \rightarrow Su = z$ i.e., $z = Su = Pu$.

Also, $\gamma(Sv_{m(n_i)} - Sv) \leq \gamma(Pv_{m(n_i)} - Qv) = \gamma(Pv_{m(n_i)} - z) \rightarrow 0$, therefore

$Sv_{m(n_i)} \rightarrow Sv = z$ i.e., $z = Sv = Pv$.

(i) is proved.

To prove (ii) by (i) $\exists z, u, v \ni P\, u = Su = z = Qv = Sv$. Since (S, P) and (S, Q) are weakly compatible and $PPv = Pv\ \forall\ v \in (S, P)$, then

$Pz = PSu = SPu = Sz = SQv = QSv = Qz$ and $Pz = PPu = Pu = z$

Thus $z = Pz = Qz = Sz$, z is fixed point for P, Q, S ■

Theorem (4.1.3):

Let M_γ be a complete modular space and $S : M_\gamma \longrightarrow M_\gamma$, $\emptyset \neq A \subset M_\gamma$ and P, Q : $A \longrightarrow A$ be two affine mappings, and there is some v$\in A$ $\gamma(Sv) < \infty$ and A is stars-shaped at $u \in A$. Assume that S is (P, Q)-non-expansive mapping and $\overline{S(A)} \subset P(A) \cap Q(A)$. If:

i) S is strongly continuous and A is weakly compact, or

ii) P or Q is strongly continuous and A is weakly compact, or

iii) $\overline{S(A)}$ is weakly compact and M_γ is opial's space.

Then **(i)** $(S, P, Q) \neq \emptyset$;

If, in addition, (S, P) and (S, Q) are weakly compatible and $PPv = Pv$ $\forall v \in C(S, P)$, then

(ii) $F(S) \cap F(P) \cap F(Q) \neq \emptyset$.

Proof:

Since A is star-shaped then there is a sequence (h_n) $(0 < h_n < 1)$ converging to 1 $\ni$ $(1 - h_n)u + h_n Sv \in A$, $\forall$ v in A.

define the mapping S_n: $A \longrightarrow A$ by

$S_n v = (1 - h_n)u + h_n Sv$

Since $\overline{S(A)} \subset M_\gamma$ and M_γ is a complete then $\overline{S(A)}$ is a complete. by similar of Theorem (4.1.2) that $\overline{S_n(A)} \subset P(A) \cap Q(A)$ $\forall$ n and S_n is (P, Q)-contraction mapping with $h_n \in (0,1)$, $\overline{S_n(A)}$ is complete, $\exists$ $v_{m(n)}$, $v_{t(n)}$, $u_n \in A \ni Pv_{m(n)} = S_n v_{m(n)} = u_n = S_n v_{t(n)} = Qv_{t(n)}$.

If (i) holds. Since $< v_{m(n)} > \subset A$ together with the weak compactness of A, $\exists\ u \in A$ and $< v_{m(n_i)} > \subset < v_{m(n)} > \ni v_{m(n_i)} \xrightarrow{w} y$ (i $\longrightarrow \infty$).By strong continuity of S that $Sv_{m(n_i)} \longrightarrow Su \in \overline{S(A)} \subset P(A) \cap Q(A)$.

$\exists\ u, v \ni Su = Pu = Qv$, and $h_n \longrightarrow 1$,

$$Pv_{m(n_i)} = Qv_{t(n_i)} = u_{n_i} = S_{n_i} v_{m(n_i)} = h_{n_i} Sv_{m(n_i)} + (1 - h_{n_i})u \to Su$$

We claim that $Su = Pu$. Since as $i \longrightarrow \infty$

$$\gamma(Su - Sv_{t(n_i)}) \le \gamma(Pu - Qv_{t(n_i)})$$, since S is (P, Q)-non-expansive

$$= \gamma(Su - Qv_{t(n_i)}) \to 0,$$

then $Sv_{t(n_i)} \to Su$.

Since $\lim\limits_{i\to\infty} S_{n_i} v_{t(n_i)} = \lim\limits_{i\to\infty}(1 - h_{n_i})u + \lim\limits_{i\to\infty} h_{n_i} \cdot \lim\limits_{i\to\infty} Sv_{t(n_i)} = Su$

Hence lim $Sv_{t(n_i)} = Su$. Thus $Sv_{t(n_i)} \to Su = Su$. Also, we claim that $Sv = Qv = Su$

Now, $\gamma(Sv_{m(n_i)} - Sv) \le \gamma(Pv_{m(n_i)} - Qv)$. Since S is (P, Q)-non-expansive

$$= \gamma(Pv_{m(n_i)} - Su) \to 0 \text{ as } i \longrightarrow \infty$$

then $Sv_{m(n_i)} \to Sv = Su$. Therefore $Su = Pu = Sv = Qv$. (i) Is proved.

If (ii) holds. Assume that P is strongly continuous, then $Qv_{t(n_i)} = Pv_{m(n_i)} \to Pu$. Since as $i \longrightarrow \infty$

$$\gamma(Su - Sv_{t(n_i)}) \le \gamma(Pu - Qv_{t(n_i)})$$, since S is (P, Q)-nonexpansive

$$= \gamma(Pu - Pv_{m(n_i)}) \to 0,$$

Then $Sv_{t(n_i)} \to Su \in \overline{S(A)} \subset (A) \cap (A)$. $\exists$ u, $v \ni Su = Pu = Qv$, and h_n $\longrightarrow 1$,

$Qv_{t(n_i)} = S_{n_i} v_{t(n_i)} = (1-h_n)u + h_n Sv_{t(n_i)} \to Su$, then $Qv_{t(n_i)} \to Su = Pu$.

By (i) that we also reach our objective.

If (iii) holds. By the weak compactness of $\overline{S(A)}$, $\exists$ $u \in A$ and

($Sv_{m(n_i)}) \subset (Sv_{m(n)}) \ni Sv_{m(n_i)} \xrightarrow{w} y$ ($i \longrightarrow \infty$).

Therefore by $h_n \longrightarrow 1$, we have

$$S_{n_i} v_{m(n_i)} = S_{n_i} v_{t(n_i)} = Pv_{m(n_i)} = Qv_{t(n_i)} = h_{n_i} Sv_{m(n_i)} + (1-h_{n_i})u \xrightarrow{w} u$$

Since weak closeness subset M_γ implies closeness in complete space M_γ, then $u \in \overline{S(A)} \subset P(A) \cap Q(A)$.

Thus $\exists$ u, $v \in A \ni u = Pu = Qv$. As (Sv_n) is bounded by the weak compactness of $\overline{S(A)}$, then

$$\gamma(Pv_{m(n)} - Sv_{m(n)}) = \gamma(h_n Sv_{m(n)} + (1-h_n)u - Sv_{m(n)})$$

$$= |1-h_n| \gamma(Sv_{m(n)} - u) \to 0 \ (n \to \infty).$$

Also, $\gamma(Qv_{t(n_i)} - Sv_{t(n)}) = \gamma(h_n Sv_{t(n)} + (1-h_n)u - Sv_{t(n)})$

$$\gamma(Qv_{t(n_i)} - Sv_{t(n)}) = |1-h_n| \gamma(Sv_{t(n)} - u) \to 0 \ (n \to \infty).$$

We claim that $Sv = y$. If not, by M_γ satisfying Opial's space, we get

$$\liminf_{i \to \infty} \gamma(Sv_{m(n_i)} - u) < \liminf_{i \to \infty} \gamma(Sv_{m(n_i)} - Sv)$$

$$\leq \liminf_{i \to \infty} \gamma(Pv_{m(n_i)} - Qv)$$,since S is (P, Q)-non-expansive

$$= \liminf_{i\to\infty} \gamma(Pv_{m(n_i)} - Sv_{m(n_i)} + Sv_{m(n_i)} - Qv)$$

$$= \liminf_{i\to\infty} \gamma(Pv_{m(n_i)} - Sv_{m(n_i)}) + \liminf_{i\to\infty} \gamma(Sv_{m(n_i)} - Qv)$$

$$= \liminf_{i\to\infty} \gamma(Sv_{m(n_i)} - u).$$

Which is a contraction. Hence $u = Sv = Qv$.Similarly, we also can show that $u = Su = Pu$. (i) Is proved. By similar of Theorem (4.1.2-ii) that

$Pz = Sz = Qz = z$ and $z \in F(S) \cap F(P) \cap F(Q)$.

