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A Study of Some Generalizations of Fibrewise Bitopological Spaces

A Thesis
Submitted to the College of Education for Pure Sciences / Ibn Al-Haitham, University of Baghdad as a Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

تَبَٰرَكَ ٱلَّذِى بِيَدِهِ ٱلۡمُلۡكُ وَهُوَ عَلَىٰ كُلِّ شَيۡءٖ قَدِيرٌ ﴿١﴾ ٱلَّذِى خَلَقَ ٱلۡمَوۡتَ وَٱلۡحَيَوٰةَ لِيَبۡلُوَكُمۡ
أَيُّكُمۡ أَحۡسَنُ عَمَلٗاۚ وَهُوَ ٱلۡعَزِيزُ ٱلۡغَفُورُ ﴿٢﴾ ٱلَّذِى خَلَقَ سَبۡعَ سَمَٰوَٰتٖ طِبَاقٗاۖ مَّا تَرَىٰ فِى
خَلۡقِ ٱلرَّحۡمَٰنِ مِن تَفَٰوُتٖۖ فَٱرۡجِعِ ٱلۡبَصَرَ هَلۡ تَرَىٰ مِن فُطُورٖ ﴿٣﴾ ثُمَّ ٱرۡجِعِ ٱلۡبَصَرَ كَرَّتَيۡنِ
يَنقَلِبۡ إِلَيۡكَ ٱلۡبَصَرُ خَاسِئٗا وَهُوَ حَسِيرٞ ﴿٤﴾ وَلَقَدۡ زَيَّنَّا ٱلسَّمَآءَ ٱلدُّنۡيَا بِمَصَٰبِيحَ
وَجَعَلۡنَٰهَا رُجُومٗا لِّلشَّيَٰطِينِۖ وَأَعۡتَدۡنَا لَهُمۡ عَذَابَ ٱلسَّعِيرِ ﴿٥﴾

صدق الله العلي العظيم

الملك (1-5)

إهداء

الى من بلغ الرسالة ... وادى الامانة ... ونصح الامة ... الى نبي الرحمة ونور العالمين سيدنا محمد صلى الله عليه واله وسلم.

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۞ وَوَصَّيْنَا الْإِنسَانَ بِوَالِدَيْهِ إِحْسَانًا حَمَلَتْهُ أُمُّهُ كُرْهًا وَوَضَعَتْهُ كُرْهًا وَحَمْلُهُ وَفِصَالُهُ ثَلَاثُونَ شَهْرًا حَتَّىٰ إِذَا بَلَغَ أَشُدَّهُ وَبَلَغَ أَرْبَعِينَ سَنَةً قَالَ رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وعلى وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَصْلِحْ لِي فِي ذُرِّيَّتِي إِنِّي تُبْتُ إِلَيْكَ وَإِنِّي مِنَ الْمُسْلِمِينَ (15)

الاحقاف (15)

الى من علمني وأرشدني في حياتي وشجعني على تحقيق حلمي ابي (رحمه الله).

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Author's Publications

[1] Y. Y. Yousif and L. A. Hussain, Fibrewise Bitopological Spaces, International Journal of Science and Research (IJSR), Vol. 6, Issue 2, pp. 978-982, February (2017).

[2] Y. Y. Yousif and L. A. Hussain, Fibrewise Pairwise Bi-Separation Axioms, 1st Science International Conference College of Science / Al Nahrain University (21-22) November 2017. (Accepted, waiting publication).

[3] Y. Y. Yousif and L. A. Hussain Fibrewise IJ-Perfect Bitopological Spaces, Ibn AL-Haitham 1st. International Scientific Conference – 2017. (Accepted, waiting publication).

Abstract

In this research, we introduce and study the concept of fibrewise bitopological spaces. We generalize some fundamental results from fibrewise topology into fibrewise bitopological space. We also introduce the concepts of fibrewise closed bitopological spaces,(resp., open, locally sliceable and locally sectionable). We state and prove several propositions concerning with these concepts. On the other hand, we extend separation axioms of ordinary bitopology into fibrewise setting. The separation axioms we extend are called fibrewise pairwise T_0 spaces, fibrewise pairwise T_1 spaces, fibrewise pairwise R_0 spaces, fibrewise pairwise Hausdorff spaces, fibrewise pairwise functionally Hausdorff spaces, fibrewise pairwise regular spaces, fibrewise pairwise completely regular spaces, fibrewise pairwise normal spaces, and fibrewise pairwise functionally normal spaces. In addition, we offer some results concerning these extended axioms. Finally, we introduce some concepts in fibrewise bitopological spaces which are fibrewise ij-bitopological spaces, fibrewise ij-closed bitopological spaces, fibrewise $ij-$compact bitopological spaces, fibrewise ij-perfect bitopological spaces, fibrewise weakly ij-closed bitopological space, fibrewise almost ij-perfect bitopological space, fibrewise ij^*-bitopological spaces. We study several theorems and characterizations concerning these concepts.

Abbreviation

(M, τ)	topological space
(M, τ_1, τ_2)	bitopological space
$\cap$	intersection of set
$\cup$	union of set
$\in$	belong of set
$\notin$	not belong of set
p_M	projection function $p: M \to B$
M_b	$p^{-1}(b) : b \in B$
M_{B^*}	$p^{-1}(B^*) : B^* \subseteq B$
$M \mid B^*$	$p^{-1}(B^*) : B^* \subseteq B$
$\emptyset$	empty set
φ	fibrewise function
$\Delta_{s \in S} \varphi_s$	diagonal of function
id_M	identity function $id_M : M \to M$
π_2	projection function of product
$\Delta: M \to M \times_B M$	diagonal embedding
$(M \times_B M, \tau_1 \times \tau_1, \tau_1 \times \tau_1)$	product of two bitopology
$\mathbb{R}$	real numbers
$\mathcal{A}$	open cover
Γ	graph function
λ	continuous function $\lambda : M_W \to [0,1]$
$\mathcal{F} < \mathcal{G}$	$\mathcal{G}$ is finer than $\mathcal{F}$
$\prod_B M_r$	product of function

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Introduction

Mathematics plays a vital and an important role in the development of civilization which mankind has witnessed since the dawn of the history up to nowadays. Mathematics has undoubtedly the big favor in accelerating the wheel of progress for producing ideas and laws helped to organize and coordinate the various natural sciences such as, Geometry, Physics, Chemistry, Biology, Astronomy, Economics and Computers, etc.

The middle of 19^{th} century witnessed an important changes in mathematics structure, especially in Geometry. For the first time, the term of (Topology) has been used in 1847 in Germany by the German scientist (Johann Benedict Listing). Topology is a Greek word, consisting of two syllables : "topo" means a place, "logos" means study. At the beginnings of 20^{th} century as for 1925 up to 1975, this branch has clearly developed and formed an integrated competence. So, the topology is a science that deals with Geometry in a different way not as it used in Euclidean Geometry. This science distinguished by flexibility concerning the mathematical shapes. It could find the suitable solutions and remove the ambiguity of many problems that scientists couldn't find the right solutions through the Euclidean Geometry.

Bitopological spaces are first introduced by Kelly [18] in (1963) followed by many researchers who developed and generalized bitopological space on different science.

The concept of fibrewise set over a given set was introduced by James in [9], [10], [11], [12], [13], [14] in 1989.

In order to begin the work in the category of fibrewise (briefly F.W.) sets over a given set, called the base set, which is denoted by B. A F.W. set over B consist of a set M with a function $p: M \to B,$ that is called the projection. The fibre over b for every point b of B is the subset $M_b = p^{-1}(b)$ of M. Perhaps, fibre will be empty since we do not require p is surjectve, also, for every subset B^* of B we considered $M_{B^*} = p^{-1}(B^*)$ as a F.W. set over B^* with the projection determined by p. The alternative notation $M \mid B^*$ is some time convenient. We considered the Cartesian product $B \times T$, for every set $T,$ like a F.W. set B by the first projection.

The functions are not only a fundamental but the most important concepts in Mathematics for having a wide applications. Thus, the mathematical scientists were interesting in inserting this vital concept within topology for finding new visions and opening a wide horizons. For this reason, the general topology idea for the continuous functions or the general fibrewise topology which deals with the topological spaces as a mapping from this space onto a one point space.

To put the foundation stone for fibrewise topological spaces, many attempts appeared during the last two decades, most of the results, obtained so far in this field can be found in the work of Dyckhoff [6] in (1972) and Niefield [28] in (1984). Some hope of this is provided by the link between fibrewise topology and topos theory, referred to by Lever [21] and [22] in (1983, 1984) and Johnstone [15] in (1981, 1984). Moreover, in Pasynkov [29] in (1984) and James [9], [10], [11], [12], [13] and [14] in (1986, 1989), we can find definitions of some fibrewise topological spaces. Also in Buhagiar [5] in (1997), we can find definitions of some topological mappings which are precisely the definitions of fibrewise topological spaces, where the codomain is the base set. In (2003), Al-Zoubi and Hdeib [42] defined countably paracompact mappings, which are the fibrewise topological analogue of countably paracompact spaces finally Y.Y.Yousif and M. A. Hussain [35] and [36] in (2017) defined the concept of fibrewise soft topological spaces .Several characterizations of countably paracompact mappings are proved. As well as, we built on some of the result in [1], [2], [8], [17],[19], [20], [23], [31], [32], [33], [37], [38], [39], [40], [41].

The purpose of this thesis is to generalize fibrewise sets on the bitopological spaces, and to generalize some other mathematical concepts. The thesis will be entitled:

"A Study of Some Generalizations of Fibrewise Bitopological Spaces"

This thesis includes four chapters:

Chapter one: In this chapter we recall some of the fundamental definitions in the general topological spaces, bitopological spaces, and some basic concepts in the fibrewise spaces.

Chapter two: We introduce new definitions by mixing between the fibrewise sets and bitopological spaces and called it "fibrewise bitopological spaces". We deal with many definitions and theorems which are generalized from general topology.

Chapter three: We study a basic concept and very important in topology which is called separation axioms in which we put new definitions of spaces, T_0, T_1, T_2, T_3, regular, normal in the light of the fibrewise bitopological space.

Chapter four: The aim of this chapter is to study compact fibrewise bitopological spaces, closed fibrewise bitopological spaces, rigid fibrewise bitopological spaces and the relationship among them and we give some basic definitions on the concept of filter and the point which is related with director filter and convergence of the filter.

Chapter 1

Preliminary Concepts

Chapter 1

Preliminary Concepts

This chapter consists of two sections. Section one contains fundamental concepts of topological spaces, Bitopological spaces, compact spaces, the concept of filters, and filter base and some examples about some of these concepts. Section two gives an explains fibrewise sets theories and some of their properties.

1.1. Fundamental Notions of Topological (bitopological) Spaces

Some basic concepts in topology which are useful for our study are given in this section.

Definition 1.1.1. [7] Let X be a nonempty set and τ be a collection of subsets of X. The collection τ is said to be a topology on X if τ satisfies the following three conditions:

(a) $\phi \in \tau$ and $X \in \tau$,

(b) τ is closed under finite intersection,

(c) τ is closed under arbitrary union.

If τ is a topology on X, then the pair (X, τ) is called a topological space or simply X is a space. The subsets of X belonging to τ are called open sets in the space and the complement of the subsets of X which belongs to τ are called closed sets in the space.

Definition 1.1.2. [7] Let (X, τ) be a topological space and $A \subseteq X$. The closure (resp., interior) of A is denoted by $Cl(A)$ (resp., $Int(A)$) and is defined as:

$$cl(A) = \bigcap \{ F \subseteq X;\ F \text{ is closed set and } A \subseteq F\}.$$

$$int(A) = \bigcup \{ O \subseteq X;\ O \text{ is open set and } O \subseteq A\}.$$

Evidently, *cl(A)* (resp., *int(A)*) is the smallest closed (resp., largest open) subset of X which contains (resp., contained in) A. Note that A is closed (resp., open) if and only if $A = cl(A)$ (resp., $A = int(A)$).

Definition 1.1.3. [7] Let (X, τ) be a topological space and $A \subseteq X$. The boundary of A is denoted by *Bd(A)* and is defined by:

$$bd(A) = cl(A) - int(A).$$

Definition 1.1.4. [4] Let (X, τ) be a topological space and $A \subseteq X$. The subspace topology on A is denoted by τ_A and is defined by:

$$\tau_A = \{A \cap O;\ O \in \tau\}.$$

The subspace topology is also called the relative topology or the induced topology or the trace topology.

Definition 1.1.5. [7] A function $f : X \to Y$ is said to be continous if the inverse image of each open set in Y is open in X.

Definition 1.1.6. [7] A function $f : X \to Y$ is said to be open if the image of each open set in X is open in Y.

Definition 1.1.7. [7] A function $f : X \to Y$ is said to be closed if the image of each closed set in X is closed in Y.

The bitopological space was first created by Kelly [18] in 1963 and after that a large number of researches have been completed to generalize the topological ideas into bitopological setting.

Next Some basic concepts in bitopological spaces which are useful for our study are given.

Definition 1.1.8. [18] A triple (M, τ_1, τ_2) where M is a non-empty set and τ_1 and τ_2 are two topologies on M is called bitopological space.

Example 1.1.9. Let M = $\{1, 2, 3\}$, $\tau_1 = \{M, \emptyset, \{1, 2\}, \{2\}\}$, $\tau_2 = \{M, \emptyset, \{1, 3\}\}$ .Then (M, τ_1, τ_2) is bitopological space.

In this work (M, τ_1, τ_2) and (N, σ_1, σ_2) (briefly, M and N) always mean bitopological spaces on which no separation axioms are supposed unless clearly stated. By τ_i-open (resp., τ_i-closed), we shall mean the open (resp., closed) set with respect to τ_i in M, where $i = 1,2$. A set A is open (resp., closed) in M if it is both τ_1-open (resp., τ_1-closed) and τ_2-open (resp., τ_2-closed).

In what follows we consider $i, j \in \{1, 2\}; i \neq j$.

Definition 1.1.10. [18] A function $f : (M, \tau_1, \tau_2) \rightarrow (N, \sigma_1, \sigma_2)$ is said to be τ_i-continuous (resp., τ_i-open, τ_i-closed), if the function $f: (M, \tau_i) \rightarrow (N, \sigma_i)$ is continuous (resp., open, closed). f is called continuous (resp., open, closed) if it is τ_i-continuous (resp., τ_i-open, τ_i-closed) for every $i = 1,2$.

Example 1.1.11. Let M = {1, 2, 3} and $N = \{a, b, c\}$ be two sets. Let τ_1 and τ_2 (resp. σ_1 and σ_2) be two topologies on M (res. N) such that τ_1 ={M, ∅, {1, 2}} and τ_2 = {M, ∅, {3}, {2, 3}}, σ_1 ={N, ∅, {a, b}}, σ_2 = {N, ∅, {c}, {a, c}}. Define $\varphi : M \rightarrow N$ such that $\varphi(1) = b, \varphi(2) = a, \varphi(3) = c$. Then φ is continuous (open and closed).

Definition 1.1.12. [27] A bitopological space (M, τ_1, τ_2) is said to be paiwise T_0 space if for every pair of points *x* and *y* such that $x \neq y$ there exists a τ_i-open set containing *x* but not containing *y* or a τ_j-open set containing *y* but not containing *x*, where $i, j = 1, 2$, $i \neq j$.

Definition 1.1.13. [16] A point x in (M, τ_1, τ_2) is called an ij-contact point of a subset $A \subseteq M$ iff for every τ_i-open neighborhood (nbd) U of x, $(\tau_{j\_}cl(U)) \cap A \neq \emptyset$. The set of all ij-contact points of A is called the ij-closure of A and is denoted by $ij - cl(A)$. $A \subset M$ is called ij-closed iff $A = ij - cl(A)$, where $i, j = 1, 2$.

Definition 1.1.14. [4] A filter $\mathcal{F}$ on a set M is a nonempty collection of nonempty subsets of M with the properties:

(a) If F_1, $F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

(b) If $F \in \mathcal{F}$ and $F \subseteq F^* \subseteq$ M, then $F^* \in \mathcal{F}$.

Definition 1.1.15. [4] A filter base $\mathcal{F}$ on a set M is a nonempty collection of nonempty subsets of M such that if F_1, $F_2 \in \mathcal{F}$ then $F_3 \subset F_1 \cap F_2$ for some $F_3 \in \mathcal{F}$.

Definition 1.1.16. [4] If $\mathcal{F}$ and $\mathcal{G}$ are filter bases on M, we say that $\mathcal{G}$ is finer than $\mathcal{F}$ (written as $\mathcal{F} < \mathcal{G}$) if for each $F \in \mathcal{F}$, there is $G \in \mathcal{G}$ such that $G \subseteq F$ and that $\mathcal{F}$ meets $\mathcal{G}$ if $F \cap G \neq \emptyset$ for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

Definition 1.1.17. [4] A filter base $\mathcal{F}$ on M is said to be ij-converges to a subset A of M (written as $\mathcal{F} \xrightarrow{ij-con}$ A) iff for every τ_i-open cover $\mathcal{U}$ of A, there is a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ and a member F of $\mathcal{F}$ such that $F \subset \cup$

$\{\tau_j - cl(U): U \in \mathcal{U}_0\}$. Also if $x \in M$, we say $\mathcal{F} \xrightarrow{ij-con} x$ iff $\mathcal{F} \xrightarrow{ij-con} \{x\}$ or equivalently, τ_j-closure of every τ_i-open nbd of x contains some members of $\mathcal{F}$.

Definition 1.1.18. [3] A function $f : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2)$ is called ij-continuous iff for any $x \in M$, there exist σ_i-open nbd V of $f\ (x)$, there exists a τ_i-open nbd U of x such that $f\ (\tau_j\text{-}cl(U)) \subseteq \sigma_j\text{-}cl(V)$, where $i, j\ = 1, 2$.

Definition 1.1.19. [3] A point x in a bitopological space (M, τ_1, τ_2) is called an ij-adherent point of a filter base $\mathcal{F}$ on M iff it is an ij-contact point of every number of $\mathcal{F}$. The set of all ij-adherent points of $\mathcal{F}$ is called the ij-adherence of $\mathcal{F}$ and is denoted by ij-ad $\mathcal{F}$, where $i, j\ = 1, 2$.

