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Semi-Analytical Iterative Methods for Non-Linear Differential Equations

A Thesis

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By

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بِسمِ اللهِ الرَّحمٰنِ الرَّحِيمِ

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We certify that we have read this thesis entitled **"Semi-Analytical Iterative Methods for Non-Linear Differential Equations"** and as examining committee, we examined the student **(Mustafa Mahmood Azeez)** in its contains and what is connected with it, and that in our opinion it is adequate for the partial fulfillment of the requirement for the degree of Master of Science in Mathematics.

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Abstract

The main aim of this thesis is to use two semi-analytical iterative methods to find the analytical solutions for some important problems in physics and engineering.

The first objective is to use an iterative method which is proposed by Temimi and Ansari namely (TAM) for finding the analytical solutions for Riccati, pantograph, and beam deformation equations. Also, the TAM has been applied for 1D, 2D and 3D nonlinear Burgers' equations and systems of equations to get the analytic solutions for each of them. The convergence of the TAM has been investigated and successfully proved for those problems.

The second objective is to use an iterative technique based on Banach contraction method (BCM) which is easy to apply in dealing with nonlinear terms. The BCM is used to solve the same physical and engineering problems mentioned above. In addition, we have been presented several comparisons among the TAM, BCM, variational iteration method (VIM) and Adomian decomposition method (ADM) with some conclusions and future works. The presented methods are efficient and have high accuracy. The TAM and BCM do not require any restrictive assumptions in the nonlinear case. In addition, both methods do not rely on using any additional restrictive assumptions as in the ADM or VIM or any other iterative methods.

The software used for the calculations in this presented thesis is MATHEMATICA® 10.0.

AUTHOR'S PUBLICATIONS

Journal Papers

1. M. A. Al-Jawary, M. M. Azeez, Efficient Iterative Method for Initial and Boundary Value Problems Appear in Engineering and Applied Sciences, International Journal of Science and Research, 6 (6) (2017) 529- 538.

2. M. A. AL-Jawary, M. M. Azeez, G. H. Radhi, Analytical and numerical solutions for the nonlinear Burgers and advection-diffusion equations by using a semi-analytical iterative method, Computers & Mathematics with Applications (under review).

3. M. A. AL-Jawary, M. M. Azeez, G. H. Radhi, Banach Contraction Method for Some Physical and Engineering Problems, (under preparation).

Contents

List of Tables

List of Figures

LIST OF SYMBOLS AND ABBREVIATIONS

A_n: Adomian Polynomials

ADM: Adomian Decomposition Method

BCM: Banach Contraction Method

BVP: Boundary Value Problem

ER: Error Remainder

HAM: Homotopy Analysis Method

IVP: Initial Value Problem

Re: Reynolds number

MER: Maximum Error Remainder

$ODEs$: Ordinary Differential Equations

$PDEs$: Partial Differential Equations

TAM: Temimi- Ansari Method

VIM: Variational Iteration Method

VIE: Volterra Integral Equation

λ: Lagrange's multiplayer

δ: Variation

$|r_n|$: Absolute Error

$\| \ \|$: Norm

INTRODUCTION

The ordinary and partial differential equations play an important role in many problems that appears in different fields of engineering and applied sciences. The past few decades have seen a significant progress to implement analytical, approximate methods for solving linear and nonlinear differential equations. For examples: Adomian decomposition method (ADM) [68], variational iteration method (VIM) [4], homotopy perturbation method (HPM) [63], differential transform method (DTM) [78] and many others. Although these methods provided or given some useful solutions, however, some drawbacks have emerged such as the calculation of the Adomian polynomials for the nonlinear problems in ADM, calculate the Lagrange multiplier in the VIM and the calculation became more complicated after several iterations, homotopy construction and solving the corresponding equations in HPM.

In this thesis, some of the ordinary and partial differential equations that appear in the problems of chemistry, physics, engineering and other applied sciences will be solved by using iterative methods.

One of these equations is the Riccati equation, which is an initial value problem of nonlinear ordinary differential equation, which plays a significant role in many fields of applied science such as random processes, optimal control, diffusion problems, network synthesis and financial mathematics [26].

The other one is the pantograph equation which is originated from the work of Ockendon and Tayler on the collection of current by the pantograph head of an electric locomotive [38]. The pantograph equations are appeared in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics [16].

Moreover, the beam deformation equation will be solved, which is a nonlinear boundary value problem (BVP) which is frequently used as mathematical models in viscoelastic, inelastic flows and deformation of beams [1].

Furthermore, the other equation that will be solved is the Burgers' equation which is a partial differential equation and firstly introduced by Harry Bateman in 1915 [27] and it was subsequently processed to become as Burgers' equation [45]. Burgers' equation has a lot of importance in engineering and physical fields, especially in problems that have the form of nonlinear equations and equation systems. The applications of Burgers' equations by mathematical scientists and researchers has become more commonplace and interesting, it has been known that this equation describes different types of phenomena such as modeling of dynamics, heat conduction, acoustic waves, turbulence and many others [5, 10, 36, 45, 56 , 61, 77]. In recent decades, some scientists and researchers used some methods and techniques to solve these types of problems [22, 39, 42, 44, 71] which will be discussed in this thesis and solved in reliable iterative methods.

Recently, Temimi and Ansari have introduced a semi–analytical iterative method namely (TAM) for solving nonlinear problems [28]. The main feature of the TAM is very effective and reliable, does not require any restrictive assumptions to deal with nonlinear terms, time saver, and has a higher convergence and accuracy. The TAM was inspired from the homotopy analysis method (HAM) [70] and it is one of the famous iterative methods that used for solving nonlinear problems [65]. Moreover, the TAM is successfully used to solve many differential equations, such as Duffing equations [51], korteweg–de vries equations [18], chemistry problems [52] and nonlinear thin film flow problems [53].

In addition, Sehgal and Bharucha-Reid proved the probabilistic version of the classical Banach contraction method (BCM) in 1972 [3]. The BCM characterized as one of the developments of Picard’s method where the ease of application clearly observed which makes it distinct from the other known iterative methods. The BCM used to solve various nonlinear functional delay equations including partial differential equations (PDE), ordinary differential equations (ODE), system of ODEs and PDEs and algebraic equations [75].

The main advantages of the BCM, it does not require any external assumptions to deal with the nonlinear terms like the TAM. It avoided the large computational work or calculated the Lagrange's multipliers in the VIM.

This thesis has been arranged as follows: in chapter one, the introductory concepts about the non-linear differential equations (ordinary and partial) and some analytic and approximate methods will be introduced to solve some scientific applications, such as ADM and VIM. Chapter two presents the basic idea of the TAM as well as an overview of the convergence of TAM, which is based on the Banach fixed-point theory. Furthermore, the TAM is successfully implemented to solve the Riccati equations, pantograph, and beam deformation equations. In addition, the TAM will be implemented to solve the Burgers and system equations in 1D, 2D and 3D to illustrate the efficiency and accuracy of the proposed method. In chapter three, the basic concepts of the BCM will be presented, and the application of the BCM for the same problems solved by TAM will be introduced. Finally, in Chapter four the conclusions and future works will be given.

CHAPTER 1

PRIMARY CONCEPTS

Chapter 1

Primary Concepts

1.1 Introduction

The importance of the non-linear differential equations highlights when dealing with the problems that are occurring in different fields such as physical and chemical sciences and engineering. Many of these problems are of a nonlinear form. It is known that there is a difficulty in to solving these nonlinear problems and require to use efficient methods to get the analytic, approximate or numerical solutions.

Moreover, the problems that will be solved in this thesis are mathematical models contain non-linear ODEs such as Riccati equation, pantograph equation, beam deformation equation and nonlinear PDEs such as Burgers' equation, that appeared in many engineering and applied sciences.

This chapter contains four sections. In section two, some definitions and theorems that will be used in the next chapters will be given. In section three, some types of ODEs are presented, such as Riccati, pantograph and beam deformation equations. In addition, some analytic and approximate methods will be introduced to solve some scientific applications, such as ADM and the VIM. Finally, in section four, the Burgers' equations and system of equations in 1D, 2D, and 3D will be introduced. Also, the 1D, 2D Burgers' equations will be given in literature by ADM and VIM.

1.2 Preliminaries

1.2.1 Initial Value Problems [6]:

Differential equations are often divided into two classes, ordinary and partial, according to the number of independent variables. A more meaningful division, however, is between initial value problems, which usually model time-dependent phenomena, and boundary value problems, which generally model steady-state systems.

A differential equation (DE) that has given conditions allows us to find the specific function that satisfies a given DE rather than a family of functions. These types of problems are called initial value problems (IVP). In physics or other sciences, frequently amounts to solving an initial value problem. For example, an initial value problem is a differential equation

$$v'(x) = f\big(x, v(x)\big), \text{ with } f\colon \theta \subset R \times R^n \to R^n,$$

where θ is an open set of $R \times R^n$, together with a point in the domain of f, $(x_0, v_0) \in \theta$ called the initial condition. A solution to an initial value problem is a function v that is a solution to the differential equation and satisfies

$$v(x_0) = v_0.$$

1.2.2 Boundary Value Problems [6]:

When imposed an ordinary or a partial differential equation the conditions are given at more than one point and the differential equation is of order two or greater, it is called a boundary value problem (BVP). For example, if the independent variable is time over the domain [0,1], a boundary value problem would specify values for $v(x)$ at both $x = 0$ and $x = 1$, whereas an initial value problem would specify a value of $v(x)$ and $v'(x)$ at time $x = 0$. In addition, there are four types of boundary value problems or boundary conditions: Dirichlet, Neunmann, Mixed, Robin boundary conditions.

The boundary value problems have a greater importance in various scientific fields. Many of researchers and scientists experiencing in their scientific researchers have a problem in finding the desired solution for those boundary value problems. These problems come with boundary conditions, which are satisfying the solution. In some cases, these problems are formulated from BVP to Volterra integral equation. The integral equation make the reaching to the solution simple and easy.

Definition 1.1: [6]

An integral equation is called Volterra integral equation (VIE) if limits of integration are functions of x rather than constants. The first kind Volterra integral equations, is:

$$f(x) = \lambda \int_{a}^{x} k(x,t)v(t)dt, \qquad (1.1)$$

where the unknown function $v(x)$ appears only inside integral sign, and the Volterra integral equations of the second kind is

$$v(x) = f(x) + \lambda \int_{a}^{x} k(x,t)v(t)dt, \qquad (1.2)$$

where the unknown function $v(x)$ appears inside and outside the integral sign.

Definition 1.2: [57]

Let (X, d) be a metric space. Then a mapping $F: X \rightarrow X$, is said to be Lipschitz if there exists a real number $k \geq 0$, such that

$$d\big(F(x), F(y)\big) \leq kd(x,y) \qquad \text{for all} \quad x, y \text{ in } X.$$

In addition F is called a contraction mapping on X if $k < 1$.

Theorem 1.1 (Banach Fixed Point Theorem): [69]

Let (X, d) be a non-empty complete metric space with a contraction mapping $F: X \to X$. Then F admits a unique fixed-point $x^* \in X$

$$F(x^*) = x^*$$

Theorem 1.2: [57]

Let F be a mapping of a complete metric space (X, d) into itself such that F^k is a contraction mapping of X for some positive integer k. Then F has a unique fixed point in X.

Proof: See [57].

1.3 Ordinary Differential Equations (ODEs)

An ordinary differential equation is a relation containing one real independent variable $x \in R = (-\infty, \infty)$, the real dependent variable v, and some of its derivatives $(v', v'', v''', \dots, v^n), v' = \frac{dv}{dx}$. ODEs arise in many contexts of mathematics and science. Various differentials, derivatives, and functions become related to each other via equations, and thus an ODE is a result that describes dynamically changing phenomena, evolution, or variation. Also, in specific mathematical fields, include geometry and analytical mechanics, and scientific fields include much of physics, chemistry and biology [4].

In this section, we have examined three specific problems in physical and engineering sciences that are governed by non-linear ODEs and they are solved by using ADM and VIM.

1.3.1 Riccati differential equation

The Riccati differential equation is named after the Italian noble man Count Jacopo Francesco Riccati (1676-1754) [26]. This equation has many applications such as network synthesis, financial mathematics [12], control the boundary arising in fluid structure interaction [31]. Consider the following nonlinear Riccati differential equation [20].

$$v'(x) = p(x) + q(x)v(x) + r(x)v^2(x), \quad v(x_0) = \alpha, \; x_0 \leq x \leq X_f, \quad (1.3)$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous functions, x_0, X_f and α are arbitrary constants, and $v(x)$ is unknown function.

A substantial amount of research work has been done to develop the solution for Riccati differential equation. The most used methods are ADM [68], HAM [34]. Taylor matrix method [59] and Haar wavelet method [79], HPM [82] combination of Laplace Adomian decomposition and Pade approximation methods [64], and many other methods available in the literature.

1.3.2 Analytical methods available in literatures to solve the Riccati differential equation

In this subsection, the ADM and VIM will be used to solve the nonlinear Riccati differential equation.

1.3.3 The Adomian decomposition method

In the early 1980s, George Adomian had introduced the ADM [22, 23, 24]. This method plays an important role in applied mathematics. The importance of the ADM is due to its strength and effectiveness, and it can easily deal with many types of ODEs, PDEs (linear and nonlinear), integral equations and the other types of equations [7]. It consists of decomposing the unknown function $v(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series.

$$v(x) = \sum_{n=0}^{\infty} v_n(x)$$

or equivalently

$$v(x) = v_0(x) + v_1(x) + v_2(x) + \cdots.$$

Where the linear components $v_n(x)$, $n \geq 0$, are evaluated in recursive manner. The decompositions method concerns itself with finding the components $v_0, v_1, v_2, \ldots$ etc.

The zero component is identified by all terms that are not included under the integral sign. Consequently, those components $v_i(x)$, $i > 1$ of the unknown function $v(x)$ are completely determined by setting the recurrence. Consider the following nonlinear differential equation

$$Lv + Nv = f(x), \tag{1.4}$$

where L and N are linear and nonlinear operators, respectively, and $f(x)$ is the source inhomogeneous term. The ADM introduces for Eq. (1.4) in the form

$$v_0(x) = f(x),$$

$$v_{n+1}(x) = \int_0^x (\sum_{n=0}^{\infty} v_n(t))dt, \quad n \geq 0 \tag{1.5}$$

For the nonlinear solution $v(x)$, and the infinite series of polynomials

$$v(x) = \sum_{n=0}^{\infty} A_n(v_0, v_1, \dots, v_n),$$

where the components $v_n(x)$ of the solution $v(x)$ will be determined, recurrently and A_n are the Adomian polynomials which are obtained from the definitional formula [24]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[f(\sum_{i=0}^{n} \lambda^i v_i)]_{\lambda=0}, \qquad n = 1,2, \dots$$

The formulas of the first several Adomian polynomials from A_0 to A_5, will be listed below as given in [24]

$$A_0 = f(v_0),$$

$$A_1 = v_1 f'(v_0),$$

$$A_2 = v_2 f'(v_0) + \frac{1}{2!} {v_1}^2 f''(v_0),$$

$$A_3 = v_3 f'(v_0) + v_1 v_2 f''(v_0) + \frac{1}{3!} {v_1}^3 f'''(v_0),$$

$$A_4 = v_4 f'(v_0) + (\frac{1}{2!} {v_2}^2 + v_1 v_3) f''(v_0) + \frac{1}{2!} {v_1}^2 v_2 f'''(v_0) + \frac{1}{4!} {v_1}^4 f''''(v_0),$$

$$A_5 = v_5 f'(v_0) + (v_2 v_3 + v_1 v_4) f''(v_0) + (\frac{1}{2!} v_1 {v_2}^2 + \frac{1}{2!} {v_1}^2 v_3) f'''(v_0) + \frac{1}{3!} {v_1}^3 v_2 f''''(v_0) + \frac{1}{5!} {v_1}^5 f'''''(v_0),$$

$$\vdots$$

and so on.

Note 1.1:

It is worth mentioning that we can use the transformation formula of converting multiple integrals to single [6]

$$\int_0^x \int_0^x \dots \int_0^x v(x_n)dx_n \dots dx_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}v(t)dt. \qquad (1.6)$$

1.3.4 Error analysis

We used the appropriate function of the maximal error remainder [35, 54] to assess the accuracy of the approximate solution.

$$ER_n(x) = L(v) - N(v) - f. \qquad (1.7)$$

The maximal error remainder is

$$MER_n = \max_{0\le x\le 1} |ER_n(x)|. \qquad (1.8)$$

The ADM will be implemented to solve the Riccati differential equation

Example 1.1 [74]:

Let us consider the Riccati equation of the form.

$$v' = -v^2 + 1, \qquad \text{with initial condition} \qquad v(0) = 0. \qquad (1.9)$$

Eq. (1.9) can be converted to the following Volterra integral equation

$$v(x) = x - \int_0^x v^2(t)dt, \qquad (1.10)$$

The Adomain method proposes that the solution $v(x)$ can be expressed by the decomposition series:

$$v(x) = \sum_{n=0}^{\infty} v_n(x),$$

and the nonlinear term v^2, be equated to:

$$v^2 = \sum_{n=0}^{\infty} A_n.$$

Therefore, Eq. (1.10) will be:

$$\sum_{n=0}^{\infty} v_n(x) = x - \int_0^x (\sum_{n=0}^{\infty} A_n \ (t))dt, \tag{1.11}$$

The following recursive relation:

$$v_0 = x,$$

$$v_{n+1} = -\int_0^x \left(\sum_{n=0}^{\infty} A_n \ (t)\right) dt, \qquad n \geq 0. \tag{1.12}$$

Now, we find Adomian polynomial A_n.

$$A_0 = f(v_0) = {v_0}^2,$$

$$A_1 = v_1 f'(v_0) = 2v_0 v_1,$$

$$A_2 = v_2 f'(v_0) + \frac{1}{2!}{v_1}^2 f''(v_0) = 2v_0 v_2 + {v_1}^2,$$

$$A_3 = v_3 f'(v_0) + v_1 v_2 f''(v_0) + \frac{1}{3!}{v_1}^3 f'''(v_0) = 2v_0 v_3 + 2v_1 v_2,$$

$$\vdots$$

Consequently, by applying the Mathematica's code, see appendix A, we get

$$v_0 = x,$$

$$v_1 = -\int_0^x A_0(t)dt = -\frac{x^3}{3},$$

$$v_2 = -\int_0^x A_1(t)dt = \frac{2x^5}{15},$$

$$v_3 = -\int_0^x A_2(t)dt = -\frac{17x^7}{315},$$

$$v_4 = -\int_0^x A_3(t)dt = \frac{62x^9}{2835},$$

$$v_5 = -\int_0^x A_4(t)dt = -\frac{1382x^{11}}{155925},$$

$$v_6 = -\int_0^x A_5(t)dt = \frac{21844x^{13}}{6081075},$$

Then, we have

$$v(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} - \frac{1382x^{11}}{155925} + \frac{21844x^{13}}{6081075}.$$

The obtained series solution in Eq. (1.9) can be used to calculate the maximal error remainder for show the highest accuracy level that we can achieve. It can be clearly seen in table 1.1 and figure 1.1 that by increasing the iterations the errors will be reduced.

Table 1.1: The maximal error remainder: MER_n by the ADM, where $n = 1, \ldots, 6$.

n	MER_n **by ADM**
1	6.65556×10^{-5}
2	3.76891×10^{-7}
3	1.96289×10^{-9}
4	9.72067×10^{-12}
5	4.65513×10^{-14}
6	2.15106×10^{-16}

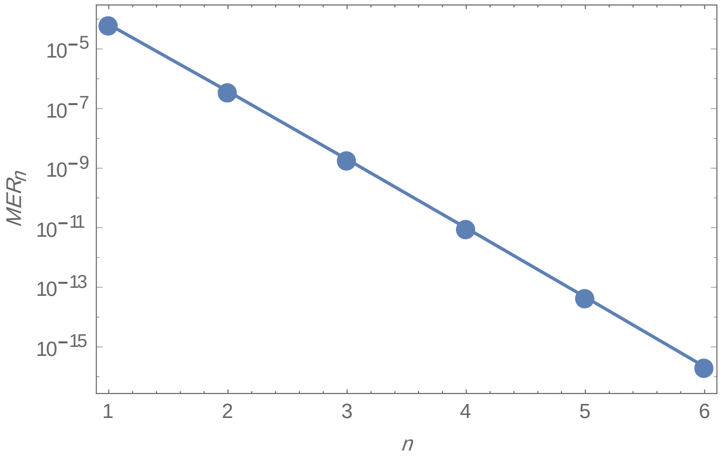

Figure 1.1: Logarithmic plots of MER_nversus n is 1 through 6 by ADM.

1.3.5 Variational Iteration Method

The method used to solve the Riccati differential equation is called the VIM [20]. The VIM established by Ji-Huan He in 1999 is now used to deal with a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations [40, 41]. The obtained series can be employed for numerical purposes if exact solution is not obtainable. To illustrate the basic concepts of the VIM, we consider the following nonlinear equation [39, 43]:

$$Lv(x) + Nv(x) = f(x), \quad x > 0, \tag{1.13}$$

where, L is a linear operator, N a nonlinear operator and $f(x)$ is the source of the inhomogeneous term. The VIM introduces functional for Eq. (1.13) in the form:

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(s)\big(Lv_n(s) + N\tilde{v}_n(s) - f(s)\big)ds, \qquad (1.14)$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory, and $\tilde{v}_n$ as a restricted variation. The Lagrange multiplier λ may be constant or a function and it is given by the general formula [8, 9]:

$$\lambda(s) = (-1)^n \frac{1}{(n-1)!}(s-x)^{n-1}. \qquad (1.15)$$

However, for fast convergence, the function $v_0(x)$ should be selected by using the initial condition for ODE as follows:

$v_0(x) = v(0)$, for first order.

$v_0(x) = v(0) + xv'(0)$, for second order.

$v_0(x) = v(0) + xv'(0) + \frac{1}{2!}x^2 v''(0)$, for third order.

$\vdots$

and so on.

The successive approximations v_{n+1}, $n \geq 0$ of the solution $v_n(x)$ will be readily upon using selective function $v_0(x)$. Consequently, the solution is given by

$$v(x) = \lim_{n\to\infty} v_n(x).$$

In what follows, the VIM will be implemented to solve the Riccati differential equation.

Example 1.2 [74]:

Rewrite the example 1.1 and solve it by VIM.

$$v' = -v^2 + 1, \text{ with initial condition } \quad v(0) = 0. \tag{1.16}$$

By using Eq. (1.14), we get

$$v_{n+1}(x) = v_n(x) + \int_0^x [\lambda(s)(v_n' + v_n^2(s) - 1)]\, ds, \tag{1.17}$$

By using the formula in Eq. (1.15) leads to $\lambda = -1$. Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional Eq. (1.17) gives the iteration formula:

$$v_{n+1}(x) = v_n(x) - \int_0^x (v_n' + v_n^2(s) - 1)\, ds. \tag{1.18}$$

The initial approximation will be:

$$v_0(x) = v(0) = 0,$$

By using equation (1.18), the following successive approximations will be obtained:

$$v_0 = 0,$$

$$v_1 = v_0 - \int_0^x (v_0' + v_0^2(s) - 1)\, ds = x,$$

$$v_2 = v_1 - \int_0^x (v_1' + v_1^2(s) - 1)\, ds = x - \frac{x^3}{3},$$

$$v_3 = v_2 - \int_0^x (v_2' + v_2^2(s) - 1)\, ds = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{x^7}{63},$$

By continuing in this way till n = 6, we have

$$v_6(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} - \frac{1382x^{11}}{155925} + \frac{20404x^{13}}{6081075} - \frac{766609x^{15}}{638512875} + \frac{4440998x^{17}}{10854718875} - \frac{3206482x^{19}}{24105934125} + \cdots, \quad (1.19)$$

Table 1.2 and Figure 1.2 illustrate the convergence of the solution through the use of the maximal error remainder which can be clearly seen, the error will be reduced by increasing the number of iterations.

Table 1.2: The maximal error remainder: MER_n by the VIM, where $n = 1, \ldots, 6$.

n	MER_n by VIM
1	0.01
2	6.65556×10^{-5}
3	2.65463×10^{-7}
4	7.56708×10^{-10}
5	1.67808×10^{-12}
6	3.04791×10^{-15}

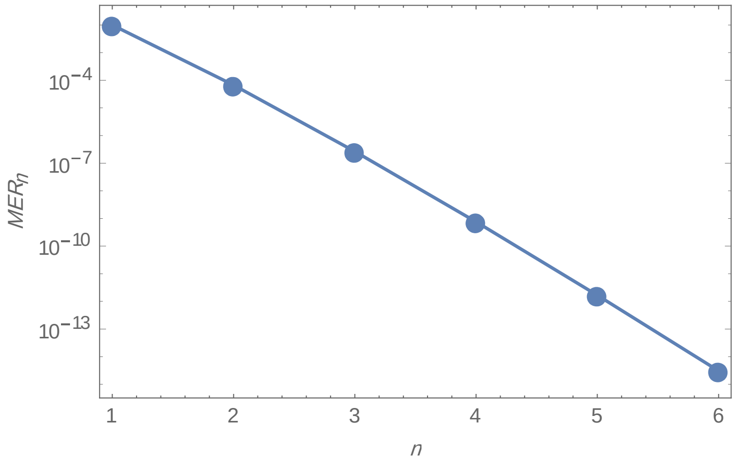

Figure 1.2: Logarithmic plots of MER_nversus n is 1 through 6 by VIM.

1.3.6 Pantograph differential equation

The pantograph is a type of delay differential equations. In 1971 Taylor and Okondon developed first the mathematical model of the pantograph [46]. Pantograph equation used in many applications, such as industrial applications [49], studies based on biology, economy, control and electrodynamics [58]. In addition, the nonlinear dynamic systems and population models. Consider the following generalized pantograph equation [11].

$$v^{(m)}(x) = \sum_{j=0}^{J} \sum_{k=0}^{m-1} p_{jk}(x) v^{(k)}\left(\alpha_j x + \beta_j\right) + f(x), \tag{1.20}$$

with initial conditions

$$\sum_{k=0}^{m-1} c_{ik} v^{(k)}(0) = \mu_i\,, \quad i = 0, 1, \dots, m-1,$$

where $p_{jk}(x)$ and $f(x)$ are analytical functions, $J \geq 1$, c_{ik}, μ_i, α_j and β_j are real or complex constants.

