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On Some Generalization of Convex Sets, Convex Functions, and Convex Optimization Problems

A Thesis

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By

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Supervised By

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بسم الله الرحمن الرحيم

يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا

الْعِلْمَ دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ

صدق الله العظيم

المجادلة 11

Supervisor's Certification

I certify that this thesis was prepared under my supervision at the department of mathematics, college of education for pure science / Ibn Al-Haitham, Baghdad university as a partial fulfillment of the requirements for the degree of master of science in mathematics.

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We certify that we have read this thesis entitled " **On Some Generalization of Convex Sets, Convex Functions, and Convex Optimization Problems**" and as examining committee, we examined the student **(Mahmood Idrees Abdulmaged)** in its contains and what is connected with it, and that in our opinion it is adequate for the partial fulfillment of the requirement for the degree of Master of Science in Mathematics.

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الأهداء

أَشكر اللهَ تعالى الذي أَتوكل عليه فهو حسبي انه نعم المولى ونعم النصير

أهدي هذا العمل المتواضع الى أَبي (رحمه الله) الذي لم يبخل علي يوماً بشيء ...

الى أُمي التي غمرتني بالمحبة والحنان ...

أَقول لهم: أنتم وهبتم لي الأمل والشغف في حب الاطلاع والمعرفة

الى أسرتي جميعاً ...

الى كل من أضاء بعلمه عقول الأخرين

أَساتذتي الكرام...

الى من علمتني وكانت سنداً لي فأَصبح علمها نوراً يضئ الطريق أمامي...أستاذتي ومشرفتي

الدكتورة صبا ناصر مجيد امتناناً وتقديراً

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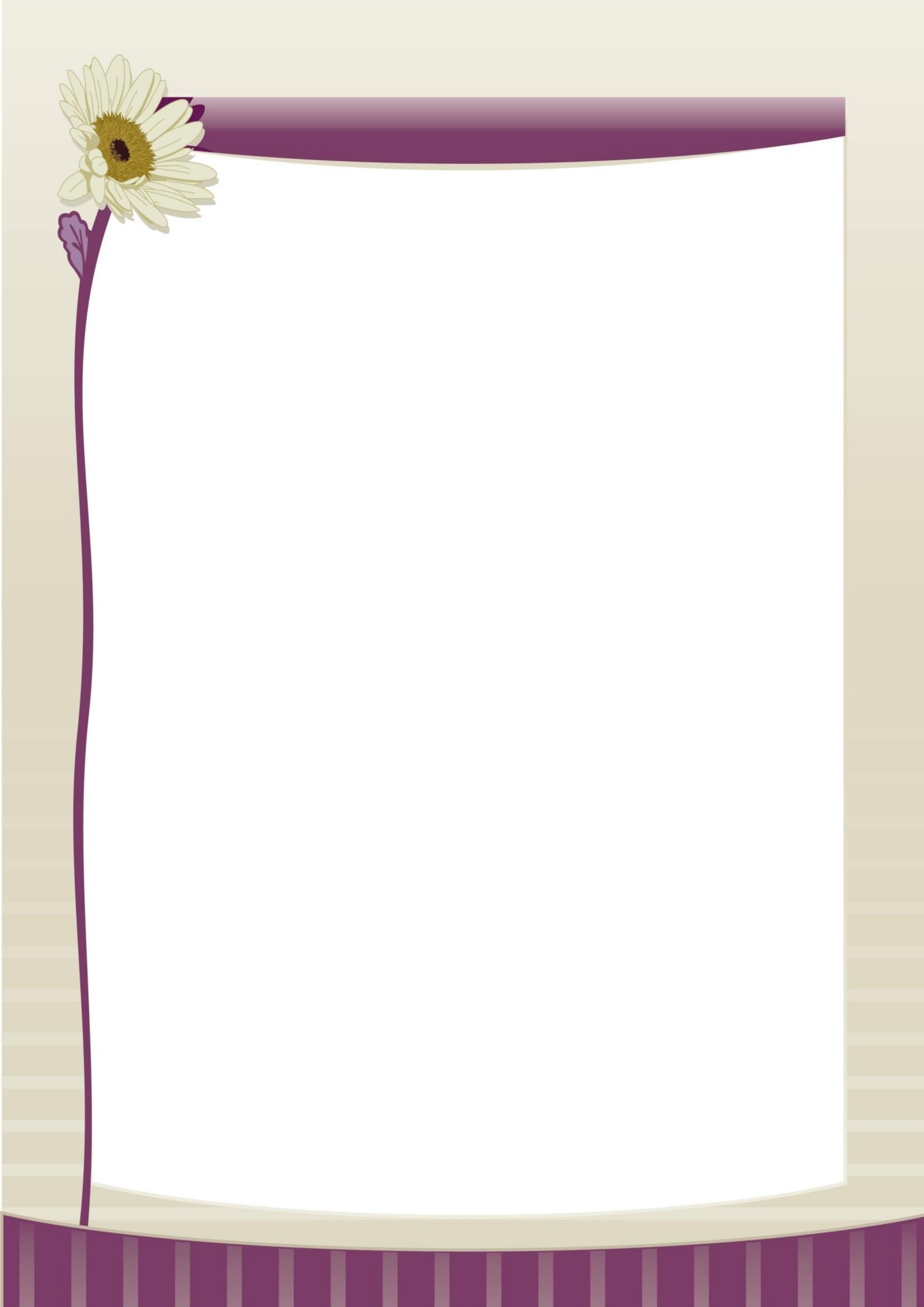

Author's Publications

[1] S. N. Majeed, M. I. Abd Al-Majeed, ‘‘On convex functions, E-convex functions and their generalizations: applications to non-linear optimization problems’’, International Journal of Pure and Applied Mathematics, 116 (3), 655-673, (2017).

[2] S. N. Majeed, M. I. Abd Al-Majeed, ‘‘Some notions on generalized convex sets and affine sets’’, 23th Scientific Specialized Conference, College of Education, University of Al-Mustansiryah, 30-40, (2017).

Contents

List of Symbols and Acronyms

x^T	The transpose of a column vector x
$\|x\|$	The norm of x
$\langle .,. \rangle$	Scalar (inner) product mapping
$B(x,r)$	Open ball with center x and radius r
$B[x,r]$	Closed ball with center x and radius r
∇f	Vector of partial derivatives of f defined on $\mathbb{R}^n$ (gradient of f)
$\nabla^2 f = H$	The matrix of second-order partial derivatives of f defined on $\mathbb{R}^n$ (Hessian)
C^1	The class of functions whose first derivatives are continuous
C^2	The class of functions whose second derivatives are continuous
p.d.	Positive definite
p.s.d.	Positive semi-definite
n.d.	Negative definite
n.s.d.	Negative semi-definite
$conv(S)$	The convex hull of a set S
$E\text{-}conv(S)$	The E-convex hull of a set S
$cone(K)$	The convex cone hull of a set K
$E\text{-}cone(K)$	The E-convex cone hull of a set K
$aff(M)$	The affine hull of a set M
$E\text{-}aff(M)$	The E-affine hull of a set M

I	Identity mapping
$epi\ f$	The epigraph of f
$E-e(f)$	The first type of an epigraph associated with an mapping E
$epi_E\ f$	The second type of an epigraph associated with an mapping E
$epi^E\ f$	The third type of an epigraph associated with an mapping E
$S_\alpha[f]$	α −level set associated with $epi\ f$
$E-S_\alpha[f]$	α −level set associated with $E-e(f)$
$S_{\alpha,E}[f]$	α −level set associated with $epi_E\ f$
$S_\alpha^E[f]$	α −level set associated with $epi^E\ f$
$argmin_s f$	The set of all feasible solutions of a minimization problem
$argmax_s f$	The set of all feasible solutions of a maximization problem
p^*	The optimal value of a minimization problem
q^*	The optimal value of a maximization problem
(NLP)	A non-linear constrained generalized optimization problem
(NLP_E)	Another type of non-linear constrained generalized optimization problem
(M-NLP_E)	A maximization non-linear constrained generalized optimization problem
■	The end of the proof

Abstract

The main aim conducted and reported in this thesis is divided into two parts. The first part is devoted to providing some properties and characterizations of generalized convex, cone, and affine sets such as (respectively, E-convex, E-cone, E-affine) sets, and the study of some properties and characterizations of generalized convex functions such as (quasi convex ,E-convex, semi E-convex, quasi semi E-convex, pseudo semi E-convex, E-quasiconvex, E-pseudoconvex) functions. The aim of the second part is to study some optimality properties and characterizations of generalized non-linear optimization problems. We consider the objective functions for non-linear optimization problems as E-convex functions or some generalized convex functions and the constraint sets as E-convex sets.

In the first part, we presented some new properties of (E-convex, E-cone, E-convex hull) sets and we introduced a new characterization for E-convex sets. We defined new sets, namely, E-convex cone hull, E-affine sets and E-affine hull, and we proved some of their properties and characterizations. Moreover, we discussed some new characterizations of convex functions, E-convex functions, and their generalizations in terms of some level sets and different forms of epigraphs which are related to these functions. Some general properties of generalized convex functions, and some differentiability properties of E-convex functions are also presented.

In the second part of this thesis and as an application of generalized convex functions in optimization problems, some optimality properties and characterizations of generalized non-linear optimization problems are discussed. In this generalized optimization problems, we used, as objective functions, E-

convex (strictly E-convex) functions and their generalizations such as E-quasi convex (strictly E-quasiconvex), and strictly quasi semi E-convex functions and the constraint sets are E-convex sets. Some E-differentiability properties for the objective functions of generalized optimization problems are also discussed in this part.

Introduction

Classical convex analysis is an important field of mathematics which plays a vital role in optimization and operation research. The main ingredient of convex analysis is related to convex sets and convex functions. The earlier definition and properties of convex sets were introduced by H. Brunn in 1887, followed by H. Minkowski in 1911. Convex analysis, in general, is developed and extensively studied in the 20th century by Fenchel [51], Brøndsted [3], Moreau [30, 31], and Rockafellar [45, 46]. It has been studied in finite dimensions (see e.g. [27, 28, 29, 31]), and in infinite dimensions [10, 23, 43]. In addition to convex functions, convex analysis field may include other types of functions with less restrictive convexity assumptions, such as quasi convex and pseudo convex functions (see [39, 41]). The latter types of functions represent generalizations of convex functions. This area of the classical convex analysis has been generalized into other kinds of convexity by many researchers. For instance, the concept of convex functions has been extended to the class of m-convex functions [21], invex functions [37], geodesic semi E-convex functions [5, 6], and B-vex functions [9] (see also [36, 55], for more recent papers on invex and B-vex functions).

Another type of generalized convexity is E-convexity introduced first by Youness in 1999 [15]. Youness introduced E- convex sets, E-convex functions, and E-convex programmings, defined in finite dimensional Euclidian space, by relaxing the definitions of the ordinary convex sets and convex functions. The effect of a mapping called $E: \mathbb{R}^n \to \mathbb{R}^n$ on a given set takes the major place in defining this type of generalized convexity. In other words, a non-empty set $S \subseteq \mathbb{R}^n$ is said to be E-convex if there exists a mapping $E: \mathbb{R}^n \to \mathbb{R}^n$ such that for every $s_1, s_2 \in S$ and for every $0 \leq \lambda \leq 1$ we have

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in S,$$

and a real valued function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be E-convex if there exists a mapping $E: \mathbb{R}^n \to \mathbb{R}^n$ such that for every $s_1, s_2 \in \mathbb{R}^n$ and for every $0 \leq \lambda \leq 1$ we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \lambda f(E(s_1)) + (1-\lambda)f(E(s_2)).$$

It was shown in [15] that many results of convex sets and convex functions hold for the wider class of E-convex sets and E-convex functions. The research on E-convexity is continued, improved, generalized, and extended in different directions. Abou Tair and Sulaiman [22] and Suneja et. al [48] studied E-convex sets and used these sets to prove some inequalities. Further study of E-convex sets are recently introduced by Grace and Thangavelu [25] in which the authors defined E-convex hull, E-cone, and E-convex cone and study some of their properties. Youness [16] studied some properties of E-convex programming and established the necessary and sufficient conditions of optimality for nonlinear E-convex programming. Stability in E-convex programming was studied by Youness [18], and very recently, Megahed et al. [1, 8] introduced duality in E-convex programming and studied optimality conditions for E-convex programming which has E-differentiable objective function. Note that some results appeared in Youness's first paper [15] are incorrect (see [12, 53, 54] for some counter examples that clarify the erroneous results in [15]). This motivates Chen to introduce new classes of E-functions and to study some of their properties [52, 53]. These functions, which are generalizations of the class of convex functions, are called semi E-convex, quasi semi E-convex, and pseudo semi E-convex functions.

The initial results of Youness inspired a great deal of subsequent work which has expanded the role of E-convexity in optimization theory. Thus, the notion of E-convex functions has been extended to new classes of generalized convex and

E-convex functions in optimization theory. For instance, some optimality properties of semi E-convex problems are introduced in [53]. E-quasiconvex functions, their related properties, and their applications to optimization problems are studied by Youness [17] and Syau and Lee [56]. Solemani-damaneh [40] defined E-pseudoconvexity functions and introduced an extensive study of E-convexity and its generalizations with applications for strict E-quasiconvexity and E-pseudoconvexity in multi-objective optimization problems. For more recent papers on E-convex function and their extensions and generalizations, see [26, 35, 42, 57].

The overall aim of this thesis is as follows:

(i) Establish new properties and characterizations of E-convex sets, E-convex functions and their extensions and generalizations.

(ii) Apply E-convex functions and some generalizations of convex functions to a non-linear optimization problems to obtain new results and optimality conditions different than the ones introduced in the literature.

This thesis starts with Chapter 1 which includes five sections of preliminary material and results that make this work self-contained. Chapters 2-4 include the main results of this work. The outline of Chapters 2-4 is summarized as follows:

Chapter two: consists of four sections. Section 2 presents further study of E-convex sets and E-convex hull. In specific, we provide a characterization for E-convex sets (see Theorem 2.2.2). In section 3, we discuss some new properties of E-cone and E-convex cone (see Propositions 2.3.1, 2.3.3-2.3.6). We provide two characterizations of E-convex cone of an arbitrary set (see Theorems 2.3.7-2.3.8). Then, we define E-convex cone hull sets and provide a new characterization of these sets (see Theorem 2.3.10). Finally, in section 4, we define E-affine sets and E-affine hull of a set and show a characterization of each of these sets (see Theorems 2.4.15-2.4.17). Some properties related to E-affine sets are also discussed (see Propositions 2.4.7, 2.4.9, 2.4.13-2.4.14). We also provide some examples to show the relationship between E-affine sets and their counterpart sets defined in the classical convex analysis, namely, affine sets (see Examples 2.4.5 and 2.4.10).

Chapter three: includes four sections. Section 2 starts with some general properties of generalized convex functions. Namely, the closedness property under addition and non-negative multiplication is proved for E-convex and semi E-convex functions (see Theorem 3.2.1). We show a composite property which satisfies, under certain conditions, for semi E-convex, E-convex, E-quasiconvex, and pseudo semi E-convex functions (see Theorems 3.2.2-3.2.4, 3.2.6). Another property we show is the supremum property of an arbitrary non-empty collection of semi E-convex, quasi semi (respectively, strictly quasi semi, strongly quasi semi) E-convex and pseudo semi E-convex functions (see Propositions 3.2.8-3.2.9). In section 3, we provide new properties and characterizations which relate convex functions and their generalizations with different α-level sets and different epigraphs associated with these functions. In other words, new relations and characterizations of semi E-convex, E-convex, and convex functions are given using the epigraph sets denoted by $epif$, $epi_E\ f$ and $epi^E\ f$ (see Propositions 3.3.12-3.3.22). In addition, new properties and characterizations of convex, quasi convex, and quasi semi E-convex functions are presented in terms of α-level sets of f denoted by $S_\alpha^E[f]$ and E-$S_\alpha[f]$ (see Propositions 3.3.2, 3.3.3, 3.3.6, 3.3.7). These α-level sets ($S_\alpha^E[f]$ and E-$S_\alpha[f]$) are, respectively, associated with the epigraphs $epi^E\ f$ and E-$e(f)$ mentioned earlier. We end this chapter, with section 4, by discussing some differentiability properties of E-convex and strictly E-convex functions (see Section 3.4). An important result in this section is characterizing E-convex and E-concave functions f by using the second derivative of f (see Theorem 3.4.7).

Chapter four: consists of three sections. In section 2, some optimality properties and characterizations of generalized non-linear optimization problems are presented. The properties and characterizations involve the existing, uniqueness, and the convexity of the global optimal solutions using (E-convex, strictly E-convex, strictly quasi semi E-convex, E-quasiconvex, and strictly E-quasiconvex) functions as the objective functions. In section 3, we study differentiability properties of the objective functions (foE) of a generalized optimization problems. In such case, the functions f are non - differentiable and are called E-differentiable.

CHAPTER ONE

PRELIMINARIES

Chapter One

Preliminaries

1.1 Introduction

In this preliminary chapter, we collect some essential definitions and properties that will make this thesis self-contained. The chapter is divided into five sections. In sections 2-4, we summarize definitions and results we need, from mathematical analysis, linear algebra, calculus, convex analysis, and E-convex analysis. It is worth mentioning that, in subsection 1.4.1, some new examples are illustrated to clarify the properties of E-convex, cone and E-cone, and E-convex cone and to discuss the relationship between them. Various concepts related to basic optimization and generalized optimization theory is introduced in section 5.

Throughout this thesis, the real line is denoted by $\mathbb{R}$ and the set of n-dimensional vectors with coordinates in $\mathbb{R}$ is referred to as $\mathbb{R}^n$. All sets considered are non-empty subsets of $\mathbb{R}^n$.

1.2 Basic Mathematical Concepts

In this section, we recall some basic and fundamental concepts and properties that are needed throughout this work. These concepts are collected from mathematical analysis, linear algebra, and advanced calculus.

Definition 1.2.1 [19, Section 2.4] A function $\|.\|: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a **norm** if the following axioms hold:

1. **nonnegativity**: $\|x\| \geq 0 \quad \forall\, x \in \mathbb{R}^n$ and $\|x\| = 0$ if and only if $x = 0$.
2. **Positive homogeneity**: $\|\lambda x\| = |\lambda|\, \|x\|\ \forall\, x \in \mathbb{R}^n$ and $\forall\, \lambda \in \mathbb{R}$.
3. **Triangle inequality**: $\|x + y\| \leq \|x\| + \|y\| \quad \forall\, x, y \in \mathbb{R}^n$.

Definition 1.2.2 [19, Section 2.4] A mapping $\langle .,. \rangle: \mathbb{R}^n$ x $\mathbb{R}^n \longrightarrow \mathbb{R}$ is referred to as **scalar (inner) product** if it satisfies the following properties.

1. **Positive definiteness**: $\langle x, x \rangle > 0 \ \ \forall x \in \mathbb{R}^n$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
2. **Symmetry**: $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in \mathbb{R}^n$.
3. **Additivity**: $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \quad \forall x, y, z \in \mathbb{R}^n$.
4. **Homogeneity**: $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n, \forall\, \lambda \in \mathbb{R}$.

Definition 1.2.3 [2, Definitions 1.13-1.14, 1.16] Fix $x \in \mathbb{R}^n$, then

1. The **open ball** with center x and radius r is denoted by $B(x, r)$ and defined by

$$B(x, r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}.$$

2. The **closed ball** with center x and radius r is denoted by $B[x, r]$ and defined by

$$B[x, r] = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}.$$

3. A set $U \subseteq \mathbb{R}^n$ is called a **neighborhood** of x if there exists an open ball $B(x,r)$ such that $B(x,r) \subseteq U$.
4. If $x \in V \subseteq \mathbb{R}^n$, then x is called an **interior** point of V if there exists an open ball $B(x,r)$ such that $B(x,r) \subseteq V$. The set of all interior points of a given set V is called the interior of V and is denoted by V^0. The set V is said to be **open** if $V = V^0$.

Definition1.2.4 [19, p.37] Let $S \subseteq \mathbb{R}^n$. A function $f: S \longrightarrow \mathbb{R}$ is said to be **continuous** at $\bar{s}$ if for any given $\epsilon > 0$ there is exists a $\delta > 0$ such that $\forall s \in S$ and $\|s - \bar{s}\| < \delta$ implies $|f(s) - f(\bar{s})| < \epsilon$.

Definition 1.2.5 [39, p.763] Let S be a subset of $\mathbb{R}^n$, $s^T = (s_1, \dots, s_n)^T \in S^0$, and let $f: S \longrightarrow \mathbb{R}$. Then f is said to be **differentiable** at s if there is a vector called the **gradient** of f at the point s, and is denoted by $\nabla f(s)$ in $\mathbb{R}^n$. The gradient vector consists of the n partial derivatives of f at s, that is,

$$\nabla f(s) = \begin{pmatrix} \frac{\partial f(s)}{\partial s_1} \\ \frac{\partial f(s)}{\partial s_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f(s)}{\partial s_n} \end{pmatrix}.$$

If the function f is defined on a set $S \subset \mathbb{R}$, then $\nabla f(s) = f'(s)$. Moreover, f is called **twice differentiable** at s, if in addition to the gradient vector, there exists $n \times n$ matrix of second-order partial derivatives of f. This matrix is called **Hessian** and is denoted by $\nabla^2 f(s)$ or $H(s)$. It is defined as follows

$$\nabla^2 f(s) = H(s) = \begin{bmatrix} \frac{\partial^2 f(s)}{\partial s_1 \partial s_1} & \cdots & \frac{\partial^2 f(s)}{\partial s_1 \partial s_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(s)}{\partial s_n \partial s_1} & \cdots & \frac{\partial^2 f(s)}{\partial s_n \partial s_n} \end{bmatrix}.$$

Hessian is a symmetric matrix which describes the local curvature of a function of many variables. If the function f defined on a set $S \subset \mathbb{R}$, then $\nabla^2 f(s) = f''(s)$.

Definition 1.2.6 [19, p.59] A continuous function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be **continuously differentiable** at $s \in \mathbb{R}^n$ if its first partial derivatives are continuous. i.e., if $\frac{\partial f}{\partial s_i}(s)$ exists and is continuous, for $i = 1, \ldots, n$. The class of functions whose first derivatives are continuous is denoted by C^1. Similarly, the function f is said to be **twice continuously differentiable** if f is continuously differentiable and all second partial derivatives of f exist and are continuous over $\mathbb{R}^n$. i.e., if $\frac{\partial^2 f}{\partial s_i s_j}(s)$ exists and is continuous, for $i, j = 1, \ldots, n$. The class of functions whose second derivatives are continuous is denoted by C^2. Note that when f is C^2, the Hessian is a symmetric matrix .

As we will see later in this thesis (see Chapters 3 and 4) that the sign of a matrix is very useful. i.e., whether the matrix is positive (semi) definite, negative (semi) definite or indefinite. The sign of the Hessian, for example, determine whether a function is E-convex or not (see Theorem 3.4.7). It is also necessary to employ Taylor's Formula (see Theorems 3.4.7, 4.2.4, 4.3.9). Hence, we state these concepts next.

Definition 1.2.7 [2, Definitions 2.9, 2.14] An $n \times n$ matrix A is said to be

1. **Positive definite** (for short, p.d.) if the quadratic form
$$x^T A\, x > 0 \quad \forall\, x \neq 0;\ x \in \mathbb{R}^n.$$
2. **Positive semi-definite** (for short, p.s.d.) if
$$x^T A\, x \geq 0 \quad \forall x \neq 0;\ x \in \mathbb{R}^n.$$

3. **Negative definite** (for short, n.d.) and **Negative semi-definite** (for short, n.s.d.) matrices if:

$$x^T A\, x < 0 \qquad \forall\, x \neq 0;\; x \in \mathbb{R}^n,$$

and

$$x^T A\, x \leq 0 \qquad \forall\, x \neq 0;\; x \in \mathbb{R}^n,$$

respectively.

4. **Indefinite** if there exist non-zero vectors $x, y \in \mathbb{R}^n$ such that

$$x^T A\, x > 0 \text{ and } y^T A\, y < 0.$$

Example 1.2.8 Clarify that the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$$

is indefinite.

