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Some Types of Fibrewise Soft Topological Spaces

A Thesis

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By

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

اللَّهُ نُورُ السَّمَاوَاتِ وَالْأَرْضِ مَثَلُ نُورِهِ كَمِشْكَاةٍ فِيهَا
مِصْبَاحٌ الْمِصْبَاحُ فِي زُجَاجَةٍ الزُّجَاجَةُ كَأَنَّهَا كَوْكَبٌ
دُرِّيٌّ يُوقَدُ مِن شَجَرَةٍ مُّبَارَكَةٍ زَيْتُونَةٍ لَّا شَرْقِيَّةٍ وَلَا
غَرْبِيَّةٍ يَكَادُ زَيْتُهَا يُضِيءُ وَلَوْ لَمْ تَمْسَسْهُ نَارٌ نُّورٌ عَلَى
نُورٍ يَهْدِي اللَّهُ لِنُورِهِ مَن يَشَاءُ وَيَضْرِبُ اللَّهُ الْأَمْثَالَ
لِلنَّاسِ وَاللَّهُ بِكُلِّ شَيْءٍ عَلِيمٌ ﴿٣٥﴾

صدق الله العظيم

النور ﴿٣٥﴾

إهداء

الى من بلغ الرسالة ... وادى الامانة ... ونصح الامة ... الى نبي الرحمة ونور العالمين سيدنا محمد صلى الله عليه واله وسلم

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۞وَوَصَّيْنَا الْإِنسَانَ بِوَالِدَيْهِ إِحْسَانًا حَمَلَتْهُ أُمُّهُ كُرْهًا وَوَضَعَتْهُ كُرْهًا وَحَمْلُهُ وَفِصَالُهُ ثَلَاثُونَ شَهْرًا حَتَّىٰ إِذَا بَلَغَ أَشُدَّهُ وَبَلَغَ أَرْبَعِينَ سَنَةً قَالَ رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَىٰ وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَصْلِحْ لِي فِي ذُرِّيَّتِي إِنِّي تُبْتُ إِلَيْكَ وَإِنِّي مِنَ الْمُسْلِمِينَ (15)

الاحقاف (15)

الى من علمني وارشدني في حياتي وشجعني على تحقيق حلمي ابي العزيز

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Author's Publications

[1] Y. Y. Yousif and M. A. Hussain, "Fibrewise Soft Near Topolgical Spaces". International Journal of Science and Research (IJSR) Volume 6 Issue 2, pp.1010-1019, February (2017).

[2] Y. Y. Yousif and M. A. Hussain," Fibrewise Soft Near Separation Axioms", The 23th Scientific Conference of Collage of Education . AL. Mustansiriyah Universtity, pp. 400-414, 26-27April (2017).

[3] Y. Y. Yousif and M. A. Hussain," Fibrewise Soft Ideal Topological Spaces ", acceptable for publication. Ibn Al-Haitham 1st. International Scientific Conference – 2017.

Abstract

In this thesis, we introduced some types of fibrewise topological spaces by using a near soft set, various related results also some fibrewise near separation axiom concepts and a fibrewise soft ideal topological spaces.

We introduced preliminary concepts of topological spaces, fibrewise topology, soft set theory and soft ideal theory.

We explain and discuss new notion of fibrewise topological spaces, namely fibrewise soft near topological spaces, Also, we show the notions of fibrewise soft near closed topological spaces, fibrewise soft near open topological spaces, fibrewise soft near compact spaces and fibrewise locally soft near compact spaces.

On the other hand, we studied fibrewise soft near forms of the more essential separation axioms of ordinary soft topology namely fibrewise soft near T_0 spaces, fibrewise soft near T_1 spaces, fibrewise soft near R_0 spaces, fibrewise soft near Hausdorff spaces, fibrewise soft near functionally Hausdorff spaces, fibrewise soft near regular spaces, fibrewise soft near completely regular spaces, fibrewise soft near normal spaces and fibrewise soft near functionally normal spaces. Too we add numerous outcomes about it.

Finally, we introduced a notion fibrewise soft ideal topological spaces and give the results related it to, Further we obtain some properties in the light of the study notions fibrewise soft ideal open topological spaces, fibrewise soft ideal closed topological spaces and fibrewise soft near ideal topological spaces.

Abbreviation

(F,E)	soft set
$\tilde{\cap}$	intersection of soft set
$\tilde{\cup}$	union of soft set
$\tilde{\in}$	belong of soft set
$\tilde{\notin}$	not belong of soft set
P_H	projection function $P: H \rightarrow B$
H_b	$P^{-1}(b) : b \in B$
$(H,\tilde{\tau},E)$	soft topological space
$H_{(N,G)}$	$P^{-1}(N,G) : (N,G) \tilde{\in} (B,\tilde{\Omega},G)$
$\mathbb{P}(H)$	power set of H
$\tilde{\mathfrak{F}}$	family of all soft closed sets
$\tilde{h}$	soft element
$(H,\tilde{\tau},E,I)$	soft topological space with a soft ideal I
$(F,E)^*$	set $\{\tilde{h} : (U,E) \tilde{\cap} (F,E) \tilde{\notin} I\}$
Cl^*	soft *_closure
$\prod_B(H_r,\tilde{\tau}_r, E_r)$	fibrewise soft topological product
id_H	identity function $id_H : H \rightarrow H$
π_2	projection function of product
$\mathbb{R}$	real numbers
Γ_E	soft covering
G	fibrewise soft graph
$(H^*,\tilde{\tau}^*,E^*)$	fibrewise soft subspaces
Δ	soft diagonal
λ	continuous function $\lambda : H_{(N,G)} \rightarrow [0,1]$

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Introduction

Topology is one of the most important branch in the modern mathematics. It is interested in plastic geometrical shapes or those shapes which are increasable or decreases with keeping some their characteristics that can be distinguished by the continuous functions between those shapes and those so called topology spaces. Topology interlaces in all the branches of mathematics. The first people who introduced the term topology are Germans in 1847 by Johan Benedict Listings. Then, the English specialists said that topologist is every one is specialized in topology. As for the modern topology is strongly depending on concepts groups theory that established by Georg Ferdinand Ludwig Philipp Cantor in the end of 19^{th} century. Topology has many branches of them points set topology, algebra topology and geometric topology.

The study of general topology is concerned with the category "TOP" of topological spaces as objects and continuous mappings as morphisms. With this in mind, a branch of general topology which has become known as general topology of continuous mappings or fibrewise general topology (briefly fibrewise topology), was initiated. Fibrewise topology is concerned most of all in extending the main notions and results concerning topological spaces to continuous mappings. I. M. James 1989 has been promoting the fibrewise viewpoint systematically in topology [30-31-32-33-34-35]. As a matter of fact in many directions interests in research on fibrewise theory are growing now. Fibrewise topological spaces theory, presented in the recent 20 years, is a new branch of mathematics developed on the basis of general topology, algebra topology and fibrewise spaces theory. It is associated with differential geometry, Lie groups and dynamical systems theory.

Despite of many theories such as theory of intuitionistic fuzzy sets [11], [12], theory of vague sets [26], theory of interval mathematics [12], [27] and theory of rough sets. There are many problems in accuracy of data in various of medicine, geometry and social sciences. However, these theories have their own difficulties. As a result of these problems, the Russian scientist Molodtsov (1999) introduced a new concept of soft set theory as a mathematical tool to process uncertainty and this theory has a rich abilities for application in many directions and just a few showed by molodtsov in his leading work. In 2011, Naz and Sharab [51] defined the structure of the topology of the soft sets and studied some important properties. In addition, Maji et al. [43] proposed several operations on soft sets, and some basic properties of these operations have been revealed so far.

In 1933, Kuratowski studied the ideal in the topological space. This topic has a wide scope for application in other branches of mathematics. Scientists have continued to study this subject in recent years [37], [28], [29] and [36].

Recently each of Hamlett, Jankovic' and Rose in (1990, 1991) developed the theme of ideals in general topology. Future, Abd El-Monsef [2] defined I-continuity for functions. The notion of soft ideal was first given by R. Sahin and A. Kucuk [51]. We built on some of the results in [3], [42], [54], [55], [56], [57], [58] and [59].

In the advanced world, there exists an appoint of view says that Topology and its application has a big attention by the researches centers and because of near-open sets play an important role in general topology and its now a subject of research for many topologist around the world. We chose some of these sets of which as far as we know did not enter in fibrewise topological spaces up to now. Therefore, we called our thesis:

"***Some Types of Fibrewise Soft Topological Spaces***"

This thesis consists of four chapters:

Chapter One: We introduced some basic concepts to be used through this thesis. These concepts are considered like a fundamental concepts in topological space, fibrewise topology, soft set theory and soft ideal theory.

Chapter Two: We have introduced the concept of soft set in the fibrewise topology, and we have developed new concepts in fibrewise topology called fibrewise soft near topological spaces, fibrewise soft near closed, fibrewise soft near open topological spaces, fibrewise soft near compact spaces and fibrewise locally soft near compact spaces.

Chapter Three: We have identified new concepts of fibrewise soft separation axioms called fibrewise soft near T_0 spaces, fibrewise soft near T_1 spaces, fibrewise soft near R_0 spaces, fibrewise soft near Hausdorff spaces, fibrewise soft near functionally Hausdorff spaces, fibrewise soft near regular spaces, fibrewise soft near completely regular spaces, fibrewise soft near normal spaces and fibrewise soft near functionally normal spaces. Also, we demonstrated the relationship between fibrewise soft near compact (resp., locally soft near compact) spaces and some fibrewise soft near separation axioms.

Chapter Four : In this chapter we introduced the concept of soft ideal in the fibrewise topology and we have developed new concepts in fibrewise topology called fibrewise soft ideal topological spaces, fibrewise soft closed and fibrewise soft open ideal topological spaces. Additionally, we studied fibrewise soft near ideal topological spaces.

Chapter 1

Preliminary Concepts

Chapter 1

Preliminary Concepts

General topology has been considered the entrance to understand topological sciences such as algebraic topology, geometric topology, digital topology and differential topology. Many works have appeared recently for example in algebra [16], [15] topology [38], [39] structure analysis [17], [18], [40] and chemistry [52].

The concept of topological structures ([24], [53], [38]) and their generalizations are very useful not only in theoretical studies but also in practical applications. This chapter is divided into three sections. Section one contains of fundamental concepts of topological space and basic definitions about near open sets. In section two, we show the concept of fibrewise topology. Finally, in section three we give an account of soft sets theories and soft ideal theory.

1.1. Fundamental Notions of Topological Spaces

Some of basic concepts in topology which are useful for our study are given in this section.

Definition 1.1.1. [24] Let H be a nonempty set and τ be a collection of a subset of H. The collection τ is said to be a topology on H if τ satisfies the following three conditions:

(a) $\phi \in \tau$ and $H \in \tau$,

(b) τ is closed under a finite intersection,

(c) τ is closed under arbitrary union.

If τ is a topology on H, then the pair (H, τ) is called a topological space or simply H is a space. The subsets of H belonging to τ are called open sets in the space and the complement of the subsets of H which belongs to τ are called closed sets in the space.

Definition 1.1.2. [24] Let (H, τ) be a topological space and $A \subseteq H$. The closure (resp., interior) of A is denoted by $Cl(A)$ (resp., $int(A)$) is defined by:

$$Cl(A) = \cap \{ F \subseteq H; F \text{ is closed set and } A \subseteq F\}$$

$$int(A) = \cup \{ O \subseteq H; O \text{ is open set and } O \subseteq A\}.$$

Evidently, $Cl(A)$ (resp., $int(A)$) is the smallest closed (resp., largest open) subset of H which contains (resp., contained in) A. Note that A is closed (resp., open) if and only if $A = Cl(A)$ (resp., $int(A)$).

Definition 1.1.3. [24]

(a) A function $\phi : H \to K$ is said to be continous if the inverse image of each open set in K is open in H.

(b) A function $\phi : H \to K$ is said to be open if the image of each open set in H is open in K.

(c) A function $\phi : H \to K$ is said to be closed if the image of each closed set in H is closed in K.

Definition 1.1.4. A subset A of a topological space (H, τ) is called

(a) α-open [48] if $A \subseteq int(Cl(int(A)))$

(b) semi-open [41] (briefly S-open) if $A \subseteq Cl(int(A))$,

(c) pre-open [44] (briefly P-open) if $A \subseteq int(Cl((A))$,

(d) b-open [23] if $A \subseteq Cl(int(A)) \cup int(Cl(A))$and

(e) β-open [1] (or semi-pre-open [6]) if $A \subseteq Cl(int(Cl(A)))$.

Definition 1.1.5. [53] For every topological space H^* and any subspace H of H^*, the function $i_H : H \to H^*$ define by $i_H(h) = h$ is called embedding of the subspace H in the space H^*. Observe that i_H is continuous, since $i_H^{-1}(U = H \cap U$, where U is open set in H^*. The embedding i_H is closed (resp., open) iff the subspace H is closed (resp., open).

Definition 1.1.16. [53] An open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ of H is *shrinkable* provided an open cover $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ exists with the property that $Cl(V_\alpha) \subset U_\alpha$ for each $\alpha \in \Lambda$. Of course $\mathcal{V}$ is called a shrinking of $\mathcal{U}$.

1.2. Fundamental Notions of Fibrewise Topology

To begin with our work in the type of fibrewise sets over a given set, named the base set. If the base set is denoted by B then a fibrewise set over B consists of a set H together with a function $P : H \to B$, named the projection function. For all point b of B the fibre over b is the subset $H_b = P^{-1}(b)$ of H; fibres may be empty because we do not require P to be surjective, in addition for all subset B^* of B we consider $H_{B^*} = P^{-1}(B^*)$ as a fibrewise set over B^* with the projection function determined by P.

Definition 1.2.1. [30] Let H and K are fibrewise sets over B, with projections $P_H : H \to B$ and $P_K : K \to B$, respectively, a function $\phi : H \to K$ is said to be fibrewise if $P_K \circ \phi = P_H$, in other words if $\phi(H_b) \subset K_b$for each point b of B.

Note that a fibrewise function $\phi : H \to K$ over B determines by restriction, a fibrewise function $\phi_{B^*} : H_{B^*} \to K_{B^*}$ over B^* for all subset B^* of B .

Definition 1.2.2. [30] Suppose that B is a topological space. The fibrewise topology on a fibrewise set H over B mean any topology on H for which the projection function P is continuous. .

1.3. Fundamental Notions of the Soft Set Theory

The soft set theory has a rich potential for applications in several directions, a few of which had been shown by Molodtsov in his pioneer work [45]. In this section, we will present some basic definitions, examples and properties about a soft set theory.

Definition 1.3.1. [45] Let H be an initial universe and E be a set of parameters. Let $\mathbb{P}(H)$ denote the power set of H and A be a non-empty subset of E. A pair (F, A) is called a soft set over $H,$ where F is a function given by $F : A \to \mathbb{P}(H)$. In other words, a soft set over H is a parameterized family of subset of the universe H. For $\varepsilon \epsilon A$, $F(\varepsilon)$ may be considered as the set of ε_approximate elements of the soft set (F, A).

Note that the set of all soft sets over H will be denoted by $S(H)$.

Example 1.3.2. [45] Suppose that there are six houses in the universe $H = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ under consideration, and $E = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of decision parameters. The $e_i (i = 1, 2, 3, 4, 5)$ stand for the parameters ‘‘expensive’’, ‘‘beautiful’’, ‘‘wooden’’, ‘‘cheap’’ and ‘‘in green surroundings’’, respectively. Consider the function F_A given by ‘‘houses (.)’’; (.) is to be filled in by one of the parameters $e_i \in E$. For instance, $F_A(e_1)$ means‘‘houses (expensive)’’, and its functional value is the set $\{u \in U : u$ is an expensive house$\}$. Suppose that $A = \{e_1, e_3, e_4\} \subseteq E$ and $F_A(e_1) = \{h_1, h_4\}$, $F_A(e_3) = H$ and $F_A(e_4) = \{h_1, h_3, h_5\}$. Then we can view the soft set (F, A) as consisting of the following collection of approximations: $(F, A) = \{(e_1, \{h_2, h_4\}), (e_3, H), (e_4, \{h_1, h_3, h_5\})\}$.

Definition 1.3.3. [43] A soft set (F, E) over H is said to be a null soft if $F(e) = \Phi$ for all $e \in E$ and this denoted by $\widetilde{\Phi}$. Also, (F, E) is said to be an absolute soft set if $F(e) = H$, for all $e \in E$ and this denoted by $\widetilde{H}$.

Definition 1.3.4. [51] The difference of two soft sets (F, E) and (G, E) over the common universe H, denoted by $(F, E) - (G, E)$ is the soft set (L, E) where for all $e \in E$, $L(e) = F(e) - G(e)$.

Definition 1.3.5. [43]

(a) The union of two soft sets of (F, A) and (G, B) over the common universe H is the soft set (L, C), where $C = A \cup B$ and for all $e \in C$,

$$L(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B \end{cases}$$

We write $(F, A)\ \widetilde{\cup}\ (G, B) = (L, C)$.

(b) The intersection (L, C) of two soft sets (F, A), (G, B) over a common universe H, denoted by $(F, A)\ \widetilde{\cap}\ (G, B)$, is defined as $C = A \cap B$, and $H(e) = F(e) \cap G(e)$ for all $e \in C$.

Definition 1.3.6. [9] The complement of a soft set (F, E), denoted by $(F, E)^c$, is defined by $(F, E) = (F^c, E)$. $F^c : E \to \mathbb{P}(H)$ is a function given by $F^c(e) = H - F(e)$, $\forall e \in E$. F^c is called the soft complement function of F. Clearly, $(F^c)^c$ is the same as F and $((F, E)^c)^c = (F, E)$.

Definition 1.3.7. [51] Let $\tilde{\tau}$ be the collection of soft sets over H, then $\tilde{\tau}$ is said to be a soft topology on H if

(a) $\widetilde{\Phi}, \widetilde{H}\ \widetilde{\in}\ \tilde{\tau}$

(b) the union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$

(c) the intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The triplet $(H, \tilde{\tau}, E)$ is called a soft topological space over H. The members of $\tilde{\tau}$ are called soft open sets in H. Also, a soft set (F, A) is called a soft closed if the complement $(F, A)^c$ belongs to $\tilde{\tau}$, and the family of all closed soft sets is denoted by $\tilde{\mathfrak{F}}$.

Example 1.3.8. [51] Let $H = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ where (F_1, E), $(F_2, E), (F_3, E)$ and (F_4, E) are soft sets over H, defined as follows

$F_1(e_1) = \{h_2\}$, $F_1(e_2) = \{h_2\}$,

$F_2(e_1) = \{h_2, h_3\}$, $F_2(e_2) = \{h_1, h_2\}$,

$F_3(e_1) = \{h_1, h_2\}$, $F_3(e_2) = \{h_2\}$,

$F_4(e_1) = H$ and $F_4(e_2) = \{h_1, h_2\}$.

Then $\tilde{\tau}$ defines a soft topology on H and hence $(H, \tilde{\tau}, E)$ is a soft topological space over H.

Definition 1.3.9. [47]

(a) A soft set (F, E) over H is said to be a soft element if $\exists e \in E$ such that $F(e)$ is a singleton, say, $\{h\}$ and $f(e') = \Phi$, $\forall e' (\neq e) \in E$. Such a soft element is denoted by F_e^h. For simplicity of notation we denote such soft element as $\tilde{h}$. Let $SE(H, E)$ be the set of all soft elements of the universal set H.

(b) A soft set (F, E) is said to be a soft neighbourhood (briefly soft nbd) of the soft set (L, E) if there exists a soft set $(G, E) \tilde{\in} \tilde{\tau}$ such that $(L, E) \tilde{\subseteq} (G, A) \tilde{\subseteq} (F, E)$. If $(L, E) = \tilde{h}$, then (F, E) is said to be a soft nbd of the soft element $\tilde{h}$. The soft neighbourhood system of soft element $\tilde{h}$, denoted by $N(\tilde{h})$, is the family of all its soft neighbourhood. The soft open neighbourhood system of soft element $\tilde{h}$, denoted by $V(\tilde{h})$, is the family of all its soft open neighbourhood.

Definition 1.3.10. [51]

(a) Let $(H, \tilde{\tau}, E)$ be a soft topological space and (F, E) be a soft set over H. Then, the soft closure of (F, E), denoted $\overline{(F, E)}$ or $Cl(F, E)$, is defined as the soft intersection of all soft closed super sets of (F, E).

Note that $\overline{(F, E)}$ is the smallest soft closed set that containing (F, E).

(b) Assume that H is an initial universe set, E is the set of parameters and $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}\}$. Then $\tilde{\tau}$ is called the soft indiscrete topology on H and $(H, \tilde{\tau}, E)$ is said to be a soft indiscrete space over H and denoted by $\tilde{\tau}_{ind}$. If $\tilde{\tau}$ is the collection of all soft sets which can be defined over H, then $\tilde{\tau}$ is called the soft discrete topology on H and $(H, \tilde{\tau}, E)$ is said to be a soft discrete space over H, and denoted by $\tilde{\tau}_{dis}$.

Definition 1.3.11. [60] Let $(H, \tilde{\tau}, E)$ be a soft topological space and (F, E) be a soft set over H. Then, the soft interior of (F, E), denoted $(F, E)^{\circ}$ or $int(F, E)$, is defined as the soft union of all soft open sets contained in (F, E).

Clearly $(F, E)^{\circ}$ is the largest soft open set contained in (F, E).

Definition 1.3.12. [19]

(a) Let $(H, \tilde{\tau}, E)$ be a soft topological space over H and K be a non-empty subset of H. Then, the collection $\tilde{\tau}_K = \{K \tilde{\cap} (F, E) : (F, E) \tilde{\in} \tilde{\tau}\}$ is called a soft subspace topology on K. Hence, $(K, \tilde{\tau}_K, E)$ is called a soft topological subspace of $(H, \tilde{\tau}, E)$.

(b) A soft basis of a soft topological space $(H, \tilde{\tau}, E)$ is a subcollection $\tilde{\mathcal{B}}$ of $\tilde{\tau}$ such that every element of $\tilde{\tau}$ can be expressed as the union of elements of $\tilde{\mathcal{B}}$. Also, if the soft element of $\tilde{\mathcal{B}}$ is soft nbds then is called soft base nbds.

Definition 1.3.13. A soft set (F, E) in a soft topological space $(H, \tilde{\tau}, E)$ is called

(a) Soft α-open set if $(F,E) \tilde{\subseteq}\ int\left(Cl\left(int(F,E)\right)\right)$ [4].

(b) Soft pre-open (briefly soft P-open) set if $(F,E)\ \tilde{\subseteq} int(Cl(F,E))$ [10].

(c) Soft sime-open (briefly soft S-open) set if $(F,E)\ \tilde{\subseteq} (Cl(int(F,E))$ [21].

(d) Soft b-openif $(F,E) \tilde{\subseteq} int(Cl((F,E))) \tilde{\cup} Cl\left(int((F,E))\right)$ [6].

(e) Soft β-open set if $(F,E) \tilde{\subseteq} Cl(int(Cl(F,E)))$ [10].

The complement of a soft α-open (resp., soft S-open, soft P-open, soft b-open and soft β-open) set is called soft α-closed (resp., soft S-closed, soft P-closed, soft b-closed and soft β-closed) set. The family for all soft α-open (resp., soft S-open, soft P-open, soft b-open and soft β-open) sets of $(H,\tilde{\tau},E)$ are greater than of $\tilde{\tau}$ and closed under making arbitrary union. We shall call these families soft near topology (briefly S. j-topology), where $j \in \{\alpha, S, P, b, \beta\}$.

Definition 1.3.14. [46] Let H and K be two non-empty sets and E be the parameter set. Let $\{f_e : H \to K, e \in E\}$ be a collection of functions. Then a function $\tilde{f} : \mathrm{SE}(H,E) \to \mathrm{SE}(K,E)$ defined by $\tilde{f}(e_{\tilde{h}}) = e_{f_e(\tilde{h})}$ is called a soft function, where $\mathrm{SE}(H,E)$ and $\mathrm{SE}(K,E)$ are sets of all soft elements of the soft sets $\widetilde{H}$ and $\widetilde{K}$ respectively.

Definition 1.3.15. [4, 6, 22] A soft function $\phi : (H,\tilde{\tau},E) \to (K,\tilde{\sigma},L)$ is said to be soft near continuous (briefly S. j-continuous) if the inverse image of every soft open set of K is a S. j-open set in H where $j \in \{\alpha, S, P, b, \beta\}$.

Definition 1.3.16. [4, 22, 6, 5] A function $\phi : (H,\tilde{\tau},E) \to (K,\tilde{\sigma},L)$ is said to be

(a) Soft near-open (briefly, S. j-open) if the image of every soft open set in H is S.j-open set in K, where $j \in \{\alpha, S, P, b, \beta\}$.

(b) Soft near-closed (briefly, S. j-closed) if the image of every soft closed set in H is S. j-closed set in K, where $j \in \{\alpha, S, P, b, \beta\}$.

Definition 1.3.17. [22, 6] Let $\phi : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ be a function. ϕ is called soft near irresolute (briefly, S. j-irresolute) if the inverse image of S. j-open set in K is S. j-open in H, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 1.3.18. [4] If $\phi : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ is a soft pre-continuous and soft semi-continuous, then ϕ is soft α-continuous.

Definition 1.3.19. [49] If (H, E) is a soft set, then the soft set $\Delta_{(H,E)}$ = $\{(h_e, h_e)$: $h_e \epsilon (H, E)\}$ is called as the diagonal of $(H, E) \times (H, E)$. Here $\Delta_{(H,E)}$ = (Δ_H, Δ_E) is defined by $((h, h), (e, e))$ = $(h,\ h)_{(e,e)}$ =(h_e, h_e).

Example 1.3.20. [49] Let $H = \{h_1, h_2, h_3\}$ be any set and $E = \{e_1, e_2\}$ be a set of parameters. In this case, the soft set (F, E) is formed by the following soft points $\{{h_1}_{e_1}, {h_1}_{e_2}, {h_2}_{e_1}, {h_2}_{e_2}, {h_3}_{e_1}, {h_3}_{e_2}\}$. Thus, the soft diagonal $\Delta_{(H,E)}$ of this soft set is $\Delta_{(H,E)}$= $\{({h_1}_{e_1}, {h_1}_{e_1}), ({h_1}_{e_2}, {h_1}_{e_2}), ({h_2}_{e_1}, {h_2}_{e_1}), ({h_2}_{e_2}, {h_2}_{e_2}), ({h_3}_{e_1}, {h_3}_{e_1}), ({h_3}_{e_2}, {h_3}_{e_3})\}$.

Definition 1.3.21. [14] Let (F, E) and (G, B) be two soft sets over H, then the Cartesian product of (F, E) and (G, B) is defined as, $(F, E) \times (G, B) = (L, E \times B)$, where $L : E \times B \rightarrow P(H \times H)$ and $L(e, b) = F(e) \times G(b)$, where $(e, b) \in E \times B$

i.e., $L(e, b) = \{(h_i, h_j)$; where $h_i \in F(e)$ and $h_j \in G(b)\}$.

The Cartesian product of three or more nonempty soft sets can be defined by generalizing the definition of the Cartesian product of two soft sets. The Cartesian product $(F_1, E) \times (F_2, E) \times \cdots \times (F_n, E)$ of the nonempty soft sets

$(F_1, E), (F_2, E), \dots, (F_n, E)$ is the soft set of all ordered n-tuple $(h_1, h_2, \dots, h_n)$, where $h_i \in F_i(e)$.

Definition 1.3.22. [13] Let $\{(H_r, \tilde{\tau}_r, E_r)\}_{r \in S}$ be a family of soft topological spaces. Then, the initial soft topology on $H = \prod_{r \in S} H_r$ generated by the family $\{(f_r, u_r)\}_{r \in S}$ is called product soft topology on H. (Here, (f_r, u_r) is the soft projection function from H to H_r, $r \in S$).

The product soft topology is denoted by $\prod_{r \in S}(H_r, \tilde{\tau}_r, E_r)$.

Definition 1.3.23. [60]

(a) A family N of soft sets is a cover of a soft set (F, E) if

$$(F, E) \tilde{\subseteq} \tilde{\cup} \{(F_i, E) : (F_i, E) \tilde{\in} N, i \in I\}.$$

It is a soft open cover if each member of N is a soft open set. A subcover of N is a subfamily of N which is also a cover.

(b) A soft topological space $(H, \tilde{\tau}, E)$ is compact if each soft open cover of $\tilde{H}$ has a finite subcover.

The notion of soft ideal was first given by R. Sahin and A. Kucuk [50]. We review the basic definitions about soft ideal theory

Definition 1.3.24. [25] A soft ideal I is a nonempty collection of soft sets over H if

(a) $(F, E) \tilde{\in} I$,$(G, E) \tilde{\subset} (F, E)$ implies $(G, E) \tilde{\in} I$

(b) $(F, E) \tilde{\in} I$, $(G, E) \tilde{\in} I$ implies $(F, E) \tilde{\cup} (G, E) \tilde{\in} I$.

A soft topological space $(H, \tilde{\tau}, E)$ with a soft ideal I called a soft ideal topological space and denoted by $(H, \tilde{\tau}, E, I)$.

Definition 1.3.25. [25] Let (F, E) be a soft set in an soft ideal topological space $(H, \tilde{\tau}, E, I)$ and $(\cdot)^*$ be a soft operator from $S(H)$ to $S(H)$. Then the soft local function of (F,E) defined by $(F,E)^*(I,\tau) = \{\tilde{h}: (U,E) \tilde{\cap} (F,E) \tilde{\notin} I$ for every $(U,E) \tilde{\in} \tilde{V}(\tilde{h})$ denoted by $(F,E)^*$ simply. Also, the soft set operator Cl^* is called a soft $^*-$closure and is defined as $Cl^*(F,E) = (F,E) \tilde{\cup} (F,E)^*$ for a soft subset (F,E).