Hence $F(S) \cap F(Q) \cap F(P) \neq \emptyset$. ■

For commuting mappings, we have:

Theorem (4.1.4):

Let M_γ complete modular space, $\emptyset \neq A \subset M_\gamma$ and $S: M_\gamma \longrightarrow M_\gamma$ be a mapping, and A is star-shaped and P, Q: $A \longrightarrow A$ be two affine mappings, and there is some $v \in A \ni \gamma(Sv) < \infty$ and S is a (P, Q)-non-expansive mapping and $\overline{S(A)} \subset (A) \cap (A)$. If (S, P) and (S, Q) are C_u-commuting, and P Q are affine, and $\forall$ S, P, Q is continuous. If either $\overline{S(A)}$ or $P(A)$ or $Q(A)$ is compact, then $F(S) \cap F(P) \cap F(Q) \neq \emptyset$.

Proof:

Since A is star-shaped $\exists$ a sequence $< h_n >$ $(0 < h_n < 1)$ converging to 1 $\ni (1 - h_n)\, u + h_n\, Sv \in A$, $\forall$ v in A.

define $_n$: $A \longrightarrow A$ as,$S_n v = (1 - h_n)\, u + h_n\, Sv$

By Theorem (4.1.2) that $\overline{S_n(A)} \subset P\ (A) \cap (A)\ \forall\ n$ and S_n is (P, Q)-contraction mapping, if (A) is compact, then $\overline{S_n(A)}$ is compact. Since

(S, P) and $(S,)$ are C_u-commuting, and P, Q are affine, then $u \in$ F$(S) \cap$ F(Q), and further, $\forall\ S_n v = Pv = Qv$, we have

$S_n\, Pv = (1 - h_n) + h_n\, S\, fv = (1 - h_n)\, Pu + h_n\, PSv = P\ ((1 - h_n) + h_n\, Sv) = PS_n v$

also,

$S_n\, Qv = (1 - h_n)u + h_n\, SQu = (1 - h_n)Qu + h_n\, QSv = Q((1 - h_n)u + h_n\, Sv) = Q\, S_n v$

namely, (S_n, P) and (S_n, Q) are weakly compatible.

By Theorem (4.1.1-ii) $\forall\ n$, $\exists$ unique $v_n \in A \ni$

$v_n = Pv_n = Qv_n = S_n\, v_n = (1 - h_n)\, u + h_n\, Sv$.

As Theorem (4.1.2-i) we get, $\exists\ z, u, v \in A$ and $(v_{n_i}) \subset (v_n) \ni Su = fu = z = Sv = Qv$ and $v_{n_i} = Pv_{n_i} = Qv_{n_i} \to z$ and $Sv_{n_i} \to z$ as $i \longrightarrow \infty$.As C_u $-$commuting of (S, P) and (S, Q) implies that weakly compatible, then

$Pz = PSu = SPu = Sz = SQv = Q\ Sv = Qz$.

By continuity of either S or P or Q that either $Sv_{n_i} \to Sz$ or $Pv_{n_i} \to Qz$ or $Qv_{n_i} \to Pz$.

Hence $z = Sz = Pz = Qz$ and F$(S) \cap$ F$(P) \cap$ F$(Q) \neq \emptyset$.
This complete the proof. ■

Corollary (4.1.1):

Let $\emptyset \neq A$ be star-shaped subset M_γ and S: $A \longrightarrow A$ a non-expansive mapping, and there is some$v \in A \ni \gamma(Sv) < \infty$, and $\overline{S(A)} \subset A$. If $\overline{S(A)}$ is compact subset M_γ, then F(S) $\neq \emptyset$.

Theorem (4.1.5):

Let M_γ be a complete modular space and S: $M_\gamma \longrightarrow M_\gamma$. Let $\emptyset \neq A \subset M_\gamma$ and P, Q : $A \longrightarrow A$ be two affine mappings,$\exists v \in A \ni \gamma(Sv) < \infty$, and A is star-shaped to S and $u \in A$. Assume that S is a (P, Q)-non-expansive mapping and $\overline{S(A)} \subset P(A) \cap Q(A)$. If ($S$, P) and (S, Q) are C_u-commuting, and S is strongly continuous, and either A or $\overline{S(A)}$ or (A) or Q(A) is weakly compact. Then F(S) $\cap$ F(P) $\cap$ F(Q)$\neq \emptyset$.

Proof:

Since A is star-shaped at u then $\exists$ a sequence (h_n) $(0 < h_n < 1)$ converging to 1 $\ni (1 - h_n)\, u + h_n\, Sv \in A$, $\forall\, v$ in A. define S_n: $A \longrightarrow A$ as, $S_n v = (1 - h_n)\, u + h_n Sv$. Since either A or $\overline{S(A)}$ or P (A) or (A) is complete and it by similar proof of Theorem (4.1.4) $\forall$ n, $\exists$ a unique $v_n \in A \ni v_n = Pv_n = Qv_n = h_n Sv_n + (1 - h_n)\, u$

By similar proof of Theorem (4.1.3-i) we have,$\exists$ z, v, $v \in A$ and $(v_{n_i}) \subset (v_n) \ni Su{=}Pu{=}z{=}Sv{=}Qv$ and $v_{n_i} = Pv_{n_i} = Qv_{n_i} \xrightarrow{w} z$ and $Sv_{n_i} \xrightarrow{w} z$ as $i \longrightarrow \infty$. Since C_u-commuting of (S, P) and (S,) implies weakly compatible, then

$$Pz = PSu = SPu = Sz = SQv = QSv = Qz$$

As S is strongly continuous together with $v_{n_i} \xrightarrow{w} z$, then $Sv_{n_i} \to Sz$. By $Sv_{n_i} \xrightarrow{w} z$, we have $z = Sz = Pz = Qz$.

Therefore $F(S) \cap F(P) \cap F(Q) \neq \emptyset$. ■

Corollary (4.1.2):

Let $\emptyset \neq A \subset M_\gamma$, $u \in M_\gamma$ and $S: M_\gamma \longrightarrow M_\gamma$ be a mapping, $P, Q : A \longrightarrow A$ be two affine mappings,$\exists v \in A \ni \gamma(Sv) < \infty$, and $P_A(u)$ is star-shaped and S and $u \in P_A(u)$ and $\overline{S(P_A(u))} \subset P(P_A(u)) \cap Q(P_A(u)) = P_A(u)$. Assume that S is a (P,Q)-non-expansive mapping on $P_A(u)$, If $\overline{S(P_A(u))}$ or $(P_A(u))$ or $Q(P_A(u))$ is compact, then

i) $\exists\ z, w, v \in A \ni Pw = Sw = z = Sv = Qv$; if (S, P) and (S,Q) are weakly compactible and $PPv = Pv\ \forall\ v \in C\ (S, P)$, then

ii) $P_A(u) \cap F(S) \cap F(P) \cap F(P) \neq \emptyset$.