Solution. Suppose that $x^T = (x_1, x_2)^T \in \mathbb{R}^2$ is a non-zero vector. Then

$$x^T A\, x = (x_1, x_2)^T \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x^T A\, x = x_1^2 + x_2^2 + 4x_1 x_2 + 4x_1 x_2$$

$$= x_1^2 + x_2^2 + 8x_1 x_2.$$

Now, if $x = (1,1)^T$, then $x^T A\, x > 0$ and when $x = (1, -1)^T$, then $x^T A\, x < 0$. Thus, from definition 1.2.7(4), the matrix A is indefinite.

Definition 1.2.9 [24, Section 2.6] Taylor formula is a series expansion of a function around a point. An n-th order Taylor series is an expansion of an n continuously differentiable real function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ around a point $x = x_0$ which is given by

$$f(x) = f(x_0) + < \nabla f(x_0), (x - x_0) > + \frac{1}{2} < (x - x_0), \nabla^2 f(x_0)(x - x_0) >$$

$$+ o(x - x_0),$$

where $o(x - x_0)$ is called the remainder term of the Taylor series.

<u>Taylor's Theorem 1.2.10</u> [2, Theorem 1.24]

<u>Truncated Taylor Series (First Order)</u>: Assume that $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ continuously differentiable on some open set and that $x_0 \in \mathbb{R}^n$. Then for every $x \in \mathbb{R}^n$.

$$f(x) = f(x_0) + < \nabla f(\xi), (x - x_0) >,$$

where ξ is some point lies on the line segment joining x and x_0 (i.e., $\xi = \lambda x + (1 - \lambda)x_0$ for some $\lambda \in [0,1]$).

<u>Truncated Taylor Series (Second Order)</u>: Assume that $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ twice continuously differentiable on some open set and that $x_0 \in \mathbb{R}^n$. Then for every $x \in \mathbb{R}^n$.

$$f(x) = f(x_0) + < \nabla f(x_0), (x - x_0) > + \frac{1}{2} < (x - x_0), \nabla^2 f(\xi)(x - x_0) >,$$

where ξ is some point lies on the line segment joining x and x_0.

<u>Definition 1.2.11</u> Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function. Then

1) f is called **linear** if and only if
   $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \forall x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. [47]
2) f is called **sublinear** if and only if
   $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \quad \forall x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. [44]
3) f is called **non-decreasing** if whenever $x, y \in \mathbb{R}^n$ such that $x \preccurlyeq y$ (i.e., $x_i \leq y_i, \ \forall i = 1, \dots, n$) we get $f(x) \leq f(y)$. In other words,

$(x-y)^T(f(x)-f(y)) \geq 0$. [4, Definition 5.2.1]

Definition 1.2.12 A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **idempotent** if $f\big(f(x)\big) = f^2(x) = f(x) \quad \forall x \in \mathbb{R}^n$. [25]

1.3 Elements of Convex Analysis

In this section, we review some concepts from convex analysis in the classical sense such as convex sets, cone, convex cone sets, affine sets, and convex functions with some of their generalizations functions.

We start first with the definition of a convex set in $\mathbb{R}^n$.

Definition 1.3.1 [44, p.10] A set $S \subset \mathbb{R}^n$ is said to be **convex** if and only if $\forall s_1, s_2 \in S$, and for every $0 \leq \lambda \leq 1$, we have $\lambda s_1 + (1-\lambda)s_2 \in S$. In this case, S is said to be closed for convex combinations.

Convex sets satisfy the relations given next.

Proposition 1.3.2 [44]

i. The intersection of two convex sets is a convex set. In general, if $\{S_i : i \in \Lambda\} \subseteq \mathbb{R}^n$ be a family of convex sets. Then $\bigcap_{i\in\Lambda} S_i$ is a convex set.
ii. Let $S \subseteq \mathbb{R}^n$ be a convex set and $a \in \mathbb{R}$. Then the set $aS = \{as : s \in S\}$ is a convex set.
iii. Let $S_1, S_2 \subseteq \mathbb{R}^n$ be convex sets. Then the Minkowski addition

$$S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$$

and the Cartesian product

$$S_1 \times S_2 = \{(s_1, s_2) \in \mathbb{R}^n \times \mathbb{R}^n : s_1 \in S_1, s_2 \in S_2\}$$ is a convex set.

Definition 1.3.3 [44, p.3] A set $M \subset \mathbb{R}^n$ is said to be **affine** if and only if $\forall x, y \in M$, and for every $\lambda \in \mathbb{R}$, we have $\lambda x + (1-\lambda) y \in M$. In this case, M is said to be closed for affine combinations.

Remark 1.3.4 Each affine set is a convex set but the converse is not true as the following example shows.

Example 1.3.5 Let $B(x,r) \subset \mathbb{R}^2$ be an open ball of center $x = (0,0)$ and radius $r = 4$. i.e., $B(x,r) = B((0,0),4) = \{y = (y_1, y_2) \in \mathbb{R}^2 : \|(y_1, y_2) - (0,0)\| < 4\}, \forall x, y \in \mathbb{R}^2$. It is clear that, $B(x,r)$ is a convex but not an affine set. i.e., Let $(y_1, y_2) = (1,1) \in B(x,r)$, $(y_1^*, y_2^*) = (\frac{1}{4}, \frac{1}{4}) \in B(x,r)$. If $\lambda = \frac{1}{2}$, then

$$\frac{1}{2}(1,1) - \frac{1}{2}(\frac{1}{4}, \frac{1}{4}) = (\frac{3}{8}, \frac{3}{8}) \in B(x,r)$$

Now, if $\lambda \in \mathbb{R}$, such that $\lambda = 4$, then we get

$$4(1,1) - 3(\frac{1}{4}, \frac{1}{4}) = (\frac{13}{4}, \frac{13}{4}) \notin B(x,r).$$

Next, we define an important set in convex analysis.

Definition 1.3.6 [44, p.13] A set $K \subset \mathbb{R}^n$ is said to be a **cone** if for every $x \in K$ and $\alpha \geq 0$ we get $\alpha x \in K$. In case that the cone is convex, then K is called **convex cone**.

Example 1.3.7

- Let $K = \{(x,y) : y = |x|\}$, then K is a cone but not convex set.
- Let $K = \{(x,y) : y \geq |x|\}$. This set is a convex cone.
- Let $K = \{(x,y) : x^2 + y^2 \leq 1\}$. This set is a convex but not cone.

Definition 1.3.8 [44, p.12] Let $S \subset \mathbb{R}^n$. The convex hull of S, denoted by $conv(S)$ is the intersection of all convex sets containing S (or, smallest convex set that contains S); that is,

$$conv(S) = \bigcap_{N \supseteq S} N; \ N \text{ are convex sets.}$$

Definition 1.3.9 [11, p.21] Let $M \subset \mathbb{R}^n$. The affine hull of M, denoted by $aff(M)$ is the intersection of all affine sets containing M (or, smallest affine set that contains M); that is,

$$aff(M) = \bigcap_{N \supseteq M} N; \ N \text{ are affine sets.}$$

Definition 1.3.10 [13, p.36] Let $K \subset \mathbb{R}^n$. The convex cone hull of K, denoted by $cone(K)$ is the intersection of all convex cone sets containing K (or, smallest convex cone set that contains K); that is,

$$cone(K) = \bigcap_{N \supseteq K} N; \ N \text{ are convex cone.}$$

Now, let us recall the definitions of convex (concave) functions.

Definition 1.3.11 [47, Definition 3.1.1] A real valued function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be **convex** if for every $x_1, x_2 \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

If for every $x_1 \neq x_2$ and $0 < \lambda < 1$

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2).$$

Then f is called **strictly convex**.

Definition 1.3.12 [47, Definition 3.1.1] A real valued function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be **concave** if $(-f)$ is convex. Mathematically, f is a concave function if for every $x_1, x_2 \in \mathbb{R}^n$, $0 \leq \lambda \leq 1$ we have

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2).$$

If for every $x_1 \neq x_2$ and $0 < \lambda < 1$

$$f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda)f(x_2).$$

Then f is called **strictly concave**.

Remark 1.3.13

1. Every linear function is convex and concave.
2. Every convex and concave function at the same time is called an **affine function.**
3. The domain of a convex function is always a convex set.

An example of a convex function defined on $\mathbb{R}^n$ is given next.

Example 1.3.14 Show that $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as $f(x) = \|x\|$ is a convex function.

Solution: let $x_1, x_2 \in \mathbb{R}^n$, $0 \leq \lambda \leq 1$ then,

$$f(\lambda x_1 + (1-\lambda)x_2) = \|\lambda x_1 + (1-\lambda)x_2\|,$$

using triangle inequality and positive homogeneity properties of the norm, the right-hand side of the above equation yields

$$\leq \lambda\|x_1\| + (1-\lambda)\|x_2\|$$

$$= \lambda f(x_1) + (1-\lambda)f(x_2)$$

Thus, $f(x)$ is a convex function.

The field of convex analysis may include other types of functions with less restrictive convexity assumptions, such as quasi convex and pseudo convex functions. The latter types of functions represent generalizations of convex functions. Next, we recall the definitions of quasi convex and pseudo convex**.**

Definition 1.3.15 A function $f: S \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be **quasi convex** if and only if S is a convex set, and for each $s_1, s_2 \in S$, $0 \leq \lambda \leq 1$, we have
$f(\lambda s_1 + (1-\lambda)s_2) \leq \max\{f(s_1), f(s_2)\}$. [39, Definition 3.5.1]
If for every $s_1 \neq s_2$ and $0 < \lambda < 1$ with $f(s_1) \neq f(s_2)$, we have
$f(\lambda s_1 + (1-\lambda)s_2) < \max\{f(s_1), f(s_2)\}$, then f is called **strictly quasi convex**. [39, Definition 3.5.5]

<u>Definition 1.3.16</u> [39, p.768] The function $f: S \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be **pseudo convex** if f is differentiable and for each $s_1, s_2 \in S$,with $\nabla f(s_1)^T(s_2 - s_1) \geq 0$, we have $f(s_2) \geq f(s_1)$. The function f is said to be **strictly pseudo convex** on S if whenever $s_1 \neq s_2$ with $\nabla f(s_1)^T(s_2 - s_1) \geq 0$, we have $f(s_2) > f(s_1)$.

1.4 Elements of E-Convex Analysis

An important class of generalized convex sets and convex functions, called E-convex sets and E-convex functions, respectively, has first introduced and studied by Youness [15]. In these classes, Youness relaxed the definitions of the classical convex sets and convex functions with respect to an mapping E. Further study of E-convex sets are recently introduced by Grace and Thangavelu [25] in which the authors defined E-convex hull, E-cone, and E-convex cone and studied some of their properties. Other types of generalized convex functions are also introduced and studied in the literature such as semi E-convex, quasi semi E-convex, pseudo semi E-convex, E-quasiconvex, and E-pseudo convex functions [17, 40, 53, 56].

In this section, we recall E-convex sets, E-convex hull, E-cone, E-convex cone, E-convex functions, and some of generalized convex functions. We review some algebraic properties of E-convex sets and add a new property (see Proposition 1.4.1.10). Many examples are added, in the next subsection, to show the relationship between E-convex sets, E-cones, and E-convex cones (see Examples 1.4.1.15-1.4.1.17). Other examples are shown to illustrate some reviewed concepts and properties (see Examples 1.4.1.19, 1.4.1.21).

1.4.1 Generalized Convex Sets

In this subsection, we recall E-convex, E-cones, and E-convex cone sets and some of their existing properties. We also add a new property (see Proposition 1.4.1.10) and various examples to show the relationships between these concepts and to clarify some properties and observations related to the E-convexity of sets (see Examples 1.4.1.4,1.4.1.12, 1.4.1.15-1.4.1.17, 1.4.1.19, 1.4.1.21). In Chapter 2, we continue studying these concepts by providing new properties and characterizations of these generalized sets.

Definition 1.4.1.1 [15] A set $S \subset \mathbb{R}^n$ is said to be ***E*-convex** if and only if there exists a mapping $E: \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall s_1, s_2 \in S$ and for every $0 \leq \lambda \leq 1$ we have $\lambda E(s_1) + (1-\lambda)E(s_2) \in S$.

Note that E-convex sets are considered as generalization of convex sets in the following sense.

Proposition 1.4.1.2 [15] Every convex set is an E-convex. (Choose $E = I$ identity mapping).

Proposition 1.4.1.3 [15]

1. If a set S is an E-convex, then $E(S) \subseteq S$.
2. If $E(S)$ is convex and $E(S) \subseteq S$, then S is an E-convex set.

The following example show that the converse of Proposition 1.4.1.2 does not hold, in general.

Example 1.4.1.4 Suppose that $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined as $E(x, y) = (0, y)$. Let

$$S = \{(x,y) \in \mathbb{R}^2 : (x,y) = \lambda_1(0,0) + \lambda_2(2,3) + \lambda_3(0,3) + \lambda_4(2,0)\}$$

$$\cup\{(x,y) \in \mathbb{R}^2 : (x,y) = \lambda_1(0,0) + \lambda_2(-2,-3) + \lambda_3(0,-3)$$

$+\lambda_4(-2,0)\}$ with $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$, $\sum_{i=1}^{4} \lambda_i = 1$.

First, we show that S is E-convex using Proposition 1.4.1.3(2)

$$E(S) = \{(x,y) \in \mathbb{R}^2 : E(x,y) = E(\lambda_1(0,0) + \lambda_2(2,3) + \lambda_3(0,3) + \lambda_4(2,0))\} \cup \{(x,y) \in \mathbb{R}^2 : E(x,y) = E(\lambda_1(0,0) + \lambda_2(-2,-3) + \lambda_3(0,-3) + \lambda_4(-2,0))\}.$$

Since E is a linear mapping, the set $E(S)$ can be written as

$$E(S) = \{(x,y) \in \mathbb{R}^2 : E(x,y) = \lambda_1 E(0,0) + \lambda_2 E(2,3) + \lambda_3 E(0,3) + \lambda_4 E(2,0)\} \cup \{(x,y) \in \mathbb{R}^2 : E(x,y) = \lambda_1 E(0,0) + \lambda_2 E(-2,-3) + \lambda_3 E(0,-3) + \lambda_4 E(-2,0)\},$$

$$E(S) = \{(x,y) \in \mathbb{R}^2 : \lambda_1(0,0) + \lambda_2(0,3) + \lambda_3(0,3) + \lambda_4(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : \lambda_1(0,0) + \lambda_2(0,-3) + \lambda_3(0,-3) + \lambda_4(0,0)\}.$$

It is clear that, $E(S)$ is a convex set and $E(S) \subseteq S$. Using Proposition 1.4.1.3(2), S is E-convex. To show S is not convex, take $(-2,0), (0,3) \in S$ and $\lambda = \frac{1}{2}$. Then

$$\frac{1}{2}(-2,0) + \frac{1}{2}(0,3) = (-1,0) + \left(0, \frac{3}{2}\right) = \left(-1, \frac{3}{2}\right) \notin S$$ as we need to show.

The following proposition provides a condition under which a convex set is an E-convex without taking $E = I$ the identity mapping.

Proposition 1.4.1.5 [25, 40] If E is a given mapping, S is a convex set, and $E(S) \subseteq S$ then S is an E-convex set.

Proposition 1.4.1.6 Let S_1 and S_2 are two E-convex sets, then

i. $S_1 \cap S_2$ is E-convex set. [15]
ii. If E is a linear mapping, then $S_1 + S_2$ is E-convex set. [15]
iii. If E is a linear mapping and $a \in \mathbb{R}$ then aS_1 is E-convex set. [25]

Remark 1.4.1.7

1. The intersection property, in the above proposition, has been extended to an arbitrary family of E-convex sets. [56, Theorem 2.1]
2. The union of two E-convex sets may not be E-convex set. [15, Example 2.3]

For the sack of completeness, we add the following property of E-convex sets which is needed in Proposition 2.3.6(iii).

Proposition 1.4.1.8 Let S_1 and S_2 be two E-convex sets, then $S_1 \times S_2 = \{(s_1, s_2): s_1 \in S_1, s_2 \in S_2\}$ is $E \times E$-convex set.

Proof. Since S_1 and S_2 are E-convex sets, then $\forall\, s_1,\ \overline{s_1} \in S_1,\ \forall\, s_2,\ \bar{s_2} \in S_2$, and $\lambda_1,\ \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 = 1$ we have

$$\lambda_1 E(s_1) + \lambda_2 E(\overline{s_1}) \in S_1 \text{ and } \lambda_1 E(s_2) + \lambda_2 E(\overline{s_2}) \in S_2 .$$

Hence, $\left(\lambda_1 E(s_1) + \lambda_2 E(\overline{s_1}),\ \lambda_1 E(s_2) + \lambda_2 E(\overline{s_2})\right) \in S_1 \times S_2$.

i.e., $\lambda_1 (E \times E)(s_1, s_2) + \lambda_2 (E \times E)(\overline{s_1}, \overline{s_2}) \in S_1 \times S_2$. Thus, $S_1 \times S_2$ is $E \times E$-convex set. ■

We pointed out in Remark 1.4.1.7 that the intersection of arbitrary E-convex sets is E-convex. This fact is used next to define the smallest E-convex set containing a fixed set.

Definition 1.4.1.9 [25] The E-convex hull of a set $S \subset \mathbb{R}^n$, denoted by E-$conv(S)$ is the smallest E-convex set contains S, that is,

$$E\text{-}conv(S) = \bigcap_{N \supseteq S} N,\ N \text{ are } E\text{-convex sets.}$$

An example of E-convex hull of a non-convex set S is given next.

Example 1.4.1.10 Let $S = (-2,3) \cup (3,6) \subset \mathbb{R}$ and let $E: \mathbb{R} \longrightarrow \mathbb{R}$ is given by $E(x) = \frac{3}{4}\, x \quad \forall x \in \mathbb{R}$. Clearly, S is not E-convex set. For instance, let $x = 2, y = 4,$ and $\lambda = 0$. Then,

$$\lambda E(x) + (1-\lambda)E(y) = 3 \notin S.$$

From Definition 1.4.1.9, $E\text{-}conv(S) = (-2,6)$ which is E-convex. i.e., $E - conv(S)$ is a smallest E-convex set in $\mathbb{R}$ contains S.

Remark 1.4.1.11 [44] From the above definition and Proposition 1.4.1.6, it is clear that

1. $E - conv(S)$ is E-convex set and $S \subseteq E\text{-}conv(S)$.
2. If S is E-convex set, then $E\text{-}conv(S) = S$.

Next, we recall the definition of E-cone as a generalization of a cone set.

Definition 1.4.1.12 [25] A set $K \subset \mathbb{R}^n$ is called ***E*-cone** if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for every $x \in K$ and $\alpha \geq 0$ we have $\alpha E(x) \in K$. If K is E-cone and E-convex set, then it is called ***E*-convex cone**.

Examples of E-convex cone set, E-convex set (not E-cone), and E-cone (not E-convex set) are given, respectively, next.

Example 1.4.1.13 Let $K \subset \mathbb{R}^2$ is defined by $K = \{\, (x,y) \in \mathbb{R}^2 : x, y \ \geq 0 \,\}$, and let $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $E(x,y) = (x, 0) \quad \forall x, y \in \mathbb{R}$.

For any $(x,y) \in K$ and $\alpha \geq 0$, $\alpha E(x,y) = (\alpha x, 0) \in K$. Thus, K is E-cone. Also, K is E-convex. Indeed, let (x_1, y_1), $(x_2, y_2) \in K$ and λ_1, $\lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$, then

$$\lambda_1 E(x_1, y_1) + \lambda_2 E(x_2, y_2) = (\lambda_1 x_1 + \lambda_2 x_2 \,, 0) \in K$$

Thus, K is E-convex cone set.

Example 1.4.1.14 Let $K \subset \mathbb{R}^2$ is defined by $K = \{ (x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1 \}$, and let $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $E(x, y) = \left(\frac{1}{2}x, \frac{1}{4}y\right)$ $\forall x, y \in \mathbb{R}$. Note that $E(K) = \{\left(\frac{1}{2}x, \frac{1}{4}y\right): -1 \leq x \leq 1, -1 \leq y \leq 1\}$ is a convex set and $E(K) \subseteq K$. From Proposition 1.4.1.3(2), K is E-convex set. To show that K is not E-cone, take for example $(1,1) \in K$ and $\alpha = 5$. Then $\alpha E(x, y) = \left(\frac{5}{2}, \frac{5}{4}\right) \notin K$.

Example 1.4.1.15 Let $K \subset \mathbb{R}^2$ is defined by

$K = \{ (x, y) \in \mathbb{R}^2 : x \leq -1, -1 \leq y \leq 1 \} \cup \{ (x, y) \in \mathbb{R}^2 : x \geq 1, -1 \leq y \leq 1 \}$, and let $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $E(x, y) = (x, 0)$. For each $(x, y) \in$ K and $\alpha \geq 0$, $\alpha E(x, y) = (\alpha x, 0) \in K$. Thus, K is E - cone. However, there exists $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $\lambda \in [0,1]$ such that, $\lambda E(x_1, y_1) + (1 - \lambda)E(x_2, y_2) \notin K$. For example, take $(-1,1), (1,1) \in K$, and $\lambda = \frac{1}{2}$. Then

$$\lambda E(-1,1) + (1 - \lambda)E(1,1) = \frac{1}{2}(-1,0) + \frac{1}{2}(1,0) = (0,0) \notin K$$

Thus, K is not E-convex.

Proposition 1.4.1.16 Every cone is an E-cone. (Take $E = I$).

Obviously, not every E-cone is a cone as we show in the following example.

Example 1.4.1.17 Consider K defined as in the Example 1.4.1.14, i.e., $K = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}$, and let $E(x, y) = (0,0)$ $\forall\, x, y \in \mathbb{R}$. We show that K is E-cone but not cone. For any $\alpha \geq 0$ and any $(x, y) \in K$, $\alpha E(x, y) = (0,0) \in K$, thus, K is E-cone. Now, if we take $\alpha = 5$, and $(x, y) = (1,1) \in K$, then

$$\alpha(x, y) = 5(1,1) = (5,5) \notin K.$$

Thus, K is not a cone.

Remark 1.4.1.18 Proposition 1.4.1.16 may not be true for an arbitrary mapping E as the next example shows.

Example 1.4.1.19 Suppose that $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined as $E(x, y) = (x^2, y^2)$ $\forall\, x, y \in \mathbb{R}$. and $K = \{(x, y) \in \mathbb{R}^2 : x \leq 0\,, y \leq 0\}$. We show that K is cone but not E-cone. For any $\alpha \geq 0$ and for any $(x, y) \in K$, we have, $\alpha(x, y) = (\alpha x, \alpha y) \in K$. Thus, K is a cone. To show K is not E-cone. Let $(x, y) = (-3, -5) \in K$ and $\alpha = 3$, then $\alpha E(x, y) = \alpha(x^2, y^2) = 3(9,25) \notin K$ as required.

1.4.2 Generalized Convex Functions

In Chapters 3 and 4, we deal with E-convex functions, some of its generalized versions, and another class of generalized convex functions, namely, semi E-convex functions. To prepare the ground for this study, we present in this section the definitions of E-convex, semi E-convex, quasi semi E-convex, pseudo semi E-convex, E-quasiconvex, and E-pseudoconvex functions. We also provide some related notions which will be used in developing our work in Chapters 3 and 4.

Let us first define E-convex function and strictly E-convex function.

Definition 1.4.2.1 [15] Let $f: S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function. Then f is referred to as ***E*-convex function** on S if and only if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is an E-convex set and for each $s_1, s_2 \in S$, and each $0 \leq \lambda \leq 1$, we have

$$f(\lambda E(s_1) + (1 - \lambda)E(s_2)) \leq \lambda f(E(s_1)) + (1 - \lambda) f(E(s_2)).$$

On the other hand, f is **strictly *E*-convex** if for each $s_1, s_2 \in S$, $s_1 \neq s_2$, and each $0 < \lambda < 1$, we have

$$f\big(\lambda E(s_1) + (1 - \lambda)E(s_2)\big) < \lambda f(E(s_1)) + (1 - \lambda) f(E(s_2)).$$

Definition 1.4.2.2 [15] A real valued function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be ***E*-concave** on S if and only if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is an E-convex set and for every $s_1, s_2 \in S$, and each $0 \leq \lambda \leq 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) \geq \lambda f(E(s_1)) + (1-\lambda)f(E(s_2)).$$

If for every $s_1, s_2 \in S$, $s_1 \neq s_2$, and $0 < \lambda < 1$

$$f\big(\lambda E(s_1) + (1-\lambda)E(s_2)\big) > \lambda f(E(s_1)) + (1-\lambda)f(E(s_2)).$$

Then f is called **strictly *E*-concave**.