Definition 1.3.26. [7]

(a) A soft subset (F,E) of an soft ideal topological space $(H,\tilde{\tau},E,I)$ is a soft I-open (briefly S.I-open) if $(F,E) \tilde{\subseteq} int(F,E)^*$.

(b) A soft subset (F,E) of an soft ideal topological space $(H,\tilde{\tau},E,I)$ is a soft I-closed (briefly S.I-closed) if it is complement is a soft I-open.

(c) A soft function $\phi : (H,\tilde{\tau},E,I) \to (K,\tilde{\sigma},L)$ is said to be an soft ideal continuous (briefly S.I-continuous) if the inverse image of every soft open set of K is a soft I-open set in H.

(d) An soft ideal topological space $(H,\tilde{\tau},E,I)$ is said to be a soft near I-compact (briefly S.j.I-compact) if for every soft near ideal open cover (briefly S.j.I-open cover) $\{(W_i,E_i): i \in \Delta\}$ of H, there exists a finite subset Δ_o of Δ such that $\widetilde{H} - \{(W_i,E_i): i \in \Delta_o\} \in I$, where $j \in \{\alpha,S,P,b,\beta\}$.[8]

Theorem 1.3.27. [7] For any soft I-open set (F,E) of a space $(H,\tilde{\tau},E,I)$, we have $(F,E)^* = (int(F,E)^*)^*$.

Lemma 1.3.28. [7] For any soft function $\phi : (H,\tilde{\tau},E,I) \to (K,\tilde{\sigma},L)$, $\phi(I)$ is an soft ideal on K.

Definition 1.3.29. A subset S of an ideal topological space (H,τ,I) is said to be

(a) α-I-open set if $S \subset int(Cl^*(int(S)))$[29].

(b) pre-I-open set (briefly P-I-open) if $S \subset int\big(Cl^*(S)\big)$ [2].

(c) sime-I-open set (briefly S-I-open) if $S \subset Cl^*(int(S))$ [29].

(d) b-I-open set if $S \subset Cl^*(int(S)) \cup int(Cl^*(S))$ [20].

Chapter 2

Fibrewise Soft Near Topological Spaces

Chapter 2

Fibrewise Soft Near Topological Spaces

The soft set is one of the newly emerging concepts and plays an important role in solving complex problems in engineering, environment and economics. The basic idea of this chapter is to generate fibrewise topology on the soft sets. Properties of this fibrewise soft topological spaces are studied and several example are given. In section one we introduced a new class of fibrewise topological spaces over B called fibrewise near topological spaces over B and counter examples are given to illustrate these concepts. In section two we studied the notions of fibrewise near closed and near open topological spaces over B. The purpose of section three is to introduce the new notions of fibrewise soft near compact and fibrewise soft locally near compact topological spaces over B.

2.1. Fibrewise Soft Near Topological Spaces.

In this section, we give a definition of fibrewise soft near topology and its related properties.

Definition 2.1.1. Assume that $(B, \tilde{\Omega}, G)$ is a soft topological space the fibrewise soft near topology (briefly, F.W.S. j-topological space) on a fibrewise set H over B mean any soft j-topology on H for which the projection function P is soft near continuous (briefly, S. j-continuous) where $j \in \{\alpha, S, P, b, \beta\}$.

Example 2.1.2. Let $H = \{h_1, h_2, h_3\}$, $B = \{a, b, c\}$, $E = \{e_1, e_2\}$, $G = \{g_1, g_2\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, $f(h_3) = \{c\}$, $f(h_1) = f(h_2) = \{a\}$, $u(e_1) =$

$\{g_1\}$, $u(e_2) = \{g_2\}$, $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, (F_1, E), (F_2, E)\}$, where (F_1, E),(F_2, E) are soft sets over $(H, \tilde{\tau}, E)$, defined as follows:
$(F_1, E) = \{(e_1, \{h_1, h_2\}), (e_2, \{h_3\})\}$,
$(F_2, E) = \{(e_1, H), (e_2, \{h_3\})\}$ and
$\tilde{\Omega} = \{\tilde{\Phi}, \tilde{B}, (M, G), (N, G)\}$, $(M, G) = \{(g_1, \{a\}), (g_2, \{c\})\}$ and $(N, G) = \{(g_1, \{a, c\}), (g_2, \{b, c\})\}$. It is clear that $H_{(M,G)} = \{(e_1, \{h_1, h_2\}), (e_2, \{h_3\})\}$ and $H_{(N,G)} = \{(\mathrm{e}_1, H), (e_2, \{h_3\})\}$ are soft j-open in $(H, \tilde{\tau}, E)$, then the projection $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a S. j-continuous, thus $(H, \tilde{\tau}, E)$ is fibrewise soft near topological space.

Remark 2.1.3. In F.W.S. topological space we work over at soft topological base space B, say. When B is a point-space the theory reduces to that of ordinary soft topology. A F.W.S. topological (resp., S. j-topological) spaces over B is just a soft topological (resp., S. j-topological) space H together with a soft continuous (resp., S. j-continuous) projection function P_{fu}: $(H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$. So the implication between F.W.S. topological spaces and the families of F.W.S. j-topological spaces are given in the following diagram where $j \in \{\alpha, S, P, b, \beta\}$.

F.W.S. topological space
$\Downarrow$
F.W.S. α-topological space ⇒ F.W.S. S-topological space
$\Downarrow$ $\Downarrow$
F.W.S. P-topological space ⇒ F.W.S. b-topological space
$\Downarrow$
F.W.S. β-topological space.

Figure 2.1.1: Implication between fibrewise soft topology and fibrewise soft *j*-topology, where $j \in \{\alpha, S, P, b, \beta\}$.

The following examples show that these implications are not reversible.

Example 2.1.4. Let $H = B = \{a, b, c, d\}$, $E = \{e_1, e_2, e_3\}$, $G = \{g_1, g_2, g_3\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to$

B and $u: E \to G$ as $f(a) = \{b\}, f(b) = \{d\}, f(c) = \{a\}$, $f(d) = \{c\}$, $u(e_1) = \{g_2\}, u(e_2) = \{g_1\}, u(e_3) = \{g_3\}$. Then $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, (F_1, E), (F_2, E), (F_3, E), \ldots, (F_{15}, E)\}$ where $(F_1, E), (F_2, E), (F_3, E), \ldots, (F_{15}, E)$ are soft sets over $(H, \tilde{\tau}, E)$, defined as follows:

$(F_1, E) = \{(e_1, \{a, b\}), (e_2, \{c\}), (e_3, \{a, c\})\}$,

$(F_2, E) = \{(e_1, \{b\}), (e_2, \{a, b\}), (e_3, \{a, b\})$

$(F_3, E) = \{(e_1, \{b\}), (e_3, \{a\})\}$,

$(F_4, E) = \{(e_1, \{a, b\}), (e_2, H), (e_3, H)\}$,

$(F_5, E) = \{(e_1, \{c\}), (e_2\{a, c\}), (e_3, \{b\})\}$,

$(F_6, E) = \{(e_2, \{c\})\}$,

$(F_7, E) = \{(e_1, H), (e_2, \{a, c\}), (e_3, H)\}$,

$(F_8, E) = \{(e_2, \{a\}), (e_3, \{b\})\}$,

$(F_9, E) = \{(e_1, \{b, c\}), (e_2, H), (e_3, \{a, b\})\}$,

$(F_{10}, E) = \{(e_1, \{b, c\}), (e_2, \{a, c\}), (e_3, \{a, b\})\}$,

$(F_{11}, E) = \{(e_1, \{a, c\}), (e_2, \{b\})\}$,

$(F_{12}, E) = \{(e_1, \{b\}), (e_2, H), (e_3, \{a, b\})\}$,

$(F_{13}, E) = \{(e_1, \{b\}), (e_2, \{a\}), (e_3, \{a, c\})\}$,

$(F_{14}, E) = \{(e_1, \{a, b\}), (e_2, \{a, c\}), (e_3, H)\}$,

$(F_{15}, E) = \{(e_1, \{b\}), (e_2, \{c\}), (e_3, \{a\})\}$,

$\tilde{\Omega} = \{\tilde{\Phi}, \tilde{B}, (F, G)\}$ and $(F, G) = \{(g_1, \{a, c, d\}), (g_2, \{a, b, d\}), (g_3, \{b, d\})\}$ and let the projection function $P_{fu}: (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a soft function. Then (F, G) is a soft open in $(B, \tilde{\Omega}, G)$ and $H_{(F,G)} = \{(e_1, \{a, b, c\}), (e_2, \{b, c, d\}), (e_3, \{a, b\})\}$ is a S. α-open but not soft open in $(H, \tilde{\tau}, E)$. Therefore, P_{fu} is a S.α-continuous but not S. continuous. Thus, $(H, \tilde{\tau}, E)$ is F.W.S. α-topological space but not F.W.S. topological space.

Example 2.1.5. Let $H = B = \{a, b, c, d\}$, $E = \{e_1, e_2, e_3\}$, $G = \{g_1, g_2, g_3\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f: H \to B$

and $u : E \to G$ as $f(a) = \{b\}, f(b) = \{d\}, f(c) = \{a\}, f(d) = \{c\}$, $f\ (b) = \{d\}$, $u(e_1) = \{g_2\}, u(e_2) = \{g_1\}, u(e_3) = \{g_3\}$. Let us consider the F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ given in Example (2.1.4); that is, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, (F_1, E), (F_2, E), (F_3, E), \dots, (F_{15}, E)\}, \tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, (M, G)\}$ and $(M, G) = \{(g_3, \{d\})\}$ and let the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a soft function. Then (M, G) is a soft open in $(B, \tilde{\Omega}, G)$ and $H_{(M,G)} = \{(e_3, \{b\})\}$ is a S. P-open but not S. α-open in $(H, \tilde{\tau}, E)$. Therefore, P_{fu} is a S. P-continuous but not S. α-continuous. Thus, $(H, \tilde{\tau}, E)$ is F.W.S.P-topological space but not F.W.S.α-topological space.

Example 2.1.6. Let $H = B = \{a, b, c, d\}$, $E = \{e_1, e_2, e_3\}$, $G = \{g_1, g_2, g_3\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u: E \to G$ as $f(a) = \{d\}, f(b) = \{d\}, f(c) = \{a\}, f(d) = \{c\}$, $u(e_1) = \{g_2\}, u(e_2) = \{g_1\}, u(e_3) = \{g_3\}$. Let us consider the F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ given in Example (2.1.4); that is, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, (F_1, E), (F_2, E), (F_3, E), \dots, (F_{15}, E)\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, (J, G)\}$ and $(J, G) = \{(g_1, \{d\}), (g_2, \{d\}), (g_3, \{b, d\})\}$ and let the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a soft function. Then (J, G) is a soft open in $(B, \tilde{\Omega}, G)$ and $H_{(J,G)} = \{(e_1, \{b\}), (e_2, \{c\}), (e_3, \{a, c\})\}$ is a S. α-open but not soft open in $(H, \tilde{\tau}, E)$. Therefore, P_{fu} is a S. S-continuous but not S. α-continuous. Thus $(H, \tilde{\tau}, E)$ is F.W.S. S-topological space but not F.W.S. α- topological space.

Example 2.1.7. Let $H = B = \{a, b, c, d\}$, $E = \{e_1, e_2, e_3\}$, $G = \{g_1, g_2, g_3\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$ as $f(a) = \{b\}, f(b) = \{d\}, f(c) = \{a\}$, $f(d) = \{c\}$, $u(e_1) = \{g_2\}, u(e_2) = \{g_1\}$, $u(e_3) = \{g_3\}$. Let us consider the F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ given in Example (2.1.4); that is, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H},$

$(F_1, E), (F_2, E), (F_3, E), \ldots, (F_{15}, E)\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, (O, G)\}$ and $(O, G) = \{(g_1, \{a, b, c\}), (g_2, \{d\}), (g_3, \{c, d\})\}$ and let the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a soft function. Then (O, G) is a soft open in (B, Ω, G) and $H_{(O,G)} = \{(e_2, \{c, a, d\}), (e_1, \{b\}), (e_3, \{d, b\})\}$ is a S. b-open but not S. S-open in $(H, \tilde{\tau}, E)$. Therefore, P_{fu} is a S. b-continuous but not S. S-continuous. Thus, $(H, \tilde{\tau}, E)$ is F.W.S. b-topological space but not F.W.S. S-topological space.

Example 2.1.8. Let $H = B = \{a, b, c, d\}$, $E = \{e_1, e_2, e_3\}$, $G = \{g_1, g_2, g_3\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S .topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$ as $f(a) = \{b\}, f(b) = \{d\}, f(c) = \{a\}, f(d) = \{c\}$, $u(e_1)$ $\{g_2\}$, $u(e_2) = \{g_1\}$, $u(e_3) = \{g_3\}$. Let us consider the F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ given in Example (2.1.4); that is, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, (F_1 E), (F_2, E), (F_3, E), \ldots, (F_{15}, E)\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, (N, G)\}$ and $(N, G) = \{(g_1, \{a, \{b\}), (g_2, \{c, d\}), (g_3, \{a, b, d\})\}$ and let the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a soft function. Then (N, G) is a soft open in $(B, \tilde{\Omega}, G)$ and $H_{(N,G)} = \{(e_2, \{a, c\}), (e_1, \{d, b\}), (e_3, \{c, a, b\})\}$. is a S. b-open but not S. P-open in $(H, \tilde{\tau}, E)$. Therefore, P_{fu} is a S. b-continuous but not S. P-continuous. Thus, $(H, \tilde{\tau}, E)$ is F.W.S. b-topological space but not F.W.S. P-topological space.

Example 2.1.9. Let $H = B = \{a, b, c, d\}$, $E = \{e_1, e_2, e_3\}$, $G = \{g_1, g_2, g_3\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$ as $f(a) = \{b\}$, $f(b) = \{d\}$, $f(c) = \{a\}$, $u(e_1) = \{g_2\}$, $u(e_2) = \{g_1\}$, $u(e_3) = \{g_3\}$. Let us consider the F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ given in Example (2.1.4); that is, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, (F_1, E), (F_2, E), (F_3, E), \ldots, (F_{15}, E)\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, (L, G)\}$ and $(L, G) = \{(g_1, \{a, b\}), (g_2, \{c, d\}),$

$(g_3, \{a, b, d\})\}$ and let the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a soft function. Then (L, G) is a soft open in $(B, \tilde{\Omega}, G)$ and $H_{(L,G)} = \{(e_1, \{a, b\}), (e_2, \{c, d\}), (e_3, \{a, c, d\})\}$ is a S. β-open but not S. b-open in $(H, \tilde{\tau}, E)$. Therefore, P_{fu} is a S. β-continuous but not S. b-continuous. Thus, $(H, \tilde{\tau}, E)$ is F.W.S. β-topological space but not F.W.S. b-topological space.

Proposition 2.1.10. A fibrewise soft set is F.W.S. α-topological space iff it is F.W.S. S-topological space and F.W.S. P-topological space.

Proof . (⇐) Let $(H, \tilde{\tau}, E)$ be a F.W.S. S-topological space and F.W.S. P-topological space over $\left(B, \tilde{\Omega}, G\right)$ then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to \left(B, \tilde{\Omega}, G\right)$ exists. To show that P_{fu} is S. α-continuous. Since $(H, \tilde{\tau}, E)$ is F.W.S. S-topological space and F.W.S. P-topological space over $\left(B, \tilde{\Omega}, G\right)$, then P_{fu} is S. S-continuous and S. P-continuous then P_{fu} is S. α-continuous by proposition (1.3.18). Thus, $(H, \tilde{\tau}, E)$ is F.W.S. α-topological space over $\left(B, \tilde{\Omega}, G\right)$.

(⇒) It obvious.

The fibrewise function and soft function is called fibrewise soft function.

Let ϕ : $(H) \to (K, \tilde{\sigma}, L)$ be a fibrewise soft function, (H) is a fibrewise set and $(K, \tilde{\sigma}, L)$ is a fibrewise topological space over $(B, \tilde{\Omega}, G)$. We can give $(H, \tilde{\tau}, E)$ the induced (resp., j-induced) soft topology, in the ordinary sense, and this is necessarily a F.W.S. topology (resp., j-topology). We may refer to it, as the induced (resp., j-induced) F.W.S. topology, where $j \in \{\alpha, S, P, b, \beta\}$, and note the following characterizations.

Proposition 2.1.11. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a fibrewise soft function, $(K, \tilde{\sigma}, L)$ a F.W.S. topological space over $(B, \tilde{\Omega}, G)$ and $(H, \tilde{\tau}, E)$ has the induced F.W.S. topology. Then for each F.W.S. topological space $(Z, \tilde{\gamma}, M)$, a fibrewise soft function $\psi: (Z, \tilde{\gamma}, M) \to (H, \tilde{\tau}, E)$ is S. j-continuous iff the composition $\phi \circ \psi : (Z, \tilde{\gamma}, M) \to (K, \tilde{\sigma}, L)$ is S. j-continuous, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. ($\Longrightarrow$) Suppose that ψ is S. j-continuous. Let $\tilde{z} \tilde{\in} Z_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and (N, L) soft open set of $(\phi \circ \psi)(\tilde{z}) = \tilde{k} \in K_{\tilde{b}}$ in $(K, \tilde{\sigma}, L)$. Since ϕ is S. continuous, then $\phi^{-1}(N, L)$ is a soft open set containing $\psi(\tilde{z}) = \tilde{h} \tilde{\in} H_{\tilde{b}}$ in $(H, \tilde{\tau}, E)$. Since ψ is S. j-continuous, then $\psi^{-1}(\phi^{-1}(N, L))$ is a S. j-open set containing $\tilde{z} \tilde{\in} Z_{\tilde{b}}$ in $(Z, \tilde{\gamma}, M)$ and $\psi^{-1}(\phi^{-1}(N, L) = (\phi \circ \psi)^{-1}(N, L)$ is a S. j-open set containing $\tilde{z} \tilde{\in} Z_{\tilde{b}}$ in $(Z, \tilde{\gamma}, M)$, where $j \in \{\alpha, S, P, b, \beta\}$.

($\Longleftarrow$) Suppose that $\phi \circ \psi$ is S. j-continuous. Let $\tilde{z} \tilde{\in} Z_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and (F, E) soft open set of $\psi(\tilde{z}) = \tilde{h} \tilde{\in} H_{\tilde{b}}$ in $(H, \tilde{\tau}, E)$. Since ϕ is open, $\phi(F, E)$ is a soft open set containing $\phi(\tilde{h}) = \phi(\psi(\tilde{z})) = \tilde{k} \tilde{\in} K_{\tilde{b}}$in $(K, \tilde{\sigma}, L)$. Since $\phi \circ \psi$ is S. j-continuous, then $(\phi \circ \psi)^{-1}\big(\phi(F, E)\big) = \psi^{-1}(F, E)$is a S. j-open set containing $\tilde{z} \tilde{\in} Z_{\tilde{b}}$ in $(Z, \tilde{\gamma}, M)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.1.12. Let $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ be a fibrewise soft function, $(K, \tilde{\sigma}, L)$ is a F.W.S. topological space over $(B, \tilde{\Omega}, G)$ and $(H, \tilde{\tau}, E)$ has the j-induced fibrewise soft topology. If for each F.W.S. topological space $(Z, \tilde{\gamma}, M)$, a fibrewise soft function $\psi: (Z, \tilde{\gamma}, M) \to (H, \tilde{\tau}, E)$ is soft j-irresolute iff the composition $\phi \circ \psi : (Z, \tilde{\gamma}, M) \to (K, \tilde{\sigma}, L)$ is S. j-continuous, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof . The proof is like to previous Proposition.

Proposition 2.1.13. Let $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ be a fibrewise soft function, $(K, \tilde{\sigma}, L)$ is a F.W.S. topological space over $(B, \tilde{\Omega}, G)$ and $(H, \tilde{\tau}, E)$ has the induced fibrewise soft topology. If for each fibrewise soft topological space $(Z,\tilde{\gamma},M)$, a fibrewise soft function $\psi : (Z,\tilde{\gamma},M) \to (H,\tilde{\tau},E)$ is a soft open, surjective iff the composition $\phi \circ \psi : (Z,\tilde{\gamma},M) \to (K,\tilde{\sigma},L)$ is a soft open.

Proof . The proof is like to the proof of Proposition (2.1.11).

2.2. Fibrewise Soft Near Closed and Soft Near Open Topological Spaces.

In this part, we explain the ideas of fibrewise soft near closed and soft near open topological spaces. Several topological properties on the obtained concepts are studied.

Definition 2.2.1. A F.W.S. topological space $(H,\tilde{\tau},E)$ over $(B,\tilde{\Omega},G)$ is called fibrewise soft j-closed (briefly, F.W.S. j-closed) if the projection function $P_{fu}$is S. j-closed, where $j \in \{\alpha, S, P, b, \beta\}$.

Example 2.2.2. Let $H = \{h_1,h_2\}$, $B = \{a,b\}$, $E = \{e_1,e_2\}$, $G = \{g_1,g_2\}$ and $(H,\tilde{\tau},E)$ be a F.W.S. topological space over $(B,\tilde{\Omega},G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_1) = \{b\}$, $f(h_2) = \{a\}$, $u(e_1) = \{g_1\}$, $u(e_2) = \{g_2\}$. $\tilde{\tau} = \{\tilde{\Phi},\tilde{H},(F_1,E)\}$, $\tilde{\mathfrak{F}} = \{\tilde{H},\tilde{\Phi},(F_1^C,E)\}$ where (F_1,E) is soft sets over $(H,\tilde{\tau},E)$, defined as follows:

$(F_1,E) =\{(e_1,\{h_1\}),(e_2,\{h_2\})\}$,

$(F_1^c,E) = \{(e_1,\{h_2\}),(e_2,\{h_1\})\}$,

$\tilde{\Omega} = \{\tilde{\Phi},\tilde{B},(M,G)\}$, $\tilde{\eta} = \left\{\tilde{\Phi},\tilde{B},(M^C,G)\right\}$ and $(M,G) = \{(g_1,\{a\}),(g_2,\{b\})\}$.

It is clear that $P_{fu}(F_1^c,E) = \{(g_1,\{a\}),(g_2,\{b\})\}$ is closed set in $(B,\tilde{\Omega},G)$, then the projection $P_{fu} : (H,\tilde{\tau},E) \to (B,\tilde{\Omega}\ ,G)$ be a S. closed, thus $(H,\tilde{\tau},E)$ is F.W.S. closed.

Proposition 2.2.3. Let $\phi : (H,\tilde{\tau},E) \to (K,\tilde{\sigma},L)$ be a soft closed fibrewise soft function, where $(H,\tilde{\tau},E)$ and $(K,\tilde{\sigma},L)$ are F.W.S. topological spaces over $(B,$

$\tilde{\Omega}, G)$. If $(K, \tilde{\sigma}, L)$ is F.W.S. j-closed, then $(H, \tilde{\tau}, E)$ is F.W.S. j-closed, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Suppose that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a soft closed fibrewise soft function and $(K, \tilde{\sigma}, L)$ is F.W.S. j-closed i.e., the projection function $P_{K(qd)} : (K, \tilde{\sigma}, L) \to (B, \tilde{\Omega}, G)$ is S. j-closed. To show that $(H, \tilde{\tau}, E)$ is F.W.S. j-closed i.e., the projection function $P_{H(fu)} : (H, \tilde{\tau}, E) \to \left(B, \tilde{\Omega}, G\right)$ is S. j-closed. Now let (F, C) be a soft closed subset of $H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, since ϕ is a soft closed, then $\phi(F, C)$ is a soft closed subset of $K_{\tilde{b}}$. Since $P_{K(qd)}$ is S. j-closed, then $P_{K(qd)}\left(\phi(F, C)\right)$ is S. j-closed at $(B, \tilde{\Omega}, G)$, but $P_{K(qd)}(\phi(F, C)) = (P_{K(qd)} \text{o } \phi)(F, C) = P_{H(fu)}(F, C)$ is S. j-closed in $(B, \tilde{\Omega}, G)$. Thus, $P_{H(fu)}$ is S. j-closed and $(H, \tilde{\tau}, E)$ is F.W.S. j-closed where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.2.4. Let $(H, \tilde{\tau}, E)$ be a F.W.S topological space over $(B, \tilde{\Omega}, G)$. Assume that (H_i, E_i) is F.W.S. j-closed for all member (H_i, E_i) of a finite soft covering of $(H, \tilde{\tau}, E)$. Then $(H, \tilde{\tau}, E)$ is F.W.S. j- closed.

Proof. Let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $\left(B, \tilde{\Omega}, G\right)$, then the projection function $P_{fu}: (H, \tilde{\tau}, E) \to \left(B, \tilde{\Omega}, G\right)$ exists. To show that P_{fu} is S. j-closed. Now, since (H_i, E_i) is F.W.S. j-closed, then the projection function $P_{i(fu)} : (H_i, \tilde{\tau}_{H_i}, E_i) \to (B, \tilde{\Omega}, G)$ is S. j-closed for all member (H_i, E_i) of a finite soft covering of $(H, \tilde{\tau}, E)$. Assume that (F, C) is a S. j-closed subset of $(H, \tilde{\tau}, E)$, then $P_{fu}(F, C) = \tilde{\cup}\left((H_i, E_i) \tilde{\cap} (F, C)\right)$ which is a finite union of soft closed sets and hence P_{fu} is S. j-closed. Thus, $(H, \tilde{\tau}, E)$ is F.W.S. j-closed, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.2.5. Let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $\left(B, \tilde{\Omega}, G\right)$. Then $(H, \tilde{\tau}, E)$ is F.W.S. j-closed iff for all fibre soft $(H_{\tilde{b}}, E_{\tilde{b}})$ of $(H, \tilde{\tau}, E)$

and all soft open set (F, E) of $(H_{\tilde{b}}, E_{\tilde{b}})$ in $(H, \tilde{\tau}, E)$, there exists a S. j-open set (F, G) of $\tilde{b}$ such that $\left(H_{(F,G)}, E_{(F,G)}\right) \tilde{\subset} (F, E)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof : $(\Longrightarrow)$ Suppose that $(H, \tilde{\tau}, E)$ is F.W.S. j-closed i.e., the projection function $P_{fu} : (H, \tilde{\tau}, E) \to \left(B, \tilde{\Omega}, G\right)$ is S. j-closed. Now, let $\tilde{b} \tilde{\in} (B, G)$ and (F, E) a soft open set of $(H_{\tilde{b}}, E_{\tilde{b}})$ in $(H, \tilde{\tau}, E)$, then $(H, E) - (F, E)$ is a soft closed in $(H, \tilde{\tau}, E)$, this implies $P_{fu}((H, E) - (F, E))$ is S. j-closed in $\left(B, \tilde{\Omega}, G\right)$, let $(F, G) = (B, G) - P_{fu}((H, E) - (F, E))$, then (F, G) a S. j-open set of $\tilde{b}$ in $(B, \tilde{\Omega}, G)$ and $\left(H_{(F,E)}, E_{(F,E)}\right) = H_{(F,E)} = (H, E) - H_{P_{fu}(H,E)-(F,G)}$ $\tilde{\subset} (F, G)$, where $j \in \{\alpha, S, P, b, \beta\}$.

$(\Longleftarrow)$ Suppose that the assumption hold and $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$. Now, let (F, C) be a soft closed subset of $(H, \tilde{\tau}, E)$ and $\tilde{b} \tilde{\in} (B, G) - P_{fu}(F, C)$ and each soft open set (F, E) of fibre soft $(H_{\tilde{b}}, E_{\tilde{b}})$ in $(H, \tilde{\tau}, E)$. By assumption there exists a S. j-open $(F, G\ )$ of $\tilde{b}$ such that $(H_{(F,G)}, E_{(F,G)}) \tilde{\subset} (F, E)$. It is easy to show that $(F, G) \tilde{\subset} (B, G) - P_{fu}(F, C)$, hence $(B, G) - P_{fu}(F, L)$ is S. j-open in $(\ B, \tilde{\Omega}, G)$ and this implies $P_{fu}(F, L)$ is S. j-closed in $(B, \tilde{\Omega}, G)$ and P_{fu} is S. j-closed. Thus, $(H, \tilde{\tau}, E)$ is F.W.S. j-closed, where $j \in \{\alpha, S, P, b, \beta\}$.

Definition 2.2.6. A F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is called a fibrewise soft near open (briefly, F.W.S. j-open) if the projection function P_{fu} is S. j-open, where $j \in \{\alpha, S, P, b, \beta\}$.

Example 2.2.7. Let $H = \{h_1, h_2\}$, $B = \{a, b\}$, $E = \{e_1, e_2\}$, $G = \{g_1, g_2\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_1) = \{b\}$, $f(h_2) = \{a\}$, $u(e_1) = \{g_1\}$, $u(e_2) = \{g_2\}$, $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, (F_1, E)\}$, where (F_1, E) is soft sets over $(H, \tilde{\tau}, E)$, defined as follows:

$(F_1, E) = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$,

$\tilde{\Omega} = \{\tilde{\Phi}, \tilde{B}, (M, G)\}$ and $(M, G) = \{(g_1, \{a\}), (g_2, \{b\})\}$. It clear that $P_{fu}(F_1, E) = \{(g_1, \{a\}), (g_2, \{b\})\}$ is soft open $(B, \tilde{\Omega}, G)$, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a S. open, thus $(H, \tilde{\tau}, E)$ is F.W.S. open.