Proof:

By Theorem (4.1.2), when $P_A(u) = A$. ■

Corollary (4.1.3):

Let M_γ be a complete modular space, $u \in M_\gamma$, $S : M_\gamma \longrightarrow M_\gamma$, $P, Q : A \longrightarrow A$ be two affine mappings,$\exists v \in A \ni \gamma(Sv) < \infty$ and $P_A(u)$ is star-shaped to S and $u \in P_A(u)$ and $\overline{S(P_A(u))} \subset P(P_A(u)) \cap Q(P_A(u)) = P_A(u)$. Assume that S is a (P,Q)-nonexpansive mappings on $P_A(u)$, if:

a) S is strongly continuous and $P_A(u)$ is weakly compact;

b) P or Q is strongly continuous and $P_A(u)$ is weakly compact;

c) $\overline{S(P_A(u))}$ is weakly compact and M_γ opial's space.

Then **(i)** $(S, P) \neq \emptyset$

If, in addition, (S, P) and (S, Q) are weakly compatible and $PPv = Pv$ $\forall$ $v \in C(S, P)$, then

(ii) $P_A(u) \cap F(S) \cap F(P) \cap F(Q) \neq \emptyset$.

Proof:

By Theorem (4.1.3), when $P_A(u) = A$. ■

Corollary (4.1.4):

Let M_γ be a complete modular space , $u \in M_\gamma$, $S : M_\gamma \longrightarrow M_\gamma$ and $P, Q : A \longrightarrow A$ be two affine mappings, $\exists v \in A \ni \gamma(Sv) < \infty$, and $P_A(u)$ is star-shaped to S and $u \in P_A(u)$ and $\overline{S(P_A(u))} \subset P(P_A(u)) \cap Q(P_A(u)) = P_A(u)$. Assume that S is a (P, Q)-non-expansive mapping on $P_A(u)$,and

$(S,)$, $(S,)$ are C_u-commuting If S is strongly continuous on $P_A(u)$ and $P_A(u)$ or $\overline{S(P_A(u))}$ or $P(P_A(u))$ or $Q(P_A(u))$ is weakly compact. Then $P_A(u) \cap F(S) \cap F(P) \cap F(Q) \neq \emptyset$.

Proof:

By Theorem (4.1.5), when $P_A(u) = A$. ■

4.2 Common Fixed Point and Invariant Best Approximation for Generalized (P, Q)-Non-expansive Mappings

In this section, we prove that there is a fixed point of S, P, Q if S is generalized (P, Q)-non-expansive mapping [22], and both (S, P), (S,) are weakly compatible. We also apply these results to derive some invariant best approximations.

Definition (4.2.1):

Let M_γ be a modular space and S, P, Q be three mappings on M_γ, we say that S is a generalized (P, Q)-contraction $\forall$ v, u in M_γ and $0 < h < 1$,

$$\gamma\,(Sv - Su) \leq h \max \begin{cases} \gamma\,(Pv - Qu),\ \gamma(Sv - Pv),\ \gamma(Su - Qu) \\ \\ \frac{1}{2}\,[\gamma(Pv - Su) + \gamma(Sv - Qu)] \end{cases}$$

when $h = 1$ then S is called a generalized (P, Q)-nonexpansive.

It is obvious that the generalized (P, Q)-contraction contains the (P, Q)-contraction. Furthermore the contraction is its main subclass also (when $P = Q = \mathrm{I}$ in (P, Q)-contraction).

Note that, in the setting of modular space the generalized (P, Q)-contraction will be:

$$\gamma(Sv - Su) \leq h \max \begin{cases} \gamma(Pv - Qu),\ \gamma(Sv - Pv),\ \gamma(Su - Qu) \\ \\ \frac{1}{2}\,[\gamma(Pv - Su) + \gamma(Sv - Qu)] \end{cases}$$

We need the following remark in modular space:

Remark (4.2.1):

Let M_γ complete modular space If $A \subset M_\gamma$ and star-shaped

S, P, Q: $A \longrightarrow A$ three mappings and $\forall\ v, u \in A$,

$$\gamma(Sv - Su) \le \max \begin{cases} \gamma(Pv - Qu), \gamma(Pv - |Sv, u|), \gamma(Qu - |Su, u|) \\ \frac{1}{2}[\gamma(Pv - |Su,u|) + \gamma(Qu - |Sv, u|)] \end{cases}$$

Then S is called (P, Q) non-expansive mapping.

Theorem (4.2.1):[41]

Let $\emptyset \neq A$ subset on metric space $M$and S, P, Q : $A \longrightarrow A$ or M be three affine mappings with $\overline{S(A)} \subset P(A) \cap Q(A)$ is (P, Q)-contraction $r \in [0,1)$ or $r \in (0,1)$. Then neither (S, P) nor (S, Q) is empty. Moreover, if both (S, P) and (S, Q) are weakly compatible, then F $(S) \cap$ F(P) $\cap$ F(Q) $\neq \emptyset$ is singleton.

An applying of the above theorem we obtain the following in modular space M_γ:

Theorem (4.2.2):

Let $\emptyset \neq A \subset M_\gamma$, and P, Q: $A \longrightarrow A$ or M_γ be two affine continuous mappings and S: $M_\gamma \longrightarrow M_\gamma$ be a continuous mapping, and A

is star-shaped to S and $u \in A$. If both (S, P) and (S, Q) are C_u-commuting, $\overline{S(A)}$ is a compact ⊂ (A) ∩ (A) and S satisfy ∀ v, $u \in A$

$$\gamma\,(Sv - Su) \le \max \begin{cases} \gamma\,(Pv - Qu),\ \gamma(Pv - |Sv, u|),\ \gamma(Qu - |Su, u|) \\ \\ \frac{1}{2}\,[\gamma(Pv - |Su, u|) + \gamma(Qu - |Sv, u|)] \end{cases}$$

Then F(S) ∩ F(P) ∩ F(Q) ≠ ∅.

Proof:

Since A has star-shaped then there is a sequence (h_n) ($0 < h_n < 1$) converging to 1 ∋$(1 - h_n)\, u + h_n\, Sv \in A$ ∀ v in A. define the mapping

S_n: $A \longrightarrow A$ as ∀ n, $S_n v = (1 - h_n)\, u + h_n$ ∀ v in A. Since $\overline{S(A)} \subset (A) \cap$ (A) to proof

$\overline{S_n(A)} \subset P\,(A) \cap (A)$ as follows;

$$\overline{S_n(A)} = \{\overline{(1 - h_n)u + h_n\, Sv}\} = (1 - h_n)u + h_n\, \{\overline{Sv : v \in A}\}$$

Since $(u) = u$ and $(u) = u$ then:

$P\,((1 - h_n)\, u + h_n\, Sv) = (1 - h_n)\, Pu + h_n\, Pv = (1 - h_n)\, u + h_n\, Pv,$

also, $Q((1 - h_n)\, u + h_n Qv) = (1 - h_n)\, Qu + h_n\, Qv = (1 - h_n)\, u + h_n\, Qv,$

∀ $v \in A$. Thus $\overline{S_n(A)} \subset (A) \cap (A)$. ∀ v, $u \in A$, and by condition (4.2.2), we have:

$\gamma(S_n v - S_n u) = \gamma(h_n Sv + (1 - h_n)u - h_n\, Su - (1 - hn)u)$

$\qquad = |h_n|\, \gamma(Sv - Su)$

$$\leq \gamma|h_n| \max \begin{cases} \gamma(Pv - Qu), \gamma(Pv - |Sv, u|), \gamma(Qu - |Su, u|) \\ \frac{1}{2}\,[\gamma\,(Pv - |Su, u|, \gamma(Qu - |Sv, u|)] \end{cases}$$

Therefore

$$\gamma(S_n v - S_n u) = \gamma|h_n| \max \begin{cases} \gamma(Pv - Qu), \gamma(Pv - S_n v), \gamma(Qu - S_n u) \\ \frac{1}{2}\,[\gamma\,(Pv - S_n u), \gamma(Qu - S_n v)] \end{cases}$$

Thus S_n is generalized (P, Q)-contraction with coefficient $r = h_n \in (0,1)$. note that (S, P) and (S, Q) are C_u-commuting, and P and Q are affine, then $u \in F(P) \cap F(Q)$. If $S_n v = Pv = Qv$, we have

$S_n Pv = (1 - h_n)\, u + h_n SPv = (1 - h_n)\, Pu + h_n\, PSv = P\,((1 - h_n) + h_n Sv) = PS_n v$.