Remark 1.4.2.3 The class of E-convex functions is broader and more general than the class of ordinary convex functions. Indeed, by taking $E = I$, every convex function is E-convex and the converse does not satisfy (for an arbitrary E) as we illustrate next.

Example 1.4.2.4 [15, Example 3.2] Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a function and let $E: \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping such that $E(x) = -x^2$. Suppose that for each $x, y \in \mathbb{R}$, $\lambda \in [0,1]$ we have $f(x) = \begin{cases} 1 & x > 0 \\ -x & x \leq 0 \end{cases}$

To prove that f is E-convex, we must show that

$$f\big(\lambda E(x) + (1-\lambda)E(y)\big) \leq \lambda f(E(x)) + (1-\lambda)f(E(y))$$

We consider three cases: first, if $x, y > 0$, then,

$$f(-\lambda x^2 - (1-\lambda)y^2) = \lambda x^2 + (1-\lambda)y^2 = \lambda f(E(x)) + (1-\lambda)f(E(y)).$$

In this case, f is E-convex

$$f\big(\lambda E(x) + (1-\lambda)E(y)\big) = \lambda f E(x) + (1-\lambda) f E(y)$$

Similarly, when $x, y \leq 0$ and $x > 0, y \leq 0$, we get f is E-convex. To show that f is not a convex function. Take $x = 2$, $y = 0$ and $\lambda = \frac{1}{2}$ then,

$$f(\lambda x + (1-\lambda)y) = f(1+0) = 1, \tag{1.1}$$

and

$$\lambda f(x) + (1-\lambda)f(y) = \frac{1}{2}f(2) + \frac{1}{2}f(0) = \frac{1}{2}. \quad (1.2)$$

From (1.1) and (1.2), we conclude

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$$

Then, f is not convex.

A new class of generalized convex functions called semi E-convex function is introduced by Chen [52, 53]. This class includes quasi semi E-convex and pseudo semi E-convex functions. Chen used these functions to improve some of the Youness's incorrect results [15, Theorems 4.2-4.3, 4.6], and to study the properties of those functions. Next, we state the definition of those functions.

Definition 1.4.2.5 [52,53] Let $f: S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function, then f is said to be

i. **Semi $\boldsymbol{E}$-convex** on S if and only if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is E-convex set and for each $s_1, s_2 \in S$, $0 \le \lambda \le 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) \le \lambda f(s_1) + (1-\lambda)f(s_2).$$

ii. **Quasi semi $\boldsymbol{E}$-convex** function if and only if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is a E-convex set and for each $s_1, s_2 \in S$ and $0 \le \lambda \le 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) \le \max\{f(s_1), f(s_2)\},$$

and f is **strictly quasi semi $\boldsymbol{E}$-convex function** if and only if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is a E-convex set and for each $s_1, s_2 \in S$ with $E(s_1) \neq E(s_2)$, and $0 < \lambda < 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) < \max\{f(s_1), f(s_2)\},$$

and f is **strongly quasi semi E-convex function** if and only if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is a E-convex set and for each $s_1, s_2 \in S$ with $s_1 \neq s_2$, and $0 < \lambda < 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) < \max\{f(s_1), f(s_2)\}.$$

iii. **Pseudo semi E-convex** on E-convex set S if there exists a positive function $b: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that if $f(s_1) < f(s_2)$ then

$$f\big(\lambda E(s_1) + (1-\lambda)E(s_2)\big) \leq f(s_2) + \lambda(\lambda - 1)b(s_1, s_2),$$

for all $s_1, s_2 \in S,$ and $0 < \lambda < 1$.

Remark 1.4.2.6

i. An E-convex function is not necessary semi E-convex function. [53, Example 4]
ii. A semi E-convex function is not necessary E-convex function. [53, Example 6]
iii. A quasi semi E-convex function is not necessary semi E-convex (respectively, E-convex) function. [53, Remark 4]
iv. Every semi E-convex function is a pseudo semi E-convex. [53, Proposition 10]

Observe, from the preceding remark, that the class of semi E-convex functions is not a generalization of the class of E-convex functions. Rather, it is a generalization of the class of convex functions when $E = I$. The following proposition confirms the last observation.

Proposition 1.4.2.7 Every convex function is a semi E-convex (respectively, quasi semi E-convex, pseudo semi E-convex) function when $E = I$.

Proof. It is easy to prove that every convex function is a semi E-convex (respectively, quasi semi E-convex), by taking $E = I$. To show every convex function is a pseudo semi E-convex, let $f: S \longrightarrow \mathbb{R}$ be a convex function defined on the convex set $S \subseteq \mathbb{R}^n$ and $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $E = I$. Since $f(x) < f(y)$ and f is a convex on S, then for every $x, y \in S$ and $0 < \lambda < 1$, we have

$$
\begin{aligned}
f(\lambda E(x) + (1-\lambda)E(y) &= f(\lambda x + (1-\lambda)y) \\
&\leq \lambda f(x) + (1-\lambda)f(y) \\
&= f(y) + \lambda\big(f(x) - f(y)\big) \\
&< f(y) + \lambda(1-\lambda)\big(f(x) - f(y)\big) \\
&= f(y) + \lambda(\lambda-1)\big(f(y) - f(x)\big) \\
&= f(y) + \lambda(\lambda-1)b(x,y),
\end{aligned}
$$

where $b(x,y) = f(y) - f(x) > 0$. Hence, f is pseudo semi E-convex. ■

Remark 1.4.2.8 The converse of the proceeding proposition may not be true. In other words,

i. A semi E-convex function is not necessary convex function. [53, Example 6]
ii. From Remark 1.4.2.6 (iii) and Proposition 1.4.2.7, a quasisemi E-convex function is not necessary convex function.

Another type of functions, namely E-quasiconvex and E-pseudoconvex, are introduced as a generalization of E-convex functions, and hence generalizations of convex functions. E-quasiconvex function is established independently by Youness [17] and Solimani [40] and its properties are studied. Some of the E-quasiconvex function properties are also studied by Syau and Lee [56]. E-pseudoconvex function, on the other hand, is defined and studied by Solimani [40]. These functions are generalizations of quasi convex and pseudo convex functions introduced earlier (see Definitions 1.3.15-1.3.16). The definition of these functions is given next

Definition 1.4.2.9 [40] Let $f: S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function, then f is said to be

i. ***E*-quasiconvex** if and only if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is an E-convex set and for each $s_1, s_2 \in S$ with $s_1 \neq s_2$, and $0 < \lambda < 1$, we have

$$f\big(\lambda E(s_1) + (1-\lambda)E(s_2)\big) \leq \max\{f(E(s_1), f(E(s_2)\};$$

and f is **strictly *E*-quasiconvex** if and only if there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is an E-convex set and for each $s_1, s_2 \in S$, $f(E(s_1)) \neq f(E(s_2))$, and each $0 < \lambda < 1$, we have

$$f\big(\lambda E(s_1) + (1-\lambda)\mathrm{E}(s_2)\big) < \max\{f(E(s_1), f(E(s_2)\}$$

ii. ***E*-pseudoconvex function** if f is differentiable, there exists a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is E-convex and for each $s_1, s_2 \in S$ such that if $\nabla f\big(E(s_2)\big)^T \big(E(s_1) - E(s_2)\big) \geq 0$, we have $f\big(E(s_1)\big) \geq f\big(E(s_2)\big)$.

Remark 1.4.2.10 It can be seen that E-quasiconvex and E-pseudoconvex are generalization of convex functions and E-convex functions in the sense that

i. From the definition of E-convex function, every E-convex function is E-quasiconvex.
ii. Every differentiable E-convex function is E-pseudoconvex. [40, Lemma 2.3]
iii. The converse of parts (i)-(ii) may not hold. [40, Example 3.4]
iv. Every convex function is E-quasiconvex where $E = I$.
v. E-quasiconvex function is not necessary convex. [40, Example 3.3]
vi. Every differentiable convex function is E-pseudoconvex when $E = I$.
vii. From [40, Example 3.4] and Proposition 1.4.1.2, E-pseudoconvex function is not necessary convex.

when studying convex functions in the classical sense, the set of points located on or above the graph of f, which is called the epigraph of f ($epi\ f$), is useful for characterizing convex functions. However, in generalized convexity (when the functions are E-convex, semi E-convex, quasi semi E −convex, etc), we deal with three different notions of epigraphs [12, 15, 53]. These epigraphs are associated with the mapping E. We list below the ordinary epigraph and its generalized versions.

Definition 1.4.2.11 Let $f: S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function, and $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given mapping. Then the ordinary epigraph is defined as

$$epi\ f = \{(s, \alpha) \in S \times \mathbb{R} : f(s) \leq \alpha\} \quad [44],$$

while the epigraphs associated with the mapping E are classified as

$$E - e(f) = \{(s, \alpha) \in S \times \mathbb{R} : f\big(E(s)\big) \leq \alpha\} \quad [15];$$

$$epi_E\ f = \{(E(s), \alpha) \in E(S) \times \mathbb{R} : f(E(s)) \leq \alpha\} \quad [12];$$

and

$$epi^E\ f = \{(E(s), \alpha) \in E(S) \times \mathbb{R} : f(s) \leq \alpha\} \quad [52].$$

Associated with each epigraph defined above, an α −level set is defined, respectively, as follows.

Definition 1.4.2.12 [40, 44, 56] Let $f, S,$ and E are defined as in Definition 1.4.2.11 and $\alpha \in \mathbb{R}$. Then

i. $S_\alpha[f] = \{s \in S: f(s) \leq \alpha\}$.
ii. $E - S_\alpha[f] = \{s \in S: f\big(E(s)\big) \leq \alpha\}$.
iii. $S_{\alpha,E}[f] = \{E(s) \in E(S): f\big(E(s)\big) \leq \alpha\}$.
iv. $S_\alpha^E[f] = \{E(s) \in E(S): f(s) \leq \alpha\}$.

1.5 Optimization Problems

Optimization is the act of obtaining the best result with available resources. For example, in economics, investors minimize risk or maximize profits, factories minimize cost. Since the risk required or the cost desired in any practical situation can be expressed as a function of certain variables, optimization can be defined as the process of finding the condition that give the maximum or minimum value of a function.

1.5.1 Mathematical Formulation of an Optimization Problem

Mathematically, optimization problem is the minimization or maximization of a function f subject to a constraint set S on its variables x. The optimization problem consists of:

- The variables x of the problem which represent all the possible decisions one can make.
- The objective function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ that we want to maximize or minimize. This objective can be the cost or the return of the system.
- The constraints set S which is the restrictions on the variables x. When the constraint set $S = \mathbb{R}^n$, the problem is said to be **unconstrained**. Otherwise, it is a **constrained** problem. The constraint set may include equality and /or inequality involving the variables.

We are ready now to give a mathematical expression for an optimization problem.

Definition 1.5.1.1 [47, p.127] The most basic form of mathematical optimization problem (or optimization problem, for short) is as follows:

$$\min f(x) \tag{1.3}$$

$$\text{subject to } g_i(x) \leq 0 \qquad i = 1, \dots, r$$

$$h_j(x) = 0 \qquad j = 1, \dots, m$$

$$x \in \mathbb{R}^n,$$

where the vector $x^T = (x_1, \dots . x_n)^T \in \mathbb{R}^n$ is the optimization variable of the problem, the function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is the **objective function (**or cost function) of the problem, and the functions $g_i, h_j: \mathbb{R}^n \longrightarrow \mathbb{R}$ are the **inequality** and **equality constraints**.

Definition 1.5.1.2 [13, p.84] The **feasible region** or the **feasible set** or a constraint set $S \subset \mathbb{R}^n$ is the set of all points that satisfy the problem's constraints. Mathematically,

$$S = \{x \in \mathbb{R}^n : h_j(x) = 0 \text{ and } g_i(x) \leq 0;\ j = 1, \dots, m \ \&\ i = 1, \dots, r\}.$$

Remark 1.5.1.3

- If there exists at least one feasible point $x \in S$, then Problem (1.3) is called feasible otherwise, if $S = \emptyset$, it is infeasible.
- If Problem (1.3) is unconstrained, then the feasible region is $S = \mathbb{R}^n$.

Below, we give two examples of constrained and unconstrained problem, respectively.

Example 1.5.1.4 Consider the following optimization problem

$$\min x^2 + 2xy$$

$$\text{subject to } x + y \leq 5$$

$$0 \leq x \leq 3$$

$$0 \leq y \leq 3$$

The optimization problem in this example is constrained and the feasible set

$$S = \{(x, y) \in \mathbb{R}^2 : x + y \leq 5, 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 3\}.$$

Example 1.5.1.5

$$\min x_1^2 + x_2^4$$
$$\text{subject to } x = (x_1, x_2) \in \mathbb{R}^2$$

This is unconstrained problem in which the feasible set $S = \mathbb{R}^2$.

Definition 1.5.1.6 [13, p.84], [47, p. 128] The set of solutions of problems (1.3) is the best solution from all feasible solutions. It is denoted by $argmin_s f$ and its elements $x^* \in S$ are called **global minimizers** or **optimal solutions**. i.e.,
x^* is global minimum point or an optimal solution of f if and only if

$$f(x^*) \leq f(x) \quad \forall x \in S.$$

Thus, $argmin_s f = \{x^* \in S : f(x^*) \leq f(x) \quad \forall x \in S\}$.
A global minimizer $x^* \in S$ is said to be **strict** when

$$f(x^*) < f(x) \quad \forall x \in S, \; x^* \neq x.$$

Moreover, the **optimal value** of Problem (1.3) is defined as

$$f(x^*) = p^* = \inf\{f(x): h_i(x) = 0, \; g_i(x) \leq 0 \text{ for all } i \text{ and for all } j\}.$$

Note that the set $argmin_s f$ may not be exist ($argmin_s f = \emptyset$) and may contains more than one minimum.

Remark 1.5.1.7 In the optimization problem (1.3)

1. It is possible to maximize the objective function (find the maximum value) instead of minimizing. In this way, Problem (1.3) can be expressed as

$$\begin{aligned} &\max f(x) && && (1.4)\\ &\text{subject to } g_i(x) \leq 0 && i = 1, \dots, r \\ &h_j(x) = 0 && j = 1, \dots, m \\ &x \in \mathbb{R}^n \end{aligned}$$

The set of the solution S will be denoted by $argmax_S f$ and x^* is a global maximum point or an optimal solution of f if and only if

$$f(x^*) \geq f(x) \quad \forall x \in S.$$

The **optimal value** of Problem (1.4) is defined as

$f(x^*) = q^* = \sup\{f(x): h_j(x) = 0, g_i(x) \leq 0$ for all i and for all $j\}$. [13, p.84]

2. $\min f(x) = -\max -f(x)$

Remark 1.5.1.8

Global minimizers (maximizers) can be difficult to find and characterize in general nonlinear function. Instead of global points, one can find a point x^* such that $f(x^*) \leq f(x)$ for all points x in a given neighborhood of x^*. This point is called a **local minimizer** point. Similarly, local maximizer point is defined.

Definition 1.5.1.9 [38, p.11] A point $x^* \in \mathbb{R}^n$ is called a **local minimizer** for Problem (1.3) if there exist $r > 0$ such that

$$f(x^*) \leq f(x) \qquad \forall x \in B(x^*, r) \cap S.$$

This definition can be extended to the definition of a local maximum by reversing the inequality above. If the inequalities in the above definition become strict then x^* is called **strict local minimizer**.

Definition 1.5.1.10 A point $x^* \in \mathbb{R}^n$ is called a **strict local minimizer** for Problem (1.3) if there exists $r > 0$ such that

$$f(x^*) < f(x) \qquad \forall x \in B(x^*, r) \cap S, x^* \neq x.$$

Note that every strict local minimum point is a local minimum but the converse is not true as we show in the next example.

Example 1.5.1.11

Consider the objective function

$$f(x) = \begin{cases} x - 1, & x \leq 3 \\ 2, & 3 < x \leq 7 \\ -2x + 16, & x > 7 \end{cases}$$

From the below graph of the function, the point $P = 5$ is a local minimum such that there exists $r > 0$ in which $f(P) = f(x)\ \forall x \in B(x^*, r) \cap (3,7]$. However, P is not strict local minimum. Note that P is also a global maximizer. In fact, this function has multiple global maximum points (i.e., global maximum is not unique). Also, there are multiple local minimum points but none of them is strict local minimum.

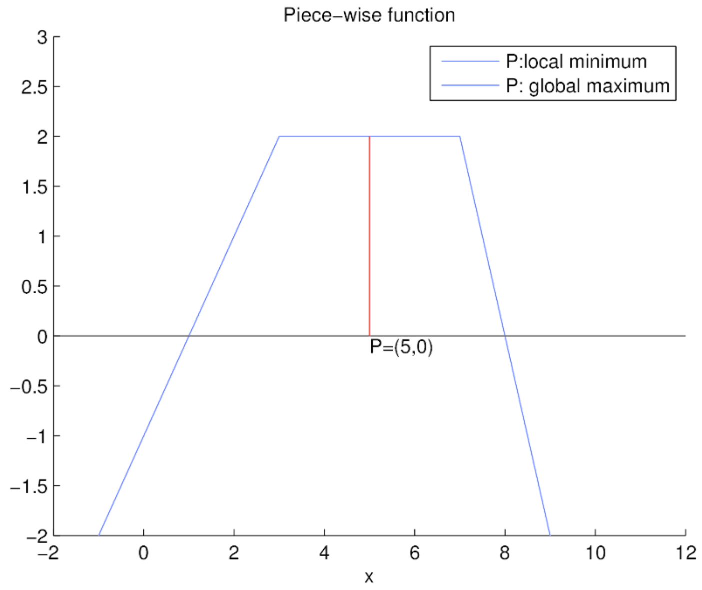

Figure: The graph of the function in Example 1.5.1.11

Remark 1.5.1.12

- Every global minimum /maximum is local minimum/maximum.
- It may not be possible to identify a global minimum by finding all local minima (global minimum may not exist) as we have seen in the above example.

1.5.2 Differentiable and Convex Optimization Problem

Problem (1.3) is said to be **differentiable** optimization problem when the functions $g_1, \dots, g_r;\ h_1, \dots, h_m$ are differentiable. If the constraints of Problem (1.3) are nonlinear, Problem (1.3) is called **nonlinear** optimization problem. When the objective function f and the constraint functions $g_1, \dots, g_r;\ h_1, \dots, h_m$ are all linear, we have a **linear** optimization problem [14, p.2], [2, p.149]. **Convex** optimization problems play an important role in optimization. There are variety of mathematical properties and tools that help to characterize and efficiently solve convex problems. In general, an optimization problem

$$\min f(x)$$
$$\text{subject to } x \in S$$

is called convex problem if the objective function f and the constraint set S are convex. [13, p.208]

The main benefit of knowing whether an optimization problem is convex is provided by the following theorem.

Theorem 1.5.2.1 Assume that we have the following convex optimization problem

$$\min f(x)$$
$$\text{subject to } x \in S \subseteq \mathbb{R}^n,$$

where $f : S \longrightarrow \mathbb{R}^n$ is a convex function defined on the convex set S. Then

1) If $x^* \in S$ be a local minimum of f. Then $x^*$is a global minimum of f. [2, Theorem 8.1]
2) If $x^* \in S$ is a strict local minimum of f. Then x^* is a unique global minimum. [7]
3) If f is a strictly convex function on the convex set S. Then $argmin_S\, f$ has only one element. [2, Theorem 8.3]
4) The set of all feasible solutions $argmin_S\, f$ is convex. [2, Theorem 8.3]

1.5.3 Generalized Optimization Problems

As for the class of convex sets and convex functions, the class of optimization problems have extended into the class of generalized optimization problems [15, 53]. Youness in his celebrity paper [15] defined two forms of non-linear constrained generalized optimization problem denoted, respectively, by (NLP) and (NLP_E) and defined as.

$$\min\ f(s)$$

$$s.t.\ s \in S,$$

and

$$\min\ (foE)(s)$$

$$s.t.\ s \in S,$$

where $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function, $S \subseteq \mathbb{R}^n$ be an E-convex set, and $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given mapping.

Remark 1.5.3.1 Problems (NLP) and (NLP_E) in which the objective function is E-convex are said to be E-convex problems [15]. Similarly, when the objective function is semi E-convex, Problems (NLP) and (NLP_E) are called semi E-

convex problems [53]. Hence, generalized optimization problems are classified according to the type of the generalized objective function.

Note that the relation between the solutions of the E-convex programming (NLP) and (NLP_E) is introduced in [15, Theorem 4.2]. Also, the characterization of the optimal solutions and some of the optimality conditions of the E-convex programming problem (NLP) is addressed in [15, Theorems 4.3, 4.5-4.6]. Later, it appears that some of Youness proceeding results are incorrect (see [53, Examples 1-3] for some counterexamples for Theorems 4.2, 4.3, 4.6 in [15]). Therefore, a new concept of semi E-convex programming is defined in [53], the relation between the solutions of the E-convex programming (NLP) and (NLP_E) is corrected, and some optimality results are introduced, for Problem (NLP), to fix the incorrect optimality results of [15].

Next, we list the main results introduced in [53] to correct Youness's results.

Theorem 1.5.3.2 [53, Theorem 5] Assume that we have (NLP) and (NLP_E) optimization problems such that the objective function f is semi E-convex function on the E-convex set S and s^* is a solution of problem (NLP_E). Then $E(s^*)$ is a solution of problem (NLP).

Theorem 1.5.3.3 Assume that we have (NLP) generalized optimization problem. Then

i. If f is an E-convex function on S, and $f\big(E(s)\big) \leq f(s) \quad \forall s \in S$, and $s^* = E(z) \in E(S)$ is a local minimum of problem (NLP). Then s^* is a global minimum of (NLP) on S. [53, Theorem 6]
ii. If f is strictly semi E-convex function on S. Then the global optimal solution of problem (NLP) is unique. [53, Theorem 7]
iii. If f is semi E-convex function on S. Then $argmin_S f$ of problem (NLP) is E-convex set. [53, Theorem 9]

In Chapter 4, we consider the optimization problem (NLP_E) for which we discuss some optimality properties for this problem when the objective functions are E-convex (strictly E-convex), strictly quasi semi E-convex, E-quasiconvex (strictly E-quasiconvex).

CHAPTER TWO

ON GENERALIZED CONVEX SETS, CONES, AND AFFINE SETS

Chapter 2

On Generalized Convex Sets, Cones, and Affine Sets

2.1 Introduction

An important class of generalized convex sets, called E-convex sets, has first introduced and studied by Youness [15]. In this class, Youness relaxed the definitions of the classical convex sets with respect to an mapping E. Some of the results introduced in [15], related to E-convex set, are recently studied by other researchers. Suneja et. al [48] studied E-convex sets and used it to prove some inequalities. Further study of E-convex sets are recently introduced by Grace and Thangavelu [25] in which the authors defined E-convex hull, E-cone, and E-convex cone and studied some of their properties.

In this chapter, we continue studying E-convex sets and E-cone by proving new properties of these sets. We give new characterizations of E-convex sets, E-convex hull, and E-convex cone. In addition, we define E-convex cone hull, E-affine set, and E-affine hull, and we discuss some of their properties and characterizations. Some examples are given to illustrate these different concepts and to clarify the relationships between them.