Definition 1.1.20. [24] A subset A in bitopological space $(M, \tau_1,\ \tau_2)$ is called ij-H-set in M iff for each τ_i-open cover $\mathcal{A}$ of A, there is a finite subcollection $\mathcal{B}$ of $\mathcal{A}$ such that $A\ \subset \cup \{\ \tau_j - cl(U)\ :\ U\ \in\ \mathcal{B}\}, i, j = 1{,}2$. A is called a pairwise-H-set iff it is a 12- and 21-H-set. If A is an ij-H-set (pairwise-H-set) and $A = M$, then the space is called an ij-QHC (resp., pairwise QHC) space, where $i, j\ = 1, 2$.

Lemma 1.1.21. [25] A subset A of a bitopological space $(M, \tau_1,\ \tau_2)$ is an ij-H-set iff for each filter base $\mathcal{F}$ on A, $(ij - ad\ \mathcal{F})\ \cap\ A\ \neq \varphi$, where $i, j\ = 1, 2$.

Proof: (⇒) Clear.

(⇐) Let $\mathcal{A}$ be a τ_i-open cover of A such that the union of τ_j-closure of any finite sub collection of $\mathcal{A}$ is not cover A. Then $\mathcal{F}$ =$\{A \backslash \cup_{\mathcal{B}}\ \tau_j$-cl (B) : $\mathcal{B}$ is finite sub collection of $\mathcal{A}\}$is a filter base on A and (ij-ad $\mathcal{F}$) $\cap\ A =\ \varphi$. This is a contradiction. Thus, A is ij-H-set.

Definition 1.1.22 [25] A topological space (M,τ) is called Urysohn space iff for each $x \neq y$ can be seperated by closed nbd.

Definition 1.1.23. [25] A bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ is said to be pairwise Urysohn space if for $x, y \in M$ with $x \neq y$, there are τ_i-open nbd U of x and τ_j-open nbd V of y such that $\tau_j - cl(U) \cap \tau_i - cl(V) = \varphi$, where $i, j = 1, 2$.

Lemma 1.1.24. [25] In a pairwise Urysohn bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ an ij-H-set is ij-closed, where $i, j = 1, 2$.

Lemma 1.1.25. [16] In a bitopological space $(\mathrm{M}, \tau_1, \tau_2)$. If $U \in \tau_j$, then $ij - cl(U) = \tau_j - cl(U)$, where $i, j = 1, 2$.

Lemma 1.1.26. [30] The bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ is pairwise Hausdorff iff $\{m\} = ij - cl\{m\}$, for each $m \in M$.

1.2. Fundamental Notions of Fibrewise Topology

In order to begin the category in the classification of fibrewise (briefly, F.W.) sets over a given set, called the base set, which say B. A F.W. set over B consists of a set M with a function $p: M \to B$, that is called the projection. The fibre over b for every point b in B is the subset $M_b = p^{-1}(b)$ of M. Perhaps, fibre will be empty since we do not require p is surjectve, also, for every subset B^* of B, we consider $M_{B^*} = p^{-1}(B^*)$ as a F.W. set over B^* with the projection determined by p. The alternative notation of M_{B^*} is sometime referred to as $M \mid B^*$. We consider the Cartesian product $B \times T$, for every set T, as a F.W. set over B by the first projection.

Definition 1.2.1. [9] If M and N are F.W. sets over B, with projections p_M and p_N, respectively. A function $\varphi: M \to N$ is said to be F.W. function if $p_N \text{ o } \varphi = p_M$, or $\varphi(M_b) \subset N_b$ for every point b of B, where $p_N : N \to B$ and $p_M: M \to B$.

Example 1.2.2. Let M = $\{1, 2, 3\}$, N = $\{2, 4, 6\}$ B = $\{a, b, c\}$, let $p_M: M \to B$ where : $p_M(1) = a, p_M(2) = b, p_M(3) = c$. Let $p_N: N \to B$ where: $p_N(2) = a, p_N(4) = c, p_N(6) = b$. Let $\varphi: M \to N$ where: $\varphi(1) = 2, \varphi(2) = 6, \varphi(3) = 4$. Then φ is a fibrewise function.

Note that a F.W. function $\varphi: M \to N$ over B is determines, by a restriction, a F.W. function $\varphi_{B^*}: M_{B^*} \to N_{B^*}$ over B^* for every subset B^* of B.

Definition 1.2.3. [9] Let (B, Λ) be a topological space. The F.W. topology on a F.W. set M over B means any topology on M for which the projection p is continuous.

Definition 1.2.4. [9] The F.W. topological space (M, τ) over (B, Λ) is called F.W. closed (resp., F.W. open) if the projection p is closed (resp., open).

Example 1.2.5. Let $B = \{1, 2, 3\}$, $\Lambda = \{B, \varphi, \{1\}, \{1, 2\}\}$. Let M be fibrewise set over B where $M = \{a, b\}$ and let $p: M \to B$ such that $p(a) = 1, p(b) = 2$. Let $\tau = \{M, \varphi, \{a\}\}$ be any topology on M. Then p is continuous and (M, τ) is F.W. topology on (B, Λ).

Definition 1.2.6. [9] The F.W. function $\varphi: M \to N$, where M and N are F.W. topological spaces over B is called

(a) continuous if for every point $m \in M_b$; $b \in B$, the inverse image of every open set of $\varphi(m)$ is an open set of m.

(b) open if for every point $m \in M_b$; $b \in B$, the image of every open set of m is an open set of $\varphi(m)$.

Example 1.2.7. Let $M = \{1,2,3\}$, $\tau = \{M, \varphi, \{3\}, \{1,3\}\}$, and let $N = \{2, 3, 5\}$, $\sigma = \{N, \emptyset, \{2,5\}, \{2\}\}$. Let $B = \{a, b, c\}$, and $\Lambda = \{B, \emptyset, \{a\}, \{a, c\}\}$. Assume that $\varphi : M \to N$ be function where $\varphi(1) = 2, \varphi(3) = 5, \varphi(2) = 3$. Let $p_M : M \to B$ such that $p_M(1) = c, p_M(2) = b, p_M(3) = a$. Let $p_N : N \to B$ such that $p_N(2) = a, p_N(3) = c, p_N(5) = a$. So p is continuous and open.

Example 1.2.8. Let $M = \{1,2,3\}$, $\tau = \{M, \emptyset, \{1\}, \{2,3\}\}$. Let $B = \{a, b, c\}$, $\Lambda = \{B, \emptyset, \{\text{b}\}, \{\text{a, c}\}\}$. Let $p : M \to B$ where $p(1) = b, p(2) = c, p(3) = a$. Let $\tau^c = \{M, \varphi, \{1\}, \{2,3\}\}$, $\Lambda^c = \{B, \emptyset, \{b\}, \{a, c\}\}$. Then p is closed (resp., open).

Definition 1.2.9. [7] Assume that we are given a topological space M, a family $\{\varphi_s\}_{s\in S}$ of continuous functions, and a family $\{N_s\}_{s\in S}$ of topological spaces where the function $\varphi_s : M \to N_s$ that transfers $x \in M$ to the point $\{\varphi_s(x)\} \in \prod_{s\in S} N_s$ is continuous, it is called the diagonal of the functions $\{\varphi_s\}_{s\in S}$ and is denoted by $\Delta_{s\in S}\varphi_s$ or $\varphi_1 \Delta\, \varphi_2\, \Delta \ldots \Delta\, \varphi_k$ if $S = \{1, 2, \ldots, k\}$.

Definition 1.2.10. [34] For every topological space M^* and any subspace M of M^*, the function $i_M : M \to M^*$ define by $i_M(x) = x$ is called embedding of the subspace M in the space M^*. Observe that i_M is continuous, since $i_M^{-1}(U) = M \cap U$, where U is open set in M^*. The embedding i_M is closed (resp., open) iff the subspace M is closed (resp., open).

Definition 1.2.11. [34] if X is topological space and x$\in X$ a nieghberhood of x is a set U which contain an open set V containig x. If A is open set and contains x we called A is open neighborhood for a point x.

Definition 1.2.12.[4] A topological space (M, τ) is called compact iff each open cover of M has a finite subcover for M.

Definition 1.2.13. [26] Let (M, τ) and (N, σ) be topological spaces. A function $f: M \to N$ is a local homeomorphism if for every point x in M there exists an open set U containing x, such that the image is open in N and the restriction is a homeomorphism.

Definition 1.2.14. [18] A bitopological space $(M,\ \tau_1, \tau_2)$ is said to be pairwise Hausdorff , if for each distinct points $x, y \in M$ there exist disjoint sets τ_i-open set U of x and τ_j-open set V of y, for $i\,, j\ =\ 1\,, 2\,, i\ \neq\ j$.

Chapter 2

Fibrewise Bitopological Spaces

Chapter 2

Fibrewise Bitopological Spaces

The aim of this chapter is to introduce a new bitopological structure which is called Fibrewise Bitopological space. We define the concept of bitopological space based on a fibrewise set. Some examples and theories related to this new structure are introduced. In section one, we defined the concept of fibrewise bitopological spaces and the notion of induced fibrewise bitopological spaces. In section two we studied the notions of fibrewise open and fibrewise closed bitopological spaces. The purpose of section three is to show the notions of Fibrewise locally sliceable and fibrewise locally sectionable bitopological spaces.

2.1. Fibrewise Bitopological Spaces

In this section we establish F.W. bitopological spaces. Several topological properties on this space are obtained and studied.

Definition 2.1.1. Let $(B,\ \Lambda_1,\ \Lambda_2)$ be a bitopological space. The F.W. bitopology on a F.W. set M over B means any bitopology on M for which the projection p is continuous.

Example 2.1.2. Let B = {a, b, c}, Λ_1 ={B, $\emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{a, c\}\}$. Let M be a fibrewise set over B where $M = \{1, 2, 3\}$. Let $\tau_1 = \{M, \varphi, \{1\}\}, \tau_2 = \{M, \emptyset\}$. Let p : $M \ \to B$ where $p(1) = a, p(2) = c = p(3)$. Then (M, τ_1, τ_2) is a fibrewise bitopology on (B, Λ_1, Λ_2).

For another example, we consider $(B,\ \Lambda_1,\ \Lambda_2)$ as a F.W. bitopological spaces over itself with the identity as a projection. Also, if we consider the bitopological product $B \times T$, for every bitopological space T, can be regarded as a

F.W. bitopological space over B, by the first projection. The latter situation can be applied for every subspace of $B \times T$.

Remarks 2.1.3.

(a) In F.W. bitopology, we work over bitopological base space $(B,\ \Lambda_1,\ \Lambda_2)$. If B is a point–space, the theory changes to that of ordinary bitopology.

(b) A F.W. bitopological spaces over B is just a bitopological space $(M,\ \tau_1, \tau_2)$ with a continuous projection $p:\ (M,\ \tau_1, \tau_2) \longrightarrow\ (B,\ \Lambda_1,\ \Lambda_2)$.

(c) The coarsest such bitopology is obtained by p, in which the τ_i −open set of $(M,\ \tau_1, \tau_2)$ is exactly the inverse image of the Λ_i −open set of $(B,\ \Lambda_1,\ \Lambda_2)$; called, the F.W. indiscrete bitopology, where i =*1, 2*.

(d) The F.W. bitopological space over $(B,\ \Lambda_1,\ \Lambda_2)$ is defined to be a F.W. set over B with F.W. bitopology.

(e) We consider the bitopological product $B \times T$, for every bitopological space T, as a F.W. bitopological spaces over B by the first projection.

Definition 2.1.4. The F.W. function $\varphi : M \to N$ where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$ are said to be:

(a) i −continuous if for every point $m \in M_b$; $b \in B$, the inverse image of every σ_i −open set of $\varphi(m)$ is τ_i −open set contain m. φ is called continuous if it is i −continuous for every $i = 1,2$.

(b) i −open if for every point $m \in M_b$; $b \in B$, the image of every τ_i −open set of m is σ_i −open set of $\varphi(m)$. φ is called open if it is i −open for every $i = 1,2$.

(c) i −closed if for every point $m \in M_b$; $b \in B$, the image of every τ_i −closed set of m is σ_i −closed set of $\varphi(m)$. φ is called closed if it is i −closed for every $i = 1,2$.

Example 2.1.5. Let $M = \{1, 2, 3\}$, $\tau_1 = \{M, \emptyset, \{1\}, \{2, 3\}\}$, and $\tau_2 = \{M, \emptyset, \{3\}\}$. Let $\mathrm{N} = \{4, 5, 6\}$, $\sigma_1 = \{N, \emptyset, \{5\}, \{4, 6\}\}$, and $\sigma_2 = \{N, \emptyset, \{4\}\}$. Let B = {a, b, c}, $\Lambda_1 = \{B, \emptyset, \{a\}, \{b, c\}, \{b\}, \{a, c\}, \{c\}, \{a, b\}\}$, $\Lambda_2 = \{B, \emptyset, \{b\}, \{a\}, \{a, b\}\}$. Define $p_M : M \to B$ such that $p_M(1) = a, p_M(2) = c, p_M(3) = b$. Define $p_N : N \to B$ such that $p_N(4) = a, p_N(5) = b, p_N(6) = c$. Let $\varphi : M \to N$ such that $\varphi(3) = 4, \varphi(1) = 5, \varphi(2) = 6$. Then φ is continuous, open, closed.

If $\varphi : M \to N$ is a F.W. function where M is a F.W. set and (N, σ_1, σ_2) is a F.W. bitopoligical space over $(B, \Lambda_1, \Lambda_2)$. We can give M the induced bitopology, in the ordinary sense and this is necessarily a F.W. bitopology. We may refer to it, therefore, like the induced F.W. bitopology and note the next characterizations.

Proposition 2.1.6. Let $\varphi : M \to N$ be a F.W. function, where (N, σ_1, σ_2) is a F.W. bitopoligical space over $(B, \Lambda_1, \Lambda_2)$ and M has an induced F.W. bitopology. Then for every F.W. bitopoligical space (Q, δ_1, δ_2) a F.W. function $\psi: (Q, \delta_1, \delta_2) \to (M, \tau_1, \tau_2)$ is continuous iff the composition $\varphi \text{ o } \psi: Q \to N$ is continuous.

Proof. $(\Rightarrow)$ Suppose that ψ is continuous. Let $q \in Q_b$; $b \in B$ and let V be σ_i-open set of $(\varphi\ o\ \psi)(q) = n \in N_b$ in N. Since φ is continuous, then $\varphi^{-1}(V)$ is τ_i −open set containing $\psi(q) = m \in M_b$ in M. Since ψ is continuous, then $\psi^{-1}(\varphi^{-1}(V))$ is a δ_i −open set containing $q \in Q_b$ in Q and $\psi^{-1}(\varphi^{-1}(V)) = (\varphi\ o\ \psi)^{-1}(V)$ is a δ_i −open set containing $q \in Q_b$ in Q, where $i = 1,2$.

$(\Leftarrow)$ Suppose that $\varphi\ o\ \psi$ is continuous. Let $q \in Q_b$; $b \in B$ and U be a τ_i −open set of $\psi(q) = m \in M_b$ in M. Since φ is open then, $\varphi(U)$ is a σ_i −open set containing $\varphi(m) = \varphi(\psi(q)) = n \in N_b$ in N. Since $\varphi o \psi$ is continuous, then $(\varphi\ o\ \psi)^{-1}(\varphi(U)) = \psi^{-1}(U)$ is a δ_i −open set containing $q \in Q_b$ in Q, where $i = 1,2$.

Proposition 2.1.7. Let $\varphi: M \to N$ be a F.W. function where, (N, σ_1, σ_2) a F.W. bitopoligical space over $(B, \Lambda_1, \Lambda_2)$ and M has an induced F.W. bitopology. Then for every F.W. bitopoligical space (Q, δ_1, δ_2), the surjectve F.W. function $\psi: (Q, \delta_1, \delta_2) \to (M, \tau_1, \tau_2)$ is open iff the composition $\varphi o \psi: (Q, \delta_1, \delta_2) \to (N, \sigma_1, \sigma_2)$ is open.

Proof. (⇒) Suppose that ψ is open. Let $q \in Q_b$; $b \in B$ and let U be a δ_i-open set of q in Q. Since ψ is open, then $\psi(U)$ is τ_i open set containing $\psi(q) = m \in M_b$ in M where $i = 1, 2$. Since φ is open, then $\varphi\big(\psi(U)\big)$ is σ_i-open set containing $\varphi(m) = n \in N_b$ in N. And $\varphi\big(\psi(U)\big) = \varphi o \psi(U)$. where $i = 1, 2$.

(⇐) Suppose that $\varphi\ o\ \psi$ is open. Let $q \in Q_b$; $b \in B$. Let U be a $\delta_i -$ open set of q in Q. Since $\varphi\ o\ \psi$ is open, then $\varphi\ o\ \psi(U)$ is σ_i-open set containing $\varphi\ o\ \psi(q) = n \in N_b$. Since φ is continuous, then $\varphi^{-1}(\varphi\ o\ \psi\ (U))$ is τ_i-open set of $\psi\ (q) = m \in M_b$ in M. But $\varphi^{-1}\big(\varphi\ o\ \psi(U)\big) = \psi(U)$, where $i = 1,2$.

Let us consider general cases of Propositions (2.1.6) and (2.1.7) as follows:

Corollary 2.1.8.

In the case of families $\{\varphi_r\}$ of F.W. functions, where $\varphi_r: M \to N_r$ with $(N_r, \sigma_{r1}, \sigma_{r2})$ F.W. bitopological space over B for every r. Specially, given a family $\{(M_r, \tau_{r1}, \tau_{r2})\}$ of F.W. bitopological space over B, the F.W. bitopological product $\prod_B M_r$ is defined to be the F.W. product with the F.W. bitopology generated by the family of projections $\pi_r: \prod_B M_r \to M_r$. Then for every F.W. bitopological space(Q, δ_1, δ_2) over B, a F.W. function $\theta: Q \to \prod_B M_r$ is continuous (resp., open). For example when $M_r = M$ for every index r we see that the diagonal $\Delta: M \to \prod_B M$ is continuous (resp., open) iff the composition $\pi_r o\Delta= id_M$ is continuous (resp., open).

2.2. Fibrewise Closed and Fibrewise Open Bitopological Spaces

In this section we introduce the F.W. closed and F.W. open bitopological spaces over B. Several topological properties on these concepts are studied.

Definition 2.2.1. The F.W. bitopoligical space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. closed if the projection p is closed.