Proposition 2.2.8. Let $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ be a soft open fibrewise function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(K, \tilde{\sigma}, L)$ is F.W.S. j-open, then $(H, \tilde{\tau}, E)$ is F.W.S. j-open, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Suppose that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is an open fibrewise soft function and $(K, \tilde{\sigma}, L)$ is F.W.S. j-open i.e., the projection function $P_{K(qd)} : (K, \tilde{\sigma}, L) \to (B, \tilde{\Omega}, G)$ is S. j-open. To show that $(H, \tilde{\tau}, E)$ is F.W.S. j-open i.e., the projection function $P_{H(fu)} : (H, \tilde{\tau}, E) \to \left(B, \tilde{\Omega}, G\right)$ is S. j-open. Now let (F, E) be a soft open subset of $H_{\tilde{b}}$, where $\tilde{b} \,\tilde{\in}\, (B, G)$, since ϕ is a soft open, then $\phi(F, E)$ is a soft open subset of $K_{\tilde{b}}$, since $P_{K(qd)}$ is S. j-open, then $P_{K(qd)}\big(\phi(F, E)\big)$ is S. j-open in $\left(B, \tilde{\Omega}, G\right)$, but $P_{K(qd)}\big(\phi(F, E)\big) = (P_{K(qd)} \circ \phi)(F, E)$ is S. j-open in $(B, \tilde{\Omega}, G)$. Thus, $P_{H(fu)}$ is S. j-open and $(H, \tilde{\tau}, E)$ is F.W.S. j-open, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.2.9. Assume that $(H_r, \tilde{\tau}_r, E_r)$ is a finite family of F.W.S. j-open spaces over $(B, \tilde{\Omega}, G)$. Then the F.W.S. topological product $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ is and F.W.S. j-open, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Let $(H_r, \tilde{\tau}_r, E_r)$ be a finite family of F.W.S. j-open. Suppose that $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ is a F.W.S. topological space over $(B, \tilde{\Omega}, G)$, then $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is exists. To show that P_{fu} is S. j-open. Now, since $(H_r, \tilde{\tau}_r, E_r)$ be a finite family of F.W.S. j-open spaces over $(B, \tilde{\Omega}, G)$, then the projection function $P_{r(fu)} : (H_r, \tilde{\tau}_r, E_r) \to \left(B, \tilde{\Omega}, G\right)$ is S. j-open for each r. Let (F, C) be a soft open subset of $(H, \tilde{\tau}, E)$, then $P_{fu}(F, C) = P_{fu}(\prod_B (H_r,$

$E_r)\ \tilde{\cap}\ (F,E))) = \prod_B P_{fu}\,((H_r,E_r)\ \tilde{\cap}\ (F,C))$ which is a finite product of S. j-open sets and hence P_{fu} is S. j-open. Thus, the F.W.S. topological product $(H,\tilde{\tau},E) = \prod_B (H_r,\tilde{\tau}_r\,,E_r)$ is a F.W.S. j-open, where $j \in \{\alpha,S,P,b,\beta\}$.

Remark 2.2.10. If $(H,\tilde{\tau},E)$ is F.W.S. open (resp., F.W.S. j-open) then the second projection $\pi_2\colon (H,\tilde{\tau},E) \times_B (K,\tilde{\sigma},L) \to (K,\tilde{\sigma},L)$ is a soft open (resp., S. j-open) for all F.W.S. topological space $(K,\tilde{\sigma},L)$. Because for every non-empty soft open (resp., soft open, S. j-open and S. j-open) set $(F,E) \times_B (F,L)\ \tilde{\subseteq}\ (H,\tilde{\tau},E) \times_B (K,\tilde{\sigma},L)$, we have $\pi_2\big((F,E) \times_B (F,L)\big) = (F,L)$ is a soft open (resp., S. j-open, soft open and S. j-open), where $j \in \{\alpha,S,P,b,\beta\}$. We will use this in the proof of the following results.

Proposition 2.2.11. Let $\phi : (H,\tilde{\tau},E) \to (K,\tilde{\sigma},L)$ be a fibrewise soft function, where $(H,\tilde{\tau},E)$ and $(K,\tilde{\sigma},L)$ are F.W.S. topological spaces over $(B,\tilde{\Omega},G)$. Let $id_H \times \phi : (H,\tilde{\tau},E) \times_B (H,\tilde{\tau},E) \to (H,\tilde{\tau},E) \times_B (K,\tilde{\sigma},L)$. If $id_H \times \phi$ is a soft open, $(H,\tilde{\tau},E)$ is F.W.S. open and $(K,\tilde{\sigma},L)$ is F.W.S. j-open. Then ϕ it self is S. j- open, where $j \in \{\alpha,S,P,b,\beta\}$.

Proof. Consider the following commutative figure.

$$\begin{tikzcd}
H \times_B H \arrow[r, "id_H \times \phi"] \arrow[d, "\pi_2"'] & H \times_B K \arrow[d, "\pi_2"] \\
H \arrow[r, "\phi"] & K
\end{tikzcd}$$

Figure2.2.1: Diagram of Proposition 2.2.11.

The projection function on the left is surjective and S. j-open, since $(K,\tilde{\sigma},L)$ is F.W.S. j-open, while the projection function on the right is S. j-open, since $(H,\tilde{\tau},E)$ is F.W.S. j-open. Therefore, $\pi_2 \circ (id_H \times \phi) = \phi \circ \pi_2$ is S. j-open, and so ϕ is S. j-open, by Proposition (2.2.8) as asserted, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.2.12. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-continuous fibrewise surjection, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E)$ is F.W. j-closed (resp., F.W.S. j-open), then $(K, \tilde{\sigma}, L)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Suppose that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is S. j-continuous fibrewise surjection and $(H, \tilde{\tau}, E)$ is F.W.S. j-closed (resp., F.W.S. j-open) i.e., the projection function $P_{H(fu)} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is S. j-closed (resp., S. j-open). To show that $(K, \tilde{\sigma}, L)$ is F.W.S. j-closed (resp., F.W.S. j-open) i.e., the projection function $P_{K(qd)} : (K, \tilde{\sigma}, L) \to (B, \tilde{\Omega}, G)$ is S. j-closed (resp., S. j-open). Let (G, E) be a soft closed (resp., soft open) subset of $K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$. Since ϕ is a soft continuous fibrewise, then $\phi^{-1}(G, E)$ is a soft closed (resp., soft open) subset of $H_{\tilde{b}}$. Since $P_{H(fu)}$ is S. j-closed (resp., S. j-open), then $(P_{H(fu)}(\phi(G, E)))$ is S. j-closed (resp., S. j-open) in $\left(B, \tilde{\Omega}, G\right)$. But $P_{H(fu)}(\phi(G, E)) = (P_{H(fu)} o \phi)(G, E) = P_{K(qd)}(G, E)$ is S. j-closed (resp., S. j-open) in $\left(B, \tilde{\Omega}, G\right)$. Thus $P_{K(qd)}$ is S. j-closed (resp., S. j-open) and $(K, \tilde{\sigma}, L)$ is F.W.S. j-closed (resp., F.W.S. j-open), where $j \in \{\alpha, S, P, \beta, b\}$.

Proposition 2.2.13. Let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Suppose that $(H, \tilde{\tau}, E)$ is F.W.S. j-closed (resp., F.W.S j-open) over $(B, \tilde{\Omega}, G)$. Then $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-closed (resp., F.W.S. j-open) over $(B^*, \tilde{\Omega}^*, G^*)$ for each subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $\left(B, \tilde{\Omega}, G\right)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Suppose that $(H, \tilde{\tau}, E)$ is a F.W.S. j-closed (resp., F.W.S. j-open) i.e., the projection function $P_{fu} : (H, \tilde{\tau}, E) \to \left(B, \tilde{\Omega}, G\right)$ is S. j-closed (resp., S. j-open). To show that $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-closed (resp., F.W.S. j-open) over $\left(B^*, \tilde{\Omega}^*, G^*\right)$ i.e., the projection function $P_{B^*(fu)} : (H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*}) \rightarrow (B^*, \tilde{\Omega}^*, G^*)$ is S. j-closed (resp., S. j-open). Now, let (N, E) be a soft closed (resp., soft open) subset of $(H, \tilde{\tau}, E)$, then $(G, E) \tilde{\cap} (H_{B^*}, E_{B^*})$ is a soft closed (resp., soft open) in subspace $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ and $P_{B^*(fu)}((G, E) \tilde{\cap} (H_{B^*}, E_{B^*})$ $= P_{fu}((G, E) \tilde{\cap} (H_{B^*}, E_{B^*}) = P_{fu}\left((G, E) \tilde{\cap} (H_{B^*}, E_{B^*}\right) = P_{fu}(G, E) \tilde{\cap} (B^*, G^*)$ which is S. j-closed (resp., S. j-open) set in $(B^*, \tilde{\Omega}^*, G^*)$. Thus $P_{B^*(fu)}$ is S. j-closed (resp., S. j-open) and $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-closed (resp., F.W.S. j-open) over $\left(B^*, \tilde{\Omega}^*, G^*\right)$, where $j \in \{\alpha, S, P, b, \beta\}$.

2.3. Fibrewise Soft Near Compact and Locally Soft Near Compact Spaces.

In this section, we study fibrewise soft near compact and fibrewise locally soft near compact spaces as a generalizations of well-known concepts soft near compact and locally soft near compact topological spaces.

Definition 2.3.1. The function $\phi : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ is called a soft near proper (briefly S. j-proper) function. If it is S. j-continuous, soft closed and for each $\left(\tilde{k}\right) \tilde{\in} K, \phi^{-1}(\tilde{k})$ is soft compact set, where $j \in \{\alpha, S, P, b, \beta\}$.

For example, let $(\mathbb{R}, \tilde{\tau}, E)$ where $\tilde{\tau}$ is the soft topology with soft basis whose members are of the formula (a, b) and $(a, b) - \mathbb{N}$, $\mathbb{N} = \{1\backslash n; n \in \mathbb{Z}^+\}$ and $E = \mathbb{N}$. Define $\phi : (\mathbb{R}, \tilde{\tau}, E) \rightarrow (\mathbb{R}, \tilde{\tau}, E)$ by $\phi\ (F, E) = (F, E)$, then ϕ is S. j-proper function, where $j \in \{\alpha, S, P, b, \beta\}$.

If a function $\phi : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ is a fibrewise and S. j-proper function, then ϕ is said to be fibrewise S. j-proper function, where $j \in \{\alpha, S, P, b, \beta\}$.

Definition 2.3.2. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is called a fibrewise soft j-compact (briefly, F.W.S. j-compact) if the projection function P_{fu} is S. j-proper, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.3.3. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is F.W.S. j-compact iff $(H, \tilde{\tau}, E)$ is a fibrewise soft closed and every fibre soft of $(H, \tilde{\tau}, E)$ is a soft compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. $(\Longrightarrow)$ Let $(H, \tilde{\tau}, E)$ be a F.W.S. j-compact space, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is S. j-proper function i.e., P_{fu} is a soft closed and for all $\tilde{b} \tilde{\in} (B, G)$, $H_{\tilde{b}}$ is a soft compact. Hence $(H, \tilde{\tau}, E)$ is a fibrewise soft closed and every fibre soft of $(H, \tilde{\tau}, E)$ is a soft compact, where $j \in \{\alpha, S, P, b, \beta\}$.

$(\Longleftarrow)$ Let $(H, \tilde{\tau}, E)$ be F.W.S. closed and every fibre soft of $(H, \tilde{\tau}, E)$ is a soft compact, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is a soft closed and it is clear that P_{fu} is S. j-continuous, and for all $\tilde{b} \tilde{\in} (B, G)$, $H_{\tilde{b}}$ is a soft compact. Hence $(H, \tilde{\tau}, E)$ is F.W.S. j-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.3.4. Let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Then $(H, \tilde{\tau}, E)$ is F.W.S. j-compact iff for all fibre soft $H_{\tilde{b}}$ of $(H, \tilde{\tau}, E)$ and each soft covering Γ_E of $H_{\tilde{b}}$ by soft open sets of H there exists a soft nbd (N, G) of $\tilde{b}$ such that a finite soft subfamily of Γ_E soft covers $H_{(N,G)}$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof . $(\Longrightarrow)$ Let $(H, \tilde{\tau}, E)$ be a F.W.S. j-compact space, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is S. j-proper function, so that $H_{\tilde{b}}$ is soft compact for all $\tilde{b} \tilde{\in} (B, G)$. Assume that Γ_E is a soft covering of $H_{\tilde{b}}$ in soft open sets of H for all $\tilde{b} \tilde{\in} (B, G)$ and let $H_{(N,G)} = \tilde{\cup}\, H_{\tilde{b}}$ for all $\tilde{b} \tilde{\in} (N, G)$.

Since $H_{\tilde{b}}$ is soft compact for each $\tilde{b} \tilde{\in} (N,G) \tilde{\in} (B,G)$ and the union of soft compact sets is a soft compact, we have $H_{(N,G)}$ is a soft compact. Thus, there exists a soft nbd (N,G) of $\tilde{b}$ such that a finite soft subfamily of Γ_E soft covers $H_{(N,G)}$, where $j \in \{\alpha, S, P, b, \beta\}$.

($\Longleftarrow$) Let $(H, \tilde{\tau}, E)$ be F.W.S. topological space over $(B, \tilde{\Omega}, G)$, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ exist. To show that P_{fu} is S. j-proper. Now, it is clear that P_{fu} is S. j-continuous and for all $\tilde{b} \tilde{\in} (B,G)$, $H_{\tilde{b}}$ is a soft compact by take $H_{\tilde{b}} = H_{(N,G)}$. By Proposition (2.2.4), we have P_{fu} is soft closed. Thus, P_{fu} is S. j-proper and $(H, \tilde{\tau}, E)$ is F.W.S. j-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.3.5. Let $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ be a j-proper, j-closed fibrewise soft function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(K, \tilde{\sigma}, L)$ is F.W.S. j-compact, then $(H, \tilde{\tau}, E)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Suppose that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a j-proper, j-closed fibrewise soft function and $(K, \tilde{\sigma}, L)$ is F.W.S. j-compact space i.e., the projection function $P_{K(qd)} : (K, \tilde{\sigma}, L) \to (B, \tilde{\Omega}, G)$ is a j-proper. To show that $(H, \tilde{\tau}, E)$ is F.W.S. j-compact space i.e., the projection function $P_{H(fu)} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is S. j-proper. Now, clear that $P_{H(fu)}$ is S. j-continuous, assume that (F, L) is a soft closed subset of $H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B,G)$. Since ϕ is a soft closed, then $\phi\,(F, L)$ is a soft closed subset of $K_{\tilde{b}}$. Since $P_{K(qd)}$ is a soft closed, then $P_{K(qd)}\big(\phi(F, L)\big)$ is a soft closed in $(B, \tilde{\Omega}, G)$. But $P_{K(qd)}\big(\phi(F, L)\big) = \big(P_{K(qd)} \text{o}\, \phi\big)(F, L) = P_{H(fu)}(F, L)$ is soft a closed in $(B, \tilde{\Omega}, G)$ so that $P_{H(fu)}$ is a soft closed. Let $\tilde{b} \tilde{\in} (B,G)$, since $P_{K(qd)}$ is S. j-proper, then $K_{\tilde{b}}$ is soft compact. Now let $\{(F,E)_i : i \in I\}$ be a family of S. j-open sets of $(H,\tilde{\tau},E)$ such that $K_{\tilde{b}} \tilde{\subset} \tilde{\cup}_{i\in I} (F,E)_i$. If $\tilde{k} \tilde{\in} K_{\tilde{b}}$, then there exist a finite subset $N(\tilde{k})$ of I such that $\phi^{-1}(\tilde{k}) \tilde{\subset} \tilde{\cup}_{i\in N(\tilde{k})} (F,E)_i$. Since ϕ is S. j-closed function, so by Proposition (2.2.5) there exists a S. j-open set $(M,L)_{\tilde{k}}$ of $(K,\tilde{\sigma},L)$ such that $\tilde{k} \tilde{\in} (M,L)_{\tilde{k}}$ and $\phi^{-1}((M,L)_{\tilde{y}}) \tilde{\subset} \tilde{\cup}_{i\in N(\tilde{y})} (F,E)_i$. Since $K_{\tilde{b}}$ is a soft compact, there exist a finite subset $\tilde{C}$ of $K_{\tilde{b}}$ such that $K_{\tilde{b}} \tilde{\subset} \tilde{\cup}_{\tilde{k}\tilde{\in}\tilde{c}} (M,L)_{\tilde{k}}$. Hence $\phi^{-1}(K_{\tilde{b}}) \tilde{\subset}\tilde{\cup}_{\tilde{k}\tilde{\in}\tilde{c}} \phi^{-1}(M,K)_{\tilde{k}} \tilde{\subset}\tilde{\cup}_{\tilde{k}\tilde{\in}\tilde{c}}\tilde{\cup}_{i\in N(\tilde{k})}$ $(F,E)_i$. Thus if $N = \tilde{\cup}_{\tilde{k}\tilde{\in}\tilde{c}} N(\tilde{k})$, then N is a finite subset of I and $\phi^{-1}(K_{\tilde{b}}) \tilde{\subset}$ $\tilde{\cup}_{i\in N} ((F,E)_i)$. Thus $\phi^{-1}(K_{\tilde{b}}) = \phi^{-1}\left(P^{-1}_{K(qd)}(\tilde{b})\right) = \left(P_{K(qd)} \mathrm{o} \phi\right)^{-1}(\tilde{b}) =$ $P^{-1}_{H(fu)}(\tilde{b}) = H_{\tilde{b}}$ and $H_{\tilde{b}} \tilde{\subset} \tilde{\cup}_{i\in N} (F,E)_i$ so that $H_{\tilde{b}}$ is a soft compact. Thus, $P_{H(fu)}$ is S. j-proper and $(H,\tilde{\tau},E)$ is a F.W.S. j-compact, where $j \in \{\alpha,S,P,b,\beta\}$.

The class of F.W.S. j-compact spaces is multiplicative, where $j \in \{\alpha,S,P,b,\beta\}$, in the following sense :

Proposition 2.3.6. Let $(H_r,\tilde{\tau}_r,E_r)$ be a family of F.W.S. j-compact spaces over $(B,\tilde{\Omega},G)$. Then the F.W.S. topological product $(H,\tilde{\tau},E) = \prod_B (H_r,\tilde{\tau}_r, E_r)$ is F.W.S. j-compact, where $j \in \{\alpha,S,P,b,\beta\}$.

Proof., Without loss of generality, for finite products a simple argument can be used. Thus, assume that $(H,\tilde{\tau},E)$ and $(K,\tilde{\sigma},L)$ are F.W.S. topological spaces over $(B,\tilde{\Omega},G)$. If $(H,\tilde{\tau},E)$ is F.W.S. j-compact, then the projection function $P_{(fu)} \times id_K: (H,\tilde{\tau},E) \times_B (K,\tilde{\sigma},L) \equiv (K,\tilde{\sigma},L)$ is S. j-proper. If $(K,\tilde{\sigma}, L)$ is and F.W.S. j-compact, then so is $(H,\tilde{\tau},E) \times_B (K,\tilde{\sigma},L)$, by Proposition (2.3.5), where $j \in \{\alpha,S,P,b,\beta\}$.

A similar result holds for finite coproducts.

Proposition 2.3.7. Assume that $(H,\tilde{\tau},E)$ is F.W.S. topological space over $(B,\tilde{\Omega},G)$. Suppose that (H_i,E_i) is F.W.S. j-compact for all member (H_i,E_i) of a finite soft covering of $(H, \tilde{\tau}, E)$. Then $(H, \tilde{\tau}, E)$ is F.W.S. j-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Now, the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ exist. To show that P_{fu} is S. j-proper. Now, it is clear that P_{fu} is S. j-continuous. Since (H_i, E_i) is F.W.S. j-compact, then the projection function $P_{i(fu)}: (H_i, E_i) \to (B, G)$ is a soft closed and for all $\tilde{b} \tilde{\in} (B, G)$, $\left(H_{i_{\tilde{b}}}, E_{i_{\tilde{b}}}\right)$ is soft compact for all member (H_i, E_i) of a finite soft covering of $(H, \tilde{\tau}, E)$. Assume that (F, L) is a soft closed subset of $(H, \tilde{\tau}, E)$, then $P_{fu}(F, L) = \tilde{\cup}\, P_{i(fu)}((H_i, E_i) \tilde{\cap} (F, L)$ which is a finite union of soft closed sets and hence P_{fu} is a soft closed. Assume that $\tilde{b} \tilde{\in} (B, G)$, then $H_{\tilde{b}} = \tilde{\cup}\, (H_i)_{\tilde{b}}$ which is a finite union of soft compact sets and hence $H_{\tilde{b}}$ is a soft compact. Thus, P_{fu} is S. j-proper and $(H, \tilde{\tau}, E)$ is F.W.S. j-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Definition 2.3.8. A F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is called fibrewise soft j-irresolute (briefly, F.W.S. j-irresolute) if the projection function P_{fu} is S. j-irresolute, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.3.9. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-continuous, S. j-irresolute fibrewise surjection, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $\left(B, \tilde{\Omega}, G\right)$. If $(H, \tilde{\tau}, E)$ is F.W.S. j-compact, then $(K, \tilde{\sigma}, L)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Suppose that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-continuous, S. j-irresolute fibrewise surjection and $(H, \tilde{\tau}, E)$ is a F.W.S. j-compact i.e., the projection function $P_{H(fu)}: (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is a S. j-proper. To show that $(K, \tilde{\sigma}, L)$ is F.W.S. j-compact i.e., the projection function $P_{K(qd)}: (K, \tilde{\sigma}, L) \to (B, \tilde{\Omega}, G)$ is S. j-proper. Now, it is clear that $P_{K(qd)}$ is S. j-continuous. Assume that (F, L) is a closed subset of $K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$. Since ϕ is S. j-continuous F.W.S. topological space over $\left(B, \tilde{\Omega}, G\right)$, then $\phi^{-1}(F, L)$ is a soft closed subset of $H_{\tilde{b}}$. Since $P_{H(fu)}$ is soft closed, then $P_{H(fu)}\left(\phi^{-1}(F, L)\right)$ is a soft closed in $\left(B, \tilde{\Omega}, G\right)$. But $P_{H(fu)}\left(\phi^{-1}(F, L)\right) = \left(P_{H(fu)} \circ \phi^{-1}\right)(F, L)$ $= P_{K(qd)}(F, L)$ is a soft closed in $\left(B, \tilde{\Omega}, G\right)$, hence $P_{K(qd)}$ is a soft closed. For any $\tilde{b} \tilde{\in} (B, G)$, we have $K_{\tilde{b}} = \phi(H_{\tilde{b}})$ is a soft compact because $H_{\tilde{b}}$ is a soft compact and the image of a soft compact subset under S. j-irresolute function is a soft compact. Thus, $P_{K(qd)}$ is S. j-proper and $(K, \tilde{\sigma}, L)$ is F.W.S. j-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.3.10. Let $(H, \tilde{\tau}, E)$ be F.W.S. j-compact space over $(B, \tilde{\Omega}, G)$. Then $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-compact space over $(B^*, \tilde{\Omega}^*, G^*)$ for all soft subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $(B, \tilde{\Omega}, G)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Suppose that $(H, \tilde{\tau}, E)$ is F.W.S. j-compact i.e., the projection function $P_{fu} : (H, \tilde{\tau}, E) \to \left(B, \tilde{\Omega}, G\right)$ is S. j-proper. To show that $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-compact space over $\left(B^*, \tilde{\Omega}^*, G^*\right)$ i.e., the projection function $P_{B^*(fu)} : (H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*}) \to \left(B^*, \tilde{\Omega}^*, G^*\right)$ is S. j-proper. Now, it is clear that $P_{B^*(fu)}$ is S. j-continuous. Assume that (F, L) is a soft closed subset of $(H, \tilde{\tau}, E)$, then $(F, L) \tilde{\cap} (H_{B^*}, E_{B^*})$ is a soft closed in a soft subspace $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ and $P_{B^*(fu)}\left((F, L) \tilde{\cap} (H_{B^*}, E_{B^*})\right) = P_{fu}(F, L) \tilde{\cap} (B^*, G^*)$ which is soft closed set in $(B^*, \tilde{\Omega}^*, G^*)$, hence $P_{B^*(fu)}$ is a soft closed. Let $\tilde{b} \tilde{\in} (B^*, G^*)$, then $(H_{B^*})_{\tilde{b}} = H_{\tilde{b}} \tilde{\cap} H_{B^*}$ which is a soft compact set in (H_{B^*}, E_{B^*}). Thus, $P_{B^*(fu)}$ is S. j-proper and $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-compact over $(B^*, \tilde{\Omega}^*, G^*)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.3.11. Let $(H,\tilde{\tau},E)$ be a F.W.S. topological space over $(B,\tilde{\Omega},G)$. Suppose that (H_{B_i},E_{B_i}) is F.W.S. j-compact over (B_i,G_i) for all member (B_i,G_i) of a soft open covering of $(B,\tilde{\Omega},G)$. Then $(H,\tilde{\tau},E)$ is F.W.S. j-compact over $(B,\tilde{\Omega},G)$, where $j \in \{\alpha,S,P,b,\beta\}$.

Proof. Suppose that (H,τ,E) is F.W.S. topological space over $(B,\tilde{\Omega},G)$, then the projection function P_{fu}: $(H,\tilde{\tau},E) \to (B,\tilde{\Omega},G)$ exist. To show that P_{fu} is S. j-proper. Now, it is clear that P_{fu} is S. j-continuous. Since (H_{B_i},E_{B_i}) is F.W.S. j-compact over (B_i,G_i), then the projection function $P_{i(fu)} : (H_{B_i},E_{B_i}) \to (B_i,G_i)$ is S. j-proper for all member (B_i,G_i) of a S. j-open covering of $(B,\tilde{\Omega},G)$. Assume that (F,L) is a soft closed subset of $(H,\tilde{\tau},E)$, then we have $P_{fu}(F,L) = \tilde{\cup}\, P_{B_i(fu)}\left(\left(H_{B_i},E_{B_i}\right) \tilde{\cap} (F,L)\right)$ which is a union of soft closed sets and hence P_{fu} is a soft closed. Let $\tilde{b} \tilde{\in} (B,G)$ then $H_{\tilde{b}} = \tilde{\cup}\, (H_{B_i})_{\tilde{b}}$ for every $\tilde{b} = \{\tilde{b}_i\} \in (B_i,G_i)$. Since $\left(H_{B_{i_{\tilde{b}}}},E_{B_{i_{\tilde{b}}}}\right)$ is soft compact in (H_{B_i},E_{B_i}) and the union of soft compact sets is a soft compact, we have $(H_{\tilde{b}},E_{\tilde{b}})$ is a soft compact. Thus, P_{fu} is a soft j-proper and $(H,\tilde{\tau},E)$ is a F.W.S. j-compact over $(B,\tilde{\Omega},G)$, where $j \in \{\alpha,S,P,b,\beta\}$.

In fact the last result is also holds for locally finite soft closed coverings, instead of soft open coverings.

Proposition 2.3.12. Assume that $\phi : (H,\tilde{\tau},E) \to (K,\tilde{\sigma},L)$ is a fibrewise soft function, where $(H,\tilde{\tau},E)$ and $(K,\tilde{\sigma},L)$ are F.W.S. topological spaces over $(B,\tilde{\Omega},G)$. If $(H,\tilde{\tau},E)$ is F.W.S. j-compact and $id_H \times \phi : (H,\tilde{\tau},E) \times_B (H,\tilde{\tau},E) \to (H,\tilde{\tau},E) \times_B (K,\tilde{\sigma},L)$ is S. j-proper and S. j-closed, then ϕ is S. j-proper, where $j \in \{\alpha,S,P,b,\beta\}$.

Proof. Consider the commutative figure shown below

$$\begin{tikzcd}
H \times_B H \arrow[r, "id_H \times \phi"] \arrow[d, "\pi_2"'] & H \times_B K \arrow[d, "\pi_2"] \\
H \arrow[r, "\phi"] & K
\end{tikzcd}$$

Figure 2.3.1: Diagram of Proposition 2.3.12.

If $(H, \tilde{\tau}, E)$ is F.W.S. j-compact, then π_2 is S. j-proper. Condition $id_H \times \phi$ is also S. j-proper and j-closed then $\pi_2 \circ (id_H \times \phi) = \phi \circ \pi_2$ is S. j-proper, and so ϕ itself is S. j-proper, where $j \in \{\alpha, S, P, b, \beta\}$.

The second new concept in this section is given by the following:

Definition 2.3.13. A F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is called fibrewise locally soft near compact (briefly, F.W. L. S. j-compact) if for all soft point $\tilde{h}$ of $H_{\tilde{b}}$, where $\tilde{b} \,\tilde{\in}\, (B, G)$, there exists a soft nbd (N, G) of $\tilde{b}$ and an soft open set $(F, E) \,\tilde{\subset}\, H_{(N,G)}$ of $\tilde{h}$ such that the closure of (F, E) in $H_{(N,G)}$ (i.e., $H_{(N,G)} \,\tilde{\cap}\, Cl(F, E)$) is F.W.S. j-compact over (N, G), where $j \in \{\alpha, S, P, b, \beta\}$.