In Section 2, we give a characterization of E-convex set in terms of the E-convex combinations of its elements (see Theorem 2.2.2). A new characterization of an E-convex hull of a set S, is also given (see Theorem 2.2.3), in terms of the set of all E-convex combinations of any finite elements of the set S. In Section 3, we prove some properties of E-cone and E-convex cone. We obtain two new characterizations of the E-convex cone set K. The first characterization (see Theorem 2.3.7) is proved in terms of the E-closdeness of K under addition and non-negative multiplications. The second characterization of E-convex cone (see Theorem 2.3.8) is proved in terms of non-negative E-linear combination of any finite elements of the considered set. Then, we define E-convex cone hull of an arbitrary set K and discuss some of its properties. E-convex cone hull of a set K is characterized using the set of all non-negative E-linear combinations of K (Theorem 2.3.10). Finally, in Section 4, we define E-affine set, explain its relationship with E-convex set, and prove some properties related to E-affine sets. As for E-convex set and E-convex cone set, we define E-affine hull and show characterizations of an E-affine set M (see Theorem 2.4.15) and the E-affine hull of an arbitrary set M (see Theorem 2.4.17). The characterization of E-affine set and E-affine hull is formulated in terms of E-affine combinations of all elements of M. Some examples are shown throughout this chapter to illustrate the aforementioned concepts and to show the relationship between them. The contents of this chapter have been published recently in [49].

2.2 Characterizations of E-Convex Set and E-Convex Hull

In this section, we study E-convex sets and E-convex hull of an arbitrary set, and we give some of their new properties and characterizations. The following definition will be employed to show a characterization of E-convex sets.

Definition 2.2.1 Let $S \subset \mathbb{R}^n$. The set of E-convex combinations of p elements of S is denoted by $S(s,p)$ and is defined as

$$S(s,p) = \{s = \sum_{i=1}^{p} \lambda_i E(s_i) : \{s_1, \dots, s_p\} \subset S,\ \lambda_i \geq 0 \text{ and } \sum_{i=1}^{p} \lambda_i = 1\}.$$

Next, we characterize E-convex set in terms of the E-convex combinations of its elements.

Theorem 2.2.2 Assume that a set $S \subset \mathbb{R}^n$ and $S(s,p)$ is the set of E-convex combinations of p elements of S defined in Definition 2.2.1 such that the mapping E appears in Definition 2.2.1 is linear and idempotent. Then S is E-convex if and only if $S(s,p) \subset S\ \ \forall p \in N$.

Proof. Assume that S is E-convex. We need to show that for each $p \in N$,

$$S(s,p) \subset S. \tag{2.1}$$

We show (2.1) by induction. If $p = 1$, then there exists $s_1 \in S$ and $\lambda_1 = 1$ such that $s = \lambda_1 E(s_1) = E(s_1) \in S(s,1)$. Since S is E-convex then, from Proposition 1.4.1.3(1), $s = E(s_1) \in S$. Let $p = 2$, then there exists $s_1, s_2 \in S$ and $\lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1$ such that

$$s = \lambda_1 E(s_1) + \lambda_2 E(s_2) \in S(s,2).$$

Since S is E-convex, then

$$s = \lambda_1 E(s_1) + \lambda_2 E(s_2) \in S.$$

Assume now (2.1) holds for $p = k$ i.e.,

$$\text{if } s \in S(s,k), \text{ then } s \in S. \tag{2.2}$$

We must prove that (2.1) is true when $p = k + 1$. Let $s \in S(s, k+1)$, this means there exists $\{s_1, \dots, s_{k+1}\} \subset S$, and there exists $\lambda_1, \dots, \lambda_{k+1} \geq 0$ such that $\sum_{i=1}^{k+1} \lambda_i = 1$. Assume that $\lambda_1 \neq 1$ and let s^* is the E $-$convex combinations of k elements of the set $\{s_2, \dots, s_{k+1}\} \subset S$. Hence,

$$s^* = \lambda_2^* E(s_2) + \cdots + \lambda_{k+1}^* E(s_{k+1}),$$

where

$$\lambda_i^* = \frac{\lambda_i}{1-\lambda_1} \geq 0 \text{ for } i = 2, \dots, k+1. \tag{2.3}$$

This yield, $\sum_{i=2}^{k+1} \lambda_i^* = 1$. Thus, $s^* \in S(s^*, k)$, and from (2.2), $s^* \in S$. Now, $s_1, s^* \in S$, from induction $\lambda_1 E(s_1) + (1-\lambda_1)E(s^*) \in S$. Since E is linear, the last statement yields

$$\lambda_1 E(s_1) + (1-\lambda_1)(\lambda_2^* E(E(s_2)) + \cdots + \lambda_{k+1}^* E(E(s_{k+1}))) \in S.$$

Substituting the values of λ_i^* from (2.3) in the expression above gives

$$\lambda_1 E(s_1) + \lambda_2 E(E(s_2)) + \cdots + \lambda_{k+1} E(E(s_{k+1})) \in S.$$

Since E is idempotent, then $E(E(s_i)) = E(s_i) \in S \ \forall i = 2, \dots, k+2$. Thus, the last expression can be expressed as

$$s = \lambda_1 E(s_1) + \lambda_2 E(s_2) + \cdots + \lambda_{k+1} E(s_{k+1}) \in S.$$

Hence, $S(s,p) \subset S$. To show the prove of the other direction, assume that $S(s,p) \subset S \ \forall p \in N$. In particular, for each $s_1, s_2 \in S$ and $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ we have $s = \lambda_1 E(s_1) + \lambda_2 E(s_2) \in S$. Hence, S is E-convex. ■

A property of E-convex hull of a set S is given next.

Theorem 2.2.3 Let $S \subset \mathbb{R}^n$ and $\mathcal{F}$ is the set of all E-convex combinations of elements of S. That is $\mathcal{F} = \bigcup_{p \in N} S(s,p)$, where $S(s,p)$ is defined as in Definition 2.2.1 and the mapping E appears in Definition 2.2.1 is linear and idempotent. Then $\mathcal{F} \subseteq E\text{-}conv(S)$. Moreover, if $E(\mathcal{F}) \subseteq \mathcal{F}$, then $E\text{-}conv(S) = \mathcal{F}$.

Proof. To show the first assertion. Assume that $s \in \mathcal{F}$, from the definition of $\mathcal{F}$, there exists $\{s_1, \dots, s_m\} \subset S$ and $\lambda_1, \dots, \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that

$$s = \sum_{i=1}^{m} \lambda_i E(s_i).$$

Since $\{s_1, \ldots, s_m\} \subset S \subset E\text{-}conv(S)$, using the E-convexity of $E\text{-}conv(S)$ and Theorem 2.2.2, every E-convex combination of $s_i{}'s$ must remain in $E\text{-}conv(S)$. Hence, $s \in E\text{-}conv(S)$ and consequently,

$$\mathcal{F} \subseteq E\text{-}conv(S) \tag{2.4}$$

Next, we must show that $E\text{-}conv(S) \subseteq \mathcal{F}$ if $E(\mathcal{F}) \subseteq \mathcal{F}$. To prove this, it is enough to show that $\mathcal{F}$ is a convex set. Indeed, if $\mathcal{F}$ is a convex set and $E(\mathcal{F}) \subseteq \mathcal{F}$. Then from Proposition 1.4.1.5, $\mathcal{F}$ is E-convex set. The last conclusion with the fact that $S \subseteq \mathcal{F}$ yield $E\text{-}conv(S) \subseteq \mathcal{F}$ as required. Let show that $\mathcal{F}$ is a convex set. Take $s, s^* \in \mathcal{F}$, then

$$s = \sum_{i=1}^{p} \lambda_i E(s_i) \quad \text{and } s^* = \sum_{i=1}^{q} \gamma_i E(s_i^*),$$

where $\{s_1, \ldots, s_p, s_1^*, \ldots, s_q^*\} \subset S$ and $\{\lambda_1, .., \lambda_p, \gamma_1, \ldots, \gamma_q\}$ are non-negative which satisfy

$$\sum_{i=1}^{p} \lambda_i = 1 \text{ and } \sum_{i=1}^{q} \gamma_i = 1\ .$$

Fix $\alpha \in [0,1]$, then the convex combination

$$\alpha s + (1-\alpha)s^* = \alpha \sum_{i=1}^{p} \lambda_i E(s_i) + (1-\alpha) \sum_{i=1}^{q} \gamma_i E(s_i^*).$$

Note that

$$\alpha \sum_{i=1}^{p} \lambda_i + (1-\alpha) \sum_{i=1}^{q} \gamma_i = 1.$$

Therefore, $\alpha s + (1-\alpha)s^* \in \mathcal{F}$. i.e., $\mathcal{F}$ is a convex set, and using the assumption $E(\mathcal{F}) \subseteq \mathcal{F}$ yield $\mathcal{F}$ is E-convex set. Because $S \subseteq \mathcal{F}$ and $S \subseteq E\text{-}conv(S)$. Then

$$E\text{-}conv(S) \subseteq \mathcal{F}. \tag{2.5}$$

From (2.4) and (2.5), we obtain $E\text{-}conv(S) = \mathcal{F}$. ∎

2.3 E-Convex Cone and E-Convex Cone Hull: Properties and Characterizations

In this section, some properties of E-cone and E-convex cone set are deduced, and two different characterizations of E-convex cone of an arbitrary set are introduced. E-convex cone hull is defined and a characterization of E-convex cone hull of an arbitrary set K is shown.

Proposition 2.3.1

i. If a set K is E-cone, then $E(K) \subseteq K$.
ii. If $E(K)$ be a convex cone and $E(K) \subseteq K$. Then K is an E-convex cone.

Proof. First, let us show (i). Let $E(x) \in E(K)$ such that $x \in K$. Since K is E-cone, then $\alpha E(x) \in K \ \ \forall \alpha \geq 0$. If $\alpha = 1$, then $\alpha E(x) = E(x) \in K$ as required. To prove (ii), it is enough to prove that K is an E-cone since K is already an E-convex by Proposition 1.4.1.3(2). Consider $x \in K$, then $E(x) \in E(K) \subseteq K$. Since $E(K)$ is a cone, then $\alpha E(x) \in E(K) \subseteq K$, for each $\alpha \geq 0$. Thus, K is an E-cone. ∎

Remark 2.3.2 The converse of Proposition 2.3.1(i) is not true in general (see Example 1.4.1.14).

For an arbitrary mapping E, the following proposition provides a condition to ensure that every cone (respectively, convex cone) is an E-cone (respectively, E-convex cone). Also, the first part of the proposition makes the converse of Proposition 2.3.1(i) holds.

Proposition 2.3.3 Let $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a given mapping. Then

i. Let K be a cone such that $E(K) \subseteq K$. Then K is an E-cone.
ii. Let K be a convex cone such that $E(K) \subseteq K$. Then K is an E-convex cone.

Proof. To show (i), If $E = I$, then using Proposition 1.4.1.16, the conclusion automatically holds. Otherwise, assume that $x \in K$ and $\alpha \geq 0$. Since $E(K) \subseteq K$, then $E(x) \in K$. Also, from the assumption, K is a cone, hence, $\alpha E(x) \in K$. This means K is E-cone. Part (ii) directly follows from Proposition 1.4.1.5 and part (i). ■

Proposition 2.3.4 Let K be E_1-cone and E_2-cone, then K is an $(E_1 o E_2)$-cone and $(E_2 o E_1)$-cone.

Proof. Assume that $x \in K$ and $\alpha \geq 0$, we must show that $\alpha(E_1 o E_2)(x) = \alpha E_1\big(E_2(x)\big) \in K$. Now, K is E_2-cone then from Proposition 2.3.1(i), $E_2(x) \in K$. Because K is E_1-cone, then using the last assertion and Definition 1.4.1.12, $\alpha E_1\big(E_2(x)\big) \in K$ as required. Similarly, one can show that K is an $(E_2 o E_1)$-cone. ■

Proposition 2.3.5

i. Let $\{K_i : i \in \Lambda\}$ be a non-empty family of E-cones, then $\bigcup_{i\in\Lambda} K_i$ is E-cone.
ii. Let $\{K_i: i \in \Lambda\}$ be a non-empty family of E-cones, then $\bigcap_{i\in\Lambda} K_i$ is E-cone.
iii. Let K be E- cone, E is a linear mapping, and $a \in \mathbb{R}$, then the set aK is E- cone.
iv. If K_1 and K_2 be two E-cones, then $K_1 \times K_2$ is $E \times E$- cone.
v. If K_1 and K_2 be two E-cones and let E is a linear mapping, then the set $K_1 + K_2$ is E- cone.

Proof. We prove part (i) and in a similar way one can show part (ii). Take an arbitrary $x \in \bigcup_{i\in\Lambda} K_i$ where K_i is E-cone for each $i \in \Lambda$. Then, for $\alpha \geq 0$, $\alpha E(x) \in K_i$ for some $i \in \Lambda$. Hence $\alpha E(x) \in \bigcup_{i\in\Lambda} K_i$. Thus, $\bigcup_{i\in\Lambda} K_i$ is E-cone. The proof of parts (iii)-(iv) proceed in a way similar to that of Proposition

1.4.1.6(iii) and Proposition 1.4.1.8, respectively, in which (the sets under considerations are E-convex), hence the proof of parts (iii)-(iv) are omitted. Let us show part (v). Let $x_1 + x_2 \in K_1 + K_2$ where $x_1 \in K_1$ and, $x_2 \in K_2$. Then, from the assumption, for $\alpha \geq 0$, we have $\alpha E(x_1 + x_2) = \alpha E(x_1) + \alpha E(x_2) \in K_1 + K_2$. Thus, $K_1 + K_2$ is an E-cone. ■

Propositions 1.4.1.6 and 1.4.1.8 together with Proposition 2.3.5 yield the following result.

Proposition 2.3.6

i. Let $\{K_i : i \in \Lambda\}$ be a non-empty family of E- convex cone sets, then $\bigcap_{i\in\Lambda} K_i$ is E- convex cone set. [25]
ii. Let K be an E- convex cone, E is a linear mapping, and $a \in \mathbb{R}$, then the set aK is E- convex cone set.
iii. If K_1 and K_2 be two E- convex cones, then $K_1 \times K_2$ is $E \times E$- convex cone set. Moreover, if E is a linear mapping then $K_1 + K_2$ is E- convex cone set.

In [25, Proposition 4.6], a characterization of E-convex cone K is shown if the image of K under the mapping E satisfies certain conditions. The following theorems give alternative characterizations of E-convex cone.

Theorem 2.3.7 A set K is E-convex cone if and only if K is E-closed (i.e., closed with respect to the mapping E) under addition and non- negative scalar multiplication.

Proof. Assume that K is E-convex cone. From the definition of E-cone, we have $\alpha E(x) \in K$, for any $\alpha \geq 0$ and for any $x \in K$. Thus, K is E-closed for non-negative scalar multiplication. Next, we show that K is an E-closed under addition. Fix $x, y \in K$ which is E- convex set, then

$$w = \frac{1}{2}E(x) + \frac{1}{2}E(y) = \frac{1}{2}\big(E(x) + E(y)\big) \in K.$$

Hence, $E(x) + E(y) = 2w \in K$ as required. For proving the opposite direction, assume that K is E-closed with respect to addition and non-negative scalar multiplication. Then, K is E −cone automatically holds. Let $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$, and $x, y \in K$ then

$$\lambda_1 E(x) \in K \text{ and } \lambda_2 E(y) \in K.$$

This yield $\lambda_1 E(x) + \lambda_2 E(y) \in K$, and hence K is E-convex set. ■

Theorem 2.3.8 Let K be a subset of $\mathbb{R}^n$ and $K(x,p)$ is the set of E-non-negative linear combinations of p elements of K. That is

$$K(x,p) = \left\{x = \sum_{i=1}^{p} \lambda_i E(x_i) \colon \{x_1, \dots, x_p\} \subset K, \lambda_i \geq 0\right\},$$

where E is linear and idempotent. Then K is E-convex cone if and only if $K(x,p) \subset K \quad \forall p \in N$.

Proof. Assume that K is E-convex cone. We need to show that for each $p \in N, p \geq 1$,

$$K(x,p) \subset K. \tag{2.6}$$

We show (2.6) by induction. Let $p = 1$ and $x \in K(x,1)$, then there exist $x_1 \in K$ and $\lambda_1 \geq 0$ such that $x = \lambda_1 E(x_1)$. Since K is E-cone, hence $\lambda_1 E(x_1) \in K$. If $p = 2$, then there exists $x_1, x_2 \in K$ and $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ such that

$$x = \lambda_1 E(x_1) + \lambda_2 E(x_2) \in K(x,2).$$

We must show that $x = \lambda_1 E(x_1) + \lambda_2 E(x_2) \in K$. Since K is E-cone, then $\lambda_1 E(x_1), \lambda_2 E(x_2) \in K$. From Theorem 2.3.7, K is E-closed under addition. Hence,

$$E(\lambda_1 E(x_1)) + E(\lambda_2 E(x_2)) = \lambda_1 E(E(x_1)) + \lambda_2 E(E(x_2)) \in K,$$

where we used the fact that E is linear. Since E is idempotent, then $E(E(x_i)) = E(x_i) \in S \ \forall i = 1,2.$ Thus, the last expression can be expressed as $x = \lambda_1 E(x_1) + \lambda_2 E(x_2) \in K$, as required. Assume now (2.6) holds for $p = k$. i.e.,

$$\text{if } x \in K(x,k), \text{ then } x \in K, \tag{2.7}$$

We must prove that (2.6) is true when $p = k + 1$. Let $x \in K(x, k + 1)$, this means there exists $\{x_1, \dots, x_{k+1}\} \subset K$, and there exists $\lambda_1, \dots, \lambda_{k+1} \geq 0$ such that

$$x = \sum_{i=1}^{k+1} \lambda_i E(x_i) \in K(x, k + 1).$$

Since K is E-cone, then $\lambda_{k+1} E(x_{k+1}) \in K$ and from (2.7), $\sum_{i=1}^{k} \lambda_i E(x_i) \in K$. From Theorem 2.3.7, K is E-closed under addition, thus

$$\sum_{I=1}^{k} E\,(\lambda_i E(x_i)) + E(\lambda_{k+1} E(x_{k+1})) \in K.$$

Using again the fact that E is linear and idempotent, we get

$$\sum_{i=1}^{k+1} \lambda_i E(E(x_i)) = \sum_{i=1}^{k+1} \lambda_i E(x_i) \in K.$$

Hence, $x = \sum_{i=1}^{k+1} \lambda_i E(x_i) \in K$. To show the prove of the other direction, assume that $K(x,p) \subset K \ \forall p \in N$. In particular, for each $x_1, x_2 \in K$ and λ_1, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ we have $x = \lambda_1 E(x_1) + \lambda_2 E(x_2) \in K$ and $\lambda_1 E(x_1) \in K$. Hence, K is E-convex cone. ∎

Next, we introduce a smallest E-convex cone that contains a certain set.

Definition 2.3.9 The E-convex cone hull of a set K, denoted by $E\text{-}cone(K)$ is the intersection of all E-convex cone sets containing K; that is, $E\text{-}cone(K) = \bigcap_{N \supseteq K} N$, N are E-convex cone sets.

The following result is analogue to the one introduced in Theorem 2.2.3 for general E-convex sets.

Theorem 2.3.10 Let $K \subset \mathbb{R}^n$ and $\mathcal{H}$ is the set of all non-negative E-linear combinations of elements of K. That is, $\mathcal{H} = \bigcup_{p \in N} K(x,p)$, where $K(x,p)$ is defined as in Theorem 2.3.8. Then $\mathcal{H} \subseteq E\text{-}cone(K)$. Moreover, if $E(\mathcal{H}) \subseteq \mathcal{H}$. Then $E\text{-}cone(K) = \mathcal{H}$.

Proof: Assume that $x \in \mathcal{H}$, from the definition of $\mathcal{H}$, there exists $\{x_1, \dots, x_m\} \subset K$ and $\lambda_1, \dots, \lambda_m \geq 0$ such that $x = \sum_{i=1}^{m} \lambda_i E(x_i)$. Since $E\text{-}cone(K)$ is the intersection of all E-convex cones containing K, then from Proposition 2.3.6 (i), $E\text{-}cone(K)$ is also E-convex cone containing K. Using the fact that $E\text{-}cone(K)$ is E-convex cone and $\{x_1, \dots, x_m\} \subset K \subset E\text{-}cone(K)$, we get from Theorem 2.3.8, every non-negative E-linear combinations of $x_i's$ must remain in $E\text{-}cone(K)$. Hence, $x \in E\text{-}cone(K)$ which yields, $\mathcal{H} \subseteq E\text{-}cone(K)$. To show $E\text{-}cone(K) \subseteq \mathcal{H}$, we follow same technique that used to prove $E\text{-}conv(S) \subseteq \mathcal{F}$ in Theorem 2.2.3. Namely, we must show that $E\text{-}cone(K) \subseteq \mathcal{H}$ if $E(\mathcal{H}) \subseteq \mathcal{H}$. To prove this, it is enough to show that $\mathcal{H}$ is a convex cone set. To show that $\mathcal{H}$ is a convex set, follow similar steps that is used in Theorem 2.2.3 to show that $\mathcal{F}$ is a convex set. Next, we show that $\mathcal{H}$ is a cone. Let $x \in \mathcal{H}$, then there exists $p \in N$ such that $x = \sum_{i=1}^{p} \lambda_i E(x_i)$, where $\{x_1, \dots, x_p\} \subset K$ and $\{\lambda_1, .., \lambda_p\}$ are non-negative scalars. Fix $\alpha \geq 0$, then the non-negative E-linear combination

$$\alpha x = \alpha \sum_{i=1}^{p} \lambda_i E(x_i) = \sum_{i=1}^{p} (\alpha \lambda_i) E(x_i) \in \mathcal{H}\ .$$

Thus, $\mathcal{H}$ is a convex cone set, and since $E(\mathcal{H}) \subseteq \mathcal{H}$, then from Proposition 2.3.3(ii), $\mathcal{H}$ is an E-convex cone set. The last conclusion with the fact that $K \subseteq \mathcal{H}$ yield $E\text{-}cone(K) \subseteq \mathcal{H}$ as required. All together, we obtain $\mathcal{H} = E\text{-}cone(K)$. ■

2.4 E-Affine Set, E-Affine Hull and Their Characterizations

In this section, we define E-affine set and E-affine hull. We study some of their properties, and discuss their characterizations.

Definition 2.4.1 A set $M \subset R^n$ is said to be ***E*-affine** if $\lambda E(x) + (1 - \lambda)\, E(y) \in M \quad \forall x, y \in M$ and $\lambda \in \mathbb{R}$.

Remark 2.4.2 It is easy to show that every E-affine set is E-convex set. The converse does not hold as we show in the next example.

Example 2.4.3 Let $S \subset \mathbb{R}^2$ is defined by $S = \{\, (x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1 \,\}$, and let $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $E(x, y) = \left(\frac{1}{2}x, \frac{1}{4}y\right) \quad \forall x, y \in \mathbb{R}$. From Example1.4.1.14, S is E-convex set. However, S is not E-affine. i.e., Let $\lambda_1 = 4$, $\lambda_2 = -3$, $(x_1, y_1) = (1,1)$, and $(x_2, y_2) = (0,0)$ then

$$4E(1,1) - 3E(0,0) = (2,1) \notin S,$$

as required.

Remark 2.4.4 As for E-convex set and E-cone, if $E = I$ the identity mapping, then every affine set is an E-affine.

An E-affine set is not necessary an affine set as we show next.

Example 2.4.5 Let $M = \{(x, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by $E(x, y) = (0, y)$, $\forall x, y \in \mathbb{R}$. Let $(x_1, y_1), (x_2, y_2) \in M$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 + \lambda_2 = 1$. Then

$$\lambda_1 E(x_1, y_1) + \lambda_2 E(x_2, y_2) = (0, \lambda_1 y_1 + \lambda_2 y_2) \in M.$$

Thus, M is E-affine set. To show that M is not affine, take $(0, -2)$, $(3,0) \in M$ and $\lambda_1 = 4$, $\lambda_2 = -3$,then

$$4(0,-2)-3(3,0)=(-9,-8)\notin M.$$

Consequently, M is not affine set.

Remark 2.4.6: From Remark 2.4.4 and Example 2.4.5, one deduces that the class of E-affine sets is a generalization of the class of affine sets.

Proposition 2.4.7

i. Let M is E-affine set, then $E(M)\subseteq M$.
ii. If $E(M)$ is affine and $E(M)\subseteq M$. Then M is an E-affine set.