Example 2.2.2. Let $\mathrm{M} = \{1,2,3\}$, $\tau_1 = \{M, \emptyset, \{1\}, \{2,3\}\}, \tau_2 = \{M, \emptyset, \{2\}, \{1, 2\}\}$, $B = \{a, b, c\}$, $\Lambda_1 = \{B, \emptyset, \{b\}, \{a, c\}\}, \Lambda_2 = \{B, \emptyset, \{c\}, \{b, c\}\}$. Let $p: M \to B$ such that $p(1) = b, p(2) = c, p(3) = a$. Then p is closed and (M, τ_1, τ_2) is F.W. closed space.

For another example is to consider trivial F.W. bitopological space with compact fibre is F.W. closed.

Proposition 2.2.3. Let $\varphi : M \to N$ be a closed F.W. function where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then M is a F.W. closed if N is a F.W. closed.

Proof. Assume that $\varphi: M \to N$ is a closed F.W. function and N is F.W. closed i.e. the projection $p_N: N \to B$ is closed. To prove that M is F.W. closed i.e. $p_M: M \to B$ is closed. Now, let $m \in M_{\mathrm{b}}$; $\mathrm{b} \in B$, and F be τ_i −closed set of m where $i = 1, 2$. Since φ is closed, then $\varphi(F)$ is $\sigma_i -$ closed set of $\varphi(m) = n \in N_{\mathrm{b}}$ in N. Since p_N is closed, hence $p_N(\varphi(F))$ is Λ_i – closed set in B.

But, $p_N \, o \, \varphi(F) = p_M(F)$ is σ_i –closed set of F. Thus, p_M is closed and M is a F.W. closed where i = 1, 2.

Proposition 2.2.4. If $(M\,,\tau_1\,,\tau_2)$ is a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that M_j is a F.W. closed for every member M_j of a finite covering of M. Then M is a F.W. closed.

Proof. Assume that M is a F.W. bitopological space over B, then the projection $p_M : M \to B$ exist. To prove that p is closed. Since M_j is F.W. closed, then the projection $p_{M_j} : M_j \to B$ is closed for every member M_j of a finite covering of M. Let F be τ_i-closed subset of M . Then $p(F) = \cup\, p_j\big(M_j \cap F\big)$ which is a finite union of closed sets and so p is closed. Thus M is F.W. closed where $i = 1, 2$.

Proposition 2.2.5. Let $(M\,,\tau_1\,,\tau_2)$ be a F.W. bitopoligical space over $(B, \Lambda_1, \Lambda_2)$. Then $(M\,,\tau_1, \tau_2)$ is a F.W. closed iff for every fibre M_b, $b \in B$ of M and every $\tau_i -$ open set U of M_b in M, there is a Λ_i −open set O of b where $M_O \subset U$, $i = 1, 2$.

Proof. (⇒) Assume that M is closed. i.e., $p: M \to B$ is closed. Now, let $b \in B$ and U be τ_i −open set of M_b where $i = 1, 2$. Thus we have $M - U$ is τ_i −closed set and $p(M - U)$ is Λ_i −closed set. Let $O = B - p(M - U)$ is Λ_i −open set of b. Hence, $M_O = p^{-1}(B - p(M - U))$ is a subset of U.

(⇐) Suppose that the other direction is hold, to show that M is closed. Let F be τ_i −closed set in M where i=1, 2. Let $b \in B - p(F)$ and every τ_{i-}open set U of M_b in M. By assumption there is Λ_i −open set O of b such that $M_O \subset U$. It's easy to show that $O \subset B - p(F)$. Hence, $B - p(F)$ is Λ_i −open set in B. Hence, $p(F)$ is a Λ_i −closed in B, p is closed, and M is F.W. closed bitopological, where i=1, 2.

Definition 2.2.6. The F.W. bitopoligical space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. open if the projection p is open.

Example 2.2.7. Let $M = \{x, y, z\}$, $\tau_1 = \{M, \emptyset, \{x\}, \{y, z\}\}$, $\tau_2 = \{M, \emptyset, \{y\}, \{x, y\}\}$. Let B= $\{a, b, c\}$, $\Lambda_1 = \{B, \emptyset, \{b\}, \{a, c\}\}$, $\Lambda_2 = \{B, \emptyset, \{c\}, \{b, c\}\}$. Let $p : M \to B$ such that $p(x) = b$, $p(y) = c$, $p(z) = a$. Then p is open and (M, τ_1, τ_2) is F.W. open space.

For another example, trivial F.W. bitopological spaces are always F.W. open.

Proposition 2.2.8. Let $\varphi: M \longrightarrow N$ be an open F.W. function where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If N is F.W. open, then M is F.W. open.

Proof. Since N is F.W. open, we have $p_N : N \to B$ is open. To prove that p_M is open. i.e., $p_M : M \to B$ is open. Let $m \in M_b$; b $\in B$, and let U be τ_i – open set of m where $i = 1, 2,$ since φ is open then $\varphi(U)$ is σ_i – open set of $\varphi(m) = n \in N_b$ in N. Also, since N is F.W. open then p_N (φ(U)) is Λ_i – open set in B. Since $p_N \ o\ \varphi(U) = p_M(U)$, then p_M is open and M is F.W. open, where $i = 1,2$.

Proposition 2.2.9. Let $\varphi: M \longrightarrow N$ be a F.W. function where $(M, \tau_1, \tau_2)$and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Assume that the product: $id_M \times \varphi: (M \times_B M, \tau_1 \times \tau_1, \tau_1 \times \tau_1) \longrightarrow (M \times_B N, \tau_1 \times \sigma_1, \tau_2 \times \sigma_1)$ is open and M is F.W. open . Then φ itself is open.

Proof. Consider the following figure:

$$\begin{tikzcd} M \times_B M \arrow[r, "id_M \times \varphi"] \arrow[d, "\pi_2"'] & M \times_B N \arrow[d, "\pi_2"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Figure 2.2.1. Diagram of Proposition 2.2.9.

The projection on the left is surjective while the projection on the right is open because M is F.W. open bitopological space. Thus, $\pi_2 o(id_M \times \varphi) = \varphi o \pi_2$ is open and thus, φ is open.

Our next three results apply equally to F.W. closed and F.W. open bitopological spaces, respectively.

Proposition 2.2.10. Let $\varphi: M \longrightarrow N$ be a surjection F.W. continuous where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then N is F.W. closed (resp., open) if M is F.W. closed (resp., open).

Proof. Suppose that M is a F.W. closed (resp., open). Then $p_M: M \longrightarrow B$ is closed (resp., open). To prove that N is a F.W. closed (resp., open) bitopological space over B. i.e., the projection $p_N : (N, \sigma_1, \sigma_2) \rightarrow (B, \Lambda_1, \Lambda_2)$ is closed (resp., open). Suppose that $\in N_b$; $b \in B$. Let V be $\sigma_i -$ closed (resp., open) set of n where $i = 1, 2$. Since φ is continuous, then $\varphi^{-1}(V)$ is $\tau_i -$closed (resp., τ_i-open) set of $\varphi^{-1}(n) = m \in M_b$ in M where $i = 1, 2$. Since p_M is closed (resp., open) then $p_M(\varphi^{-1}(V))$ is closed (resp., open) set in B. But, $p_M\left(\varphi^{-1}(V)\right) = p_N(V)$. Thus p_N is closed (resp., open), and N is F.W. closed (resp., open).

Proposition 2.2.11. If (M, τ_1, τ_2) is a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that M is F.W. closed (resp., open) over B. Then M_{B^*} is a F.W. closed (resp., open) over B^* for every subspace B^* of B.

Proof. Assume that M is F.W. closed (resp., open) so that the projection $p: M \to B$ is closed (resp., open). To prove that M_{B^*} is closed (resp., open). i.e., the projection p_{B^*}:$M_{B^*} \longrightarrow B^*$ is closed (resp., open). Let $m \in M \mid B^*$, G be τ_{i-} closed (resp., τ_i-open) set of m, where $i = 1, 2$. G $\cap M_{B^*}$ is τ_{iB^*} $-$closed (resp., τ_{iB^*} $-$open) set of M_{B^*}. $p_{B^*}(G \cap M_{B^*}) = p(G \cap M_{B^*}) = p(G) \cap p(M_{B^*}) = p(G) \cap B^*$ which is Λ_{iB^*} $-$closed (resp., Λ_{iB^*}-open) set in B^*. p_{B^*} is closed (resp., open). Thus, M_{B^*} is F.W. closed (resp., open), where $i = 1,2$.

Proposition 2.2.12. Let (M, τ_1, τ_2) be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that $(M_{B_j}, \tau_{1B_j}, \tau_{2B_j})$ is a F.W. closed (resp., open) bitopoligical spaces over $(B_j, \Lambda_{1B_j}, \Lambda_{2B_j})$ for every member of a Λ_{iB_j}-open covering of B. Then M is a F.W. closed (resp., open) bitopoligical space over B, where $i = 1, 2$.

Proof. Assume that M is F.W. bitopological space over B then, the projection $p: M \to B$ exist .To prove that p is closed (resp., open). Since M_{Bj} is closed (resp., open) over B_j for every member Λ_i $-$open covered of B where $i = 1, 2$, then the projection $p_{Bj} : M_{Bj} \to B_j$ is closed (resp., open). Now, let F be τ_i-closed (resp., τ_i-open) set of M_b ; $b \in B$, $p(F) = \cup P_{Bj}(F \cap M_{Bj})$ which is a finite union of Λ_i –closed (resp., open) sets of B. Thus, p is closed (resp., open) and M is closed (resp., open) F.W. bitopological space over B, where $i = 1,2$.

Actually, the proceeding proposition is true in locally finite closed covering see Theorem (1.1.11) and Corollary (1.1.12) in [7].

There are several subclasses of the class of F.W. open bitopological spaces which induced many important examples and have interesting properties.

2.3. Fibrewise Locally Sliceable and Fibrewise Locally Sectionable Bitopological Spaces

In this section, we generalize F.W. locally sliceable and F.W. locally sectionable bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. some topological properties related to these concepts are studied.

Definition 2.3.1. The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called locally sliceable if for every point $m \in M_b$; $b \in B$, there exist a Λ_i $-$open set W of b and a section $s: W \to M_W$ such that $s(b) = m$, for $i = 1$ or 2.

Example 2.3.2. Let M = {1, 2}, $\tau_1 = \{M, \emptyset, \{1\}\}, \tau_2 = \{M, \emptyset, \{2\}\}$. Let B = {a, b}, $\Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p : M \to B$ where $p\,(1) = a, p\,(2) = b$. We have $M_a = \{1\}$, $M_b = \{2\}$. Let $s_1 : \{a\} \to \{1\}$ where $s_1(a) = 1$, $s_2 : \{b\} \to \{2\}$ where $s_2(b) = 2$ .Then M is a F.W. locally sliceable bitopological space.

The condition leads to p is open for if U is a τ_i $-$open set of m in M, then $s^{-1}(M_W \cap U) \subset p(U)$ is a Λ_i $-$open set of b in W, and hence, in B, where $i = 1, 2$. The class of locally sliceable bitopological space is finitely multiplicative.

Proposition 2.3.3. Let $\{ (M_r, \tau_{r1}, \tau_{r2}) \}_{r=1}^{k}$ be a finite family of locally sliceable bitopological space over $(B, \Lambda_1, \Lambda_2)$. The F.W. bitopological product $M = \prod_B M_r$ is locally sliceable.

Proof. Let $m = (m_r)$ be a point of M_b ; $b \in B$, so that $m_r = \pi_r(m)$ for every index r. Since M_r is a locally sliceable bitopological space, there is a Λ_i $-$open set W_r of b and a section $s_r: W_r \to M_r \mid W_r$ where $s_r(b) = m_r$.

Then the intersection $W = W_1 \cap ... \cap W_n$ is a Λ_i −open set of b and a section $s: W \to M_W$ is given by $(\pi_r \ o \ s)$ (w) $=s_r(w)$ for every index r and every point $w \in W$, where $i = 1, 2$.

Proposition 2.3.4. Let $\varphi: M \to N$ be a continuous F.W. surjection, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M is locally sliceable, then N is so.

Proof. Let $n \in N_b$; $b \in B$. Then $n = \varphi(m)$, for some $m \in M_b$. If M is locally sliceable then, there is a Λ_i −open set W of b and a section $s: W \to M_W$ where $s(b) = m$. Then $\varphi os: W \to N_W$ is a section such that $s(b) = n$, where $i = 1, 2$, as required.

Definition 2.3.5. The F.W. bitopological space $(M\ , \tau_1\ , \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is called F.W. discrete if the projection p is a local homeomorphism.

Example 2.3.6. Let $\mathrm{M} = \{1, 2\}$, $\tau_1 = \big\{M, \emptyset, \{1\}\big\}$, $\tau_2 = \big\{M, \emptyset, \{2\}\big\}$. Let $\mathrm{B} = \{a, b\}$, $\Lambda_1 = \big\{B, \emptyset, \{a\}\big\}$, $\Lambda_2 = \{B, \emptyset, \{b\}\}$. Let p : M→ B where: p(1)=a, p(2) = b. We have $M_a = \{1\}$, $M_b = \{2\}$. Let $s_1 : \{a\} \to \{1\}$ such that $s_1(a) = 1$, $s_2 : \{b\} \to \{2\}$ such that $s_2(b) = 2$. Then, p is local homeomorphism, and thus M is F.W. discrete.

Remark 2.3.7. It is not difficult to show examples of different F.W. discrete bitopologies on the same F.W. set which are equivalent, as F.W. bitopologies. For this reason, we must be careful not to say the F.W. discrete bitopology.

This means, we recall, that for every point b of B and every point m of M_b there is a $\tau_i -$ open set V of m in M and a Λ_i −open set W of b in B where p maps V homeomorphically onto W. In that case we say that W is evenly covered by V, where $i = 1, 2$. It is clear that F.W. discrete bitopological spaces are locally sliceable therefor is F.W. open.

The class of F.W. discrete bitopological spaces are finitely multiplicative.

Proposition 2.3.8. Let $\{(M_r, \tau_{r1}, \tau_{r2})\}_{r=1}^{k}$ be a finite family of F.W. discrete bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then the F.W. bitopological product $(M = \prod_B M_r, \tau_1, \tau_2)$ is F.W. discrete.

Proof. Given a point $m \in M_b$; $b \in B$, then there is for every index r a $\tau_i -$ open set U_r of $\pi_r(m)$ in M_r, where the projection $p_r = p \, o \, \pi_r^{-1}$ maps U_r homeomorphically onto the Λ_i −open $p_r(U_r) = W_r$ of b. Then, the $\tau_i -$ open $\prod_B U_r$ of m is mapped homeomorphically onto the intersection $W = \cap W_r$ which is a Λ_i −open of b, where $i = 1, 2$.

An attractive characterization of F.W. discrete bitopological spaces are given by the following proposition.

Proposition 2.3.9. If (M, τ_1, τ_2) is F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then, M is F.W. discrete iff:

(a) M is F.W. open

(b) The diagonal embedding $\Delta: M \to M \times_B M$ is open

Proof. $(\Leftarrow)$ Suppose that (a) and (b) are satisfied. Let $m \in M_b$; $b \in B$, then $\Delta(m) = (m, m)$ admits a $\tau_i \times \tau_i$ −open set in $M \times_B M$ which is entirely contained in $\Delta\,(M)$. Without real lacking in general, we may suppose the

$\tau_i \times \tau_i$ −open set is of the form $U \times_B U$, where U is a τ_i − open set of m in M. Then $p|U$ is a homeomorphism. Therefore, M is F.W. discrete where $i = 1, 2$.

(⇒) Assume that M is F.W. discrete. We have already seen that M is a F.W. open. To prove that Δ is open, it is sufficient to show that $\Delta(M)$ is $\tau_i \times \tau_{i-}$open in $M \times_B M$. So, let $m \in M_b$; $b \in B$, and let U be a τ_i − open set of m in M, where $W = p(U)$ is a Λ_i −open set of b in B and p maps U homeomorphically onto W. Then, $U \times_B U$ is contained in $\Delta(M)$ since if not, then there exist distinct $\xi, \xi^* \in M_W$, where $w \in W$ and $\xi, \xi^* \in U$, which is absurd.

Open subset of F.W. discrete bitopological spaces are also F.W. discrete. Actually, we have the following results.

Proposition 2.3.10. Assume that $\varphi: M \to N$ is a continuous F.W. injection, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. open bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If N is F.W. discrete then M is so.

Proof. Consider the diagram shown below.

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\varphi"'] & M \times_B M \arrow[d, "\varphi \times \varphi"] \\ N \arrow[r, "\Delta"] & N \times_B N \end{tikzcd}$$

Figure 2.3.1. Diagram of Proposition 2.3.10.

Since φ is continuous so is $\varphi \times \varphi$. Now $\Delta(N)$ is $\sigma_i \times \sigma_i$ −open in $N \times_B N$, by Proposition (2.3.8.). Since N is a F.W. discrete, then $\Delta(M)$=$\Delta\left(\left(\varphi^{-1}(N)\right)\right) = (\varphi \times \varphi)^{-1}\big(\Delta(N)\big)$ is a $\tau_i \times \tau_i$ −open in $M \times_B M$. Thus, the conclusion follows from Proposition (2.3.9.) where $i = 1, 2$.

Proposition 2.3.11. Assume that $\varphi : M \to N$ be an open F.W. surjection, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. open bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M is a F.W. discrete, then N is so.

Proof. From figure 2.3.1, with the assumption on φ, if M is a F.W. discrete then $\Delta(M)$ is an $\tau_i \times \tau_i$ −open in $M \times_B M$, by Proposition (2.3.9.). Hence $\Delta(N) = \Delta\left(\left(\varphi(M)\right)\right) = (\varphi \times \varphi)\big(\Delta(M)\big)$ is an $\sigma_i \times \sigma_i$ −open in $N \times_B N$. Thus the conclusion follows again from Proposition (2.3.9.), where $i = 1, 2$.

Proposition 2.3.12. If $\varphi, \psi : M \to N$ is a continuous F.W. functions, where (M, τ_1, τ_2) is a F.W. bitopological and (N, σ_1, σ_2) is a F.W. discrete bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then the coincidence set $K(\varphi, \psi)$ of φ and ψ is open in M.