Remark 2.3.14. F.W.S j-compact spaces are necessarily F.W.L.S. j-compact by taken $W = B$ and $H_W = H$. On the other hand the in opposition is not true for example, assume that $(H, \tilde{\tau}_{dis}, E)$ where H and E is infinite set and $\tilde{\tau}$ is discrete soft topology, then $(H, \tilde{\tau}_{dis}, E)$ F.W.L.S. j-compact over $(\mathbb{R}, \tilde{\Omega}, G)$, because for all $\tilde{h} \,\tilde{\in}\, H_{\tilde{b}}$, where $\tilde{b} \,\tilde{\in}\, (\mathbb{R}, G)$, there exists a soft nbd (N, G) of $\tilde{b}$ and an soft open $(F, E) \,\tilde{\subset}\, H_{(N,G)}$ of $\tilde{h}$ such that $Cl(F, E) = (F, E)$ in $H_{(N,G)}$ is F.W.S. j-compact over (N, G). But $(H, \tilde{\tau}, E)$ is not F.W.S. j-compact space over $(\mathbb{R}, \tilde{\Omega}, G)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 2.3.15. Assume that $\phi : (H, \tilde{\tau}, E) \to (H^*, \tilde{\tau}^*, E^*)$ is a closed fibrewise soft embedding, where $(H, \tilde{\tau}, E)$ and $(H^*, \tilde{\tau}^*, E^*)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H^*, \tilde{\tau}^*, E^*)$ is F.W.L.S. j-compact, then so is $(H, \tilde{\tau}, E)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Let $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$,. Since $(H^*, \tilde{\tau}^*, E^*)$ is F.W.L.S. j-compact there exists a soft nbd (N, G) of $\tilde{b}$ and a soft open $(F, E) \subset H^*_{(N,G)}$ of $\phi(\tilde{h})$ such that the closure $H^*_{(N,G)}$ of (F, E) in $(H^*_{(N,G)}, \tilde{\tau}^*_{(N,G)}, E^*_{(N,G)})$ is F.W.S. j-compact over (N, G). Then $\phi^{-1}(F, E) \tilde{\subset} H_{(N,G)}$ is a soft open set of $\tilde{h}$ such that the closure $H_{(N,G)} \tilde{\cap} Cl\big(\phi^{-1}(F, E)\big) = \phi^{-1}(H^*_{(N,G)} \tilde{\cap} Cl(F, E))$ of $\phi^{-1}(F, E)$ in $H_{(N,G)}$ is F.W.S. j-compact over (F, E). Thus, $(H, \tilde{\tau}, E)$ is F.W.L.S. j-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

The class of F.W.L.S. j-compact spaces is finitely multiplicative, where $j \in \{\alpha, S, P, b , \beta\}$, in the following sense.

Proposition 2.3.16. Let $(H_r, \tilde{\tau}_r, E_r)$ be finite family of F.W.L.S. j-compact spaces over $(B, \tilde{\Omega}, G)$. Then the F.W.S. topological product $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ is F.W.L.S. j-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. The proof is similar to that of Proposition (2.3.6).

Chapter 3

Fibrewise Soft Near Separation Axioms

Chapter 3

Fibrewise Soft Near Separation Axioms

The separation axioms are axioms only in the sense that, where defining the notion of topological space, one could add these conditions as extra axioms to get a more restricted notion of what a topological space is. The modern approach is to fix once and for all the axiomatization of topological space and then speak of kinds of topological spaces. However, the term "separation axiom" has stuck. In section one, we define and explore several properties of fibrewise soft near T_0 spaces, fibrewise soft near T_1spaces, fibrewise soft near R_0spaces, fibrewise soft near Hausdorff spaces and fibrewise soft near functionally Hausdorff spaces. In section two, we introduce the concepts of fibrewise soft near regular spaces, fibrewise soft near completely regular spaces, fibrewise soft near normal spaces and fibrewise soft near functionally normal spaces. Also we give several results concerning it. In section three, we shall discuss relationships between fibrewise soft near compact (resp., fibrewise soft locally near compact) spaces and some fibrewise soft near separation axioms.

3.1. Fibrewise Soft Near T_0, Soft Near T_1 and Soft Near Hausdorff spaces

The concepts of soft near open sets have an important role in fibrewise soft separation axioms. By using these concepts we can construct many several fibrewise soft separation axioms. Now we introduce the versions of fibrewise soft near T_0 and near T_1 spaces as follows.

Definition 3.1.1. Assume that $(H, \tilde{\tau}, E)$ is F.W.S. topological space over $\left(B, \tilde{\Omega}, G\right)$. Then $(H, \tilde{\tau}, E)$ is called fibrewise soft near T_0 (briefly, F.W.S. j-T_0)

if whenever $\tilde{h}_1, \tilde{h}_2 \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and $\tilde{h}_1 \neq \tilde{h}_2$, either there exists a S. j-open set of $\tilde{h}_1$ which does not contain $\tilde{h}_2$ in $(H, \tilde{\tau}, E)$ or vice versa, where $j \in \{\alpha, S, P, b, \beta\}$.

Remark 3.1.2.

(a) $(H, \tilde{\tau}, E)$ is F.W.S. j-T_0 space iff each fiber soft $H_{\tilde{b}}$ is j-T_0 space, where $j \in \{\alpha, S, P, b, \beta\}$.

(b) The soft subspaces of F.W.S. j-T_0spaces are F.W.S. j-T_0 spaces, where $j \in \{\alpha, S, P, b, \beta\}$.

(c) The F.W.S. topological products of F.W.S. j-T_0 spaces with the family of F.W.S. j-irresolute projections are F.W.S. j-T_0 spaces, where $j \in \{\alpha, S, P, b, \beta\}$.

Of course one can formulate a fibrewise soft version of the near T_1 (briefly, F.W.S. j-T_1) space in a similar fashion. Let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Then $(H, \tilde{\tau}, E)$ is called F.W.S. j-T_1 if whenever $\tilde{h}_1, \tilde{h}_2 \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and $\tilde{h}_1 \neq \tilde{h}_2$, there exists a S. j-open sets (F, E), (N, E) in $(H, \tilde{\tau}, E)$ such that $\tilde{h}_1 \tilde{\in} (F, E)$, $\tilde{h}_2 \tilde{\notin} (F, E)$ and $\tilde{h}_2 \tilde{\in} (F, G)$, $\tilde{h}_2 \tilde{\notin} (F, G)$, where $j \in \{\alpha, S, P, b, \beta\}$. But it turns out that there exists no real use for this in what we are going to do. Instead we make some use of another axiom "The axiom is that every soft near open set contains the closure of each of its points", and use the term soft near R_0 space. This is true for soft near T_1 spaces of course and for soft near regular spaces. Thinking of it as a weak form of soft near regularity. For example, soft indiscrete spaces are soft near R_0 spaces. The fibrewise soft version of the near R_0 axiom is as follows.

Definition 3.1.3. The F.W.S. topological space $(H,\tilde{\tau},E)$ over $(B,\tilde{\Omega},G)$ is called fibrewise soft near R_0 (briefly, F.W.S. j- R_0) if for all soft point $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B,G)$, and each S. j-open set (F,E) of $\tilde{h}$ in $(H,\tilde{\tau},E)$, there exists a soft nbd (N,G) of $\tilde{b}$ in (B,G) such that the closure of $\{\tilde{h}\}$ in $H_{(N,G)}$ is contained in (F,E) (i.e., $H_{(N,G)} \tilde{\cap}\ Cl\{\tilde{h}\} \tilde{\subset} (F,E)$), where $j \in \{\alpha,S,P,b,\beta\}$.

For example, $B \times T$ is F.W.S. j-R_0 space for all j-R_0 spaces T, where $\left(B,\tilde{\Omega},G\right)$, $(T,\tilde{\tau},E)$ is a F.W.S. topological space over $(B,\tilde{\Omega},G)$ and $j \in \{\alpha,S,P,b,\beta\}$.

Remark 3.1.4.

(a) The nbds. of $\tilde{h}$ are given by a fibrewise basis it is sufficient if the condition in Definition (3.1.3) is satisfied for all fibrewise soft basic nbds.

(b) If $(H,\tilde{\tau},E)$ is F.W.S. j-R_0 space over $(B,\tilde{\Omega},G)$, then $(H_{B^*},\tilde{\tau}_{B^*},E_{B^*})$ is F.W.S. j-R_0 space over $(B^*,\tilde{\Omega}^*,G^*)$ for each subspace $(B^*,\tilde{\Omega}^*,G^*)$ of $(B,\tilde{\Omega},G)$, where $j \in \{\alpha,S,P,b,\beta\}$.

The soft subspaces of F.W.S. j-R_0 spaces are F.W.S. j-R_0 spaces, where $j \in \{\alpha,S,P,b,\beta\}$. In fact we have.

Proposition 3.1.5. Assume that $\phi : (H,\tilde{\tau},E) \rightarrow (H^*,\tilde{\tau}^*,E^*)$ is a F.W.S. j-irresolute embedding, where $(H,\tilde{\tau},E)$ and $(H^*,\tilde{\tau}^*,E^*)$ are F.W.S. topological spaces over $(B,\tilde{\Omega},G)$. If $(H^*,\tilde{\tau}^*,E^*)$ is F.W.S. j-R_0, then $(H,\tilde{\tau},E)$ is so, where $j \in \{\alpha,S,P,b,\beta\}$.

Proof. Assume that $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B,G)$, and (F,E) is a S. j-open set of $\tilde{h}$ in $(H,\tilde{\tau},E)$. Then $(F,E) = \phi^{-1}(F^*,E^*)$, where (F^*,E^*) is a S. j-open set of $\tilde{h}^* = \phi(\tilde{h})$ in $(H^*,\tilde{\tau}^*,E^*)$. Since $(H^*,\tilde{\tau}^*,E^*)$ is F.W.S. j-R_0 there exists a soft nbd (N,G) of $\tilde{b}$ such that $H^*_{(N,G)} \tilde{\cap}\ Cl\left\{\tilde{h}^*\right\} \tilde{\subset} (F^*,E^*)$. Then

$H_{(N,G)} \tilde{\cap} Cl\{\tilde{h}\} \tilde{\subseteq} ϕ^{-1}\big(H^*_{(N,G)} \tilde{\cap} Cl\{\tilde{h}^*\}\big) = ϕ^{-1}(F^*, E^*) = (F, E)$ and so $(H, \tilde{\tau}, E)$ is F.W.S. j-R_0, where $j \in \{\alpha, S, P, b, \beta\}$.

The class of F.W.S. j-R_0 spaces is finitely multiplicative, where $j \in \{\alpha, S, P, b, \beta\}$, in the following sense.

Proposition 3.1.6. Assume that $\{H_r, \tilde{\tau}_r, E_r\}$ is a finite family of F.W.S. j-R_0 spaces over $(B, \tilde{\Omega}, G)$. Then the F.W.S. topological product $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-R_0, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$. Consider a S. j-open set $(F, E) = \prod_B (F_r, E_r)$ of $\tilde{h}$ in $(H, \tilde{\tau}, E)$, where (F_r, E_r) is a S. j-open set of $\prod_r(\tilde{h}) = \tilde{h}_r$ in $(H_r, \tilde{\tau}_r, E_r)$ for each index r. Since $(H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-R_0 there exists a soft nbd (N_r, G_r) of $\tilde{h}$ in (B, G) such that $\big((H_r, E_r) - (N_r, G_r)\big) \tilde{\cap} Cl\{\tilde{h}\} \tilde{\subseteq} (F_r, E_r)$. Then the intersection (N, G) of the (N_r, G_r) is a soft nbd of $\tilde{b}$ such that $((H, E) - (N, G)) \tilde{\cap} Cl\{\tilde{h}\} \tilde{\subseteq} (F, E)$ and so $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-R_0, where $j \in \{\alpha, S, P, b, \beta\}$.

The same conclusion holds for infinite fibrewise soft products provided each of the factors are fibrewise soft nonempty.

Proposition 3.1.7. Let $ϕ : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ be a soft closed, soft j-irresolute fibrewise surjection function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E)$ is F.W.S. j-R_0, then $(K, \tilde{\sigma}, L)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{k} \tilde{\in} K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and (O, L) is a S. j-open set of $\tilde{k}$ in $(K, \tilde{\sigma}, L)$. Pick $\tilde{h} \tilde{\in} ϕ^{-1}(\tilde{k})$. Then $(F, E) = ϕ^{-1}(O, L)$ is a S. j-open set of $\tilde{h}$. Since $(H, \tilde{\tau}, E)$ is F.W.S. j-R_0 there exists a soft nbd (N, G) of $\tilde{b}$ such that $H_{(N,G)} \tilde{\cap} Cl\{\tilde{h}\} \tilde{\subseteq} (F, E)$. Then $K_{(N,G)} \tilde{\cap} ϕ\big(Cl\{\tilde{h}\}\big) \tilde{\subseteq} ϕ(F, L) = (O, L)$.

Since ϕ is soft closed, then $\phi\big(Cl\{\tilde{h}\}\big) = Cl\big(\phi\{\tilde{h}\}\big)$. Therefore $K_{(N,G)} \tilde{\cap}\, Cl(\phi\{\tilde{h}\}) \tilde{\subset} (O, L)$ and so $(K, \tilde{\sigma}, L)$ is F.W.S. j-R_0, where $j \in \{\alpha, S, P, b, \beta\}$.

Now we introduce the version of fibrewise soft near Hausdorff spaces as follows.

Definition 3.1.8. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $\big(B, \tilde{\Omega}, G\big)$ is called fibrewise soft near Hausdorff (briefly F.W.S. j-Hausdorff) if whenever $\tilde{h}_1, \tilde{h}_2 \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and $\tilde{h}_1 \neq \tilde{h}_2$, there exists disjoint S. j-open sets $(F_1, E), (F_2, E)$ of $\tilde{h}_1, \tilde{h}_2$ respectively in $(H, \tilde{\tau}, E)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Example 3.1.9. Let $H = \{h_1, h_2, h_3\}$, $B = \{a, b\}$, $E = \{e_1, e_2\}$, $G = \{g_1, g_2\}$ and let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_1) = \{a\}$, $f(h_2) = f(h_3) = \{b\}$, $u(e_1) = \{g_1\}$, $u(e_2) = \{g_2\}$. Then $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, (F_1, E), (F_2, E), (F_3, E)\}$ where $(F_1, E), (F_2, E), (F_3, E)$ are soft sets over $(H, \tilde{\tau}, E)$, defined as follows:

$(F_1, E) = \{(e_1, \{h_1\})\}$,

$(F_2, E) = \{(e_2, \{h_2\})\}$,

$(F_3, E) = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$,

$\tilde{\Omega} = \{\tilde{\Phi}, \tilde{B}, (M, G)\}$, $(M, G) = \{(g_1, \{a\})\}$ and then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ be a S. j-continuous, then there exists disjoin soft open $\{(e_1, \{h_1\})\}$, $\{(e_2, \{h_2\})\}$ such that $\tilde{h}_1 \tilde{\in} \{(e_1, \{h_1\})\}$ and $\tilde{h}_2 \tilde{\in} \{(e_2, \{h_2\})\}$. Also, $(H, \tilde{\tau}, E)$ is F.W.S. j-Hausdorff space, where $j \in \{\alpha, S, P, b, \beta\}$.

Remark 3.1.10. If $(H, \tilde{\tau}, E)$ is F.W.S. j-Hausdorff space over $(B, \tilde{\Omega}, G)$ then $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-Hausdorff over $(B^*, \tilde{\Omega}^*, G^*)$ for each soft subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $(B, \tilde{\Omega}, G)$. In particular the fibres soft of $(H, \tilde{\tau}, E)$ are j-Hausdorff spaces.

However a F.W.S. topological space with soft j-Hausdorff fibres is not necessarily soft j-Hausdorff:

Example 3.1.11. Assume that $H = \{h_1, h_2, h_3\}$, $B = \{a, b\}$, $E = \{e_1, e_2\}$, $G = \{g_1, g_2\}$ and let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_1) = \{a\}$, $f(h_2) = f(h_3) = \{b\}$, $u(e_1) = \{g_1\}$, $u(e_2) = \{g_2\}$. Then $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$ where $(F_1, E), (F_2, E), (F_3, E), (F_4, E)$ are soft sets over $(H, \tilde{\tau}, E)$, defined as follows:

$(F_1, E) = \{(e_1, \{h_1\}), (e_2, \{h_1, h_2\})\}$,

$(F_2, E) = \{(e_1, \{h_1\}), (e_2, \{h_1, h_3\})\}$,

$(F_3, E) = \{(e_1, \{h_1\}), (e_2, \{h_1\})\}$,

$(F_4, E) = \{(e_1, \{h_1\}), (e_2, H)\}$,

$\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, (M, G)\}$ and $(M, G) = \{(g_1, \{a\}), (g_2, B)\}$, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is a S. j-continuous. Since we have : $H_{\tilde{b}} = \{\tilde{h}_2, \tilde{h}_3\}$, $\tilde{\tau}_{\tilde{b}} = \{\widetilde{\Phi}, \widetilde{H}_{\tilde{b}}, \{(e_2, \{h_2\})\}, \{(e_2, \{h_3\})\}, \{(e_2, \{h_2, h_3\})\}\}$, then there exists disjoint soft open sets $\{(e_2, \{h_2\})\}$, $\{(e_2, \{h_3\})\}$ and where $\tilde{h}_2 \tilde{\in} \{(e_2, \{h_2\})\}$, $\tilde{h}_3 \tilde{\notin} \{(e_2, \{h_3\})\}$ and $\tilde{h}_3 \tilde{\in} \{(e_2, \{h_3\})\}$, $\tilde{h}_2 \tilde{\in} \{(e_2, \{h_2\})\}$. Since we have : $H_{\tilde{b}} = \{\tilde{h}_2, \tilde{h}_3\}$ is F.W.S. Hausdorff. But $(H, \tilde{\tau}, E)$ is not F.W.S Hausdorff since $\tilde{h}_2$ and $\tilde{h}_3 \tilde{\in} (H, \tilde{\tau}, E)$, $\tilde{h}_2 \neq \tilde{h}_3$ and there exists no disjoint soft open sets of $\tilde{h}_2$ and $\tilde{h}_3$.

Proposition 3.1.12. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is F.W.S. j-Hausdorff iff the soft diagonal embedding $\Delta : (H, \tilde{\tau}, E) \to (H, \tilde{\tau}, E) \times_B (H, \tilde{\tau}, E)$ is S. j-closed, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. $(\Longrightarrow)$ Assume that $\tilde{h}_1, \tilde{h}_2 \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and $\tilde{h}_1 \neq \tilde{h}_2$. Since $\Delta(H, \tilde{\tau}, E)$ is S. j-closed in $(H, \tilde{\tau}, E) \times_B (H, \tilde{\tau}, E)$, then $(\tilde{h}_1, \tilde{h}_2)$, being a point of the soft complement, admits a fibrewise product S. j-open set

$(F_1, E) \times_B (F_2, E)$ which does not meet $\Delta(H, \tilde{\tau}, E)$, and then (F_1, E), (F_2, E) are disjoint S. j-open sets of $\tilde{h}_1, \tilde{h}_2$, where $j \in \{\alpha, S, P, b, \beta\}$.

$(\Longleftarrow)$ The reverse direction is similar.

The soft subspaces of F.W.S. j-Hausdorff spaces are F.W.S. j-Hausdorff spaces, where $j \in \{\alpha, S, P, b, \beta\}$. In fact we have.

Proposition 3.1.13. Assume that $\phi : (H, \tilde{\tau}, E) \to (H^*, \tilde{\tau}^*, E^*)$ is a S. j-irresolute embedding fibrewise function, where $(H, \tilde{\tau}, E)$ and $(H^*, \tilde{\tau}^*, E^*)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H^*, \tilde{\tau}^*, E^*)$ is F.W.S. j-Hausdorff, then is so $(H, \tilde{\tau}, E)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{h}_1, \tilde{h}_2 \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and $\tilde{h}_1 \neq \tilde{h}_2$. Then $\phi(\tilde{h}_1), \phi(\tilde{h}_2) \tilde{\in} H^*_{\tilde{b}}$ are distinct, since $(H^*, \tilde{\tau}^*, E^*)$ is F.W.S. j-Hausdorff, there exists a S. j-open sets $(F_1^*, E^*), (F_2^*, E^*)$ of $\phi(\tilde{h}_1), \phi(\tilde{h}_2)$ in $(H^*, \tilde{\tau}^*, E^*)$ which are disjoint. Their inverse images $\phi^{-1}(F_1^*, E^*), \phi^{-1}(F_2^*, E^*)$ are S. j-open sets of $\tilde{h}_1, \tilde{h}_2$ in $(H, \tilde{\tau}, E)$ which are disjoint and so $(H, \tilde{\tau}, E)$ is F.W.S. j-Hausdorff, where $j \in \{\alpha, S, P, b, \beta\}$.

Remark 3.1.14. The graph of a soft function $\tilde{f} : (H, E) \to (K, L)$, is the subset of the Cartesian product $H \times K$ defined by

$$G(\tilde{f}) = \{(\tilde{h}, \tilde{k})\ ;\ \tilde{k} = \tilde{f}(\tilde{h})\ \}.$$

The fibrewise soft graph of a fibrewise soft function $\tilde{f}$ of a fibrewise soft space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ to a fibrewise soft space $(K, \tilde{\sigma}, L)$ over $(B, \tilde{\Omega}, G)$, we mean the soft subset of the fiberwise soft Cartesian product $H \times_B K$ defined by

$$G_B(\tilde{f}) = \{(\tilde{h}, \tilde{k}) \tilde{\in} H \times_B K;\ \tilde{k} = \tilde{f}(\tilde{h})\}.$$

Proposition 3.1.15. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-irresolut fibrewise function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(K, \tilde{\sigma}, L)$ is F.W.S. j-Hausdorff, then the fibrewise soft graph $G : (H, \tilde{\tau}, E) \to (H, \tilde{\tau}, E) \times_B (K, \tilde{\sigma}, L)$ of ϕ is a S. j-closed embedding, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. The fibrewise soft graph is defined in the same way as the ordinary soft graph, but with values in the fibrewise soft product, so that the figure shown below is commutative.

$$\begin{tikzcd} H \arrow[r, "G"] \arrow[d, "\phi"'] & H \times_B K \arrow[d, "\phi \times id_K"] \\ K \arrow[r, "\Delta"] & K \times_B K \end{tikzcd}$$

Fig. (3.1.1): Diagram of Proposition (3.1.15).

Since $\Delta(K, \tilde{\sigma}, L)$ is S. j-closed in $(K, \tilde{\sigma}, L) \times_B (K, \tilde{\sigma}, L)$, by Proposition (3.1. 12), so $G(H, \tilde{\tau}, E) = (\phi \times id_K)^{-1}(\Delta(K, \tilde{\sigma}, L))$ is S. j-closed in $(H, \tilde{\tau}, E) \times_B (K, \tilde{\sigma}, L)$, where $j \in \{\alpha, S, P, b, \beta\}$, as asserted.

The class of F.W.S. j-Hausdorff spaces is multiplicative, where $j \in \{\alpha, S, P, b, \beta\}$, in the following sense.

Proposition 3.1.16. Assume that $\{H_r, \tilde{\tau}_r, E_r\}$ is a family of F.W.S. j-Hausdorff spaces over $(B, \tilde{\Omega}, G)$. Then the fibrewise soft topological product $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ with the family of F.W.S. j-irresolute projection $\pi_r : (H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r) \to (H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-Hausdorff, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{h}_1, \tilde{h}_2 \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and $\tilde{h}_1 \neq \tilde{h}_2$.Then $\pi_r(\tilde{h}_1) \neq \pi_r(\tilde{h}_2)$ for some index r. Since $(H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-Hausdorff, then there exists a S. j-open sets $(F_{1_r}, E_r), (F_{2_r}, E_r)$ of $\pi_r(\tilde{h}_1), \pi_r(\tilde{h}_2)$ in $(H_r, \tilde{\tau}_r, E_r)$ which are soft disjoint. Since π_r is S. j-irresolute, then the inverse images $\pi_r^{-1}(F_{1r}, E_r), \pi_r^{-1}(F_{2r}, E_r)$ are disjoint S. j-open sets of $\tilde{h}_1, \tilde{h}_2$ in $(H, \tilde{\tau}, E)$, where $j \in \{\alpha, S, P, b, \beta\}$.

The soft near functionally version of the fibrewise soft near Hausdorff axiom is stronger than the non soft near functional version but its properties are farily similar. Here and elsewhere we use the closed unit interval $[0, 1]$ in the real line.

Definition 3.1.17. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is called fibrewise soft near functionally (briefly, F.W.S. j-functionally) Hausdorff if whenever $\tilde{h}_1, \tilde{h}_2 \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and $\tilde{h}_1 \neq \tilde{h}_2$, there exists a soft nbd (N, G) of $\tilde{b}$ and disjoint S. j-open sets $(F, E), (V, E)$ of $\tilde{h}_1, \tilde{h}_2$ in $(H, \tilde{\tau}, E)$ and a continuous function $\lambda : H_{(N,G)} \to [0,1]$ such that $H_{\tilde{b}} \tilde{\cap} (F, E) \tilde{\subseteq} \lambda^{-1}(0)$, $H_{\tilde{b}} \tilde{\cap} (V, E) \tilde{\subseteq} \lambda^{-1}(1)$ and $j \in \{\alpha, S, P, b, \beta\}$.

Example 3.1.18. Assume that $H = \{h_1, h_2, h_3\}$, $B = \{a, b\}$, $E = \{e\}$, $G = \{g\}$ and let $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_1) = \{a\}$, $f(h_2) = f(h_3) = \{b\}$, $u(e) = \{g\}$. $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$ where (F_1, E) , $(F_2, E), (F_3, E), (F_4, E)$ are soft sets over $(H, \tilde{\tau}, E)$, defined as follows:

$(F_1, E) = \{(e, \{h_2\})\}$,

$(F_2, E) = \{(e, \{h_2, h_3\})\}$,

$(F_3, E) = \{(e, \{h_1, h_2\})\}$, $(F_4, E) = \{(e, H)\}$, $(F_5, E) = \{(e, \{h_3\})\}$,

$\tilde{\Omega} = \{\tilde{\Phi}, \tilde{B}, (M, G)\}$ and $(M, G) = \{(g, \{b\})\}$. It clear that $H_{(M,G)} = \{(e, \{h_2, h_3\})\}$, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is a S. j-continuous. Let $\lambda : H_{(N,G)} \to [0,1]$ where $\lambda(\tilde{h}_2) = 0$, $\lambda(\tilde{h}_3) = 1$, $\tilde{\tau}_{H_{(N,G)}} = \{\tilde{\Phi}, \tilde{H}_{(N,G)}, \{(e, \{h_2\})\}, \{(e, \{h_3\})\}\}$, then λ is continuous function, and there exist disjoin soft open $\{(e, \{h_2\})\}$ and $\{(e, \{h_3\})\}$ such that $H_{\tilde{b}} \tilde{\cap} \{(e, \{h_2\})\}$

$\tilde{\subseteq} \lambda^{-1}(0)$, $H_{\tilde{b}} \tilde{\cap} \{(e, \{h_3\})\} \tilde{\subseteq} \lambda^{-1}$. Also, $(H, \tilde{\tau}, E)$ is F.W.S. j-functionally Hausdorff space, where $j \in \{\alpha, S, P, b, \beta\}$.

Remark 3.1.19. If $(H, \tilde{\tau}, E)$ is F.W.S. j-functionally Hausdorff space over $(B, \tilde{\Omega}, G)$, then $(H^*, \tilde{\tau}^*, E^*) = (H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-functionally Hausdorff over $(B^*, \tilde{\Omega}^*, G^*)$ for each soft subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $(B, \tilde{\Omega}, G)$. In particular the fibres soft of $(H, \tilde{\tau}, E)$ are soft j-functionally Hausdorff spaces, where $j \in \{\alpha, S, P, b, \beta\}$.

The soft subspaces of F.W.S. j-functionally Hausdorff spaces are F.W.S. j-functionally Hausdorff spaces, where $j \in \{\alpha, S, P, b, \beta\}$. In fact we have.

Proposition 3.1.20. Let $\phi : (H, \tilde{\tau}, E) \to (H^*, \tilde{\tau}^*, E^*)$ be a S. j-irresolute embedding fibrewise function, where $(H, \tilde{\tau}, E)$ and $(H^*, \tilde{\tau}^*, E^*)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H^*, \tilde{\tau}^*, E^*)$ is F.W.S. j-functionally Hausdorff, then is so $(H, \tilde{\tau}, E)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Moreover the class of F.W.S. j-functionally Hausdorff spaces is multiplicative, where $j \in \{\alpha, S, P, b, \beta\}$, as stated in.

Proposition 3.1.21. Let $\{H_r, \tilde{\tau}_r, E_r\}$ be a family of F.W.S. j-functionally Hausdorff spaces over $(B, \tilde{\Omega}, G)$. Then the F.W.S. topological product $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ with the family of F.W.S. j-irresolute projection $\pi_r : (H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r) \to (H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-functionally Hausdorff , where $j \in \{\alpha, S, P, b, \beta\}$.

The proofs of Propositions (3.1.20) and (3.1.21) are similar to those for the corresponding results in the non-functional case and will therefore be omitted.

3.2. Fibrewise Soft Near Regular and Soft Near Normal Spaces

We now proceed to consider the fibrewise soft version of the higher soft near separation axioms, starting with fibrewise soft near regularity and fibrewise soft near completely regularity.

Definition 3.2.1. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is called fibrewise soft near regular (briefly, F.W.S. j-regular), if for each soft point $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and for each S. j-open set (F, E) of $\tilde{h}$ in $(H, \tilde{\tau}, E)$, there exists a soft nbd (N, G) of $\tilde{b}$ in $(B, \tilde{\Omega}, G)$ and a soft open set (V, E) of $\tilde{h}$ in $(H_{(N,G)}, \tilde{\tau}_{(N,G)}, E_{(N,G)})$ such that the closure soft of (F, E) in $H_{(N,G)}$ is contained in (V, E) (i.e., $H_{(N,G)} \tilde{\cap}\ Cl(F, E) \tilde{\subseteq} (V, E)$), where $j \in \{\alpha, S, P, b, \beta\}$.