Proof. Let us show (i). Since M is E-affine set, then for any $x,y\in M$ and $\lambda\in\mathbb{R}$ we have $\lambda E(x)+(1-\lambda)\,E(y)\in M$. Thus, for $\lambda=1$, we get $\lambda E(x)+(1-\lambda)\,E(y)=E(x)\in M$. Hence, $E(M)\subseteq M$. For proving (ii), let $x,y\in M$, then $E(x),E(y)\in E(M)$. Thus, for each $\lambda\in\mathbb{R}$, we have $\lambda E(x)+(1-\lambda)\,E(y)\in E(M)\subseteq M$. The last statement holds because $E(M)$ is affine. Thus, M is E-affine set.

Remark 2.4.8 The converse of Proposition 2.4.7(i) does not hold (see Example 2.4.3).

For an arbitrary mapping E, the following proposition provides a condition to ensure that every affine set is an E-affine.

Proposition 2.4.9 Let $E:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be a given mapping. If M is an affine set and $E(M)\subseteq M$ then M is E-affine set.

Proof: Assume that $x,y\in M$. Since $E(M)\subseteq M$ and M is an affine set, then $E(x),E(y)\in M$ and for each $\lambda_1,\ \lambda_2\in\mathbb{R}$ such that $\lambda_1+\lambda_2=1$ we have $\lambda_1 E(x)+\lambda_2\,E(y)\in M$. Hence, M is E-affine set. ■

In the next example, we show that for a given mapping $E \neq I$, an affine set may not be an E-affine.

Example 2.4.10 Suppose that $M = \{(0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$ and $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, be defined as $E(x, y) = (y, x)$.

Let $(0, y_1), (0, y_2) \in M$ and $\lambda \in \mathbb{R}$. We show that

$$\lambda(0, y_1) + (1 - \lambda)(0, y_2) \in M,$$

$\lambda(0, y_1) + (1 - \lambda)(0, y_2) = (0, \lambda y_1 + (1 - \lambda) y_2) \in M$. Hence, M is an affine set. To show that M is not E-affine set, take $(x_1, y_1) = (0,4)$, $(x_2, y_2) = (0,8)$, and let $\lambda = \frac{1}{2}$. Then,

$$\lambda E((x_1, y_1)) + (1 - \lambda) E((x_2, y_2)) = \frac{1}{2}(4,0) + \frac{1}{2}(8,0)$$

$$= (6,0) \notin M$$

Then, M is not E-affine.

Remark 2.4.11 The union of two E-affine sets is not E-affine set as it demonstrated in the next example.

Example 2.4.12 Define $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\mathrm{E}(x, y) = (2x, y)$. Let $M_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $M_2 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. We show that M_1 is E-affine set. Suppose that $(0, y_1), (0, y_2) \in M_1$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 + \lambda_2 = 1$ then

$$\lambda_1 E(0, y_1) + \lambda_2 E(0, y_2) = (0, \lambda_1 y_1 + \lambda_2 y_2) \in M_1.$$

Similarly, we can show that M_2 is E-affine set.

Now, take $(0,3) \in M_1$, $(4,0) \in M_2$, and $\lambda_1 = -5$, $\lambda_2 = 6$. Then

$$-5E(0,3) + 6E(4,0) = (48, -15) \notin M_1 \cup M_2,$$

thus $M_1 \cup M_2$ is not E-affine set.

Proposition 2.4.13 Let M be $E_1$and E_2-affine sets, then M is an $(E_1 o E_2)$ and $(E_2 o E_1)$-affine set.

Proof. Assume that $x, y \in M$ and $\lambda_1,\ \lambda_2 \in \mathbb{R}$ such that $\lambda_1 + \lambda_2 = 1$. Since M is E_2-affine set, then form Proposition 2.4.7(1), $E_2(x)$ and $E_2(y) \in M$. Now, because M is E_1-affine set, then using the last assertion, we get

$$\lambda_1 E_1\big(E_2(x)\big) + \lambda_2\, E_1\big(E_2(y)\big) = \lambda_1 (E_1 o E_2)(x) + \lambda_2 (E_1 o E_2)(y) \in M.$$

Hence, M is an $(E_1 o E_2)$-affine set. In the same way, we can show that M is $E_2 o E_1$-affine set. ∎

Proposition 2.4.14

i. Let $\{M_i: i \in \Lambda\}$ be a non-empty family of E-affine sets, then $\bigcap_{i\in\Lambda} M_i$ is E- affine set.
ii. Let M be E-affine set, E is a linear mapping, and $a \in R$, then the set aM is E- affine set.
iii. If M_1 and M_2 be two E-affine sets, then $M_1 \times M_2$ is an $E \times E$- affine set.
iv. If M_1 and M_2 be two E- affine set and let E is a linear mapping, then $M_1 + M_2$ is E- affine set.

Proof. The proof follows in the same way as for the proof of [15, Proposition 2.4, Lemma 2.2], [25, Proposition 3.2], and Proposition 1.4.1.8. The only difference is that, for E-affine sets, we take λ_1, λ_2 belongs to $\mathbb{R}$ (rather than to the closed interval [0,1]). ∎

Theorem 2.4.15 Let M be a subset of $\mathbb{R}^n$ and $M(x,p)$ is the set of E-affine combinations of p elements of M. That is
$M(x,p) = \{x = \sum_{i=1}^{p} \lambda_i E(x_i) : \{x_1, .., x_p\} \subset M,\ \lambda_i \in \mathbb{R}$ and $\sum_{i=1}^{p} \lambda_i = 1\}$ such that the mapping E given in $M(x,p)$ is linear and idempotent. Then M is E-affine set if and only if $M(x,p) \subseteq M \quad \forall p \in N$.

Proof. The proof is analogous to the one of Theorem 2.2.2. ■

In the same way E-convex hull and E-convex cone hull was defined, one can introduce E-affine hull as follows.

Definition 2.4.16 The E-affine hull of a set M, denoted by $E\text{-}aff(M)$ is the intersection of all E-affine sets containing M; that is,

$$E\text{-}aff(M) = \bigcap_{N \supseteq M} N,\ N \text{ are } E\text{-affine sets.}$$

Theorem 2.4.17 Let $M \subset \mathbb{R}^n$ and $\mathcal{M}$ is the set of all E-affine combinations of the elements of M. That is

$$\mathcal{M} = \bigcup_{p \in N} M(x,p),$$

where $M(x,p)$ is defined in Theorem 2.4.15. Then $\mathcal{M} \subseteq E\text{-}aff(M)$. Moreover, if $E(\mathcal{M}) \subseteq \mathcal{M}$. Then $E\text{-}aff(M) = \mathcal{M}$.

Proof. The proof follows in a way similar to that of Theorem 2.2.3. First, we prove $\mathcal{M} \subseteq E\text{-}aff(M)$. Assume that $x \in \mathcal{M}$, from the definition of $\mathcal{M}$, there exists $\{x_1, \dots, x_m\} \subset M$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ with $\sum_{i=1}^{m} \lambda_i = 1$ such that $x = \sum_{i=1}^{m} \lambda_i E(x_i)$. Since $\{x_1, \dots, x_m\} \subset M \subset E\text{-}aff(M)$, then using Theorem 2.4.15 and the fact that $E\text{-}aff(M)$ is E-affine set, every E-affine combination of $x_i's$ must remain in $E\text{-}aff(M)$. Hence, $x \in E\text{-}aff(M)$ and consequently,

$$\mathcal{M} \subseteq E\text{-}aff(M). \tag{2.8}$$

Next, we must show that $E\text{-}aff(M) \subseteq \mathcal{M}$ if $E(\mathcal{M}) \subseteq \mathcal{M}$. As we have shown in Theorem 2.2.3, it is enough to show that $\mathcal{M}$ is an affine set. That is, if $\mathcal{M}$ is an affine set and $E(\mathcal{M}) \subseteq \mathcal{M}$. Then from Proposition 2.4.9, $\mathcal{M}$ is E-affine set. The last conclusion with the fact that $M \subseteq \mathcal{M}$ yield $E\text{-}aff(M) \subseteq \mathcal{M}$ as required. To show that $\mathcal{M}$ is an affine set. Let $x, y \in \mathcal{M}$, then

$$x = \sum_{i=1}^{p} \lambda_i E(x_i) \quad \text{and } y = \sum_{i=1}^{s} \gamma_i E(y_i),$$

where $\{x_1, \dots, x_p, y_1, \dots, y_s\} \subset S$ and $\{\lambda_1, \dots, \lambda_p, \gamma_1, \dots, \gamma_s\}$ are real numbers such that $\sum_{i=1}^{p} \lambda_i = 1$ and $\sum_{i=1}^{s} \gamma_i = 1$. Fix $\alpha \in \mathbb{R}$, then the affine combination

$$\alpha x + (1-\alpha)y = \sum_{i=1}^{p} (\alpha\lambda_i)E(x_i) + \sum_{i=1}^{s} (1-\alpha)\gamma_i E(y_i) .$$

Therefore, $\alpha x + (1-\alpha)y \in \mathcal{M}$. i.e., $\mathcal{M}$ is an affine set, and using the assumption $E(\mathcal{M}) \subseteq \mathcal{M}$ yield $\mathcal{M}$ is E −affine set. Since $M \subseteq \mathcal{M}$ and $M \subseteq E\text{-}aff(M)$.Then

$$E\text{-}aff(M) \subseteq \mathcal{M}. \tag{2.9}$$

From (2.8) and (2.9), we obtain $E\text{-}aff(M) = \mathcal{M}$. ■

CHAPTER THREE

ON CONVEX FUNCTIONS, E-CONVEX FUNCTIONS AND THEIR GENERALIZATIONS

Chapter Three

On Convex Functions, E-Convex Functions and Their Generalizations

3.1 Introduction

E-convex functions is considered as an important generalization of convex functions. This class of functions is established by Youness [15] in the same paper in which E-convex set is first defined. As for E-convex set, the mapping E played an essential role in the definition of E-convex function. Despite the importance of Youness's first paper [15] on E-conexity, some of the results appeared in this paper are incorrect (see [54]). This motivates Chen to introduce new classes of E-convex functions called semi E-convex, quasi semi E-convex functions and pseudo semi E-convex functions. These functions are generalization of the class of convex functions. Using those functions, Chen improved Youness's incorrect results and study some properties of those functions [52, 53]. Another class of functions which are generalization of E-convex function is independently studied by different authors. This class includes E-quasiconvex and E-pseudoconvex functions [17, 40, 56].

In this chapter, we introduce and prove some general properties and differentiability properties of generalized convex functions and E-convex functions, respectively. We also provide some characterizations of convex functions, E-convex functions, and their generalizations functions mentioned above. In Section 2, a variety of properties related to E-convex functions and some of generalized convex functions mentioned above are introduced. In Section 3, some new characterizations of convex function, E-convex function, and their generalizations are discussed in terms of different level sets and different forms of epigraphs which are related to these functions. Namely, we introduce some new properties and characterizations of convex function, quasi convex functions, and quasi semi E-convex functions in terms of some α-level sets of f (see Propositions 3.3.2, 3.3.3, 3.3.6, 3.3.7). We also show some new properties and characterizations of semi E-convex function, E-convex function, and convex function using the epigraph sets $epif$, $epi_E\, f$ and $epi^E\, f$ (see Propositions 3.3.12-3.3.22). Finally, in Section 4, some differentiability properties of E-convex functions are discussed. The contents of section 3.3 have been published recently in [50, Section 2].

For the sake of brevity in writing the statements of the properties in this chapter, we refer to the following assumption.

Assumption A. Let $f: S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function, and $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given mapping.

Remark: For simplicity in appearance, in the rest of the thesis, we omit in the proofs and calculations the parentheses from $E(x)$, and writing it instead as Ex whenever it seems convenient.

3.2 Some Properties of Generalized Convex Functions

In this section, we discuss some basic properties of E-convex functions and their generalization E-quasiconvex functions. Similar properties are also shown for some generalized convex functions which are semi E-convex, pseudo semi E-convex, and quasi semi E-convex functions.

We start first by showing that the set of E-convex functions (respectively, semi E-convex) functions is closed under addition and nonnegative scalar multiplication. Same property holds for classical convex functions.

Theorem 3.2.1 Let $f, g: S \longrightarrow \mathbb{R}$ are two functions such that S and E are defined as in Assumption A and S is an E-convex set. Then

1. If f and g are E-convex on S, then $\alpha f + \beta g$ is an E-convex on S for all $\alpha, \beta \geq 0$.
2. If f and g are semi E-convex on S, then $\alpha f + \beta g$ is a semi E-convex on S for all $\alpha, \beta \geq 0$.

Proof. We prove (1) and in a similar manner one can show (2). Let $x, y \in S$, and $\lambda \in [0,1]$. Set $z = \lambda E(x) + (1-\lambda)E(y) \in S$. Then

$$(\alpha f + \beta g)(z) = \alpha f(z) + \beta g(z).$$

Using the E-convexity of f and g and the above equality, we obtain

$$\begin{aligned}(\alpha f + \beta g)(z) = \alpha f(z) + \beta g(z) \leq \alpha\lambda f(Ex) + \alpha(1-\lambda)f(Ey) \\ + \beta\lambda g(Ex) + \beta(1-\lambda)g(Ey), \\ = \lambda(\alpha f + \beta g)(Ex) + (1-\lambda)(\alpha f + \beta g)(Ey).\end{aligned}$$

Hence, $\alpha f + \beta g$ is an E-convex on S. ■

Theorem 3.2.2 Let f, S and E are defined as in assumption A such that f is a semi E-convex on the E-convex set S. Assume also that G: $\mathbb{R} \longrightarrow \mathbb{R}$ is a convex non-decreasing function. Then Gof is a semi E-convex function.

Proof. Let $x, y \in S$, and $0 \leq \lambda \leq 1$. Since, f is semi E-convex on the E-convex set S, then $\lambda E(x) + (1-\lambda)E(y) \in S$ and

$$f(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f(x) + (1-\lambda)f(y),$$

$\mathrm{G}\left(f\left(\lambda E(x)+(1-\lambda)E(y)\right)\right) \leq \mathrm{G}\left(\lambda f(x)+(1-\lambda)f(y)\right).$

The last inequality holds because G is a non-decreasing function. Using the convexity assumption of G, the right-hand side of the last inequality yields,

$$\mathrm{G}(f(\lambda E(x)+(1-\lambda)E(y)) \leq \lambda \mathrm{G}(f(x))+(1-\lambda)\mathrm{G}(f(y)),$$

i.e., $(\mathrm{G}of)(\lambda E(x)+(1-\lambda)E(y)) \leq \lambda(\mathrm{G}of)(x)+(1-\lambda)(\mathrm{G}of)(y)$.

Thus, $\mathrm{G}of$ is a semi E-convex on S. ■

An analogous to the above property is followed when f is E-convex function.

Theorem 3.2.3 Let f, S and E are defined as in assumption A such that f is an E-convex on the E-convex set S. Assume also that $\mathrm{G}: \mathbb{R} \longrightarrow \mathbb{R}$ is a convex non-decreasing function. Then $\mathrm{G}of$ is an E-convex function.

Proof. The proof follows in exactly same steps as in the above Theorem. The only difference occurs in applying the definition of E-convex function rather than the definition of semi E-convex function. ■

Theorem 3.2.4 Let f, S and E are defined as in assumption A such that f is an E-quasiconvex on the E-convex set S. Let $\mathrm{G}: \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing function. Then $\mathrm{G}of$ is an E-quasiconvex function on S.

Proof. Let $x, y \in S$, and $0 \leq \lambda \leq 1$. From the definition of f and S, we have

$$f(\lambda E(x)+(1-\lambda)E(y)) \leq \max\{f(E(x)), f(E(y))\},$$

and $\lambda E(x)+(1-\lambda)E(y) \in S$. Since G is non-decreasing function then,

$\mathrm{G}(f(\lambda E(x)+(1-\lambda)E(y)) \leq \mathrm{G}(\max\{f(E(x), fE(y)\})$. That is,

$(\mathrm{G}of)(\lambda E(x)+(1-\lambda)E(y) \leq \max\{\mathrm{G}(f(E(x)), \mathrm{G}(fE(y))\}$,

$$= \max\{(\mathrm{G}of)(E(x)), (\mathrm{G}of)(E(y))\}.$$

Hence, $\mathrm{G}of$ is E-quasi convex on S. ■

Corollary 3.2.5 Let f, S and E are defined as in assumption A such that f is an E-convex on the E-convex set S. Assume also that $\mathrm{G}: \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing function. Then $\mathrm{G}of$ is an E-quasiconvex function on S.

Proof. From [40, p. 3339], every E-convex function on S is E-quasiconvex on S. Consequently, f is E-quasiconvex on S. Using now Theorem 3.2.4, we obtain $\mathrm{G}of$ is E-quasiconvex on S. ■

The composite property is also held if f is a pseudo semi E-convex as we show next.

Theorem 3.2.6 Let f, S and E are defined as in assumption A such that f is a pseudo semi E-convex on the E-convex set S. Assume also that $\mathrm{G}: \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing strictly positive sublinear mapping and $b: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly positive function. Then $\mathrm{G}of$ is a pseudo semi E-convex.

Proof. Let $x, y \in S$, $\lambda \in (0,1)$. From the definition of f we have, if $f(x) < f(y)$ then $f(\lambda E(x) + (1-\lambda)E(y)) \leq f(y) + \lambda(\lambda - 1)b(x, y)$. Since G is a non-decreasing function, then, using the last expression, if $(\mathrm{G}of)(x) < (\mathrm{G}of)(y)$ we get

$$(\mathrm{G}of)(\lambda E(x) + (1-\lambda)E(y)) \leq \mathrm{G}[f(y) + \lambda(\lambda - 1)b(x, y)].$$

From the assumption, G is a sublinear mapping. Thus, the right-hand side of the last inequality yields,

$$(\mathrm{G}of)(\lambda E(x) + (1-\lambda)E(y)) \leq (\mathrm{G}of)(y) + \lambda(\lambda - 1)(Gob)(x, y)$$

$$= (\mathrm{G}of)(y) + \lambda(\lambda - 1)\bar{b}(x, y),$$

where $\bar{b}(x,y) = (\mathrm{G}ob)(x,y)$. Since G and b are strictly positive functions, then $\bar{b}(x,y)$ is a strictly positive. Hence, we obtain the required conclusion. ■

Remark 3.2.7: It was shown in [56, Theorems 3.6-3.7] that the supremum of an arbitrary non-empty collection of E-convex (respectively, E-quasiconvex) bounded above functions $\{f_i : i \in \Lambda\}$ on E- convex set S is E-convex (respectively, E-quasiconvex) on S. Similar property is given in [53, Proposition 2] for semi E-convex functions f_i for each $i \in \Lambda$. The latter proposition is given in [53] without proof. We give its proof next.

Proposition 3.2.8 Let $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is semi E-convex and bounded from above on an E-convex set $S \subset \mathbb{R}^n$ with the same map $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for all $i \in \Lambda$. Then, $f = \sup_{i\in\Lambda} f_i$ is a semi E-convex on S.

Proof. Since f_i is a semi E-convex, $\forall i \in \Lambda$, then, for each $x, y \in S$ and $0 \leq \lambda \leq 1$ we have

$$f_i(\lambda E(x) + (1-\lambda)E(y)) \leq \lambda f_i(x) + (1-\lambda) f_i(y) \qquad \forall i \in \Lambda.$$

Taking the supremum to the right-hand side of the inequality above, we get

$$f_i\big(\lambda E(x) + (1-\lambda)E(y)\big) \leq \sup_{i\in\Lambda} [\lambda f_i(x) + (1-\lambda) f_i(y)] \qquad \forall i \in \Lambda.$$

Then, $\sup_{i\in\Lambda} f_i\big(\lambda E(x) + (1-\lambda)E(y)\big) \leq \sup_{i\in\Lambda}[\lambda f_i(x) + (1-\lambda) f_i(y)\,]$.

From the assumption and the fact that $\sup M$ and $\sup N$ are finite, then $\sup(M + N) = \sup M + \sup N$, the last inequality yields,

$$f(\lambda E(x) + (1-\lambda)E(y) \leq \lambda \sup f_i(x) + (1-\lambda) \sup f_i(y)$$

$$= \lambda f(x) + (1-\lambda) f(y).$$

Then, we get f is a semi E −convex. ■

Similar result is formulated next, if the functions f_i are quasi semi (respectively, strictly quasi semi / strongly quasi semi) E-convex or pseudo semi E-convex, for each $i \in \Lambda$.

Proposition 3.2.9 Let $f_i\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is bounded from above for each $i \in \Lambda$ and S is E-convex set with the same map $E\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Define, $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f = \sup_{i\in\Lambda} f_i$. Then

1. If f_i is a quasi semi E-convex on S for each $i \in \Lambda$, then f is a quasi semi E-convex.
2. If f_i is strictly quasi semi E-convex on S, for each $i \in \Lambda$. Then, f is strictly quasi semi E-convex.
3. If f_i is strongly quasi semi E-convex on S, for each $i \in \Lambda$. Then, f is strongly quasi semi E-convex.
4. If f_i pseudo semi E-convex bounded above functions on S, for each $i \in \Lambda$ and $b_i\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly positive, $\forall i \in \Lambda$ such that $b(x, y) = \sup\limits_{i\in\Lambda} b_i(x, y)$ exists in $\mathbb{R}$. Then f is pseudo semi E-convex on S.

Proof. To show (1), f_i is a quasi semi E-convex, $\forall i \in \Lambda$. Then, for each $x, y \in S$ and $0 \le \lambda \le 1$ we have

$$f_i(\lambda E(x) + (1-\lambda)E(y)) \le \max\{f_i(x), f_i(y)\} \qquad \forall i \in \Lambda.$$

Taking the supremum for the right-hand side and then for the left-hand side, we get

$$\sup_{i\in\Lambda} f_i\big(\lambda E(x) + (1-\lambda)E(y)\big) \le \sup_{i\in\Lambda}\{\max\{f_i(x), f_i(y)\}\},$$

$$\begin{aligned} f(\lambda E(x) + (1-\lambda)E(y)) &\le \max\left\{\sup_{i\in\Lambda} f_i(x), \sup_{i\in\Lambda} f_i(y)\right\}, \\ &\le \max\{f(x), f(y)\}. \end{aligned}$$

The last inequality yields, f is a quasi semi E-convex. The proof of Parts (2) and (3) proceed in a way similar to the proof of part (1). The only different is that we require strictly inequality with $f(x) \neq f(y)$, when f is strictly quasi semi E-convex, and $x \neq y$ in case, f is strongly quasi semi E-convex. Finally, we show Part (4), let $\{f_i : i \in \Lambda\}$ is an arbitrary nonempty collection of bounded above pseudo semi E-convex on S. By the definition of f_i, we have

If $f_i(x) < f_i(y)$, then for all $i \in \Lambda$

$$f_i\big(\lambda E(x) + (1-\lambda)E(y)\big) \leq f_i(y) + \lambda(\lambda - 1)b_i(x,y)\,,$$

i.e., $\sup_{i\in\Lambda} f_i\big(\lambda E(x) + (1-\lambda)E(y)\big) \leq \sup_{i\in\Lambda}\{f_i(y) + \lambda(\lambda - 1)b_i(x,y)\}$.

That is,

$$\sup_{i\in\Lambda} f_i\big(\lambda E(x) + (1-\lambda)E(y)\big) \leq \sup_{i\in\Lambda} f_i(y) + \lambda(\lambda - 1)\sup_{i\in\Lambda} b_i(x,y).$$

Now from the assumption, $b(x,y) = \sup_{i\in\Lambda} b_i(x,y)$.

This yield,

$$f\big(\lambda E(x) + (1-\lambda)E(y)\big) \leq f(y) + \lambda(\lambda - 1)b(x,y),$$

where $b(x,y)$ is strictly positive function. Thus, f is pseudo semi E-convex on S. ∎

3.3 Some Characterizations of Convex Function, E-Convex Function and Their Generalizations

In this section, we provide some relations and characterizations of convex functions, quasi convex functions, and quasi semi E-convex functions using the α −level sets $S_\alpha^E[f]$ and $E - S_\alpha[f]$ of a function f. Note that, some characterizations of E-convex function and its generalizations (semi E-convex functions, quasi semi E-convex functions, and E-quasiconvex functions) are given using $S_\alpha[f]$ and $S_{\alpha,E}[f]$ (see [53, Proposition 4, Proposition 6] and [40, Theorem 3.10, Theorems 3.12-3.14]). We also introduce new relations and characterizations of semi E-convex function, E-convex function, and convex function using the epigraph sets $epif$, $epi_E\ f$ and $epi^E\ f$.