Proof. The coincidence set is precisely $\Delta^{-1}(\varphi \times \psi)^{-1}\big(\Delta(N)\big)$, where:

$$M \xrightarrow{\Delta} M \times_B M \xrightarrow{\varphi \times \psi} N \times_B N \xleftarrow{\Delta} N$$

Figure 2.3.2. Diagram of Proposition 2.3.12.

Hence the required result follows at once from Proposition (2.3.9.). In particular, take $= N$, $\varphi = id_M$, and $\psi = sop$ where s is a section. We conclude that s is an open embedding when M is a F.W. discrete.

Proposition 2.3.13. If $\varphi : M \to N$ is a continuous F.W. functions, where (M, τ_1, τ_2) is a F.W. open and (N, σ_1, σ_2) is a F.W. discrete bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then, the F.W. graph $\Gamma : M \to M \times_B N$of φ is an open embedding.

Proof. The F.W. graph is defined in the same way as the ordinary graph, but with values in the F.W. bitopological product. Therefore, the diagram shown below is commutative.

$$
\begin{tikzcd}
M \arrow[r, "\Gamma"] \arrow[d, "\varphi"'] & M \times_B N \arrow[d, "\varphi \times id_N"] \\
N \arrow[r, "\Delta"] & N \times_B N
\end{tikzcd}
$$

Figure 2.3.3. Diagram of Proposition 2.3.13.

Since $\Delta(N)$ is an $\sigma_i \times \sigma_i$ −open in $N \times_B N$, by Proposition (2.3.9.), $\Gamma(M) = (\varphi \times id_N)^{-1}(\Delta(N))$ is an $\tau_i \times \sigma_i$ −open in $M \times_B N$, where $i = 1, 2,$ as asserted.

Remark 2.3.14. If (M, τ_1, τ_2) is a F.W. discrete bitopological space over $(B, \Lambda_1, \Lambda_2)$ then for every point $m \in M_b$; $b \in B$, there is a Λ_i −open set W of b and a unique section $s: W \to M_W$ exist satisfying $s(b) = m$. We may refer to s as the section through m.

Definition 2.3.15. The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called locally sectionable if every point $b \in B$, admits an Λ_i −open set W and a section $s: W \to M_W$, where $i = 1$ or 2.

Example 2.3.16. Let M $= \{1, 2\}$, $\tau_1 = \{M, \emptyset, \{1\}\}, \tau_2 = \{M, \emptyset, \{2 *\}\}$. Let B $= \{a, b\}$, $\Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let p : M$\to B$ where p(1) = a, p(2) = b. We have $M_a = \{1\}$, $M_b = \{2\}$. Let $s_1 : \{a\} \to \{1\}$ where $s_1(a) = 1$, $s_2 : \{b\} \to \{2\}$ where $s_2(b) = 2$. Thus (M, τ_1, τ_2) is locally sectionable.

Remark 2.3.17. The F.W. non-empty locally sliceable bitopological spaces are locally sectionable, but the converse is false. In fact, locally sectionable bitopological spaces are not necessarily F.W. open. For example, take $M = (-1,1] \subset \mathbb{R}$ with (M, τ_1, τ_2): $\tau_1 = \tau_2$, the natural projection onto $B = \mathbb{R} \mid \mathbb{Z}$; $(B, \Lambda_1, \Lambda_2) : \Lambda_1 = \Lambda_2$.
The class of locally sectionable bitopological spaces is finitely multiplicative as we show next.

Proposition 2.3.18. If $\{(M_r, \tau_{r1}, \tau_{r2})\}$ is a finite family of locally sectionable bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then the F.W. bitopological product $M = \prod_B M_r$ is locally sectionable.

Proof. Given a point b of B, there exist an Λ_i −open set W_r of b and a section $s_r : W_r \to M_r \mid W_r$ for every index r. Since there are finite number of indices, the intersection W of the Λ_i −open sets W_r is also a Λ_i −open set of b, and a section $s: W \to (\prod_B M_r)_W$ is given by $\pi_r \, o \, s(w) = s_r(w)$, for $w \in W$, where $i = 1, 2$.

Our last two results apply equally well to every of the above three propositions.

Proposition 2.3.19. If (M, τ_1, τ_2) is a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Suppose that (M, τ_1, τ_2) is locally sliceable, F.W. discrete or locally sectionable over $(B, \Lambda_1, \Lambda_2)$. Then so is M_{B^*} over B^* for every Λ_i −open set B^* of B, where $i = 1, 2$.

Proposition 2.3.20. Let (M, τ_1, τ_2) be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Assume that M_{B_j} is a locally sliceable F.W. discrete or locally sectionable over B_j for every member B_j of an Λ_i −open covering of B. So is M over B, such that, $i = 1, 2$.

Chapter 3

Fibrewise Pairwise Separation Axioms

Chapter 3

Fibrewise Pairwise Separation Axioms

In this chapter, we define fibrewise bitopological space in the more important concept in topology which is the separation axioms. In Section one, we define and study the concepts of fibrewise pairwise T_0 spaces, fibrewise pairwise T_1 spaces, fibrewise pairwise R_0 spaces, fibrewise pairwise Hausdorff spaces, fibrewise pairwise functionally Hausdorff spaces. Some basic properties of these spaces are investigated. In Section two, we introduce the concepts of fibrewise pairwise regular spaces, fibrewise pairwise completely regular spaces, fibrewise pairwise normal spaces and fibrewise pairwise functionally normal spaces. Also, we give several results concerning them. Some of results in this chapter stated for the case of fibrewise topological space (see [1], [23]).

3.1. Fibrewise Pairwise T_0, Pairwise T_1, Pairwise R_0 and Pairwise Hausdorff Spaces.

The concepts of open sets have an important role in F.W. separation axioms. By using these concepts, we can construct many F.W. separation axioms. Now, we introduce the versions of F.W. pairwise T_0, F.W. pairwise T_1, F.W. pairwise R_0, and F.W. pairwise Hausdorff spaces as follows.

Definition 3.1.1. Let (M, τ_1, τ_2) be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then M is called a F.W. pairwise T_0 if whenever $x, y \in M_b$; $b \in B$ and $x \neq y$, either there exists a τ_i-open set U of x which does not contains y in M or τ_j-open set V of y which does not contains x in M, where $i, j = 1, 2 \ , i \neq j$.

Example 3.1.2. Let $M = \{1, 2, 3\}$, $\tau_1 = \{M, \emptyset, \{1\}, \{3\}, \{1, 3\}\}$, $\tau_2 = \{M, \emptyset, \{2\}, \{2, 1\}, \{2, 3\}\}$. Let $B = \{a, b\}$, $\Lambda_1 = \{B, \emptyset, \{a\}\}$, $\Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \to B$ where $p(1) = a$, $p(2) = b$, $p(3) = a$. Then M is a F.W. pairwise T_0 space.

Remark 3.1.3.

(a) (M, τ_1, τ_2) is a F.W. pairwise T_0 space iff each fiber M_b is a pairwise T_0 space.

Proof: Let $x, y \in M_b$, $b \in B$ such that $x \neq y$, so $x, y \in M$. Since M is T_0 ,there exist τ_i -open set U contain x and $y \notin U$ or τ_j -open set V contain y and $x \notin V$. Hence $U \cap M_b \in \tau_{ib}$ and $V \cap M_b \in \tau_{jb}$ and $(U \cap M_b) \cap (V \cap M_b) = (U \cap V) \cap M_b = \phi \cap M_b = \phi$. So M_b is pairwise T_0 space.

(b) Subspaces of F.W. pairwise T_0 spaces are F.W. pairwise T_0 spaces.

Proof: Let N be a subset of F.W. pairwise T_0 spaces, Let $x, y \in N_b$, $b \in B$ such that $x \neq y$, then $x, y \in M_b$, $b \in B$ and since M is T_0, then ether there exist τ_i -open set U contain x, $y \notin U$ or τ_j -open set V contain y and $x \notin V$. Since $U \cap N \in \tau_N, V \cap N \in \tau_N$ and $x \in U \cap N, y \notin U \cap N$ or $y \in V \cap N, x \notin V \cap N$, there for N is F.W. pairwise T_0 spaces.

(c) The F.W. bitopological products of F.W. pairwise T_0 spaces with the family of F.W. pairwise projections are F.W. pairwise T_0 spaces.

Proof: Let $\{(M_r, \tau_{1r}, \tau_{2r})\}$ be a finite family of F.W. topological spaces, let $x, y \in M_b$, $b \in B$ such that $x \neq y$, then $\pi_r(x) = x_r$ and $\pi_r(y) = y_r$ for some index r. Since M_r is F.W. pairwise T_0 for all r, then ether there exist τ_{ir} -open set U_r contain x_r, $y_r \notin U_r$ or τ_{jr} -open set V_r contain y_r and $x_r \notin V_r$. Since π_r is continuous, then the inverse images of U_r and V_r are open in M and $x \in U, y \notin U$ or $y \in V, x \notin V$. Hence M is F.W. pairwise T_0 space.

In a similar way, we can introduced the definition of F.W. pairwise T_1 space. Let (M, τ_1, τ_2) be a F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then, M is called a F.W. pairwise T_1 if whenever $x, y \in M_b$; $b \in B$ and $x \neq y$, there exist a τ_i -open sets U, and a τ_j -open set V in M such that $x \in U$, $y \notin U$ and $x \notin V, y \in V, i, j = 1, 2, i \neq j$. But it turns out that there is no real use for this in what we are going to do. In its place, we formulate some use of a new axiom. The axiom is that "every τ_i- open set contains the τ_j-closure of each of its points", and use the word pairwise R_0 space. This is correct for pairwise T_1 spaces and for pairwise regular spaces. Thinking of it like a weak structure of pairwise regularity. For example, indiscrete spaces are pairwise R_0 spaces. The F.W. version of the pairwise R_0 axiom is defined as the following.

Definition 3.1.4. A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. pairwise R_0 space if for every $x \in M_b$; $b \in B$, and every τ_i-open set V of x in M, there exists a Λ_i −nbd W of b in B such that V is contains the τ_j- closure of $\{x\}$ in M_W (i.e., $M_W \cap \tau_j\text{-} Cl\{x\} \subset V$) where $i, j = 1, 2$, $i \neq j$.

Example 3.1.5. Let $M = \{1, 2, 3\}, \tau_1 = \{M, \emptyset, \{1\}, \{3\}, \{1, 3\}\}, \tau_2 = \{M, \emptyset, \{2\}, \{2, 1\}, \{2, 3\}\}$. Let $B = \{a, b\}$, $\Lambda_1 = \{B, \emptyset, \{a\}\}$, $\Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \to B$ where $p(1) = a, p(2) = b, p(3) = a$. Then, M is F.W. pairwise R_0.

For another example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is a F.W. pairwise R_0 space for all pairwise R_0 spaces T.

Remark 3.1.6.

(a) The nbds of x are given by a F.W. basis it is enough if the condition in Definition (3.1.4.) is satisfied for every F.W. basic nbds.

(b) If (M, τ_1, τ_2) is a F.W. pairwise R_0 space over $(B, \Lambda_1, \Lambda_2)$, then for each subspace $(B^*, \Lambda_1^*, \Lambda_2^*)$ of $(B, \Lambda_1, \Lambda_2)$, $(M_B^*, \tau_1^*, \tau_2^*)$ is a F.W. pairwise R_0 space over B^*.

Proposition 3.1.7. Let $\varphi\ : M \to M^*$ be F.W. embedding function, where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise R_0 then so is M.

Proof. Let V be a τ_i-open set of x in M where $x \in M_b$; b $\in B$. Then $V = \varphi^{-1}(V^*)$, where V^* is a τ_i^*-open set of $x^* = \varphi(x)$ in M^*. Because M^* is a F.W. pairwise R_0 then we have a nbd W of b in B, where $M_W^* \cap \tau_j^*\text{-}Cl\{x^*\} \subset V^*$. Hence, $M_W \cap \tau_j\text{-}Cl\{x\} \subset \varphi^{-1}(M_W^* \cap \tau_j^*\text{-}Cl\{x^*\}) \subset \varphi^{-1}(V^*) = V$, and hence M is a F.W. pairwise R_0 where $i, j = 1, 2, , i \neq j$.

The class of F.W. pairwise R_0 spaces is finitely multiplicative as we show in the following.

Proposition 3.1.8. If $\{(M_r, \tau_{1r}, \tau_{2r})\}$ is a finite family of F.W. pairwise R_0 spaces over B. Then the F.W. bitopological product $M = \prod_B M_r$ is a F.W. pairwise R_0.

Proof. Let $x \in M_b$; $b \in B$. Consider a τ_i-open set $V = \prod_B V_r$ of x in M, where V_r is a τ_{ir}-open set of $\pi_r(x) = x_r$ in M_r for each index r. Since M_r is a F.W. pairwise R_0, then we have a nbd W_r of b in B where $(M_r \mid W_r) \cap \tau_{jr}\text{-}Cl\{x_r\} \subset V_r$. Then, we regard W as a nbd of b where W is an intersection of W_r and $M_W \cap \tau_j\text{-}Cl\{x\} \subset V$ and hence $M = \prod_B M_r$ is F.W. pairwise R_0 where i, j =1,2, , $i \neq j$.

Similar conclusion holds for infinite F.W. products provided all that of the factors is F.W. nonempty.

Proposition 3.1.9. Let $\varphi: M \to N$ is closed, continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over B. If M is F.W. pairwise R_0 then so is N.

Proof. Assume that V is an σ_i-open set of y in N, where $y \in N_b$; $b \in B$, choose $x \in \varphi^{-1}(y)$. Then $U = \varphi^{-1}(V)$ is a τ_i-open set of x in M. Since M is F.W. pairwise R_0, then we have a nbd W of b in B , where $M_W \cap \tau_j - cl\{x\} \subset U$. Therefore $N_W \cap \varphi(\tau_j - cl\{x\}) \subset \varphi(U) = V$. Because φ is closed, $\varphi(\tau_j - cl\{x\}) = \sigma_j - cl(\varphi\{x\})$. Hence, $N_W \cap \sigma_j - cl(\varphi\{x\}) \subset V$ and N is F.W. pairwise R_0 where $i, j = 1, 2,\ i \neq j$.

Now we introduce the concept of F.W. pairwise Hausdorff spaces.

Definition 3.1.10. A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. pairwise Hausdorff if whenever $x, y \in M_b$; $b \in B$ and $x \neq y$, there exist a disjoint pair of τ_i-open set U of x and τ_j-open set V of y in M, where $i, j = 1, 2,\ i \neq j$.

Example 3.1.11. Let $M = \{1,2,3\}, \tau_1 = \{M, \emptyset, \{1\}, \{3\}, \{1,3\}\}, \tau_2 = \{M, \emptyset, \{2\}, \{2,1\}, \{2,3\}\}$. Let $B = \{a, b\}, \Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \to B$ where $p(1) = a, p(2) = b, p(3) = a$. Then M is a F.W. pairwise T_2.

Another example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise Hausdorff space for any pairwise Hausdorff spaces T.

Remark 3.1.12. If (M, τ_1, τ_2) is F.W. pairwise Hausdorff space over $(B, \Lambda_1, \Lambda_2)$ then M_B^* is F.W. pairwise Hausdorff over B^* for every subspace B^* of B. Especially, the fibers of (M, τ_1, τ_2) are pairwise Hausdorff spaces.

On the other hand, a F.W. bitopological space with pairwise Hausdorff fibres is not necessarily pairwise Hausdorff.

Example 3.1.13. Let, $\tau_1 = \{M, \emptyset, \{1\}, \{1,2\}\}, \tau_2 = \{M, \emptyset, \{1\}, \{1,3\}\}$, where $M = \{1,2,3\}$, Let $B = \{a, b\}$, $\Lambda_1 = \{B, \emptyset, \{a\}\}$, $\Lambda_2 = \{B, \emptyset\}$. Let $p: M \to B$ where $p\,(1) = a\,, p\,(2) = b = p\,(3)$. Then, we have $M_b = \{2,3\}$, $\tau_{1M_b} = \{M_b, \emptyset, \{2\}\}, \tau_{2M_b} = \{M_b, \emptyset, \{3\}\}$. Then, there exist τ_{1M_b}-open set $U = \{2\}$ where $2 \in U$, and there exist τ_{2Mb} open set $V = \{3\}$ where $3 \in V$ where $U \cap V = \emptyset$. But M is not pairwise Hausdorff since 2 and $3 \in M$ and $2 \neq 3$, and there is no disjoint pair of open sets of 2 and 3.

Proposition 3.1.14. The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is F.W. pairwise Hausdorff iff the diagonal embedding $\Delta\colon\ M \to M \times_B M$ is $\tau_i \times_B \tau_i$ -closed.

Proof. (⇒) Let $x, y \in M_b$; b ∈ B and $x \neq y$. Since $\Delta(M)$ is $\tau_i \times_B \tau_i$-closed in $M \times_B M$, then (x, y) a point of the complement admits a F.W. product $\tau_i \times_B \tau_j$-open set $U \times_B V$ which does not meet $\Delta(M)$. Then U, V are disjoint pair of x, y where U is τ_i-open set of x, and V is τ_j-open set of y such that $i, j = 1, 2,\ i \neq j$.

(⇐) Let $(x, y) \in M \times_B M - \Delta(M)$, so $(x, y) \notin \Delta(M)$, and $x \neq y$ since M is F.W. pairwise T_2 space then there exist disjoint pair τ_i-open set U of x and τ_j-open set V of y, so $U \times_B V$ is $\tau_i \times_B \tau_j$-open set in $M \times_B M$. Hence $M \times M - \Delta(M)$ is $\tau_i \times_B \tau_i$ is open and $\Delta(M)$ is $\tau_i \times_B \tau_i$ closed.

Subspaces of F.W. pairwise Hausdorff spaces are F.W. pairwise Hausdorff spaces. Actually, we have the following proposition.

Proposition 3.1.15. Assume that $\varphi : M \to M^*$ is embedding F.W. function, where (M, τ_1, τ_2) and $(M^*, {\tau^*}_1, {\tau^*}_2)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise Hausdorff then so is M.

Proof. Let $x, y \in M_b$; $b \in B$ and $x \neq y$. Then $\varphi(x), \varphi(y) \in {M_b}^*$ are distinct. Since M^* is a F.W. pairwise Hausdorff, then we have a τ_i^*-open sets U^* of $\varphi(x)$ and a τ_j^*-open set V^* of $\varphi(y)$ in M^* which are disjoint. Because φ is continuous, the inverse images $\varphi^{-1}(U^*) = U$ and $\varphi^{-1}(V^*) = V$ such that U is a τ_i-open set of x and V is a τ_j-open set of y in M such that V and U are disjoint. Hence, M is a F.W. pairwise Hausdorff where $i, j = 1, 2, i \neq j$.