Pantograph equation was solved by many authors either analytically or numerically. For instance, Yusuf oglu [81] proposed an efficient algorithm for solving generalized pantograph equations with the linear functional argument, the authors of investigated an exponential approximation to obtain an approximate solution of generalized pantograph-delay differential equations [73]. High-order pantograph equations with initial conditions have been also studied by Taylor method [62], ADM [14], DTM [78] and HPM [16]. Recently, Doha et al proposed and developed a new Jacobi rational-Gauss collocation method for solving the generalized pantograph equations on a semi-infinite domain [15].

1.3.7 VIM for solving the pantograph differential equation

Example 1.3 [11]:

Let us deal with the following pantograph differential equation,

$$v'' = \frac{3}{4}v + v\left(\frac{x}{2}\right) - x^2 + 2,$$

with the initial conditions $v(0) = 0,\ v'(0) = 0.$ (1.21)

By using Eq. (1.14), we get:

$$v_{n+1}(x) = v_n(x) + \int_0^x \left[\lambda(s)\left(v_n'' - \frac{3}{4}v_n - v_n\left(\frac{s}{2}\right) - s^2 + 2\right)\right] ds. \quad (1.22)$$

Using the formula in Eq. (1.15) leads to λ = $(s - x)$. Substituting this value of the Lagrange multiplier λ = $(s - x)$ into the functional Eq. (1.22) gives the iteration formula:

$$v_{n+1}(x) = v_n(x) + \int_0^x \left[(s - x)\left(v_n'' - \frac{3}{4}v_n - v_n\left(\frac{s}{2}\right) - s^2 + 2\right)\right] ds,$$

The initial approximation will be:

$$v_0(x) = v(0) + xv'(0) = 0, \quad (1.23)$$

By using Eq. (1.23), the following successive approximations will be obtained

$$v_0 = 0,$$

$$v_1 = v_0 + \int_0^x \left[(s - x)\left(v_0'' - \frac{3}{4}v_0 - v_0\left(\frac{s}{2}\right) - s^2 + 2\right)\right] ds = x^2 - \frac{x^4}{12},$$

$$v_2 = v_1 + \int_0^x \left[(s - x)\left(v_1'' - \frac{3}{4}v_1 - v_1\left(\frac{s}{2}\right) - s^2 + 2\right)\right] ds = x^2 - \frac{13x^6}{5760},$$

$$v_3 = v_2 + \int_0^x \left[(s-x)\left(v_2'' - \frac{3}{4}v_2 - v_2\left(\frac{s}{2}\right) - s^2 + 2\right)\right] ds,$$

$$= x^2 - \frac{91x^8}{2949120},$$

$$v_4 = v_3 + \int_0^x \left[(s-x)\left(v_3'' - \frac{3}{4}v_3 - v_3\left(\frac{s}{2}\right) - s^2 + 2\right)\right] ds,$$

$$= x^2 - \frac{17563x^{10}}{67947724800},$$

and so on. Then:

$v(x) = x^2 -$ small term,

This series converges to the exact solution [11],

$$v(x) = \lim_{n\to\infty} v_n(x) = x^2.$$

1.3.8 Beam deformation equation

In the last two decades with the rapid development of applied science, there has appeared mathematical model of the beam deformation equation which is used in structural engineering and fluid mechanics.

Recently, many analytical methods are used to solve nonlinear beam deformation equation, such as HPM [55], VIM [1], and ADM [25].

According to the classical beam theory in [1], the function $v = v(x)$ represents the configuration of the deformed beam. The length of the beam is $L = 1$, where $x = 0$ at the left side and $x = 1$ at the right side and $f = f(x)$ is a load which causes the deformation [1], as it is shown in figure 1.3.

Let us consider the nonlinear beam deformation problem as a general fourth order boundary value problem of the form [1],

$$v^{(4)}(x) = f(x, v, v', v'', v'''), \tag{1.24}$$

with the boundary conditions:

$$v(a) = \alpha_1, \qquad v'(a) = \alpha_2,$$

$$v(b) = \beta_1, \qquad v'(b) = \beta_2,$$

where f is a continuous function on [a,b] and the parameters α_1, α_2, β_1 and β_2 are finite real arbitrary constants. Eq. (1.24) used as mathematical models in engineering.

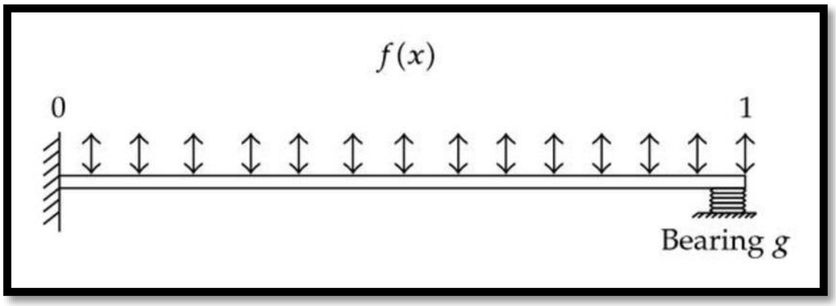

Figure 1.3: Beam on the elastic bearing.

In this subsection, the following problem of the beam deformation will be solved by using the VIM.

1.3.9 VIM for solving the nonlinear beam deformation problem

Example 1.4 [1]:

Let us consider the following form of the beam deformation equation,

$$v^{(4)} = v^2 + f(x),$$

subject to the boundary conditions

$$v(0) = 0, \qquad v'(0) = 0, \qquad v(1) = 1, \qquad v'(1) = 1. \tag{1.25}$$

Where

$$f(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,$$

By using Eq. (1.14), we have:

$$v_{n+1}(x) = v_n(x) + \int_0^x \left[\lambda(s)\left(v_n^{(4)} - v_n^2 - f(s)\right)\right] ds. \tag{1.26}$$

By using the formula in Eq. (1.15) leads to $\lambda = \frac{(s-x)^3}{3!}$. Substituting this value of the Lagrange multiplier $\lambda = \frac{(s-x)^3}{3!}$ into the functional Eq. (1.26) gives the iteration formula:

$$v_{n+1}(x) = v_n(x) + \int_0^x \left[\frac{(s-x)^3}{3!}\left(v_n^{(4)} - v_n^2 - f(s)\right)\right] ds. \tag{1.27}$$

Let us assumed that an initial approximation has the form:

$$v_0(x) = ax^3 + bx^2 + cx + d, \tag{1.28}$$

where a, b, c, and d are unknown constants to be further determined.

By using the iteration formula (1.27), the following first-order approximation is obtained:

$$v_1 = v_0 + \int_0^x \left[\frac{(s-x)^3}{3!}\left(v_0^{(4)} - v_0^2 - f(s)\right)\right] ds,$$

$$= d + cx + bx^2 + ax^3 + \left(-2 + \frac{d^2}{24}\right)x^4 + \left(1 + \frac{cd}{60}\right)x^5$$

$$+ \left(\frac{c^2}{360} + \frac{bd}{180}\right)x^6 + \left(\frac{bc}{420} + \frac{ad}{420}\right)x^7$$

$$+ \left(-\frac{1}{420} + \frac{b^2}{1680} + \frac{ac}{840}\right)x^8 + \frac{abx^9}{1512} + \left(\frac{1}{630} + \frac{a^2}{5040}\right)x^{10}$$

$$-\frac{x^{11}}{1980} - \frac{x^{12}}{2970} + \frac{x^{13}}{4290} - \frac{x^{14}}{24024}, \qquad (1.29)$$

Imposing the boundary conditions (1.25), into v_1, results in the following values:

$$a = -0.00687154, \quad b = 2.00593, \quad c = 0, \quad d = 0. \qquad (1.30)$$

The following first-order approximate solution is then achieved:

$$v_1 = 2.005929514398968x^2 - 0.006871538792438514x^3 - 2x^4$$

$$+ x^5 + 0.0000141388194862374 6x^8$$

$$- 0.0000091162847044245 09x^9$$

$$+ 0.0015873109559613 84x^{10} - \frac{x^{11}}{1980} - \frac{x^{12}}{2970} + \frac{x^{13}}{4290}$$

$$-\frac{x^{14}}{24024}. \qquad (1.31)$$

Similarly, the following second-order approximation may be written:

$$v_2 = v_1 + \int_0^x \left[\frac{(s-x)^3}{3!}\left(v_1^{(4)} - v_1^2 - f(s)\right)\right] ds, \qquad (1.32)$$

Imposing the boundary conditions, (1.25), into v_2, yields

$$a = -8.27907 \times 10^{-7}, \quad b = 2.000000763, \quad c = 0, \quad d = 0. \qquad (1.33)$$

The following second-order approximate solution is obtained:

$$v_2 = 2.0000007627556786x^2 - 8.279069998454606 \times 10^{-7}x^3 - 2x^4 + x^5 + 1.816085295168468 \times 10^{-9}x^8 - \cdots. \quad (1.34)$$

In order to check the accuracy of the approximate solution, we calculate the absolute error, $|r_n| = |v(x) - v_n(x)|$ where $v(x) = x^5 - 2x^4 + 2x^2$ is the exact solution and $v_n(x)$ is the approximate solution. In table 1.3 the absolute error of VIM with n = 1, 2 is presented.

Table 1.3: Results of the absolute errors for VIM.

$\boldsymbol{x}$	$\boldsymbol{\lvert r_1 \rvert}$ **for VIM**	$\boldsymbol{\lvert r_2 \rvert}$ **for VIM**
0	0	0
0.1	5.24236×10^{-5}	6.79965×10^{-9}
0.2	1.82208×10^{-4}	2.3887×10^{-8}
0.3	3.48134×10^{-4}	4.62946×10^{-8}
0.4	5.09092×10^{-4}	6.90558×10^{-8}
0.5	6.24721×10^{-4}	8.72068×10^{-8}
0.6	6.57823×10^{-4}	9.58101×10^{-8}
0.7	5.81138×10^{-4}	9.01332×10^{-8}
0.8	3.92709×10^{-4}	6.67054×10^{-8}
0.9	1.45715×10^{-4}	2.80674×10^{-8}
1.0	3.3447×10^{-8}	0

It can be seen clearly from table 1.3, that by increasing the iterations, the errors will decreasing.

1.4 Partial Differential Equations (PDEs)

PDE is an equation that contains the dependent variable, and its partial derivatives. In the ordinary differential equations, the dependent variable $v = v(x)$ depends only on one independent variable x. Unlike the ODEs, the dependent variable in the PDEs, such as $v = v(x, t)$ or $v = v(x, y, t)$, must depend on more than one independent variable. If $v = v(x, t)$, then the function v depends on the independent variable x, and on the time variable t [7]. In addition, it is well known that most of the phenomena that arise in mathematical, chemical, physics and engineering fields can be described by partial differential equations. For example, the heat flow and the wave propagation phenomena, physical phenomena of fluid dynamics and many other models described by partial differential equations.

In this section, we have examined the Burgers' equations governed by nonlinear PDEs and will be introduced by using the ADM and the VIM.

1.4.1 Burgers' equation

The Burgers equation was first presented by Bateman [27] and treated later by Johannes Martinus Burgers (1895-1981) then it is widely named as Burgers' equation [45]. The introduction of the applications of Burgers' equations by mathematical scientists and researchers has become more important and interesting, especially in problems that have the form of nonlinear equations and systems of equations. It has been known that this equation describes different types of phenomena such as modeling of dynamics, heat conduction, acoustic waves and many others [5, 10, 36, 45, 61]. In addition, the 2D and 3D Burgers' equations have played an important role in many physical applications such as modeling of gas dynamics and shallow water waves [37, 48].

We consider the 1D, 2D and (1+3)-D Burgers' equations as follows [13, 76, 80],

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=k\left(\frac{\partial^2 v}{\partial x^2}\right), \qquad 0\le x\le 1, \qquad t>0. \qquad (1.35)$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=k\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \quad 0\le x,y\le 1, \qquad t>0. \qquad (1.36)$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=k\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \quad 0\le x,y,z\le 1, \qquad t>0. \qquad (1.37)$$

In addition, the system of coupled Burger's equation,

$$\frac{\partial v}{\partial t}-2v\frac{\partial v}{\partial x}+\frac{\partial (vu)}{\partial x}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}\right),$$

$$\frac{\partial u}{\partial t}-2u\frac{\partial u}{\partial x}+\frac{\partial (vu)}{\partial x}=\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}\right), \qquad 0\le x\le 1, \qquad t>0, \qquad (1.38)$$

and the system of 2D Burgers' equation,

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+u\frac{\partial v}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right),$$

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \quad 0\le x,y\le 1, \quad t>0, \qquad (1.39)$$

and the system of 3D Burgers' equation,

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

$$\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-w\frac{\partial u}{\partial z}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

$$0\leq x,y,z\leq 1, \qquad t>0, \qquad (1.40)$$

where $k=\frac{1}{Re}$ is an arbitrary constant (*Re*: Reynolds number) for simulating the physical phenomena of wave motion, and thus determines the behavior of the solution. If the viscosity is k = 0, then the equation is called the equation inviscid Burgers'. It is the lowest approximation of the one-dimensional system.

Recently, a number of researchers have solved the Burgers' equations. For example, Marinca and Herisaun proposed the optimal homotopy asymptotic method (OHAM) [66], Daftardar-Jafari method (DJM) in [32]. In addition, considerable attention has been given to ADM for solving Burgers' equation [5]. Also, Laplace decomposition method (LDM) [60]. Moreover, HPM was proposed by He [47], and many others [10, 61, 76].

1.4.2 Exact and numerical solutions for nonlinear Burgers' equation by ADM

In this subsection, the ADM is used to get an exact solution for the 1D Burgers' equation and numerical solution for the 2D Burgers' equation. As given in [7, 30].

Example 1.5 [33]:

Let us consider the following 1D Burgers' equation.

$$v_t+vv_x=v_{xx}, \quad \text{with initial condition} \quad v(x,0)=2x. \qquad (1.41)$$

Applying the integration on both sides of Eq. (1.41) from 0 to t and using the initial condition at $v(x,0)=2x$, we have

$$v(x,t) = 2x + \int_0^t (-vv_x + v_{xx})dt. \tag{1.42}$$

By using the decomposition series for the linear term $v(x,t)$ and the series of Adomian polynomials for the nonlinear term vv_x give

$$\sum_{n=0}^{\infty} v_n(x,t) = 2x - \int_0^t \left(\sum_{n=0}^{\infty} A_n\right) dt + \int_0^t \left(\sum_{n=0}^{\infty} v_{nxx}\right) dt. \tag{1.43}$$

Identifying the zeroth component $v_0(x,t)$ by the term that arise from the initial condition and following the decomposition method, we obtain the recursive relation

$$v_0(x,t) = 2x,$$

$$v_{n+1}(x,t) = -\int_0^t \left(\sum_{n=0}^{\infty} A_n\right) dt + \int_0^t \left(\sum_{n=0}^{\infty} v_{nxx}\right) dt, \qquad n \geq 0. \tag{1.44}$$

The Adomian polynomials for the nonlinear term vv_x have been derived in the form

$$A_0 = v_{0x}v_0,$$

$$A_1 = v_{0x}v_1 + v_{1x}v_0,$$

$$A_2 = v_{0x}v_2 + v_{1x}v_1 + v_{2x}v_0,$$

$$A_3 = v_{0x}v_3 + v_{1x}v_2 + v_{2x}v_1 + v_{3x}v_0,$$

$$A_4 = v_{0x}v_4 + v_{1x}v_3 + v_{2x}v_2 + v_{3x}v_1,$$

$$\vdots$$

Consequently, we obtain:

$$v_0(x,t) = 2x,$$

$$v_1(x,t) = -\int_0^t A_0 dt + \int_0^t (v_{0xx}) dt = -4tx,$$

$$v_2(x,t) = -\int_0^t A_1 dt + \int_0^t (v_{1xx}) dt = 8t^2 x,$$

$$v_3(x,t) = -\int_0^t A_2 dt + \int_0^t (v_{2xx}) dt = -16t^3 x,$$

$$v_4(x,t) = -\int_0^t A_3 dt + \int_0^t (v_{3xx}) dt = 32t^4 x,$$

$$\vdots$$

Summing these iterates gives the series solution

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t),$$

$$= 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - 64xt^5 + 128xt^6 - 256xt^7 + 512xt^8 - \cdots,$$

This series converges to the exact solution [33]:

$$v(x,t) = \frac{2x}{1+2t},$$

$$= 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - 64xt^5 + 128xt^6 - 256xt^7 + 512xt^8 - 1024xt^9 + 2048xt^{10} - \cdots.$$

Example 1.6 [80]:

Consider the following 2D Burgers' equation,

$$v_t + vv_x + vv_y = k\left(v_{xx} + v_{yy}\right), \quad k > 0, \ (x, y) \in [0,1] \times [0,1], \quad (1.45)$$

with initial condition $v(x, y, 0) = sin(2\pi x)cos(2\pi y)$.

Applying the integration on both sides of Eq. (1.45) from 0 to t and using the initial condition at $v(x, y, 0) = sin(2\pi x)cos(2\pi y)$, we have

$$v(x, y, t) = sin(2\pi x)cos(2\pi y) + \int_0^t \left(-vv_x - vv_y + k\left(v_{xx} + v_{yy}\right)\right) dt. \quad (1.46)$$

By using the decomposition series for the linear term $v(x, y, t)$ and the series of Adomian polynomials for the nonlinear term vv_x, vv_y give

$$\sum_{n=0}^{\infty} v_n(x, y, t) = sin(2\pi x)cos(2\pi y) - \int_0^t \left(\sum_{n=0}^{\infty} A_n\right) dt - \int_0^t \left(\sum_{n=0}^{\infty} B_n\right) dt + \int_0^t \left(\sum_{n=0}^{\infty} k(v_{nxx} + v_{nyy})\right) dt, \quad (1.47)$$

identifying the zeroth component $v_0(x, y, t)$ by the term that arises from the initial condition and following the ADM, we obtain the recursive relation

$$v_0(x, y, t) = sin(2\pi x)cos(2\pi y),$$

$$v_{n+1}(x,y,t) = -\int_0^t \left(\sum_{n=0}^{\infty} A_n\right) dt$$

$$-\int_0^t \left(\sum_{n=0}^{\infty} B_n\right) dt + \int_0^t \left(\sum_{n=0}^{\infty} k(v_{nxx} + v_{nyy})\right) dt\, n \geq 0. \quad (1.48)$$

The Adomian polynomials for the nonlinear term vv_x, vv_y have been derived in the form

$$A_0 = v_{0x}v_0,$$

$$A_1 = v_{0x}v_1 + v_{1x}v_0,$$

$$A_2 = v_{0x}v_2 + v_{1x}v_1 + v_{2x}v_0,$$

$$A_3 = v_{0x}v_3 + v_{1x}v_2 + v_{2x}v_1 + v_{3x}v_0,$$

and

$$B_0 = v_{0y}v_0,$$

$$B_1 = v_{0y}v_1 + v_{1y}v_0,$$

$$B_2 = v_{0y}v_2 + v_{1y}v_1 + v_{2y}v_0,$$

$$B_3 = v_{0y}v_3 + v_{1y}v_2 + v_{2y}v_1 + v_{3y}v_0,$$

consequently, we obtain:

$$v_0(x,y,t) = sin(2\pi x)cos(2\pi y),$$

$$v_1(x,y,t) = -\int_0^t A_0 dt - \int_0^t B_0 dt + \int_0^t k(v_{0xx} + v_{0yy}) dt,$$

$$= t\big(-8\pi^2 kcos(2\pi y)sin(2\pi x) - 2\pi cos(2\pi x)cos(2\pi y)^2 sin(2\pi x) + 2\pi sin(2\pi x)sin(2\pi y)\big),$$

$$v_2(x,y,t) = -\int_0^t A_1 dt - \int_0^t B_1 dt + \int_0^t k(v_{1xx} + v_{1yy})dt,$$

$$= -2\pi^2 t^2 cos(2\pi y)sin(2\pi x) + 32\pi^4 t^2 k^2 cos(2\pi y)sin(2\pi x) + \cdots,$$

$$v_3(x,y,t) = -\int_0^t A_2 dt - \int_0^t B_2 dt + \int_0^t k(v_{2xx} + v_{2yy})dt,$$

$$= 16\pi^4 t^3 kcos(2\pi y)sin(2\pi x) - \frac{256}{3}\pi^6 t^3 k^3 cos(2\pi y)sin(2\pi x) + \cdots,$$

$$v_4(x,y,t) = -\int_0^t A_3 dt - \int_0^t B_3 dt + \int_0^t k(v_{3xx} + v_{3yy})dt,$$

$$= \frac{2}{3}\pi^4 t^4 cos(2\pi y)sin(2\pi x) - 64\pi^6 t^4 k^2 cos(2\pi y)sin(2\pi x) + \cdots,$$

then

$$v(x,y,t) = cos(2\pi y)sin(2\pi x) - 2\pi^2 t^2 cos(2\pi y)sin(2\pi x) + \frac{2}{3}\pi^4 t^4 cos(2\pi y)sin(2\pi x) + 16\pi^4 t^3 kcos(2\pi y)sin(2\pi x) + 32\pi^4 t^2 k^2 cos(2\pi y)sin(2\pi x) - \cdots.$$

The obtained series solution in Eq. (1.45) can be used to calculate the maximal error remainder to show the highest accuracy level that we can achieve. This can be clearly seen in table 1.4 and figure 1.4.

Table 1.4: The maximal error remainder: MER_n by the ADM.

n	MER_n by ADM
1	1.61705×10^{-2}
2	2.98775×10^{-5}
3	4.9464×10^{-8}
4	7.1962×10^{-11}

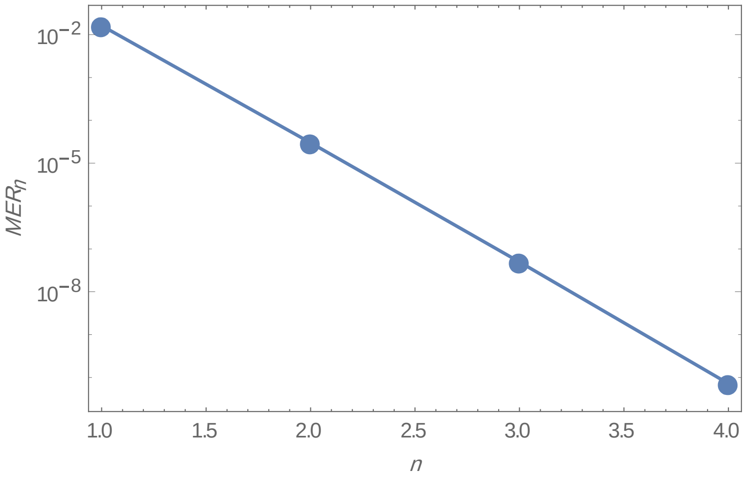

Figure 1.4: Logarithmic plots of MER_nversus n is 1 through 4 by ADM**.**

1.4.3 The VIM to solve the 2D Burgers' equation

In this subsection, the VIM is used to get a numerical solution for the 2D Burgers' equation. As given in [50].

Example 1.7 [80]:

Rewrite the example 1.6 and solve it by VIM,

$$v_t + vv_x + vv_y = k\left(v_{xx} + v_{yy}\right), \tag{1.49}$$

with initial condition $v(x, y, 0) = sin(2\pi x)cos(2\pi y)$.

By using Eq. (1.14), we get:

$$
\begin{aligned}
v_{n+1}&(x,y,t) \\
&= v_n(x,y,t) \\
&+ \int_0^t \left[\lambda(s)\big(v_n(x,y,s)_s + v_n(x,y,s)v_n(x,y,s)_x\right. \\
&+ v_n(x,y,s)v_n(x,y,s)_y - k(v_n(x,y,s)_{xx} \\
&\left.+ v_n(x,y,s)_{yy})\big)\right] ds. \qquad (1.50)
\end{aligned}
$$

Using Eq. (1.15) leads to $\lambda = -1$. Substituting this value of the Lagrange multiplier into the functional Eq. (1.50) gives the following iteration formula:

$$
\begin{aligned}
v_{n+1}&(x,y,t) \\
&= v_n(x,y,t) \\
&- \int_0^t \big(v_n(x,y,s)_s + v_n(x,y,s)v_n(x,y,s)_x \\
&+ v_n(x,y,s)v_n(x,y,s)_y - k(v_n(x,y,s)_{xx} \\
&+ v_n(x,y,s)_{yy})\big) ds. \qquad (1.51)
\end{aligned}
$$

By applying Mathematica's code (see appendix B), we get the following initial approximation

$$v_0(x,y,t) = v(x,y,0) = sin(2\pi x)cos(2\pi y),$$

By using Eq. (1.51) the obtained successive approximations will be

$$v_0 = sin(2\pi x)cos(2\pi y),$$

$$v_1 = v_0 - \int_0^t \left(v_{0_s} + v_0 v_{0_x} + v_0 v_{0_y} - k(v_{0_{xx}} + v_{0_{yy}})\right) ds,$$

$$
\begin{aligned}
&= cos(2\pi y)sin(2\pi x) - t(8\pi^2 k cos(2\pi y)sin(2\pi x) \\
&\quad + 2\pi cos(2\pi x)cos(2\pi y)^2 sin(2\pi x) - 2\pi sin(2\pi x)sin(2\pi y)),
\end{aligned}
$$

$$v_2 = v_1 - \int_0^t \left(v_{1_s} + v_1 v_{1_x} + v_1 v_{1_y} - k(v_{1_{xx}} + v_{1_{yy}})\right) ds,$$

$$= cos(2\pi y)sin(2\pi x) - 2\pi^2 t^2 cos(2\pi y)sin(2\pi x)$$
$$+ 32\pi^4 t^2 k^2 cos(2\pi y)sin(2\pi x)$$
$$+ 40\pi^3 t^2 k cos(2\pi x)cos(2\pi y)^2 sin(2\pi x) - \cdots,$$

continuing in this way till n = 6, we have

$$v_6 = cos(2\pi y)sin(2\pi x) - 2\pi^2 t^2 cos(2\pi y)sin(2\pi x)$$
$$+ \frac{2}{3}\pi^4 t^4 cos(2\pi y)sin(2\pi x) + \cdots, \quad (1.52)$$

Table 1.5 and Figure 1.5 illustrate the convergence of the solution through the use of the maximal error remainder. Clearly seen, the errors will be decreased as the number of iterations will be increased.