The following definition is needed in this section.

Definition 3.3.1 [20, 34] Let S_1 and S_2 be two subsets of $\mathbb{R}^n$. Then S_1 is said to be **slack 2-convex** with respect to S_2 (for short, S_1 is s. 2-convex w.r.t. S_2) if, for every for every $s_1, s_2 \in S_1 \cap S_2$ and every $0 \leq \lambda \leq 1$ such that $(1-\lambda)s_1 + \lambda s_2 \in S_2$, we get $(1-\lambda)s_1 + \lambda s_2 \in S_1$.

The next two propositions give sufficient conditions for $S_\alpha^E[f]$ to be a convex set and a s. 2-convex w.r.t. $E(S)$, respectively.

Proposition 3.3.2 Let f, S, and E are defined as in assumption A such that f is convex on the convex set S, E is a linear mapping, and $E(S)$ is a convex set. Then $S_\alpha^E\ [f]$ is a convex set, for all $\alpha \in \mathbb{R}$.

Proof. Let $\alpha \in \mathbb{R}$ and $E(s_1), E(s_2) \in S_\alpha^E[f]$, then $E(s_1), E(s_2) \in E(S)$ and $f(s_1) \leq \alpha, f(s_2) \leq \alpha$. Since $E(S)$ is a convex set, then

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in E(S). \tag{3.1}$$

For each $0 \leq \lambda \leq 1$. Using (3.1) and the linearity of E,

$$\lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in E(S). \tag{3.2}$$

This means that $\lambda s_1 + (1-\lambda)s_2 \in S$. From the convexity of f we have

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda)f(s_2) \leq \alpha. \tag{3.3}$$

By (3.2) and (3.3), we get $\lambda E(s_1) + (1-\lambda)E(s_2) \in S_\alpha^E[f]$. ■

Proposition 3.3.3 Let f, S, and E are defined as in assumption A. If f is a convex function on the convex set S, and E is a linear mapping. Then $S_\alpha^E[f]$ is a s. 2-convex w. r. t. $E(S)$, for all $\alpha \in \mathbb{R}$.

Proof. Let $\alpha \in \mathbb{R}$. Assume that $E(s_1), E(s_2) \in S_\alpha^E[f] \cap E(S)$ such that for each $0 \leq \lambda \leq 1$, we have $\lambda E(s_1) + (1-\lambda)E(s_2) \in E(S)$. Since $E(s_1), E(s_2) \in S_\alpha^E[f]$, then $s_1, s_2 \in S$, and $f(s_1) \leq \alpha$, $f(s_2) \leq \alpha$. By the linearity of E,

$$\lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in E(S). \tag{3.4}$$

This means $\lambda s_1 + (1-\lambda)s_2 \in S$. Since f is a convex function, then

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda)f(s_2) \leq \alpha. \tag{3.5}$$

From (3.4) and (3.5), $\lambda E(s_1) + (1-\lambda)E(s_2) \in S_\alpha^E[f]$, which implies, $S_\alpha^E[f]$ is a s. 2-convex w. r. t. $E(S)$. ■

Remark 3.3.4 If the set $S_\alpha^E[f]$ is convex or s. 2-convex w. r. t. $E(S)$, it is not necessary that f is a convex function as we show in the following example.

Example 3.3.5 Let $S = [-10,10] \subseteq \mathbb{R}$, $E: \mathbb{R} \longrightarrow \mathbb{R}$ be a linear mapping such that $E(x) = \frac{1}{2}x$ for each $x \in \mathbb{R}$ and define a function $f: S \longrightarrow \mathbb{R}$ as $f(x) = x^3$ for each $x \in S$. It is clear that f is not a convex function on S. However, the level sets

$$S_\alpha^E[f] = \left\{\frac{1}{2}x \in [-5,5]: x^3 \leq \alpha\right\} = \{x \in [-10,10]: x^3 \leq \alpha\}$$

are either empty sets or intervals. In either case, $S_\alpha^E[f]$ is convex, for all $\alpha \in \mathbb{R}$. Also, since $E(S)$ and $S_\alpha^E[f]$ are convex sets, then for each $E(s_1), E(s_2) \in S_\alpha^E[f] \cap E(S)$ such that

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in E(S),$$

for every $0 \leq \lambda \leq 1$, we have

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in S_\alpha^E[f],$$

for all $\alpha \in \mathbb{R}$. i.e., $S_\alpha^E[f]$ is a s. 2-convex w. r. t. $E(S)$.

The following proposition proposes a necessary and sufficient condition for f to be a quasiconvex.

Proposition 3.3.6 Let f, S, and E are defined as in assumption A. If E is a linear mapping, S is a convex set. Then $S_\alpha^E[f]$ is a convex set, for all $\alpha \in \mathbb{R}$ if and only if f is a quasi convex on S.

Proof. First, we prove f is a quasi convex on S. Let $s_1, s_2 \in S$, and set $\alpha = \max\{f(s_1), f(s_2)\}$. Let $E(s_1), E(s_2) \in S_\alpha^E[f]$ which is a convex set, then for each $0 < \lambda < 1$.

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in S_\alpha^E[f]. \tag{3.6}$$

Using (3.6), and the linearity of E, we get

$$\lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in S_\alpha^E[f] \subseteq E(S).$$

Then,

$$\lambda s_1 + (1-\lambda)s_2 \in S.$$

and $f(\lambda s_1 + (1-\lambda)s_2) \leq \alpha = \max\{f(s_1), f(s_2)\}$. Hence, f is a quasi convex on S. Let us show the other direction and obtain $S_\alpha^E[f]$ is a convex set, for all $\alpha \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ and $E(s_1), E(s_2) \in S_\alpha^E[f]$, then $s_1, s_2 \in S$ and $f(s_1) \leq \alpha$,

$f(s_1) \leq \alpha$. Since S is convex and E is linear, then, for each $0 \leq \lambda \leq 1$, $\lambda s_1 + (1-\lambda)s_2 \in S$ and

$$\lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in E(S). \tag{3.7}$$

From the assumption, f is a quasi convex on S, thus,

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \max\{f(s_1), f(s_2)\} \leq \alpha. \tag{3.8}$$

From (3.7) -(3.8), we conclude $S_\alpha^E[f]$ is a convex set. ■

A necessary and sufficient condition for the level set $E - S_\alpha[f]$ of a function f to be E-convex is given next.

Proposition 3.3.7 Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}^n$ is E-convex set, and $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Then foE is a quasi semi E-convex on S if and only if $E - S_\alpha[f]$ is E-convex set, for all $\alpha \in \mathbb{R}$.

Proof. let $\alpha \in \mathbb{R}$, and $s_1, s_2 \in E - S_\alpha[f]$, then $f\big(E(s_1)\big) \leq \alpha$, and $f(E(s_2)) \leq \alpha$. We intend to show that $\lambda E(s_1) + (1-\lambda)E(s_2) \in E - S_\alpha[f]$, for each $0 \leq \lambda \leq 1$. Since foE is a quasi semi E −convex on the E −convex set S, then

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in S.$$

and

$$(foE)(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \max\{(foE)(s_1),\ (foE)(s_2)\} \leq \alpha.$$

Therefore, $\lambda E(s_1) + (1-\lambda)E(s_2) \in E - S_\alpha[f]$. To show the reverse direction, let $s_1, s_2 \in E - S_\alpha[f]$, and $0 \leq \lambda \leq 1$. Set $\alpha = \max\{(foE)(s_1), (foE)(s_2)\}$. Since $E - S_\alpha[f]$ is E-convex, then $\lambda E(s_1) + (1-\lambda)E(s_2) \in E - S_\alpha[f]$ such that

$$f(E(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \alpha = \max\{(foE)(s_1), (foE)(s_2)\}.$$

i.e, $(foE)(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \max\{(foE)(s_1), (foE)(s_2)\}$. Thus, foE is a quasi semi E-convex on S. ■

In general, the epigraphs defined in Definition 1.4.2.11 are not equal, e.g. (see [53]). We start first with the below proposition which shows the relationship between $epif$ and $E - e(f)$, $epi^E f$, and $epi_E f$, respectively. The first part of this proposition has been proved in [52, Theorem 2.2]. We get same conclusion but under weaker condition than the one assumed in [52].

Proposition 3.3.8 Let $f, S,$ and E are defined as in assumption A such that $f\big(E(s)\big) \leq f(s) \quad \forall s \in S$, then

1. $epif \subset E - e(f)$.
2. $epi^E f \subset epi_E f$.

Proof. To show (1), let $(s, \alpha) \in epif$, from the definition of $epif$ and the assumption, $f(E(s)) \leq f(s) \leq \alpha$ which implies that $(s, \alpha) \in E - e(f)$. For proving (2), suppose that $(E(s), \alpha) \in epi^E f$, then $(E(s), \alpha) \in E(S) \times \mathbb{R}$ such that $f(s) \leq \alpha$. Since $f\big(E(s)\big) \leq f(s)$, then $f\big(E(s)\big) \leq \alpha$. Thus, $(E(s), \alpha) \in epi_E f$ as required. ■

Proposition 3.3.9 Let $f, S,$ and E are defined as in assumption A such that $E(S) \subseteq S$, then $epi_E\, f \subset epi\, f$.

Proof. Let $(E(x), \alpha) \in epi_E\, f$, thus

$$(E(x), \alpha) \in E(S) \times \mathbb{R} \quad \text{and } f\big(E(x)\big) \leq \alpha.$$

Since $E(S) \subseteq S$, then $(E(x), \alpha) \in S \times \mathbb{R}$. Thus, $(E(x), \alpha) \in epi\, f$. ■

Remark 3.3.10 In the preceding proposition, if $E: S \longrightarrow S$ then $epi_E\, f \subset epi\, f$ automatically holds [52]. However, if $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the assumption $E(S) \subseteq S$ is essential for proving $epi_E\, f \subset epi\, f$. If we ignore this assumption, then the conclusion of Proposition 3.3.9 may not hold. For example, let $S = [-6,6] \subset \mathbb{R}$

and $E:\mathbb{R} \longrightarrow \mathbb{R}$ defined as $E(x) = 2x$, for all $x \in \mathbb{R}$. Let $f: S \longrightarrow \mathbb{R}$ such that $f(x) = x^2$ for all $x \in S$. Clearly, $E(S) = [-12,12] \nsubseteq S$. i.e., S is not E-convex set [15]. Moreover, $epi_E\, f \not\subset epi\, f$. Indeed, let $(E(6), 150) \in epi_E\, f$, i.e., $f\big(E(6)\big) = 144 < 150$ but $E(6) = 12 \notin S$. Thus, $(E(6), 150) \notin epif$.

In classical analysis, one of the possible characterization of convex functions is given in terms of $epi\, f$ as in the following proposition.

Proposition 3.3.11 [44, p.21] let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$. Then $epif$ is a convex set if and only if f is convex.

Youness [15] has provided a characterization of E-convex function using E-$e(f)$ (see [15, Theorem 3.1]). Unfortunately, this characterization has some erroneous (see [12, Counterexample 2.1] for a counter example). This motivates Chen [53, Proposition 9] and [52, Theorems 2.4-2.5, 2.8] to provide some characterizations of semi E-convex function f in terms of $epif, E - e(f)$, $epi_E f$, and $epi^E f$. Duca and Lupsa [12, Theorems 3.1-3.5], on the other hand, relate E-convex function f with $epif$ and $epi_E\, f$. In what follow, we give new relations and characterizations of semi E-convex functions, E-convex functions, and convex functions in terms of $epif$, $epi_E\, f$ and $epi^E\, f$. We start first with sufficient conditions for f to be semi E-convex function using the epigraph $epi^E f$. Another sufficient condition, for this result, is shown in [52, Theorem 2.8].

Proposition 3.3.12 Let f, S, and E are defined as in assumption A, if S is E-convex, $f\big(E(s)\big) \le f(s)$ $\forall s \in S$, and $epi^E f$ is a convex set. Then f is a semi E-convex function.

Proof. Let $s_1, s_2 \in S$ such that $\big(E(s_1), f(s_1)\big)$, $(E(s_2), f(s_2)) \in epi^E f$ which is a convex set. Thus, for each $0 \le \lambda \le 1$ we have

$$(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda f(s_1) + (1-\lambda)f(s_2)) \in epi^E f \subseteq E(S) \times \mathbb{R}.$$

Since $\lambda E(s_1)+(1-\lambda)E(s_2)\in E(S)$, then $\exists\, w\in S$, such that $E(w)=\lambda E(s_1)+(1-\lambda)E(s_2)$ and

$$f(w)\leq \lambda f(s_1)+(1-\lambda)f(s_2). \tag{3.9}$$

From the assumption and the inequality in (3.9),

$$f\big(\lambda E(s_1)+(1-\lambda)E(s_2)\big)=(f\big(E(w)\big)$$

$$\leq f(w)\leq \lambda f(s_1)+(1-\lambda)f(s_2)\,.$$

Hence, f is a semi E-convex function. ■

<u>Proposition 3.3.13</u> Let f, S, and E are defined as in assumption A, if S is E-convex, $E(S)$ is convex, $f\big(E(s)\big)\leq f(s)\quad \forall s\in S,$ and $epif$ is a s. 2-convex w. r. t. $E(S)\times\mathbb{R}$. Then f is a semi E-convex function.

Proof. The conclusion directly follows from [12, Theorem 3.4] and [53, Proposition 5]. Indeed, since S is E-convex set, $E(S)$ is a convex set, and $epif$ is a s. 2-convex w. r. t. $E(S)\times\mathbb{R}$, then using [12, Theorem 3.4], f is an E-convex function. Applying [53, Proposition 5], the last conclusion with the assumption $f\big(E(s)\big)\leq f(s)\quad \forall\, s\in S$ yield f is a semi E-convex function. ■

A necessary condition for f to be semi E- convex on S is given next.

<u>Proposition 3.3.14</u> Let f, S, and E are defined as in assumption A. Assume that S is E-convex set and f is semi E- convex on S. Then $epif$ is a s. 2-convex w. r. t. $E(S)\times\mathbb{R}$.

Proof. Suppose that $(s_1,\alpha),(s_2,\beta)\in\ epif\cap(E(S)\times\mathbb{R})$ such that $\lambda(s_1,\alpha)+(1-\lambda)(s_2,\beta)\in E(S)\times\mathbb{R}$. Let $0\leq\lambda\leq 1,$ we must prove that

$$(\,\lambda s_1+(1-\lambda)s_2,\ \lambda\alpha+(1-\lambda)\beta)\in epif.$$

Because $(s_1,\alpha),(s_2,\beta)\in\ epif$, then $f(s_1)\leq\alpha$ and $f(s_2)\leq\beta$. We also have $(s_1,\alpha),(s_2,\beta)\in E(S)\times\mathbb{R}$ and S is E-convex then from Proposition 1.4.1.3(1).

$$s_1, s_2 \in E(S) \subseteq S \text{ and } \lambda s_1 + (1-\lambda)s_2 \in E(S) \subseteq S. \tag{3.10}$$

From the first inclusion in (3.10), there exists $s, w \in S$ such that

$$s_1 = E(s) \text{ and } s_2 = E(w). \tag{3.11}$$

Since f is a semi E-convex on S, thus, from (3.10) and (3.11),

$$\begin{aligned} f(\lambda s_1 + (1-\lambda)s_2) &= f\big(\lambda E(s) + (1-\lambda)E(w)\big) \\ &\leq \lambda f\big(E(s)\big) + (1-\lambda) f\big(E(w)\big) \\ &= \lambda f(s_1) + (1-\lambda) f(s_2) \\ &\leq \lambda\alpha + (1-\lambda)\beta. \end{aligned} \tag{3.12}$$

From (3.10) and (3.12), $(\lambda s_1 + (1-\lambda)s_2, \lambda\alpha + (1-\lambda)\beta) \in epif$. ■

The next proposition provides a necessary condition for f to be E-convex function using the set $epi_E f$. The sufficient condition is given in [12, Theorem 3.1].

Proposition 3.3.15 Let $f, S,$ and E are defined as in assumption A. If $E(S)$ is a convex set and f is an E-convex function on the E-convex set S. Then $epi_E f$ is a convex set.

Proof. Assume that $(E(s_1), \alpha), (E(s_2), \beta) \in epi_E f$. From the definition of $epi_E f$, we have $f(E(s_1)) \leq \alpha, f(E(s_2)) \leq \beta$ and $E(s_1), E(s_2) \in E(S)$. Since $E(S)$ is a convex set, it follows that, for each $0 \leq \lambda \leq 1$, we have

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in E(S). \tag{3.13}$$

Since f is an E-convex function, then

$$\begin{aligned} f(\lambda E(s_1) + (1-\lambda)E(s_2)) &\leq \lambda f(E(s_1)) + (1-\lambda) f(E(s_2)) \\ &\leq \lambda\, \alpha + (1-\lambda)\beta. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14), we get $(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda\, \alpha + (1-\lambda)\beta) \in epi_E f$. i.e., $epi_E f$ is a convex set. ■

Combining the preceding Proposition and [12, Theorem 3.1], we obtain the following result.

Proposition 3.3.16 Let $f, S,$ and E are defined as in assumption A. Assume that $E(S)$ is a convex set and S is an E-convex set. Then $epi_E f$ is a convex set if and only if f is an E-convex function on S.

Another necessary condition for f to be an E-convex function using the set $epi_E f$ is given next.

Proposition 3.3.17 Let $f, S,$ and E are defined as in assumption A. If f *is* an E-convex on the E-convex set, S. Then epi_Ef is a s. 2-convex w. r. t. $E(S) \times \mathbb{R}$.

Proof. Assume that $(E(s_1), \alpha), (E(s_2), \beta) \in epi_E f \cap (E(S) \times \mathbb{R})$ such that, for each $0 \le \lambda \le 1$, we have

$$(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda\alpha + (1-\lambda)\beta) \in E(S) \times \mathbb{R}.$$

From the last assertion and the E-convexity of S, we have

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in E(S) \subseteq S. \tag{3.15}$$

Since f is E-convex, then

$$f\big(\lambda E(s_1) + (1-\lambda)E(s_2)\big) \le \lambda f\big(E(s_1)\big) + (1-\lambda) f(E(s_2))$$

$$\le \lambda\alpha + (1-\lambda)\beta. \tag{3.16}$$

From (3.15) and (3.16), it follows that

$(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda\alpha + (1-\lambda)\beta) \in epi_E f$, as required. ■

The converse of the preceding proposition is satisfied when $E(S)$ is a convex set (see [12, Theorem 3.2]). Consequently, the following proposition follows.

Proposition 3.3.18 Let f, S, and E are defined as in assumption A. Assume that $E(S)$ is a convex set and S is an E-convex set. Then $epi_E f$ is a s. 2-convex w. r. t. $E(S) \times \mathbb{R}$ if and only if f is an E-convex function on S.

The following two propositions give necessary conditions for f to be a convex function using, the convexity and the slack 2-convexity of the set $epi^E f$, respectively.

Proposition 3.3.19 Let f, S, and E are defined as in assumption A. If $E(S)$ is a convex set, f is a convex function on the convex set S, and E is a linear mapping. Then $epi^E f$ is a convex set.

Proof. Suppose that $(E(s_1), \alpha), (E(s_2), \beta) \in epi^E f$ and $0 \leq \lambda \leq 1$. We must show that $(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda\alpha + (1-\lambda)\beta) \in epi^E f$. From the definition of $epi^E f$, we have $f(s_1) \leq \alpha, f(s_2) \leq \beta$ and $E(s_1), E(s_2) \in E(S)$. Since $E(S)$ is a convex set and E is a linear mapping, then

$$\lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in E(S), \qquad (3.17)$$

where $\lambda s_1 + (1-\lambda)s_2 \in S$. Since f is a convex function on S, then

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda)f(s_2) \leq \lambda\alpha + (1-\lambda)\beta. \qquad (3.18)$$

Thus, from (3.17) and (3.18), $(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda\alpha + (1-\lambda)\beta) \in epi^E f$, and hence, $epi^E f$ is a convex set. ■

Proposition 3.3.20 Let f, S, and E are defined as in assumption A. Assume that f is a convex on the convex set S, E is a linear mapping, and $f(E(s)) \leq f(s)$ for all $s \in S$. Then $epi^E f$ is a s. 2-convex w. r. t. $E(S) \times \mathbb{R}$.

Proof. $(E(s_1),\alpha),(E(s_2),\beta)\in epi^E f\cap(E(S)\times\mathbb{R})$ such that, for $0\le\lambda\le 1$, we have $(\lambda E(s_1)+(1-\lambda)E(s_2),\lambda\alpha+(1-\lambda)\beta)\in E(S)\times\mathbb{R}$, i.e.,

$$\lambda E(s_1)+(1-\lambda)E(s_2)\in E(S). \tag{3.19}$$

Since $(E(s_1),\alpha),(E(s_2),\beta)\in epi^E f$, then $f(s_1)\le\alpha$ and $f(s_2)\le\beta$. Because E is a linear mapping, hence, $\lambda E(s_1)+(1-\lambda)E(s_2)=E(\lambda s_1+(1-\lambda)s_2)\in E(S)$. Thus, $\lambda s_1+(1-\lambda)s_2\in S$. Using the last two assertions and the assumption $f\big(E(s)\big)\le f(s)$ for all $s\in S$, we get

$$\begin{aligned} f\big(\lambda E(s_1)+(1-\lambda)E(s_2)\big) &= f\big(E(\lambda s_1+(1-\lambda)s_2)\big)\\ &\le f(\lambda s_1+(1-\lambda)s_2)\\ &\le \lambda\alpha+(1-\lambda)\beta. \end{aligned} \tag{3.20}$$

By (3.19) and (3.20), we obtain $(\lambda E(s_1)+(1-\lambda)E(s_2),\lambda\alpha+(1-\lambda)\beta)\in epi^E f$, as we want to prove. ■

The next proposition suggests a sufficient condition for f to be a convex function using the set $epi^E f$.

Proposition 3.3.21 Let $f,S,$ and E are defined as in assumption A. If E is a linear mapping, S is a convex set, and $epi^E f$ is a s. 2-convex w. r. t. $E(S)\times\mathbb{R}$. Then f is a convex function.

Proof. Let $s_1,s_2\in S$ and $0\le\lambda\le 1$. Let $(E(s_1),f(s_1)),(E(s_2),f(s_2))\in(E(S)\times\mathbb{R})\cap epi^E f$ such that whenever

$$\big(\lambda E(s_1)+(1-\lambda)E(s_2),\lambda f(s_1)+(1-\lambda)f(s_2)\big)\in E(S)\times\mathbb{R}.$$

then $\big(\lambda E(s_1)+(1-\lambda)E(s_2),\lambda f(s_1)+(1-\lambda)f(s_2)\big)\in epi^E f$.

Since E is a linear mapping, the last statement yields

$$\big(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda f(s_1) + (1-\lambda)f(s_2)\big)$$

$$= (E(\lambda s_1 + (1-\lambda)s_2), \lambda f(s_1) + (1-\lambda)f(s_2)) \in epi^E f.$$

This means $f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda)f(s_2)$. From the convexity of S, $\lambda s_1 + (1-\lambda)s_2 \in S$. Therefore, f is a convex function. ■

From Propositions 3.3.20 and 3.3.21, the following result deduces.

Proposition 3.3.22 Let $f, S,$ and E are defined as in assumption A. If E is a linear mapping, S is a convex set, and $f\big(E(s)\big) \leq f(s)$ for all $s \in S$. Then $epi^E f$ is a s. 2-convex w. r. t. $E(S) \times \mathbb{R}$ if and only if f is a convex function.

3.4 Differentiability Properties of E-convex Functions

The differentiability of E-convex functions has been briefly studied by Youness [16]. It is also discussed recently by Soleimani-damaneh [40]. Both researchers presented some characteristics of differentiable of E-convex functions by using different approaches (for more details, see Lemmas 3.1-3.2 in [16], and their counter parts Lemmas 2.3-2.4 stated in [40], respectively). According to Lemmas 2.3-2.4 [40], Soleimani-damaneh [40, Proposition 2.5] provided a characterization of differentiable E-convex functions with respect to E-monotonicity of the gradient of the differentiable function. In this section, we prove the characterizations of E-convex (respectively, E-concave) functions using different assumptions (see Theorems 3.4.1-3.4.2). We also obtain differentiability properties for strictly E-convex (respectively, E-concave) functions. Some test criteria of E-convexity and E-concavity of a function are presented in this section with an illustration example (see Theorem 3.4.7 and Example 3.4.8).