Example 3.2.2. Let $H = \{h_1, h_2, h_3\}$, $B = \{a, b\}$, $E = \{e\}$, $G = \{g\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_3) = \{a\}$, $f(h_1) = f(h_2) = \{b\}$, $u(e) = \{g\}$. $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, (F_1, E), (F_2, E)\}$ where (F_1, E), (F_2, E) are soft sets over $(H, \tilde{\tau}, E)$, defined as follows : $(F_1, E) = \{(e, \{h_3\})\}$, $(F_2, E) = \{(e, \{h_1, h_2\})\}$, $\Omega = \{\tilde{\Phi}$, $\tilde{B}, (M, G), (N, G)\}$ and $(M, G) = \{(g, \{a\})\}$, $(N, G) = \{(g, \{b\})\}$. It clear that $H_{(M,G)} = \{(e, \{h_3\})\}$, $H_{(N,G)} = \{(e, \{h_1, h_2\})\}$ is soft open in $(H, \tilde{\tau}, E)$, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is a S. j-continuous and , $(V, E) = \{(e, \{h_1, h_2\})\}$ and $(F, E) = \{(e, \{h_3\})\}$ such that $H_{(N,G)} \tilde{\cap}\ Cl(F, E)$ $\tilde{\subseteq}$ (V, E). Also, $(H, \tilde{\tau}, E)$ is F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$.

Remark 3.2.3.

(a) The nbds. of $\tilde{h}$ are given by a fibrewise soft basis it is sufficient if the condition in Definition (3.2.1) is satisfied for all fibrewise soft basic nbds.

(b) If $(H, \tilde{\tau}, E)$ is F.W.S. j-regular space over $\left(B, \tilde{\Omega}, G\right)$, then $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-regular space over $\left(B^*, \tilde{\Omega}^*, G^*\right)$ for all subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $\left(B, \tilde{\Omega}, G\right)$, where $j \in \{\alpha, S, P, b, \beta\}$.

The soft subspaces of F.W.S. j-regular spaces are F.W.S. j-regular spaces, where $j \in \{\alpha, S, P, b, \beta\}$. In fact we have.

Proposition 3.2.4. Assume that $\phi : (H, \tilde{\tau}, E) \to (H^*, \tilde{\tau}^*, E^*)$ is a F.W.S. j-irresolute embedding function, where $(H, \tilde{\tau}, E)$ and $(H^*, \tilde{\tau}^*, E^*)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H^*, \tilde{\tau}^*, E^*)$ is F.W.S. j-regular, then $(H, \tilde{\tau}, E)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and (F, E) is a S. j-open set of $\tilde{h}$ in $(H, \tilde{\tau}, E)$. Then $(F, E) = \phi^{-1}(F^*, E^*)$, where (F^*, E^*) is a S. j-open set of $\tilde{h}^* = \phi(\tilde{h})$ in $(H^*, \tilde{\tau}^*, E^*)$. Since $(H^*, \tilde{\tau}^*, E^*)$ is F.W.S. j-regular there exists a soft nbd (N, G) of $\tilde{h}$ in $(B, \tilde{\Omega}, G)$ and a soft open set (V^*, E^*) of $\tilde{h}^*$ in $H^*_{(N,G)}$ such that $H^*_{(N,G)} \tilde{\cap} Cl(V^*, E^*) \tilde{\subset} (F^*, E^*)$. Then $(V, E) = \phi^{-1}(V^*, E^*)$ is a soft open set of $\tilde{h}$ in $H_{(N,G)}$ such that $H_{(N,G)} \tilde{\cap} Cl(V, E) \tilde{\subset} (F, E)$, and so $(H, \tilde{\tau}, E)$ is F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$.

The class of F.W.S. j-regular spaces is fibrewise soft multiplicative, where $j \in \{\alpha, S, P, b, \beta\}$, in the following sense.

Proposition 3.2.5. Assume that $\{H_r, \tilde{\tau}_r, E_r\}$ are a finite family of F.W.S. j-regular spaces over $(B, \tilde{\Omega}, G)$. Then the F.W.S. topological product $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$. Consider a S. j-open set $(F, E) = \prod_B (F_r, E_r)$ of $\tilde{h}$ in $(H, \tilde{\tau}, E)$, where (F, E) is a S. j-open set of $\pi_r(\tilde{h}) = \tilde{h}_r$ in $(H_r, \tilde{\tau}_r, E_r)$ for each index r. Since $(H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-regular then there exists a soft nbd (N_r, G_r) of $\tilde{b}$ in $(B, \tilde{\Omega}, G)$ and a soft open set (V_r, E_r)

of $\tilde{h}_r$ in $(H_r, E_r) - (N_r, G_r)$ such that the closure soft $((H_r, E_r) - (N_r, G_r)) \tilde{\cap}\, Cl(V_r, E_r)$ of (V_r, E_r) in $\big((H_r, E_r) - (N_r, G_r)\big)$ is in (F_r, E_r). Then the intersection (N, G) of the (N_r, G_r) is a soft nbd of $\tilde{b}$ and $(V, E) = \prod_B (V_r, E_r)$ is a soft open set of $\tilde{h}$ in $H_{(N,G)}$ such that the closure soft $H_{(N,G)} \tilde{\cap}\, Cl(V, G)$ of (V, E) in $(H_{(N,G)}, \tilde{\tau}_{(N,G)}, E_{(N,G)})$ is contained in (F, E), and so $(H, \tilde{\tau}, E) = \prod_B (H_r, \tilde{\tau}_r, E_r)$ is F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$.

The same conclusion holds for infinite fibrewise soft products provided each of the factors are fibrewise soft non-empty.

Proposition 3.2.6. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a soft open, soft closed and S. j-irresolute fibrewise surjection function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. The $(H, \tilde{\tau}, E)$ is F.W.S. j-regular iff $(K, \tilde{\sigma}, L)$ so is, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof.(⇒) Assume that $\tilde{k} \tilde{\in} K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and (U, L) is a S. j-open set of $\tilde{k}$ in $(K, \tilde{\sigma}, L)$. Pick $\tilde{h} \tilde{\in} \phi^{-1}(\tilde{k})$. Then $(F, E) = \phi^{-1}(U, L)$ is a S. j-open set of $\tilde{h}$. Since $(H, \tilde{\tau}, E)$ is F.W.S. j-regular there exists a soft nbd (N, G) of $\tilde{b}$ and a soft open set (F^*, E^*) of $\tilde{h}$ such that $H_{(N,G)} \tilde{\cap}\, Cl(F^*, E^*) \tilde{\subset} (F, E)$. Then $K_{(N,G)} \tilde{\cap}\, \phi(Cl(F^*, E^*)) \tilde{\subset} \phi(F, E) = (U, L)$. Since ϕ is a soft closed, then $\phi(Cl(F^*, E^*)) = Cl(\phi(F^*, E^*))$ and since ϕ is a soft open, then $\phi(F^*, E^*)$ is a soft open set of $\tilde{k}$. Thus $(K, \tilde{\sigma}, L)$ is F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$, as asserted.

(⇐) Assume that $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and (F, E) is a S. j-open set of $\tilde{h}$ in $(H, \tilde{\tau}, E)$. Pick $\tilde{k} \tilde{\in} \phi(\tilde{h})$. Then $\phi(F, E) = (U, L)$ is a S. j-open set of $\tilde{k}$. Since $(K, \tilde{\sigma}, L)$ is F.W.S. j-regular there exists a soft nbd (N, G) of $\tilde{b}$ and a soft open set (U^*, L^*) of $\tilde{k}$ such that $K_{(N,G)} \tilde{\cap}\, Cl(U^*, L^*) \tilde{\subset} (U, L)$. Then $H_{(N,G)} \tilde{\cap}\, \phi^{-1}(Cl(U^*, L^*)) \tilde{\subset} \phi^{-1}(U, L) = (F, E)$. Since ϕ is soft closed, then

$\phi^{-1}\big(Cl(U^*, L^*)\big) = Cl(\phi^{-1}(U^*, L^*))$ and since ϕ is S. j-irresolute, then $\phi^{-1}(U^*, L^*)$ is a soft open set of $\tilde{h}$. Thus $(H, \tilde{\tau}, E)$ is F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$.

The soft near functionally type of the fibrewise soft near regularity axiom is stronger than the non soft near functionally type. On the other hand its properties are justly like. In the ordinary theory the word completely regular is every time used as an alternative of functionally regular also we are expanding this procedure to the fibrewise soft theory.

Definition 3.2.7. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is named fibrewise soft near completely (briefly, F.W.S. j-completely) regular condition for all soft point $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and for all S. j-open set (F, E) of $\tilde{h}$ there exists a soft nbd (N, G) of $\tilde{b}$, a soft open set (V, E) of $\tilde{h}$ in $H_{(N,G)}$ and a continuous function $\lambda : H_{(N,G)} \to [0,1]$ such that $H_{\tilde{b}} \tilde{\cap} (V, E) \tilde{\subseteq} \lambda^{-1}(0)$ and $H_{(N,G)} \tilde{\cap} (H_{(N,G)}, E_{(N,G)}) - (F, E)) \tilde{\subseteq} \lambda^{-1}(1)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Example 3.2.8. Let $H = \{h_1, h_2, h_3, h_4\}$, $B = \{a, b\}$, $E = \{e_1, e_2\}$, $G = \{g_1, g_2\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_1) = f(h_3) = \{a\}$, $f(h_2) = = f(h_4) = \{b\}$, $u(e_1) = \{g_1\}$, $u(e_2) = \{g_2\}$. $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, (F_1, E)\}$ where (F_1, E) is soft set over $(H, \tilde{\tau}, E)$, defined as follows : $(F_1, E) = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2, h_4\})\}$, $\tilde{\Omega} = \{\tilde{\Phi}, \tilde{B}, (M, G)\}$ and $(M, G) = \{(g_1, \{a\}), (g_2, \{b\})\}$. It is clear that $H_{(M,G)} = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2, h_4\})\}$, then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is a S. j-continuous, $H_{\tilde{a}} = \{\tilde{h}_1, \tilde{h}_3\}$, $\tilde{h} = \tilde{h}_1$, $(F, E) = \{(e_1\{h_1, h_3\})\}$, $(N, G) = \{(g_1, \{a\})\}$, $H_{(N,G)} = \{\tilde{h}_1, \tilde{h}_3\}$, $\tilde{\tau}_{H_{(N,G)}} = \{\tilde{H}_{(N,G)}, \tilde{\Phi}\}$ and $(U, E) = \tilde{H}_{(N,G)}$. Let $\lambda : H_{(N,G)} \to [0,1]$ such that $\lambda\big(\tilde{h}_1\big) = 0 =$

$\lambda(\tilde{h}_3)$. λ is continuous and $H_{\tilde{b}} \tilde{\cap} (U,E) \tilde{\subseteq} \lambda^{-1}(0)$, $H_{(N,G)} \tilde{\cap} \left(H_{(N,G)} - (F,E)\right) \tilde{\subseteq} \lambda^{-1}(1)$. Similar if $\tilde{h} = \tilde{h}_3$. $H_{\tilde{b}} = \{\tilde{h}_2, \tilde{h}_4\}$ and $\tilde{h} = \tilde{h}_2$, $(F,E) = \{(e_2, \{h_2, h_4\})\}$, $(N,G) = \{(g_2, \{b\})\}$, $H_{(N,G)} = \{\tilde{h}_2, \tilde{h}_4\}$, $\tilde{\tau}_{H_{(N,G)}} = \{\widetilde{\Phi}, \widetilde{H}_{(N,G)}\}$ and $(U,E) = \widetilde{H}_{(N,G)}$ and $\lambda : H_{(N,G)} \to [0,1]$ such that $\lambda(\tilde{h}_2) = 0 = \lambda(\tilde{h}_4)$. λ is continuous, $H_{\tilde{b}} \tilde{\cap} (U,E) \tilde{\subseteq} \lambda^{-1}(0)$ and $H_{(N,G)} \tilde{\cap} (H_{(N,G)} - (F,E)) \tilde{\subseteq} \lambda^{-1}(1)$. Similar if $\tilde{h} = \tilde{h}_4$. Also, $(H, \tilde{\tau}, E)$ is F.W.S.j-completely regular, where $j \in \{\alpha, S, P, b, \beta\}$.

Remark 3.2.9.

(a) The soft nbds. of $\tilde{h}$ are given by a fibrewise soft basis it is sufficient if the condition in Definition (3.2.7) is satisfied for all fibrewise soft basic nbds.

(b) If $(H, \tilde{\tau}, E)$ is F.W.S. j-completely regular space over $(B, \tilde{\Omega}, G)$, then $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-completely regular space over $(B^*, \tilde{\Omega}^*, G^*)$ for all subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $(B, \tilde{\Omega}, G)$, where $j \in \{\alpha, S, P, b, \beta\}$.

The soft subspaces of F.W.S. j-completely regular spaces are F.W.S. j-completely regular spaces, where $j \in \{\alpha, S, P, b, \beta\}$. In fact we have.

Proposition 3.2.10. Let $\phi : (H, \tilde{\tau}, E) \to (H^*, \tilde{\tau}^*, E^*)$ be a F.W.S. j-irresolute embedding, where $(H, \tilde{\tau}, E)$ and $(H^*, \tilde{\tau}^*, E^*)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H^*, \tilde{\tau}^*, E^*)$ is F.W.S. j-completely regular, then so is $(H, \tilde{\tau}, E)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. The proof is like to the proof of Proposition (3.1.5).

The class of F.W.S. j-completely regular spaces is finitely multiplicative, where $j \in \{\alpha, S, P, b, \beta\}$, in the following sense.

Proposition 3.2.11. Assume that $\{H_r, \tilde{\tau}_r, E_r\}$ is a finite family of F.W.S. j-completely regular spaces over $(B, \tilde{\Omega}, G)$. Then the fibrewise soft topological product $(H,\tilde{\tau},E)=\prod_B(H_r,\tilde{\tau}_r,E_r)$ is F.W.S. j-completely regular, where $j\in\{\alpha,S,P,b,\beta\}$.

Proof. Assume that $\tilde{h}\tilde{\in}H_{\tilde{b}}$, where $\tilde{b}\tilde{\in}(B,G)$. Consider a F.W.S. j-open set $\prod_B(F_r,E_r)$ of $\tilde{h}$ in $(H,\tilde{\tau},E)$, where (F_r,E_r) is a S. j-open set of $\pi_r(\tilde{h})=\tilde{h}_r$ in $(H_r,\tilde{\tau}_r,E_r)$ for each index r. Since $(H_r,\tilde{\tau}_r,E_r)$ is F.W.S. j-completely regular there exists a soft nbd (N_r,G_r) of $\tilde{b}$ and a soft open set (V,E) of $\tilde{h}_r$ in $(H_r,\tilde{\tau}_r,E_r)$ and a continuous function $\lambda_r : H_{(N_r,G_r)}\rightarrow[0,1]$ such that $(H_r)_{\tilde{b}}$ $\tilde{\cap}$ $(V,E)\tilde{\subset}\lambda_r^{-1}(0)$ and $H_{(N_r,G_r)}$ $\tilde{\cap}$ $(H_{(N_r,G_r)}-(F_r,E_r))$ $\tilde{\subset}$ $\lambda_r^{-1}(1)$. Then the intersection (N,G) of the (N_r,G_r) is a soft nbd of $\tilde{b}$ and $\lambda : H_{(N,G)}\rightarrow$ [0,1] is a continuous function where

$$\lambda(\xi)=inf_r=1,2,\dots,n\{\lambda_r(\xi_r)\}\text{ for }\xi=(\xi_r)\in H_{(N,G)}.$$

Since $H_{\tilde{b}}\,\tilde{\cap}\,\pi_r^{-1}(V,E)\tilde{\subset}\pi_r^{-1}[(H_r)_{\tilde{b}}\,\tilde{\cap}\,(V,E))]\tilde{\subset}\pi_r^{-1}(\lambda_r^{-1}(0))=(\lambda_r\circ\pi_r)^{-1}$ and $H_{(N,G)}\,\pi_r^{-1}\left(H_{(N_r,G_r)},E_{(N_r,G_r)}\right)-(F_r,E_r)\tilde{\subset}\pi_r^{-1}[H_{(N_r,G_r)}\,\tilde{\cap}\,(H_{(N_r,G_r)},E_{(N_r,G_r)})-(F_r,E_r)]\tilde{\subset}\pi_r^{-1}(\lambda_r^{-1}(1))=(\pi_r\circ\lambda_r)^{-1}(1)$, where $j\in\{\alpha$ $S,P,b,\beta\}$, this proves the result.

Lemma 3.2.12. Assume that $\phi:(H,\tilde{\tau},E)\rightarrow(K,\tilde{\tau},L)$ is a soft open and a soft closed fibrewise surjection function, where $(H,\tilde{\tau},E)$ and $(K,\tilde{\sigma},L)$ are F.W.S. topological spaces over $(B,\tilde{\Omega},G)$. Assume that $\gamma:(H,\tilde{\tau},E)\rightarrow(\mathbb{R},\tilde{\vartheta},O)$ is a S. j-continuous real-valued function which is a fibrewise soft bound above, in the sense that γ is bounded above on each fiber soft of H. Then $\varphi:(K,\tilde{\sigma},L)\rightarrow(\mathbb{R},\tilde{\vartheta},O)$ is S. j-continuous, where

$$\varphi(\mu)=\mathrm{Sup}_{\zeta\in\phi^{-1}(\mu)}\gamma(\zeta).$$

The similar conclusion holds for infinite fibrewise soft products provided that all of the factors are fibrewise soft non-empty.

Proposition 3.2.13. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a soft open and a soft closed and S. j-irresolute fibrewise surjection function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E)$ is F.W.S. j-completely regular, then $(K, \tilde{\sigma}, L)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{k} \tilde{\in} K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, assume that $(U_{\tilde{k}}, L_{\tilde{k}})$ is a S. j-open set of $\tilde{k}$. Pick $\tilde{h} \tilde{\in} H_{\tilde{b}}$, so that $(F_{\tilde{h}}, E_{\tilde{h}}) = \phi^{-1}(U_{\tilde{k}}, L_{\tilde{k}})$ is a S. j-open set of $\tilde{h}$. Since $(H, \tilde{\tau}, E)$ is F.W.S. j-completely regular there exists a soft nbd (N, G) of $\tilde{b}$ and a soft open set $(V_{\tilde{h}}, E_{\tilde{h}})$ of $\tilde{h}$ in $H_{(N,G)}$ and a continuous function $\lambda : H_{(N,G)} \to [0,1]$ such that $H_{\tilde{b}} \tilde{\cap} (V_{\tilde{h}}, E_{\tilde{h}}) \tilde{\subseteq} \lambda^{-1}(0)$ and $H_{(N,G)} \tilde{\cap} (H_{(N,G)} - (F_{\tilde{h}}, E_{\tilde{h}})) \tilde{\subseteq} \lambda^{-1}(1)$. Using Lemma (3.2.12) we obtain a continuous function $\omega : K_{(N,G)} \to [0,1]$ such that $K_{\tilde{b}} \tilde{\cap} (M_{\tilde{k}}, L_{\tilde{k}}) \tilde{\subseteq} \omega^{-1}(0)$ and $K_{(N,G)} \tilde{\cap} (K_{(N,G)}, L_{(N,G)}) - (U_{\tilde{k}}, L_{\tilde{k}}) \tilde{\subseteq} \omega^{-1}(1)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Now we introduce the version of fibrewise soft near normal space as follows.

Definition 3.2.14. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is named a fibrewise soft near normal (briefly, F.W.S. j-normal) if for all soft point $\tilde{b}$ of (B, G) and each pair (C, E),(S, E) of disjoint soft closed sets of H, there exists a soft nbd (N, G) of $\tilde{b}$ and a pair of disjoint S. j-open sets (F, E), (V, E) of $H_{(N,G)} \tilde{\cap} (C, E), H_{(N,G)} \tilde{\cap} (S, E)$ in $H_{(N,G)}$, where $j \in \{\alpha, S, P, b, \beta\}$.

Example 3.2.15. Let $H = \{h_1, h_2\}$, $B = \{a, b\}$, $E = \{e_1, e_2\}$, $G = \{g_1, g_2\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_1) = \{a\}$, $f(h_2) = \{b\}$, $u(e_1) = \{g_1\}$, $u(e_2) = \{g_2\}$. $\tilde{\tau} = \{ \tilde{\Phi}, \tilde{H}, (F_1, E)\}$ where (F_1, E) is soft sets over $(H, \tilde{\tau}, E)$, defined as follows : $(F_1, E) = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$, $\tilde{\Omega} = \{\tilde{\Phi}, \tilde{B}, (M, G)\}$,

$(M,G)=\{(g_1,\{a\}),(g_2,\{b\})\}$. It is clear that $H_{(M,G)}=\{(e_1,\{h_1\}),(e_2,\{h_2\})\}$ is soft j-open then the projection function $P_{fu}:(H,\tilde{\tau},E)\to(B,\tilde{\Omega},G)$ be a S. j-continuous, $(C,E)=\{(e_1,\{h_1\})\}$, $(S,E)=\{(e_2,\{h_2\})\}$, $H_{\tilde{a}}=\{\tilde{h}_1\}$, $H_{\tilde{b}}=\{\tilde{h}_2\}$, $H_{\tilde{a}}\tilde{\cap}(C,E)=\{\tilde{h}_1\}$, $H_{\tilde{a}}\tilde{\cap}(S,E)=\tilde{\Phi}$, $H_{\tilde{b}}\tilde{\cap}(C,E)=\tilde{\Phi}$, $H_{\tilde{b}}\tilde{\cap}(S,E)=\{\tilde{h}_2\}$, $(F,E)=\{(e_1\{h_1\})\}$, $(U,E)=\{(e_2,\{h_1\})\}$. Therefore, $(H,\tilde{\tau},E)$ F.W.S. j-normal, where $j\in\{\alpha,S,P,b,\beta\}$.

Remark 3.2.16. If $(H,\tilde{\tau},E)$ is a F.W.S. j-normal space over $(B,\tilde{\Omega},G)$ then $(H_{B^*},\tilde{\tau}_{B^*},E_{B^*})$ is a F.W.S. j-normal space over $\left(B^*,\tilde{\Omega}^*,G^*\right)$ for each soft subspace $\left(B^*,\tilde{\Omega}^*,G^*\right)$ of $\left(B,\tilde{\Omega},G\right)$, where $j\in\{\alpha,S,P,b,\beta\}$.

The soft closed subspaces of F.W.S. j-normal spaces are F.W.S. j-normal, where $j\in\{\alpha,S,P,b,\beta\}$. In fact we have.

Proposition 3.2.17. Assume that $\phi:(H,\tilde{\tau},E)\to(H^*,\tilde{\tau}^*,E^*)$ is a soft closed F.W.S. j-irresolute embedding, where $(H,\tilde{\tau},E)$ and $(H^*,\tilde{\tau}^*,E^*)$ are F.W.S. topological spaces over $(B,\tilde{\Omega},G)$. If $(H^*,\tilde{\tau}^*,E^*)$ is F.W.S. j-normal, then $(H,\tilde{\tau},E)$ is so, where $j\in\{\alpha,S,P,b,\beta\}$.

Proof. Assume that $\tilde{b}$ is a soft point of (B,G) and $(C,E),(S,E)$ are disjoint soft closed sets of H. Then $\phi(C,E)$, $\phi(S,E)$ are disjoint soft closed sets of H^*. Since $(H^*,\tilde{\tau}^*,E^*)$ is F.W.S. j-normal there exists a soft nbd (N,G) of $\tilde{b}$ and a pair of disjoint S. j-open sets $(V,E),(U,E)$ of $H^*_{(N,G)}\tilde{\cap}\phi(C,E)$, $H^*_{(N,G)}\tilde{\cap}\phi(S,E)$. Then $\phi^{-1}(V,E)$ and $\phi^{-1}(U,E)$ are disjoint S. j-open sets of $H_{(N,G)}\tilde{\cap}(C,E),H_{(N,G)}\tilde{\cap}(S,E)$ in $H_{(N,G)}$, where $j\in\{\alpha,S,P,b,\beta\}$.

Definition 3.2.18. A soft function $P_{fu}:(H,E)\to(B,G)$ where $f:H\to B$ and $u:E\to G$ is called a soft near biclosed (briefly, S. j-biclosed) function if the image of every S. j-closed set in H is S. j-closed set in K, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 3.2.19. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a soft j-biclosed continuous fibrewise surjection function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E)$ is F.W.S. j-normal, then $(K, \tilde{\sigma}, L)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that (B, G) has a soft point of $\tilde{b}$ and (C, L), (S, L) are disjoint soft closed sets of $(K, \tilde{\sigma}, L)$. Then $\phi^{-1}(C, L), \phi^{-1}(S, L)$ are disjoint soft closed sets of $(H, \tilde{\tau}, E)$. Since $(H, \tilde{\tau}, E)$ is F.W.S. j-normal there exists a soft nbd (N, G) of $\tilde{b}$ and a pair of disjoint S. j-open sets $(F, E), (U, E)$ of $H_{(N,G)} \tilde{\cap}\ \phi^{-1}(C, L)$ and $H_{(N,G)} \tilde{\cap}\ \phi^{-1}(S, L)$. Since ϕ is soft j-biclosed the sets $\left(K_{(N,G)}, L_{N,G}\right) - \phi\left(\left(H_{(N,G)}, E_{(N,G)}\right) - (F, E)\right)$ and $\left(K_{(N,G)}, L_{(N,G)}\right) - \phi((H_{(N,G)}, E_{(N,G)}) - (U, E))$ are S. j-open in $(K_{(N,G)}, \tilde{\sigma}_{(N,G)}, L_{(N,G)})$, and form a disjoint pair of a S. j-open sets of $K_{(N,G)} \tilde{\cap} (C, E)$, $K_{(N,G)} \tilde{\cap} (S, E)$ in $(K_{(N,G)}, \tilde{\sigma}_{(N,G)}, L_{(N,G)})$, where $j \in \{\alpha, S, P, b, \beta\}$.

Finally, we introduce the version of a fibrewise soft near functionally normal space as follows.

Definition 3.2.20. The F.W.S. topological space $(H, \tilde{\tau}, E)$ over $(B, \tilde{\Omega}, G)$ is called a fibrewise soft near functionally (briefly, F.W.S. j-functionally) normal if for all soft point $\tilde{b}$ of (B, G) and all pair $(C, E), (S, E)$ of disjoint soft closed sets of $(H, \tilde{\tau}, E)$ there exists a soft nbd (N, G) of $\tilde{b}$ and a pair of disjoint S. j-open sets $(F, E), (U, E)$ and a continuous function $\lambda : H_{(N,G)} \to [0,1]$ such that $H_{(N,G)} \tilde{\cap} (C, E) \tilde{\cap} (F, E) \tilde{\subseteq} \lambda^{-1}(0)$ and $H_{(N,G)} \tilde{\cap} (S, E) \tilde{\cap} (U, E) \tilde{\subseteq} \lambda^{-1}(1)$ in $H_{(N,G)}$, where $j \in \{\alpha, S, P, b, \beta\}$.

Example 3.2.21. Let $H = \{h_1, h_2, h_3, h_4\}$, $B = \{a, b\}$, $E = \{e_1, e_2\}$, $G = \{g_1, g_2\}$ and $(H, \tilde{\tau}, E)$ be a F.W.S. topological space over $(B, \tilde{\Omega}, G)$. Define $f : H \to B$ and $u : E \to G$, such that $f(h_1) = f(h_2) = \{a\}$, $f(h_3) = f(h_4) = \{b\}$, $u(e_1) = \{g_1\}$, $u(e_2) = \{g_2\}$. $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, (F_1, E)\}$ where (F_1, E) is soft sets over $(H, \tilde{\tau}, E)$, defined as follows : $(F_1, E) = \{(e_1, \{h_1, h_2\}), (e_2, \{h_3, h_4\})\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \widetilde{B}, (M, G)\}$, $(M, G) = \{(g_1, \{a\}), (g_2, \{b\})\}$. It is clear that $H_{(M,G)} = \{ (e_1, \{h_1, h_2\}), (e_2, \{h_3, h_4\})\}$ is soft j-open in$(H, \tilde{\tau}, E)$ then the projection function $P_{fu} : (H, \tilde{\tau}, E) \to (B, \tilde{\Omega}, G)$ is a S. j-continuous, $(C, E) = \{(e_1, \{h_1, h_2\})\}$, $(S, E) = \{(e_2, \{h_3, h_4\})\}$, $\tilde{b} = \tilde{a}$, soft nbd of $\tilde{a}$ is $(N, G) = (g_1, \{a\})$, $H_{\tilde{a}} = \{\tilde{h}_1, \tilde{h}_3\}$, $\tilde{h} = \tilde{h}_1$, $(F, E) = \{(e_1, \{h_1, h_3\})\}$, $(N, G) = \{(g_1, \{a\})\}$, $H_{(N,G)} = \{\tilde{h}_1, \tilde{h}_2\}$, $\tilde{\tau}_{H_{(N,G)}} = \{\widetilde{\Phi}, \widetilde{H}_{(N,G)}\}$, $(U, E) = \{(e_1, \{h_3, h_4\})\}$ and $(V, E) = \{((e_1, \{h_1, h_2\})\}$. Let $\lambda : H_{(N,G)} \to [0,1]$ such that $\lambda(\tilde{h}_1) = 1 = \lambda(\tilde{h}_2)$. λ is continuous, $H_{(N,G)} \tilde{\cap} (C, E) \tilde{\cap} (U, E) = \widetilde{\Phi} \tilde{\subseteq} \lambda^{-1}(0)$ and $H_{(N,G)} \tilde{\cap} (S, E) \tilde{\cap} (V, E) = \{\tilde{h}_1, \tilde{h}_2\} \tilde{\subseteq} \lambda^{-1}(1)$. Let $\tilde{b} = \tilde{b}$, soft nbd of $\tilde{b}$ is $(N, G) = (g_2, \{b\})$, $H_{(N,G)} = \{\tilde{h}_3, \tilde{h}_4\}$, $\tilde{\tau}_{H_{(N,G)}} = \{\widetilde{H}_{(N,G)}, \widetilde{\Phi}\}$. Let $\lambda : H_{(N,G)} \to [0,1]$ such that $\lambda(\tilde{h}_3) = 1 = \lambda(\tilde{h}_4)$. λ is continuous, $H_{(N,G)} \tilde{\cap} (C, E) \tilde{\cap} (U, E) = \{\tilde{h}_3, \tilde{h}_4\} \tilde{\subseteq} \lambda^{-1}(0)$ and $H_{(N,G)} \tilde{\cap} (S, E) \tilde{\cap} (V, E) = \widetilde{\Phi} \tilde{\subseteq} \lambda^{-1}(1)$. Also, $(H, \tilde{\tau}, E)$ is F.W.S. j-functionally normal, where $j \in \{\alpha, S, P, b, \beta\}$.