Also $S_n\, Qv = (1 - h_n)\, u + h_n SQv = (1 - h_n)\, Qu + h_n\, Q\, Sv = Q\,((1 - h_n) + h_n Sv) = Q\, S_n v$

namely, (S_n, P), (S_n, Q) are weakly compatible. As $\overline{S(A)}$ is compact, then $\overline{S(A)}$ is complete. By theorem (4.2.1) that $\forall\, n$, $\exists$ a unique $v_n \in A \ni$

$v_n = Pv_n = Qv_n = h_n Sv_n + (1 - h_n)\, u$.

By the compactness of $\overline{S(A)}$ $\exists \left(v_{n_i}\right) \subset (v_n)$ and $u \in A \ni$

$v_{n_i} = pv_{n_i} = qv_{n_i} = v_{n_i}\ Sv_{n_i} + (1 - h_{n_i}) \rightarrow u\ (i \longrightarrow \infty)$. The continuity of S and p and q imply $Sv_{n_i} \longrightarrow Su$ and $pv_{n_i} \longrightarrow pu$ and $qv_{n_i} \longrightarrow qu$.

Hence $u = Su = Pu = Qu$. Therefore $F(S) \cap F(P) \cap F(Q) \neq \emptyset$. This finishes the proof. ■

Corollary (4.2.1):

Let $\emptyset \neq A$ star-shaped subset M_γ and S: $A \longrightarrow A$ a non-expansive mapping, $\exists v \in A \ni \gamma(Sv) < \infty$, and $\overline{S(A)} \subset A$. If $\overline{S(A)}$ is compact subset A, then F $(S) \neq \emptyset$.

To illustrate Theorem (4.2.3), we give the following example:

Example (4.2.1):

Let M_γ = □ and A = [0, 1] with γ (v) = $|v|$ for ∈ □ . Let P, Q: $A \longrightarrow A$ as (v) = (v) = $\frac{1}{3}v^2 \forall\ v \in A$ and S: $A \longrightarrow A$ by $Sv = \frac{2}{9}v^2$ for all v $\in A$. Then S is a generalized $(P,)$-non-expansive mapping since

$$\gamma(Sv\text{-}Su) = \frac{2}{9}\gamma(S^2 - u^2) = \frac{2}{3} \cdot \frac{1}{3}\gamma(S^2 - u^2) = \frac{2}{3}\gamma(Pv - Qu)$$

On the other hand, A (P,S) = F(Q,S) = {0} so, F(P) ∩ F(Q) ∩ F(S) = {0}.

Theorem (4.2.3):

Let M_γ be a complete modular space, and S: $M_\gamma \longrightarrow M_\gamma$ be a weakly continuous mapping. Let $\emptyset \neq A \subset M_\gamma$ and A is star-shaped to S and $u \in A$, P, Q: $A \longrightarrow A$ be two weakly continuous affine mappings. Assume that $\overline{S(A)}$ is weakly compact subset P (A) ∩ (A). If both (S, P) and $(S,)$ are C_u-commuting, and S satisfy condition (4.2.2) then F(S) ∩ F(P) ∩ F(Q) ≠ ∅.

Proof:

Since A is star-shaped ∃ a sequence (h_n) < (0 < h_n < 1) converging to 1 ∋$(1 - h_n)\, u + h_n\, Sv \in A$, ∀ v in A.

define the mapping $S_n: A \longrightarrow A$ as follows:

$\forall n$, $S_n v = (1 - h_n)\, u + h_n \ \ \forall\, v$ in A.

By the proof of Theorem (4.2.2) there is a common approximate fixed sequence $(v_n) \in \overline{S(A)}$ of S, P, Q. Since P, Q, S are weakly continuous and $\overline{S(A)}$ is weakly compact, then the weak cluster u of (v_n) is a common fixed point of S, P, Q. The proof is completed. ■

As an application to the above common fixed points, we have the following results in best approximation:

Corollary (4.2.2):

Let M_γ be complete modular space, $\emptyset \neq A \subset M_\gamma$, $u \in M_\gamma$, and S: $M_\gamma \longrightarrow M_\gamma$ be a continuous mapping and P, Q: $A \longrightarrow A$ be two continuous mappings. $\emptyset \neq P_A(u)$ is star-shaped to S and $u \in P_A(u)$ and $\overline{S(P_A(u))}$ is compact subset of $P_A(u)$, $P(P_A(u)) \cap Q(P_A(u)) = P_A(u)$, P and u are affine on $P_A(u)$. If (S, P), (S, Q) are C_u-commuting and $\forall\ v \in P_A(u) \cup \{u\}$,

$$\gamma(Sv - Su) \leq \begin{cases} \gamma(Pv - Qu) & if\ u = u \\ \max \gamma(Pv - Qu), \gamma(Pv - |Sv, u|), \gamma(Qu - |Su, u|) & \\ \frac{1}{2}\,[\gamma(Pv - |Su, u|) + \gamma(Qu - |Sv, u|)] & if\ u \in P_A(u) \end{cases}$$

Then $P_A(u) \cap F(S) \cap F(P) \cap F(Q) \neq \emptyset$.

Proof:

Since $\overline{S(P_{\mathrm{A}}(u))} \subset \mathrm{P_A}(u) = P(\mathrm{P_A}(u)) \cap Q(\mathrm{P_A}(u)) = \mathrm{P_A}(u)$ is compact , the results follows from Theorem (4.2.2), when $\mathrm{P_A}(u) = A$. ■

Corollary (4.2.3):

Let $\emptyset \neq A \subset M_\gamma$ with $S(\partial A) \subset A$ and $u \in \mathrm{F}(S) \cap \mathrm{F}(P) \cap \mathrm{F}(Q)$,

S, P, : $A \longrightarrow A$ are three weakly continuous mappings. $\emptyset \neq \mathrm{P_A}(u)$ is star-shaped, and weakly compact, $P(\mathrm{P_A}(u)) \cap Q(\mathrm{P_A}(u)) = \mathrm{P_A}(u)$, P and Q are affine. If (S, P), $(S,)$ are C_u-commuting on $\mathrm{P_A}(u)$ satisfy condition (4.2.3)

$\forall\, v \in \mathrm{P_A}(u) \cup \{u\}$, then $\mathrm{P_A}(u) \cap \mathrm{F}(S) \cap \mathrm{F}(P) \cap \mathrm{F}(Q) \neq \emptyset$.