Proposition 3.1.16. Let $\varphi : M \to N$ be a continuous F.W. function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If N is F.W. pairwise Hausdorff, then the F.W. graph $\Gamma : M \to M \times_B N$ of φ is a $\tau_i \times_B \sigma_j$- closed embedding.

Proof. The F.W. graph is defined in a similar way to the ordinary graph, but with values in the F.W. product. Hence, the figure shown below is commutative.

$$\begin{tikzcd}
M \arrow[r, "\Gamma"] \arrow[d, "\varphi"'] & M \times_B N \arrow[d, "\varphi \times_B id_N"] \\
N \arrow[r, "\Delta"] & N \times_B N
\end{tikzcd}$$

Figure 3.1.1: Diagram of Proposition 3.1.16.

Since $\Delta(N)$ is a $\sigma_i \times_B \sigma_i$-closed in $N \times_B N$, by Proposition (3.1.14.), then $\Gamma(M) = (\varphi \times id_N)^{-1}(\Delta(N))$ is a $\tau_i \times_B \sigma_j$-closed in $M \times_B N$, as asserted, where $i, j = 1, 2\ , i \neq j$.

The category of F.W. pairwise Hausdorff spaces is multiplicative, in the following sense.

Proposition 3.1.17. Assume that $\{(M_r, \tau_{1r}, \tau_{2r})\}$ is a family of F.W. pairwise Hausdorff spaces over $(B, \Lambda_1, \Lambda_2)$. The F.W. bitopological product $M = \prod_B M_r$ is a F.W. pairwise Hausdorff.

Proof. Let $x,\ y\ \in\ M_b;\ b\ \in\ B$ and $x\ \neq\ y$. Then $\pi_r(x) = x_r\ \neq\ \pi_r(y) = y_r$ for some index *r*. Because M_r is F.W. pairwise Hausdorff, then we have a τ_{ir}-open set U_r of x_r, and a τ_{jr}-open set V_r of y_r in M_r where U_r and V_r are disjoint. Because π_r is continuous, the inverse images U and V are disjoint τ_i-open and τ_j-open sets, respectively, of x, y in M, where $i, j = 1, 2\ , i \neq j$.

The pairwise functionally version of the F.W. pairwise Hausdorff axiom is stronger than the non-pairwise functional version but their properties are similar. From now on, we denote by I the closed unit interval [*0*, *1*] in the real line ℝ.

Definition 3.1.18. A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is F.W. pairwise functionally Hausdorff if for every $x, y \in M_b$; b $\in B$ and $x\ \neq\ y$, there exists a nbd W of b in B and disjoint pair τ_i-open sets U of x and τ_j-open set V of y in M and a continuous function $\lambda: M_W \to I$ such that $M_b \cap\ U \subset \lambda^{-1}(0)$ and $M_b\ \cap\ V \subset \lambda^{-1}(1)$ where $i, j = 1, 2,\ i \neq j$.

Example 3.1.19. Let $M = \{2, 4, 6\}$, $\tau_1 = \{M, \emptyset, \{2\}, \{6\}, \{2, 6\}\}$, let $\tau_2 = \{M, \emptyset, \{4\}, \{2, 4\}, \{4, 6\}\}$. Let $B = \{a, b\}, \Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \to B$ where $p(2) = a = p(6), p(4) = b$. Hence, M is F.W. pairwise Hausdorff and $M_a = \{2, 6\}$ and $2 \neq 6$, so $W = \{a\}$. Let, $\lambda: M_W \to I$ where $\lambda(2) = 0$, $\lambda(6) = 1, \tau_{1M_W} = \{M_W, \emptyset, \{2\}, \{6\}\}$, $\tau_{2M_W} = \{M_W, \emptyset, \{2\}, \{6\}\}$.

Thus λ is continuous and $M_b \cap U \subset \lambda^{-1}(0)$ and $M_b \cap V \subset \lambda^{-1}(1)$. There for, M is F.W. pairwise functionally Hausdorff.

For another example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise functionally Hausdorff space for each pairwise functionally Hausdorff spaces *T*.

Remark 3.1.20. If (M, τ_1, τ_2) is F.W. pairwise functionally Hausdorff space over $(B, \Lambda_1, \Lambda_2)$ then M_B^* is F.W. pairwise functionally Hausdorff over B^* for every subspace B^* of *B*. In particular, the fibers of M are pairwise functionally Hausdorff spaces.

Subspaces of F.W. pairwise functionally Hausdorff spaces are F.W. pairwise functionally Hausdorff spaces. Actually, we have the following result.

Proposition 3.1.21. Assume that $\varphi : M \to M^*$ is a embedding F.W. function, where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise functionally Hausdorff then so is M.

Proof. Let $x, y \in M_b$ and $x \neq y; b \in B$. Then $\varphi(x) = x^*$, $\varphi(y) = y^* \in M_b^*$, $x^* \neq y^*$. Since M^* is F.W. pairwise functionally Hausdorff, then we have a nbd W of b in B and disjoint pair of τ_i^*-open set U^* of x^* and τ_j^*-open set V^* of y^* and a continuous function $\lambda^*: M^* \mid W \to I$ such that $M_b^* \cap U^* \subset (\lambda^*)^{-1}(0)$ and $M_b^* \cap V^* \subset (\lambda^*)^{-1}(1)$. Now, since φ is continuous, then $\varphi^{-1}(U^*) = U$ and $\varphi^{-1}(V^*) = V$ are disjoint pair of τ_i-open set of x and τ_j-open set of y, respectively and the continuous function λ where $\lambda = \lambda^* o\, \varphi: M_W \to I$ such that $M_b \cap U \subset \lambda^{-1}(0)$ and $M_b \cap V \subset \lambda^{-1}(1)$, where $i, j = 1, 2\ , i \neq j$.

Furthermore the category of F.W. pairwise functionally Hausdorff spaces is multiplicative, as in the following proposition.

Proposition 3.1.22. Assume that $\{(M_r, \tau_{1r}, \tau_{2r})\}$ is a family of F.W. pairwise functionally Hausdorff spaces over $(B, \Lambda_1, \Lambda_2)$. The F.W. bitopological product $M = \prod_B M_r$ is F.W. pairwise functionally Hausdorff.

Proof. Let $x, y \in M_b; b \in B,$ and $x \neq y$. Then, $\pi_r(x) = x_r,\ \pi_r(y) = y_r \in (M_r)_b$ for some index r where $x_r \neq y_r$. Since M_r is F.W. pairwise functionally Hausdorff, then we have a nbd W_r of b in B and disjoint pair of τ_{ir}-open set U_r of x_r, and τ_{jr}-open set V_r of y_r and a continuous function $\lambda: M_r \mid W_r \to I$ such that $(M_r)_b \cap U_r \subset \lambda^{-1}(0)$ and $(M_r)_b \cap V_r \subset \lambda^{-1}(1)$. Now, the intersection of W_r is a nbd W of b in B, and since π_r is continuous, then $\pi_r^{-1}(U_r) = U$ and $\pi_r^{-1}(V_r) = V$ are disjoint pair of τ_i-open set of x and τ_j-open set of y, respectively, and the continuous function Ω where $\Omega = \lambda o \pi_r: M_W \to I$ where $M_b \cap U \subset \Omega^{-1}(0)$ and $M_b \cap V \subset \Omega^{-1}(1)$ where $i, j = 1, 2$, $i \neq j$.

3.2. Fibrewise Pairwise Regular and Pairwise Normal Spaces

In this section we consider the F.W. Concept advanced pairwise separation axioms. Namely, F.W. pairwise regularity and F.W. pairwise completely regularity.

Definition 3.2.1. The F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. pairwise regular if for every $x \in M_b$; $b \in B$, and for every τ_i-open set V of x in M, there exists a nbd W of b in B, and a τ_i- open set U of x in M_W such that V is containing the τ_j-closure of U in M_W $\left(i.e., M_W \cap \tau_j - cl(U) \subset V\right)$, where $i, j = 1, 2$, $i \neq j$.

Example 3.2.2. Let $M = \{1, 2, 3\}$, $\tau_1 = \{M, \emptyset, \{3\}\}, \tau_2 = \{M, \emptyset, \{1, 2\}\}$. Let $B = \{a, b\}$, $\Lambda_1 = \{B, \emptyset, \{a\}\}$, $\Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \to B$ such that $p\,(1) = b = p\,(2), p\,(3) = a$. Then M is F.W. pairwise regular.

We can consider another example, trivial F.W. spaces with pairwise regular fibre are F.W. pairwise regular.

Remark 3.2.3.

(a) The nbds of x are given by a F.W. basis it is enough if the condition in Definition (3.2.1) is satisfied for every F.W. basic nbds.

(b) If (M, τ_1, τ_2) is F.W. pairwise regular space over $(B, \Lambda_1, \Lambda_2)$ then $(M_B^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise regular space over $(B^*, \Lambda_1^*, \Lambda_2^*)$ for every sub-space B^* of B.

Subspaces of F.W. pairwise regular spaces are F.W. pairwise regular spaces. Actually we have the following proposition.

Proposition 3.2.4. Assume that $\varphi : M \to M^*$ is embedding F.W. function, where (M, τ_1, τ_2) and $(M^*, {\tau^*}_1, {\tau^*}_2)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise regular then so is M.

Proof. Let V be a τ_i-open set of x in M where $x \in M_b$; $b \in B$. Then $V = \varphi^{-1}(V^*)$, where V^* is a τ_i^*-open set of $x^* = \varphi(x)$ in M_b^*. Because M^* is F.W. pairwise regular, then we have a nbd W of b in B and a τ_i^*-open set U^* of x^* in M_W^* where $M_W^* \cap \tau_j^* - cl(U^*) \subset V^*$. Then $U = \varphi^{-1}(U^*)$ is a τ_i-open set of x in M_W such that $M_W \cap \tau_j - cl(U) \subset V$. Hence, M is F.W. pairwise regular, where $i, j = 1, 2,\ i \neq j$ as required.

The class of F.W. pairwise regular spaces is F.W. multiplicative as in the following proposition.

Proposition 3.2.5. Assume that $\{(M_r, \tau_{1r}, \tau_{2r})\}$ is a finite family of F.W. pairwise regular spaces over B. The F.W. bitopological product $M = \prod_B M_r$ is F.W. pairwise regular.

Proof. Consider a τ_i-open set $V = \prod_B V_r$ of x in M, where $x \in M_b$; $b \in B$ and V_r is a τ_{ir}-open set of $\pi_r(x) = x_r$ in M_r for each index r. Since M_r is F.W. pairwise regular we have a nbd W_r of b in B, and a τ_{ir}-open set U_r of x_r in $M_r \mid W_r$ such that the τ_{jr}-closure of U_r in $M_r \mid W_r$ is contained in V_r. (i. e. $(M_r \mid W_r) \cap \tau_{jr} - cl(U_r) \subset V_r$). Then we regard W as a nbd of b in B, where W is the intersection of W_r, and $U = \prod_B U_r$ is a τ_i-open set of x in M_W where the τ_j-closure of U in M_W is contained in V. (i. e. $M_W \cap \tau_j - cl(U) \subset V$). Hence, $M = \prod_B M_r$ is F.W. pairwise regular, where $, j = 1, 2$, $i \neq$ j.

Similar conclusion holds for infinite F.W. products provided that every of the factors is F.W. non-empty.

Proposition 3.2.6. Assume that $\varphi : M \to N$ is a closed, open and continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over B. Then M is F.W. pairwise regular iff N is F.W. pairwise regular.

Proof.(⇒) Let V be a σ_i-open set of y in N where $y \in N_b$; $b \in B$, choose $x \in \varphi^{-1}(y)$. Then $U = \varphi^{-1}(V)$ is a τ_i-open set of x in M. Because M is F.W. pairwise regular , we have a nbd W of b in B, and a τ_i- open set U^* of x such that $M_W \cap \tau_j - cl(U^*) \subset U$. Then $N_W \cap \varphi\left(\tau_j - cl(U^*)\right) \subset V$. Because φ is closed, $\varphi\left(\tau_j - cl(U^*)\right) = \sigma_j - cl\big(\varphi(U^*)\big)$, and because φ is open, then

$\varphi(U^*)$ is a σ_i-open set of y. Hence, N is F.W. pairwise regular, where $i, j = 1, 2,\ i \neq j$, as asserted.
(⇐) By a similar way of the first direction.

The pairwise functionally version of the F.W. pairwise regularity axiom is stronger than the non-pairwise functionally version. However, their properties are similar. In the ordinary theory, the word completely regular is used instead of functionally regular. We widen this usage to the F.W. theory.

Definition 3.2.7. A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. pairwise completely regular if for every $x \in M_b$; $b \in B$, and for every τ_i-open set V of x there exists a nbd W of b in B and a τ_j-open set U of x in M_W and a continuous function $\lambda: (M_W, \tau_{1W}, \tau_{2W}) \to I$ such that $M_b \cap U \subset \lambda^{-1}(0)$ and $M_W \cap (M_W - V) \subset \lambda^{-1}(1)$, where $i, j = 1, 2, i \neq j$.

Example 3.2.8. Let $M = \{1, 2, 3, 4\}$, $\tau_1 = \{M, \emptyset, \{1, 3\}\}, \tau_2 = \{M, \emptyset, \{2, 4\}\}$. Let $B = \{a, b\}$, $\Lambda_1 = \{B, \emptyset, \{a\}\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let p: $M \to B$ such that $p(1) = a = p(3)$, $p(2) = b = p(4)$, $M_a = \{1, 3\}$, let x $= 1, V = \{1, 3\}, W = \{a\}$, $M_W = \{1, 3\}$, $\tau_{1M_W} = \{M_W, \emptyset\} = \tau_{2M_W}$, let $U = M_W$. Let $\lambda: M_W \to I$ such that $\lambda(1) = 0 = \lambda(3)$. λ is continuous and $M_b \cap U \subset \lambda^{-1}(0), M_W \cap (M_W - V) \subset \lambda^{-1}(1)$ Similar if $x = 3$. $M_b = \{2, 4\}$, let x$= 2, V = \{2, 4\}, W = \{b\}$, $M_W = \{2, 4\}$, $\tau_{1M_W} = \{M_W, \emptyset\} = \tau_{2M_W}$, let $U = M_W$. Let $\lambda: M_W \to I$ such that $\lambda(2) = 0 = \lambda(4)$. λ is continuous and $M_b \cap U \subset \lambda^{-1}(0), M_W \cap (M_W - V) \subset \lambda^{-1}(1)$. Similarly if $x = 4$

For another example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise completely regular space for every pairwise completely regular spaces T.

Remark 3.2.9.

(a) The nbds of x are given by a F.W. basis it is enough if the condition in Definition (3.2.7.) is satisfied for every F.W. basic nbds.

(b) If (M, τ_1, τ_2) is F.W. pairwise completely regular space over $(B, \Lambda_1, \Lambda_2)$ then $(M_B^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise completely regular space over $(B^*, \Lambda_1^*, \Lambda_2^*)$ for every subspace B^* of B.

Subspaces of F.W. pairwise completely regular spaces are F.W. pairwise completely regular spaces. In fact, we have the following result.

Proposition 3.2.10. Assume that $\varphi: M \to M^*$ is embedding F.W. function, where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M^* is F.W. pairwise completely regular then so is M.

Proof. Let V be a τ_i-open set of x in M where $x \in M_b; b \in B$, then $\varphi(x) = x^* \in M_b^*$ and $V = \varphi^{-1}(V^*)$ is a τ_i^*-open set of x^*. Because M^* is F.W. pairwise completely regular, then we have a nbd W of b in B and τ_j^*-open set U^* of x^* and a continuous function $\lambda: M_W^* \to I$ such that $M_b^* \cap U^* \subset \lambda^{-1}(0)$ and $M_W^* \cap (M_W^* - V^*) \subset \lambda^{-1}(1)$. Now, because φ is continuous, then $\varphi^{-1}(U^*) = U$ is τ_i-open set of x in M_W and the continuous function $\Omega = \lambda o \varphi$ such that $\Omega : M_W \to I$ and $M_b \cap U \subset \Omega^{-1}(0)$ and $M_W \cap (M_W - V) \subset \Omega^{-1}(1)$ where $i, j = 1, 2, i \neq j$.

The class of F.W. pairwise completely regular spaces is finitely multiplicative, as we show next.

Proposition 3.2.11. Assume that $\{(M_r, \tau_{1r}, \tau_{2r})\}$ is a finite family of F.W. pairwise completely regular spaces over $(B, \Lambda_1, \Lambda_2)$. The F.W. bitopological product $M = \prod_B M_r$ is F.W. pairwise completely regular.

Proof. Let $x \in M_b$; $b \in B$. Consider a F.W. τ_i-open set $\prod_B V_r$ of x in M, where V_r is a τ_{ir}-open set of $\pi_r(x) = x_r$ in M_r for all index r. Because M_r is F.W. pairwise completely regular, we have a nbd W_r of b in B, and a τ_{jr}-open set U of x_r in M_r and a continuous function $\lambda_r: (M_r)_W \to I$ where $(M_r)_b \cap U \subset \lambda_r^{-1}(0)$ and $(M_r)_W \cap ((M_r)_W - V_r) \subset \lambda_r^{-1}(1)$. Then we regard W as a nbd of b in B where W is the intersection of W_r and $\lambda : M_W \to I$ is a continuous function where

$$\lambda(\xi) = inf_{r=1,2,3,\dots,n}\{\lambda_r \xi_r\} \text{ for } \xi = (\xi_r) \in M_W.$$

Since $(M_r)_b \cap \pi_r{}^{-1}(U) \subset \pi_r{}^{-1}[(M_r)_b \cap U] \subset \pi_r{}^{-1}\left(\lambda^{-1}{}_r(0)\right)(\lambda_r o \pi_r)^{-1}(0)$ and $(Mr)_W \cap \pi_r{}^{-1}((M_r)_w - V_r) \subset \pi_r{}^{-1}[(M_r)_W \cap ((M_r)_W - V_r)] \subset \pi_r{}^{-1} \left(\lambda_r{}^{-1}(1)\right) = (\lambda_r o \pi_r)^{-1}(1)$ where $i, j = 1, 2, i \neq j$.