Remark 3.2.22. If $(H, \tilde{\tau}, E)$ is F.W.S. j-functionally normal space over $(B, \tilde{\Omega}, G)$, then $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is F.W.S. j-functionally normal space over $(B^*, \tilde{\Omega}^*, G^*)$ for all subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $(B, \tilde{\Omega}, G)$, where $j \in \{\alpha, S, P, b, \beta\}$.

The soft closed subspaces of F.W.S. j-functionally normal spaces are F.W.S. j-functionally normal, where $j \in \{\alpha, S, P, b, \beta\}$. In fact we have.

Proposition 3.2.23. Assume that $\phi : (H, \tilde{\tau}, E) \to (H^*, \tilde{\tau}^*, E^*)$ is a soft closed F.W.S. j-irresolute embedding, where $(H, \tilde{\tau}, E)$ and $(H^*, \tilde{\tau}^*, E^*)$ are F.W.S.

topological spaces over $(B,\tilde{\Omega},G)$. If $(H^*,\tilde{\tau}^*,E^*)$ is F.W.S. j-functionally normal, then $(H,\tilde{\tau},E)$ is so, where $j\in\{\alpha,S,P,b,\beta\}$.

Proof. Assume that $\tilde{b}$ is a soft point of (B,G) and (C,E), (S,E) is disjoint soft closed sets of $(H,\tilde{\tau},E)$. Then $\phi(C,E),\phi(S,E)$ are disjoint soft closed sets of $(H^*,\tilde{\tau}^*,E^*)$. Since $(H^*,\tilde{\tau}^*,E^*)$ is F.W.S. j-functionally normal there exists a soft nbd (N,G) of $\tilde{b}$ and a pair of disjoint S. j-open sets (F,E), (U,E) and a continuous function $\lambda : H^*_{(N,G)} \to[0,1]$ such that $H^*_{(N,G)}\,\tilde{\cap}\,\phi(C,E)\,\tilde{\cap}\,(F,E)\,\tilde{\subseteq}\,\lambda^{-1}(0)$ and $H^*_{(N,G)}\,\tilde{\cap}\,\phi(S,E)\,\tilde{\cap}\,(U,E)\,\tilde{\subseteq}\,\lambda^{-1}(1)$ in $(H^*_{(N,G)},\tilde{\tau}^*_{(N,G)},E^*_{(N,G)})$. Then $\omega=\lambda\circ\phi$ is a continuous function $\lambda : H_{(N,G)}\to[0,1]$ such that $H_{(N,G)}\,\tilde{\cap}\,(C,E)\,\tilde{\cap}\,\phi^{-1}(F,E)\ \omega^{-1}(0)$ and $H_{(N,G)}\,\tilde{\cap}\,(S,E)\,\tilde{\cap}\,\phi^{-1}(U,E)\,\tilde{\subseteq}\,\omega^{-1}(1)$ in $(H_{(N,G)},\tilde{\tau}_{(N,G)},E_{(N,G)})$, where $j\in\{\alpha,S,P,b,\beta\}$, as required.

Definition 3.2.24. A function $P_{fu}:(H,E)\to(B,G)$ where $f:H\to B$ and $u:E\to G$ is called a soft near-biopen (briefly, S. j-biopen) if the image of every S. j-open set in H is S. j-open set in K, where $j\in\{\alpha,S,P,b,\beta\}$.

Proposition 3.2.25. Assume that $\phi:(H,\tilde{\tau},E)\to(K,\tilde{\sigma},L)$ is a soft j-biopen, soft closed and continuous fibrewise surjection function, where $(H,\tilde{\tau},E)$ and $(K,\tilde{\sigma},L)$ are F.W.S. topological spaces over $(B,\tilde{\Omega},G)$. If $(H,\tilde{\tau},E)$ is F.W.S j-functionally normal, then $(K,\tilde{\sigma},L)$ is so, where $j\in\{\alpha,S,P,b,\beta\}$.

Proof. Assume that $\tilde{b}$ is a point of (B,G) and (C,E), (S,E) is disjoint soft closed sets of $(K,\tilde{\sigma},L)$. Then $\phi^{-1}(C,E),\phi^{-1}(S,E)$ are disjoint soft closed sets of $(H,\tilde{\tau},E)$. Since $(H,\tilde{\tau},E)$ is F.W.S. j-functionally normal there exists a soft nbd (N,G) of $\tilde{b}$ and a pair of disjoint S. j-open sets (F,E), (U,E) and a continuous function $\lambda:H_{(N,G)}\to[0,1]$ such that $H_{(N,G)}\,\tilde{\cap}\,\phi^{-1}(C,E)\,\tilde{\cap}\,(F,E)\,\tilde{\subseteq}\,\lambda^{-1}(0)$ and $H_{(N,G)}\,\tilde{\cap}\,\phi^{-1}(S,E)\,\tilde{\cap}\,(U,E)\,\tilde{\subseteq}\,\lambda^{-1}(1)$ in $H_{(N,G)}$. Now a function $\omega: K_{(N,G)}\to[0,1]$ is given by

$$\omega(k) = sup_{h \in \phi^{-1}(k)} \lambda(h)\,;\ k \in K_{(N,G)}.$$

Since ϕ is S. j-biopen and soft closed, as well as it is a continuous, it follows that ω is continuous. Since $K_{(N,G)} \tilde{\cap} (C,E) \tilde{\cap} \phi(F,E) \tilde{\subseteq} \omega^{-1}(0)$ and $K_{(N,G)} \tilde{\cap} (S,E) \tilde{\cap} \phi(U,E) \tilde{\subseteq} \omega^{-1}(1)$ in $H_{(N,G)}$, where $j \in \{\alpha, S, P, b, \beta\}$. This proves the proposition.

3.3 Fibrewise Soft Near Compact (Resp., Locally Soft Near Compact) Spaces and Some Fibrewise Soft Near Separation Axioms

Now we give a series of results in which give relationships between fibrewise soft near compactness (or fibrewise locally soft near compactness in a number of cases) and some fibrewise soft near separation axioms.

Proposition 3.3.1. Assume that $(H, \tilde{\tau}, E)$ is F.W.L.S. j-compact and F.W.S. j-regular over $(B, \tilde{\Omega}, G)$. Then for all soft point $\tilde{h}$ of $H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and all S. j-open set (F, E) of $\tilde{h}$ in $(H, \tilde{\tau}, E)$, there exists a soft open set (U, E) of $\tilde{h}$ in $(H_{(N,G)}, \tilde{\tau}_{(N,G)}, E_{(N,G)})$ such that the closure $H_{(N,G)} \tilde{\cap} Cl(F,E)$ of (F,E) in $(H_{(N,G)}, \tilde{\tau}_{(N,G)}, E_{(N,G)})$ is F.W.S. j-compact over (N, G) and contained in (V, E), where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Since $(H, \tilde{\tau}, E)$ is F .W.L.S. j-compact there exists a soft nbd (N^*, G^*) of $\tilde{b}$ in $(B, \tilde{\Omega}, G)$ and a soft open set (F^*, E^*) of $\tilde{h}$ in $(H_{(N^*,G^*)}, \tilde{\tau}_{(N^*,G^*)}, E_{(N^*,G^*)})$ such that the closure $H^*_{(N,G)} \tilde{\cap} Cl(F^*, E^*)$ of (F^*, E^*) in $(H_{(N^*,G^*)}, \tilde{\tau}_{(N^*,G^*)}, E_{(N^*,G^*)})$ is F.W.S. j-compact over (N^*, G^*). Since is F.W.S. j-regular there exists a soft nbd $(N, G) \tilde{\subseteq} (N^*, G^*)$ of $\tilde{b}$ and a soft open set (F, E) of $\tilde{h}$ in $(H_{(N,G)}, \tilde{\tau}_{(N,G)}, E_{(N,G)})$ such that the closure $H_{(N,G)} \tilde{\cap} Cl(F,E)$ of (F,E) in $(H_{(N,G)}, \tilde{\tau}_{(N,G)}, E_{(N,G)})$ remains contained in $H_{(N,G)} \tilde{\cap} (F^*, E^*) \tilde{\cap} (U, E)$. At this time $H_{(N,G)} \tilde{\cap} Cl(F^*, E^*)$ is F.W.S. j-compact over (N, G), since $H^*_{(N,G)} \tilde{\cap} Cl(F^*, E^*)$ is F.W.S. j-compact over (N^*, G^*), and $H_{(N,G)} \tilde{\cap} Cl(N, G)$

is soft closed in $H_{(N,G)}\ \widetilde{\cap}\ Cl(N^*,G^*)$. Hence $(H_{(N,G)}\ \widetilde{\cap}\ Cl(F,E))$ is F.W.S. j-compact over (N,G) and contained in (U,E), where $j\in\{\alpha,S,P,b,\beta\ \}$, as required.

Proposition 3.3.2. Assume that $\phi:(H,\tilde{\tau},E)\rightarrow(K,\tilde{\sigma},L)$ is a soft open, S. j-irresolute fibrewise surjection function, where $(H,\tilde{\tau},E)$ and $(K,\tilde{\sigma},L)$ are F.W.S. topological spaces over $(B,\tilde{\Omega},G)$. If $(H,\tilde{\tau},E)$ is F.W. L. S. j-compact and F.W.S. j-regular, then so is $(K,\tilde{\sigma},L)$, where $j\in\{\alpha,S,P,b,\beta\}$.

Proof. Assume that $\tilde{k}$ exist a soft point of $K_{\tilde{b}}$, where $\tilde{b}\ \widetilde{\in}\ (B,G)$, and (U,E) is a S. j-open set of $\tilde{k}$ in $(K,\tilde{\sigma},L)$. Choice any soft point $\tilde{h}$ of $\phi^{-1}(\tilde{h})$. Then $\phi^{-1}(U,E)$ is a S. j-open set of $\tilde{h}$ in $(H,\tilde{\tau},E)$. Since $(H,\tilde{\tau},E)$ is F.W. L. S. j-compact there exists a soft nbd (N,G) of $\tilde{b}$ in $(B,\tilde{\Omega},G)$ and a soft open set (F,E) of $\tilde{h}$ in $H_{(N,G)}$ such that the closure $H_{(N,G)}\ \widetilde{\cap}\ Cl(F,E)$ of (F,E) in $H_{(N,G)}$ is F.W.S. j-compact over (N,G) and is contained in $\phi^{-1}(U,E)$. Then $\phi(F,E)$ is a soft open set of $\tilde{k}$ in $K_{(N,G)}$, since ϕ is a soft open, and the closure $K_{(N,G)}\ \widetilde{\cap}\ Cl(\phi(F,E))$ of $\phi(F,E)$ in $K_{(N,G)}$ is F.W.S. j-compact over (N,G) and contained in (N,G) , where $j\in\{\alpha,S,P,b,\beta\}$, as required.

Proposition 3.3.3. Let $(H,\tilde{\tau},E)$ be F.W.S. locally j-compact and F.W.S. j-regular over $\big(B,\tilde{\Omega},G\big)$. Let C is a soft compact subset of $H_{\tilde{b}}$, where $\tilde{b}\ \widetilde{\in}\ (B,G$), and (U,E) is a S. j-open set of C in $(H,\tilde{\tau},E)$. Then there exists a soft nbd (N,G) of $\tilde{b}$ in $\big(B,\tilde{\Omega},G\big)$ and a soft open set (F,E) of C in $\big(H_{(N,G)},\tilde{\tau}_{(N,G)},E_{(N,G)}\big)$ such that the soft closure $H_{(N,G)}\ \widetilde{\cap}\ Cl(F,E)$ of (F,E) in $H_{(N,G)}$ is F.W.S. j-compact over (N,G) and contained in (U,E), where $j\in\{\alpha,S,P,b,\beta\}$.

Proof. Since $(H,\tilde{\tau},E)$ is F.W.L.S. j-compact there exists for all soft point $\tilde{h}$ of C a soft nbd $(N_{\tilde{h}},G_{\tilde{h}})$ of $\tilde{b}$ in $(B,\tilde{\Omega},G)$ and a soft open set $(F_{\tilde{h}},E_{\tilde{h}})$ of $\tilde{h}$ in

$H_{(N_{\tilde{h}},G_{\tilde{h}})}$ such that the closure $H_{(N_{\tilde{h}},G_{\tilde{h}})} \cap Cl(F_{\tilde{h}}, E_{\tilde{h}})$ of $(F_{\tilde{h}}, E_{\tilde{h}})$ in $H_{(N_{\tilde{h}},G_{\tilde{h}})}$ is F.W.S. j-compact over $(N_{\tilde{h}}, G_{\tilde{h}})$ and contained in (U, E). The family $\{(F_{\tilde{h}}, E_{\tilde{h}}); \tilde{h} \tilde{\in} C\}$ founds a soft covering of the soft compact C using soft open sets of H. Extract a finite soft subcovering indexed by $\tilde{h}_1, \dots, \tilde{h}_n$, say. Take (N, G) to be the intersection $(N_{\tilde{h}_1}, G_{\tilde{h}_1}) \tilde{\cap} \dots \tilde{\cap} (N_{\tilde{h}_n}, G_{\tilde{h}_n})$, and take (F, E) to be the restriction to $H_{(N,G)}$ of the union $(F_{\tilde{h}_1}, E_{\tilde{h}_1}) \tilde{\cup} \dots \tilde{\cup} (F_{\tilde{h}_n}, E_{\tilde{h}_n})$. Then (N, G) is a soft nbd of $\tilde{b}$ in $(B, \tilde{\Omega}, G)$ and (F, E) is a soft open set of C in $H_{(N,G)}$ such that the soft closure $H_{(N,G)} \tilde{\cap} Cl(F, E)$ of (F, E) in $H_{(N,G)}$ is F.W.S. j-compact over (N, G) and contained in (U, E), where $j \in \{\alpha, S, P, b, \beta\}$, as required.

Proposition 3.3.4. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-proper, S. j-irresolute fibrewise surjection function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E)$ is F.W.S. locally j-compact and F.W.S. j-regular, then $(K, \tilde{\sigma}, L)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{k} \tilde{\in} K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and (U, E) is a S. j-open set of $\tilde{k}$ in $(K, \tilde{\sigma}, L)$. Then $\phi^{-1}(U, E)$ is a S. j-open set of $\phi^{-1}(\tilde{k})$ in $(H, \tilde{\tau}, E)$. suppose that $(H, \tilde{\tau}, E)$ is a F.W.L.S. j-compact. Since $\phi^{-1}(\tilde{k})$ soft compact, using Proposition (3.3.3) there exists a soft nbd (N, G) of $\tilde{b}$ in $(B, \tilde{\Omega}, G)$ and a soft open set (F, E) of $\phi^{-1}(\tilde{k})$ in $H_{(N,G)}$ such that the closure $H_{(N,G)} \tilde{\cap} Cl(F, E)$ of (F, E) in $H_{(N,G)}$ is F.W.S. j-compact over (N, G) and contained in $\phi^{-1}(U, E)$. Since ϕ is soft closed there exists a soft open set (F^*, E^*) of $\tilde{k}$ in $K_{(N,G)}$ such that $\phi^{-1}(F^*, E^*) \tilde{\subset} (F, E)$. Then the closure $K_{(N,G)} \tilde{\cap} Cl(F^*, E^*)$ of (F^*, E^*) in $K_{(N,G)}$ is contained in $\phi(K_{(N,G)} \tilde{\cap} Cl(F, E))$ and so is F.W.S. j-compact over (N, G). Since $K_{(N,G)} \tilde{\cap} Cl(F^*, E^*)$ is contained in (U, E) this proves that $(K, \tilde{\sigma}, L)$ is a F.W.S. locally j-compact, where $j \in \{\alpha, S, P, b, \beta\}$, as asserted.

Proposition 3.3.5. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-countinuous fibrewise function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(K, \tilde{\sigma}, L)$ is F.W.S. j-Hausdorff, then the fibrewise soft graph $G : (H, \tilde{\tau}, E) \to (H, \tilde{\tau}, E) \times_B (K, \tilde{\sigma}, L)$ of ϕ is a soft closed embedding, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 3.3.6. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-continuous fibrewise function, where $(H, \tilde{\tau}, E)$ is F.W.S. j-compact space and $(K, \tilde{\sigma}, L)$ is F.W.S. Hausdorff space over $(B, \tilde{\Omega}, G)$. Then ϕ is S. j-proper, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Consider the figure shown below, where r is the standard F.W.S. topological equivalence and G is the fibrewise soft graph of ϕ .

$$\begin{tikzcd} H \arrow[r, "G"] \arrow[d, "\phi"'] & H \times_B K \arrow[d, "P \times id_K"] \\ K \arrow[r, "r"] & B \times_B K \end{tikzcd}$$

Fig. (3.3.1): Diagram of Proposition (3.3.6).

Now G soft closed embedding, by Lemma (3.3.5), since $(K, \tilde{\sigma}, L)$ is F.W.S. Hausdorff. So G is S. j-proper. And P_{fu} is S. j-proper and so $P_{fu} \times id_K$ is S. j-proper. Therefore $(P \times id_K) \circ G = r \circ \phi$ is S. j-proper and so ϕ is S. j-proper, since r is a F.W.S. topological equivalence, where $j \in \{\alpha, S, P, b, \beta\}$.

Corollary 3.3.7. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-continuous fibrewise injection, where $(H, \tilde{\tau}, E)$ is F.W.S. j-compact space and $(K, \tilde{\sigma}, L)$ is F.W.S. Hausdorff space over $(B, \tilde{\Omega}, G)$. Then ϕ is soft closed embedding, when $j \in \{\alpha, S, P, b, \beta\}$.

The corollary is often used in the case when ϕ is surjective to show that ϕ is a F.W.S.topological equivalence.

Proposition 3.3.8. Assume that $\phi : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ is a S. j-proper fibrewise surjection function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E)$ is a F.W.S. Hausdorff, then $(K, \tilde{\sigma}, L)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Since ϕ is a j-proper surjection function so is $\phi \times \phi$, in the following figure below. The diagonal $\Delta(H, \tilde{\tau}, E)$ is soft closed, since $(H, \tilde{\tau}, E)$ is a F.W.S. Hausdorff, hence $\big((\phi \times \phi) \circ \Delta\big)(H, \tilde{\tau}, E) = (\Delta \circ \phi)(H, \tilde{\tau}, E)$ is a soft closed. But $(\Delta \circ \phi)(H, \tilde{\tau}, E) = \Delta(K, \tilde{\sigma}, L)$, since ϕ is surjective function, and so $(K, \tilde{\sigma}, L)$ is a F.W.S. Hausdorff, where $j \in \{\alpha, S, P, b, \beta\}$.

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\phi"'] & H \times_B H \arrow[d, "\phi \times \phi"] \\
K \arrow[r, "\Delta"] & H \times_B K
\end{tikzcd}$$

Figure 3.3.2: Diagram of Proposition 3.3.8.

Proposition 3.3.9. Assume that $(H, \tilde{\tau}, E)$ is a F.W.S. j-compact and F.W.S. Hausdorff space over $(B, \tilde{\Omega}, G)$. Then $(H, \tilde{\tau}, E)$ is a F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and (F, E) is a S. j-open set of $\tilde{h}$ in $(H, \tilde{\tau}, E)$. Since $(H, \tilde{\tau}, E)$ is a F.W.S. Hausdorff there exists for all soft point $\tilde{h}^* \tilde{\in} H_{\tilde{b}}$ such that $\tilde{h}^* \tilde{\notin} (F, E)$ a soft open set $(U_{\tilde{h}^*}, E_{\tilde{h}^*})$ of $\tilde{h}$ and a soft open set $(U^*_{\tilde{h}^*}, E^*_{\tilde{h}^*})$ of $\tilde{h}^*$ which do not intersect. Now the family of soft open sets $(U^*_{\tilde{h}^*}, E^*_{\tilde{h}^*})$, for $\tilde{h}^* \tilde{\in} ((H, E) - (F, E))_{\tilde{b}}$, forms a soft covering of $((H, E) - (F, E))_{\tilde{b}}$. Since $((H, E) - (F, E))$ is S. j-closed in $(H, \tilde{\tau}, E)$ and therefore F.W.S. j-compact there exists, by Proposition (2.3.4), a soft nbd

(N,G) of $\tilde{b}$ in (B,G) such that $H_{(N,G)} - (H_{(N,G)} \tilde{\cap} (F,E))$ is soft covered by a finite subfamily, indexed by $\tilde{h}_1^*, \dots, \tilde{h}_n^*$, say. Then the intersection $(U,E) = \left(U_{\tilde{h}_1}^*, E_{\tilde{h}_1}^*\right) \tilde{\cap} \dots \tilde{\cap} (U_{\tilde{h}_n}^*, E_{\tilde{h}_n}^*)$ is a soft open set of $\tilde{h}$ which does not meet the open set $(U^*, E^*) = ({U^*}_{\tilde{h}_1}^*, {E^*}_{\tilde{h}_1}^*) \tilde{\cup} \dots \tilde{\cup} ({U^*}_{\tilde{h}_n}^*, {E^*}_{\tilde{h}_n}^*)$ of $\left(H_{(N,G)}, E_{(N,G)}\right) - ((H_{(N,G)}, E_{(N,G)}) \tilde{\cap} (F,E))$. Therefore the soft closure $H_{(N,G)} \tilde{\cap} Cl(U,E)$ of $H_{(N,G)} \tilde{\cap} (U,E)$ in $(H_{(N,G)}, E_{(N,G)})$ is contained in (F,E), where $j \in \{\alpha, S, P, b, \beta\}$, equally asserted.

We extend this last result to.

Proposition 3.3.10. Assume that $(H, \tilde{\tau}, E)$ is a F.W.S. locally j-compact and F.W.S. Hausdorff space over $(B, \tilde{\Omega}, G)$. Then $(H, \tilde{\tau}, E)$ is a F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $\tilde{h} \tilde{\in} H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B,G)$, and (U,E) is a S. j-open set of $\tilde{h}$ in $(H, \tilde{\tau}, E)$. Assume that (N,G) is a soft nbd of $\tilde{b}$ in (B,G) and (F,E) is a soft open set of $\tilde{b}$ in $H_{(N,G)}$ such that the soft closure $H_{(N,G)} \tilde{\cap} Cl(F,E)$ of (F,E) in $(H_{(N,G)}, \tilde{\tau}_{(N,G)}, E_{(N,G)})$ is a F.W.S. j-compact over $(B, \tilde{\Omega}, G)$. Then $H_{(N,G)} \tilde{\cap} Cl(F,E)$ is a F.W.S. j-regular over (N,G), by Proposition (3.3.9), since $H_{(N,G)} \tilde{\cap} Cl(F,E)$ is a F.W.S. Hausdorff over (N,G). So there exists a soft nbd $(N^*, G^*) \tilde{\cap} (N,G)$ of $\tilde{b}$ in (B,G) and a soft open set (F^*, E^*) of $\tilde{h}$ in $H_{(N^*,G^*)}$ such that the soft closure $H_{(N^*,G^*)} \tilde{\cap} Cl(F^*, E^*)$ of (F^*, E^*) in $H_{(N^*,G^*)}$ is contained in $(F,E) \tilde{\cap} (U,E) \tilde{\subseteq} (U,E)$, where $j \in \{\alpha, S, P, b, \beta\}$, as required.

Proposition 3.3.11. Assume that $(H, \tilde{\tau}, E)$ is a F.W.S. j-regular space over $(B, \tilde{\Omega}, G)$ and $(Y, \tilde{\delta}, D)$ is a F.W.S. j-compact subset of $(H, \tilde{\tau}, E)$. Assume that $\tilde{b}$ is a soft point of (B,G) and (U,E) is a S. j-open set of $Y_{\tilde{b}}$ in $(H, \tilde{\tau}, E)$. Then there exists a soft nbd (N,G) of $\tilde{b}$ in (B,G) and a soft open set (F,E) of

$Y_{(N,G)}$ in $H_{(N,G)}$ such that the soft closure $H_{(N,G)} \tilde{\cap}\, Cl(F,E)$ of (F,E) in $H_{(N,G)}$ is contained in (U,E), where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. We may suppose that $Y_{(N,G)}$ is non-empty since otherwise we can take $(F,E) = H_{(N,G)}$, where $(N,G) = (B,G) - P_{fu}((H,E) - (U,E))$. Since (U,E) is a S. j-open set of each soft point $\tilde{h}$ of $Y_{\tilde{b}}$ there exists, by F.W.S. j-regularity, a soft nbd $(N_{\tilde{h}}, G_{\tilde{h}})$ of $\tilde{b}$ and a soft open set $(F_{\tilde{h}}, E_{\tilde{h}}) \tilde{\cap}\, H_{(N_{\tilde{h}}, G_{\tilde{h}})}$ of $\tilde{h}$ such that the soft closure $H_{(N_{\tilde{h}}, G_{\tilde{h}})} \tilde{\cap}\, Cl(F_{\tilde{h}}, E_{\tilde{h}})$ of $(F_{\tilde{h}}, E_{\tilde{h}})$ in $H_{(N_{\tilde{h}}, G_{\tilde{h}})}$ is contained in (U,E). The family of soft open sets $\{H_{(N_{\tilde{h}}, G_{\tilde{h}})} \tilde{\cap}\, (F_{\tilde{h}}, E_{\tilde{h}}); \tilde{h} \tilde{\in} Y_{\tilde{b}}\}$ soft covers $Y_{\tilde{b}}$ and so there exists a soft nbd (N^*, G^*) of $\tilde{b}$ and a finite subfamily indexed by $\tilde{h}_1, \dots, \tilde{h}_n$, say, which soft covers $Y_{\tilde{b}}$. Then the conditions are satisfied with $(N,G) = (N^*, G^*) \tilde{\cap}\, (N_{\tilde{h}_1}, G_{\tilde{h}_1}) \tilde{\cap} \dots \tilde{\cap}\, (N_{\tilde{h}_n}, G_{\tilde{h}_n})$, $(F,E) = \left(F_{\tilde{h}_1}, E_{\tilde{h}_1}\right) \tilde{\cup} \dots \tilde{\cup}\, (F_{\tilde{h}_n}, E_{\tilde{h}_n})$.

Corollary 3.3.12. Assume that $(H, \tilde{\tau}, E)$ is a F.W.S. j-compact and F.W.S. j-regular space over $(B, \tilde{\Omega}, G)$. Then $(H, \tilde{\tau}, E)$ is F.W.S. j-normal, where $j \in \{\alpha, ,S, P, b, \beta\}$.

Proposition 3.3.13. Assume that $(H, \tilde{\tau}, E)$ is a F.W.S. j-regular space over $(B, \tilde{\Omega}, G)$ and $(Y, \tilde{\delta}, D)$ is a F.W.S. j-compact subset of $(H, \tilde{\tau}, E)$. Assume that $\{(U_i, E_i)\, ;\, i = 1, \dots, n\}$ is a soft covering of $Y_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B,G)$ by S. j-open sets of $(H, \tilde{\tau}, E)$. Then there exists a soft nbd (N,G) of $\tilde{b}$ and a soft covering $\{(F_i, E_i); i = 1, \dots, n\}$ of $Y_{(N,G)}$ by soft open sets of $H_{(N,G)}$ such that the soft closure $H_{(N,G)} \tilde{\cap}\, Cl(F_i, E_i)$ of (F_i, E_i) in $H_{(N,G)}$ is contained in (U_i, E_i) for all i, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Write $(U,E) = (U_2, E_2) \tilde{\cup} \dots \tilde{\cup}\, (U_n, E_n)$, so that $(H,E) - (U,E)$ is S. j-closed in $(H, \tilde{\tau}, E)$. Hence $(Y, \tilde{\delta}, D) \tilde{\cap}\, ((H,E) - (U,E))$ is S. j-closed in $(Y, \tilde{\delta}, D)$ and so F.W.S. j-compact. Applying the preceding consequence to the S. j-open set (U_1, E_1) of $Y_{\tilde{b}} \tilde{\cap} ((H_{\tilde{b}}, E_{\tilde{b}}) - (U_{\tilde{b}}, E_{\tilde{b}}))$ we get a soft nbd (N, G) of $\tilde{b}$ and a soft open set (F, E) of $Y_{(N,G)} \tilde{\cap} \left(H_{(N,G)}, E_{(N,G)}\right) - (U_{(N,G)}, E_{(N,G)}))$ such that $H_{(N,G)} \tilde{\cap}\, Cl(F, E) \tilde{\subseteq} (U_1, E_1)$. Now $(Y, \tilde{\delta}, D) \tilde{\cap} (U, E)$ and $(Y, \tilde{\delta}, D) \tilde{\cap} ((H, E) - (U, E))$ soft cover Y, hence (U, E) and (F, E) soft cover $Y_{(N,G)}$. Thus $(F, E) = (F_1, E_1)$ is the first step in the shrinking process. We continue by repeating the argument for $\{(F_1, E_1), (U_2, E_2), \ldots, (U_n, E_n)\}$, so as to shrink (U_2, E_2), and so on, where $j \in \{\alpha, S, P, b, \beta\}$. Hence the result is obtained.