Corollary (4.2.3):

Let$\emptyset \neq A \subset M_\gamma$ with $S(\partial A \cap A) \subset A$ and $u \in \mathrm{F}(S) \cap \mathrm{F}(P) \cap \mathrm{F}(Q)$ and S, P, Q: $A \longrightarrow A$ be two continuous mappings. $\emptyset \neq \mathrm{P_A}(u)$ is star-shaped and compact, $P(\mathrm{P_A}(u)) \cap Q(\mathrm{P_A}(u)) = \mathrm{P_A}(u)$, P and Q are affine on $\mathrm{P_A}(u)$. If (S, P), $(S,)$ are C_u-commuting on $\mathrm{P_A}(u)$ and S satisfy condition (4.2.3) $\forall\, v \in \mathrm{P_A}(u) \cup \{u\}$. Then $\mathrm{P_A}(u) \cap \mathrm{F}(S) \cap \mathrm{F}(P) \cap \mathrm{F}(Q) \neq \emptyset$.

4.3 Invariant Best Approximation for (P, Q)-Non-expansive Mappings with (w)-Convexity

In this section, some existence results on best approximation are proved without star-shaped and affine mapping.

Theorem (4.3.1):

Let M_γ be a modular space with (w)-convex structure. Let S, h, : $M_\gamma \longrightarrow M_\gamma$ and $A \subseteq M_\gamma \ni S(\partial A) \subset A$. Let $v_0 \in F(S) \cap F(Q)$. If S is (h,Q)-non-expansive mapping on $P_A(v_0) \cup \{v_0\}$.$\exists v \in A \ni \gamma(Sv) < \infty$ Assume that (S,h) and (S,Q) are weakly compatible on $P_A(v_0)$ and $h(P_A(v_0)) \subseteq P_A(v_0)$, $Q(P_A(v_0)) \subseteq P_A(v_0)$ and $\overline{S(P_A(v_0))} \subset h(P_A(v_0)) \cap Q(P_A(v_0))$. If $\overline{S(P_A(u))}$ or $h(P_A(v_0))$ or $Q(P_A(v_0))$ is compact and $hhv = hv$ where $v \in (S,h)$ then $P_A(v_0) \cap F(S) \cap F(h) \cap F(Q) \neq \varnothing$.

Proof:

Let $P_A(v_0) = D$.S: $D \longrightarrow D$, let $u \in D$ then $hu \in D$ $h(D) \subseteq D$. Since $D \subseteq \partial A$ by Lemma (3.2.1), therefore $u \in \partial A$ and $(\partial A) \subseteq A$ then $Su \in A$.Now, since $Sv_0 = v_0 = Qv_0$ and S is a $(h,)$-non-expansive mapping, we have

$$\begin{aligned}\gamma(Su - v_0) &= \gamma(Su - Sv_0)\\ &\leq \gamma(hu - Qv_0)\\ &= \gamma(hu - v_0)\end{aligned}$$

Thus $\gamma(Su - v_0) \leq \gamma(hu - v_0) = \gamma(v_0, A)$.Implies Su is also closest to v_0, so $Su \in D$. Choose $h_n \in (0,1) \ni \langle h_n \rangle \longrightarrow 1$. Then define S_n as

$S_n(v) = P_{Sv}(h_n)$ $\forall$ $v \in$ D and by Definition (3.3.1) condition (iv) then S_n is a well-defined map from D into D, $\forall$ n. Thus S_n, h, Q:D $\longrightarrow$ D and $\forall$ v, $u \in$ D,

$$\begin{aligned}\gamma(S_n v - S_n u) &= \gamma(P_{Sv}(h_n) - P_{Su}(h_n)) \\ &\leq [\phi(h_n)]\gamma(Sv - Su) \\ &\leq [\phi(h_n)]\gamma(hv - Qu)\end{aligned}$$

Therefore $\gamma(S_n v - S_n u) \leq [\phi(h_n)]\ \gamma(hv - Qu)$

Hence S_n is (h,)-contraction. Since $\{\overline{S_n v}\} = \{\overline{P_{Sv}(h_n)}\} \subseteq \{\overline{Sv}\}$ $\forall$ $v \in$ D, and $\overline{S\,(\text{D})} \subset h(\text{D}) \cap Q(\text{D})$ then $\overline{S_n(\text{D})} \subset h(\text{D}) \cap Q(\text{D})$. Since $\overline{S\,(\text{D})}$ is compact and by definition (3.3.1-iv) then $\overline{S_n(\text{D})}$ is compact, therefore $\overline{S_n(\text{D})}$ is complete. Now, By Theorem (4.1.1-i), $\forall$ $v_{m(n)}$, $v_{t(n)}$, $u_n \in$ D $\ni$

$hv_{m(n)} = S_n v_{m(n)} = u_n = S_n\, v_{t(n)} = Qv_{t(n)}$

Since either $\overline{S\,(\text{D})}$ or h(D) or Q(D) is compact $\exists$ $< u_{n_i} > \subset < u_n >$ and $u \in$ D $\ni$ $h\ v_{m(n_i)} = S_{n_i} v_{m(n_i)} = S_{n_i} v_{t(n_i)} = Q\, v_{t(n_i)} = u_{n_i} \rightarrow u$ as ($i \longrightarrow \infty$).

Thus $P_{Sv_{m(n_i)}}(h_{n_i}) = P_{Sv_{t(n_i)}}(h_{n_i}) \rightarrow u$ as ($i \longrightarrow \infty$) and

$$p_{Sv_{m(n_i)}}(h_{n_i}) \subset Sv_{m(n_i)}, p_{Sv_{t(n_i)}}(h_{n_i}) \subset Sv_{t(n_i)}$$

Also, $Sv_{t(n_i)}, Sv_{m(n_i)} \subset \overline{S\,(\text{D})}$ .Hence $u \in \overline{S\,(\text{D})} \subset h(\text{D}) \cap Q(\text{D})$.

$\exists$ v, $w \in$ D $\ni$ $y = hw = Qv$. As $i \longrightarrow \infty$,

$\gamma(Sw - Sv_{t(n_i)}) \leq \gamma(hw - Qv_{t(n_i)}) = \gamma(u - Qv_{t(n_i)}) \rightarrow 0,$ therefore $Sv_{t(n_i)} \rightarrow Sw$.

Now, $\lim_{i\to\infty} P_{Sv_{t(n_i)}}(h_{n_i}) = P_{Sw}(1) = Sw$, but $P_{Sv_{t(n_i)}}(h_{n_i}) \to u$, hence $Sw = u$,

also $\gamma(Sv_{m(n_i)} - Sv) \leq \gamma(hv_{m(n_i)} - Qv) = \gamma(u - Qv) \to 0$.

Thus $\lim_{i\to\infty} P_{Sv_{m(n_i)}}(h_{n_i}) = P_{Sv}(1) = Sv$, but $P_{Sv_{m(n_i)}}(h_{n_i}) \to u$. Hence $u = Sv$.

Now, (S,h) and (S,Q) are weakly compatible and $hhv = hv$ for all $v \in$ A (S,h), then $hu = hSw = Shw = Su = SQv = QSv = Qu$, and $hu = hhw = hw = u$. Thus $u = hu = Qu = Su$.

Hence $P_A(v_0) \cap F(S) \cap F(h) \cap F(Q) \neq \emptyset$. ■

Theorem (4.3.2):

Let M_γ a complete modular space with (w)-convex structure, $u \in M_\gamma$, and $S, h, Q: M_\gamma \longrightarrow M_\gamma$ three mappings,$\exists v \in A \ni \gamma(Sv) < \infty$, and $A \subseteq M_\gamma$. $\emptyset \neq P_A(u)$ and $\overline{S(P_A(u))} \subset P_A(u)$ and $h(P_A(u)) \cap Q(P_A(u)) = P_A(u)$. If S is $(h,)$-non-expansive mapping on $P_A(u) \cup \{u\}$ and either $\overline{S(P_A(u))}$ or $Q(P_A(u))$ or $h(P_A(u))$ is compact then

i. $\exists\, u, w, v \in P_A(u) \ni hw = Sw = u = Qv = Sv$.