Table 1.5: The maximal error remainder: MER_n by the VIM.

n	MER_n by VIM
1	1.61705×10^{-2}
2	2.52036×10^{-5}
3	3.56783×10^{-8}
4	4.53593×10^{-11}

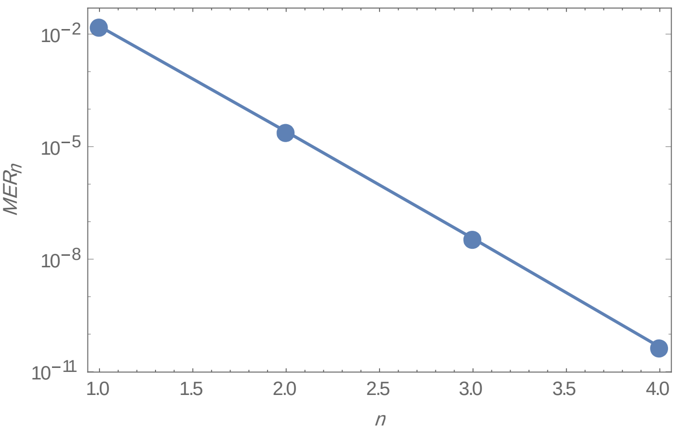

Figure 1.5: Logarithmic plots of MER_nversus n is 1 through 4 by V1M.

CHAPTER 2

SEMI-ANALYTICAL ITERATIVE METHOD FOR SOLVING SOME ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Chapter 2

Semi-Analytical Iterative Method for Solving Some Ordinary and Partial Differential Equations

2.1 Introduction

Temimi and Ansari have proposed recently a semi-analytical iterative method namely (TAM) to solve various linear and nonlinear ODEs and PDEs [28]. For examples, solving nonlinear ordinary differential equations [29], the Korteweg-de Vries Equations [18], Duffing equations [51], the nonlinear thin film flow problems [53], and some chemistry problems [52].

The main advantages of the TAM, it does not required any restrictive assumptions for nonlinear terms since the ADM required the so-called Adomian polynomial, avoided the large computational work or using any other parameters or restricted assumptions that appear in other iterative methods such as VIM and HPM. In addition, the TAM for solving non-linear ODEs and PDEs have successfully implemented.

This chapter is organized as follows: in section two, the basic idea of the TAM is presented. In section three, an overview of the convergence of the TAM based on the Banach fixed-point theorem will be given. In section four, some types of non-linear ODEs equations will be solved by TAM, namely the Riccati equation, pantograph equation and beam deformation equation. Moreover, the Burgers' equations will be solved and the convergence will be proved.

2.2 The basic idea of the TAM

We start by pointing out that the nonlinear differential equation can be written in operator form as**:**

$$L\big(v(x)\big) + N\big(v(x)\big) + f(x) = 0, \quad x \in D,$$

with boundary conditions $\quad B\left(v, \frac{dv}{dx}\right) = 0, \ x \in \mu. \qquad (2.1)$

Where x denotes the independent variable, $v(x)$ is an unknown function, $f(x)$ is a known function, μ is the boundary of the domain D, L is a linear operator, N is a nonlinear operator and B is a boundary operator. The main requirement here is that L is the linear part of the differential equation, but it is possible to take some linear parts and add them to N as needed. The method works in the following steps, starts by assuming that $v_0(x)$ is an initial guess of the solution to the problem [28].

$$L(v_0(x)) + f(x) = 0, \qquad B\left(v_0, \frac{dv_0}{dx}\right) = 0. \qquad (2.2)$$

To generate the next iteration of the solution, we solve the following problem:

$$L(v_1(x)) + N\big(v_0(x)\big) + f(x) = 0, \qquad B\left(v_1, \frac{dv_1}{dx}\right) = 0. \qquad (2.3)$$

Thus, we have a simple iterative procedure which is effectively the solution of a linear set of problems i.e.

$$L(v_{n+1}(x)) + N\big(v_n(x)\big) + f(x) = 0, \qquad B\left(v_{n+1}, \frac{dv_{n+1}}{dx}\right) = 0. \qquad (2.4)$$

It is noted that each of the $v_n(x)$, are solutions to Eq. (2.1). Thus, evaluating more approximate terms provides better accuracy.

2.3 The convergence of the TAM

The Banach fixed-point theorem also known as the contraction mapping principle is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces. Also, provides a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892–1945), and was first stated by him in 1922 [69]. In this section, some basic concepts and the main theorem of the convergence will be presented.

Theorem 2.1: [33]

Suppose that $(X, \|\ \|_X)$ and $(Y, \|\ \|_Y)$ be Banach spaces and $T: X \to Y$, is a contraction nonlinear mapping, that is

$$\forall\, v, v^* \in X\,; \quad \|T(v) - T(v^*)\|_X \leq k\|v - v^*\|_Y, \qquad 0 < k < 1.$$

According to theorem 1.1, having the fixed point v, that is $T(v) = v$, the sequence generated by the TAM will be regarded as

$$v_n = T(v_{n-1}), \qquad v = \lim_{n \to \infty} v_n,$$

and suppose that $v_0 \in B_r(v)$ where $B_r(v) = \{v^* \in X,\ \|v^* - v\| < r\}$ then we have the following statements:

1. $\|v_n - v\| \leq k^n \|v_0 - v\|$,

2. $v_n \in B_r(v)$,

3. $\lim_{n \to \infty} v_n = v$,

2.4 Application of the TAM with convergence for the ODE and PDE

In this section, some types of non-linear ODEs equations will be solved by TAM, namely the Riccati equation, pantograph equation and beam deformation equation. Moreover, the Burgers' equations will be solved and the convergence will be proved.

2.4.1 Exact and numerical solutions for nonlinear Riccati equation by TAM

The TAM will be implemented to solve the Riccati differential equation which is a nonlinear ODE of first order. So, to verify the accuracy of the TAM in solving this kind of problems, the following examples will be discussed.

Example 2.1 [19]:

Let us consider the following Riccati differential equation,

$$v' = e^{x} - e^{3x} + 2e^{2x}v - e^{x}v^{2}, \text{ with initial condition } v(0) = 1. \qquad (2.5)$$

By applying the TAM by first distributing the equation as,

$$L(v) = v', \quad N(v) = -2e^{2x}v + e^{x}v^{2} \quad \text{and} \quad f(x) = -e^{x} + e^{3x}.$$

Thus, the initial problem will be

$$L\big(v_0(x)\big) = e^{x} - e^{3x}, \qquad v_0(0) = 1. \qquad (2.6)$$

By integrating both sides of Eq. (2.6) from 0 to *x*, we obtain

$$\int_0^x v_0'(t)\,dt = \int_0^x (e^{t} - e^{3t})\,dt, \qquad v_0(0) = 1.$$

Therefore, we have

$$v_0(x) = \frac{1}{3} + e^x - \frac{e^{3x}}{3}.$$

The second iteration can be given as

$$L\big(v_1(x)\big) + N\big(v_0(x)\big) + f(x) = 0, \qquad v_1(0) = 1. \tag{2.7}$$

By using simple manipulation, one can solve Eq. (2.7) as follows:

$$\int_0^x v_1'(t)\,dt = \int_0^x \left(e^t - e^{3t} + 2e^{2t}v_0(t) - e^t {v_0}^2(t)\right)dt,\ v_1(0) = 1.$$

Then, we obtain

$$v_1(x) = \frac{1}{14} + \frac{8e^x}{9} + \frac{e^{4x}}{18} - \frac{e^{7x}}{63}.$$

We turn the function $v_1(x)$ by using Maclaurin series expansion to exponential function, we get

$$v_1(x) = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{23x^4}{24} - \frac{209x^5}{120}.$$

Applying the same process, we get the second iteration $v_2(x)$,

$$L\big(v_2(x)\big) + N\big(v_1(x)\big) + f(x) = 0, \qquad v_2(0) = 1.$$

Then, we have

$$v_2(x) = \frac{11}{9720} + \frac{195e^x}{196} + \frac{e^{2x}}{126} - \frac{e^{3x}}{243} - \frac{e^{5x}}{630} + \frac{e^{6x}}{486} + \frac{e^{8x}}{3528} - \frac{5e^{9x}}{6804}$$
$$+ \frac{e^{12x}}{6804} - \frac{e^{15x}}{59535}.$$

We can write the function $v_2(x)$ in the following series form,

$$v_2(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{79x^7}{5040} - \frac{3919x^8}{40320} - \frac{116479x^9}{362880}.$$

By continuing in this way, we will get a series of the form:

$$v(x) = \lim_{n\to\infty} v_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots,$$

This series converges to the following exact solution [19].

$$v(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots.$$

Suppose that T: $[0,1] \rightarrow R$, then $v_n = T(v_{n-1})$ and $0 \le x \le 1$.

According to the theorem 2.1 for nonlinear mapping T, a sufficient condition for convergence of the TAM is strictly contraction T, the exact solution is $v = v(x) = e^x$, therefore, we have

$$\|v_0 - v\| = \left\|\frac{1}{3} + e^x - \frac{e^{3x}}{3} - e^x\right\|,$$

$$\|v_1 - v\| = \left\|\frac{1}{14} + \frac{8e^x}{9} + \frac{e^{4x}}{18} - \frac{e^{7x}}{63} - e^x\right\| \le k\left\|\frac{1}{3} + e^x - \frac{e^{3x}}{3} - e^x\right\|$$
$$= k\|v_0 - v\|,$$

But, $\forall\, x \in [0,1]$, $0 < k < 1$, when $x = \frac{1}{2}$,

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \le k = 0.19551 < 1,$$

Then, we have

$\|v_1 - v\| \leq k\ \|v_0 - v\|,$

$\|v_2 - v\| = \left\| \frac{11}{9720} + \frac{195e^x}{196} + \frac{e^{2x}}{126} - \frac{e^{3x}}{243} - \frac{e^{5x}}{630} + \frac{e^{6x}}{486} + \frac{e^{8x}}{3528} - \frac{5e^{9x}}{6804} + \frac{e^{12x}}{6804} - \frac{e^{15x}}{59535} - e^x \right\| \leq k\|v_1 - v\| \leq kk\|v_0 - v\| = k^2\|v_0 - v\|,$

Since, $\forall\ x \in [0,1]$, $0 < k < 1,$ when $x = \frac{1}{2}$,

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \leq k^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \leq k = 0.0580412 < 1,$$

Thus, we get

$\|v_2 - v\| \leq k^2\|v_0 - v\|,$

Similarly, we have

$\|v_3 - v\| \leq k^3\|v_0 - v\|,$

By continuing in this way, we get:

$\|v_n - v\| \leq k^n\|v_0 - v\|,$

Therefore,

$\lim_{n\to\infty} \|v_n - v\| \leq \lim_{n\to\infty} k^n\ \|v_0 - v\| = 0,$ and $\lim_{n\to\infty} k^n = 0,$ we drive

$\lim_{n\to\infty} \|v_n - v\| = 0$, then $\lim_{n\to\infty} v_n = v = e^x$, which is the exact solution [19].

Example 2.2:

Let us recall example 1.1

$$v' = -v^2 + 1, \quad \text{with initial condition} \quad v(0) = 0. \tag{2.8}$$

By applying the TAM by first distributing the equation as

$$L(v) = v', \quad N(v) = v^2 \quad \text{and} \quad f(x) = -1$$

Thus, the initial problem will be

$$L\big(v_0(x)\big) = 1, \qquad v_0(0) = 0, \tag{2.9}$$

By integrating both sides of Eq. (2.9) from 0 to *x*, we have

$$\int_0^x v_0'(t)\,dt = \int_0^x (1)\,dt, \qquad v_0(0) = 0,$$

So, we get

$$v_0(x) = x.$$

The second iteration can be given as

$$L\big(v_1(x)\big) + N\big(v_0(x)\big) + f(x) = 0, \qquad v_1(0) = 0, \tag{2.10}$$

By using simple manipulation, one can solve Eq. (2.10) as follows

$$\int_0^x v_1'(t)\,dt = \int_0^x (-{v_0}^2(t) + 1)\,dt, \qquad v_1(0) = 0.$$

Then, we obtain

$$v_1(x) = x - \frac{x^3}{3}.$$

Applying the same process to get the second iteration $v_2(x)$,

$$L\big(v_2(x)\big) + N\big(v_1(x)\big) + f(x) = 0, \qquad v_2(0) = 0,$$

We have

$$v_2(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{x^7}{63}.$$

Continuing in this way till $n = 6$, but for brevity, they are not listed.

The obtained series solution in Eq. (2.8) can be used to calculate the maximal error remainder to show the highest accuracy level that we can achieve. This can be clearly seen in table 2.1 and figure 2.1

Table 2.1: The maximal error remainder: MER_n by the TAM.

n	MER_n by TAM
1	6.65556×10^{-5}
2	2.65463×10^{-7}
3	7.56708×10^{-10}
4	1.67806×10^{-12}
5	2.97852×10^{-15}
6	1.24033×10^{-16}

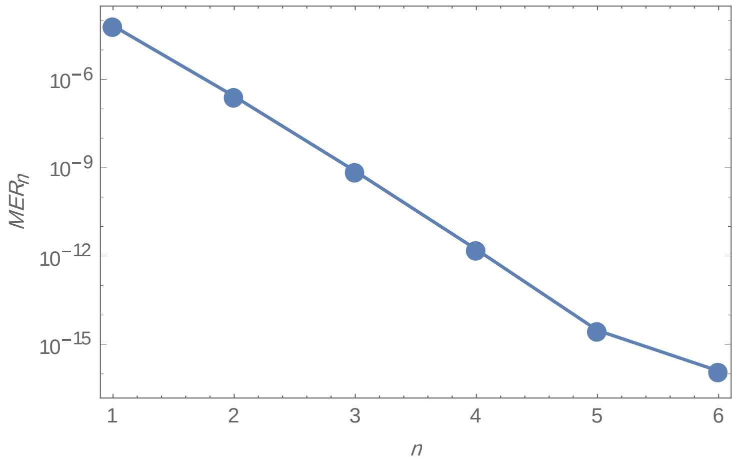

Figure 2.1: Logarithmic plots of MER_nversus n is 1 through 6 by TAM.

2.4.2 TAM for solving linear pantograph equation

In this subsection, the TAM will be applied for second order linear pantograph equation which is used in many applications, such as industrial applications, studies based on biology, economy and others. The following example presents a solution for pantograph equation by using the TAM.

Example 2.3 [11]:

Resolve example 1.3 by using TAM.

$$v'' = \frac{3}{4}v + v\left(\frac{x}{2}\right) - x^2 + 2,$$

with initial conditions $v(0) = 0\,,\quad v'(0) = 0.$ (2.11)

To solve Eq. (2.11) by TAM, distribute the equation as follows

$$L(v) = v'',\;\; N(v) = -\frac{3}{4}v - v\left(\frac{x}{2}\right) \;\text{ and }\; f(x) = x^2 - 2.$$

Thus, the initial problem which needs to be solved is

$$L\big(v_0(x)\big) = -x^2 + 2, \qquad v_0(0) = 0, \quad {v_0}'(0) = 0. \tag{2.12}$$

Integrate both sides of Eq. (2.12) twice from 0 to x and substitute the initial conditions, we get

$$\int_0^x\int_0^x {v_0}''(t)dtdt = \int_0^x\int_0^x (-t^2 + 2)dtdt\,, \quad v_0(0) = 0, \quad {v_0}'(0) = 0.$$

Then, we obtain

$$v_0(x) = x^2 - \frac{x^4}{12}.$$

The second iteration can be carried as

$$L\big(v_1(x)\big) + N\big(v_0(x)\big) + f(x) = 0, \quad v_1(0) = 0, \qquad v_1{}'(0) = 0. \qquad (2.13)$$

By integrating both sides of Eq. (2.13) twice from 0 to x, we obtain

$$\int_0^x \int_0^x v_1''(t)\, dtdt = \int_0^x \int_0^x \left(-t^2 + 2 + \frac{3}{4} v_0(t) + v_0\left(\frac{t}{2}\right)\right) dtdt,$$

with initial conditions $v_1(0) = 0, \qquad v_1{}'(0) = 0.$

Thus, we get

$$v_1(x) = x^2 - \frac{13x^6}{5760}.$$

The next iteration is

$$L\big(v_2(x)\big) + N\big(v_1(x)\big) + f(x) = 0\,, \quad v_2(0) = 0, \qquad v_2{}'(0) = 0.$$

Thus, we have

$$v_2(x) = x^2 - \frac{91x^8}{2949120}\,.$$

The next iteration is

$$L\big(v_3(x)\big) + N\big(v_2(x)\big) + f(x) = 0, \quad v_3(0) = 0, \qquad v_3{}'(0) = 0.$$

Then, we get

$$v_3(x) = x^2 - \frac{17563x^{10}}{67947724800}\,.$$

By continuing in this way, we will get:

$v(x) = x^2 -$ small term,

This series converges to the exact solution [11]:

$$v(x) = \lim_{n \to \infty} v_n(x) = x^2.$$

Suppose that T: $[0,1] \to R$, then $v_n = T(v_{n-1})$ and $0 \le x \le 1$.
Following the same procedure as for the Riccati equation, the exact solution for pantograph equation is $v = v(x) = x^2$, therefore, we have

$$\|v_0 - v\| = \left\| x^2 - \frac{x^4}{12} - (x^2) \right\|,$$

$$\|v_1 - v\| = \left\| x^2 - \frac{13x^6}{5760} - (x^2) \right\| \le k \left\| x^2 - \frac{x^4}{12} - (x^2) \right\| = k\|v_0 - v\|,$$

But, $\forall\, x \,\epsilon\, [0,1]$, $0 < k < 1,$ when $x = \frac{1}{4}$,

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \le k = 0.00169271 < 1,$$

Then, we have

$$\|v_1 - v\| \le k\,\|v_0 - v\|,$$

$$\|v_2 - v\| = \left\| x^2 - \frac{91x^8}{2949120} - (x^2) \right\| \le k\|v_1 - v\| \le kk\|v_0 - v\| = k^2\|v_0 - v\|,$$

Since, $\forall\, x \,\epsilon\, [0,1]$, $0 < k < 1,$ when $x = \frac{1}{4}$,

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \le k^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \le k = 0.00120267 < 1,$$

Thus, we get

$$\|v_2 - v\| \leq k^2\|v_0 - v\|,$$

Similarly, we have

$$\|v_3 - v\| \leq k^3\|v_0 - v\|,$$

By continuing in this way, we get:

$$\|v_n - v\| \leq k^n\|v_0 - v\|,$$

Therefore,

$\lim_{n \to \infty} \|v_n - v\| \leq \lim_{n \to \infty} k^n \|v_0 - v\| = 0$, and $\lim_{n \to \infty} k^n = 0$, then

$\lim_{n \to \infty} \|v_n - v\| = 0$, then $\lim_{n \to \infty} v_n = v = x^2$, which is the exact solution [11].

2.4.3 Solving the nonlinear beam equation by using the TAM

In this subsection, the TAM will be implemented to solve fourth order nonlinear beam deformation problem which is a boundary value problems that are considered one of more important equations because they are widely used in different scientific specializations. These boundary value problems appear in applied mathematics, engineering, several branches of physics and others.

Example 2.4:

Let us recall example 1.4

$$v^{(4)} = v^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,$$

with boundary conditions $v(0) = v'(0) = 0, \quad v(1) = v'(1) = 1.$ (2.14)

By applying the TAM for Eq. (2.14), we get

$$L(v) = v^{(4)}, \qquad N(v) = -v^2,$$

$$f(x) = x^{10} - 4x^9 + 4x^8 + 4x^7 - 8x^6 + 4x^4 - 120x + 48.$$

Thus, the initial problem which needs to be solved is

$$L\big(v_0(x)\big) + f(x) = 0, \;\; v_0(0) = {v_0}'(0) = 0, \; v_0(1) = {v_0}'(1) = 1. \quad (2.15)$$

By integrating both sides of (2.15) from 0 to x four times, one can obtain

$$\int_0^x \int_0^x \int_0^x \int_0^x {v_0}^{(4)}(t)\, dt\, dt\, dt\, dt = \int_0^x \int_0^x \int_0^x \int_0^x -f(t)\, dt\, dt\, dt\, dt,$$

Then, we have

$$v_0(x) = v_0(0) + x v_0'(0) + \frac{x^2}{2} v_0''(0) + \frac{x^3}{6} v_0'''(0) \\ - \int_0^x \int_0^x \int_0^x \int_0^x f(t)\, dt\, dt\, dt\, dt, \quad (2.16)$$

Imposing the boundary conditions (2.15), into Eq. (2.16), and by applying Mathematica's code, see appendix C, the results are as follows.

$$v_0(x) = \frac{718561x^2}{360360} + \frac{4019x^3}{540540} - 2x^4 + x^5 - \frac{x^8}{420} + \frac{x^{10}}{630} - \frac{x^{11}}{1980} - \frac{x^{12}}{2970} \\ + \frac{x^{13}}{4290} - \frac{x^{14}}{24024}.$$

The second iteration can be carried through and we have

$$L\big(v_1(x)\big) + N\big(v_0(x)\big) + f(x) = 0,$$

$$v_1(0) = {v_1}'(0) = 0, \;\; v_1(1) = {v_1}'(1) = 1. \quad (2.17)$$

By integrating both sides of (2.17) from 0 to x four times, one can obtain

$$\int_0^x\int_0^x\int_0^x\int_0^x {v_1}^{(4)}(t)\,dt\,dt\,dt\,dt$$
$$= \int_0^x\int_0^x\int_0^x\int_0^x (v_0^2(t) - f(t))\,dt\,dt\,dt\,dt,$$

Then, we have

$$v_1(x) = v_1(0) + xv_1'(0) + \frac{x^2}{2}v_1''(0) + \frac{x^3}{6}v_1'''(0)$$
$$+ \int_0^x\int_0^x\int_0^x\int_0^x (v_0^2(t) - f(t))\,dt\,dt\,dt\,dt, \tag{2.18}$$

Similarly, by imposing the boundary conditions $(2.17),$ into Eq. (2.18), the results are as follows:

$$v_1(x) = \frac{4720792684282308505367359x^2}{2360410309588661890560000} + \frac{185532483278679268 39x^3}{1180205154794330945280000} - 2x^4 + x^5 - \frac{3107407679x^8}{218163673728000} + \frac{2887896659x^9}{294520959532800} + \frac{7018307521x^{10}}{1472604797664000} - \frac{22553x^{11}}{4281076800} + \frac{4019x^{12}}{3210807600} - \frac{718561x^{14}}{1818030614400} - \frac{4019x^{15}}{3718698984000} + \frac{1799641x^{16}}{4958265312000} - \frac{10117717x^{17}}{85587287385600} - \frac{68671x^{18}}{655004750400} + \frac{127713533x^{19}}{1941434080185600} + \frac{11676571x^{20}}{7550021422944000} - \frac{2086087x^{21}}{186529941037440} + \frac{11827x^{22}}{3321402084000} - \frac{19x^{23}}{49945896000} + \frac{x^{24}}{61856071200} - \frac{x^{25}}{111891780000} - \frac{1153x^{26}}{672022030680000} + \frac{289x^{27}}{112699346760000} - \frac{2857x^{28}}{5522267991240000} - \frac{191x^{30}}{1525197826152000} - \frac{x^{31}}{38914512436800} + \frac{x^{32}}{498105759191040}.$$

The next iteration is

$$L\big(v_2(x)\big) + N\big(v_1(x)\big) + f(x) = 0,$$

with boundary conditions $v_2(0) = {v_2}'(0) = 0, v_2(1) = {v_2}'(1) = 1.$ (2.19)

By solving Eq. (2.19), we get

$$v_2(x) = 1.9999999795270846x^2 + 2.745035293882598 \times 10^{-8}x^3 - 2x^4 + x^5 - 2.817792217763172 \times 10^{-8}x^8 + 2.079400216725163 \times 10^{-8}x^9 + 9.392717549543088 \times 10^{-9}x^{10} + O[x]^{11}. \quad (2.20)$$

In order to check the accuracy of the approximate solution, we calculate the absolute error, where $v(x) = x^5 - 2x^4 + 2x^2$ is the exact solution and $v_n(x)$ is the approximate solution. In table 2.2 the absolute error of TAM with n = 1, 2 is presented.

Table 2.2: Results of the absolute errors for TAM.

$\boldsymbol{x}$	$\lvert \boldsymbol{r_1} \rvert$ **for TAM**	$\lvert \boldsymbol{r_2} \rvert$ **for TAM**
0	0	0
0.1	1.02627×10^{-7}	1.77279×10^{-10}
0.2	3.47659×10^{-7}	5.99375×10^{-10}
0.3	6.41401×10^{-7}	1.1028×10^{-9}
0.4	8.93924×10^{-7}	1.53129×10^{-9}
0.5	1.02777×10^{-6}	1.752×10^{-9}
0.6	9.93141×10^{-7}	1.68284×10^{-9}
0.7	7.86445×10^{-7}	1.32351×10^{-9}
0.8	4.64662×10^{-7}	7.76486×10^{-10}
0.9	1.471×10^{-7}	2.44241×10^{-10}
1.0	2.22045×10^{-16}	1.11022×10^{-16}

It can be seen clearly from table 2.2, that by increasing the iterations, the errors will decreasing.

In order to discuss the convergence for TAM, suppose that $T{:}\ [0,1] \rightarrow R$, then $v_n = T(v_{n-1})$ and $0 \leq x \leq 1$.