A similar conclusion holds for infinite F.W. products if all of the factors is F.W. non-empty.

Lemma 3.2.12. Assume that $\varphi: M \to N$ is a closed, open F.W. surjection function, where M and N are F.W. bitopological spaces over B. Let $\alpha : M \to \mathbb{R}$ be a continuous real-valued function which is F.W. bounded above, in the sense that α is bounded above on each fibre of M. Then, $\beta: N \to \mathbb{R}$ is continuous, where:

$$\beta(\eta) = \sup_{\xi \in \varphi^{-1}(\eta)} \alpha(\xi)$$

Proposition 3.2.13. Assume that $\varphi : M \to N$ is a closed, open and continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If M is F.W. pairwise completely regular then so is N.

Proof. Let V_y be a σ_i-open set of y in N where $y \in N_b$; $b \in B$. Choose $x \in \varphi^{-1}(y)$ such that $V_x = \varphi^{-1}(V_y)$ is a τ_i-open set of x. Because M is F.W. pairwise completely regular, we have a nbd W of b in B, and a τ_j-open set U_x of x in M_W and a continuous function $\lambda : M_W \to I$ such that $M_b \cap U_x \subset \lambda^{-1}(0)$ and $M_W \cap (M_W - V_x) \subset \lambda^{-1}(1)$. Using Lemma (3.2.12.), we get a continuous function $\Omega : N_W \to I$ such that $N_b \cap U_y \subset \Omega^{-1}(0)$ and $N_W \cap \left(N_W - V_y\right) \subset \Omega^{-1}(1)$, where $i, j = 1, 2, i \neq j$.

Next, we define the version of F.W. pairwise normal space.

Definition 3.2.14. A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. pairwise normal if for every $b \in B$ and every disjoint pair of τ_i-closed set H, and τ_j-closed set K of M , there exists a nbd W of b in B and a disjoint pair of τ_j-open set U, and τ_i-open set V of $M_W \cap H, M_W \cap K$ in M_W, where $i, j = 1, 2, i \neq j$.

Example 3.2.15. Let $M = \{1, 2\}$, $\tau_1 = \{M, \varphi, \{1\}\}, \tau_2 = \{M, \varphi, \{2\}\}$. H=$\{1\}, K = \{2\}$. Let $B = \{a, b\}$, $\Lambda_1 = \{B, \varphi, \{a\}\}, \Lambda_2 = \{B, \varphi, \{b\}\}$. Let $p : M \to B$ where $p(1) = a, p(2) = b$. We have $M_a = \{1\}$, $M_b = \{2\}$, where the nbd of a is $\{a\}$, and the nbd of b is $\{b\}$, $M_a \cap H = \{1\}, M_a \cap K = \varphi$, $M_b \cap H = \varphi, M_b \cap K = \{2\}$. Let V $= \{2\}, U = \{1\}$. So, M is F.W. pairwise normal.

Remark 3.2.16. If (M, τ_1, τ_2) is a F.W. pairwise normal space over $(B, \Lambda_1, \Lambda_2)$, then for each subspace B^* of B and $(M_B^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise normal space over $(B^*, \Lambda_1^*, \Lambda_2^*)$.

Closed subspaces of F.W. pairwise normal spaces are F.W. pairwise normal. Actually, we have.

Proposition 3.2.17. Assume that $\varphi: M \to M^*$ is a closed, embedding F.W. function where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over B. If $(M^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise normal then so is (M, τ_1, τ_2).

Proof. Let H and K be disjoint pair of τ_i-closed and τ_j-closed sets of M and let $b \in B$. Then $\varphi(H)$ and $\varphi(K)$ are disjoint pair of ${\tau_i}^*$-closed set and ${\tau_j}^*$-closed set of M^*. Since M^* is F.W. pairwise normal then, we have a nbd W of b in B and a ${\tau_j}^*$-open set U^* and ${\tau_i}^*$-open set V^* of $M_W^* \cap \varphi(H), M_W^* \cap \varphi(K)$, in M_W^*. Since φ is continuous, then $\varphi^{-1}(U^*) = U$ and $\varphi^{-1}(V^*) = V$ are disjoint pair of τ_j-open and τ_i-open sets of $M_W \cap H$, $M_W \cap K$ in M_W, where $i, j = 1, 2, \;\; i \neq j$.

Proposition 3.2.18. Let $\varphi: M \to N$ be a closed continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. Then (M, τ_1, τ_2) is F.W. pairwise normal iff (N, σ_1, σ_2) is F.W. pairwise normal .

Proof. (⇒) Let H and K be disjoint pair of σ_i-closed and σ_j-closed sets of N and let $b \in B$. Then, $\varphi^{-1}(H)$ and $\varphi^{-1}(K)$ are disjoint pair of τ_i-closed and τ_j-closed sets of M. Because M is F.W. pairwise normal, then we have a nbd W of b in B and a disjoint pair of τ_j-open set and τ_i-open set U, V of $M_W \cap \varphi^{-1}(H)$ and $M_W \cap \varphi^{-1}(K)$. Since φ is closed then, the sets $N_W - \varphi(M_W - U)$ and $N_W - \varphi(M_W - V)$ are open in N_W, and structure a disjoint pair of σ_j- open, σ_i-open sets of $N_W \cap H$, $N_W \cap K$ in N_W, as required, where $i, j = 1, 2 \, , i \neq j$.

(⇐) By similar way of first direction.

Lastly, we define the version of F.W. pairwise functionally normal space.

Definition 3.2.19. A F.W. bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. pairwise functionally normal if for every $b \in B$ and every disjoint pair of τ_i-closed set H, and τ_j-closed set K of M , there exists a nbd W of b in B and a disjoint pair of τ_j-open set U, and τ_i-open set V and a continuous function $\lambda : M_W \to I$ such that $M_W \cap H \cap U \subset \lambda^{-1}(0)$ and $M_W \cap K \cap V \subset \lambda^{-1}(1)$ in M_W, where $i, j = 1, 2, i \neq j$.

Example 3.2.20. Let $\mathrm{M} = \{1, 2, 3, 4\}$, $\tau_1 = \big\{M, \emptyset, \{1, 2\}\big\}, \tau_2 = \big\{M, \emptyset, \{3, 4\}\big\}$. Let $B = \{a, b\}$, $\Lambda_1 = \big\{B, \emptyset, \{a\}\big\}, \Lambda_2 = \{B, \emptyset, \{b\}\}$. Let $p: M \to B$ such that $p\,(1) = a = p\,(2)\,, p\,(3) = b = p\,(4)$. Let $H = \{3, 4\}, K = \{1, 2\}$. Let $b = a,$ nbd of a is $\mathrm{W} = \{a\}$, $M_W = \{1, 2\}$, $\tau_{1M_W} = \{M_W, \emptyset\} = \tau_{2M_W}$ let $U = \{3, 4\}, V = \{1, 2\}$. Let $\lambda : M_W \to I$ such that $\lambda(1) = 1 = \lambda(2)$, λ is continuous and $M_W \cap H \cap U = \emptyset \subset \lambda^{-1}(0), M_W \cap K \cap V = \{1, 2\} \subset \lambda^{-1}(1)$. Let $b = b$, nbd of b is $W = \{b\}$, $M_W = \{3, 4\}$ $\tau_{1M_W} = \{M_W, \emptyset\} = \tau_{2M_W}$. Let $\lambda : M_W \to I$ such that $\lambda(3) = 0 = \lambda(4)$, λ is continuous and $M_W \cap H \cap U = \{3, 4\} \subset \lambda^{-1}(0), M_W \cap K \cap V = \subset \lambda^{-1}(1)$. So M is F.W. pairwise functionally normal.

For another example, $(B, \Lambda_1, \Lambda_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise functionally normal space when T is pairwise functionally normal space.

Remark 3.2.21. If (M, τ_1, τ_2) is F.W. pairwise functionally normal space over $(B, \Lambda_1, \Lambda_2)$ then for every subspace B^* of B we have $(M_B^*, \tau_1^*, \tau_2^*)$ is F.W. pairwise functionally normal space over $(\mathrm{B}^*, \Lambda_1^*, \Lambda_2^*)$.

Closed subspaces of F.W. pairwise functionally normal spaces are F.W. pairwise functionally normal. Actually we have.

Proposition 3.2.22. Assume that $\varphi: M \to M^*$ is a closed, embedding F.W. function where (M, τ_1, τ_2) and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over B. If M^* is F.W. pairwise functionally normal then so is M.

Proof. Let H and K be disjoint pair of τ_i-closed and τ_j-closed sets of M and let $b \in B$. Then $\varphi(H), \varphi(K)$ are disjoint pair of ${\tau_i}^*$-closed set and ${\tau_j}^*$- closed set of M^*. Since M^* is F.W. pairwise functionally normal, we have a nbd W of b in B and a disjoint pair of τ_j^*-open set U and τ_i^*-open set V and a continuous function $\lambda : M_W^* \to I$ such that $M_W^* \cap \varphi(H) \cap U \subset \lambda^{-1}(0)$ and $M_W^* \cap \varphi(K) \cap V \subset \lambda^{-1}(1)$ in M_W^*. Since φ is continuous, then $\varphi^{-1}(U), \varphi^{-1}(V)$ are τ_j-open set, τ_i-open set and the function, $\Omega = \lambda\ o\ \varphi$ is a continuous, $\Omega : M_W \to I$ such that $M_W \cap H \cap \varphi^{-1}(U) \subset \Omega^{-1}(0)$ and $M_W \cap K \cap \varphi^{-1}(V) \subset \Omega^{-1}(1)$ in M_W as required where $i, j = 1, 2$, $i \neq j$.

Proposition 3.2.23. Assume that $\varphi: M \to N$ is a closed, open and continuous F.W. surjection function, where (M, τ_1, τ_2) and (N, σ_1, σ_2) are F.W. bitopological spaces over $(B, \Lambda_1, \Lambda_2)$. If (M, τ_1, τ_2) is F.W. pairwise functionally normal then so is (N, σ_1, σ_2).

Proof. Let H, K be disjoint pair of σ_i-closed and σ_j-closed sets of N and let $b \in B$. Then $\varphi^{-1}(H), \varphi^{-1}(K)$ are disjoint pair of τ_i-closed and τ_j-closed sets of M. Because M is F.W. pairwise functionally normal, then we have a nbd W of b in B and a disjoint pair of τ_j-open set and τ_i-open set U, V and a continuous function $\lambda : M_W \to I$ such that $M_W \cap \varphi^{-1}(H) \cap U \subset \lambda^{-1}(0)$ and $M_W \cap \varphi^{-1}(K) \cap V \subset \lambda^{-1}(1)$ in M_W. Hence, a function $\Omega: N_W \to I$ is given by $\Omega(\mathrm{y}) = \sup_{\mathrm{x} \in \varphi^{-1}(\mathrm{y})} \lambda(x); \mathrm{y} \in \mathrm{N_W}$. Because φ is open and closed, in addition to continuous, it leads to that Ω is continuous. Hence, $N_W \cap H \cap \varphi(U) \subset \Omega^{-1}(0)$ and $N_W \cap K \cap \varphi(V) \subset \Omega^{-1}(1)$ in M_W where $i, j = 1, 2, i \neq j$.

Chapter 4

Fibrewise IJ-Perfect Bitopological Spaces

Chapter 4

Fibrewise IJ-Perfect Bitopological Spaces

In many times, it has been mixed between topological spaces and some of the basic concepts to get a new topological structure. In this chapter, we shall give a new definition for fibrewise bitopological space in the light of compactness to get a space which has big importance characteristics in topology. Filter concept is considered as one of the rich concepts in topology for having a notable role in the modern directions for topology.

4.1. Fibrewise IJ-Perfect Bitopological Spaces

Definition 4.1.1. Let $(B,\ \Lambda_1,\ \Lambda_2)$ be a bitopological space. A F.W. ij-bitopology on a F.W. set M over B means any bitopology on M for which the projection p is ij-continuous, where $i, j\ =\ 1, 2$.

Definition 4.1.2. A function $f\ :\ (\mathrm{M}, \tau_1, \tau_2)\ \rightarrow\ (\mathrm{N}, \sigma_1, \sigma_2)$ is called ij-closed if the image of each ij-closed set in M is ij-closed set in N, where $i, j\ = 1, 2$.

Theorem 4.1.3. A function $f: (M, \tau_1, \tau_2) \rightarrow\ (N, \sigma_1, \sigma_2)$ is ij-closed iff $\mathrm{ij}-\mathrm{cl}(\mathrm{f}(\mathrm{A}))\ \subset\ f(\mathrm{ij}-\mathrm{cl}(\mathrm{A}))$ for each $A\ \subset\ M$, where $i, j\ = 1, 2$.

Proof: $(\Rightarrow)$ Suppose that f is ij-closed. Let $A \subset M$, since f is ij-closed then $f(\mathrm{ij}-\mathrm{cl}(\mathrm{A}))$ is ij-closed set in N, since $\mathrm{ij}-\mathrm{cl}(\mathrm{A})$ is closed set in M. so, $\mathrm{ij}-\mathrm{cl}(\mathrm{f}(\mathrm{A}))\ \subset f(\mathrm{ij}-\mathrm{cl}(\mathrm{A}))$.

$(\Leftarrow)$ Suppose that A is ij-closed set in M, so $A = \mathrm{ij}-\mathrm{cl}(\mathrm{A})$, but we have $\mathrm{ij}-\mathrm{cl}(\mathrm{f}(\mathrm{A})) \subset f(\mathrm{ij}-\mathrm{cl}(\mathrm{A}))$, thus $ij -\mathrm{cl}\big(f(A)\big) \subset f(A)$. Thus, $f(A)$ is ij-closed in M. Therefore f is ij-closed.

Definition 4.1.4. A filter base $\mathcal{F}$ on bitopological space (M, τ_1, τ_2) is said to be ij-directed toward a set $A \subseteq M$, written as $\mathcal{F} \xrightarrow{ij-d} A$, iff every filter base $\mathcal{G}$ finer than $\mathcal{F}$ has an ij −adherent point in A, i.e. $(ij - ad\ \mathcal{G}) \cap A \neq \varphi$. We write $\mathcal{F} \xrightarrow{ij-d} x$ to mean $\mathcal{F} \xrightarrow{ij-d} \{x\}$, where $x \in M$, where $i, j = 1, 2$.

Now, we introduce a characterization of ij −adherent point x of a filter base $\mathcal{F}$.

Theorem 4.1.5. A point x in bitopological space (M, τ_1, τ_2) is an ij-adherent point of a filter base $\mathcal{F}$ on M iff there exists a filter base $\mathcal{F}^*$ finer than $\mathcal{F}$ such that $\mathcal{F}^* \xrightarrow{ij-con} x$, where $i, j = 1, 2$.

Proof: (⇒) Let x be an ij −adherent point of a filter base $\mathcal{F}$ on M, so it is an ij −contact point of every number of $\mathcal{F}$. This yields, for every τ_i-open nbd U of x, we have $\tau_j - cl(U) \cap F \neq \varphi$ for every number F in $\mathcal{F}$. Consequently, $\tau_j - cl(U)$ contains a some member of any filter base $\mathcal{F}^*$ finer than $\mathcal{F}$, such that $\mathcal{F}^* \xrightarrow{ij-con} x$.

(⇐) Suppose that x is not an ij −adherent point of a filter base $\mathcal{F}$ on M, then there exists $F \in \mathcal{F}$ such that x is not an ij −contact of F. Hence, there exists an τ_i-open nbd U of x such that $\tau_j - cl(U) \cap F = \varphi$. Denote by $\mathcal{F}^*$ the family of sets $F^* = F \cap (M - \tau_j\text{-}cl(U))$ for $F \in \mathcal{F}$, then the sets F^* are nonempty. Also $\mathcal{F}^*$ is a filter base and indeed it is finer than $\mathcal{F}$. This is, given $F_1^* = F_1 \cap (M \setminus \tau_j\text{-}cl(U))$ and $F_2^* = F_2 \cap (M \setminus \tau_j\text{-}cl(U))$, there is an $F_3 \subseteq F_1 \cap F_2$ and this gives $F_3^* = F_3 \cap (M \setminus \tau_j\text{-}cl(U)) \subseteq F_1 \cap F_2 \cap (M \setminus \tau_j\text{-}cl(U)) = F_1 \cap (M \setminus \tau_j\text{-}cl(U)) \cap F_2 \cap (M \setminus \tau_j\text{-}cl(U))$. By construction $\mathcal{F}^*$ is not ij-convergent to x. This is a contradiction, and thus, x is an ij-adherent point of a filter base $\mathcal{F}$ on M.

Theorem 4.1.6. Let $\mathcal{F}$ be a filter base on bitopological space (M, τ_1, τ_2). Let $x \in M$, then $\mathcal{F} \xrightarrow{ij-con} x$ iff $\mathcal{F} \xrightarrow{ij-d} x$, where $i, j = 1, 2$.

Proof: (⇐) If $\mathcal{F}$ does not ij-converge to x, then there exists a τ_i-open nbd U of x such that $F \not\subset \tau_j\text{-}cl(U)$, for all $F \in \mathcal{F}$. Then $\mathcal{G} = \{(M\text{-}\tau_j\text{-}cl(U) \cap F : F \in \mathcal{F}\}$ is a filter base on M finer than $\mathcal{F}$, and clearly $x \notin ij$-adherence of $\mathcal{G}$. Thus, $\mathcal{F}$ cannot be ij-directed towards x which is contradiction. Hence, $\mathcal{F}$ is ij-converge to x.

(⇒) Clear.

Definition 4.1.7. A function $f : (M, \tau_1, \tau_2) \to (\mathrm{N}, \sigma_1, \sigma_2)$ is said to be ij-perfect iff for each filter base $\mathcal{F}$ on $f\,(M)$, such that $\mathcal{F}$ ij-directed towards some subset A of $f(M)$, the filter base $f^{-1}(\mathcal{F})$ is ij-directed towards f^{-1}(A) in M. f is called pairwise ij-perfect iff f is 12 and 21-perfect, where $i, j = 1, 2$.

Definition 4.1.8. The F.W. bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ over bitopological space $(B, \Lambda_1, \Lambda_2)$ is called F.W. ij-perfect iff the projection p is ij-perfect, where $i, j = 1, 2$.

In the following theorem we show that only points of N could be sufficient for the subset A in Definition (4.1.7.) and hence ij-direction can be replaced in view of Theorem (4.1.5.) by ij-convergence.