If in addition, (S,h) and (S,Q) are weakly compatible and $hhv = hv\ \forall\ v \in C(S,h)$, then

ii. $P_A(u) \cap F(S) \cap F(h) \cap F(Q) \neq \emptyset$.

Proof:

Let $P_A(v_0) = D$, since $h(D) \cap Q(D) = D$ and $\overline{S(D)} \subset D$ then

S, h, Q:D$\longrightarrow$D, choose $h_n \in (0,1) \ni (h_n) \longrightarrow 1$, define S_n as $S_n(v) = P_{Sv}(h_n)\ \forall\ v \in$ D, and by Definition (3.3.1-iv) then S_n is a well-defined map by D $\to$ D $\forall\ n$.Thus S_n, h, Q : D $\longrightarrow$ D and $\forall\ v, u \in$ D

$$
\begin{aligned}
\gamma(S_n v - S_n u) &= \gamma(P_{Sv}(h_n) - P_{Su}(h_n)) \\
&\le [\phi(h_n)]\ \gamma(Sv - Su) \\
&\le [\phi(h_n)]\ \gamma(hv - Qu)
\end{aligned}
$$

Therefore $\gamma(S_n v - S_n u) \le [\phi(h_n)]\ \gamma(hv - Qu)$

Hence S_n is (h,)-contraction. Since $\overline{S\,(\mathrm{D})}$ is compact and by definition (3.3.1-iv) then $\overline{S_n(\mathrm{D})}$ is compact, therefore $\overline{S_n(\mathrm{D})}$ is complete.

Now, by Theorem (4.1.1-i), $\exists\ v_{m(n)}, v_{t(n)}, u_n \in$ D $\ni$

$hv_{m(n)} = S_n\, v_{m(n)} = u_n = S_n\, v_{t(n)} = Qv_{t(n)}$

since either $\overline{S_n(\mathrm{D})}$ or h(D) or Q(D) is compact $\exists\ < u_{n_i} > \subseteq < u_n >$ and $u \in$ D $\ni$

$hv_{m(n_i)} = S_{n_i} v_{m(n_i)} = S_{n_i} v_{t(n_i)} = Q\, v_{t(n_i)} = u_{n_i} \to u$ as ($i \longrightarrow \infty$).

Thus $P_{Sv_{m(n_i)}}(h_{n_i}) = P_{Sv_{t(n_i)}}(h_{n_i}) \to u$ as ($i \longrightarrow \infty$) and

$P_{Sv_{m(n_i)}}(h_{n_i}) \subset Sv_{m(n_i)}, P_{Sv_{t(n_i)}}(h_{n_i}) \subset Sv_{t(n_i)}$

Also, $Sv_{t(n_i)}, Sv_{m(n_i)} \subset \overline{S\,(\mathrm{D})}$ .Hence $u \in \overline{S\,(\mathrm{D})} \subset h(\mathrm{D}) \cap Q(\mathrm{D})$.

$\exists\ w, v \in A \ni u = hw = Qv$. As $i \longrightarrow \infty$,

$\gamma(Sw - Sv_{t(n_i)}) \le \gamma(hw - Qv_{t(n_i)}) = \gamma(u - Qu_{t(n_i)}) \to 0$, therefore $Sv_{t(n_i)} \to Sw$.

Now, $\lim_{i\to\infty} P_{Sv_{t(n_i)}}(h_{n_i}) = P_{Sw}(1) = Sw$, but $P_{Sv_{t(n_i)}}(h_{n_i}) \to u$, hence $Sw = u$, also $\gamma(Sv_{m(n_i)} - Sv) \leq \gamma(hv_{m(n_i)} - Qv) = \gamma(u - Qv) \to 0$.

Thus $\lim_{i\to\infty} P_{Sv_{m(n_i)}}(h_{n_i}) = P_{Sv}(1) = Sv$, but $P_{Sv_{m(n_i)}}(h_{n_i}) \to u$. Hence $u = Sv$.

Therefore $Sw = hw = u = Sv = Qv$. (i) Proved.

Subsequently, we show (ii). Since (S,h) and (S,Q) are weakly compatible and $hhv = hv$ $\forall$ $v \in A$ (S,h), then $hu = h\ Sw = Shw = Su = SQv = QSv = Qu$, and $hu = hhw = hw = u$. Thus $u = hu = Qu = Su$.

Hence $P_A(u) \cap F(S) \cap F(h) \cap F(Q) \neq \emptyset$. ■

Theorem (4.3.3):

Let M_γ be a complete modular space M_γ with (w)-convex structure, $u \in M_\gamma$, $\exists v \in A \ni \gamma(Sv) < \infty$, and $S, h, Q : M_\gamma \longrightarrow M_\gamma$, $A \subseteq M_\gamma$. $\emptyset \neq P_A(u)$ and $\overline{S(P_A(u))} \subset P_A(u)$ and $h(P_A(u)) \cap Q(P_A(u)) = P_A(u)$. S is a $(h,)$-non-expansive mapping on $P_A(u) \cup \{u\}$. If:

a) S is strongly continuous and $P_A(u)$ is weakly compact

b) h or Q is strongly continuous and $P_A(u)$ is weakly compact

c) $\overline{S(P_A(u))}$ is weakly compact and M_γ Opial's space. Then

i. $C(S,h,Q) \neq \emptyset$

If in addition, (S,h) and (S,Q) are weakly compatible and $hhv = hv$ $\forall$ $v \in C(S,h)$, then

ii. $P_A(u) \cap F(S) \cap F(h) \cap F(Q) \neq \emptyset$.

Proof:

Let $P_A(v_0) = D$, since $h(D) \cap Q(D) = D$ and $\overline{S(D)} \subset D$ then S, h, Q :D$\longrightarrow$D, Since $\overline{S(D)}$ is complete, let $< h_n >$ and S_n defined as in theorem (4.3.2). Then a similar proof $\exists\ v_{m(n)}, v_{t(n)}, u_n \in D \ni$

$hv_{m(n)} = S_n\ v_{m(n)} = u_n = S_n\ v_{t(n)} = Q\ v_{t(n)}$

if the condition (a) holds. Since $< v_{m(n)} > \subset D$ together with weak compactness of D $\exists\ u \in D$ and $< v_{m(n_i)} > \subset < v_{m(n)} > \ni\ v_{m(n_i)} \xrightarrow{w} u$ $(i \longrightarrow \infty)$. By strong continuity of S that $S v_{m(n_i)} \longrightarrow Su \in \overline{S(D)} \subset h(D) \cap Q(D)$.

$\exists\ w, v \in D \ni Su = hw = Qv$, and noticing $h_n \longrightarrow 1$, and

$S_{n_i} v_{m(n_i)} = P_{Sv_{m(n_i)}}(h_{n_i}) \to P_{Su}(1) = Su$ as $(i \longrightarrow \infty)$.

Hence, $h v_{m(n_i)} = S_{n_i} v_{m(n_i)} = u_{n_i} = S_{n_i} v_{t(n_i)} = Q v_{t(n_i)} \longrightarrow Su$ as $(i \longrightarrow \infty)$.

We claim that $Sw = Su = hw$. Indeed, since as $i \longrightarrow \infty$

$\gamma(Sw - Sv_{t(n_i)}) \leq \gamma(hw - Qv_{t(n_i)}) = \gamma(Su - Qv_{t(n_i)}) \to 0,$

then $Sv_{t(n_i)} \to Sw$.