The convergence issue can be done as follows:

Since, the exact solution is $v = v(x) = x^5 - 2x^4 + 2x^2$, therefore, we have

$\|v_0 - v\| = \|v_0 - (x^5 - 2x^4 + 2x^2)\|$,

$\|v_1 - v\| = \|v_1 - (x^5 - 2x^4 + 2x^2)\| \leq k\|v_0 - (x^5 - 2x^4 + 2x^2)\|$
$= k\|v_0 - v\|$,

But, $\forall\, x \in [0,1],\ \ 0 < k < 1,$ when $x = \frac{1}{3}$,

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \leq k = 0.99894 < 1,$$

Then, we have

$\|v_1 - v\| \leq k\ \|v_0 - v\|$,

$\|v_2 - v\| = \|v_2 - (x^5 - 2x^4 + 2x^2)\| \leq k\|v_1 - v\| \leq kk\|v_0 - v\| =$
$k^2\|v_0 - v\|$,

Since, $\forall\, x \in [0,1],\ \ 0 < k < 1,$ when $x = \frac{1}{3}$,

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \leq k^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \leq k = 0.99472 < 1,$$

Thus, we get

$\|v_2 - v\| \leq k^2\|v_0 - v\|$,

Similarly, we have

$\|v_3 - v\| \leq k^3 \|v_0 - v\|$,

By continuing in this way, we get

$\|v_n - v\| \leq k^n \|v_0 - v\|$,

Therefore,

$\lim_{n\to\infty} \|v_n - v\| \leq \lim_{n\to\infty} k^n \|v_0 - v\| = 0$, and $\lim_{n\to\infty} k^n = 0$, then

$\lim_{n\to\infty} \|v_n - v\| = 0$, then $\lim_{n\to\infty} v_n = v = x^5 - 2x^4 + 2x^2$, which is the exact solution [1].

2.4.4 Solving the nonlinear Burgers' equation by using TAM

This subsection produces an implementation for the TAM to solve the first order nonlinear Burgers' equation and finding the exact solutions for different types of Burgers' equations in 1D, 2D and (1+3)-D.

Example 2.5:

Resolve example 1.5 by using TAM [33].

$$v_t + vv_x = v_{xx}, \quad \text{with initial condition } v(x,0) = 2x. \tag{2.21}$$

By applying TAM for Eq. (2.21), we get

$$L(v) = v_t, \quad N(v) = vv_x - v_{xx} \quad \text{and} \quad f(x,t) = 0.$$

Thus, the initial problem which needs to be solved is

$$L\big(v_0(x,t)\big) = 0, \qquad v_0(x,0) = 2x. \tag{2.22}$$

By using a simple manipulation, one can solve Eq. (2.22) as follows:

$$\int_0^t v_{0t}(x,t)\,dt = 0, \qquad v_0(x,0) = 2x.$$

Then, we get

$$v_0(x,t) = 2x.$$

The second iteration can be carried through and is given as

$$L\big(v_1(x,t)\big) + N\big(v_0(x,t)\big) + f(x,t) = 0, \qquad v_1(x,0) = 2x. \tag{2.23}$$

By integrating both sides of Eq. (2.23) from 0 to *t*, we get

$$\int_0^t v_{1t}(x,t)\,dt = \int_0^t \big(-v_0(x,t)v_{0x}(x,t) + v_{0xx}(x,t)\big)\,dt\,,\ v_1(x,0) = 2x.$$

Thus,

$$v_1(x) = 2x - 4xt.$$

The next iteration is

$$L\big(v_2(x,t)\big) + N\big(v_1(x,t)\big) + f(x,t) = 0, \quad v_2(x,0) = 2x.$$

Then, we have

$$v_2(x) = 2x - 4xt + 8xt^2 - \frac{16xt^3}{3}.$$

The next iteration is

$$L\big(v_3(x,t)\big) + N\big(v_2(x,t)\big) + f(x,t) = 0, \qquad v_3(x,0) = 2x.$$

Then, we get

$$v_3(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + \frac{64xt^4}{3} - \frac{64xt^5}{3} + \frac{128xt^6}{9} - \frac{256xt^7}{63}.$$

By continuing in this way, we will get series of the form:

$$v(x,t) = \lim_{n \to \infty} v_n(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - \cdots,$$

This series converges to the exact solution [33]:

$$v(x,t) = \frac{2x}{1+2t} = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - 64xt^5 + \cdots.$$

To check the convergence of the proposed method, let us suppose that $T: R \times [0,\frac{1}{2}) \to R^2$, then $v_n = T(v_{n-1})$ and $t \leq \frac{k}{2}$, $0 < k < 1$.

According to the theorem 2.1 for nonlinear mapping *T*, a sufficient condition for convergence of the TAM is strictly contraction *T*, the exact solution is $v = v(x,t) = \frac{2x}{1+2t}$, therefore, we have

$$\|v_0 - v\| = \left\|2x - \frac{2x}{1+2t}\right\|,$$

$$\|v_1 - v\| = \left\|2x - 4xt - \frac{2x}{1+2t}\right\| \leq k\left\|2x - \frac{2x}{1+2t}\right\| = k\|v_0 - v\|,,$$

But, $\forall\, t \in \left[0,\frac{k}{2}\right)$, $0 < k < 1$, when $t = \frac{1}{4}$, $x = 1$,

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \leq k = 0.5 < 1,$$

Then, we get

$\|v_1 - v\| \leq k\ \|v_0 - v\|,$

Also,

$$\|v_2 - v\| = \left\|2x - 4xt + 8xt^2 - \frac{16xt^3}{3} - \frac{2x}{1+2t}\right\| \leq k\|v_1 - v\| \leq kk\|v_0 - v\| = k^2\|v_0 - v\|,$$

Since, $\forall\ t\ \epsilon\ \left[0, \frac{k}{2}\right), 0 < k < 1,$ when $t = \frac{1}{4},\ x = 1,$

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \leq k^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \leq k = 0.353553 < 1,$$

Then, we have

$\|v_2 - v\| \leq k^2\|v_0 - v\|,$

Similarly, we have

$\|v_3 - v\| \leq k^3\|v_0 - v\|,$

By continuing in this way, we get

$\|v_n - v\| \leq k^n\|v_0 - v\|,$

Therefore,

$\lim_{n \to \infty} \|v_n - v\| \leq \lim_{n \to \infty} k^n\ \|v_0 - v\| = 0$, and $\lim_{n \to \infty} k^n = 0$, then

$\lim_{n \to \infty} \|v_n - v\| = 0$, then $\lim_{n \to \infty} v_n = v = \frac{2x}{1+2t}$, which is the exact solution [33].

Example 2.6:

Let us recall example 1.6

$$v_t + vv_x + vv_y = k\left(v_{xx} + v_{yy}\right), \tag{2.24}$$

with initial condition $v(x, y, 0) = sin(2\pi x)cos(2\pi y)$.

Applying the TAM by first distributing the equation as

$$L(v) = v_t, \quad N(v) = vv_x + vv_y - k\left(v_{xx} + v_{yy}\right) \text{ and } f(x, y, t) = 0.$$

Thus, the initial problem which needs to be solved is

$$L\left(v_0(x, y, t)\right) = 0, \qquad v_0(x, y, 0) = sin(2\pi x)cos(2\pi y). \tag{2.25}$$

By using simple manipulation, one can solve Eq. (2.25) as follows:

$$\int_0^t v_{0t}(x, y, t)\, dt = 0, \quad v_0(x, y, 0) = sin(2\pi x)cos(2\pi y),$$

hence, we get

$$v_0(x, y, t) = sin(2\pi x)cos(2\pi y).$$

The second iteration can be carried through and given as

$$L\left(v_1(x, y, t)\right) + N\left(v_0(x, y, t)\right) + f(x, y, t) = 0,$$

with initial condition $v_1(x, y, 0) = sin(2\pi x)cos(2\pi y)$. (2.26)

By integrating both sides of Eq. (2.26) from 0 to t, we get

$$\int_0^t v_{1t}(x,y,t)\,dt$$
$$= \int_0^t \big(-v_0(x,y,t)v_{0x}(x,y,t) - v_0(x,y,t)v_{0y}(x,y,t) + k(v_{0xx}(x,y,t) + v_{0yy}(x,y,t))\big)\,dt,$$

with initial condition $v_1(x,y,0) = sin(2\pi x)cos(2\pi y)$. (2.27)

Thus,

$$v_1(x,y,t) = cos(2\pi y)sin(2\pi x) - \frac{1}{2}\pi t sin(4\pi x) + \frac{1}{2}\pi t sin(4\pi y) - 4\pi^2 tk sin(2\pi x - 2\pi y) - 4\pi^2 tk sin(2\pi x + 2\pi y) - \frac{1}{2}\pi t sin(4\pi x + 4\pi y).$$

The next iteration is

$$L\big(v_2(x,y,t)\big) + N\big(v_1(x,y,t)\big) + f(x,y,t) = 0,$$

with initial condition $v_2(x,y,0) = sin(2\pi x)cos(2\pi y)$. (2.28)

Then, we have

$$v_2(x,y,t) = cos(2\pi y)sin(2\pi x) + \frac{1}{2}\big(-\pi sin(4\pi x) + \pi sin(4\pi y) - 8\pi^2 k sin(2\pi x - 2\pi y) - 8\pi^2 k sin(2\pi x + 2\pi y) - \pi sin(4\pi x + 4\pi y)\big)t + \cdots.$$

By continuing in this process till $n = 4$, we can get

$$v_4(x,y,t) = cos(2\pi y)sin(2\pi x) + \frac{1}{2}(-\pi sin(4\pi x) + \pi sin(4\pi y) - 8\pi^2 ksin(2\pi x - 2\pi y) - 8\pi^2 ksin(2\pi x + 2\pi y) - \pi sin(4\pi x + 4\pi y))t + \cdots. \quad (2.29)$$

Table 2.3 and Figure 2.2 illustrate the convergence of the solution through the use of the maximal error remainder. Clearly seen, the errors will be decreased as the number of iterations will be increased.

$$MER_n = \max_{\substack{0\le x\le 1\\ 0\le y\le 1}} |ER_n(x,y)|, \quad n = 1, \dots, m, \text{ when } k = 0.1,\ t = 0.0001.$$

Table 2.3: The maximal error remainder: MER_n by the TAM, where $n = 1, \dots, 4$.

n	MER_n by TAM
1	5.99981×10^{-3}
2	2.89069×10^{-6}
3	1.12531×10^{-8}
4	4.54397×10^{-11}

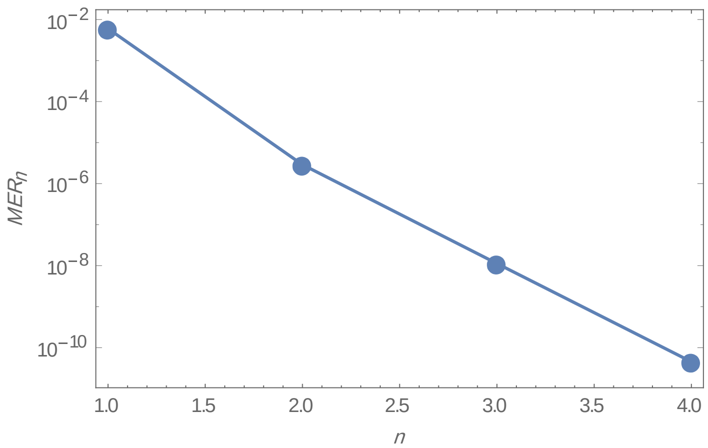

Figure 2.2: Logarithmic plots of MER_nversus n is 1 through 4 by TAM.

Example 2.7 [76]:

Let us consider the following 2D Burgers' equation.

$$v_t = vv_x + v_{xx} + v_{yy}, \text{ with initial condition } v(x,y,0) = x + y. \qquad (2.30)$$

By using the TAM, we get the following:

$$L(v) = v_t, \quad N(v) = -vv_x - v_{xx} - v_{yy} \quad \text{and} \quad f(x,y,t) = 0.$$

Thus, the initial problem which needs to be solved is

$$L\big(v_0(x,y,t)\big) = 0, \qquad v_0(x,y,0) = x + y. \qquad (2.31)$$

By using a simple manipulation, one can solve Eq. (2.31) as follows:

$$\int_0^t v_{0t}(x,y,t)\,dt = 0, \qquad v_0(x,y,0) = x + y.$$

Then, we get

$$v_0(x,y,t) = x + y.$$

The second iteration can be carried through and is given as

$$L\big(v_1(x,y,t)\big) + N\big(v_0(x,y,t)\big) + f(x,y,t) = 0,$$

with initial condition $\quad v_1(x,y,0) = x + y. \qquad (2.32)$

By integrating both sides of Eq. (2.32) from 0 to *t*, we get

$$\int_0^t v_{1t}(x,y,t)\,dt$$
$$= \int_0^t \Big(v_0(x,y,t)v_{0x}(x,y,t) + v_{0xx}(x,y,t) + v_{0yy}(x,y,t)\Big)\,dt,$$

with initial condition $v_1(x,y,0) = x + y$.

Thus,

$$v_1(x,y,t) = x + y + (x+y)t.$$

The next iteration is

$$L\big(v_2(x,y,t)\big) + N\big(v_1(x,y,t)\big) + f(x,y,t) = 0, \quad v_2(x,y,0) = x + y.$$

Then, we have

$$v_2(x,y,t) = (x+y) + (x+y)t + (x+y)t^2 + \frac{1}{3}(x+y)t^3.$$

By continuing in this way, we will get series of the form:

$$v(x,y,t) = \lim_{n\to\infty} v_n(x,y,t) = (x+y) + (x+y)t + (x+y)t^2 + (x+y)t^3 + (x+y)t^4 + \cdots.$$

This series converges to the exact solution [76]:

$$\begin{aligned} v(x,y,t) &= \frac{x+y}{1-t}, \\ &= (x+y) + (x+y)t + (x+y)t^2 + (x+y)t^3 + (x+y)t^4 \\ &\quad + (x+y)t^5 + \cdots. \end{aligned}$$

In order to prove the convergence of TAM for 2D problem, let us assume that $T\colon R^2 \times [0,1) \to R^3$, then $v_n = T(v_{n-1})$, $0 \le t < 1$.

By following similar steps as for example 2.5, since the exact solution is $v = v(x,y,t) = \frac{x+y}{1-t}$, therefore, we have

$$\|v_0 - v\| = \left\|x + y - \frac{x+y}{1-t}\right\|,$$

$$\|v_1 - v\| = \left\| x + y + (x+y)t - \frac{x+y}{1-t} \right\| \le k \left\| x + y - \frac{x+y}{1-t} \right\|$$
$$= k\|v_0 - v\|,$$

But, $\forall\ t \in [0,1)$, when $t = \frac{1}{2}$, $x = 1$, $y = 1$,

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \le k = 0.5 < 1,$$

Then, we get

$$\|v_1 - v\| \le k\, \|v_0 - v\|,$$

Also,

$$\|v_2 - v\| = \left\| (x+y) + (x+y)t + (x+y)t^2 + \frac{1}{3}(x+y)t^3 - \frac{x+y}{1-t} \right\|$$
$$\le k\|v_1 - v\| \le kk\|v_0 - v\| = k^2\|v_0 - v\|,$$

Since, $\forall\ t \in [0,1)$, when $t = \frac{1}{2}$, $x = 1$, $y = 1$,

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \le k^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \le k = 0.456435 < 1,$$

Thus,

$$\|v_2 - v\| \le k^2\|v_0 - v\|,$$

Similarly, we have

$$\|v_3 - v\| \le k^3\|v_0 - v\|,$$

By continuing in this way, we get:

$$\|v_n - v\| \le k^n\|v_0 - v\|,$$

Therefore,

$\lim_{n\to\infty} \|v_n - v\| \leq \lim_{n\to\infty} k^n \|v_0 - v\| = 0$, and $\lim_{n\to\infty} k^n = 0$, then

$\lim_{n\to\infty} \|v_n - v\| = 0$, then $\lim_{n\to\infty} v_n = v = \frac{x+y}{1-t}$, which is the exact solution [76].

Example 2.8 [76]:

Let us consider, the following (1+3)-D Burgers' equation with the following initial condition.

$$v_t = v v_x + v_{xx} + v_{yy} + v_{zz},$$

with initial condition $v(x, y, z, 0) = x + y + z.$ (2.33)

Now, applying the proposed method by first distributing the equation as

$$L(v) = v_t, \qquad N(v) = -v v_x - v_{xx} - v_{yy} - v_{zz} \quad \text{and} \quad f(x, y, z, t) = 0.$$

Thus, the initial problem which needs to be solved is

$$L\big(v_0(x, y, z, t)\big) = 0, \qquad v_0(x, y, z, 0) = x + y + z. \tag{2.34}$$

By using a simple manipulation, one can solve Eq. (2.34) as follows:

$$\int_0^t v_{0t}(x, y, z, t)\, dt = 0, \qquad v_0(x, y, z, 0) = x + y + z.$$

Then, we get

$$v_0(x, y, z, t) = x + y + z.$$

The second iteration can be carried through and is given as

$$L\big(v_1(x, y, z, t)\big) + N\big(v_0(x, y, z, t)\big) + f(x, y, z, t) = 0,$$

with initial condition $v_1(x,y,z,0) = x + y + z$. (2.35)

By integrating both sides of Eq. (2.35) from 0 to t, we get

$$\int_0^t v_{1t}(x,y,z,t)\,dt$$
$$= \int_0^t \Big(v_0(x,y,z,t)v_{0x}(x,y,z,t) + v_{0xx}(x,y,z,t)$$
$$+ v_{0yy}(x,y,z,t) + v_{0zz}(x,y,z,t)\Big)\,dt,$$

with initial condition $v_1(x,y,z,0) = x + y + z$.

Thus,

$$v_1(x,y,z,t) = x + y + z + (x + y + z)t.$$

The next iteration is

$$L\big(v_2(x,y,z,t)\big) + N\big(v_1(x,y,z,t)\big) + f(x,y,z,t) = 0,$$

with initial condition $v_2(x,y,z,0) = x + y + z$.

Then, we have

$$v_2(x,y,z,t)$$
$$= (x + y + z) + (x + y + z)t + (x + y + z)t^2$$
$$+ \frac{1}{3}(x + y + z)t^3.$$

The next iteration is

$$L\big(v_3(x,y,z,t)\big) + N\big(v_2(x,y,z,t)\big) + f(x,y,z,t) = 0,$$

with initial condition $v_3(x,y,z,0) = x + y + z$.

Then, we get

$$v_3(x,y,z,t)$$
$$= (x+y+z) + (x+y+z)t + (x+y+z)t^2 + (x+y+z)t^3$$
$$+\frac{2}{3}(x+y+z)t^4 + \frac{1}{3}(x+y+z)t^5.$$

By continuing in this way, we will get series of the form:

$$v(x,y,z,t) = \lim_{n\to\infty} v_n(x,y,z,t),$$
$$= (x+y+z) + (x+y+z)t + (x+y+z)t^2 + (x+y+z)t^3$$
$$+ (x+y+z)t^4 + (x+y+z)t^5 + \cdots,$$

This series converges to the exact solution [76]:

$$v(x,y,z,t) = \frac{x+y+z}{1-t},$$

$$= (x+y+z) + (x+y+z)t + (x+y+z)t^2 + (x+y+z)t^3$$
$$+ (x+y+z)t^4 + (x+y+z)t^5 + (x+y+z)t^6 + \cdots.$$

For convergence issue, suppose that $T\colon R^3 \times [0,1) \to R^4$, then $v_n = T(v_{n-1})$ and $0 \le t < 1$, similar procedure can be followed, the exact solution is $v = v(x,y,z,t) = \frac{x+y+z}{1-t}$, therefore, we have

$$\|v_0 - v\| = \left\|x+y+z - \frac{x+y+z}{1-t}\right\|,$$

$$\|v_1 - v\| = \left\|x+y+z + (x+y+z)t - \frac{x+y+z}{1-t}\right\|$$
$$\le k\left\|x+y+z - \frac{x+y+z}{1-t}\right\| = k\|v_0 - v\|,$$

But, $\forall\, t \in [0,1)$, when $t = \frac{1}{3},\ x = 1,\ y = 1,\ z = 1,$

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \le k = 0.333333 < 1,$$

Then, we get

$\|v_1 - v\| \le k \, \|v_0 - v\|,$

Moreover,

$$\|v_2 - v\| = \left\| (x+y+z) + (x+y+z)t + (x+y+z)t^2 + \frac{1}{3}(x+y+z)t^3 - \frac{x+y+z}{1-t} \right\| \le k\|v_1 - v\| \le kk\|v_0 - v\| = k^2\|v_0 - v\|,$$

Since, $\forall\, t \in [0,1)$, when $t = \frac{1}{3},\ x = 1,\ y = 1,\ z = 1,$

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \le k^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \le k = 0.293972 < 1,$$

Thus,

$\|v_2 - v\| \le k^2\|v_0 - v\|,$

Similarly, we have

$\|v_3 - v\| \le k^3\|v_0 - v\|,$

By continuing in this way, we get

$\|v_n - v\| \le k^n\|v_0 - v\|,$

Therefore,

$\lim_{n \to \infty} \|v_n - v\| \le \lim_{n \to \infty} k^n \, \|v_0 - v\| = 0$, and $\lim_{n \to \infty} k^n = 0$, then

$\lim_{n\to\infty} \|v_n - v\| = 0$, then $\lim_{n\to\infty} v_n = v = \frac{x+y+z}{1-t}$, which is the exact solution[76].

2.4.5 Solving the nonlinear system of Burgers' equations by the TAM

Systems of partial differential equations have attracted much attention in studying evolution equations describing wave propagation, in investigating the shallow water waves [37, 48], examining the chemical reaction–diffusion model [48]. Moreover, systems of Burgers' equations have a lot of importance in engineering and physical fields, engineering, chemistry, finance and in different fields in science [5, 10, 36, 45, 61]. To solve many researchers have been interested it by various techniques such as the lattice Boltzmann method (LBM) [21], LADM [17], ADM [72] and the HAM [2]. In this section, the TAM will be implemented to solve the Burgers' equation and systems of Burgers' equations in 1D, 2D and 3D.

Example 2.9:

Let us consider, the following system of coupled Burgers' equation [50],

$$\begin{aligned} &u_t - u_{xx} - 2uu_x + (uv)_x = 0, \\ &v_t - v_{xx} - 2vv_x + (uv)_x = 0, \end{aligned} \qquad x \in [-\pi, \pi], \quad t > 0,$$

Subject to the initial conditions:

$$u(x,0) = sin(x), \quad v(x,0) = sin(x). \tag{2.36}$$

To solve the system of Eq. (2.36) by using the TAM, we have

$$L_1(u) = u_t, \quad N_1(u) = -u_{xx} - 2uu_x + (uv)_x \text{ and } g(x,t) = 0,$$

$$L_2(v) = v_t, \quad N_2(v) = -v_{xx} - 2vv_x + (uv)_x \text{ and } f(x,t) = 0.$$

Thus, the initial problem which needs to be solved is

$$L_1\big(u_0(x,t)\big) = 0, \qquad u_0(x,0) = sin(x),$$

$$L_2\big(v_0(x,t)\big) = 0, \qquad v_0(x,0) = sin(x). \tag{2.37}$$

By using simple manipulation, one can solve Eq. (2.37) as follows:

$$\int_0^t u_{0t}(x,t)\,dt = 0, \qquad u_0(x,0) = sin(x),$$

$$\int_0^t v_{0t}(x,t)\,dt = 0, \qquad v_0(x,0) = sin(x).$$

Therefore, we have

$$u_0(x,t) = sin(x),$$

$$v_0(x,t) = sin(x).$$

The second iteration can be carried through and is given as

$$L_1\big(u_1(x,t)\big) + N_1\big(u_0(x,t)\big) + f(x,t) = 0, \quad u_1(x,0) = sin(x),$$

$$L_2\big(v_1(x,t)\big) + N_2\big(v_0(x,t)\big) + g(x,t) = 0, \quad v_1(x,0) = sin(x). \tag{2.38}$$

By integrating both sides of Eq. (2.38) from 0 to *t*, we get

$$\int_0^t u_{1t}(x,t)\,dt$$

$$= \int_0^t \Big[u_{0xx}(x,t) + 2u_0(x,t)u_{0x}(x,t) - \big(u_0(x,t)v_0(x,t)\big)_x\Big]dt, \quad u_1(x,0) = sin(x),$$

$$\int_0^t v_{1t}(x,t)\,dt$$

$$= \int_0^t \Big[v_{0xx}(x,t) + 2v_0(x,t)v_{0x}(x,t)$$

$$-\big(u_0(x,t)v_0(x,t)\big)_x\Big]dt\,, \quad v_1(x,0) = sin(x).$$

Thus, we get

$$u_1(x,t) = sin(x) - tsin(x),$$

$$v_1(x,t) = sin(x) - tsin(x).$$

The next iteration is

$$L_1\big(u_2(x,t)\big) + N_1\big(u_1(x,t)\big) + g(x,t) = 0, \qquad u_2(x,0) = sin(x),$$

$$L_2\big(v_2(x,t)\big) + N_2\big(v_1(x,t)\big) + f(x,t) = 0, \qquad v_2(x,0) = sin(x). \quad (2.39)$$

Once again, by taking the integration to both sides of problem (2.39), we have

$$\int_0^t u_{2t}(x,t)\,dt$$

$$= \int_0^t \Big[u_{1xx}(x,t) + 2u_1(x,t)u_{1x}(x,t)$$

$$-\big(u_1(x,t)v_1(x,t)\big)_x\Big]dt\,, \quad u_2(x,0) = sin(x),$$

$$\int_0^t v_{2t}(x,t)\,dt$$

$$= \int_0^t \Big[v_{1xx}(x,t) + 2v_1(x,t)v_{1x}(x,t)$$

$$-\big(u_1(x,t)v_1(x,t)\big)_x\Big]dt\,, \quad v_2(x,0) = sin(x).$$

Therefore, we get

$$u_2(x,t) = sin(x) - tsin(x) + \frac{1}{2}t^2 sin(x),$$

$$v_2(x,t) = sin(x) - tsin(x) + \frac{1}{2}t^2 sin(x).$$

By continuing in this way, we will get series of the form:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t),$$

$$= sin(x) - tsin(x) + \frac{1}{2}t^2 sin(x) - \frac{1}{6}t^3 sin(x) + \cdots,$$

$$v(x,t) = \lim_{n\to\infty} v_n(x,t),$$

$$= sin(x) - tsin(x) + \frac{1}{2}t^2 sin(x) - \frac{1}{6}t^3 sin(x) + \cdots.$$

This series converges to the exact solution [50],

$$u(x,t) = e^{-t} sin(x),$$

$$v(x,t) = e^{-t} sin(x).$$

The convergence, suppose that $u_n = T(u_{n-1})$ and $v_n = T(v_{n-1})$, $0 \le t < 1$.