Proposition 3.3.14. Assume that $\phi : (H, \tilde{\tau}, E) \to (K, \tilde{\sigma}, L)$ is a S. j-proper, S. j-irresolute fibrewise surjection function, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are F.W.S. topological spaces over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E)$ is F.W.S. j-regular, then $(K, \tilde{\sigma}, L)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Assume that $(H, \tilde{\tau}, E)$ is F.W.S. j-regular. Assume that $\tilde{k}$ is a soft point of $K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$, and (U, E) is a S. j-open set of $\tilde{k}$ in $(K, \tilde{\sigma}, L)$. Then $\phi^{-1}(U, E)$ is a S. j-open set of the soft compact $\phi^{-1}(\tilde{k})$ in $(H, \tilde{\tau}, E)$. Using Proposition (3.3.13), therefore, there exists a soft nbd (N, G) of $\tilde{b}$ in $(B, \tilde{\Omega}, G)$ and a soft open set (F, E) of $\phi^{-1}(\tilde{k})$ in $H_{(N,G)}$ such that the soft closure $H_{(N,G)} \tilde{\cap}\, Cl(F, E)$ of (F, E) in $H_{(N,G)}$ is contained in $\phi^{-1}(U, E)$. Now since $\phi_{(N,G)}$ is a soft closed there exists a soft open set (U^*, E^*) of $\tilde{k}$ in $K_{(N,G)}$ such that $\phi^{-1}(U^*, E^*) \tilde{\subseteq} (F, E)$, and then the soft closure $H_{(N,G)} \tilde{\cap}\, Cl(U^*, E^*)$ of (U^*, E^*) in $H_{(N,G)}$ is contained in (U, E) since $Cl(U^*, E^*) = Cl(\phi(\phi^{-1}(U^*, E^*)) = \phi(Cl(\phi^{-1}(U^*, E^*))) \tilde{\subseteq} \phi(Cl(F, E)) \tilde{\subseteq} \phi(\phi^{-1}(U, E)) \tilde{\subseteq} (U, E)$. Thus $(K, \tilde{\sigma}, L)$ is a F.W.S. j-regular, where $j \in \{\alpha, S, P, b, \beta\}$, as asserted.

Chapter 4

Fibrewise Soft Ideal Topological Spaces

Chapter 4

Fibrewise Soft Ideal Topological spaces

In this chapter, we shall study the fibrewise soft ideal topological spaces over B, we discuss relation between fibrewise soft ideal topological spaces over B and fibrewise soft topological spaces over B. In section one, We offer a new concept of fibrewise soft ideal topological spaces over B and distinguish it from fibrewise soft topological spaces over B. Some basic properties of this spaces are investigated. In section two, we introduced the concepts of the fiberwise soft ideal closed and a soft ideal open topological over B. Many of the propositions concerning with these concepts are provided. In section three, we define and study new notions of fibrewise soft ideal topological spaces over B namely fibrewise soft near ideal topological spaces over B and counter examples are given to illustrate these notions .

4.1. Fibrewise Soft Ideal Topological Spaces

In this section, we introduced a definition of fibrewise soft ideal topology and its related properties.

Definition 4.1.1. Assume that $(B, \tilde{\Omega}, G)$ is a soft topology . The fibrewise soft ideal topological space (briefly, F.W.S.I-topological space) on a fibrewise set H over B mean any soft ideal topology on H for which the projection function is S.I-continuous.

The F.W.S.I-topological space and F.W.S. topological space are independent and the following examples show that.

Example 4.1.2. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S. I-topological space over $(B, \tilde{\Omega}, G)$ given as follows : $H = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, \{(e, \{h_3\})\}, \{(e, \{$

$h_1, h_2\})\}, \{(e, \{h_1, h_2, h_3\})\}\}$, $I = \{\tilde{\Phi}, \{(e, \{h_1\})\}\}$, $B = \{a, b, c, d\}$, $G = \{g\}$, $\tilde{\Omega} = \{\tilde{\Phi}, \tilde{B}, (M, G)\}$ and $(M, G) = \{(g, \{b, c, d\})\}$. Furthermore, define $f: H \to B$, such that $f(h_1) = \{a\}$, $f(h_2) = \{b\}$, $f(h_3) = \{c\}$, $f(h_4) = \{d\}$ and $u : E \to G$, $u(e) = \{g\}$. It is clear that $H_{(M,G)} = \{(e, \{h_1, h_2, h_3\})\}$ is S.I-open in $(H, \tilde{\tau}, E, I)$ but not soft open in $(H, \tilde{\tau}, E)$, then the soft function $P_{fu}: (H, \tilde{\tau}, E,\ I) \to (B, \tilde{\Omega}, G)$ is a S. I-continuous but is not a soft continuous. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S. I-topological space but not a F.W.S. topological space.

Example 4.1.3. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S. I-topological space over $(B, \tilde{\Omega}, G)$ given as follows : $H = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$, $\tilde{\tau} = \{\tilde{\Phi}, \tilde{H}, \{(e, \{h_4\})\}, \{(e, \{h_1, h_3\})\}, \{(e, \{h_1, h_3, h_4\})\}\}$, $I = \{\tilde{\Phi}, \{(e, \{h_3\})\}, \{e, \{h_4\})\}, \{(e, \{h_3, h_4\})\}\}$, $B = \{a, b, c, d\}$, $G = \{g\}$, $\tilde{\Omega} = \left\{\tilde{\Phi}, \tilde{B}, (M, G)\right\}$ and $(M, G) = \{(g, \{a, c, d\})\}$. Furthermore, define $f : H \to B$, such that $f(h_1) = \{a\}$, $f(h_2) = \{b\}$, $f(h_3) = \{c\}$, $f(h_4) = \{d\}$ and $u : E \to G$, $u(e) = \{g\}$. It is clear that $H_{(M,G)} = \{(e, \{h_1, h_3, h_4\})\}$ is soft open in $(H, \tilde{\tau}, E)$ but not S.I-open in $(H, \tilde{\tau}, E, I)$, then the projection function $P_{fu}: (H, \tilde{\tau}, E,\ I) \to (B, \tilde{\Omega}, G)$ is a soft continuous but is not a S.I-continuous. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S. topological space but not a F.W.S.I-topological space.

Proposition 4.1.4. The following statmants are equivalent.

(a) $(H, \tilde{\tau}, E, I)$ is a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$,

(b) For each soft element in $(H, \tilde{\tau}, E)$ and each soft open in $(B, \tilde{\Omega}, G)$ containing the image of a soft element, there exist an S.I-open of $(H, \tilde{\tau}, E, I)$ containing soft element such that the image of an S.I-open containing in soft open,

(c) For each soft element in $(H, \tilde{\tau}, E)$ and each soft open in $(B, \tilde{\Omega}, G)$ containing the image of a soft element, $(H_{(M,G)})^*$ is nbd of soft element.

Proof . (a) ⇒ (b) Suppose that $(H,\tilde{\tau},E,I)$ is a F.W.S.I-topological space over $(B,\tilde{\Omega},G)$, then the projection function $P_{fu}:(H,\tilde{\tau},E,I)\to(B,\tilde{\Omega},G)$ is S.I-continuous. To proof for each soft element in $(H,\tilde{\tau},E)$ and each soft open in $(B,\tilde{\Omega},G)$ containing the image of a soft element, there exist an S.I-open of $(H,\tilde{\tau},E,I)$ containing soft element such that the image of an S.I-open containing in a soft open, since $(M,G)\tilde{\in}\Omega$ containing $P_{fu}(\tilde{h})$, then by (a) $H_{(M,G)}$ is an S.I-open in H. By taking $(F,E)=H_{(M,G)}$ which containing $P_{fu}(\tilde{h})$, thus $P_{fu}(F,E)\tilde{\subset}(M,G)$.

(b) ⇒(c) Since the soft open in $(B,\tilde{\Omega},G)$ containing the image of a soft element, then by (b) there exists (F,E) is an S.I -open of $(H,\tilde{\tau},E,I)$ containing $P_{fu}(\tilde{h})$, such that $P_{fu}(F,E)\tilde{\subset}(M,G)$. So $\tilde{h}\tilde{\in}(F,E)\tilde{\subset}int(F,E)^*\tilde{\subset}int(H_{(M,G)})^*\tilde{\subset}(H_{(M,G)})^*$. Hence $(H_{(M,G)})^*$ is a nbd of $\tilde{h}$.

(c)⇒(a) Since the soft open in $(B,\tilde{\Omega},G)$ containing the image of a soft element, then by (c) $(H_{(M,G)})^*$ is a soft nbd of $\tilde{h}$, then there exist a soft set $(F,E)\tilde{\in}\tilde{\tau}$ such that $\tilde{h}\tilde{\in}(F,E)\tilde{\subset}(H_{(M,G)})^*$, then $H_{(M,G)}$ is an S.I-open, then P_{fu} is S.I-continuous, thus $(H,\tilde{\tau},E,I)$ is a F.W.S.I. topological spaces.

Proposition 4.1.5. The $(H,\tilde{\tau},E,I)$ is a F.W.S.I-topological space over $(B,\tilde{\Omega},G)$ if and only if the graph soft function $g:(H,\tilde{\tau},E)\to(H,\tilde{\tau},E)\times(B,\tilde{\Omega},G)$, defined by $g(\tilde{h})=(\tilde{h},f(\tilde{h}))$, for each $\tilde{h}\tilde{\in}(H,E)$, is an S.I-continuous.

Proof. (⇒) Suppose that $(H,\tilde{\tau},E,I)$ is a F.W.S.I-topological space over $(B,\tilde{\Omega},G)$, then the projection function $P_{fu}:(H,\tilde{\tau},E,I)\to(B,\tilde{\Omega},G)$ is an S.I-continuous. To show that g is S.I-continuous, let $\tilde{h}\tilde{\in}(H,E)$ and $(W,E\times G)$ be any soft open of $H\times B$ containing $g(\tilde{h})=(\tilde{h},f(\tilde{h}))$. Thus we have a basic soft open set $(F,E)\times(M,G)$ such that $g(\tilde{h})=\left(\tilde{h},f(\tilde{h})\right)\tilde{\in}(F,E)\times(M,G)\tilde{\subseteq}(W,E\times G)$. Since $(H,\tilde{\tau},E,I)$ is a F.W.S.I-topological space, then

$P_{fu}$is an S.I-continuous, we have an S.I-open set (L,E) of $(H,\tilde{\tau},E)$ containing $(\tilde{h})$ such that $P_{fu}(L,E)\,\tilde{\subseteq}\,(M,G)$. Because $(L,E)\,\tilde{\cap}\,(F,E)$ is a soft ideal open of $(H,\tilde{\tau},E,I)$ and $(L,E)\,\tilde{\cap}\,(F,E)\,\tilde{\subseteq}\,(F,E)$ then $g((F,E)\,\tilde{\cap}\,(L,E))\,\tilde{\subseteq}\,(F,E)\times(M,G)\,\tilde{\subseteq}\,(W,E\times G)$. This explain that g is S.I-continuous.

($\Leftarrow$) Suppose that g is S.I-continuous. To show that $(H,\tilde{\tau},E,I)$ is F.W.S.I-topological spaces i.e., the projection function $P_{fu}:(H,\tilde{\tau},E,\ I)\to\left(B,\tilde{\Omega},G\right)$ is S.I-continuous, let $\tilde{h}\,\tilde{\in}\,(H,E)$ and (M,G) be any soft open set of $(B,\tilde{\Omega},G)$ containing $P_{fu}(\tilde{h})$. Then $H\times(M,G)$ is a soft open in $(H,\tilde{\tau},E)\times\left(B,\tilde{\Omega},G\right)$. Because g is S.I-continuous, we have (F,E) is an S.I-open in $(H,\tilde{\tau},E,I)$ containing $\tilde{h}$ such that $g(F,E)\,\tilde{\subseteq}\,H\times(M,G)$. Hence, we get $P_{fu}(F,E)\,\tilde{\subseteq}\,(M,G)$. This explains that P_{fu} is S.I-continuous. Thus $(H,\tilde{\tau},E,I)$ is a F.W.S.I-topological space.

Proposition 4.1.6. If $(H,\tilde{\tau},E,I)$ is a F.W.S. I-topological space over $(B,\tilde{\Omega},G)$. Then $(H_{B^*},\tilde{\tau}_{B^*},E_{B^*},I_{B^*})$ is a F.W.S. I-topological space over $(B^*,\tilde{\Omega}^*,G^*)$ for each open subspace $(B^*,\tilde{\Omega}^*,G^*)$ of $\left(B,\tilde{\Omega},G\right)$.

Proof. suppose that $(H,\tilde{\tau},E,I)$ is a F.W.S.I-topological space over $\left(B,\tilde{\Omega},G\right)$ then there exist the projection function $P_{fu}:(H,\tilde{\tau},E,I)\to\left(B,\tilde{\Omega},G\right)$ is S.I-continuous. To show that $(H_{B^*},\tilde{\tau}_{B^*},E_{B^*},I_{B^*})$ is F.W.S. I-topological over $(B^*,\tilde{\Omega}^*,G^*)$ i.e., the projection function $P_{B^*(fu)}:(H_{B^*},\tilde{\tau}_{B^*},E_{B^*},I_{B^*})\to(B^*,\tilde{\Omega}^*,G^*)$ is S.I-continuous. Assume (M,G) is a soft open subset of $\left(B,\tilde{\Omega},G\right)$. Then $(M,G)\,\tilde{\cap}\,(B^*,G^*)$ is soft open in subspace $\left(B^*,\tilde{\Omega}^*,G^*\right)$ and so $H_{(M,G)}\,\tilde{\subseteq}\,int(H_{(M,G)})^*$, then $H_{(M,G)}\,\tilde{\cap}\,(H_{B^*},E_{B^*})\,\tilde{\subseteq}\,int\left(H_{(M,G)}\right)^*\,\tilde{\cap}\,(H_{B^*},E_{B^*})$. This $(H_{B^*},E_{B^*})_{(M,G)}=H_{(M,G)}\,\tilde{\cap}\,(H_{B^*},E_{B^*})\,\tilde{\subseteq}\,(H_{B^*},E_{B^*})_{(M,G)}\,\tilde{\subseteq}\,int((H_{B^*},,E_{B^*})\,\tilde{\cap}\,(H_{(M,G)}))^*=int((H_{B^*},E_{B^*})_{(M,G)})^*$. This we have that $(H_{B^*},E_{B^*})_{(M,G)}$ is S.I-open of $(H_{B^*},\tilde{\tau}_{B^*},E_{B^*},I_{B^*})$. This shows that $P_{B^*(fu)}$ is S.I-continuous, then $(H_{B^*},\tilde{\tau}_{B^*},E_{B^*},I_{B^*})$ is a F.W.S.I-topological space.

Proposition 4.1.7. Assume that $(H, \tilde{\tau}, E, I)$ is a F.W.S. I-topological space over $(B, \tilde{\Omega}, G)$ and for all member (H_i, E_i) of a infinite soft open covering of $(H, \tilde{\tau}, E)$. Then $(H_{i_{B^*}}, E_{i_{B^*}}, I_{i_{B^*}})$ is a F.W.S.I-topological spaces over $(B^*, \tilde{\Omega}^*, G^*)$ for each open subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $\left(B, \tilde{\Omega}, G\right)$.

Proof. The proof is like to previous proposition.

Definition 4.1.8. A F.W.S.I-topological space $(H, \tilde{\tau}, E, I)$ over $\left(B, \tilde{\Omega}, G\right)$ is called fibrewise soft I-irresolute (briefly, F.W.S. I-irresolute) if the projection function P_{fu} is a soft I-irresolute.

Proposition 4.1.9. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$ and $(H_{(M,G)^*}) \tilde{\subseteq} (H_{(M,G)})^*$ for each soft subset (M, G) of $\left(B, \tilde{\Omega}, G\right)$. Then $(H, \tilde{\tau}, E, I)$ is a F.W.S.I-irresolute.

Proof. Suppose that $(H, \tilde{\tau}, E, I)$ is a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$, then there exist the projection function $P_{fu} : (H, \tilde{\tau}, E, I) \to \left(B, \tilde{\Omega}, G\right)$ is an S.I-continuous. To show that $(H, \tilde{\tau}, E, I)$ is a F.W.S.I-irresolute, i.e., there exist the projection function $P_{fu} : (H, \tilde{\tau}, E, I) \to \left(B, \tilde{\Omega}, G, J\right)$ is an S. I-irresolute. Let (M, G) be any S. I-open set of $(B, \tilde{\Omega}, G, J)$. By Theorem (1.3.27), we have $(M, G)^* = (int(M, G)^*)^*$. Therefore, we have $H_{(M,G)^*} = H_{(int(M,G)^*)^*}$, such that $(H_{(M,G)})^* \tilde{\subseteq} (H_{int(M,G)^*})^*$, $H_{(M,G)} \tilde{\subseteq} H_{int(M,G)^*}$. so $H_{(M,G)}$ is S. I-open in $(H, \tilde{\tau}, E, I)$, then P_{fu} is an S. I-irresolute, thus $(H, \tilde{\tau}, E, I)$ is a F.W.S.I-irresolute.

Remark 4.1.10. Let $\phi : (H, \tilde{\tau}, E, I) \to (K, \tilde{\sigma}, L)$ be a fibrewise soft ideal function and $\psi : (K, \tilde{\sigma}, L, M) \to (Z, \tilde{\vartheta}, C)$ is a fibrewise soft ideal function, where $(H, \tilde{\tau}, E, I), (K, \tilde{\sigma}, L, M), (Z, \tilde{\vartheta}, C, N)$ are F.W.S.I-topological space over

$(B, \tilde{\Omega}, G)$ then the composition ψo$\phi : (H, \tilde{\tau}, E, I) \to (Z, \tilde{\vartheta}, C)$ is not fibrewise soft ideal function, in general, like shown by the next example.

Example 4.1.11 Let $H = Z = \{h_1, h_2, h_3\}$, $K = \{h_1, h_2, h_3, h_4\}$, $B = \{a, b, c\}$, $E = L = C = \{e\}$, $G = \{g\}$, with soft topologies, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, \{(e, \{h_1\})\}\}$, $\tilde{\sigma} = \widetilde{\Phi}, \widetilde{K}, \{(e, \{h_1, h_3\})\}\}$, $\tilde{\vartheta} = \{\widetilde{\Phi}, \tilde{Z}, \{(e, \{h_3\})\}, \{(e, \{h_2, h_3\})\}\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}\}$, $I = \widetilde{\Phi}, \{(e, \{h_3\})\}\}$, $M = \{\widetilde{\Phi}, \{(e, \{h_1\})\}\}$. Let $(H, \tilde{\tau}, E, I)$, $(K, \tilde{\sigma}, L, M)$ and $(Z, \tilde{\vartheta}, C, N)$ be F.W.S.I-topological spaces over $(B, \tilde{\Omega}, G)$. Define the identity function f from H, Z to B, let u be a identify function from E, C to G and define $q : K \to B$, $d : L \to G$, such that $q(h_1) = \{a\}$, $q(h_2) = \{q\}$, $(h_4) = \{b\}$, $q(h_3) = \{c\}$, $d(e) = \{g\}$. Then the soft functions $P_{H(fu)} : (H, \tilde{\tau}, E, I) \to (B, \tilde{\Omega}, G)$, $P_{K(qd)} : (K, \tilde{\sigma}, L, M) \to \left(B, \tilde{\Omega}, G\right)$ and $P_{z(tv)} : (Z, \tilde{\vartheta}, C, N) \to (B, \tilde{\Omega}, G)$ are an S.I-continuous. And define identity functions $\phi : (H, \tilde{\tau}, E, I) \to (K, \tilde{\sigma}, L)$, $\psi : (K, \tilde{\sigma}, L, M) \to (Z, \tilde{\vartheta}, C)$ define as : $\psi\left(\tilde{h}_{1_{\tilde{b}}}\right) = \tilde{h}_{1_{\tilde{b}}}$, $\psi\left(\tilde{h}_{2_{\tilde{b}}}\right) = \psi\left(\tilde{h}_{4_{\tilde{b}}}\right) = \tilde{h}_{2_{\tilde{b}}}$, $\psi\left(\tilde{h}_{3_{\tilde{b}}}\right) = \tilde{h}_{3_{\tilde{b}}}$ where $\tilde{b} \tilde{\in} (B, G)$. It clear that both ϕ and ψ are S.I-continuous function. However, the composition function ϕoψ is not S.I-continuous function because $\{(e, \{h_3\})\}$ is an S. open in $(Z, \tilde{\vartheta}, C)$, but $(\phi\text{o}\psi)^{-1}\{(e, \{h_3\})\} = \{(e, \{h_3\})\}$ is not an S.I-open in $(H, \tilde{\tau}, E, I)$**.**

Proposition 4.1.12. Let $\phi : (H, \tilde{\tau}, E, I) \to (K, \tilde{\sigma}, L)$ be a fibrewise soft function, where $(K, \tilde{\sigma}, L, M)$ a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$ and $(H, \tilde{\tau}, E, I)$ has the induced F.W.S. I-topology, the following are hold :

(a) If ϕ is a soft continuous and for each F.W.S. I-topological space $(Z, \tilde{\gamma}, C, N)$, a fibrewise soft function $\psi : (Z, \tilde{\gamma}, M, C) \to (H, \tilde{\tau}, E)$ is S.I-continuous then the composition $\phi \circ \psi: (Z, \tilde{\gamma}, C, N) \to (K, \tilde{\sigma}, L)$ is S.I-continuous.

(b) If ϕ is an S.I-continuous and for each F.W.S. I-topological space $(Z, \tilde{\gamma}, C, N)$, a fibrewise soft function $\psi : (Z, \tilde{\gamma}, M, C) \to (H, \tilde{\tau}, E, I)$ is S. I-irresolute then the composition $\phi \circ \psi : (Z, \tilde{\gamma}, C, N) \to (K, \tilde{\sigma}, L)$ is S.I-continuous.

Proof. (a) Suppose that ψ is an S.I-continuous. Let $\tilde{z} \tilde{\in} Z_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and (N, L) soft open set of $(\phi \circ \psi)(\tilde{z}) = \tilde{k} \tilde{\in} K_{\tilde{b}}$ in $(K, \tilde{\sigma}, L)$. since ϕ is soft continuous, then $\phi^{-1}(N, L)$ is a soft open set containing $\psi(\tilde{z}) = \tilde{h} \tilde{\in} H_{\tilde{b}}$ in $(H, \tilde{\tau}, E)$. Since ψ is S.I-continuous, then $\psi^{-1}(\phi^{-1}(N, L))$ is a S.I-open set containing $\tilde{z} \in Z_{\tilde{b}}$ in $(Z, \tilde{\gamma}, C)$ and $\psi^{-1}(\phi^{-1}(N, L)$=$(\phi \circ \psi)^{-1}(N, L)$ is a S.I-open set containing $\tilde{z} \in Z_{\tilde{b}}$ in $(Z, \tilde{\gamma}, C, N)$.

(b) The proof is similar to the proof of (a).

4.2. Fibrewise Soft Ideal Closed and Soft Ideal Open Topological Spaces

during this part, we explain the ideas of a fibrewise soft ideal closed and soft ideal open topological spaces. Several topological properties on the obtained concepts are studied.

Definition 4.2.1. A function $\phi : (H, \tilde{\tau}, E, I) \to (K, \tilde{\sigma}, L)$ is said to be

(a) A soft ideal open (briefly, S. I-open) function if the image of every S.I-open set in H is a soft open set in K.

(b) A soft ideal closed (briefly, S. I-closed) function if the image of every S.I-closed set in H is a soft closed set in K.

Definition 4.2.2. A F.W.S.I-topological space $(H, \tilde{\tau}, E, I)$ over $(B, \tilde{\Omega}, G)$ is called fibrewise soft ideal closed (briefly, F.W.S. I-closed) if the projection function P_{fu} is a soft I-closed.

Definition 4.2.3. A F.W.S.I-topological space $(H, \tilde{\tau}, E, I)$ over $(B, \tilde{\Omega}, G)$ is called fibrewise soft ideal open (briefly, F.W.S. I-open) if the projection function P_{fu} is a soft I-open .

The F.W.S.I-open and F.W.S. open are this two concepts are independent.

Example 4.2.4. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$ given as follows : $H = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, \{(e, \{h_1, h_2\})\}, \{(e, \{h_1, h_2, h_3\})\}\}$, $I = \{\widetilde{\Phi}, \{(e, \{h_3\})\}, \{(e, \{h_4\})\}, \{(e, \{h_3, h_4\})\}\}$, $B = \{a, b, c, d\}$, $G = \{g\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, \{(g, \{a, b\})\}, \{(g, \{a, b, d\})\}\}$ and $f : H \to B$, such that $f(h_1) = \{a\}$, $f(h_2) = \{b\}$, $f(h_3) = \{c\}$, $f(h_4) = \{d\}$ and $u : E \to G$, $u(e) = \{g\}$. Then projection function $P_{fu} : (H, \tilde{\tau}, E, I) \to (B, \tilde{\Omega}, G)$ is an S. I – open but is not a soft open. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S. I-open but not a F.W.S. open.

Example 4.2.5. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$ given as follows : $H = \{h_1, h_2, h_3\}$, $E = \{e\}$, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, \{(e, \{h_1\})\}, \{(e, \{h_1, h_2\})\}\}$, $I = \{\widetilde{\Phi}, \{(e, \{h_1\})\}\}$, $B = \{a, b, c\}$, $G = \{g\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, \{(g, \{a\})\}$, $f : H \to B$, such that $f(h_1) = \{a\}$, $f(h_2) = \{b\}$, $f(h_3) = \{c\}$ and $u : E \to G$, $u(e) = \{g\}$. It is clear that $P_{fu}(\{(e, \{h_1\})\})$ is soft open in $\left(B, \tilde{\Omega}, G\right)$ but not S.I-open, then projection function $P_{fu} : (H, \tilde{\tau}, E, I) \to (B, \tilde{\Omega}, G)$ is a soft open but is not an S. I-open. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S. open but not a F.W.S.I-open.

Proposition 4.2.6. The $(H, \tilde{\tau}, E, I)$ is a F.W.S.I-open over $(B, \tilde{\Omega}, G)$ then for each soft element in $(H, \tilde{\tau}, E)$ and each soft nbd $(F, E)^*$ of the soft element, there exists a soft open of $(B, \tilde{\Omega}, G)$ containing the image of a soft element such that a soft open containing in image of a soft nbd.

Proof. Suppose that $(H, \tilde{\tau}, E, I)$ is a F.W.S.I-open, then the projection function P_{fu} $(H, \tilde{\tau}, E, I) \to \left(B, \tilde{\Omega}, G\right)$ is an S.I-open function. To proof for each soft element and each soft nbd (F, E) of the soft element, there exists a soft open of $\left(B, \tilde{\Omega}, G\right)$ containing the image of a soft element such that a soft open containing in image of a soft nbd, since for each soft element $\tilde{h} \tilde{\in} (H, E)$ and each soft nbd $(F, E)^*$ of $\tilde{h}$, then there exists an S.I-open (U, E) in $(H, \tilde{\tau}, E)$ such that $\tilde{h} \tilde{\in} (U, E) \tilde{\subset} (F, E)^*$. Since P_{fu} is an S.I-open, $(M, G) = P_{fu}(U, E)$ is a soft open of $(B, \tilde{\Omega}, G)$ and $P_{fu}(\tilde{h}) \tilde{\in} (M, G) \tilde{\subset} P_{fu}(F, E)^*$.

Proposition 4.2.7. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S. I-topological space over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E, I)$ is a F.W.S. I-open (resp., F.W.S.I-closed) if and for all fibre soft $(H_{\tilde{b}}, E_{\tilde{b}})$ of $(H, \tilde{\tau}, E)$ and all S.I-open (resp., S.I-open) subset (F, E) of $(H_{\tilde{b}}, E_{\tilde{b}})$ in $(H, \tilde{\tau}, E)$, there exists a soft open (soft closed) subset (F, G) of $\tilde{b}$ such that $(H_{(F,G)}, E_{(F,G)}) \tilde{\subset} (F, E)$.

Proof. (⇒)Suppose that $(H, \tilde{\tau}, E, I)$ is a F.W.S.I-open (resp., F.W.S.I-closed) i.e., the projection function $P_{fu} : (H, \tilde{\tau}, E, I) \rightarrow (B, \tilde{\Omega}, G)$ is an S.I-open (resp., soft I-closed). Then, let $\tilde{b} \tilde{\in} (B, G)$ and (F, E) S.I-open (resp., soft I-closed) subset of $(H_{\tilde{b}}, E_{\tilde{b}})$ in $(H, \tilde{\tau}, E)$, then (H, E) _(F, E) is a soft I-closed (resp., S.I-open) in $(H, \tilde{\tau}, E)$, this implies $P_{fu}((H, E) - (F, E))$ is a soft open (resp., soft closed) in $(B, \tilde{\Omega}, E)$, let $(F, G) = (B, G) - P_{fu}((H, E) - (F, E))$, then (F, G) a soft open (resp., soft closed) subset of $\tilde{b}$ in $(B, \tilde{\Omega}, G)$ and $(H_{(F,E)}, E_{(F,E)}) = H_{(F,G)} = (H, E) - H_{(P_{fu}((H,E)-(F,E))} \tilde{\subset} (F, E)$.

(⇐)Suppose that the assumption hold and there exist the projection function $P_{fu} : (H, \tilde{\tau}, E, I) \rightarrow (B, \tilde{\Omega}, G)$ is a soft I-closed. Now, let (F, C) be an S. I-closed subset of $(H, \tilde{\tau}, E, I)$ and $\tilde{b} \tilde{\in} (B, G) - P_{fu}(F, C)$ and each S.I-open set (F, E) of fibre soft $(H_{\tilde{b}}, E_{\tilde{b}})$ in $(H, \tilde{\tau}, E, I)$. By assumption we have a soft open (F, G) of $\tilde{b}$ such that $(H_{(F,G)}, E_{(F,G)}) \tilde{\subset} (F, E)$. It is easy to show that $(F, G) \tilde{\subset} (B, G) - P_{fu}(F, C)$, hence $(B, G) - P_{fu}(F, C)$ is a soft open in $(B, \tilde{\Omega}, G)$ and this implies $P_{fu}(F, C)$ is a soft closed in $(B, \tilde{\Omega}, G)$ and P_{fu} is a soft I-closed. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S. I-closed. For an S.I-open function, we can prove similarly.