Now, as $i \longrightarrow \infty$ $S_{n_i} v_{t(n_i)} = P_{Sv_{t(n_i)}}(h_{n_i}) \to P_{Sw}(1) = Sw$. Then $Sw = Su$.

also, we claim that $Sv = Su = Qv$. Indeed, since as $i \longrightarrow \infty$

$\gamma(Sv_{m(n_i)} - Sv) \leq \gamma(hv_{m(n_i)} - Qv) = \gamma(hv_{m(n_i)} - Su) \to 0,$

then $Sv_{m(n_i)} \to Sv = Su$. (i) is proved.

If condition (b) holds. Assuming that h is strongly continuous, then $Q v_{t(n_i)} = h v_{m(n_i)} \longrightarrow hu$. Since $i \longrightarrow \infty$,

$\gamma(Su - Sv_{t(n_i)}) \leq \gamma(hu - Qv_{t(n_i)}) \to 0, Sv_{t(n_i)} \to Su \in \overline{S(\mathrm{D})} \subset h(\mathrm{D}) \cap$ $qhq(\mathrm{D})$. $\exists\, w, v \in$ D such that $Su = hw = Qv$.

Now, $Q v_{t(n_i)} = S_{n_i} v_{t(n_i)} = P_{Sv_{t(n_i)}}(h_{n_i}) \to P_{Su}(1) = Su$ as ($i \longrightarrow \infty$), then $Q v_{t(n_i)} \longrightarrow Su = hu$, we claim that $Sw = Su = hw$. Indeed, since as $i \longrightarrow \infty$

$\gamma(Sw - Sv_{t(n_i)}) \leq \gamma(hw - Qv_{t(n_i)}) = \gamma(Su - Qv_{t(n_i)}) \to 0,$ then $Sv_{t(n_i)} \to Sw$

Since as $i \longrightarrow \infty$ $S_n v_{t(n_i)} = P_{Sv_{t(n_i)}}(h_{n_i}) \to P_{Sw}(1) = Sw$. Then $Sw = Su$.

Also, we claim that $Sv = Su = Qv$. Indeed, since as $i \longrightarrow \infty$

$\gamma(Sv_{m(n_i)} - Sv) \leq \gamma(hv_{m(n_i)} - Qv) = \gamma(hv_{m(n_i)} - Su) \to 0,$

then $Sv_{m(n_i)} \to Sv = Su$ and

$S_{n_i} v_{m(n_i)} = P_{Sv_{m(n_i)}}(h_{n_i}) \to P_{Sv}(1) = Sv$ as $i \longrightarrow \infty$, then $Sv = Su$. (i) is proved

If condition (c) holds. By the weak compactness of $\overline{S(\mathrm{D})}$, $\exists\, u \in$ D and $< Sv_{m(n_i)} > \subset < Sv_{m(n)} > \ni Sv_{m(n_i)} \xrightarrow{w} u \quad (i \longrightarrow \infty)$.

Therefore by $h_{n_i} \longrightarrow 1$, we have

$S_{n_i} v_{m(n_i)} = S_{n_i} v_{t(n_i)} = h v_{m(n_i)} = Q v_{t(n_i)} = P_{Sv_{t(n_i)}}(h_{n_i}) \xrightarrow{w} u$, ($i \longrightarrow \infty$).

Since weak closeness subset M_γ implies closeness in complete modular space M_γ, then $u \in \overline{S(\mathrm{D})} \subset h(\mathrm{D}) \cap Q(\mathrm{D})$. Thus $\exists\, w, v \in \mathrm{D} \ni u = h\,w = Qv$. As $(i \longrightarrow \infty)$,

$$\gamma(hv_{m(n_i)} - Sv_{m(n_i)}) = \gamma(P_{Sv_{t(n_i)}}(h_{n_i}) - Sv_{m(n_i)}) \xrightarrow{w} 0$$

We claim that $Sv = u$. If not, by M_γ satisfying Opial's space, we get

$$\begin{aligned}
\liminf_{i\to\infty} \gamma(Sv_{m(n_i)} - u) &< \liminf_{i\to\infty} \gamma(Sv_{m(n_i)} - Sv) \\
&\le \liminf_{i\to\infty} \gamma(hv_{m(n_i)} - Qv) \\
&= \liminf_{i\to\infty} \gamma(hv_{m(n_i)} - u) \\
&\le \liminf_{i\to\infty} \gamma(hv_{m(n_i)} - Sv_{m(n_i)} + Sv_{m(n_i)} - u) \\
&\le \liminf_{i\to\infty} \gamma(hv_{m(n_i)} - Sv_{m(n_i)}) + \liminf_{i\to\infty} \gamma(Sv_{m(n_i)} - u) \\
&\le \liminf_{i\to\infty} \gamma(Sv_{m(n_i)} - u)
\end{aligned}$$

Thus $\liminf_{i\to\infty} \gamma(Sv_{m(n_i)} - u) < \lim_{i\to\infty} \inf \gamma(Sv_{m(n_i)} - u)$ which is a contradiction.

Hence $u = Sv$, also we claim that $Sw = u$. Since the weak compactness of $\overline{S(\mathrm{D})}$, $\exists\, u' \in \mathrm{D}$ and $< Sv_{t(n_i)} > \subset < Sv_{t(n)} > \ni Sv_{t(n_i)} \xrightarrow{w} u'$, $(i \longrightarrow \infty)$, therefore by $h_{n_i} \longrightarrow 1$, we have

$S_{n_i} v_{t(n_i)} = P_{Sv_{t(n_i)}}(h_{n_i}) \xrightarrow{w} P_{u'}(1) = u'$, but $S_{n_i} v_{t(n_i)} \longrightarrow u$ then $u = u'$.

Now, as $i \longrightarrow \infty$, $\gamma(Qv_{t(n_i)} - Sv_{t(n_i)}) = \gamma(S_n v_{t(n_i)} - Sv_{t(n_i)}) \xrightarrow{w} 0$.

Similarly, $u = Sw = hw$. (i) Proved.

By similar proof of Theorem (4.3.2-ii) that $Su = hu = Qu = u$.

Hence $\mathrm{P_A}(u) \cap \mathrm{F}(S) \cap \mathrm{F}(h) \cap \mathrm{F}(Q) \neq \emptyset$.

Theorem (4.3.4):

Let M_γ be a complete modular space with (w)-convex structure, $u \in M_\gamma$, $A \subseteq M_\gamma$, and S, h, $Q : M_\gamma \longrightarrow M_\gamma$, three mappings. $\exists v \in A \ni \gamma(Sv) < \infty, \emptyset \neq P_A(u)$ and $\overline{S(P_A(u))} \subset P_A(u)$ and $h(P_A(u)) \cap Q(P_A(u)) = P_A(u)$ and S is (h,Q) – non-expansive on $P_A(u) \cup \{u\}$ and S or h or Q is continuous, and if (S,h) and (S,Q) are C_u-commuting on $P_A(u)$. If either $\overline{S(P_A(u))}$ or $h(P_A(u))$ or $Q(P_A(u))$ is compact, and h and Q have star-shaped then $P_A(u) \cap F(S) \cap F(h) \cap F(Q) \neq \emptyset$.