$$u = \lim_{n\to\infty} u_n, \qquad v = \lim_{n\to\infty} v_n.$$

Similar procedure can be followed, the exact solution is $u = u(x,t) = e^{-t} sin(x)$, and $v = v(x,t) = e^{-t} sin(x)$, we have

$$\|u_0 - u\| = \|sin(x) - e^{-t} sin(x)\|,$$

$$\|v_0 - v\| = \|sin(x) - e^{-t} sin(x)\|.$$

Also,

$$\|u_1 - u\| = \|sin(x) - tsin(x) - e^{-t}sin(x)\| \leq k_1\|sin(x) - e^{-t}sin(x)\| = k_1\|u_0 - u\|,$$

$$\|v_1 - v\| = \|sin(x) - tsin(x) - e^{-t}sin(x)\| \leq k_2\|\sin(x) - e^{-t}\sin(x)\| = k_2\|v_0 - v\|.$$

But, $\forall\, t \in [0,1)$, when $t = \frac{1}{4},\ x = \frac{\pi}{2}$,

$$\frac{\|u_1 - u\|}{\|u_0 - u\|} \leq k_1 = 0.130203 < 1,$$

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \leq k_2 = 0.130203 < 1.$$

Then, we get

$\|u_1 - u\| \leq k_1\|u_0 - u\|,$

$\|v_1 - v\| \leq k_2\|v_0 - v\|.$

Also,

$$\|u_2 - u\| = \left\|sin(x) - tsin(x) + \frac{1}{2}t^2sin(x) - e^{-t}sin(x)\right\| \leq k_1\|u_1 - u\| \leq k_1k_1\|u_0 - u\| = {k_1}^2\|u_0 - u\|,$$

$$\|v_2 - v\| = \left\|sin(x) - tsin(x) + \frac{1}{2}t^2sin(x) - e^{-t}sin(x)\right\| \leq k_2\|v_1 - v\| \leq k_2k_2\|v_0 - v\| = {k_2}^2\|v_0 - v\|.$$

Since, $\forall\, t \in [0,1)$, when $t = \frac{1}{4},\ x = \frac{\pi}{2}$,

$$\frac{\|u_2 - u\|}{\|u_0 - u\|} \leq {k_1}^2 \Rightarrow \sqrt{\frac{\|u_2 - u\|}{\|u_0 - u\|}} \leq k_1 = 0.105226 < 1,$$

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \le {k_2}^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \le k_2 = 0.105226 < 1.$$

Then, we have

$\|u_2 - u\| \le {k_1}^2 \|u_0 - u\|,$

$\|v_2 - v\| \le {k_2}^2 \|v_0 - v\|.$

Similarly, we have

$\|u_3 - u\| \le {k_1}^3 \|u_0 - u\|,$

$\|v_3 - v\| \le {k_2}^3 \|v_0 - v\|.$

By continuing in this way, we get:

$\|u_n - u\| \le {k_1}^n \|u_0 - u\|,$

$\|v_n - v\| \le {k_2}^n \|v_0 - v\|.$

Therefore,

$\lim\limits_{n \to \infty} \|u_n - u\| \le \lim\limits_{n \to \infty} {k_1}^n \|u_0 - u\| = 0$, and $\lim\limits_{n \to \infty} {k_1}^n = 0$, then

$\lim\limits_{n \to \infty} \|u_n - u\| = 0$, then $\lim\limits_{n \to \infty} u_n = u = e^{-t} sin(x)$, which is the exact solution.

Similarly, we get

$\lim\limits_{n \to \infty} \|v_n - v\| \le \lim\limits_{n \to \infty} {k_2}^n \|v_0 - v\| = 0$, and $\lim\limits_{n \to \infty} {k_2}^n = 0$, then

$\lim_{n\to\infty} \|v_n - v\| = 0$, then $\lim_{n\to\infty} v_n = v = e^{-t}sin(x)$, which is the exact solution [50].

Example 2.10 [30]:

Let us consider, the following system 2D Burgers' equation.

$$u_t + uu_x + vu_y - u_{xx} - u_{yy} = 0,$$

$$v_t + uv_x + vv_y - v_{xx} - v_{yy} = 0,$$

Subject to the initial conditions:

$$u(x,y,0) = x + y, \quad v(x,y,0) = x - y. \tag{2.40}$$

To solve the system of Eq. (2.40) by using the TAM

$$L_1(u) = u_t,\ \ N_1(u) = uu_x + vu_y - u_{xx} - u_{yy} \text{ and } g(x,y,t) = 0,$$

$$L_2(v) = v_t,\ \ N_2(v) = uv_x + vv_y - v_{xx} - v_{yy} \text{ and } f(x,y,t) = 0.$$

Thus, the initial problem which needs to be solved is

$$L_1\big(u_0(x,y,t)\big) = 0, \qquad u_0(x,y,0) = x + y,$$

$$L_2\big(v_0(x,y,t)\big) = 0, \qquad v_0(x,y,0) = x - y. \tag{2.41}$$

By using a simple manipulation, one can solve Eq. (2.41) as follows:

$$\int_0^t u_{0t}(x,y,t)\,dt = 0, \qquad u_0(x,y,0) = x + y,$$

$$\int_0^t v_{0t}(x,y,t)\,dt = 0, \qquad v_0(x,y,0) = x - y.$$

Then, we get

$$u_0(x,y,t) = (x+y),$$

$$v_0(x,y,t) = (x-y).$$

The second iteration can be carried through and is given as

$$L_1\big(u_1(x,y,t)\big) + N_1\big(u_0(x,y,t)\big) + f(x,y,t) = 0, u_1(x,y,0) = x+y,$$

$$L_2\big(v_1(x,y,t)\big) + N_2\big(v_0(x,y,t)\big) + g(x,y,t) = 0, v_1(x,y,0) = x-y. \quad (2.42)$$

By integrating both sides of Eq. (2.42) from 0 to t, we get

$$\int_0^t u_{1t}(x,y,t)\,dt$$
$$= \int_0^t \big[-u_0(x,y,t)u_{0x}(x,y,t) - v_0(x,y,t)u_{0y}(x,y,t)$$
$$+ u_{0xx}(x,y,t) + u_{0yy}(x,y,t)\big]\,dt\,, u_1(x,y,0) = x+y,$$

$$\int_0^t v_{1t}(x,y,t)\,dt$$
$$= \int_0^t \big[-u_0(x,y,t)v_{0x}(x,y,t) - v_0(x,y,t)v_{0y}(x,y,t)$$
$$+ v_{0xx}(x,y,t) + v_{0yy}(x,y,t)\big]\,dt\,, v_1(x,y,0) = x-y.$$

Thus, we have

$$u_1(x,y,t) = (x+y) - 2tx,$$

$$v_1(x,y,t) = (x-y) - 2ty.$$

The next iteration is

$$L_1\big(u_2(x,y,t)\big) + N_1\big(u_1(x,y,t)\big) + g(x,y,t) = 0,$$

$$u_2(x,y,0) = x + y,$$

$$L_2\big(v_2(x,y,t)\big) + N_2\big(v_1(x,y,t)\big) + f(x,y,t) = 0,$$

$$v_2(x,y,0) = x - y. \tag{2.43}$$

Once again, by taking the integration on both sides of problem (2.43), we have

$$\int_0^t u_{2t}(x,y,t)\,dt$$
$$= \int_0^t \big[-u_1(x,y,t)u_{1x}(x,y,t) - v_1(x,y,t)u_{1y}(x,y,t)$$
$$+ u_{1xx}(x,y,t) + u_{1yy}(x,y,t)\big]\,dt\,, \quad u_2(x,y,0) = x + y,$$

$$\int_0^t v_{2t}(x,y,t)\,dt$$
$$= \int_0^t \big[-u_1(x,y,t)v_{1x}(x,y,t) - v_1(x,y,t)v_{1y}(x,y,t)$$
$$+ v_{1xx}(x,y,t) + v_{1yy}(x,y,t)\big]\,dt\,, \quad v_2(x,y,0) = x - y.$$

Therefore, we get

$$u_2(x,y,t) = (x+y) - 2tx + 2(x+y)t^2 - \frac{4t^3x}{3},$$

$$v_2(x,y,t) = (x-y) - 2ty + (2x-2y)t^2 - \frac{4t^3y}{3}.$$

By continuing in this way, we will get a series of the form:

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t),$$
$$= (x+y) - 2xt + 2(x+y)t^2 - 4xt^3 + 4(x+y)t^4 - 8xt^5$$
$$+ 8(x+y)t^6 - 16xt^7 + 16(x+y)t^8 - \cdots,$$

$$v(x,y,t) = \lim_{n\to\infty} v_n(x,y,t),$$
$$= (x-y) - 2yt + (2x-2y)t^2 - 4yt^3 + (4x-4y)t^4 - 8yt^5$$
$$+ (8x-8y)t^6 - 16yt^7 + (16x-16y)t^8 - \cdots.$$

This series converges to the exact solution [30]:

$$u(x,y,t) = \frac{x-2xt+y}{1-2t^2},$$

$$= (x+y) - 2xt + 2(x+y)t^2 - 4xt^3 + (4x+4y)t^4 - 8xt^5$$
$$+ (8x+8y)t^6 - 16xt^7 + (16x+16y)t^8 - 32xt^9 + \cdots,$$

$$v(x,y,t) = \frac{x-2yt-y}{1-2t^2},$$

$$= (x-y) - 2yt + (2x-2y)t^2 - 4yt^3 + (4x-4y)t^4 - 8yt^5$$
$$+ (8x-8y)t^6 - 16yt^7 + (16x-16y)t^8 - 32yt^9 + \cdots.$$

For convergence issue, suppose that $u_n = T(u_{n-1})$, $v_n = T(v_{n-1})$, $0 \le t < 1$,

$$u = \lim_{n\to\infty} u_n, \qquad v = \lim_{n\to\infty} v_n .$$

Since, the exact solution is $u = u(x,y,t) = \frac{x-2xt+y}{1-2t^2}$, and $v = v(x,y,t) = \frac{x-2yt-y}{1-2t^2}$, therefore, we have

$$\|u_0 - u\| = \left\| x + y - \left(\frac{x-2xt+y}{1-2t^2}\right) \right\|,$$

$$\|v_0 - v\| = \left\| x - y - \left(\frac{x-2yt-y}{1-2t^2}\right) \right\|.$$

Also,

$$\|u_1 - u\| = \left\| x + y - 2tx - \left(\frac{x - 2xt + y}{1 - 2t^2}\right) \right\|$$
$$\le k_1 \left\| x + y - \left(\frac{x - 2xt + y}{1 - 2t^2}\right) \right\| = k_1 \|u_0 - u\|,$$

$$\|v_1 - v\| = \left\| x - y - 2ty - \left(\frac{x-2yt-y}{1-2t^2}\right) \right\| \le k_2 \left\| x - y - \left(\frac{x-2yt-y}{1-2t^2}\right) \right\| =$$
$$k_2 \|v_0 - v\|.$$

But, $\forall\ t \in [0,1)$, when $t = \frac{1}{4},\ x = 1,\ y = 1,$

$$\frac{\|u_1 - u\|}{\|u_0 - u\|} \le k_1 = 0.75 < 1,$$

and when $t = \frac{1}{2},\ x = 0,\ y = 1,$

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \le k_2 = 0.666667 < 1.$$

Then, we get

$\|u_1 - u\| \le k_1\ \|u_0 - u\|,$

$\|v_1 - v\| \le k_2\ \|v_0 - v\|.$

Also,

$$\|u_2 - u\| = \left\| x + y - 2tx + 2(x+y)t^2 - \frac{4t^3x}{3} - \left(\frac{x-2xt+y}{1-2t^2}\right) \right\| \le$$
$$k_1 \|u_1 - u\| \le k_1 k_1 \|u_0 - u\| = {k_1}^2 \|u_0 - u\|,$$

$$\|v_2 - v\| = \left\| x - y - 2ty + (2x - 2y)t^2 - \frac{4t^3y}{3} - \left(\frac{x-2yt-y}{1-2t^2}\right) \right\| \le$$
$$k_2 \|v_1 - v\| \le k_2 k_2 \|v_0 - v\| = {k_2}^2 \|v_0 - v\|.$$

Since, $\forall\ t \epsilon\ [0,1)$, when $t = \frac{1}{4},\ x = 1,\ y = 1,$

$$\frac{\|u_2 - u\|}{\|u_0 - u\|} \leq {k_1}^2 \Rightarrow \sqrt{\frac{\|u_2 - u\|}{\|u_0 - u\|}} \leq k_1 = 0.228218 < 1,$$

and when $t = \frac{1}{2},\ x = 0,\ y = 1,$

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \leq {k_2}^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \leq k_2 = 0.625485 < 1.$$

Thus,

$\|u_2 - u\| \leq {k_1}^2 \|u_0 - u\|,$

$\|v_2 - v\| \leq {k_2}^2 \|v_0 - v\|.$

Similarly, we have

$\|u_3 - u\| \leq {k_1}^3 \|u_0 - u\|,$

$\|v_3 - v\| \leq {k_2}^3 \|v_0 - v\|.$

By continuing in this way, we have

$\|u_n - u\| \leq {k_1}^n \|u_0 - u\|,$

$\|v_n - v\| \leq {k_2}^n \|v_0 - v\|.$

Therefore,

$\lim\limits_{n \to \infty} \|u_n - u\| \leq \lim\limits_{n \to \infty} {k_1}^n \|u_0 - u\| = 0$, and $\lim\limits_{n \to \infty} {k_1}^n = 0$, then

$\lim\limits_{n \to \infty} \|u_n - u\| = 0$, then $\lim\limits_{n \to \infty} u_n = u = \frac{x - 2xt + y}{1 - 2t^2}$, which is the exact solution,

Similarly, we have

$\lim_{n\to\infty} \|v_n - v\| \leq \lim_{n\to\infty} {k_2}^n \|v_0 - v\| = 0$, and $\lim_{n\to\infty} {k_2}^n = 0$, then

$\lim_{n\to\infty} \|v_n - v\| = 0$, then $\lim_{n\to\infty} v_n = v = \frac{x-2yt-y}{1-2t^2}$, which is the exact solution [30].

Example 2.11:

Let us consider, the following system of 3D Burgers' equation,

$$\begin{aligned}
&u_t - uu_x - vu_y - wu_z - u_{xx} - u_{yy} - u_{zz} = g(x,y,z,t),\\
&v_t + uv_x + vv_y + wv_z + v_{xx} + v_{yy} + v_{zz} = f(x,y,z,t),\\
&w_t + uw_x + vw_y + ww_z + w_{xx} + w_{yy} + w_{zz} = h(x,y,z,t),
\end{aligned}$$

$$g(x,y,z,t) = e^t(x+y+z) - e^t\big(1 + e^t(x+y+z)\big),$$

$$f(x,y,z,t) = -e^t(x+y+z) - e^t(1 + e^t(x+y+z)),$$

$$h(x,y,z,t) = e^t(x+y+z) + e^t(1 + e^t(x+y+z)), \qquad (2.44)$$

Subject to the initial conditions:

$$u(x,y,z,0) = x+y+z,$$

$$v(x,y,z,0) = -(x+y+z),$$

$$w(x,y,z,0) = 1 + x + y + z.$$

To solve the system of Eq. (2.44) by using the TAM, we have

$$L_1(u) = u_t, \quad N_1(u) = -uu_x - vu_y - wu_z - u_{xx} - u_{yy} - u_{zz},$$

$$-\,g(x,y,z,t) = -e^t(x+y+z) + e^t\big(1 + e^t(x+y+z)\big),$$

$$L_2(v) = v_t, \quad N_2(v) = uv_x + vv_y + wv_z + v_{xx} + v_{yy} + v_{zz},$$

$$-f(x,y,z,t) = e^t(x+y+z) + e^t(1+e^t(x+y+z)),$$

$$L_3(w) = w_t, \quad N_3(w) = uw_x + vw_y + ww_z + w_{xx} + w_{yy} + w_{zz},$$

$$-h(x,y,z,t) = -e^t(x+y+z) - e^t(1+e^t(x+y+z)).$$

Thus, the initial problem which needs to be solved is

$$L_1\big(u_0(x,y,z,t)\big) = e^t(x+y+z) - e^t\big(1+e^t(x+y+z)\big),$$

$$u_0(x,y,z,0) = x+y+z,$$

$$L_2\big(v_0(x,y,z,t)\big) = -e^t(x+y+z) - e^t\big(1+e^t(x+y+z)\big),$$

$$v_0(x,y,z,0) = -(x+y+z),$$

$$L_3\big(w_0(x,y,z,t)\big) = e^t(x+y+z) + e^t\big(1+e^t(x+y+z)\big),$$

$$w_0(x,y,z,0) = 1+x+y+z. \tag{2.45}$$

By using simple manipulation, one can solve Eq. (2.45) as follows:

$$\int_0^t u_{0t}(x,y,z,t)\,dt = \int_0^t \big[e^t(x+y+z) - e^t\big(1+e^t(x+y+z)\big)\big]\,dt,$$

$$u_0(x,y,z,0) = x+y+z,$$

$$\int_0^t v_{0t}(x,y,z,t)\,dt = \int_0^t \big[-e^t(x+y+z) - e^t\big(1+e^t(x+y+z)\big)\big]\,dt,$$

$$v_0(x,y,z,0) = -(x+y+z),$$

$$\int_0^t w_{0t}(x,y,z,t)\,dt = \int_0^t [e^t(x+y+z) + e^t(1+e^t(x+y+z))]\,dt,$$

$$w_0(x,y,z,0) = 1 + x + y + z.$$

Then, we get

$$u_0(x,y,z,t)$$
$$= 1 - e^t + \frac{x}{2} + e^t x - \frac{1}{2}e^{2t}x + \frac{y}{2} + e^t y - \frac{1}{2}e^{2t}y + \frac{z}{2} + e^t z$$
$$- \frac{1}{2}e^{2t}z,$$

$$v_0(x,y,z,t)$$
$$= 1 - e^t + \frac{x}{2} - e^t x - \frac{1}{2}e^{2t}x + \frac{y}{2} - e^t y - \frac{1}{2}e^{2t}y + \frac{z}{2} - e^t z$$
$$- \frac{1}{2}e^{2t}z,$$

$$w_0(x,y,z,t)$$
$$= e^t - \frac{x}{2} + e^t x + \frac{1}{2}e^{2t}x - \frac{y}{2} + e^t y + \frac{1}{2}e^{2t}y - \frac{z}{2} + e^t z$$
$$+ \frac{1}{2}e^{2t}z.$$

We turn the function $u_0(x,y,z,t)$, $v_0(x,y,z,t)$ and $w_0(x,y,z,t)$ by using Taylor series expansion to exponential function, we get

$$u_0(x,y,z,t)$$
$$= (x+y+z) - t + \frac{1}{2}(-1-x-y-z)t^2 + \frac{1}{6}(-1-3x-3y$$
$$-3z)t^3 + \frac{1}{24}(-1-7x-7y-7z)t^4 + \frac{1}{120}(-1-15x-15y$$
$$-15z)t^5 + O[t]^6,$$

$$
\begin{aligned}
v_0(x,y,z,t) \\
&= (-x-y-z) + (-1-2x-2y-2z)t + \frac{1}{2}(-1-3x-3y \\
&- 3z)t^2 + \frac{1}{6}(-1-5x-5y-5z)t^3 + \frac{1}{24}(-1-9x-9y \\
&- 9z)t^4 + \frac{1}{120}(-1-17x-17y-17z)t^5 + O[t]^6,
\end{aligned}
$$

$$
\begin{aligned}
w_0(x,y,z,t) \\
&= (1+x+y+z) + (1+2x+2y+2z)t \\
&+ \frac{1}{2}(1+3x+3y+3z)t^2 + \frac{1}{6}(1+5x+5y+5z)t^3 \\
&+ \frac{1}{24}(1+9x+9y+9z)t^4 + \frac{1}{120}(1+17x+17y+17z)t^5 \\
&+ O[t]^6.
\end{aligned}
$$

The second iteration can be carried through and is given as

$$
\begin{aligned}
&L_1\big(u_1(x,y,z,t)\big) + N_1\big(u_0(x,y,z,t)\big) + g(x,y,z,t) = 0, \\
&\quad u_1(x,y,z,0) = x+y+z, \\
&L_2\big(v_1(x,y,z,t)\big) + N_2\big(v_0(x,y,z,t)\big) + f(x,y,z,t) = 0, \\
&\quad v_1(x,y,z,0) = -(x+y+z), \\
&L_3\big(w_1(x,y,z,t)\big) + N_3\big(w_0(x,y,z,t)\big) + h(x,y,z,t) = 0, \\
&\quad w_1(x,y,z,0) = 1+x+y+z. \qquad (2.46)
\end{aligned}
$$

By integrating both sides of Eq. (2.46) from 0 to *t*, we turn the function $u_1(x,y,z,t)$, $v_1(x,y,z,t)$ and $w_1(x,y,z,t)$ by using Taylor series expansion to exponential function, we get

$$u_1(x,y,z,t) = (x+y+z)+(x+y+z)t+\frac{1}{2}(-2-x-y-z)t^2+\frac{1}{6}(-3 - 5x-5y-5z)t^3+\frac{1}{24}(-2-13x-13y-13z)t^4+\frac{1}{120}(9 - 23x-23y-23z)t^5+O[t]^6,$$

$$v_1(x,y,z,t) = (-x-y-z)+(-1-2x-2y-2z)t+\frac{1}{2}(-1-3x-3y - 3z)t^2+\frac{1}{6}(-1-5x-5y-5z)t^3+\frac{1}{24}(-1-9x-9y - 9z)t^4+\frac{1}{120}(-1-17x-17y-17z)t^5+O[t]^6,$$

$$w_1(x,y,z,t) = (1+x+y+z)+(1+2x+2y+2z)t+\frac{1}{2}(1+3x+3y + 3z)t^2+\frac{1}{6}(1+5x+5y+5z)t^3+\frac{1}{24}(1+9x+9y+9z)t^4 + \frac{1}{120}(1+17x+17y+17z)t^5+O[t]^6.$$

The next iteration is

$$L_1\big(u_2(x,y,z,t)\big)+N_1\big(u_1(x,y,z,t)\big)+g(x,y,z,t)=0,$$

$$u_2(x,y,z,0)=x+y+z,$$

$$L_2\big(v_2(x,y,z,t)\big)+N_2\big(v_1(x,y,z,t)\big)+f(x,y,z,t)=0,$$

$$v_2(x,y,z,0)=-(x+y+z),$$

$$L_3\big(w_2(x,y,z,t)\big)+N_3\big(w_1(x,y,z,t)\big)+h(x,y,z,t)=0,$$

$$w_2(x,y,z,0) = 1 + x + y + z.$$

Similarly, we have

$$u_2(x,y,z,t) = (x+y+z) + (x+y+z)t + \frac{1}{2}(x+y+z)t^2 + \frac{1}{6}(-4-3x - 3y - 3z)t^3 + \frac{1}{24}(-15-23x-23y-23z)t^4 + \frac{1}{120}(-16 - 75x - 75y - 75z)t^5 + O[t]^6,$$

$$v_2(x,y,z,t) = (-x-y-z) + (-x-y-z)t + \frac{1}{2}(-x-y-z)t^2 + \frac{1}{6}(-2 - 3x - 3y - 3z)t^3 + \frac{1}{24}(-7-11x-11y-11z)t^4 + \frac{1}{120}(-14-31x-31y-31z)t^5 + O[t]^6,$$

$$w_2(x,y,z,t) = (1+x+y+z) + (x+y+z)t + \frac{1}{2}(x+y+z)t^2 + \frac{1}{6}(2 + 3x + 3y + 3z)t^3 + \frac{1}{24}(7+11x+11y+11z)t^4 + \frac{1}{120}(14 + 31x + 31y + 31z)t^5 + O[t]^6.$$

By continuing in this way, we will get series of the form:

$$u(x,y,z,t) = \lim_{n\to\infty} u_n(x,y,z,t),$$

$$= (x+y+z) + (x+y+z)t + \frac{1}{2}(x+y+z)t^2 + \frac{1}{6}(x+y+z)t^3 + \cdots,$$

$$v(x,y,z,t) = \lim_{n\to\infty} v_n(x,y,z,t),$$

$$= (-x-y-z) + (-x-y-z)t + \frac{1}{2}(-x-y-z)t^2$$

$$+\frac{1}{6}(-x-y-z)t^3 + \cdots,$$

$$w(x,y,z,t) = \lim_{n\to\infty} w_n(x,y,z,t),$$

$$= (1+x+y+z) + (x+y+z)t + \frac{1}{2}(x+y+z)t^2$$

$$+\frac{1}{6}(x+y+z)t^3 + \cdots.$$

This series converges to the exact solution:

$$u(x,y,z,t) = e^t(x+y+z),$$

$$= (x+y+z) + (x+y+z)t + \frac{1}{2}(x+y+z)t^2$$

$$+\frac{1}{6}(x+y+z)t^3 + \frac{1}{24}(x+y+z)t^4 + \cdots,$$

$$v(x,y,z,t) = -e^t(x+y+z),$$

$$= (-x-y-z) + (-x-y-z)t + \frac{1}{2}(-x-y-z)t^2$$

$$+\frac{1}{6}(-x-y-z)t^3 + \frac{1}{24}(-x-y-z)t^4 + \cdots,$$

$$w(x,y,z,t) = 1 + e^t(x+y+z),$$

$$= (1+x+y+z) + (x+y+z)t + \frac{1}{2}(x+y+z)t^2$$

$$+\frac{1}{6}(x+y+z)t^3 + \frac{1}{24}(x+y+z)t^4 + \cdots.$$

Finally, the convergence can be done in the following steps, suppose that $u_n = T(u_{n-1}),\ v_n = T(v_{n-1})$ and $w_n = T(w_{n-1}),\ 0 \le t \le 1.$

$$u = \lim_{n\to\infty} u_n\,, \qquad v = \lim_{n\to\infty} v_n\,, \qquad w = \lim_{n\to\infty} w_n\,.$$