Theorem 4.1.9. Let $(\mathrm{M}, \tau_1, \tau_2)$ be a F.W. bitopological space over bitopological space (B, Λ_1, Λ_2). Then the following are equivalent:

(a) $(\mathrm{M}, \tau_1, \tau_2)$ is F.W. ij-perfect bitopological space.

(b) For each filter base $\mathcal{F}$ on $p(M)$, which is ij-convergent to a point b in B, $M_{\mathcal{F}} \xrightarrow{ij-d} M_b$.

(c) For any filter base $\mathcal{F}$ on M, ij-ad $p(\mathcal{F}) \subset p$ (ij-ad $\mathcal{F}$).

Proof: (a)⇒(b) Follows from Theorem (4.1.6.).

(b)⇒(c) Let $b \in$ ij-ad p $(\mathcal{F})$. Then by Theorem (4.1.5.), there is a filter base $\mathcal{G}$ on $p(M)$ finer than p $(\mathcal{F})$ such that $\xrightarrow{ij-con} b$. Let $\mathcal{U}$ ={$M_{\mathcal{G}}$ ∩ F : $G \in \mathcal{G}$ and $F \in \mathcal{F}\}$. Then $\mathcal{U}$ is a filter base on M finer than $M_{\mathcal{G}}$. Since $\mathcal{G} \xrightarrow{ij-d} b$, by Theorem (4.1.6.) and p is ij-perfect, $M_{\mathcal{G}} \xrightarrow{ij-d} M_b$. $\mathcal{U}$ being finer than $M_{\mathcal{G}}$, we have M_b ∩ (ij-ad $\mathcal{U}$) $\neq \varphi$. It is then clear that M_b ∩ (ij-ad $\mathcal{F}$) $\neq \varphi$. Thus $b \in p\ (ij - ad\ \mathcal{F})$.

(c)⇒(a) Let $\mathcal{F}$ be a filter base on $p(M)$ such that it is ij-directed towards some subset A of $p(M)$. Let $\mathcal{G}$ be a filter base on M finer than $M_{\mathcal{F}}$. Then $p(\mathcal{G})$ is a filter base on $p(M)$ finer than $\mathcal{F}$ and hence $A \cap (ij - ad\ p(\mathcal{G})) \neq \varphi$. Thus, by (c), A ∩ p (ij − ad $\mathcal{G}$) ≠ φ such that $M_A \cap (ij - ad\ \mathcal{G}) \neq \varphi$. This shows that $M_{\mathcal{F}}$ is ij-directed towards M_A. Hence, p is ij-perfect.

Definition 4.1.10. The function $f: (M, \tau_1, \tau_2) \rightarrow (N, \sigma_1, \sigma_2)$ is called ij-compact function if it is ij −continuous, ij −closed and for each filter base $\mathcal{F}$ in N then $f^{-1}(\mathcal{F})$ is filter base in M, where $i, j = 1, 2$.

Definition 4.1.11. The F.W. ij −bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1, \Lambda_2)$ is called F.W. ij −compact iff the projection p is ij −compact, where $i, j = 1, 2$.

For example the bitopological product $B \times_B T$ is F.W. ij-compact over B, for all ij −compact space T, where $i, j = 1, 2$.

Definition 4.1.12. The F.W. ij-bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ over $(\mathrm{B}, \Lambda_1, \Lambda_2)$ is called F.W. ij-closed if and only if the projection p is ij-closed, where $i, j = 1, 2$.

Theorem 4.1.13. If the F.W. bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ over $(B, \Lambda_1, \Lambda_2)$ is ij-perfect, then it is ij-closed, where $i, j = 1, 2$.

Proof: Assume that M is a F.W. ij-perfect bitopological space over B, then the projection $p_M : M \to B$ is ij-perfect, to prove that it is ij-closed, by [(4.1.9.) (a)⇒(c)] for any filter base $\mathcal{F}$ on M ij-ad p $(\mathcal{F}) \subset$ p (ij-ad $(\mathcal{F})$), by Theorem (4.1.3.) f is ij-closed if $\mathrm{ij}-\mathrm{cl}\, f(\mathrm{A}) \subset f(\mathrm{ij}-\mathrm{cl}(\mathrm{A})$ for all $A \subset M$, therefore p is ij-closed where $\mathcal{F} = \{A\}$.

4.2. Fibrewise IJ-perfect Bitopological Spaces and IJ-Rigidity

In this section, we introduce the notion of ij-perfect bitopological, ij-rigidity spaces and investigate some of their basic properties.

Definition 4.2.1. A subset A of bitopological space (M, τ_1, τ_2) is said to be ij-rigid in M iff for each filter base $\mathcal{F}$ on M with $(\mathrm{ij}-\mathrm{ad}\,\mathcal{F}) \cap \mathrm{A} = \varphi$, there is a τ_i-open set U and $F \in \mathcal{F}$ such that $A \subset U$ and τ_j-cl(U) $\cap\ \mathrm{F} = \varphi$, or equivalently, iff for each filter base $\mathcal{F}$ on M and whenever $\mathrm{A} \cap (\mathrm{ij}-\mathrm{ad}\,\mathcal{F}) = \varphi$, then for some $\mathrm{F} \in \mathcal{F}$, $A \cap (ij - cl(F)) = \varphi$, where $i, j = 1, 2$.

Theorem 4.2.2. If (M, τ_1, τ_2) is a F.W. ij-closed bitopological space over $(B, \Lambda_1, \Lambda_2)$ such that each M_b where $b \in B$ is ij-rigid in M, then $(\mathrm{M}, \tau_1, \tau_2)$ is a F.W. ij-perfect, where $i, j = 1, 2$.

Proof: Assume that M is a F.W. ij-closed bitopological space over B, then the projection $p_M : M \to B$ exist. To prove that it is ij-perfect, let $\mathcal{F}$ be a filter base on $p\ (M)$ such that $\mathcal{F} \xrightarrow{ij-con} b$ in B, for some $b\ \in\ B$. If $\mathcal{G}$ is a filter base on M finer than the filter base $M_{\mathcal{F}}$, then $p(\ \mathcal{G})$ is a filter base on B, finer than $\mathcal{F}$. Since $\mathcal{F} \xrightarrow{ij-d} b$ by Theorem (4.1.5.), $b\ \in$ ij-ad $p\ (\mathcal{G})$, i.e, $b\ \in \cap$ $\{ij - ad\ p(G) : G\ \in \mathcal{G}\}$ and hence $b\ \in \cap\ \{p(ij - ad\ (G) : G\ \in \mathcal{G}\}$ by Theorem (4.1.3.). Since p is ij-closed, then $M_b\ \cap$ ij-ad $(\ G) \neq \varphi$, for all $G\ \in \mathcal{G}$. Hence, for all $U \in \tau_i$ with $M_b \subset$ U, τ_j-cl(U) $\cap\ G \neq \varphi$, for all $G\ \in\ \mathcal{G}$. Since, M_b is ij-rigid, it then follows that $M_b \cap$ (ij-ad $\mathcal{G}$)$\neq\ \varphi$. Thus $M_{\mathcal{F}} \xrightarrow{ij-d} M_b$. Hence by Theorem [(4.1.9.) (b)⇒(a)], p is ij-perfect.

Theorem 4.2.3. If the F.W. ij-bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1,\ \Lambda_2)$ is ij-perfect, then it is ij-closed and for each $b \in B, M_b$ is ij-rigid in M, where $i, j\ = 1, 2$.

Proof: Assume that M is a F.W. ij-bitopological space over B, then the projection $p_M : M\ \rightarrow\ B$ exist and it is ij-continuous. Since p is an ij-perfect so it is ij-closed. To prove the other part, let $b\ \in\ B$, and suppose $\mathcal{F}$ is a filter base on M such that (ij-ad $\mathcal{F}$) $\cap\ M_b$= φ. Then $b\ \notin\ p$ (ij-ad $\mathcal{F}$). Since p is ij-perfect, by Theorem [(4.1.9) (a)⇒(c)], b $\notin$ ij $-$ ad p($\mathcal{F}$). Thus there exists an $F \in \mathcal{F}$ such that $b\ \notin$ ij-ad $p(F)$. There exists an Λ_i-open nbd V of b such that $\Lambda_j - cl(V)\ \cap\ p(F)\ =\ \varphi$. Since p is ij-continuous, for each $x\ \in\ M_b$ we shall get a τ_i-open nbd U_x of x such that $p(\tau_j - cl(U_x)\ \subset\ \Lambda_j - cl(V)\ \subset$ $B - p(F)$. Then $p(\tau_j - cl(U_x)\ \cap\ p(F) =\ \varphi$, so that τ_j-cl-$(U_x) \cap F =\ \varphi$. Then $x\ \notin$ ij-cl(F), for all $x\ \in M_b$, so that $M_b\ \cap$ (ij-cl(F)) = φ, Hence M_b is ij-rigid in M.

Corollary 4.2.4. A F.W. ij-bitopological space (M, τ_1, τ_2) over $(B, \Lambda_1,\ \Lambda_2)$ is ij-perfect iff it is ij-closed and each M_b, where $b \in B$ is ij-rigid in M, where $i, j\ = 1, 2$.

Next we show that the above theorem remains valid if F.W. ij-closedness bitopological space replaced by a strictly weaken condition which we shall called F.W. weak ij-closedness bitopological space. Thus we define as follows.

Definition 4.2.5. A function $f\,:\,(M,\tau_1,\tau_2)\rightarrow(N,\sigma_1,\sigma_2)$ is said to be weakly ij-closed if for every $y\in f(M)$ and every τ_i-open set U containing $f^{-1}(y)$ in M, there exists a σ_i- open nbd V of y such that f^{-1} (σ_j-cl(V)) $\subset\tau_j$-$cl(U)$, where $i,j\ =1,2$.

Definition 4.2.6. The F.W. ij-bitopological space $(\mathrm{M},\tau_1,\tau_2)$ over $(\mathrm{B},\Lambda_1,\Lambda_2)$ is called F.W. weakly ij-closed iff the projection p is weakly ij-closed, where $i,j\ =1,2$.

Theorem 4.2.7. The F.W. ij-closed bitopological space $(M,\tau_1,\ \tau_2)$ over $(\mathrm{B},\Lambda_1,\Lambda_2)$ is weakly ij-closed, where $i,j\ =1,2$.

Proof: Assume that M is a F.W. ij-closed bitopological space over B, then the projection $p_M\,:\,M\ \rightarrow\ B$ exist and to prove its weakly ij-closed. Let $b\ \in\ p(M)$ and let U be a τ_i-open set containing M_b in M. Now, by Lemma (1.1.25.) $\tau_j-cl\left(M-\tau_j-cl(U)\right)=ij-cl\left(M-\tau_j-cl(U)\right)$ and hence by theorem (4.1.3.) and since p is ij-closed, we have ij-cl $p(M-\ \tau_j-cl(U))\ \subset\ p[ij-cl(M-\ \tau_j-cl(U)]$. Now since $b\notin\ p[ij-cl(M-\ \tau_j-cl(U)]$, $b\notin\ ij-cl\,p(M-\tau_j\text{-}cl(U))$ and thus there exists an σ_i-open nbd V of b in B such that $\sigma_j\text{-}cl(V)\,\cap\,p(M-\tau_j-cl(U))\ =\ \varphi$ which implies that $\mathrm{M}_{(\sigma_j-\mathrm{cl(V)})}\cap(M-\tau_j-cl(U))\ =\ \varphi$ i.e., $\mathrm{M}_{(\sigma_j-\mathrm{cl(V)})}\subset\tau_j$-cl (U), and thus p is weakly ij-closed.

A F.W. weakly ij-closed is not necessarily to be F.W. ij-closed and the following example show this.

Example 4.2.8. Let τ_1, τ_2, Λ_1 and Λ_2 be any topologies and p : (M, τ_1, τ_2) → (B, Λ_1, Λ_2) be a constant function, then p is weakly ij-closed for $i, j = 1, 2$ and $(i \neq j)$. Now, let $M = B = \mathbb{R}$. If Λ_1 or Λ_2 is the discrete topology on B, then p : (M, τ_1, τ_2) → (B, Λ_1, Λ_2) given by $p(x) = 0$, for all $x \in M$, is neither 12-closed nor 21-closed, irrespectively of the topologies τ_1, τ_2 and Λ_2 (or Λ_1).

Theorem 4.2.9. Let (M, τ_1, τ_2) be F.W. ij-bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then (M, τ_1, τ_2) is F.W. ij-perfect if :

(a) (M, τ_1, τ_2) is F.W. weakly ij-closed bitopological space, and

(b) M_b is ij-rigid, for each $b \in B$.

Proof: Assume that M is a F.W. ij-bitopological space over B satisfying the conditions (a) and (b), then the projection $p_M : M \to B$ exist. To prove that p is ij-perfect we have to show in view of Theorem (4.2.2.) that p is ij-closed. Let $b \in ij - cl\, p(A)$, for some non-null subset A of M, but $b \notin p\,(ij - cl(A))$. Then $\mathcal{H} = \{A\}$ is a filter base on M and (ij-ad $\mathcal{H}$) ∩ M_b = φ. By ij-rigidity of M_b, there is a τ_i-open set U containing M_b such that $\tau_j - cl(U) \cap A = \varphi$. By weak ij-closedness of p, there exists an Λ_i –open nbd V of b such that $M_{(\Lambda_j - \mathrm{cl(V)})} \subset \tau_j$-cl(U), which implies that $M_{(\Lambda_j - \mathrm{cl(V)})} \cap A = \varphi$, i.e., $(\Lambda_j - cl(V)) \cap p(A) = \varphi$, which is impossible since $b \in ij - cl\, p(A)$. Hence $b \in p(ij - cl(A))$. So f is ij-closed.

Theorem 4.2.10. If (M, τ_1, τ_2) is F.W. ij- perfect bitopological space over $(B, \Lambda_1, \Lambda_2)$ and $B^* \subset B$ is an ij-H-set in B, then M_{B^*} is an ij-H-set in M, where $i, j = 1, 2$.

Proof: Assume that M is a F.W. ij-perfect bitopological space over B, then the projection $p_M : M \to B$ exist. Let $\mathcal{F}$ be a filter base on M_{B^*}, then $p(\mathcal{F})$ is a filter base on B^*. Since B^* is an ij-H-set in $B, B^* \cap ij - ad\, p(\mathcal{F}) \neq \varphi$ by Lemma (1.1.21.). By Theorem [(4.1.9) (a)⇒(c)], $B^* \cap p(ij - ad\ (F)) \neq \varphi$, so that $M_{B^*} \cap$ ij-ad $(\mathcal{F}) \neq \varphi$. Hence by Lemma (1.1.21.), M_{B^*} is an ij-H-set in M.

The converse of the above theorem is not true, is shown in the next example.

Example 4.2.11. Let $M = B = \mathbb{R}, \tau_1$ and τ_2 be the cofinite and discrete topologies on M and Λ_1, Λ_2 respectively denote the indiscrete and usual topologies on B. Suppose $p : (M, \tau_1, \tau_2) \to (B, \Lambda_1, \Lambda_2)$ is the identity function. Each subset of either of (M, τ_1, τ_2) and $(B, \Lambda_1, \Lambda_2)$ is a 12-set. Now, any non-void finite set $A \subset M$ is 12-closed in M, but $p(A)$ (i.e., A) is not 12-closed in B (in fact, the only 12-closed subsets of B are B and φ).

Definition 4.2.12. A function $f : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2)$ is said to be almost ij-perfect if for each ij-H-set K in N, $f^{-1}(K)$ is an ij-H-set in M, where $i, j = 1, 2$.

Definition 4.2.13. The F.W. ij-bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ over $(\mathrm{B}, \Lambda_1, \Lambda_2)$ is called F.W. almost ij-perfect iff the projection p is almost ij-perfect, where $i, j = 1, 2$.

By analogy to Theorem (4.2.2.), a sufficient condition for a function to be almost ij-perfect, is proved as follows.

Theorem 4.2.14. Let (M, τ_1, τ_2) be F.W. ij- bitopological space over $(B, \Lambda_1, \Lambda_2)$ such that:

(a) M_b is ij-rigid, for each $b \in B$, and

(b) (M, τ_1, τ_2) is F.W. weakly ij-closed bitopological space.

Then (M, τ_1, τ_2) is F.W. almost ij-perfect bitopological space.

Proof: Assume that M is a F.W. ij-bitopological space over B, then the projection $p_M : M \to B$ exist and it is ij-continuous. Let B^* be an ij-H-set in B and let $\mathcal{F}$ be a filter base on M_{B^*}. Now $p(\mathcal{F})$ is a filter base on B^* and so by Lemma (1.1.21.), $(ij - ad\, p(\mathcal{F})) \cap B^* \neq \varphi$. Let $b \in [ij - ad\, p(\mathcal{F})] \cap B^*$. Suppose that $\mathcal{F}$ has no ij-ad point in M_{B^*} so that (ij-ad $(\mathcal{F})$) $\cap$ $M_b = \varphi$. Since M_b is ij-rigid, there exists an $F \in \mathcal{F}$ and a τ_i-open set U containing M_b such that $F \cap \tau_j - cl(U) = \varphi$. By weak ij-closedness of p, there is a Λ_i-open nbd V of b such that $M_{(\Lambda_j - \mathrm{cl(V)})} \subset \tau_j - cl(U)$ which implies that $M_{(\Lambda_j - \mathrm{cl(V)})} \cap F = \varphi$, i.e., Λ_j-$cl(V) \cap p(F) = \varphi$, which is a contradiction. Thus by Lemma (1.1.21.), M_{B^*} is an ij-H-set in M and hence p is almost ij-perfect.

4.3. Application of Fibrewise IJ-Perfect Bitopological Spaces

We now give some applications of fibrewise ij-perfect bitopological spaces. The following characterization theorem for an ij-continuous function is recalled to this end.

Theorem 4.3.1. A bitopological space (M, τ_1, τ_2) is F.W. ij-bitopological space over $(B, \Lambda_1, \Lambda_2)$ iff $p(ij - cl(A)) \subset ij - cl(p(A))$, for each $A \subset M$, where $i, j = 1, 2$.