Proof:

Let $P_A(u) = D$ and let $< h_n > \subset (0,1) \ni \lim_{n\to\infty} h_n = 1$. $\forall$ n, define S_n by $S_n(v) = P_{Sv}(h_n)$ $\forall$ $v \in D$. By similar proof of Theorem (4.3.2) that $\overline{S_n(D)} \subset h(D) \cap Q(D)$ $\forall$ n and S_n is (h,Q)-contraction mapping. Since (S,h) and (S,Q) are C_u-commuting, and h and Q have star-shaped, and furthermore, $\forall$ $S_nv = hv = Qv$, we have

$S_nhv = {}_{(hv)}(h_n) = P_{h(Sv)}(h_n) = h(S_n(v)) = hS_nv$. Thus $S_nhv = hS_nv$, also

$S_nQv = {}_{(Qv)}(h_n) = P_{Q(Sv)}(h_n) = P(S_n(v)) = QS_nv$. Thus $S_nQv = QS_nv$. Namely, (S_n,h) and (S_n,Q) are weakly compatible.

By Theorem (4.1. 1-ii) $\forall$ n, $\forall$ a unique $v_n \in D \ni v_n = hv_n = Qv_n = S_nv_n = P_{Sv_n}(h_n)$.

By similar as Theorem (4.3.2-i) implies $\exists$ w, u, $v \in D$ and $<v_{n_i}> \subset$

$< v_n > \ni Sw = hw = u = Sv = Qv$, and $v_{n_i} = hv_{n_i} = Qv_{n_i} \to u$.

Now, as $i \longrightarrow \infty$ $\gamma(Sv_{n_i} - Sw) = \gamma(Sw - Sv_{n_i}) \leq \gamma(hw - Qv_{n_i}) \to 0$.

Hence, $Sv_{n_i} \longrightarrow Sw = u$ as $(i \longrightarrow \infty)$. As C_u-commuting of (S,h) and (S,Q) implies weakly compatible, then $hu = hSw = Shw = Su = SQv = QSv = Qu$.

by continuity of either S or h or Q that either $Sv_{n_i} \longrightarrow Su$ or $hv_{n_i} \longrightarrow hu$ or $Qv_{n_i} \longrightarrow Qu$. Hence $u = Su = hu = Qu$.

Therefore $P_A(u) \cap F(S) \cap F(h) \cap F(Q) \neq \emptyset$. This completes the proof. ■

Theorem (4.3.5):

Let M_γ be a complete modular space M_γ with (w)-convex structure, $u \in M_\gamma$, and $A \subseteq M_\gamma$, and S, h, $Q : M_\gamma \longrightarrow M_\gamma$, are three mappings. $\exists v \in A \ni \gamma(Sv) < \infty$, $\emptyset \neq P_A(u)$ and $\overline{S(P_A(u))} \subset P_A(u)$ and $h(P_A(u)) \cap Q(P_A(u)) = P_A(u)$ and S is a (h,Q)-non-expansive mapping on $P_A(u) \cup \{u\}$ and (S,h) and (S,Q) are C_u-commuting on $P_A(u)$ and S is strongly continuous, and $P_A(u)$ or $\overline{S(P_A(u))}$ or $h(P_A(u))$ or $q(P_A(u))$ is weakly compact. If h and Q have star-shaped then $P_A(u) \cap F(S) \cap F(h) \cap F(Q) \neq \emptyset$.

Proof:

Let $P_A(v_0) = D$, let $< h_n >$ and S_n be defined as in Theorem (4.3.4). Then a similar proof shows that $\forall\, n$, $\exists$ unique $v_n \in D \ni v_n = hv_n = Qv_n = S_n v_n = P_{Sv_n}(h_n)$. By the similar as Theorem (4.3.3-i) implies $\exists\, w, v, u \in D$ and $< v_{n_i} > \subset < v_n > \ni$

$Sw = hw = u = Sv = Pv$, and $v_{n_i} = hv_{n_i} = Qv_{n_i} = S_{n_i} v_{n_i} \xrightarrow{w} u$ and $\gamma(Sw - Sv_{n_i}) \leq \gamma(hw - Qv_{n_i}) \to 0$ and $Sv_{n_i} \xrightarrow{w} Sw = u$ as $(i \longrightarrow \infty)$.

Since C_u-commuting of (S,h) and (S,Q) implies weakly compatible, then $hu = hSw = Shw = Su = SQv = QSv = Qy$.

as S is strongly continuous together with $v_{n_i} \xrightarrow{w} u$, then $Sv_{n_i} \longrightarrow Su$. By $Sv_{n_i} \xrightarrow{w} u$, we have $u = Su = hu = Qu$. Thus $P_A(u) \cap F(S) \cap F(h) \cap F(Q) \neq \emptyset$. This completes the proof■

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

5-1 Conclusions:

We'll list our work as follows:

1- we have been reform many concepts in the setting of modular spaces, such as, weak convergence, dual of modular space, uniformly convex modular space, demi-closeness, proximinal set, … .

2- we have been prove that

-the relation between convergence and weak convergence,

-the completeness of dual space,

-the set of best approximations is non-empty, closed and bounded,

-the existences of best approximation for usc set-valued mapping, ..

-the existences of fixed points and its application in best approximation in some modular spaces,

-the existences of fixed points, common fixed points and coincidences points for non-expansive mappings, p- non-expansive mappings and (p,q)- non-expansive mappings for commuting and non-commuting mappings complete modular spaces,

-also, we have been employing these results to have best approximations.

3- this work requires the employment of convexity property, here, we use some of its generalizations, like, star-shapness property, affinenss property and w −convex structure.

4- Some of our results are a generalization of what is proved in the references.

5-2 Future Work:

Consider M be a linear space and $A \subseteq M$. A mapping$S$: $A \rightarrow 2^M$ is called $(KKM - map)$ if co$\{x_0, x_1, \dots, x_n\} \subseteq \bigcup_{i=0}^{n} Tx_i$ for each finite subset $\{x_0, x_1, \dots, x_n\}$ of A [19]

We suggest a study about best approximations in modular spaces via $(KKM - map)$ and give a version of Proll's theorem some other results.

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مستخلص

لقد كرس هذا البحث لدراسة خواص مجموعة التقريبات الافضل وتطبيق بعض مبرهنات النقطة الصامدة او مبرهنات نقطة التطابق للحصول على التقريب الافضل الثابت في فضاءات الوحدات (**Modular spaces**) لقد ضمنت فكرة الحصول على هذه النتائج في اربعة محاور. المحور الاول هو لأعادة صياغة بعض المفاهيم في حالة فضاء الوحدات, على سبيل المثال, التقارب القوي والتقارب الضعيف وثنائي فضاء الوحدات وغيرها ثم البرهنت بعض العبارات الضرورية والمتعلقة بالعمل. المحور الثاني يتضمن اعطاء مبرهنات من نمط مبرهنة بريزاسكي-منااردس (Brosowski-Minardus)حول التقريب الافضل الثابت. من جهة اخرى, وخصص المحور الثالث لتطبيق مبرهنات النقطة الصامدة المشتركة ومبرهنات نقطة التطابق وبأستخدم خاصية المحدبة (**w-convex**) للحصول على نتائج اخرى. أخير في المحور الرابع تم البرهنة على وجود مثل هذه التقريبات بالاعتماد على التطبيقات اللامتمددة المنفردة (**non-expansive-mapping**) او ذات القيم المتعددة والتطبيقات اللامتمددة والتطبيقات اللامتمددة المعممة (**(P,Q)-non-expansive-mapping and generalized**).

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

كلية التربية للعلوم الصرفة/ ابن الهيثم

قسم الرياضيات

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رساله مقدمة إلى مجلس كلية التربية للعلوم الصرفة /ابن الهيثم جامعة بغداد كجزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قبل

كرار عماد عبد الساده اللهيبي

بإشراف

ا.م.د.سلوى سلمان عبد

1439 هـ   2018 م

رسالة ماجستير

للطالب كرار عماد اللهيبي

وبأشراف

ا.م.د.سلوى سلمان عبد