Since, the exact solution is

$$u = u(x,y,z,t) = e^t(x+y+z),$$

$$v = v(x,y,z,t) = -e^t(x+y+z),$$

$$w = w(x,y,z,t) = 1 + e^t(x+y+z).$$

Therefore, we have

$$\|u_0 - u\| = \|u_0 - e^t(x+y+z)\|,$$

$$\|v_0 - v\| = \|v_0 + e^t(x+y+z)\|,$$

$$\|w_0 - w\| = \|w_0 - (1 + e^t(x+y+z))\|.$$

Also,

$$\|u_1 - u\| = \|u_1 - e^t(x+y+z)\| \le k_1\|u_0 - e^t(x+y+z)\|$$
$$= k_1\|u_0 - u\|,$$

$$\|v_1 - v\| = \|v_1 + e^t(x+y+z)\| \le k_2\|v_0 + e^t(x+y+z)\|$$
$$= k_2\|v_0 - v\|,$$

$$\|w_1 - w\| = \left\|w_1 - \left(1 + e^t(x+y+z)\right)\right\|$$
$$\le k_3\left\|w_0 - \left(1 + e^t(x+y+z)\right)\right\| = k_3\|w_0 - w\|\,.$$

But, $\forall\, t \in [0,1],$ when $t = 1,\ x = 1,\ y = 1,\ z = 1,$

$$\frac{\|u_1 - u\|}{\|u_0 - u\|} \le k_1 = 0.82038 < 1,$$

$$\frac{\|v_1 - v\|}{\|v_0 - v\|} \leq k_2 = 0.673461 < 1,$$

$$\frac{\|w_1 - w\|}{\|w_0 - w\|} \leq k_3 = 0.673461 < 1.$$

Then, we get

$\|u_1 - u\| \leq k_1\|u_0 - u\|,$

$\|v_1 - v\| \leq k_2\|v_0 - v\|,$

$\|w_1 - w\| \leq k_3\|w_0 - w\|.$

Also,

$$\begin{aligned}\|u_2 - u\| &= \|u_2 - e^t(x+y+z)\| \leq k_1\|u_1 - u\| \leq k_1k_1\|u_0 - u\| \\ &= {k_1}^2\|u_0 - u\|,\end{aligned}$$

$$\begin{aligned}\|v_2 - v\| &= \|v_2 + e^t(x+y+z)\| \leq k_2\|v_1 - v\| \leq k_2k_2\|v_0 - v\| \\ &= {k_2}^2\|v_0 - v\|,\end{aligned}$$

$$\begin{aligned}\|w_2 - w\| &= \left\|w_2 - \left(1 + e^t(x+y+z)\right)\right\| \leq k_3\|w_1 - w\| \leq k_3k_3\|w_0 - w\| \\ &= {k_3}^2\|w_0 - w\|.\end{aligned}$$

Since, $\forall\ t \epsilon [0,1]$, when $t = 1,\ x = 1,\ y = 1,\ z = 1,$

$$\frac{\|u_2 - u\|}{\|u_0 - u\|} \leq {k_1}^2 \Rightarrow \sqrt{\frac{\|u_2 - u\|}{\|u_0 - u\|}} \leq k_1 = 0.807042 < 1,$$

$$\frac{\|v_2 - v\|}{\|v_0 - v\|} \leq {k_2}^2 \Rightarrow \sqrt{\frac{\|v_2 - v\|}{\|v_0 - v\|}} \leq k_2 = 0.672965 < 1,$$

$$\frac{\|w_2 - w\|}{\|w_0 - w\|} \le {k_3}^2 \Rightarrow \sqrt{\frac{\|w_2 - w\|}{\|w_0 - w\|}} \le k_3 = 0.672965 < 1.$$

Then, we get

$\|u_2 - u\| \le {k_1}^2 \|u_0 - u\|,$

$\|v_2 - v\| \le {k_2}^2 \|v_0 - v\|,$

$\|w_2 - w\| \le {k_3}^2 \|w_0 - w\|.$

Similarly, we have

$\|u_3 - u\| \le {k_1}^3 \|u_0 - u\|,$

$\|v_3 - v\| \le {k_2}^3 \|v_0 - v\|,$

$\|w_3 - w\| \le {k_3}^3 \|w_0 - w\|.$

By continuing in this way, we get:

$\|u_n - u\| \le {k_1}^n \|u_0 - u\|,$

$\|v_n - v\| \le {k_2}^n \|v_0 - v\|,$

$\|w_n - w\| \le {k_3}^n \|w_0 - w\|.$

Therefore,

$\lim\limits_{n \to \infty} \|u_n - u\| \le \lim\limits_{n \to \infty} {k_1}^n \|u_0 - u\| = 0$, and $\lim\limits_{n \to \infty} {k_1}^n = 0$, then

$\lim\limits_{n \to \infty} \|u_n - u\| = 0$, then $\lim\limits_{n \to \infty} u_n = u = e^t(x + y + z)$, which is the exact solution,

Similarly, we have

$\lim\limits_{n\to\infty} \|v_n - v\| \leq \lim\limits_{n\to\infty} {k_2}^n \|v_0 - v\| = 0$, and $\lim\limits_{n\to\infty} {k_2}^n = 0$, then

$\lim\limits_{n\to\infty} \|v_n - v\| = 0$, then $\lim\limits_{n\to\infty} v_n = v = -e^t(x+y+z)$, which is the exact solution,

Also,

$\lim\limits_{n\to\infty} \|w_n - w\| \leq \lim\limits_{n\to\infty} {k_3}^n \|w_0 - w\| = 0$, and $\lim\limits_{n\to\infty} {k_3}^n = 0$, then

$\lim\limits_{n\to\infty} \|w_n - w\| = 0$, then $\lim\limits_{n\to\infty} w_n = w = 1 + e^t(x+y+z)$, which is the exact solution.

CHAPTER 3

BANACH CONTRACTION METHOD FOR SOLVING SOME ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Chapter 3

Banach Contraction Method for Solving Some Ordinary and Partial Differential Equations

3.1 Introduction

Various analytic and semi-analytic iterative methods have been used to solve many variant problems in different fields of the applied sciences have significantly improved the iterative techniques. One of these methods is Banach contraction method (BCM) based on using Banach contraction to provide the required solution for different nonlinear kinds of functional equations [75]. The BCM characterized as one of the developments of Picard’s method where the ease of application clearly observed which makes it distinct from the other known iterative methods.

In this chapter, the BCM will be used to solve some important problems in physics and engineering which already solved by TAM in chapter two. The first part of this chapter will be presented the analytic solutions for Riccati, pantograph, and beam deformation equations. Moreover, the BCM will be used to solve 1D, 2D and 3D of Burgers’ equations and systems of equations. The BCM does not required any additional restrictions to deal with nonlinear terms.

The work in this chapter is organized as follows: In section two, the basic idea of the BCM will be presented. Section three discusses the application of the BCM for the Riccati, pantograph, and beam deformation equations. In section four, the BCM will be applied to solve Burgers’ equations and system of equations in 1D, 2D and 3D.

3.2 Contractions: Definition and Example

Definition 3.1: [3]

Let (X, d) be a metric space. Then a mapping $f: X \to X$,

(a) A point $x \in X$ is called a fixed point of f if $x = f(x)$.

(b) f is called contraction if there exists a fixed constant $k < 1$ such that

$$d\big(f(x), f(y)\big) \leq kd(x, y), \qquad \text{for all} \quad x, y \text{ in } X.$$

A contraction mapping is also known as Banach contraction.

Example 3.1 [3]:

(a) Consider the usual metric space (R, d). Define

$$f(x) = \frac{x}{a} + b, \qquad \text{for all } x \in R.$$

Then, f is contraction on R if $a > 1$ and the solution of the equation,

$x - f(x) = 0$ is $x = \frac{ab}{a-1}$.

(b) Consider the Euclidean metric space (R^2, d). Define

$$f(x, y) = (\frac{x}{a} + b, \frac{y}{c} + b), \quad \text{for all } (x, y) \in R^2.$$

Then, f is contraction on R^2 if $a, c > 1$. Now, solving the equation,

$f(x, y) = (x, y)$ for a fixed point, we get is $x = \frac{ab}{a-1}$ and $y = \frac{cb}{c-1}$.

Using induction, one can easily get the following concerning iterates of a contraction mapping.

If f is a contraction mapping on a metric space (X,d) with contraction constant k, then f^n (where the superscript represents the nth iterate of f) is also a contraction on X with constant k^n.

3.3 Banach Contraction Principle

In 1922 Banach published his fixed point theorem also known as Banach's Contraction Principle uses the concept of Lipschitz mapping. Also, it has long been used in analysis to solve various kinds of differential and integral equations [67].

Theorem 3.1 (Banach Contraction Principle): [75]

Let (X,d) be a complete metric space and $f: X \to X$ be a contraction mapping. Then f has a unique fixed point x_0 and for each $x \in X$, we have

$$\lim_{n\to\infty} f^n(x) = x_0,$$

Moreover for each $x \in X$, we have

$$d(f^n(x), x_0) \leq \frac{k^n}{1-k} d(f(x), x).$$

3.3.1 Solution of differential equations using Banach contraction principle

Let $f(x,v)$ be a continuous real-valued function on $[a,b]\times[c,d]$. The Cauchy initial value problem is to find a continuous differentiable function v on $[a,b]$ satisfying the differential equation [67],

$$v'(x) = f\big(x, v(x)\big), \qquad v(x_0) = v_0\,. \tag{3.1}$$

Consider the Banach space $C[a,b]$ of continuous real-valued functions with supremum norm defined by

$$\|v\| = \sup\{\, |v(x)| : x \in [a,b]\}.$$

Integrate both sides of Eq. (3.1), we obtain an integral equation

$$v(x) = v_0 + \int_{x_0}^{x} f\big(t, v(t)\big)\, dt. \tag{3.2}$$

The problem (3.1) is equivalent the problem solving the integral equation (3.2). We define an integral operator $F: C[a,b] \to C[a,b]$ by

$$Fv(x) = v_0 + \int_{x_0}^{x} f\big(t, v(t)\big)\, dt.$$

Thus, a solution of Cauchy initial value problem (3.1) corresponds with a fixed point of F. One may easily check that if F is contraction, then the problem (3.1) has a unique solution.

Now our purpose is to impose certain conditions on f under which the integral operator F is Lipschitz with $k < 1$.

Theorem 3.2: [67]

Let $f(x,v)$ be a continuous function of $Dom(f) = [a,b] \times [c,d]$ such that f is Lipschitz with respect to v, that is there exists $k > 0$ such that

$$|f(x,v) - f(x,u)| \le k|v-u|, \text{ for all } v,u \in [c,d] \text{ and for } x \in [a,b].$$

Suppose $(x_0, v_0) \in int(Dom(f))$. Then for sufficiently small $h > o$, there exists a unique solution of the problem (3.1).

3.4 The basic idea of the BCM

Let us consider the general functional equation [75],

$$v = N(v) + f \tag{3.3}$$

where $N(v)$ is a nonlinear operator and f is a known function.

Define successive approximations as

$$v_0 = f,$$

$$v_1 = v_0 + N(v_0),$$

$$v_2 = v_0 + N(v_1),$$

$$v_3 = v_0 + N(v_2),$$

$$\vdots$$

By continuing, we have

$$v_n = v_0 + N(v_{n-1}), \qquad n = 1,2,3, \dots . \tag{3.4}$$

If N^k is contraction for operator some positive integer k, then $N(v)$ has a unique fixed-point and hence the sequence defined by Eq. (3.4) is convergent in view of theorem 1.2, and the solution of Eq. (3.3) is given by

$$v = \lim_{n \to \infty} v_n \tag{3.5}$$

3.5 The BCM to solve Riccati, pantograph and beam deformation equations

In this section, three types of non-linear equations namely Riccati equation, pantograph equation, and beam deformation equation will be solved by the BCM.

3.5.1 Exact and numerical solutions for nonlinear Riccati equation by BCM

The BCM will be implemented to solve the Riccati differential equation which is a nonlinear ODE of first order, the following examples will be solved.

Example 3.2:

Let us recall example 2.1:

$$v' = e^{x} - e^{3x} + 2e^{2x}v - e^{x}v^{2}, \text{ with initial condition } v(0) = 1. \qquad (3.6)$$

Eq. (3.6) can be converted to the following Voltera integral equation

$$v(x) = f(x) + N(v), \qquad (3.7)$$

where

$$f(x) = \frac{1}{3} + e^{x} - \frac{e^{3x}}{3} \quad \text{and} \quad N(v) = \int_0^x (2e^{2t}v - e^{t}v^{2})\, dt.$$

By implementing the BCM for Eq. (3.7), we have

$$v_n(x) = v_0(x) + \int_0^x (2e^{2t}v_{n-1}(t) - e^{t}v_{n-1}^{2}(t))\, dt,\ n = 1,2, \ldots .$$

Then, we get

$$v_0(x) = f(x) = \frac{1}{3} + e^x - \frac{e^{3x}}{3},$$

$$v_1(x) = v_0(x) + \int_0^x (2e^{2t}v_0(t) - e^t v_0^2(t))\,dt,$$

$$= \frac{1}{14} + \frac{8e^x}{9} + \frac{e^{4x}}{18} - \frac{e^{7x}}{63},$$

The function $v_1(x)$ can be written by using Taylor series expansion,

$$v_1(x) = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{23x^4}{24} - \frac{209x^5}{120} - \frac{1639x^6}{720} - \frac{12161x^7}{5040} - \frac{87863x^8}{40320} - \frac{625969x^9}{362880} - \frac{4425479x^{10}}{3628800} + O[x]^{11},$$

$$v_2(x) = v_0(x) + \int_0^x (2e^{2t}v_1(t) - e^t v_1^2(t))\,dt,$$

$$= \frac{11}{9720} + \frac{195e^x}{196} + \frac{e^{2x}}{126} - \frac{e^{3x}}{243} - \frac{e^{5x}}{630} + \frac{e^{6x}}{486} + \frac{e^{8x}}{3528} - \frac{5e^{9x}}{6804} + \frac{e^{12x}}{6804} - \frac{e^{15x}}{59535},$$

Similarly, by using Taylor series expansion, we have

$$v_2(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{79x^7}{5040} - \frac{3919x^8}{40320} - \frac{116479x^9}{362880} - \frac{2735039x^{10}}{3628800} + O[x]^{11},$$

Continuing in this way, we get a series of the form:

$$v(x) = \lim_{n\to\infty} v_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots, \quad (3.8)$$

This series converges to the following exact solution [19]:

$$v(x) = e^x.$$

Example 3.3 [74]:

Rewrite the example 1.1 and solve it by BCM.

$$v' = -v^2 + 1, \quad \text{with initial condition} \quad v(0) = 0. \tag{3.9}$$

Eq. (3.9) can be converted to the following Voltera integral equation

$$v(x) = f(x) + N(v), \tag{3.10}$$

where

$$f(x) = x \;\; \text{and} \;\; N(v) = -\int_0^x (v^2)\, dt.$$

By applying the BCM for Eq. (3.10), we have

$$v_n(x) = v_0(x) - \int_0^x \left(v_{n-1}^2(t)\right) dt,\; n = 1,2, \dots .$$

Thus, we have

$$v_0(x) = f(x) = x,$$

$$v_1(x) = v_0(x) - \int_0^x \left(v_0^2(t)\right) dt = x - \frac{x^3}{3},$$

$$v_2(x) = v_0(x) - \int_0^x \left(v_1^2(t)\right) dt = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{x^7}{63},$$

$$v_3(x) = v_0(x) - \int_0^x \left(v_2^2(t)\right) dt,$$

$$= x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{38x^9}{2835} - \frac{134x^{11}}{51975} + \frac{4x^{13}}{12285} - \frac{x^{15}}{59535},$$

By continuing in this manner till $n = 6$, we have

$$v_6(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} - \frac{1382x^{11}}{155925} + \frac{21844x^{13}}{6081075} - \frac{909409x^{15}}{638512875} + \frac{543442x^{17}}{986792625} - \frac{77145154x^{19}}{371231385525} + \cdots, \quad (3.11)$$

The obtained series solution in Eq. (3.9) can be used to calculate the maximal error remainder for show the highest accuracy level that we can achieve. This can be clearly seen in table 3.1 and figure 3.1

Table 3.1: The maximal error remainder: MER_n by the BCM,

where $n = 1, \ldots, 6$.

n	MER_n by BCM
1	6.65556×10^{-5}
2	2.65463×10^{-7}
3	7.56708×10^{-10}
4	1.67808×10^{-12}
5	3.04791×10^{-15}
6	3.46945×10^{-18}

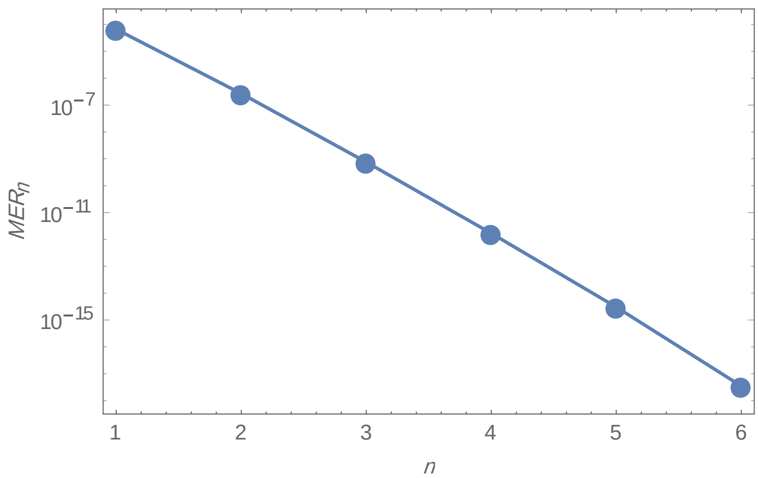

Figure 3.1: Logarithmic plots of MER_nversus n is 1 through 6 by BCM.

3.5.2 BCM for solving linear pantograph equation

In this subsection, the BCM will be applied for second order linear pantograph equation of order two. The following example presents a solution for pantograph equation by using the BCM.

Example 3.4:

Resolve example 1.3 by using BCM [11].

$$v'' = \frac{3}{4}v + v\left(\frac{x}{2}\right) - x^2 + 2, \tag{3.12}$$

with initial conditions $v(0) = 0\,,\ v'(0) = 0.$

Integrate both sides of Eq. (3.12) twice from 0 to x and using the initial condition at $x = 0$, we get

$$v(x) = x^2 - \frac{x^4}{12} + \int_0^x \int_0^x (\frac{3}{4}v(t) + v\left(\frac{t}{2}\right))\ dtdt, \tag{3.13}$$

By using the formula in Eq. (1.6), we get

$$v(x) = x^2 - \frac{x^4}{12} + \int_0^x (x-t)\left(\frac{3}{4}v(t) + v\left(\frac{t}{2}\right)\right) dt, \tag{3.14}$$

By applying the BCM for Eq. (3.14), we get

$$v(x) = f(x) + N(v),$$

where

$$f(x) = x^2 - \frac{x^4}{12} \quad \text{and} \quad N(v) = \int_0^x (x-t)\left(\frac{3}{4}v(t) + v\left(\frac{t}{2}\right)\right) dt,$$

$$v_n(x) = v_0(x) + \int_0^x (x-t)\left(\frac{3}{4}v_{n-1}(t) + v_{n-1}\left(\frac{t}{2}\right)\right)dt, n = 1,2, \dots .$$

Thus, by applying the Mathematica's code, see appendix D, we have

$$v_0(x) = f(x) = x^2 - \frac{x^4}{12},$$

$$v_1(x) = v_0(x) + \int_0^x (x-t)\left(\frac{3}{4}v_0(t) + v_0\left(\frac{t}{2}\right)\right)dt = x^2 - \frac{13x^6}{5760},$$

$$v_2(x) = v_0(x) + \int_0^x (x-t)\left(\frac{3}{4}v_1(t) + v_1\left(\frac{t}{2}\right)\right)dt = x^2 - \frac{91x^8}{2949120},$$

$$v_3(x) = v_0(x) + \int_0^x (x-t)\left(\frac{3}{4}v_2(t) + v_2\left(\frac{t}{2}\right)\right)dt,$$

$$= x^2 - \frac{17563x^{10}}{67947724800},$$

By continuing in this way, we get

$$v(x) = x^2 - \text{small term}, \tag{3.15}$$

This series converges to the exact solution [11]:

$$v(x) = \lim_{n\to\infty} v_n(x) = x^2.$$

3.5.3 Solving the nonlinear beam equation by using BCM

In this subsection, the BCM will be implemented to solve fourth order nonlinear beam deformation problem.

Example 3.5:

Resolve example 1.4 by using BCM [1].

$$v^{(4)} = v^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,$$

with the boundary conditions $v(0) = v'(0) = 0,\ v(1) = v'(1) = 1.$ (3.16)

By integrating both sides of Eq. (3.16) four times from 0 to x, we obtain

$$v(x) = v_0(x) + \int_0^x \int_0^x \int_0^x \int_0^x (v^2 - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48)\, dt\, dt\, dt\, dt\,. \quad (3.17)$$

Let us assume the following initial component,

$$v_0(x) = ax^3 + bx^2 + cx + d,$$

By using Eq. (1.6), we obtain the following

$$v(x) = v_0(x) + \int_0^x \frac{(x-t)^3}{6} (v^2 - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48)\, dt, \quad (3.18)$$

By implementing the BCM:

$$v_n(x) = v_0(x) + \int_0^x \frac{(x-t)^3}{6} (v_{n-1}^2(t) - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48)\, dt, \qquad n = 1,2,3,\dots. \quad (3.19)$$

Then, we get

$$v_0(x) = f(x) = ax^3 + bx^2 + cx + d,$$

$$v_1(x) = v_0(x) + \int_0^x \frac{(x-t)^3}{6}(v_0^2(t) - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48)\, dt,$$

$$= d + cx + bx^2 + ax^3 - 2x^4 + \frac{d^2x^4}{24} + x^5 + \frac{1}{60}cdx^5 + \frac{c^2x^6}{360} + \frac{1}{180}bdx^6 + \frac{1}{420}bcx^7 + \frac{1}{420}adx^7 - \frac{x^8}{420} + \frac{b^2x^8}{1680} + \frac{1}{840}acx^8 + \frac{abx^9}{1512} + \frac{x^{10}}{630} + \frac{a^2x^{10}}{5040} - \frac{x^{11}}{1980} - \frac{x^{12}}{2970} + \frac{x^{13}}{4290} - \frac{x^{14}}{24024}, \quad (3.20)$$

Imposing the boundary conditions (3.16), into $v_1(x)$, we get

$$a = -0.00687154, \qquad b = 2.00593, \qquad c = 0, \qquad d = 0. \qquad (3.21)$$

The following first-order approximate solution is achieved:

$$v_1(x) = 2.005929514398968x^2 - 6.871538792438514 \times 10^{-3}x^3 - 2x^4 + x^5 + 1.413881948623746 \times 10^{-5}x^8 - 9.116284704424509 \times 10^{-6}x^9 + 1.587310955961384 \times 10^{-3}x^{10} - \frac{x^{11}}{1980} - \frac{x^{12}}{2970} + \frac{x^{13}}{4290} - \frac{x^{14}}{24024}. \qquad (3.22)$$

$$v_2(x) = v_0(x) + \int_0^x \frac{(x-t)^3}{6}(v_1^2(t) - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48)\, dt,$$

$$= d + cx + bx^2 + ax^3 - 2x^4 + \frac{d^2x^4}{24} + x^5 + \frac{1}{60}cdx^5 + \frac{c^2x^6}{360} + \frac{1}{180}bdx^6 + \frac{1}{420}bcx^7 + \frac{1}{420}adx^7 - \frac{x^8}{420} + \frac{b^2x^8}{1680} + \frac{1}{840}acx^8 - \frac{dx^8}{420} + \frac{d^3x^8}{20160} + \frac{abx^9}{1512} - \frac{cx^9}{756} + \cdots. \tag{3.23}$$

Once again, by imposing the boundary conditions (3.16) in the second component $v_2(x)$, the obtained coefficients will be:

$$a = -8.27907 \times 10^{-7}, \quad b = 2, \quad c = 0, \quad d = 0.$$

In order to check the accuracy of the approximate solution, we calculate the absolute error, where $v(x) = x^5 - 2x^4 + 2x^2$ is the exact solution and $v_n(x)$ is the approximate solution. In table 3.2 the absolute error of BCM with $n = 1, 2$. It can be seen clearly from table 3.2 by increasing the iteration, the error will be decreasing.

Table 3.2: Results of the absolute errors for BCM.

x	$\lvert r_1 \rvert$ for BCM	$\lvert r_2 \rvert$ for BCM
0	0	0
0.1	5.24236×10^{-5}	6.79965×10^{-9}
0.2	1.82208×10^{-4}	2.3887×10^{-8}
0.3	3.48134×10^{-4}	4.62946×10^{-8}
0.4	5.09092×10^{-4}	6.90558×10^{-8}
0.5	6.24721×10^{-4}	8.72068×10^{-8}
0.6	6.57823×10^{-4}	9.58101×10^{-8}
0.7	5.81138×10^{-4}	9.01332×10^{-8}
0.8	3.92709×10^{-4}	6.67054×10^{-8}
0.9	1.45715×10^{-4}	2.80674×10^{-8}
1.0	3.3447×10^{-8}	0

3.5.4 Solving the nonlinear Burgers' equation by using the BCM

In this subsection, we implement the BCM to solve the first order nonlinear Burgers' equation and finding the exact solutions for different types of Burgers' equations in 1D, 2D and (1+3)-D.