Theorem 3.4.1 Let $f, S,$ and E are defined as in assumption A such that f is differentiable on $S \subseteq \mathbb{R}^n$ and S is an open E-convex set. Then

i. If f is E-convex on S, then

$$f(Ey) \geq f(Ex) + < \nabla f(Ex), Ey - Ex > \qquad \forall x, y \in S. \qquad (3.21)$$

ii. If f is E-concave on S, then

$$f(Ey) \leq f(Ex) + < \nabla f(Ex), Ey - Ex > \qquad \forall x, y \in S. \qquad (3.22)$$

Proof. Let us show (i). Since S is E-convex and f is differentiable on S, then in particular, f is differentiable on $E(S) \subseteq S$. If $E(x) = E(y)$, then the gradient inequality directly satisfied. Consider now $x, y \in S$ such that $E(x) \neq E(y)$ and $\lambda \in (0,1]$, then using the E-convexity of f, we have

$$f(\lambda Ey + (1-\lambda)Ex) \leq \lambda f(Ey) + (1-\lambda) f(Ex). \qquad (3.23)$$

That is,

$$f\big(Ex + \lambda(Ey - Ex)\big) \leq f(Ex) + \lambda\big(f(Ey) - f(Ex)\big).$$

Re-arranging the last inequality yields,

$$\frac{f\big(Ex + \lambda(Ey - Ex)\big) - f(Ex)}{\lambda} \leq f(Ey) - f(Ex).$$

Taking the limit to both sides of the above inequality ($as\ \lambda \longrightarrow 0^+$) yields,

$$\lim_{\lambda \longrightarrow 0^+} \frac{f\big(Ex + \lambda(Ey - Ex)\big) - f(Ex)}{\lambda} \leq f(Ey) - f(Ex).$$
(3.24)

The left-hand side of the inequality (3.24) is the directional derivative of f at $E(x)$ in the direction of $(Ey - Ex)$. Thus, (3.24) becomes

$$< \nabla f(Ex), Ey - Ex > \ \leq f(Ey) - f(Ex).$$

Re-arranging last expression, we get

$$f(Ey) \geq f(Ex) + < \nabla f(Ex), Ey - Ex >.$$

Part (ii) which assumes that f is E-concave function, and obtaining the inequality (3.22), proceeds in a way similar to part (i) where we use the definition of E-concave function in (3.23) instead of using the definition of E-convex function. ■

The reverse direction of Theorem 3.4.1(i) has been proved in [40, Lemma 2.4] if S is a convex set and E is linear. We give an alternative assumption to prove the other direction of Theorem 3.4.1(i-ii).

Theorem 3.4.2 Let $f, S,$ and E are defined as in assumption A. Assume that f is differentiable on the E-convex set S, $E(S)$ is convex and

i. $f(Ey) \geq f(Ex) + < \nabla f(Ex), Ey - Ex > \quad \forall x, y \in S.$ (3.25)

Then f is E-convex on S.

ii. $f(Ey) \leq f(Ex) + < \nabla f(Ex), Ey - Ex > \quad \forall x, y \in S.$ (3.26)

Then f is E-concave on S.

Proof. For proving (i). Take arbitrary $x_1, x_2 \in S$ such that S is a E-convex set, and let $\lambda \in [0,1]$. Define $z = \lambda E x_1 + (1-\lambda) E x_2 \in S$. Since $E(x_1), E(x_2) \in E(S)$ and $E(S)$ is convex, then

$$z = \lambda E x_1 + (1-\lambda) E x_2 \in E(S).$$

Hence, there exists $s \in S$ such that $E(s) = z = \lambda E x_1 + (1-\lambda) E x_2$

Apply (3.25) with $Ey = Ex_1$ and $Ex = Es$ yields,

$$f(Ex_1) \geq f(Es) + < \nabla f(Es), Ex_1 - Es >. \tag{3.27}$$

Similarly, apply (3.25) with $Ey = Ex_2$ and $Ex = Es$ we get,

$$f(Ex_2) \geq f(Es) + < \nabla f(Es), Ex_2 - Es >. \tag{3.28}$$

We multiply (3.27) by λ and (3.28) by $(1-\lambda)$, and sum the two inequalities up

$\lambda f(Ex_1) + (1-\lambda)f(Ex_2) \geq f(Es)$

$$+ < \nabla f(Ex), \lambda Ex_1 + (1-\lambda)Ex_2 - Es >.$$

The last inequality yields

$$\lambda f(Ex_1) + (1-\lambda)f(Ex_2) \geq f(\lambda Ex_1 + (1-\lambda)(Ex_2)).$$

Hence, f is E-convex as required. Part (ii) which assumes the inequality (3.26), and establishing f is E-concave function, follows in a way similar to part (i) where we reverse each inequality in the proof of part (i). ■

Theorems 3.4.1 and 3.4.2 can be extended to give a characterization to a differentiable strictly E-convex (respectively, E-concave) function in terms of its strictly gradient inequality as we show next.

Theorem 3.4.3 Let f, S, and E are defined as in assumption A. Assume that f is a differentiable function on the E-convex set S and $E(S)$ is a convex set. Then

i. f is strictly E-convex if and only if for all $x, y \in S$ such that $x \neq y$ we have $f(Ey) > f(Ex) + < \nabla f(Ex), Ey - Ex >$.
ii. f is strictly E-concave if and only if for all $x, y \in S$ such that $x \neq y$ we have $f(Ey) < f(Ex) + < \nabla f(Ex), Ey - Ex >$.

Proof. We show (ii) and in a similar pattern one can prove (i). Assume that the function is f strictly E-concave on S, hence f is E-concave. From Theorem 3.4.1(ii), the inequality (3.22) follows, for all $x, y \in S$. i.e.,

$$f(Ey) \leq f(Ex) + < \nabla f(Ex), Ey - Ex >.$$

Assume that there exists $E(x), E(y)$ such that

$$f(Ey) = f(Ex) + < \nabla f(Ex), Ey - Ex >. \tag{3.29}$$

Since f is strictly E-concave, then for any $\lambda \in (0,1)$

$$f(\lambda Ey + (1-\lambda)Ex) = f\big(Ex + \lambda(Ey - Ex)\big)$$

$$> \lambda f(Ey) + (1-\lambda) f(Ex). \tag{3.30}$$

Substitute $f(Ey)$ in (3.29) into the right-hand side of (3.30), we get

$$f\big(Ex + \lambda(Ey - Ex)\big) > f(Ex) + \lambda < \nabla f(Ex), Ey - Ex >. \tag{3.31}$$

Also, apply (3.22) with $Ey = Ex + \lambda(Ey - Ex)$ which yields

$$f(Ex + \lambda(Ey - Ex)) \leq f(Ex) + \lambda < \nabla f(Ex), Ey - Ex >. \tag{3.32}$$

Combining (3.31) and (3.32) yields

$$f(Ex) + \lambda < \nabla f(Ex), Ey - Ex > \; < f\big(Ex + \lambda(Ey - Ex)\big)$$

$$\leq f(Ex) + \lambda < \nabla f(Ex), Ey - Ex >,$$

which is a contradiction. Therefore, for all $x, y \in S$, we have that

$$f(Ey) < f(Ex) + < \nabla f(Ex), Ey - Ex >.$$

The proof of the other direction proceeds in a way similar to Theorem 3.4.2(ii).

■

The following theorem provides a necessary and sufficient conditions for f to be E-convex function using the gradient test of f

Theorem 3.4.4 Let f, S, and E are defined as in assumption A. Let f is a differentiable function on the open E-convex set S. Then

i. If f is E-convex on S then for all $x, y \in S$

$$< \nabla f(Ex) - \nabla f(Ey), Ex - Ey > \geq 0.$$

That is, $\nabla f(Ex)$ is increasing for all $x \in S$.

ii. If $E(S)$ is a convex set and for all $x, y \in S$

$$< \nabla f(Ex) - \nabla f(Ey), Ex - Ey > \geq 0.$$

Then f is E-convex on S.

Proof. For the proof of (i), see [40, Proposition 2.5(i)]. To show (ii), let $x_1, x_2 \in S$ and $\lambda \in [0,1]$. If $Ex_1 = Ex_2$ or $\lambda = 0$ or $\lambda = 1$ then, using the definition of E-convex functions, we get f is E-convex. Assume that $Ex_1 < Ex_2$ and set $x = \lambda Ex_1 + (1-\lambda)Ex_2$ for some $\lambda \in (0,1)$. i.e.

$$x - Ex_1 = (1- \lambda)(Ex_2 - Ex_1). \qquad (3.33)$$

Note that, since S is E-convex set and $E(S)$ is a convex set, then $x \in S$ and $x \in E(S)$, respectively. The last assertion means there exists $s \in S$ such that $x = E(s) \in E(S)$. Hence, (3.33) becomes $E(s) - Ex_1 = (1- \lambda)(Ex_2 - Ex_1)$. Apply the Mean Value Theorem for f,

$$\nabla f(Es)^T(Ex_2 - Ex_1) = f(Ex_2) - f(Ex_1) ,$$

i.e.

$$f(Ex_2) = f(Ex_1) + < \nabla f(Es), (Ex_2 - Ex_1) >. \qquad (3.34)$$

Apply $< \nabla f(Ex) - \nabla f(Ey), Ex - Ey > \geq 0$ with $Ey = Ex_1$ and $Ex = Es$,

$$< \nabla f(Es) - \nabla f(Ex_1), Es - Ex_1 > \geq 0. \qquad (3.35)$$

Substitute $(Es - Ex_1)$ from (3.33) into (3.35), we obtain

$$(1- \lambda) < \nabla f(Es) - \nabla f(Ex_1), Ex_2 - Ex_1 > \geq 0. \qquad (3.36)$$

Dividing both sides by $(1- \lambda)$ and re-arranging (3.36), we get

$$< \nabla f(Es), Ex_2 - Ex_1 > \geq < \nabla f(Ex_1), Ex_2 - Ex_1 >. \qquad (3.37)$$

Combining (3.34) and (3.37), we obtain

$$f(Ex_2) = f(Ex_1) + < \nabla f(Es), (Ex_2 - Ex_1) >$$

$$\geq f(Ex_1)+< \nabla f(Ex_1), Ex_2 - Ex_1 >.$$

Thus, from Theorem 3.4.2(i), f is an E-convex function. ■

Remark 3.4.5 Part (ii) of the preceding theorem has been proved [40, Proposition 2.5(ii)] wherever E is linear and S is a convex set.

Theorem 3.4.6 Let $f, S,$ and E are defined as in assumption A such that f is a differentiable function on the open E-convex set S. Then

i. If f is strictly E-convex on S then for all $x, y \in S$

$$< \nabla f(Ex) - \nabla f(Ey), Ex - Ey > > 0.$$

That is, $\nabla f(Ex)$ is strictly increasing for all $x \in S$.

ii. If $E(S)$ is a convex set and for all $x, y \in S$

$$< \nabla f(Ex) - \nabla f(Ey), Ex - Ey > > 0.$$

Then f is strictly E-convex on S.

Proof: The proof proceeds in a way like that of Theorem 3.4.4. ■

To detect E-convexity (respectively, E-concavity) of f using the second derivative of f, we have the following result.

Theorem 3.4.7 Let $f: S \longrightarrow \mathbb{R}$ is a twice continuously differentiable function on an open E-convex set S and $E(S)$ is a convex set. Then

i. f is E-convex on S if and only if $H(Ex) = \nabla^2 f(Ex)$ is a p.s.d. for all $x \in S$.
ii. If $H(Ex) = \nabla^2 f(Ex)$ is a p.d. for all $x \in S$, then f is strictly E-convex on S.

iii. f is E-concave on S if and only if $H(Ex) = \nabla^2 f(Ex)$ is n.s.d. for all $x \in S$.

iv. If $H(Ex) = \nabla^2 f(Ex)$ is a n.d. for all $x \in S$, then f is strictly E-concave on S.

Proof. We prove (i) and in a similar manner one can show (iii). Suppose there exists $x_1 \in S$ such that $H(Ex_1)$ is not p.s.d. Our aim to show that f is not E-convex. From the assumption, there exists $x_2 \in S$ such that

$$(Ex_2 - Ex_1)^T H(Ex_1)(Ex_2 - Ex_1) < 0. \tag{3.38}$$

Let $x = \lambda Ex_1 + (1-\lambda)Ex_2,$ for some $\lambda \in (0,1)$. Since S is E-convex set and $E(S)$ is a convex set such that $Ex_1, Ex_2 \in E(S)$, then $x = \lambda Ex_1 + (1-\lambda)Ex_2 \in S$ and there exists $s \in S$ such that $x = E(s) \in E(S)$.

Using now second order truncated Taylor's series, we have

$$f(Ex_2) = f(Ex_1) + < \nabla f(Ex_1), (Ex_2 - Ex_1) >$$

$$+\frac{1}{2}(Ex_2 - Ex_1)^T H(Es)(Ex_2 - Ex_1). \tag{3.39}$$

Choose $x = Es$ sufficiently close to Ex_1, we can use $f \in C^2$(continuity of second order patrials) such that $\frac{1}{2}(Ex_2 - Ex_1)^T H(Es)(Ex_2 - Ex_1) < 0$ where the last inequality follows from (3.38). Therefore, (3.39) becomes

$$f(Ex_2) < f(Ex_1) + < \nabla f(Ex_1), (Ex_2 - Ex_1) >.$$

By Theorem 3.4.1(i), this contradicts the E-convexity of f on S as required to show. Next, we prove the reverse direction, let $x_1, x_2 \in S$ and $H(Ex)$ is p.s.d. for all $x \in S$. Let $x = \lambda Ex_1 + (1-\lambda)Ex_2$, for some $\lambda \in (0,1)$. Using the same argument used above, we conclude that $x \in S$ and there exists $s \in S$ such that $x = E(s) \in E(S)$. From second order truncated Taylor's series, we have

$$f(Ex_2) = f(Ex_1) + < \nabla f(Ex_1), (Ex_2 - Ex_1) >$$

$$+\frac{1}{2}(Ex_2 - Ex_1)^T H(Es)(Ex_2 - Ex_1).$$

Since $H(x) = H(Es)$ is a p.s.d., the last term is non-negative. Hence

$$f(Ex_2) \geq f(Ex_1) + < \nabla f(Ex_1), (Ex_2 - Ex_1) >.$$

Hence, using Theorem 3.4.2(i), f is E-convex over S. The proof of (ii) and (iv) is similar. Hence, it is enough to prove (iv). Let $x_1, x_2 \in S$ and $H(Ex)$ is n.d. for all $x \in S$. Let $x = \lambda Ex_1 + (1-\lambda)Ex_2$, for some $\lambda \in (0,1)$. Using the same argument used above, we conclude that $x \in S$ and there exists $s \in S$ such that $x = E(s) \in E(S)$. From second order truncated Taylor's series, we have

$$f(Ex_2) = f(Ex_1) + < \nabla f(Ex_1), (Ex_2 - Ex_1) >$$

$$+\frac{1}{2}(Ex_2 - Ex_1)^T H(Es)(Ex_2 - Ex_1).$$

Since $H(x) = H(Es)$ is a n.d., the last term is non-positive. Hence

$$f(Ex_2) < f(Ex_1) + < \nabla f(Ex_1), (Ex_2 - Ex_1) >.$$

Hence, using Theorem 3.4.3(ii), f is strictly E-concave over S. ■

To illustrate the proceeding theorem, we consider the following example.

Example 3.4.8 Assume that $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $E: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are defined as $f(x,y,z) = x^2 + 2y + 3z^2$ and $E(x,y,z) = (x^2, y^2, z^2)$ respectively. Test E-convexity/ E-concavity of f.

Solution. We follow Theorem 3.4.7 to detect E- convexity/ E-concavity of f. The gradient is $\nabla f(x,y,z) = \begin{pmatrix} 2x \\ 2 \\ 6z \end{pmatrix}$, and the Hessian

$$\nabla^2 f(x,y,z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\nabla^2 f\big(E(x,y,z)\big) = \nabla^2 f(x^2,y^2,z^2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

E-convexity of f can be checked by testing the sign of

$$\mathrm{E(v)}^{\mathrm{T}}\nabla^2 f\big(E(x,y,z)\big)E(\mathrm{v}) \text{ for each } \mathrm{v}^T = (\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3)^T \neq (0,0,0)^T.$$

Now,

$$E(\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3)^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \mathrm{E}(\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3) \quad ,$$

$$(\mathrm{v}_1^2,\mathrm{v}_2^2,\mathrm{v}_3^2)^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \mathrm{v}_1^2 \\ \mathrm{v}_2^2 \\ \mathrm{v}_3^2 \end{pmatrix}.$$

Then we get, $2\mathrm{v}_1^4 + 6\mathrm{v}_3^4 \geq 0$. Hence, $\nabla^2 f\big(E(x,y,z)\big)$ is p.s.d. which yields f is E-convex on $\mathbb{R}^3$.

CHAPTER FOUR

APPLICATIONS OF GENERALIZED CONVEXITY TO NON-LINEAR PROGRAMMING

Chapter 4

Applications of Generalized Convexity to Non-Linear Programming

4.1 Introduction

As we mentioned in Chapter 1, the non-linear constrained optimization is extended into generalized optimization problem. In such problems, the constraint set is E-convex and the objective functions is either E-convex function or one of the generalized convex functions discussed earlier in subsection 1.4.2. Generalized non-linear constrained problems are studied by many researchers. Youness was the first to introduce E-convex optimization problem and studied some of its properties and optimality results. He continued with his collaborators studying different aspects of E-convex problems such as establishing the necessary and sufficient conditions of optimality, the study of the stability in E-convex programming, developing some duality properties in E-convex programming, and studying optimality conditions for E-convex programming which has E-differentiable objective functions (for more details see [1, 8, 16, 18]). Due to some incorrect results related to E-convex programmings introduced in [15], Chen in [53] defined semi E-convex problems and proved some optimality properties for E-convex and semi E-convex problems. Applications of E-quasiconvex optimization problems are studied by Youness [17] and Syau [56], and strict E-quasiconvexity and E-pseudoconvexity multi-objective optimization problems are studied by Solemani-damaneh [40].

As an application of generalized convex functions in optimization problems, we study in section 2 of this chapter, some optimality properties and characterizations of generalized non-linear optimization problems using E-convex (respectively, strictly E-convex) functions and some generalized convex functions such as E-quasiconvex (respectively, strictly E-quasiconvex) functions, and strictly quasi semi E-convex functions. In section 3, we study differentiability properties of foE. In such case, the function f is called E-differentiable which is non-differentiable. Most of the contents of section 4.2 have been published recently in [50, Section 3].

4.2 Some Results of Generalized Convex Programming

In this section, we consider some applications of E-convex (strictly E-convex) functions, strictly quasi semi E-convex functions, E-quasiconvex (strictly E-quasiconvex) functions in optimization programming problem. Namely, we give some characterizations of the optimal solutions of a generalized non-linear optimization problem using the generalized convex functions mentioned above. To start, consider the non-linear constrained optimization problem (NLP_E) defined in subsection 1.5.3 as follows.

$$min\ (foE)(s)$$

$$s.t.\ s \in S,$$

where $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function, $S \subseteq \mathbb{R}^n$, and $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given mapping. Equivalently, problem (NLP_E) can be expressed as

$$\min f(Es)$$

$$s.t. s \in S.$$

Definition 4.2.1 The set of all optimal solutions (or global minimum) of problem (NLP_E) is denoted by $argmin_S foE$ and is defined as

$$argmin_S foE = \{s^* \in S: f(Es^*) \leq f(Es) \quad \forall s \in S\}.$$

A global minimum $s^* \in S$ is said to be strict when

$$f(Es^*) < f(Es) \quad \forall s \in S, s^* \neq s.$$

A point $s^* \in \mathbb{R}^n$ is called a local minimizer for problem (NLP_E) if there is exists

$r > 0$ such that $f(Es^*) \leq f(Es) \quad \forall s \in B(s^*, r) \cap S,$

Definition 4.2.1 can be extended to the case when the optimization problem (NLP_E*)* is to maximize the objective function by reversing the inequalities above. For the rest of this section, the function f, the set S, and the mapping E are defined as in problem (NLP_E).

Remark 4.2.2 For the rest of this thesis and wherever it is needed, we assume that the set of minimums (respectively, maximums) optimal solutions is a non-empty set.

The following result provides conditions under which each local minimum of problem (NLP_E) is a global minimum.

Theorem 4.2.3 Let s^* is a local minimum of problem (NLP_E) where f is E-convex on S which is E-convex set, $E(S)$ is a convex set, and E is a linear mapping. Then s^* is a global minimum.

Proof. Assume that $s^* \in S$ is a local minimum, then there exists $r > 0$ such that $f(Es^*) \leq f(Es) \quad \forall\, s \in W = S \cap B(s^*, r)$. (4.1)

It is enough to show that $f(Es^*) \leq f(Es) \quad \forall\, s \in S \backslash W$. Consider any $s \in S \backslash W$ such that s lies on the extended line formed from s^* and y. In other words, $s \notin B(s^*, r)$ and for a fixed value of λ, we define $y = \lambda Es + (1 - \lambda)Es^*$ where $\lambda = \frac{r}{\|s - s^*\|} < 1$. Since S is E-convex and E is linear, then

$$y = \lambda Es + (1 - \lambda)Es^* = E(\lambda s + (1 - \lambda)s^*) \in S.$$

On the other hand, using the convexity of $E(S)$, we have $y = E(\lambda s + (1 - \lambda)s^*) \in E(S)$. Thus, there exists $z \in S$ such that

$$z = \lambda s + (1 - \lambda)s^* \in S. \tag{4.2}$$

Using the expressions for z and λ, we get $\| z - s^* \| = r$. i.e., $z \in B[s^*, r]$. The last conclusion together with (4.2) yields $z \in W = S \cap B(s^*, r)$. Since s^* is a local minimum then from (4.1)

$$\begin{aligned} f(Es^*) \leq f(Ez) = f(y) &= f(\lambda Es + (1 - \lambda)Es^*) \\ &\leq \lambda f(Es) + (1 - \lambda) f(Es^*), \end{aligned}$$

where the last inequality follows because f is E-convex function. By re-arranging last inequality, we get $\lambda f(Es^*) \leq \lambda f(Es)$ which yields

$$f(Es^*) \leq f(Es) \qquad \forall\, s \in S \backslash W. \tag{4.3}$$

From (4.1) and (4.3), $f(Es^*) \leq f(Es) \quad \forall s \in S$. Therefore, x^* is a global minimum of problem (NLP_E). ■

The following theorem provides a necessary and sufficient condition for a global minimum when the function f is differentiable and E-convex in (NLP_E) problem.

Theorem 4.2.4 Assume that f is a continuously differentiable E-convex on an E-convex set S, and E is a linear mapping. Then problem (NLP_E) has a global minimum $s^* \in S$ if and only if,

$$< \nabla f(Es^*), Es - Es^* > \geq 0 \qquad \forall s \in S.$$

Proof. Consider s^* global minimum of (NLP_E). Take any $s \in S$, since S is an E-convex then for any $\lambda \in (0,1)$, we have $\lambda E(s) + (1-\lambda)E(s^*) \in S$. Since s^* is a global minimum, then

$$f(Es^*) \leq f(\lambda E(s) + (1-\lambda)E(s^*)) = f(Es^* + \lambda(Es - Es^*)).$$

Using first order truncated Taylor Theorem.

$$f(Es^*) \leq f(Es^*) + < \nabla f(Es^*), \lambda(Es - Es^*) >.$$

Divide by λ and take $\lambda \longrightarrow 0$, the last inequality yields

$$< \nabla f(Es^*), (Es - Es^*) > \geq 0.$$

Since s is an arbitrary point then, $\forall s \in S$

$$< \nabla f(Es^*), (Es - Es^*) > \geq 0$$

as required. To prove the other direction, let $s^* \in S$ then for each $s \in S$, we have $< \nabla f(Es^*), (Es - Es^*) > \geq 0$. We need to show that s^* is a global minimum. By the E-convexity of f we have for each $s \in S$

$$f(Es) - f(Es^*) \geq < \nabla f(Es^*), (Es - Es^*) > \geq 0.$$

i.e., for each $s \in S$, $f(Es^*) \leq f(Es)$. Thus, s^* is a global minimum. ■

Next, we give a sufficient condition to obtain unique optimal solution of (NLP_E) using strictly E-convex function f.