Proof: **(⇒):** Assume that M is a F.W. ij-bitopological space over B, then the projection $p_M : M \to B$ exist and it is ij-continuous. Suppose that $x \in ij - cl(A)$ and V is Λ_i-open nbd of $f(x)$. Since p is ij-continuous, there exists an τ_i-open nbd U of x such that $p(\tau_j - cl(U)) \subset \Lambda_j - cl(V)$. Since τ_j-cl $(U) \cap$

$A \neq \varphi$, then Λ_j-$cl(V) \cap p(A) \neq \varphi$. So, $p(x) \in ij - cl(p(A))$. This shows that $p(ij - cl(A)) \subset ij - cl(p(A))$.

(⇐) Clear.

Theorem 4.3.2. Let (M, τ_1, τ_2) be a F.W. ij-perfect bitopological space over $(B, \Lambda_1, \Lambda_2)$. Then M_A preserves ij-rigidity, where $i, j = 1, 2$.

Proof: Assume that M is a F.W. ij-bitopological space over B, then the projection $p_M : M \to B$ exist and it is ij-continuous. Let A be an ij-rigid set in B and let $\mathcal{F}$ be a filter base on M such that $M_A \cap$ (ij- ad($\mathcal{F}$)) = φ. Since p is ij-perfect and $A \cap p(ij - ad(\mathcal{F}))$ = φ by Theorem [(4.1.9.) (a)⇒(c)] we get $A \cap (ij - ad\, p(\mathcal{F})) = \varphi$. Now A being an ij-rigid set in B, there exists an $F \in \mathcal{F}$ such that $A \cap ij - clp(F) = \varphi$. Since p is ij-continuous, by Theorem (4.3.1.) it follows that $A \cap p(ij - cl(F)) = \varphi$. Thus $M_A \cap (ij - cl(F)) = \varphi$. This proves that M_A is ij-rigid.

In order to investigate the conditions under which a F.W. almost ij-perfect bitopological space may be F.W. ij-perfect bitopological space, we introduce the following definition.

Definition 4.3.3. A function $f: (M, \tau_1, \tau_2) \to (\mathrm{N}, \sigma_1, \sigma_2)$ is said to be ij^*-continuous iff for any σ_j-open nbd V of $f(x)$, there exists a τ_i-open nbd U of x such that $f(\tau_j$-$cl(U)) \subset \sigma_i$-$cl(V)$, where $i, j = 1, 2$.

Definition 4.3.4. The F.W. ij-bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ over $(\mathrm{B}, \Lambda_1, \Lambda_2)$ is called F.W. ij^*-bitopological space iff the projection p is ij^*-continuous, where $i, j = 1, 2$.

The relevance of the above definition to the characterization of F.W. ij-perfect bitopological space is quite apparent from the following result.

Theorem 4.3.5. If $(\mathrm{M}, \tau_1, \tau_2)$ is F.W. ij^*-bitopological space on a pairwise Urysohn space $(\mathrm{B}, \Lambda_1, \Lambda_2)$, then it is F.W. ij-perfect bitopological space iff for every filter base $\mathcal{F}$ on M, if $p(\mathcal{F}) \xrightarrow{ij-con} b$ wher $b \in B$, then $ij - ad\, \mathcal{F} \neq \varphi$, where $i, j \ = 1, 2$.

Proof: (⇒) Let $(\mathrm{M}, \tau_1, \tau_2)$ be a F.W. $i\, j^*$-bitopological space on a pairwise Urysohn space $(\mathrm{B}, \Lambda_1, \Lambda_2)$, then there is a ij^*-continuous projection function $p: (\mathrm{M}, \tau_1, \tau_2) \to (\mathrm{B}, \Lambda_1, \Lambda_2)$ and $p(\mathcal{F}) \xrightarrow{ij-con.} b$ where $b \in B$, for a filter base $\mathcal{F}$ on M. Then $M_{p(\mathcal{F})} \xrightarrow{ij-dir.} M_b$. Since $\mathcal{F}$ is finer than $M_{p(\mathcal{F})}$, $M_b \cap ij - ad\, \mathcal{F} \neq \varphi$, so that $ij - ad\, \mathcal{F} \neq \varphi$.

(⇐) Suppose that for every filter base $\mathcal{F}$ on M, $p(\mathcal{F}) \xrightarrow{ij-con.} b$ where $b \in B$ implies $ij - ad\, \mathcal{F} \neq \varphi$. Let $\mathcal{G}$ be a filter base on B such that $\mathcal{G} \xrightarrow{ij-con.} b$, and suppose that $\mathcal{G}^*$ is a filter base on M such that $\mathcal{G}^*$ is finer than $M_{\mathcal{G}}$. Then $p(\mathcal{G}^*)$ is finer than $\mathcal{G}$. So $p(\mathcal{G}^*) \xrightarrow{ij-con.} b$. Hence $ij - ad\, \mathcal{G}^* \neq \varphi$. Let $z \in B$ such that $z \neq b$. Then since B is pairwise Urysohn, there exist a Λ_i-open nbd U of b and Λ_j-open nbd V of z such that $\left(\Lambda_j - cl\ (U)\right) \cap \left(\Lambda_i - cl(V)\right) = \varphi$. Since $p(\mathcal{G}^*) \xrightarrow{ij-con.} b$, there exist a $G \in \mathcal{G}^*$ such that $p(G) \subset \Lambda_j - cl(U)$. Now, since p is ij^*-continuous, corresponding to each $x \in M_z$ there is a τ_i-open nbd W of x such that $p\left(\tau_j - cl(W)\right) \subset \Lambda_i - cl(V)$. Thus $\Lambda_j - cl(W) \cap G = \varphi$. It follows that $M_z \cap ij - \mathcal{G}^* = \varphi$, for each $z \in B - \{b\}$. Consequently $M_b \cap ij - ad\, \mathcal{G}^* \neq \varphi$, and p is ij- perfect and hence $(\mathrm{M}, \tau_1, \tau_2)$ is F.W. ij^*-bitopology.

Definition 4.3.6. A bitopological space $(\mathrm{M}, \tau_1, \tau_2)$ is said to be locally ij-QHC iff for every $x \in M$, there is a τ_i-open nbd of x, which is an ij-H-set, where $i, j \ = 1, 2$.

Corollary 4.3.7. Let $(\mathrm{M}, \tau_1, \tau_2)$ be a F.W. ij^*-bitopological space over ij-QHC on a pairwise Urysohn bitopological space $(B, \Lambda_1, \Lambda_2)$, then $(\mathrm{M}, \tau_1, \tau_2)$ is F.W. ij-perfect bitopological space, where $i, j = 1, 2$.

Theorem 4.3.8. Let $(\mathrm{M}, \tau_1, \tau_2)$ be a F.W. ij^*-bitopological space over locally ij-QHC on a Urysohn space $(B, \Lambda_1, \Lambda_2)$, then $(\mathrm{M}, \tau_1, \tau_2)$ is F.W. ij^*-bitopological space iff it is F.W. almost ij-perfect, where $i, j = 1, 2$.

Proof: (⇒) If $(\mathrm{M}, \tau_1, \tau_2)$ is F.W. ij^*-bitopological space, then by corollary (4.3.7.), it is F.W. almost ij-perfect.

(⇐) Let $(\mathrm{M}, \tau_1, \tau_2)$ is F.W. almost ij-perfect, then there exist almost ij-perfect projection function $p: (\mathrm{M}, \tau_1, \tau_2) \to (\mathrm{B}, \Lambda_1, \Lambda_2)$, and let $\mathcal{F}$ be any filter base on M and let $p(\mathcal{F}) \xrightarrow{ij-con.} b$ where $b \in B$. There are an ij-H-set B^* in B and Λ_i-open nbd V of b such that $b \in V \subseteq B^*$. Let $\mathcal{H} = \{\Lambda_j - cl(U) \cap p(F) \cap B^*; F \in \mathcal{F}$ and U is a Λ_i-open nbd of $b\}$. By Lemma (1.1.24.), B^* is ij-closed and hence no member of $\mathcal{H}$ is void. In fact, if not, let for some Λ_i-open nbd U of b and some $F \in \mathcal{F}$, $\Lambda_j - cl(U) \cap p(F) \cap B^* = \varphi$. Then $W = U \cap V$ since $y \in U \cap V \in \Lambda_i$ and $\Lambda_j - cl(W) = ij - cl(W) \subset ij - cl(B^*) = B^*$ by Lemma (1.1.25.). Now $\varphi = \Lambda_j - cl(W) \cap p(F) \cap B^* = \Lambda_j - cl(W) \cap p(F)$, which is not possible, since $p(\mathcal{F}) \xrightarrow{ij-con.} b$. Thus $\mathcal{H}$ is filter base on B, and is clearly finer than $p(\mathcal{F})$, so that $\mathcal{H} \xrightarrow{ij-con.} b$. Also $\mathcal{G} = \{M_H \cap F: H \in \mathcal{H}$ and $F \in \mathcal{F}\}$ is clearly a filter on M_{B^*}. Since p is almost ij-perfect, M_{B^*} is an ij-H-set and hence $ij - ad\, \mathcal{G} \cap M_{B^*} \neq \varphi$. Thus $ij - ad\, \mathcal{F} \neq \varphi$. Thus p is ij-perfect and by Theorem (4.3.5.) $(\mathrm{M}, \tau_1, \tau_2)$ is F.W. ij^*-bitopological space.

The following characterization theorem for a F.W. ij-bitopological space is recalled to this end.

Theorem 4.3.9. A F.W. set M over $(B, \Lambda_1, \Lambda_2)$ is F.W. ij-bitopological space iff $p(ij\text{-}cl(A)) \subset ij\text{-}clp(A)$ for each $A \subset M$, where $i, j\ = 1, 2$.

Proof: Since M is a F.W. set over B, then there is projection p where $p: M \to B$. Now we have to prove that p is ij-continuous. But it directly by Theorem (4.3.1.).

Theorem 4.3.10. If $(\mathrm{M}, \tau_1, \tau_2)$ is a F.W. ij-perfect bijective bitopological space with M is a pairwise Hausdorff space on $(B, \Lambda_1,\ \Lambda_2)$, Then B is also pairwise Hausdorff.

Proof: Let $b_1,\ b_2 \in B$ such that $b_1 \neq b_2$. Since p is onto, then $M_{b1}, M_{b2} \in M$ and since p is one to one, then $M_{b1} \neq M_{b2}$. Since p is ij-perfect, so by Theorem (4.1.13) it is ij-closed. By Lemma (1.1.26.) we have $\{\ M_{b1}\} = ij - cl\{M_{b1}\}$ and $\{\ M_{b2}\} = ij - cl\{M_{b2}\}$. Since p is pairwise Hausdorff. Now $p(ij - cl\{\ M_{b1}\}) = ij - cl\{b_1\}$ and $p(ij - cl\{\ M_{b2}\}) = ij - cl\{b_2\}$ since p is ij-closed. This mean $\{b_1\} = ij - cl\{b_1\}$ and $\{b_2\} = ij - cl\{b_2\}$. Hence B is pairwise Hausdorff.

Our next theorem give a characterization of an important class of F.W. bitopological space viz. the ij-QHC spaces in terms of F.W. ij-perfect bitopological space.

Theorem 4.3.11. For a bitopological space $(\mathrm{M}, \tau_1, \tau_2)$, the following statement are equivalent:

a) M is ij-QHC

b) The F.W. $(\mathrm{M}, \tau_1, \tau_2)$ is ij-perfect bitopological space with constant projection over B^* where B^* is a singleton with two equal bitopologies viz. the unique bitopology on B^*.

c) The F.W. $(\mathrm{B} \times \mathrm{M}, Q_1, Q_2)$ is ij-perfect bitopological space over $(B, \Lambda_1, \Lambda_2)$, where $Q_i = \Lambda_i \times \tau_j$. $i, j = 1, 2$ and $i \neq j$.

Proof: (a) ⇒ (b) Let $p: (\mathrm{M}, \tau_1, \tau_2) \rightarrow (B^*, \Lambda_1, \Lambda_2)$ is a constant projection over B^* where B^* is a singleton with two equal bitopologies viz the unique bitopology on B^*. P is clearly ij-closed. Also, M_{B^*}, i.e. M is obviously ij-rigid since B^* is ij-QHC. Then by Theorem (4.2.2.) p is ij-perfect.

(b) ⇒ (a) Follows from Theorem (4.3.2.).

(a) ⇒ (c) Suppose that $(\mathrm{B} \times \mathrm{M}, Q_1, Q_2)$ is F.W. bitopological space over $(B, \Lambda_1, \Lambda_2)$ where $Q_i = \Lambda_i \times \tau_j$, $i, j = 1, 2$ and $i \neq j$, then there is a projection $p = \pi_i: (\mathrm{B} \times \mathrm{M}, Q_1, Q_2) \rightarrow (B, \Lambda_1, \Lambda_2)$. We show that π_i is ij- closed and for each $b \in B, M_B$ is ij-rigid in $B \times M$. Then the result will follow from Theorem (4.2.2). Let $A \subset B \times M$ and $a \notin \pi_i\big(ij - cl(A)\big)$. For each $m \in M, (a, m) \notin ij - cl(A)$, so that there exist a Λ_j-open nbd G_m of a and a τ_i-open nbd H_m of m such that $[Q_i - cl(G_m \times H_m)] \cap A = \varphi$. Since M is ij-QHC, $\{a\} \times M$ is a ij –H-set in $B \times M$. Thus there exist finitely many elements $m_1, m_2, m_3, \dots, m_n$ with $\{a\} \times M \subset \bigcup_{k=1}^{n} Q_i - cl\big(G_{m_k} \times H_{m_k}\big)$. Now, $a \in \bigcap_{k=1}^{n} G_{m_k} = G$ which is a Λ_i-open nbd of a such that $(\Lambda_i - cl(G) \cap \pi_i(A) = \varphi$. Hence $a \notin ij - cl\pi_i(A)$ and thus $ij - cl\pi_i(A) \subset \pi\big(ij - cl(A)\big)$. So π is ij-closed, by Theorem (4.1.3.). Next, let $b \in B$. To show that $(B \times M)_b = {\pi_i}^{-1}$(b) to be ij-rigid in $B \times M$. Let $\mathcal{F}$ be a filter base on $B \times M$ such that ${\pi_i}^{-1}$(b)$\cap\, ij - ad\, \mathcal{F} = \varphi$. For each $m \in M, (b, m) \notin ij - ad\, \mathcal{F}$. Thus there exist Λ_j-open nbd U_m of b in B, a τ_i −open nbd V_m of m in M and an $F_m \in \mathcal{F}$ such that $Q_i - cl(U_m \times V_m) \cap F_m = \varphi$. As show above, there exist finitely many elements $m_1, m_2, m_3, \dots, m_n$ of M such that $\{b\} \times M \subset \bigcup_{k=1}^{n} Q_i - cl\big(U_{m_k} \times V_{m_K}\big)$. Putting $U = \bigcap_{k=1}^{n} U_{m_k}$ and choosing $F \in \mathcal{F}$ with $F \subset \bigcap_{k=1}^{n} F_{m_k}$, we get $\{b\} \times M \subset U \times M \subset Q_j$ such that $Q_i - cl(U \times M) \cap$

$F = \varphi$. Thus $(ij - cl(F)) \cap [{\pi_i}^{-1}$(b) $]= \varphi$. Hence ${\pi_i}^{-1}$(b) is ij-rigid in $B \times M$.

(c)⇒(a) Taking $B^* = B$, we have that $p = \pi_i : B^* \times B \times \rightarrow B^*$ is ij-perfect . Therefore by Theorem. (4.2.10.) $B^* \times M$ is an ij-H-set and Hence M is ij-QHC.

Conclusions

The main purpose of the present work is the starting point for the applications of abstract topological structures in fibrewise theory by using bitopological systems. We believe that fibrewise bitopological structure will be an important base for modification of knowledge extraction and processing.

We used separation axioms concept in fibrewise bitopological space to introduce a new notion namely fibrewise pairwise separation axioms. The suggested methods of fibrewise pairwise separation axioms open way for constructing new types of fibrewise topologies.

Finally, the generalization of fibrewise bitopology in the ij-perfect space are introduced, we believe such generalization will be useful in compact bitopology, as well as soft bitopology.

Future Works

The following are some open problems for the future works:

In the future we can use the concepts fibrewise bitopological spaces in define fibrewise soft bitopological spaces, also we can define fibrewise soft bitopological-T_i where *i=1,2,3,4*. On the other hand we can discuss the relation between fibrewise soft bitopological spaces and fibrewise soft j-bitopological spaces, where $j \in \{\alpha, S, P, b, \beta\}$. Furthermore, we will study fibrewise bitopological digital (resp., di, tri, nano, filte, girll, fuzzy) topological spaces.

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المستخلص

قدمنا في هذه الرسالة دراسة حول بعض الفضاءات التوبولوجية الثنائية في نظرية المجموعات الليفية وبعض النتائج المتعلقة بها والمفاهيم الاساسية مثل الفضاءات التوبولوجية الثنائية الليف القابلة للتقطيع و الفضاءات التوبولوجية الثنائية الليف القابلة للتجزئة.

كما وتطرقنا لبديهيات الفصل في الفضاءات التوبولوجية الثنائية الليفية مثل T_0 , T_1 , R_0 والهاوسدورف والفضاءات التوبولوجية الثنائية الليف هاوسدورف داليا والفضاءات التوبولوجية الثنائية الليف المنتظمة و الفضاءات التوبولوجية الثنائية الليف المنتظمة بالكامل و الفضاءات التوبولوجية الثنائية الليف الطبيعية و الفضاءات التوبولوجية الثنائية الليف الطبيعية داليا.

ثم تناولنا مفهوم التراص في الفضاءات التوبولوجية الثنائية الليف والعلاقة بينه وبين مفهوم الفضاءات التوبولوجية الثنائية الليف الصلبة و الفضاءات التوبولوجية الثنائية الليف المغلقة الفضاءات التوبولوجية الثنائية الليف ضعيفة الانغلاق.

كما وتمت دراسة مفهوم المرشحات والمرشحات الاساسية والمرشح الموجه واقتراب المرشح الى نقطة معينة في الفضاءات التوبولوجية الثنائية الليف.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بغداد

كلية التربية للعلوم الصرفة / ابن الهيثم

قسم الرياضيات

دراسة بعض تعميمات الفضاءات التوبولوجية الليفية الثنائية

رسالة

مقدمة إلى كلية التربية للعلوم الصرفة / ابن الهيثم ، جامعة بغداد
كجزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

من قبل

لواء علي حسين

بإشراف

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