Example 3.6:

Rewrite the example 1.5 and solve it by BCM [33].

$$v_t + vv_x = v_{xx}, \text{ with initial condition } \quad v(x,0) = 2x. \tag{3.24}$$

Integrating both sides of Eq. (3.24) from 0 to t and using the initial condition at $t = 0$, we get

$$v(x,t) = 2x + \int_0^t (-vv_x + v_{xx})\, dt, \tag{3.25}$$

Applying the BCM for Eq. (3.25), we get

$$v(x,t) = f(x,t) + N(v),$$

where

$$f(x,t) = 2x \quad \text{and} \quad N(v) = \int_0^t (-vv_x + v_{xx})\, dt.$$

$$v_n(x,t) = v_0(x,t) + \int_0^t \left(-v_{n-1}(x,t)\big(v_{n-1}(x,t)\big)_x + \big(v_{n-1}(x,t)\big)_{xx}\right) dt,$$

$$n = 1,2,\dots. \tag{3.26}$$

Then, we have

$$v_0 = f(x,t) = 2x,$$

$$v_1 = v_0 + \int_0^t (-v_0 v_{0x} + v_{0xx})\, dt = 2x - 4tx,$$

$$v_2 = v_0 + \int_0^t (-v_1 v_{1x} + v_{1xx})\, dt = 2x - 4tx + 8t^2x - \frac{16t^3x}{3},$$

By continuing in this way, the result is

$$v(x,t) = \lim_{n \to \infty} v_n(x,t)\,,$$
$$= 2\mathrm{x} - 4\mathrm{xt} + 8\mathrm{xt}^2 - 16\mathrm{xt}^3 + 32\mathrm{xt}^4 - 64\mathrm{xt}^5 + \cdots, \qquad (3.27)$$

This series converges to the following exact solution [33]:

$$v(x,t) = \frac{2x}{1+2t} = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - 64xt^5 + \cdots.$$

Example 3.7:

Recall example 1.6:

$$v_t + vv_x + vv_y = k\big(v_{xx} + v_{yy}\big), \qquad (3.28)$$

with initial condition $v(x,y,0) = sin(2\pi x)cos(2\pi y)$.

By taking the integration on both sides of equation (3.28) from 0 to t, we obtain

$$v(x,y,t) = sin(2\pi x)cos(2\pi y)$$
$$+ \int_0^t \Big(-vv_x - vv_y + k\big(v_{xx} + v_{yy}\big)\Big)\, dt, \qquad (3.29)$$

Now, by applying the BCM for Eq. (3.29), we have

$$v(x,y,t) = f(x,y,t) + N(v),$$

where

$$f(x, y, t) = sin(2\pi x)cos(2\pi y) \text{ and}$$

$$N(v) = \int_0^t \left(-vv_x - vv_y + k\left(v_{xx} + v_{yy}\right)\right) dt.$$

$$\begin{aligned} v_n(x, y, t) \\ &= v_0(x, y, t) \\ &+ \int_0^t \left(-v_{n-1}(x, y, t)v_{n-1}(x, y, t)_x - v_{n-1}(x, y, t)v_{n-1}(x, y, t)_y\right. \\ &\left.+ k(v_{n-1}(x, y, t)_{xx} + v_{n-1}(x, y, t)_{yy})\right) dt, \; n = 1,2, \ldots. \quad (3.30) \end{aligned}$$

So, we get

$$v_0 = sin(2\pi x)cos(2\pi y),$$

$$v_1 = v_0 + \int_0^t \left(-v_0 v_{0x} - v_0 v_{0y} + k\left(v_{0xx} + v_{0yy}\right)\right) dt,$$

$$\begin{aligned} &= cos(2\pi y)sin(2\pi x) + t(-8\pi^2 k cos(2\pi y)sin(2\pi x) \\ &\quad - 2\pi cos(2\pi x)cos(2\pi y)^2 sin(2\pi x) + 2\pi sin(2\pi x)sin(2\pi y)), \end{aligned}$$

$$v_2 = v_0 + \int_0^t \left(-v_1 v_{1x} - v_1 v_{1y} + k\left(v_{1xx} + v_{1yy}\right)\right) dt,$$

$$\begin{aligned} &= cos(2\pi y)sin(2\pi x) - 2\pi^2 t^2 cos(2\pi y)sin(2\pi x) \\ &\quad - 8\pi^2 tk cos(2\pi y)sin(2\pi x) + \cdots, \end{aligned}$$

continue till $n = 4$, we get

$$v_4 = v_0 + \int_0^t \left(-v_3 v_{3x} - v_3 v_{3y} + k\left(v_{3xx} + v_{3yy}\right)\right) dt,$$

$$= cos(2\pi y)sin(2\pi x) - 2\pi^2 t^2 cos(2\pi y)sin(2\pi x)$$
$$+\frac{2}{3}\pi^4 t^4 cos(2\pi y)sin(2\pi x) - \cdots,$$

The obtained series solution in Eq. (3.28) can be used to calculate the maximal error remainder to check the accuracy of the obtained approximate solution. It can be clearly seen in table 3.3 and figure 3.2 that by increasing the iterations the errors will be reduced.

Table 3.3: The maximal error remainder: MER_n by the BCM.

n	MER_n by BCM
1	1.61705×10^{-2}
2	2.52036×10^{-5}
3	3.56783×10^{-8}
4	4.5361×10^{-11}

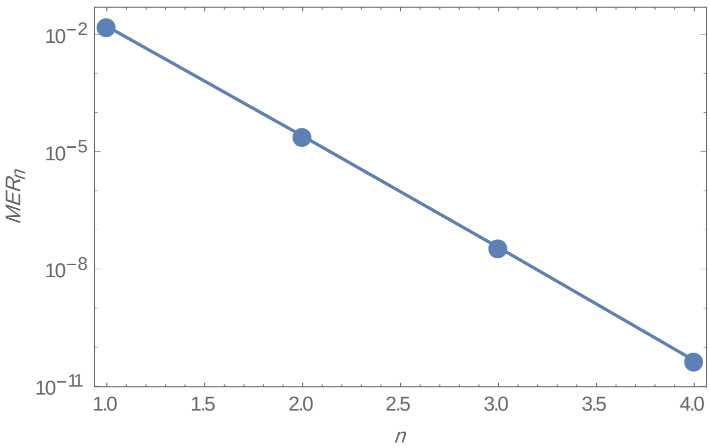

Figure 3.2: Logarithmic plots of MER_nversus n is 1 through 4 by BCM.

Example 3.8:

Recall example 2.7:

$$v_t = vv_x + v_{xx} + v_{yy}, \text{ with initial condition } \quad v(x,y,0) = x + y. \quad (3.31)$$

By taking the integration for both sides of Eq. (3.31) from 0 to t, we obtain

$$v(x,y,t) = x + y + \int_0^t (vv_x + v_{xx} + v_{yy})\,dt, \tag{3.32}$$

Implementing the BCM for Eq. (3.32), we get

$$v(x,y,t) = \ f(x,y,t) + N(v),$$

where

$$f(x,y,t) = x + y \quad \text{and} \quad N(v) = \int_0^t \left(vv_x + v_{xx} + v_{yy}\right) dt.$$

Thus, we have

$$\begin{aligned} v_n(x,y,t) \\ &= v_0(x,y,t) \\ &+ \int_0^t \left(v_{n-1}(x,y,t)v_{n-1}(x,y,t)_x + v_{n-1}(x,y,t)_{xx}\right. \\ &\left. + v_{n-1}(x,y,t)_{yy}\right) dt, \qquad n = 1,2,\ldots. \end{aligned} \tag{3.33}$$

Thus, we get

$$v_0 = f(x,y,t) = x + y,$$

$$v_1 = v_0 + \int_0^t \left(v_0 v_{0x} + v_{0xx} + v_{0yy}\right) dt \ = x + y + (x+y)t,$$

$$v_2 = v_0 + \int_0^t \left(v_1 v_{1x} + v_{1xx} + v_{1yy}\right) dt,$$

$$= x + y + (x+y)t + (x+y)t^2 + \frac{1}{3}(x+y)t^3,$$

By continuing in this way, we have a series of the form:

$$v(x,y,t) = \lim_{n\to\infty} v_n(x,y,t),$$

$$= (x+y) + (x+y)t + (x+y)t^2 + (x+y)t^3 + \cdots, \qquad (3.34)$$

This series is convergent to the following exact solution [76]:

$$v(x,y,t) = \frac{x+y}{1-t} = (x+y) + (x+y)t + (x+y)t^2 + (x+y)t^3 + \cdots.$$

Example 3.9:

Recall example 2.8:

$$v_t = vv_x + v_{xx} + v_{yy} + v_{zz},$$

with initial condition $v(x,y,z,0) = x+y+z$. (3.35)

Applying the integration on both sides of Eq. (3.35) from 0 to t and using the initial condition at t = 0, we get

$$v(x,y,z,t) = x+y+z + \int_0^t (vv_x + v_{xx} + v_{yy} + v_{zz})\, dt. \qquad (3.36)$$

$$f(x,y,z,t) = x+y+z \quad \text{and} \quad N(v) = \int_0^t \left(vv_x + v_{xx} + v_{yy} + v_{zz}\right) dt.$$

Now, applying the BCM for Eq. (3.36), we have

$$v(x,y,z,t) = f(x,y,z,t) + N(v),$$

where

$$f(x,y,z,t) = x+y+z \quad \text{and} \quad N(v) = \int_0^t \left(vv_x + v_{xx} + v_{yy} + v_{zz}\right) dt.$$

Thus, we get

$$v_n(x,y,z,t) = v_0(x,y,z,t) + \int_0^t \Big(v_{n-1}(x,y,z,t)\big(v_{n-1}(x,y,z,t)\big)_x + \big(v_{n-1}(x,y,z,t)\big)_{xx} + \big(v_{n-1}(x,y,z,t)\big)_{yy} + \big(v_{n-1}(x,y,z,t)\big)_{zz}\Big)\,dt, n = 1,2,3, \dots .$$

Thus, we have

$$v_0 = f(x,y,z,t) = x + y + z,$$

$$v_1 = v_0 + \int_0^t \left(v_0 v_{0x} + v_{0xx} + v_{0yy} + v_{0zz}\right) dt,$$

$$= x + y + z + (x + y + z)t,$$

$$v_2 = v_0 + \int_0^t \left(v_1 v_{1x} + v_{1xx} + v_{1yy} + v_{1zz}\right) dt,$$

$$= x + y + z + (x + y + z)t + (x + y + z)t^2 + \frac{1}{3}(x + y + z)t^3,$$

and so on, therefore we get the following series:

$$v(x,y,z,t) = \lim_{n\to\infty} v_n(x,y,z,t),$$
$$= x + y + z + (x + y + z)t + (x + y + z)t^2 + (x + y + z)t^3 + \cdots,$$

This series converges to the exact solution [76]:

$$v(x,y,z,t) = \frac{x + y + z}{1 - t},$$

$$= x + y + z + (x + y + z)t + (x + y + z)t^2 + (x + y + z)t^3 + \cdots.$$

3.5.5 Solving the nonlinear system of Burgers' equations by using BCM

In this subsection, the BCM will be implemented to solve the first order nonlinear Burgers' equation for the exact solutions for different types of systems of Burgers' equations in 1D, 2D and 3D.

Example 3.10:

Rewrite the example 2.9 and solve it by BCM [50].

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0,$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0,$$

Subject to the initial conditions:

$$u(x,0) = sin(x), \quad v(x,0) = sin(x). \tag{3.37}$$

By solving Eq. (3.37), we get

$$u(x,t) = sin(x) + \int_0^t (u_{xx} + 2uu_x - (uv)_x)\, dt,$$

$$v(x,t) = sin(x) + \int_0^t (v_{xx} + 2vv_x - (uv)_x)\, dt, \tag{3.38}$$

To solve the system of Eq. (3.38) by using of the BCM, we have

$$u(x,t) = g(x,t) + N_1(u), \qquad v(x,t) = f(x,t) + N_2(v),$$

where

$$g(x,t) = sin(x) \quad \text{and} \quad N_1(u) = \int_0^t (u_{xx} + 2uu_x - (uv)_x)\, dt,$$

$$f(x,t) = sin(x) \quad \text{and} \quad N_2(v) = \int_0^t (v_{xx} + 2vv_x - (uv)_x)\, dt.$$

Thus

$$u_n(x,t) = u_0(x,t) + \int_0^t \Big(\big(u_{n-1}(x,t)\big)_{xx} + 2u_{n-1}(x,t)\big(u_{n-1}(x,t)\big)_x - \big(u_{n-1}(x,t)v_{n-1}(x,t)\big)_x \Big)\, dt, \qquad n = 1,2,\dots .$$

$$v_n(x,t) = v_0(x,t) + \int_0^t \Big(\big(v_{n-1}(x,t)\big)_{xx} + 2v_{n-1}(x,t)\big(v_{n-1}(x,t)\big)_x - \big(u_{n-1}(x,t)v_{n-1}(x,t)\big)_x \Big)\, dt, \qquad n = 1,2,\dots . \tag{3.39}$$

Then,

$$u_0 = g(x,t) = sin(x),$$

$$v_0 = f(x,t) = sin(x),$$

$$u_1 = u_0 + \int_0^t (u_{0xx} + 2u_0u_{0x} - (u_0v_0)_x)\, dt = sin(x) - tsin(x),$$

$$v_1 = v_0 + \int_0^t (v_{0xx} + 2v_0v_{0x} - (u_0v_0)_x)\, dt = sin(x) - tsin(x),$$

$$u_2 = u_0 + \int_0^t (u_{1xx} + 2u_1u_{1x} - (u_1v_1)_x)\, dt,$$

$$= sin(x) - tsin(x) + \frac{1}{2}t^2 sin(x),$$

$$v_2 = v_0 + \int_0^t (v_{1xx} + 2v_1v_{1x} - (u_1v_1)_x)\, dt,$$

$$= sin(x) - tsin(x) + \frac{1}{2}t^2 sin(x),$$

Continuing in this way, we get:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)\,,$$
$$= sin(x) - tsin(x) + \frac{1}{2}t^2 sin(x) - \frac{1}{6}t^3 sin(x) + \frac{1}{24}t^4 sin(x) - \cdots,$$

$$v(x,t) = \lim_{n\to\infty} v_n(x,t)\,,$$
$$= sin(x) - tsin(x) + \frac{1}{2}t^2 sin(x) - \frac{1}{6}t^3 sin(x) + \frac{1}{24}t^4 sin(x) - \cdots.$$

This series converges to the exact solution [50]:

$$u(x,t) = e^{-t} sin(x),$$

$$v(x,t) = e^{-t} sin(x).$$

Example 3.11:

Rewrite the example 2.10 and solve it by BCM [30].

$$u_t + uu_x + vu_y - u_{xx} - u_{yy} = 0,$$

$$v_t + uv_x + vv_y - v_{xx} - v_{yy} = 0,$$

Subject to the initial conditions:

$$u(x,y,0) = x + y, \quad v(x,y,0) = x - y. \tag{3.40}$$

Integrating both sides of Eq. (3.40) from 0 to t and using the initial conditions at t = 0, we get

$$u(x,y,t) = x + y + \int_0^t (-uu_x - vu_y + u_{xx} + u_{yy})\,dt,$$

$$v(x,y,t) = x - y + \int_0^t (-uv_x - vv_y + v_{xx} + v_{yy})\,dt, \tag{3.41}$$

To solve the system of Eq. (3.41) by using of the BCM

$$u(x,y,t) = g(x,y,t) + N_1(u), \qquad v(x,y,t) = f(x,y,t) + N_2(v)$$

where

$$g(x,y,t) = x + y \quad \text{and} \quad N_1(u) = \int_0^t (-uu_x - vu_y + u_{xx} + u_{yy})\,dt,$$

$$f(x,y,t) = x - y \quad \text{and} \quad N_2(v) = \int_0^t (-uv_x - vv_y + v_{xx} + v_{yy})\,dt.$$

Thus

$$\begin{aligned} u_n(x,y,t) &\\ &= u_0(x,y,t) \\ &+ \int_0^t \Big(-u_{n-1}(x,y,t)\big(u_{n-1}(x,y,t)\big)_x \\ &- v_{n-1}(x,y,t)\big(u_{n-1}(x,y,t)\big)_y + \big(u_{n-1}(x,y,t)\big)_{xx} \\ &+ \big(u_{n-1}(x,y,t)\big)_{yy}\Big)\,dt, \qquad n = 1,2,\ldots, \end{aligned}$$

$$\begin{aligned} v_n(x,y,t) &\\ &= v_0(x,y,t) \\ &+ \int_0^t \Big(-u_{n-1}(x,y,t)\big(v_{n-1}(x,y,t)\big)_x \\ &- v_{n-1}(x,y,t)\big(v_{n-1}(x,y,t)\big)_y + \big(v_{n-1}(x,y,t)\big)_{xx} \\ &+ \big(v_{n-1}(x,y,t)\big)_{yy}\Big)\,dt, \qquad n = 1,2,\ldots. \end{aligned} \tag{3.42}$$

Therefore, we have

$$u_0 = g(x,y,t) = x + y,$$

$$v_0 = f(x,y,t) = x - y,$$

$$u_1 = u_0 + \int_0^t (-u_0u_{0x} - v_0u_{0y} + u_{0xx} + u_{0yy})\,dt = x + y - 2xt,$$

$$v_1 = v_0 + \int_0^t (-u_0 v_{0x} - v_0 v_{0y} + v_{0xx} + v_{0yy})\, dt \ = x - y - 2yt,$$

$$u_2 = u_0 + \int_0^t \left(-u_1 u_{1x} - v_1 u_{1y} + u_{1xx} + u_{1yy}\right) dt,$$

$$= x + y - 2xt + 2(x+y)t^2 - \frac{4xt^3}{3},$$

$$v_2 = v_0 + \int_0^t \left(-u_1 v_{1x} - v_1 v_{1y} + v_{1xx} + v_{1yy}\right) dt,$$

$$= x - y - 2yt + (2x - 2y)t^2 - \frac{4yt^3}{3},$$

Continuing in this way, we get the following forms:

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t)\,,$$
$$= x + y - 2xt + 2(x+y)t^2 - 4xt^3 + 4(x+y)t^4 - 8xt^5 + \cdots,$$

$$v(x,y,t) = \lim_{n\to\infty} v_n(x,y,t)\,,$$
$$= x - y - 2yt + (2x-2y)t^2 - 4yt^3 + (4x-4y)t^4 - 8yt^5 + \cdots.$$

These forms converges to the exact solutions [30]:

$$u(x,y,t) = \frac{x - 2xt + y}{1 - 2t^2},$$

$$= x + y - 2xt + 2(x+y)t^2 - 4xt^3 + 4(x+y)t^4 - 8xt^5 + \cdots,$$

$$v(x,y,t) = \frac{x - 2yt - y}{1 - 2t^2},$$

$$= x - y - 2yt + (2x-2y)t^2 - 4yt^3 + (4x-4y)t^4 - 8yt^5 + \cdots.$$

Example 3.12:

Recall example 2.11:

$$
\begin{aligned}
&u_t - uu_x - vu_y - wu_z - u_{xx} - u_{yy} - u_{zz} = g(x,y,z,t),\\
&v_t + uv_x + vv_y + wv_z + v_{xx} + v_{yy} + v_{zz} = f(x,y,z,t),\\
&w_t + uw_x + vw_y + ww_z + w_{xx} + w_{yy} + w_{zz} = h(x,y,z,t),
\end{aligned}
$$

where

$$g(x,y,z,t) = e^t(x+y+z) - e^t\big(1 + e^t(x+y+z)\big),$$

$$f(x,y,z,t) = -e^t(x+y+z) - e^t(1 + e^t(x+y+z)),$$

$$h(x,y,z,t) = e^t(x+y+z) + e^t(1 + e^t(x+y+z)), \tag{3.43}$$

Subject to the initial conditions:

$$u(x,y,z,0) = x+y+z,$$

$$v(x,y,z,0) = -(x+y+z),$$

$$w(x,y,z,0) = 1+x+y+z.$$

Integrating both sides of in Eq. (3.43) from 0 to t and using the initial conditions at t = 0, we get

$$
\begin{aligned}
&u(x,y,z,t)\\
&\quad = 1 - e^t + \frac{x}{2} + e^t x - \frac{1}{2}e^{2t}x + \frac{y}{2} + e^t y - \frac{1}{2}e^{2t}y + \frac{z}{2} + e^t z\\
&\quad - \frac{1}{2}e^{2t}z + \int_0^t (uu_x + vu_y + wu_z + u_{xx} + u_{yy} + u_{zz})\, dt,
\end{aligned}
$$

$$v(x,y,z,t) = 1 - e^{t} + \frac{x}{2} - e^{t}x - \frac{1}{2}e^{2t}x + \frac{y}{2} - e^{t}y - \frac{1}{2}e^{2t}y + \frac{z}{2} - e^{t}z - \frac{1}{2}e^{2t}z + \int_0^t \left(-uv_x - vv_y - wv_z - v_{xx} - v_{yy} - v_{zz}\right) dt,$$

$$w(x,y,z,t) = e^{t} - \frac{x}{2} + e^{t}x + \frac{1}{2}e^{2t}x - \frac{y}{2} + e^{t}y + \frac{1}{2}e^{2t}y - \frac{z}{2} + e^{t}z + \frac{1}{2}e^{2t} + \int_0^t (-uw_x - vw_y - ww_z - w_{xx} - w_{yy} - w_{zz})\, dt, \qquad (3.44)$$

To solve the system of Eq. (3.44) by using of the BCM, we have

$$u_0 = 1 - e^{t} + \frac{x}{2} + e^{t}x - \frac{1}{2}e^{2t}x + \frac{y}{2} + e^{t}y - \frac{1}{2}e^{2t}y + \frac{z}{2} + e^{t}z - \frac{1}{2}e^{2t}z,$$

$$v_0 = 1 - e^{t} + \frac{x}{2} - e^{t}x - \frac{1}{2}e^{2t}x + \frac{y}{2} - e^{t}y - \frac{1}{2}e^{2t}y + \frac{z}{2} - e^{t}z - \frac{1}{2}e^{2t}z,$$

$$w_0 = e^{t} - \frac{x}{2} + e^{t}x + \frac{1}{2}e^{2t}x - \frac{y}{2} + e^{t}y + \frac{1}{2}e^{2t}y - \frac{z}{2} + e^{t}z + \frac{1}{2}e^{2t},$$

By writing the functions $u_1,\ v_1,\ w_1$ by using Taylor series expansions, we get

$$u_1 = (x+y+z) + (x+y+z)t + \frac{1}{2}(-2-x-y-z)t^2 + \frac{1}{6}(-3-5x - 5y - 5z)t^3 + \frac{1}{24}(-2-13x-13y-13z)t^4 + \frac{1}{120}(9 - 23x - 23y - 23z)t^5 + O[t]^6,$$

$$v_1 = (-x-y-z) + (-x-y-z)t + \frac{1}{2}(-x-y-z)t^2 + \frac{1}{2}(-1-x-y-z)t^3 + \frac{1}{24}(-12-13x-13y-13z)t^4 + \frac{1}{40}(-13-19x-19y-19z)t^5 + O[t]^6,$$

$$\begin{aligned} w_1 &= (1+x+y+z)+(x+y+z)t+\frac{1}{2}(x+y+z)t^2 \\ &\quad +\frac{1}{2}(1+x+y+z)t^3+\frac{1}{24}(12+13x+13y+13z)t^4 \\ &\quad +\frac{1}{40}(13+19x+19y+19z)t^5+O[t]^6. \end{aligned} \tag{3.45}$$

In a similar way, we have

$$\begin{aligned} u_2 &= (x+y+z)+(x+y+z)t+\frac{1}{2}(x+y+z)t^2+\frac{1}{6}(-4-3x-3y \\ &\quad -3z)t^3+\frac{1}{24}(-15-23x-23y-23z)t^4+\frac{1}{120}(-16-75x \\ &\quad -75y-75z)t^5+O[t]^6, \end{aligned}$$

$$\begin{aligned} v_2 &= (-x-y-z)+(-x-y-z)t+\frac{1}{2}(-x-y-z)t^2+\frac{1}{6}(-2-3x \\ &\quad -3y-3z)t^3+\frac{1}{24}(-7-11x-11y-11z)t^4+\frac{1}{120}(-14 \\ &\quad -31x-31y-31z)t^5+O[t]^6, \end{aligned}$$

$$\begin{aligned} w_2 &= (1+x+y+z)+(x+y+z)t+\frac{1}{2}(x+y+z)t^2 \\ &\quad +\frac{1}{6}(2+3x+3y+3z)t^3+\frac{1}{24}(7+11x+11y+11z)t^4 \\ &\quad +\frac{1}{120}(14+31x+31y+31z)t^5+O[t]^6. \end{aligned} \tag{3.46}$$

By continuing, we have the following formulas:

$$\begin{aligned} u(x,y,z,t) &= \lim_{n\to\infty} u_n(x,y,z,t), \\ &= x+y+z+(x+y+z)t+\frac{1}{2}(x+y+z)t^2 \\ &\quad +\frac{1}{6}(x+y+z)t^3+\cdots, \end{aligned}$$

$$\begin{aligned} v(x,y,z,t) &= \lim_{n\to\infty} v_n(x,y,z,t), \\ &= -x-y-z+(-x-y-z)t+\frac{1}{2}(-x-y-z)t^2 \\ &\quad +\frac{1}{6}(-x-y-z)t^3+\cdots, \end{aligned}$$

$$w(x,y,z,t) = \lim_{n\to\infty} w_n(x,y,z,t)\,,$$
$$= 1 + x + y + z + (x+y+z)t + \frac{1}{2}(x+y+z)t^2$$
$$+ \frac{1}{6}(x+y+z)t^3 + \cdots.$$

Those forms converge to the following exact solutions:

$$u(x,y,z,t) = e^t(x+y+z),$$

$$v(x,y,z,t) = -e^t(x+y+z),$$

$$w(x,y,z,t) = 1 + e^t(x+y+z).$$

CHAPTER 4

CONCLUSIONS AND FUTURE WORKS

Chapter 4

Conclusions and Future Works

4.1 Introduction

The main objective of this thesis has been achieved by solving some of non-linear ODE and PDE by different iterative methods. This purpose has been obtained by implementing two iterative methods, which are so-called the semi-analytic method of TAM and BCM. In addition, the comparison between the proposed methods and other methods such as ADM and VIM will be presented. Also, some conclusions, and future works will be given.