Theorem 4.2.5 Assume that f is strictly E-convex on an E-convex set S, $E(S)$ is a convex set, and E is a linear mapping. Then the global optimal solution of problem (NLP_E) is unique.

Proof. Let $s_1^*, s_2^* \in S$ be two different global optimal solutions of problem (NLP_E), then $f(Es_1^*) = f(Es_2^*)$. Since S is E-convex, $E(S)$ is convex and E is linear, then for each $0 < \lambda < 1$, the E-convex combination

$$\lambda Es_1^* + (1-\lambda)Es_2^* = E(\lambda s_1^* + (1-\lambda)s_2^*) \in E(S) \subseteq S,$$

and hence, there exists $z \in S$ such that $z = \lambda s_1^* + (1-\lambda)s_2^* \in S$ where $z \neq s_1^*$ and $z \neq s_2^*$. Since f is strictly E-convex on the E-convex set S, we have

$$f(Ez) = f(\lambda Es_1^* + (1-\lambda)Es_2^*) < \lambda f(Es_1^*) + (1-\lambda)f(Es_2^*) = f(Es_2^*).$$

This means, z is a global optimal solution which is a contradiction. Thus, the global minimum is unique. ■

Another two sufficient conditions for a unique optimal solution of problem (NLP_E) are given next.

Theorem 4.2.6 Assume that f is strictly quasi semi E-convex, $E(S)$ is a convex set, E is linear and fixed with respect to the global optimal solution. Then the global optimal solution of (NLP_E) is unique.

Proof. Let $s_1^*, s_2^* \in S$ be two global optimal solutions of problem (NLP_E) such that $s_1^* \neq s_2^*$, then $f(Es_1^*) = f(Es_2^*)$. Since $E(s_1^*) = s_1^*$ and $E(s_2^*) = s_2^*$, the last equality yields $f(s_1^*) = f(s_2^*)$. Now f is strictly quasi semi E-convex on the E-convex set S, then for each $0 < \lambda < 1$, we have

$$f(\lambda Es_1^* + (1-\lambda)Es_2^*) < \max\{f(s_1^*), f(s_2^*)\} = f(s_1^*) = f(E(s_1^*)).$$

By the linearity of E, the left-hand side of the inequality above can be written as

$$f(E(\lambda s_1^* + (1-\lambda)s_2^*)) = f(\lambda Es_1^* + (1-\lambda)Es_2^*) < f(E(s_1^*)). \qquad (4.4)$$

The expression above entails a contradiction. Indeed, because $E(S)$ is convex and S is E-convex, then $E(\lambda s_1^* + (1-\lambda)s_2^*) \in E(S) \subseteq S$. i.e., there exists $z = \lambda s_1^* + (1-\lambda)s_2^* \in S$ where $z \neq s_1^*$ and $z \neq s_2^*$. Using this fact with the inequality in (4.4), we conclude z is a global optimal solution which is a contradiction. Thus, $s_1^* = s_2^*$ as required. ■

Theorem 4.2.7 Consider problem (NLP_E) in which S is E-convex set, $E(S)$ is a convex set, and E is a linear mapping.

i. If f is E-quasiconvex on S, then the set of optimal solutions $argmin_S foE$ is a convex set.
ii. If f is strictly E-quasiconvex, then $argmin_S foE$ is singleton (i.e., optimal solution is unique).

Proof. To prove (i), let $s_1^*, s_2^* \in argmin_S foE$, such that $s_1^* \neq s_2^*$. Thus,

$f(Es_1^*) = f(Es_2^*) \leq f(Es)$, for all $s \in S$. Since S is E convex, f is E-quasiconvex on S and E is linear, then for each $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} f\big(E(\lambda s_1^* + (1-\lambda)s_2^*)\big) &= f(\lambda Es_1^* + (1-\lambda)Es_2^*) \\ &\leq \max\{f(Es_1^*), f(Es_2^*)\} \\ &= f(Es_1^*) \leq f(Es). \end{aligned} \tag{4.5}$$

Since $s_1^*, s_2^* \in S$ which is E-convex and $E(S)$ is convex, and E linear. Then for each $0 \leq \lambda \leq 1$, we get

$E(\lambda s_1^* + (1-\lambda)s_2^*) = \lambda E(s_1^*) + (1-\lambda)E(s_2^*) \in E(S) \subseteq S$. Hence,

$$\lambda s_1^* + (1-\lambda)s_2^* \in S. \tag{4.6}$$

From (4.5) and (4.6), $\lambda s_1^* + (1-\lambda)s_2^* \in argmin_S foE$. Thus, $argmin_S foE$ is a convex set. To show (ii). Let $s_1^*, s_2^* \in S$ be two global optimal solutions of problem (NLP_E) such that $s_1^* \neq s_2^*$, then $f(Es_1^*) = f(Es_2^*)$. Since S is E-

convex, E is linear, and f is strictly E-quasiconvex, then for each $0 < \lambda < 1$ we have

$$\begin{aligned} f(E(\lambda s_1^* + (1-\lambda)s_2^*)) &= f(\lambda E s_1^* + (1-\lambda)E s_2^*) \\ &< \max\{f(Es_1^*), f(Es_2^*)\} \\ &= f(Es_1^*) < f(Es). \end{aligned} \quad (4.7)$$

The rest of the prove follows as in the Theorem 4.2.6 where (4.7) and the convexity of $E(S)$ provide an optimal solution $z = \lambda s_1^* + (1-\lambda)s_2^* \in S$ such that $z \neq s_1^*$ and $z \neq s_2^*$. From (4.7), we have $f(Ez) < f(Es_1^*)$ which contradicts the optimality of s_1^* of problem (NLP_E). ■

The conclusions of the preceding Theorem can be also obtained if the function f is E-convex (respectively, strictly E-convex) as we show next.

Corollary 4.2.8 Consider problem (NLP_E) in which S is E-convex set, $E(S)$ is a convex set, and E is a linear mapping.

i. If f is E-convex on S, then the set of optimal solutions $argmin_S foE$ is a convex set.
ii. If f is strictly E-convex, then $argmin_S foE$ is singleton.

Proof. From [40, p.3339], every E-convex (respectively, strictly E-convex) function is E-quasiconvex (respectively, strictly E-quasiconvex). Thus, the conclusions of (i)-(ii) directly follow. ■

Theorem 4.2.9 Consider problem (NLP_E) in which f is a differentiable E-convex function on the E-convex set S, and E is a linear and idempotent mapping. Then $agrmin_S(foE)$ is an E-convex.

Proof. Take an arbitrary $s^* \in agrmin_S(foE)$, we have from Theorem 3.4.4(i), for each $s \in S$

$$< \nabla f(Es) - \nabla f(Es^*), Es - Es^* > \geq 0,$$

Re-arranging the last inequality and apply Theorem 4.2.4, we get

$$< \nabla f(Es), Es - Es^* > \geq < \nabla f(Es^*), Es - Es^* > \geq 0 \qquad \forall s \in S.$$

i.e., $< \nabla f(Es), Es - Es^* > \geq 0 \qquad \forall s \in S.$

Apply Theorem 4.2.4 again to get

$$agrmin_S(foE) = \bigcap_{s\in S} \{u \in S: < \nabla f(Es), Es - Eu > \geq 0\} = \bigcap H_i,$$

where $H_i = \{u \in S: < \nabla f(Es), Es - Eu > \geq 0\}$. Since E is linear and idempotent, then applying Proposition 3.3 in [25], each H_i is E-convex. This yield, $agrmin_S(foE)$ is an E-convex. ■

Theorem 4.2.10 Consider the following maximization non-linear problem (M-NLP_E)

$$max(foE)(s)$$

$$\text{s.t.}\, s \in S,$$

where $S \subseteq \mathbb{R}^n$ an E −convex set, $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is E-convex on S, and $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given linear mapping. Assume that $E(S)$ is a convex set $f\big(E(s)\big) \leq f(s)$ for all $s \in S$, and the set of optimal solutions of problem *(M-NLP_E)* is a non-empty. i.e., $argmax_S foE = \{s^* \in S: f(Es^*) \geq f(Es) \quad \forall s \in S\} \neq \emptyset$. Then, the maximal optimal solutions of foE occur on the boundary of S.

Proof. By a contrary, assume that the maximum exists at a point s^* belongs to the interior of S. That is, $f(Es^*) \geq f(Es) \quad \forall s \in S$ and $s^* \in S^{\circ}$. Draw a line passing through $s^*$and cutting the boundary of S at $s_1$and s_2. Since S is E −convex, then for some $0 < \lambda < 1$, we have $s^* = \lambda E(s_1) + (1-\lambda)E(s_2) \in S$. We also have $E(S)$ is convex and E is linear, then

$$s^* = \lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in E(S).$$

Thus, there exists $z \in S$ such that $z = \lambda s_1 + (1-\lambda)s_2 \in S$. Since f is E-convex on S, then

$$f(Es^*) \leq f(s^*) = f(Ez) \leq \lambda f(Es_1) + (1-\lambda)f(Es_2), \quad (4.8)$$

where in the left most inequality, we used the assumption $f\big(E(s)\big) \leq f(s)$ for all $s \in S$. Now, we have two possibilities. If $f(Es_1) \leq f(Es_2)$, then

$$\lambda f(Es_1) + (1-\lambda)f(Es_2) \leq \lambda f(Es_2) + (1-\lambda)f(Es_2) = f(Es_2).$$

Using (4.8), we get $f(Es^*) \leq f(Es_2)$, yielding s^* is not a global maximum which is a contradiction. Similarly, if $f(Es_2) \leq f(Es_1)$, we get $f(Es^*) \leq f(Es_1)$, a contradiction. Hence, the maximum point must occur at the boundary of S. ■

4.3 E-Differentiability Properties of E-Convex Functions

E-convex functions which are non-differentiable can be transformed into differentiable using a mapping $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. This class of functions is referred to as E-differentiable functions and is defined by Megahed et al [8] as follows.

‘‘Let $f: S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and let $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a mapping. A function f is said to be E- differentiable at s^* if and only if f is non-differentiable function at s^* and (foE) is a differentiable function at s^*’’.

Megahed et al [8] apply Fritz-John and Kuhn-Tucker conditions to obtain a solution of a generalized optimization problem (NLP) with a non-differentiable objective function f. In this section, we consider (NLP_E) problem in which the function f is non-differentiable (i.e., foE is differentiable). We apply all the differentiability properties in section 3.4 for the case when f is E-differentiable in problem (NLP_E).

To clarify the definition of E-differentiable function, we recall the following example.

Example 4.3.1 [8] Consider the real valued function $f(x) = |x|$ and the mapping $E: \mathbb{R} \longrightarrow \mathbb{R}$ such that $E(x) = x^2$. It is clear that the function f is non-differentiable at the point $x = 0$ and the function $(foE)(x) = f(Ex) = x^2$ is a differentiable function at $x = 0$. Hence, f is an E-differentiable function.

Remark 4.3.2 The proof of the next Theorems is similar to that of Theorems 3.4.1-3.4.4 and Theorems 3.4.6-3.4.7 in section 3.4. We only replace each f in the proceeding theorems by foE. Next, we show the detailed proof of next theorem which is similar to the proof of Theorems 3.4.1-3.4.2.

Theorem 4.3.3 Consider problem (NLP_E) in which f is E-differentiable on S is an open E-convex set. Then

i. If foE is E-convex on S, then

$$(foE)(Ey) \geq (foE)(Ex) + < \nabla(foE)(Ex), Ey - Ex > \qquad \forall x, y \in S.$$

ii. If $E(S)$ is convex and

$$(foE)(Ey) \geq (foE)(Ex) + < \nabla(foE)(Ex), Ey - Ex > \qquad \forall x, y \in S.$$

Then, foE is E-convex on S.

iii. If foE is E-concave on S, then

$$(foE)(Ey) \leq (foE)(Ex) + < \nabla(foE)(Ex), Ey - Ex > \qquad \forall x, y \in S.$$

iv. If $E(S)$ is convex and

$$(foE)(Ey) \leq (foE)(Ex) + < \nabla(foE)(Ex), Ey - Ex > \qquad \forall x, y \in S.$$

Then, foE is E-concave on S.

Proof. Let us show (i). If $E(x) = E(y)$, then the gradient inequality directly satisfied. Consider now $x, y \in S$ such that $E(x) \neq E(y)$ and $\lambda \in (0,1]$, then using the E-convexity of f, we have

$$(foE)(\lambda Ey + (1-\lambda)Ex) \leq \lambda(foE)(Ey) + (1-\lambda)(foE)(Ex).$$

That is,

$$(foE)\big(Ex + \lambda(Ey - Ex)\big) \leq (foE)(Ex) + \lambda\big((foE)(Ey) - (foE)(Ex)\big).$$

Re-arranging the last inequality yields,

$$\frac{(foE)\big(Ex + \lambda(Ey - Ex)\big) - foE(Ex)}{\lambda} \leq (foE)(Ey) - (foE)(Ex).$$

Taking the limit to both sides of the above inequality ($as\ \lambda \longrightarrow 0^+$) yields,

$$\lim_{\lambda \longrightarrow 0^+} \frac{(foE)\big(Ex+\lambda(Ey-Ex)\big)-(foE)(Ex)}{\lambda} \leq (foE)(Ey) - (foE)(Ex).$$
(4.9)

The left-hand side of the inequality (4.9) is the directional derivative of foE at $E(x)$ in the direction of $(Ey - Ex)$. Thus, (4.9) becomes

$$< \nabla(foE)(Ex), Ey - Ex > \ \leq (foE)(Ey) - (foE)(Ex).$$

Re-arranging last expression, we get

$$(foE)(Ey) \geq (foE)(Ex) + < \nabla(foE)(Ex), Ey - Ex >. \qquad (4.10)$$

To show (ii), take arbitrary $x_1, x_2 \in S$ such that S is a convex set, and let $\lambda \in [0,1]$. Define $z = \lambda Ex_1 + (1-\lambda)Ex_2 \in S$. Since $E(x_1), E(x_2) \in E(S)$ and $E(S)$ is convex, then

$$z = \lambda Ex_1 + (1-\lambda)Ex_2 \in E(S).$$

Hence, there exists $s \in S$ such that $E(s) = z = \ \lambda Ex_1 + (1-\lambda)Ex_2$.

Apply (4.10) with $Ey = Ex_1$ and $Ex = Es$ yields,

$$(foE)(Ex_1) \geq (foE)(Es) + < \nabla(foE)(Es), Ex_1 - Es >. \tag{4.11}$$

Similarly, apply (4.10) with $Ey = Ex_2$ and $Ex = Es$ we get,

$$(foE)(Ex_2) \geq (foE)(Es) + < \nabla(foE)(Es), Ex_2 - Es >. \tag{4.12}$$

We multiply (4.11) by λ and (4.12) by $(1 - \lambda)$, and sum the two inequalities up

$$\lambda(foE)(Ex_1) + (1 - \lambda)(foE)(Ex_2)$$

$$\geq (foE)(Es) + < \nabla(foE)(Ex), \lambda Ex_1 + (1 - \lambda)(Ex_2) - Es >.$$

The last inequality yields

$$\lambda(foE)(Ex_1) + (1 - \lambda)(foE)(Ex_2) \geq (foE)(\lambda Ex_1 + (1 - \lambda)(Ex_2)).$$

Hence, foE is E-convex as required. The proof of parts (iii) which assumes that foE is E-concave function proceeds in a way similar to part (i) where we use the definition of E-concave function instead of using the definition of E-convex function. Finally, part (iv) follows in a way similar to part (ii) where we reverse each inequality in the proof of part (iv). ■

Theorem 4.3.3 can be extended to give a characterization to the E-differentiable strictly E-convex (respectively, E-concave) functions in terms of their strictly gradient inequalities.

Theorem 4.3.4 Consider problem (NLP_E) in which f is E-differentiable on S is an open E-convex set such that $E(S)$ is a convex set. Then

i. foE is strictly E-convex if and only if for all $x, y \in S$ such that $x \neq y$ we have

$$(foE)(Ey) > (foE)(Ex) + < \nabla(foE)(Ex), Ey - Ex > .$$

ii. $foE$is strictly E-concave if and only if for all $x, y \in S$ such that $x \neq y$ we have

$$(foE)(Ey) < (foE)(Ex) + < \nabla(foE)(Ex), Ey - Ex > \ .$$

The following theorem provides a necessary and sufficient conditions for foE to be E-convex function using the gradient test of foE.

Theorem 4.3.5 Consider problem (NLP_E) in which f is E-differentiable on S is an open E-convex set. Then

i. If foE is E-convex on S then for all $x, y \in S$

$$< \nabla(foE)(Ex) - \nabla(foE)(Ey), Ex - Ey > \geq 0.$$

That is, $\nabla(foE)(Ex)$ is increasing for all $x \in S$.

ii. If $E(S)$ is a convex set and for all $x, y \in S$

$$< \nabla(foE)(Ex) - \nabla(foE)(Ey), Ex - Ey > \geq 0.$$

Then foE is E-convex on S.

Theorem 4.3.5 can be applied for the case when foE is strictly E-convex function

Theorem 4.3.6 Consider problem (NLP_E) in which f is E-differentiable on S is an open E-convex set. Then

i. If (foE) is strictly E-convex on S then for all $x, y \in S$

$$< \nabla(foE)(Ex) - \nabla(foE)(Ey), Ex - Ey > > 0.$$

That is, $\nabla(foE)(Ex)$ is strictly increasing for all $x \in S$.

ii. If $E(S)$ is a convex set and for all $x, y \in S$

$$< \nabla(foE)(Ex) - \nabla(foE)(Ey), Ex - Ey >> 0.$$

Then, foE is strictly E-convex on S.

To detect E-convexity (respectively, E-concavity) of (foE) using the second derivative of (foE), we have the following result

Theorem 4.3.7 Consider problem (NLP_E) in which f is twice continuously E-differentiable on S is an open E-convex set such that $E(S)$ is a convex set. Then

i. foE is E-convex on S if and only if $H(Ex) = \nabla^2(foE)\,(Ex)$ is a p.s.d. for all $x \in S$.
ii. If $H(Ex) = \nabla^2(foE)\,(Ex)$ is a p.d. for all $x \in S$ then foE is strictly E-convex on S.
iii. foE is E-concave on S if and only if $H(Ex) = \nabla^2(foE)\,(Ex)$ is n.s.d. for all $x \in S$.
iv. If $H(Ex) = \nabla^2(foE)\,(Ex)$ is a n.d. for all $x \in S$ then foE is strictly E-concave on S.

Example 4.3.8 Assume that $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $E: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be such that $f(x,y) = -x^2 - \frac{1}{y^2}$ and $E(x,y) = \left(x, y^{-\frac{1}{2}}\right)$. Test E-convexity/E-concavity of foE.

Solution. $(foE)(x,y) = -x^2 - y$. Note that

$$\nabla f(x,y) = \begin{pmatrix} -2x \\ 2y^{-3} \end{pmatrix} \text{ and } \nabla(foE)(x,y) = \begin{pmatrix} -2x \\ -1 \end{pmatrix}.$$

Since, f is not differentiable at $(x,0)$ while foE is differentiable at $(x,0)$, then f is E-differentiable at $(x,0)$. The Hessian of foE at $E(x,y)$ is

$$\nabla^2(foE)(E(x,y)) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, $E(v_1, v_2)^T \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} E(v_1, v_2)$ for each $(v_1, v_2)^T \neq (0,0)^T$

$$(v_1, v_2^{\frac{-1}{2}}) \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2^{\frac{-1}{2}} \end{pmatrix} = -2v_1^2 \leq 0$$

From Theorem 4.3.7 (iii), $\nabla^2(foE)\ (Ex)$ is n.s.d. Thus, foE is E-concave on $\mathbb{R}^2$.

FUTURE WORK

Future Work

We conclude this thesis by mentioning some open problems, which suggest possible future research directions.

1) Considering unconstrained non-linear generalized optimization problem (NLP_E) and studying the first and second necessary and sufficient optimality conditions for this problem.
2) Develop the dual structure of the generalized optimization problem (NLP_E). In specific, study each of Fenchel and conjugate duality of this problem.
3) Study the development of constraint qualifications to obtain strong duality and zero duality gap for the (NLP_E) problem.
4) Develop duality properties for special kinds of separable E-convex optimization problems and obtain constraint qualifications to prove strong duality for this separable problem.

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المستخلص

هناك نوع من التعميمات المهمة للمجموعات المحدبة ، الدوال المحدبة ، ومشاكل الأمثلية المحدبة تسمى المجموعات المحدبة ، الدوال المحدبة ، ومشاكل الأمثلية المحدبة من النوع E والتي عُرّفت ودُرسّت من قبل يونس وباحثين اخرين. في هذا النوع من المجموعات والدوال، قام يونس بتعريف المجموعة المحدبة والدالة المحدبة بالنسبة الى دالة تسمى E.

ان الهدف الرئيسي لهذه الرسالة هو تقديم دراسة خواص ومكافئات جديدة لكل من المجموعات والدوال المحدبة وتعميماتها من النوع E ، بالاضافة الى اعطاء خواص جديدة لمشاكل الامثلية المحدبة وتعميماتها من النوع E.

في الفصل الثاني لهذه الرسالة قمنا بدراسة خواص ومكافئات جديدة للمجموعة المحدبة من نوع E والمخروط من نوع E. قمنا ايضاً بتعريف مجموعات جديدة والمسماة بمجموعة الانغلاق المخروطي المحدب من نوع E والمجموعات التآلفية من نوع E ومجموعات الانغلاق التآلفية من نوع E وكذلك قمنا بدراسة بعض خواص ومكافئات هذه المجموعات. واخيراً قمنا بأعطاء بعض الأمثلة لتوضيح المفاهيم المستعرضة أعلاه ولتوضيح العلاقة فيما بينهم.

في الفصل الثالث قمنا بدراسة بعض الخواص المتنوعة لبعض تعميمات الدوال المحدبة كالدوال المحدبة من النوع E ، الدوال شبه المحدبة من النوع E، الدوال التابعة شبة المحدبة من النوع E والدوال

الكاذبة شبه المحدبة من النوع E، كما قمنا بدراسة بعض خواص الدوال المحدبة من النوع E القابلة للاشتقاق. قمنا ايضا بربط الدوال المحدبة وبعض تعميماتها بأنواع مختلفة من مجاميع ال epigraphs لهذه الدوال و مجاميع المستوى level sets المرتبطة بمجاميع ال epigraphs

.

في الفصل الرابع قمنا بدراسة مشاكل الأمثلية المعممة والتي تكون فيها دالة الهدف إما دالة محدبة من النوع E أو احدى دوال التعميم للدالة المحدبة أو الدالة المحدبة من النوع E مثل الدوال التابعة شبة المحدبة من النوع E والدالة التابعة المحدبة من النوع E ، أما مجموعة القيود فهي مجموعة محدبة من النوع E . الدراسة شملت إمكانية وجود الحل الأمثل ومتى يكون وحيد وكذلك متى تكون مجموعة الحل الأمثل مجموعة محدبة. وكذلك قمنا بدراسة مشاكل الأمثلية المعممة عندما تكون الدالة fغير قابلة للأشتقاق ولكن في foE هي دالة قابلة للأشتقاق.

جمهورية العراق

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جامعة بغداد

كلية التربية للعلوم الصرفة /ابن الهيثم

حول تعميمات المجاميع المحدبة والدوال المحدبة ومشاكل الأمثلية المحدبة

رسالة

مقدمة إلى كلية التربية للعلوم الصرفة / ابن الهيثم – جامعة بغداد

وهي جزء من متطلبات نيل شهادة ماجستير في علوم الرياضيات

من قبل

محمود إدريس عبد المجيد

بإشراف

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