Proposition 4.2.8. Let $\phi : (H, \tilde{\tau}, E, I) \to (K, \tilde{\sigma}, L)$ be a fibrewise S.I-open function, where $(K, \tilde{\sigma}, L, M)$ a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$) and $(H, \tilde{\tau}, E, I)$ has the induced F.W.S. I-topology and for each F.W.S. I-topological space $(Z, \tilde{\gamma}, C, N)$, a fibrewise soft function $\psi: (Z, \tilde{\gamma}, C, N) \to (H, \tilde{\tau}, E)$ is S.I-open then the composition $\phi \circ \psi: (Z, \tilde{\gamma}, C, N) \to (K, \tilde{\sigma}, L)$ is S.I-open.

Proof. Suppose that ψ is an S.I-open. Let $\tilde{z} \tilde{\in} Z_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$ and (N, L) S.I-open set in $(Z, \tilde{\gamma}, C, N)$. Since ψ is S.I-open, then $\psi(N, L)$ is a soft open set containing $\psi(\tilde{z}) = \tilde{h} \tilde{\in} H_{\tilde{b}}$ in $(H, \tilde{\tau}, E)$. Since ϕ is S.I-open, then $\psi(\phi(N, L))$ is a soft open set containing $(\phi o \psi)(\tilde{z}) = \tilde{k} \tilde{\in} K_{\tilde{b}}$ in $(K, \tilde{\sigma}, L)$ and $\psi(\phi(N, L) = (\phi o \psi)(N, L)$ is a soft open set containing $\tilde{k} \in K_{\tilde{b}}$ in $(K, \tilde{\sigma}, L)$.

4.3. Fibrewise Soft Near Ideal Topological spaces

In this section, we study fibrewis soft near ideal topological spaces, counter examples are given to illustrate these concepts and we introduced a notion fibrewise soft near I-compact, and, we studied the properties related to fibrewise soft ideal topological spaces and we introduced a definition of fibrewise soft near ideal connected and its related properties.

Definition 4.3.1. Assume that $(B, \tilde{\Omega}, G)$ is a soft topology. The fibrewise soft near ideal topological space (briefly, F.W.S.j-I-topological space) on a fibrewise set H over B mean any soft near ideal topology on H for which the projection function is S.j-I-continuous.

Remark 4.3.2. In F.W.S.I-topological space we work over a soft topological base space B, say. When B is a point-space the theory reduces to that of ordinary soft topology. A F.W.S.I-topological (resp., S. j-I-topological) space over B is just a soft ideal topological (resp., S. j-I-topological) space H

together with a soft ideal continuous (resp., S. j-I-continuous) projection $P_{fu}:(H,\tilde{\tau},E,I)\to\left(B,\tilde{\Omega},G\right)$. So the implication between F.W.S.I- topological spaces and the families of F.W.S. j-I-topological spaces are given in the following diagram, where $j\in\{\alpha,S,P,b,\beta\}$.

$$\begin{array}{lll}
\text{F.W.S.I-topological space} & & \\
\Downarrow & & \\
\text{F.W.S.}\alpha\text{-I-topological space} & \Rightarrow & \text{F.W.S. S-I-topological space} \\
\Downarrow & & \Downarrow \\
\text{F.W.S. P-I-topological space} & \Rightarrow & \text{F.W.S. b-I-topological space} \\
 & & \Downarrow \\
 & & \text{F.W.S. }\beta\text{-I-topological space.}
\end{array}$$

Figure 4.1.1: Implication between fibrewise soft I-topology and fibrewise soft *j*-I-topology, where $j\in\{\alpha,S,P,b,\beta\}$.

The following examples show that these implications are not reversible.

Example 4.3.3. Let $(H,\tilde{\tau},E,I)$ be a F.W.S. I-topological space over $(B,\tilde{\Omega},G)$ given as follows : $H=\{h_1,h_2,h_3,h_4\}$, $E=\{e\}$, $\tilde{\tau}=\{\widetilde{\Phi},\widetilde{H},\{(e,\{h_1\})\},\{(e,\{h_2\})\},\{(e,\{h_1,h_2\})\},\{(e,\{h_1,h_2,h_4\})\}\}$, $I=\{\widetilde{\Phi},\{(e,\{h_1\})\},\{(e,\{h_3\})\},\{(e,\{h_1,h_3\})\}$, $B=\{a,b,c,d\}$, $G=\{g\}$, $\tilde{\Omega}=\{\widetilde{\Phi},\tilde{B},\{(g,\{a,b\})\}\}$ and $f:H\to B$, such that $f(h_1)=\{a\}$, $f(h_2)=\{b\}$, $f(h_3)=\{c\}$, $f(h_4)=\{d\}$ and $u:E\to G$, $u(e)=\{g\}$. It is clear that $H_{\{(g,\{a,b\})\}}=\{(e,\{h_1,h_2\})\}$ is S.α-I- open but not S.I-open in $(H,\tilde{\tau},E,I)$, then the projection function $P_{fu}:(H,\tilde{\tau},E,\ I)\to(B,\tilde{\Omega},G)$ is an S. α-I-continuous but is not an S. I-continuous. Thus, $(H,\tilde{\tau},E,I)$ is a F.W.S.α-I-topological space but not a F.W.S.I-topological space.

Example 4.3.4. Let $(H,\tilde{\tau},E,I)$ be a F.W.S. I-topological space over $(B,\tilde{\Omega},G)$ given as follows : $H=\{h_1,h_2,h_3,h_4\}$, $E=\{e\}$, $\tilde{\tau}=\{\widetilde{\Phi},\widetilde{H},\{(e,\{h_1\})\},\{(e,\{$

$h_2, h_3\})\}, \{(e, \{h_1, h_2, h_3\})\}\}$, $I = \{\widetilde{\Phi}, \{(e, \{h_1\})\}, \{(e, \{h_4\})\}, \{(e, \{h_1, h_4\})\}\}$, $B = \{a, b, c, d\}$, $G = \{g\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, \{(g, \{a, b, d\})\}\}$ and $f : H \to B$, such that $f(h_1) = \{a\}$, $f(h_2) = \{b\}$, $f(h_3) = \{c\}$, $f(h_4) = \{d\}$ and $u : E \to G$, $u(e) = \{g\}$. It is clear that $H_{\{(g,\{a,b,d\})\}} = \{(e, \{h_1, h_2, h_4\})\}$ is S.P-I-open but not S.α-I-open in $(H, \tilde{\tau}, E, I)$, then the projection function $P_{fu} : (H, \tilde{\tau}, E, I) \to (B, \tilde{\Omega}, G)$ is an S. P-I-continuous but is not an S. α-I-continuous. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S.P-I-topological space but not a F.W.S.α-I-topological space.

Example 4.3.5. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S. I-topological space over $(B, \tilde{\Omega}, G)$ given as follows : $H = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, \{(e, \{h_2\})\}, \{(e, \{h_1, h_3\})\}, \{(e, \{h_1, h_2, h_3\})\}\}$, $I = \{\widetilde{\Phi}, \{(e, \{h_1\})\}, \{(e, \{h_4\})\}, \{(e, \{h_1, h_4\})\}\}$, $B = \{a, b, c, d\}$, $G = \{g\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, \{(g, \{a, b, d\})\}\}$ and $f : H \to B$, such that $f(h_1) = \{a\}$, $f(h_2) = \{b\}$, $f(h_3) = \{c\}$, $f(h_4) = \{d\}$ and $u : E \to G$, $u(e) = \{g\}$. It is clear that $H_{\{(g,\{a,b,d\})\}} = \{(e, \{h_1, h_2, h_4\})\}$ is S.S-I-open but not S.α-I-open in $(H, \tilde{\tau}, E, I)$, then the projection function $P_{fu} : (H, \tilde{\tau}, E, I) \to (B, \tilde{\Omega}, G)$ is an S. S-I-continuous but is not an S. α-I-continuous. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S. S-I-topological space but not a F.W.S.α-I-topological space.

Example 4.3.6. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$ given as follows : $H = \{h_1, h_2, h_3, h_4\}$, $E = \{e\}$, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, \{(e, \{h_1\})\}, \{(e, \{h_4\})\}, \{(e, \{h_1, h_4\})\}\}$, $I = \{\widetilde{\Phi}, \{(e, \{h_3\})\}\}$, $B = \{a, b, c, d\}$, $G = \{g\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, \{(g, \{a, d\})\}, \{(g, \{a\})\}, \{(g, \{d\})\}\}$ and $f : H \to B$, such that $f(h_1) = \{d\}$, $f(h_2) = \{b\}$, $f(h_3) = \{d\}$, $f(h_4) = \{b\}$ and $u : E \to G$, $u(e) = \{g\}$. Then the projection function $P_{fu} : (H, \tilde{\tau}, E, I) \to (B, \tilde{\Omega}, G)$ is an S. b-I-continuous but is not an S. P-I-continuous. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S. b-I-topological space but not a F.W.S.P-I-topological space.

Example 4.3.7. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S. I-topological space over $(B, \tilde{\Omega}, G)$ given as follows : $H = \{h_1, h_2, h_3\}$, $E = \{e\}$, $\tilde{\tau} = \{\widetilde{\Phi}, \widetilde{H}, \{(e, \{h_1, h_2\})\}\}$, $I = \{\widetilde{\Phi}, \{(e, \{h_3\})\}\}$, $B = \{a, b, c\}$, $G = \{g\}$, $\tilde{\Omega} = \{\widetilde{\Phi}, \tilde{B}, \{(g, \{a, b\})\}\}$ and $f : H \to B$, such that $f(h_1) = \{a\}$, $f(h_2) = \{c\}$, $f(h_3) = \{b\}$ and $u : E \to G$, $u(e) = \{g\}$. It is clear that $H_{\{(g,\{a,b\})\}} = \{(e, \{h_1, h_3\})\}$ is S. b-I-open but not S.S-I-open in $(H, \tilde{\tau}, E, I)$, then the projection function $P_{fu} : (H, \tilde{\tau}, E,\ I) \to (B, \tilde{\Omega}, G)$ is an S. b-I-continuous but is not an S.S-I-continuous. Thus, $(H, \tilde{\tau}, E, I)$ is a F.W.S. b-I-topological space but not a F.W.S. S-I-topological space.

Proposition 4.3.8. Let $\phi : (H, \tilde{\tau}, E, I) \to (K, \tilde{\sigma}, L)$ be a fibrewise soft ideal function, where $(K, \tilde{\sigma}, L, M)$ a F.W.S. j-I-topological space over $(B, \tilde{\Omega}, G)$) and $(H, \tilde{\tau}, E, I)$ has the induced F.W.S. j-I-topology. If ϕ is a soft continuous and for each F.W.S. j-I-topological space $(Z, \tilde{\gamma}, C, N)$, a fibrewise projection function $\psi: (Z, \tilde{\gamma}, M, C) \to (H, \tilde{\tau}, E)$ is S. j-I-continuous then the composition $\phi \circ \psi: (Z, \tilde{\gamma}, C, N) \to (K, \tilde{\sigma}, L)$ is S. j-I-continuous, where $j \in \{\alpha, S, P, b, \beta\}$.
Proof. The proof is similar to that of Proposition (4.1.12) (a).

Proposition 4.3.9. If $(H, \tilde{\tau}, E, I)$ is a F.W.S. j-I-topological space over $(B, \tilde{\Omega},$). Then $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*}, I_{B^*})$ is F.W.S. j-I-topological space over $(B^*, \tilde{\Omega}^*, G^*)$ for each open subspace $(B^*, \tilde{\Omega}^*, G^*)$ of $\left(B, \tilde{\Omega}, G\right)$, where $j \in \{\alpha, S, P, b, \beta\}$.
Proof. The proof is similar to that of Proposition (4.1.6).

Proposition 4.3.10. The $(H, \tilde{\tau}, E, I)$ is a F.W.S. j-I-topological space over $(B, \tilde{\Omega}, G)$ if and only if the graph soft function $g : (H, \tilde{\tau}, E) \to (H, \tilde{\tau}, E) \times (B, \tilde{\Omega}, G)$, defined by $g(\tilde{h}) = (\tilde{h}, f(\tilde{h}))$, for each $\tilde{h} \tilde{\in} (H, E)$, is an S. j-I-continuous, where $j \in \{\alpha, S, P, b, \beta\}$.
Proof. The proof is similar to that of Proposition (4.1.5).

Definition 4.3.11. The F.W.S.I-topological space $(H, \tilde{\tau}, E, I)$ over $(B, \tilde{\Omega}, G)$ is named fibrewise soft near I-compact (briefly, F.W.S. j-I-compact) if the soft ideal topological space $(H, \tilde{\tau}, E, I)$ is a soft j-I-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 4.3.12. Let $\phi: (H, \tilde{\tau}, E, I) \to (K, \tilde{\sigma}, L, J)$ be an S. j-I-irresolute fibrewise surjective, $(H, \tilde{\tau}, E, I)$ and $(K, \tilde{\sigma}, L, J)$ are F.W.S.I-topological spaces over $\left(B, \tilde{\Omega}, G\right)$. If $(H, \tilde{\tau}, E, I)$ is F.W.S. j-I-compact, then $(K, \tilde{\sigma}, L, J)$ is so, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Let $(H, \tilde{\tau}, E, I)$ be a F.W.S. j-I-compact. To show that $(K, \tilde{\sigma}, L, J)$ is F.W.S.j-I-compact, since let $\{(W_i, E_i): i \in \Delta\}$ an S. I-open cover of $K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$. Then $\phi^{-1}\{(W_i, E_i): i \in \Delta\}$ is an S. I-open cover of $H_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$. By the hypothesis, there exists a finite subset Δ_o of Δ such that $\tilde{H} - \tilde{\cup} \{\phi^{-1}(W_i, E_i) : i \in \Delta_o\} \tilde{\in} I$. Therefore, $\phi \{\tilde{H} - \tilde{\cup} \{\phi^{-1}(W_i, E_i): i \in \Delta_o\} \tilde{\in} J$ which explains that is $(K, \tilde{\sigma}, L, J)$ an S. j-I-compact. Thus $(K, \tilde{\sigma}, L, J)$ a F.W.S. j-I-compact, where $j \in \{\alpha, S, P, b, \beta\}$.

Definition 4.3.13. The F.W.S. topological space $(H, \tilde{\tau}, E, I)$ over $\left(B, \tilde{\Omega}, G\right)$ is called fibrewise soft near ideal connected (briefly F.W.S. j-I-connected) if $H_{\tilde{b}}$ where $\tilde{b} \tilde{\in} (B, G)$ is not the union of two disjoint non-empty S. j-I-open sets of $(H, \tilde{\tau}, E, I)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proposition 4.3.14. Assume that $\phi : (H, \tilde{\tau}, E, I) \to (K, \tilde{\sigma}, L, J)$ is a S. j-I-irresolute fibrewise surjection function, where $(H, \tilde{\tau}, E, I)$ and $(K, \tilde{\sigma}, L, J)$ are F.W.S.I-topological spaces over $(B, \tilde{\Omega}, G)$. If $(H, \tilde{\tau}, E, I)$ is F.W.S. j-I-connected, then so is $(K, \tilde{\sigma}, L, J)$, where $j \in \{\alpha, S, P, b, \beta\}$.

Proof. Let $(K, \tilde{\sigma}, L, J)$ be a F.W.S.I-topological space over $(B, \tilde{\Omega}, G)$, then P_{fu} is an S. I-continuous. Suppose that $(K, \tilde{\sigma}, L, J)$ is not a F.W.S.j-I-connected, then there exist non-empty disjoint S. j-I-open subset (F, L), (G, L) of $(K, \tilde{\sigma}, L, J)$ such that $(F, L) \tilde{\cup} (G, L) = K_{\tilde{b}}$, where $\tilde{b} \tilde{\in} (B, G)$. Since ϕ is an S. j-I-irresolute, then $\phi^{-1}(F, L)$, $\phi^{-1}(G, L)$ are non-empty disjoint in $(H, \tilde{\tau}, E, I)$. Moreover, $\phi^{-1}(F, L) \tilde{\cup} \phi^{-1}(G, L) = H_{\tilde{b}}$. This shows that $(H, \tilde{\tau}, E, I)$ is not a F.W.S. j-I-connected. This is a contradiction and hence $(K, \tilde{\sigma}, L, J)$ is a F.W.S. j-I-connected, where $j \in \{\alpha, S, P, b, \beta\}$.

Conclusions

The purpose of the present work is to put a starting point for the some types of fibrewise topological spaces and develop a theory of fibrewise topology and construct new concepts of fibrewise topology.

Also, we introduce the concepts of fibrewise soft topological spaces, several topological property on the obtained concepts are studies. The suggested methods of soft near open the way for constructing new types fibrewise soft topological spaces and we hope to connect this result in the contest of study the relationship and comparison between fibrewise soft near compact and locally soft near compact spaces which will be our goal in future work.

Finally, we used soft near open concept to give a new approach for defining of separation axioms in fibrewise soft topological spaces namely fibrewise near separation axioms. In our future work, we hope to connect and comparison the fibrewise soft near topological spaces and fibrewise soft near separation axioms.

Future Works

The following are some open problems for the future works:

In the future we can use the concepts bitopological spaces in define fibrewise soft bitopological spaces, also we can define fibrewise soft bi-T_i where $i=1,2,3,4$. On the other hand we can discuss the relation between fibrewise soft bitopological spaces and fibrewise soft j-bitopological spaces, where $j \in \{\alpha, S, P, b, \beta\}$. Furthermore, we will study fibrewise soft digital (resp., di, tri, nano, filte, girll, fuzzy) topological spaces.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud,"β-open sets and β-continuous mappings", Bull. Fac. Sc. Assuit Univ., Vol.12, 77-90, (1983).

[2] M. E. Abd El-Monsef, E. F. Lashien and A. A. Nasef, "On I-open sets and I-continuous functions", Kyungpook Math. J., vol. 32 (1), 21-30, (1992).

[3] A. A. Abo Khadra, S. S. Mahmoud and Y. Y. Yousif, "fibrewise near topological spaces", Journal of Computing, USA, Vol. 4, Issue 5, 223-230 May (2012).

[4] M. Akdag, A. Ozkan, "Soft α-open sets and soft α-continuous functions", Abstr. Anal. Appl. Art ID 891341, 1-7 (2014).

[5] M. Akdag and A. Ozkan, "Soft β-open sets and soft β-continuous functions", Scientific World Journal, vol. 2014, Article ID 843456 , 6 pages, (2014).

[6] M. Akdag and A. Ozkan, "Soft b-open sets and soft b-continuous functions" , Math Sci 8:124 DOI 10.1007/s40096-014-0124-7, (2014).

[7] M. Akdağ and F. Erol, "Soft I-Sets and Soft I-Continuity of Functions", Gazi Univ. Journal of Science, 27(3), 923-932, (2014).

[8] M. Akdağ and F. Erol, Y.Bingö "Soft Sime-I-Continuity Functions", Gazi Univ. Journal of Science, 28(1):37- 44, (2015).

[9] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, "On some new operations in soft set theory", Computer and Mathematics with Applications, 57, 1547-1553, (2009).

[10] I. Arockiarani, A. Arokialancy, "Generalized soft gb-closed sets and soft gsb-closed sets in soft topological spaces", Int. J. Math. Aarch. 4(2), 1–7 (2013).

[11] K. Atanassov, "Intuitionistic fuzzy sets", Fuzzy Sets and Systems 20, 87-96, (1986).

[12] K. Atanassov, "Operators over interval valued intuitionistic fuzzy sets", Fuzzy Sets and Systems 64, 159-174, (1994).

[13] Aygunoglu and H. Aygun, "Some notes on soft topological spaces", Neural Comput and Applic., Neural Comput. and Applic, 21(1), 113-119, (2012) .

[14] K. V. Babitha, J. J. Sunil "Soft set relations and functions". Comput Math Appl; 60: 1840-1849, (2010).

[15] G. Birkhoff, "Lattic theory", American Math. Soc., New York, NY, USA, 1st edition, (1940).

[16] G. Birkhoff, "Lattic theory", American Math. Soc., New York, NY, USA, 3rd edition, (1967).

[17] S. Bonnevay, M. Lamure, C. Largeron and N. Nicoloyannis, "A pre-topological approach for structuring data in non metric spaces", Electronic Notes Discrete Mathematics, Vol. 2, (1999).

[18] S. Bonnevay, M. Lamure, C. Largeron and N. Nicoloyannis, "A pre topological-based algorithm for structuring data in spaces endowed with weak structure", in Ordinal Symbolic Data Analysis' 98, Amherst (USA), (1998).

[19] N. Çağman, S. Karataş Enginoğlu, "Soft topology", Computers and Mathematics with Applicatons,. 62:351-358, (2011).

[20] A. Caksu Guler and G. Aslim, "b-I-open sets and decomposition of continuity via idealization", Proceedings of Institute of Mathematics and Mechanics. National Academy of Sciences of Azerbaijan, vol. 22, 27–32, (2005).

[21] B. Chen, "Soft semi-open sets and related properties in soft topological spaces", Appl. Math. Inf. Sci. 7, No. 1, 287-294, (2013).

[22] M. J. Das, PK, "On soft topological space via semiopen and semiclosed soft sets", Math. J. 54, 221-236, (2014).

[23] A. A. El-Atik,"A study of some types of mappings on topological spaces", M. Sc. Thesis, Tanta Univ., Egypt, (1997).

[24] R. Englking, "Outline of general topology", Amsterdam, (1989).

[25] Feng, Y. B. Zhao, "Soft semirings", Computer and Mathematics with Applications, 56, 2621- 2628, (2008).

[26] W. L. Gau and D.J. Buehrer, "Vague sets", IEEE Trans. System Man Cybernet 23 (2), 610-614, (1993).

[27] M. B. Gorzalzany, "A method of inference in approximate reasoning based on interval-valued fuzzy sets", Fuzzy Sets and Systems 21, 1-17, (1987).

[28] T. R. Hamlett and D. Jankovic, "Ideals in topological spaces and the set operator Ψ", Bollettino U.M.I., vol. 7, 863-874, (1990).

[29] E. Hatır and T. Noiri, "On decompositions of continuity via idealization", Acta. Math. Hungar., 96, 341-349, (2002).

[30] I. M. James, "Fibrewise Topology", Cambridge University Press, London, (1989).

[31] I. M. James, "General topology and homotopy theory ", Springer-Verlag, New York, (1984).

[32] I. M. James, "General topology over a base, Aspect of Topology", London Math. Soc. Lecture Notes, Vol.93, (1984).

[33] I. M. James, "Sepaces", Bull. London Math. Soc., Vol.18, 529-559, (1984).

[34] I. M. James, "Topological and uniform spaces", Springer-Verlag, New York, (1987).

[35] I. M. James, "Uniform spaces over a base", J. London Math. Soc., Vol.32(2), 328-336, (1985).

[36] G. Kale and A.Caksu Guler, "On Soft Ideal Topological Spaces", Neural Computing and Applications, 1-14.

[37] K. Kuratowski, "Topology", Vol. I, Academic Press, New York, (1966).

[38] C. Kuratowski, "Topologies", Warsaw, (1952).

[39] C. Largeron and S. Bonneray, "A pre-topological approach for Structure al Analysis", Inf. Sciences, Vol.144, 169-185, (2002).

[40] C. Largeron and S. Bonneray, "Une method de structuration par recherché defermes minimaux-application a la modification de flux de migrations

intervilles, Societe Francophone de Classification", Societe Francophone de Classification' 1997, Lyon, France, Vol.111-118, July (1997).

[41] N. Levine,"Semi-open sets and semi-continuity in topological spaces", Amer. Math. Monthly, Vol.70, 36-41, (1963).

[42] S. S. Mahmoud and Y. Y. Yousif, "fibrewise near separation axioms", International Mathematical Forum, Hikari Ltd, Bulgaria, Vol.7, No.35, 1725-1736, (2012).

[43] P. K. Maji, R. Biswas and A. R. Roy, "Soft set theory", Computer and Math. with Appl., 555-562, (2003).

[44] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, "On pre continuous and weak pre continuous mappings", Proc. Math. Phys. Soc. Egypt, Vol.53, 47-53, (1982).

[45] D. Molodtsov, "Soft set theory-First results", Comput. Math. Appl. 37(4–5), 19–31 (1999).

[46] S. Mondal, M. Chiney, S. K. Samanta, "Urysohn’s lemma and Tietze’s extension theorem in soft topology", Annals of Fuzzy Mathematics and Informatics, Vol. 2, No. 1, 1—130, July (2015).

[47] S. K. Nazmul and S. K. Samanta, "Neighbourhood properties of soft topological spaces", Annals of Fuzzy Mathematics and Informatics, Vol. 6, No. 1, 1–15, July (2013).

[48] O. Njastad, "On some classes of nearly open sets", Pacific J. Math., Vol.15, 961-970, (1965).

[49] T. Y. Ozturk, "A new approach to soft uniform spaces", Turk J Math; DOI: 10.3906/mat-1506-98, (2015).

[50] R. Sahin and A. Kucuk, "Soft filters and their convergence properties", Ann. Fuzzy Math. Inform. 6(3), 529–543, (2013).

[51] M. Shabir and M. Naz, "On soft topological spaces", Comput. Math. Appl. 61, 1786-1799, (2011).

[52] B. M. R. Stadler and P. F. Stadler,"Generalized topological spaces in evolutionary theory and combinatorial chemistry", J. Chem. Inf. Comput. Sci., Vol.42, 577-585, (2002).

[53] S. Willard, "General topology", Addison Wesley Publishing Company, Inc, USA, (1970).

[54] Y. Y. Yousif, "Some result on fibrewise Lindelöf and locally Lindelöf Topological space", Ibn Al-haitham Journal science, Vol. 22, No. 3, 191-198, (2009).

[55] Y. Y. Yousif, "Some result on fibrewise Topological space", Ibn Al-haitham Journal for pure and Applied science. University of Baghdad – Collage of Education - Ibn Al-haitham Vol. 21, No. 2, 118-132, (2008).

[56] Y. Y. Yousif, L. A. Hussain, "Fibrewise bitopolgical Spaces", International Journal of Science and Research (IJSR) Vol. 6, Issue 2, 978-982, February (2017).

[57] Y. Y. Yousif, M. A. Hussain, "Fibrewise Soft Near Topolgical Spaces", International Journal of Science and Research (IJSR), Vol. 6 Issue 2, 1010-1019, February (2017).

[58] Y. Y. Yousif, M. A. Hussain, "Fibrewise Soft Near Separation Axiom", The 23th Scientific Conference of Collage of Education . AL. Mustansiriyah Universtity, pp. 400-414, 26-27April (2017).

[59] Y. Y. Yousif and M. A. Hussain," Fibrewise Soft Ideal Topological Spaces ", acceptable for publication. Ibn Al-Haitham 1st. International Scientific Conference – 2017.

[60] I. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, "Remarks on soft topological spaces", Annals of Fuzzy Mathematics and Informatics, 3(2), 171-185, (2012).

المستخلص

في هذه الرسالة قدمنا بعض انواع الفضاءات التبولوجية الليفية باستخدام المجموعة الناعمة القريبة ومختلف النتائج المتعلقة بها وكذلك بعض مفاهيم بديهيات الفصل والمجموعة المثالية الناعمة.

قدمنا المفاهيم الاولية للفضاء التبولوجي و التولوجية الليفية ونظرية مجموعة الناعمة وكذلك درسنا مفاهيم النظرية المثالية الناعمة.

شرحنا وناقشنا فكرة جديدة من الفضاءات التوبولوجية الليفية اسميناها الفضاءات التبولوجية القريبة الناعمة الليفية وكذلك بينا مفاهيم الفضاءات التبولوجية المفتوحة القريبة الناعمة الليفية والفضاءات التبولوجية المغلقة القريبة الناعمة الليفية والفضاءات المتراصة القريبة الناعمة الليفية والفضاءات المتراصة القريبة الناعمة المحلية الليفية .

على الجانب الاخر درسنا النسخة القريبة الناعمة الليفية لبديهيات الفصل المهمة في التبولوجية الناعمة الاعتيادية أسميناها الفضاءات T_0 القريبة الناعمة الليفية، الفضاءات T_1 القريبة الناعمة الليفية، الفضاءات R_0 القريبة الناعمة الليفية، الفضاءات هاوسدورف القريبة الناعمة الليفية، الفضاءات هاوسدورف الدالية القريبة الناعمة الليفية، الفضاءات الاعتيادية القريبة الناعمة الليفية، الفضاءات الاعتيادية الكاملة القريبة الناعمة الليفية، الفضاءات الطبيعية القريبة الناعمة الليفية، الفضاءات الطبيعية الدالية القريبة الناعمة الليفية. كذلك اعطينا العديد من النتائج المتعلقة بهذه المفاهيم.

واخيرا، قدمنا فكرة الفضاءات التبولوجية المثالية الناعمة الليفية واعطينا النتائج المتعلقة بها ، وعلاوة على ذلك حصلنا على بعض الخصائص في ضوء دراسة مفاهيم الفضاءات التبولوجية المفتوحة المثالية الناعمة الليفية و الفضاءات التبولوجية المغلقة المثالية الناعمة الليفية و الفضاءات التبولوجية المثالية القريبة الناعمة الليفية.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بغداد

كلية التربية للعلوم الصرفة / ابن الهيثم

قسم الرياضيات

بعض انواع الفضاءات التبولوجية الليفية الناعمة

رسالة

مقدمة إلى كلية التربية للعلوم الصرفة / ابن الهيثم ، جامعة بغداد

كجزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

من قبل

محمد عبد الحسين غافل

بإشراف

م.د.يوسف يعكوب يوسف

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