4.2 Conclusions

From the present study one can conclude the following:

1. The two iterative methods are efficient and reliable to find the exact solutions for Riccati and pantograph equations, different types of Burgers' equations and equations systems in 1D, 2D and 3D. The efficiency and accuracy of the TAM has been proved by studying the convergence.

2. The two proposed methods did not require any resulted assumption to deal with the nonlinear terms unlike ADM was need the so-called Adomain polynomial in nonlinear case. In addition, the VIM required the Lagrange multiplier.

3. We take example for nonlinear Riccati equation it was solved in chapter 1, chapter 2 and chapter 3

$$v' = -v^2 + 1, \quad \text{with initial condition} \quad v(0) = 0. \tag{4.1}$$

When comparing the results of TAM and BCM with those of the ADM and VIM, the numerical solutions obtained by BCM are more accurate. The maximal error remainders will be decreased when there is more obtaining of iterations which are clarified in the figure 4.1 and table 4.1.

Table 4.1: Comparative results of the maximal error remainder: $MER_n$for TAM, BCM, VIM and ADM, where $n = 1, \dots ,6$.

n	MER_n by TAM	MER_n by BCM	$MER_n$by VIM	$MER_n$by ADM
1	6.65556×10^{-5}	6.65556×10^{-5}	0.01	6.65556×10^{-5}
2	2.65463×10^{-7}	2.65463×10^{-7}	6.65556×10^{-5}	3.76891×10^{-7}
3	7.56708×10^{-10}	7.56708×10^{-10}	2.65463×10^{-7}	1.96289×10^{-9}
4	1.67806×10^{-12}	1.67808×10^{-12}	7.56708×10^{-10}	9.72067×10^{-12}
5	2.97852×10^{-15}	3.04791×10^{-15}	1.67808×10^{-12}	4.65513×10^{-14}
6	1.24033×10^{-16}	3.46945×10^{-18}	3.04791×10^{-15}	2.15106×10^{-16}

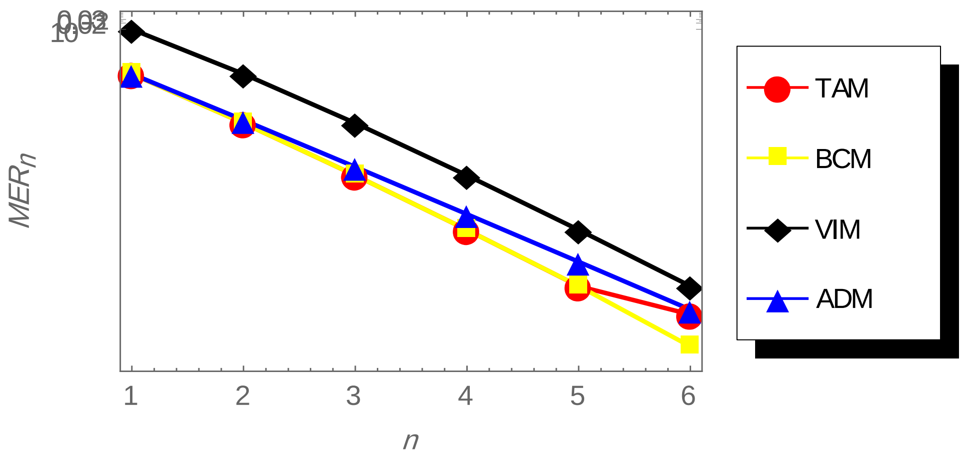

Figure 4.1: Comparison of the maximal error remainder for TAM, BCM, VIM and ADM.

4. We take example for nonlinear beam equation it was solved in chapter 1, chapter 2 and chapter 3

$$v^{(4)} = v^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,$$

with the boundary conditions $v(0) = v'(0) = 0,\ v(1) = v'(1) = 1.$ (4.2)

In comparison the results by the TAM and BCM with the some existing methods such as VIM and HPM [1], it is observed in general that the approximate solutions obtained by the TAM converge faster without any restricted assumptions and possesses high-order accuracy. As given in table 4.2, we compare the absolute error of TAM, with $n = 2$ together with the VIM, and the HPM [1].

Table 4.2: Comparative results for the absolute errors of the TAM, BCM, VIM and HPM.

x	$\|r_2\|$ for TAM	$\|r_2\|$ for BCM	$\|r_2\|$ for VIM	$\|r_2\|$for HPM[1]
0	0	0	0	0
0.1	1.77279×10^{-10}	6.79965×10^{-9}	6.79965×10^{-9}	1.896×10^{-7}
0.2	5.99375×10^{-10}	2.3887×10^{-8}	2.3887×10^{-8}	6.4171×10^{-7}
0.3	1.1028×10^{-9}	4.62946×10^{-8}	4.62946×10^{-8}	1.18180×10^{-6}
0.4	1.53129×10^{-9}	6.90558×10^{-8}	6.90558×10^{-8}	1.6405×10^{-6}
0.5	1.752×10^{-9}	8.72068×10^{-8}	8.72068×10^{-8}	1.8703×10^{-6}
0.6	1.68284×10^{-9}	9.58101×10^{-8}	9.58101×10^{-8}	1.7815×10^{-6}
0.7	1.32351×10^{-9}	9.01332×10^{-8}	9.01332×10^{-8}	1.3816×10^{-6}
0.8	7.76486×10^{-10}	6.67054×10^{-8}	6.67054×10^{-8}	7.958×10^{-7}
0.9	2.44241×10^{-10}	2.80674×10^{-8}	2.80674×10^{-8}	2.437×10^{-7}
1.0	1.11022×10^{-16}	0	0	6.0×10^{-10}

It can be observed clearly from table 4.2, the absolute error for TAM is lower than BCM, VIM and HPM.

5. We take example for nonlinear 2D Burgers' equation it was solved in chapter 1, chapter 2 and chapter 3

$$v_t + vv_x + vv_y = k\left(v_{xx} + v_{yy}\right), \tag{4.3}$$

with initial condition $v(x, y, 0) = sin(2\pi x)cos(2\pi y)$.

The TAM has also achieved the rapid convergence of our approximate solution by the sequence of the curves of error remainder functions. Also it does not require any restricted assumptions like the ADM and VIM, which are clarified in the mentioned figure 4.2 and table 4.3.

Table 4.3: Comparative results the maximal error remainder: MER_n for TAM, BCM, VIM and ADM, where $n = 1, \ldots, 4$.

n	MER_n by TAM	MER_n by BCM	MER_n by VIM	MER_n by ADM
1	5.99981×10^{-3}	1.61705×10^{-2}	1.61705×10^{-2}	1.61705×10^{-2}
2	2.89069×10^{-6}	2.52036×10^{-5}	2.52036×10^{-5}	2.98775×10^{-5}
3	1.12531×10^{-8}	3.56783×10^{-8}	3.56783×10^{-8}	4.9464×10^{-8}
4	4.54397×10^{-11}	4.5361×10^{-11}	4.53593×10^{-11}	7.1962×10^{-11}

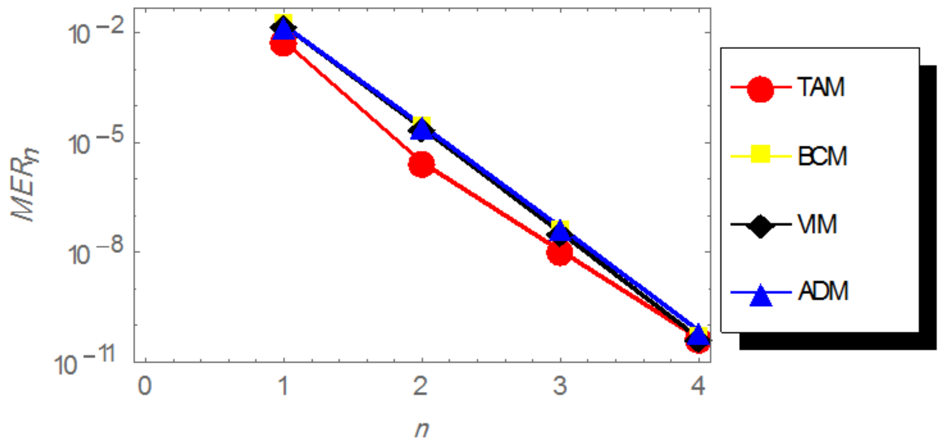

Figure 4.2: Comparison of the maximal error remainder for TAM, BCM, VIM and ADM.

6. The analytic solutions of these different problems are excellent as compared to the results of other iterative methods. In nonlinear equations that emerge much of the time to express phenomenon the proposed methods are effective and strong enough to give the reliable results.

4.3 Future works

In this section some of future works will be suggested

1) Solving the Emden–Fowler equation by using TAM

$$v'' + \frac{k}{x} v' + f(x)g(v) = 0, \qquad v(0) = v_0, \quad v'(0) = 0, \qquad (4.4)$$

where $f(x)$ and $g(v)$ are some given functions of x and v, respectively, and k is called the shape factor.

2) Using BCM for solving a Thomas-Fermi equation

$$v'' = \frac{v^{\frac{3}{2}}}{\sqrt{x}}, \qquad v(0) = 1, \quad v(\infty) = 0, \qquad (4.5)$$

3) Solving the Blasius equation by using TAM

$$v''' + \frac{1}{2} v v'' = 0, \quad v(0) = 0, \quad v'(0) = 1, \quad v''(0) = \alpha, \quad \alpha > 0. \quad (4.6)$$

4) Solving the sine-Gordon equation by using TAM

$$v_{tt} - c^2 v_{xx} + \alpha \sin(v) = 0, \quad v(x,0) = f(x), \quad v_t(x,0) = g(x), \quad (4.7)$$

where c and α are constants. It is clear that this equation adds the nonlinear term $\sin(v)$ to the standard wave equation.

5) Using BCM for solving the telegraph equation

$$v_{xx} = a v_{tt} + b v_t + c v, \quad v(x,0) = f(x), \quad v_t(x,0) = g(x), \quad (4.8)$$

where $v = v(x,t)$ is the resistance, and a, b and c are constants related to the inductance, capacitance and conductance of the cable respectively.

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APPENDICES

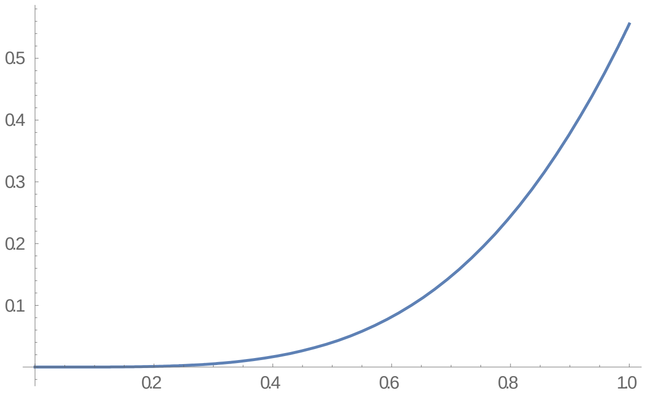
Appendix A: Code of Mathematica for chapter one (the ADM)

****

Code of example (1.1)

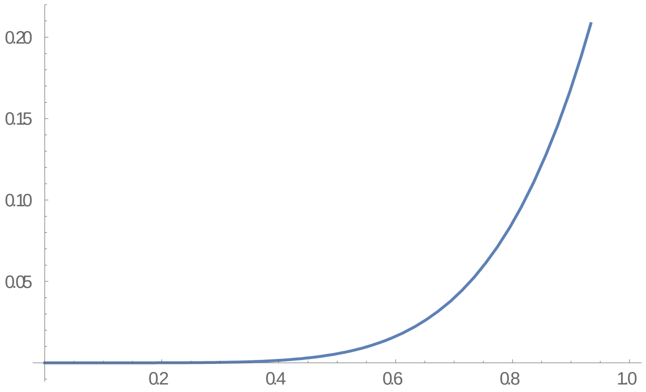
****

```
(* v'=-v^2 +1, v(0)=0 *)

tt=AbsoluteTime[];

zz=SessionTime[];

v0=x;

v1= -Integrate[v0^2/. x→t,{t,0,x}]

-x^3/3

v2= -Integrate[2v0*v1/. x→t,{t,0,x}]

2x^5/15

v3= -Integrate[2v0*v2+v1^2 /. x→t,{t,0,x}]

-17x^7/315

v4= -Integrate[2v0*v3+2v1*v2 /. x→t,{t,0,x}];

v5= -Integrate[2v0*v4+2v1*v3+v2^2 /. x→t,{t,0,x}];

v6= -Integrate[2v0*v5+2v2*v3+2v1*v4 /. x→t,{t,0,x}];

v=v0+v1+v2+v3+v4+v5+v6
x-x^3/3+2x^5/15-17x^7/315+62x^9/2835-1382x^11/155925+ 21844x^13/6081075

r1=Abs[D[v0+v1,{x,1}]+(v0+v1)^2-1];

Plot[r1,{x,0,1}]
```



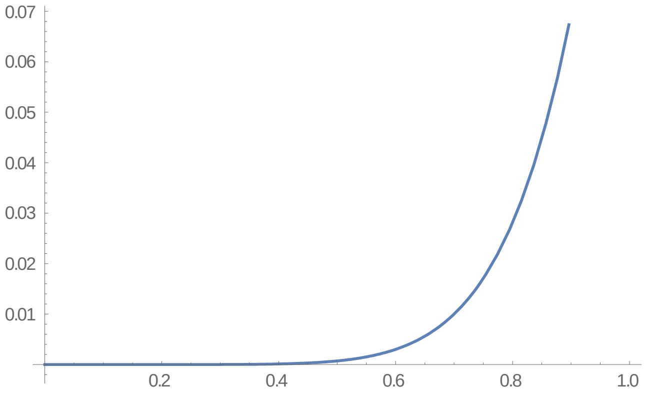
r2=Abs[D[v0+v1+v2,{x,1}]+(v0+v1+v2)^2-1];

Plot[r2,{x,0,1}]



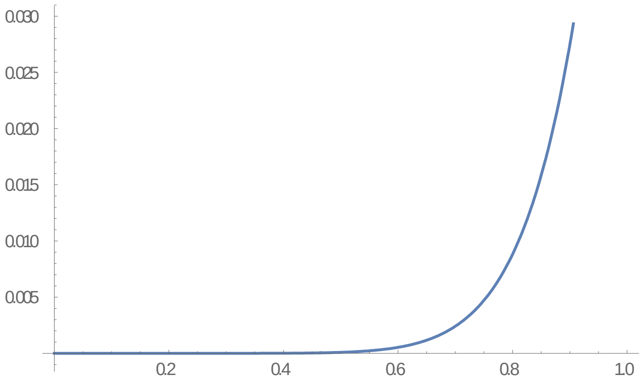
r3=Abs[D[v0+v1+v2+v3,{x,1}]+(v0+v1+v2+v3)^2-1];

Plot[r3,{x,0,1}]



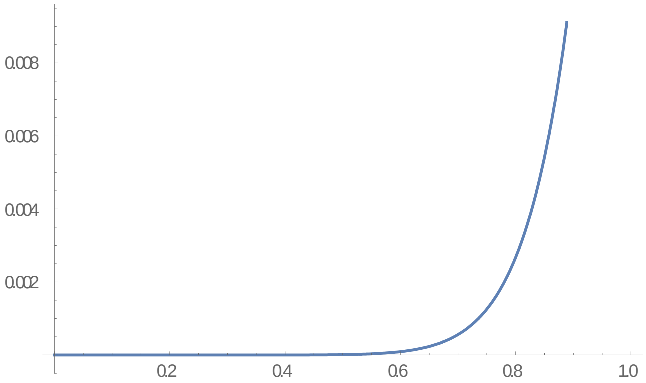
r4=Abs[D[v0+v1+v2+v3+v4,{x,1}]+(v0+v1+v2+v3+v4)^2-1];

Plot[r4,{x,0,1}]



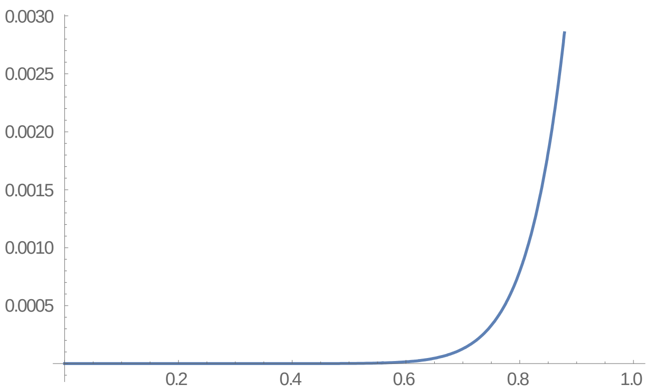
r5=Abs[D[v0+v1+v2+v3+v4+v5,{x,1}]+(v0+v1+v2+v3+v4+v5)^2-1];

Plot[r5,{x,0,1}]



r6=Abs[D[v0+v1+v2+v3+v4+v5+v6,{x,1}]+(v0+v1+v2+v3+v4+v5+v6)^2-1];

Plot[r6,{x,0,1}]



x ={0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1};

r1;r2;r3;r4;r5;r6;

Y1=Max[r1]

0.0000665556

Y2=Max[r2]

3.76891×10^{-7}

Y3=Max[r3]

1.96289×10^{-9}

Y4=Max[r4]

9.72067×10^{-12}

Y5=Max[r5]

4.65513×10^{-14}

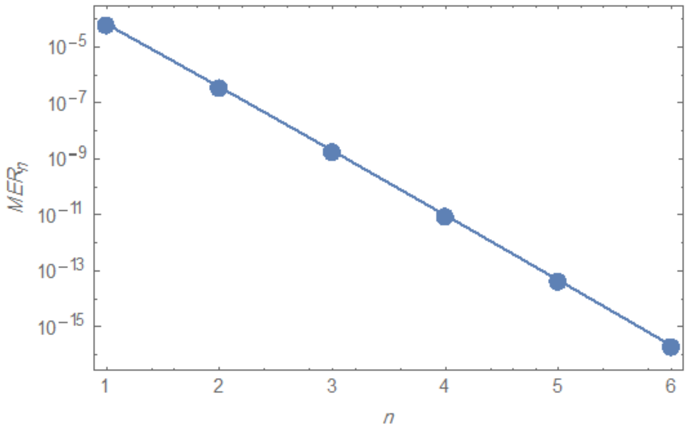
Y6=Max[r6]

2.15106×10^{-16}

Y={Y1,Y2,Y3,Y4,Y5,Y6};

n={1,2,3,4,5,6};

```
ListLog-
Plot[{{1,Y1},{2,Y2},{3,Y3},{4,Y4},{5,Y5},{6,Y6}},Joined→True,PlotRan
ge→All,Frame→True,
Aes→True,FrameLabel→{Row[{Style["n",FontSlant→Italic]}],Row[{Style[
"MERn",FontSlant→Italic]}]},PlotMarkers→{Automatic,15}]
```



Appendix B: Code of Mathematica for chapter one (the VIM)

****

Code of example (1.7)

****

```
(* D[v,{t,1}]+v*D[v,{x,1}]+ v*D[v,{y,1}]= k (D[v,{x,2}]+D[v,{y,2}]),
v(x,y,0)= sin(2πx)cos(2πy) *)

tt=AbsoluteTime[];

zz=SessionTime[];

v0= sin(2πx)cos(2πy);

v1=v0-Integrate[(D[v0,{t,1}]+v0*D[v0,{x,1}]+ v0*D[v0,{y,1}]-k
(D[v0,{x,2}]+D[v0,{y,2}]),{t,0,t}];
v2=v1-Integrate[(D[v1,{t,1}]+v1*D[v1,{x,1}]+ v1*D[v1,{y,1}]-k
(D[v1,{x,2}]+D[v1,{y,2}]),{t,0,t}];
v3=v2-Integrate[(D[v2,{t,1}]+v2*D[v2,{x,1}]+ v2*D[v2,{y,1}]-k
(D[v2,{x,2}]+D[v2,{y,2}]),{t,0,t}];
v4=v3-Integrate[(D[v3,{t,1}]+v3*D[v3,{x,1}]+ v3*D[v3,{y,1}]-k
(D[v3,{x,2}]+D[v3,{y,2}]),{t,0,t}];
r1=Abs[D[v1,{t,1}]+v1*D[v1,{x,1}]+ v1*D[v1,{y,1}]-k
(D[v1,{x,2}]+D[v1,{y,2}])];
r2=Abs[D[v2,{t,1}]+v2*D[v2,{x,1}]+ v2*D[v2,{y,1}]-k
(D[v2,{x,2}]+D[v2,{y,2}])];
r3=Abs[D[v3,{t,1}]+v3*D[v3,{x,1}]+ v3*D[v3,{y,1}]-k
(D[v3,{x,2}]+D[v3,{y,2}])];
r4=Abs[D[v4,{t,1}]+v4*D[v4,{x,1}]+ v4*D[v4,{y,1}]-k
(D[v4,{x,2}]+D[v4,{y,2}])];


k =0.1;
t =0.0001;
x ={0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1};

y ={0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1};
```

r1;r2;r3;r4;

Y1=Max[r1]

0.0161705

Y2=Max[r2]

0.0000252036

Y3=Max[r3]

3.56783×10^{-8}

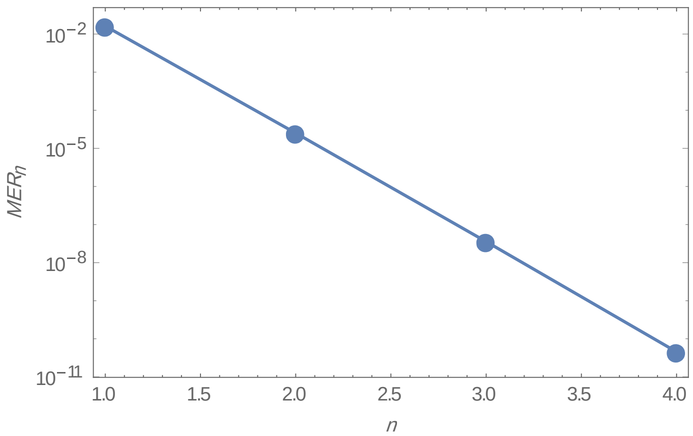
Y4=Max[r4]

4.53593×10^{-11}

Y={Y1,Y2,Y3,Y4};

n={1,2,3,4};

ListLog-Plot[{{1,Y1},{2,Y2},{3,Y3},{4,Y4}},Joined→True,PlotRange→All,Frame→True,Aes→True,FrameLabel→{Row[{Style["n",FontSlant→Italic]}],Row[{Style["MER_n",FontSlant→Italic]}]},PlotMarkers→{Automatic,15}]



Appendix C: Code of Mathematica for chapter two (the TAM)

****

Code of example (2.4)

****

```
(* v""=v^2-x^10+4x^9-4x^8-4x^7+8x^6-4x^4+120x-48, v(0)= v'(0)=0, v(1)= v'(1)=1 *)

(* L(v)= v"",  N(v)= v^2  and f(x)=-(-x^10+4x^9-4x^8-4x^7+8x^6-4x^4+120x-48) *)

tt=AbsoluteTime[];

zz=SessionTime[];

v01[x_]=v0[x]/.First@DSolve[{v0""[x]= -x^10+4x^9-4x^8-4x^7+8x^6-4x^4+120x-48,
v0[0]=0, v0'[0]=0, v0[1]=1, v0'[1]=1}, v0, x];

Expand[v01[x]]

718561x^2/360360+4019x^3/540540-2x^4+x^5-x^8/420+x^10/360-x^11/1980-
x^12/2970+x^13/4290-x^14/24024

v11[x_]=v1[x]/.First@DSolve[{v1""[x]= (v01[x])^2 -x^10+4x^9-4x^8-4x^7+8x^6-
4x^4+120x-48, v1[0]=0, v1'[0]=0, v1[1]=1, v1'[1]=1}, v1, x];

v22[x_]=v2[x]/.First@DSolve[{v2""[x]= (v11[x])^2 -x^10+4x^9-4x^8-4x^7+8x^6-
4x^4+120x-48, v2[0]=0, v2'[0]=0, v2[1]=1, v2'[1]=1}, v2, x];

v33[x_]=v3[x]/.First@DSolve[{v3""[x]= (v22[x])^2 -x^10+4x^9-4x^8-4x^7+8x^6-
4x^4+120x-48, v3[0]=0, v3'[0]=0, v3[1]=1, v3'[1]=1}, v3, x];

v44[x_]=v4[x]/.First@DSolve[{v4""[x]= (v33[x])^2 -x^10+4x^9-4x^8-4x^7+8x^6-
4x^4+120x-48, v3[0]=0, v4'[0]=0, v4[1]=1, v4'[1]=1}, v4, x];

v55[x_]=v5[x]/.First@DSolve[{v5""[x]= (v44[x])^2 -x^10+4x^9-4x^8-4x^7+8x^6-
4x^4+120x-48, v5[0]=0, v5'[0]=0, v5[1]=1, v5'[1]=1}, v5, x];
```

```
v66[x_]=v6[x]/.First@DSolve[{v6''''[x]= (v55[x])^2 -x^10+4x^9-4x^8-4x^7+8x^6-
4x^4+120x-48, v6[0]=0, v6'[0]=0, v6[1]=1, v6'[1]=1}, v6, x];

x={0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1};

v=x^5-2x^4+2x^2;

r1=Abs[v-v11[x]]

{ 0, 1.02627×10^-7, 3.47659×10^-7, 6.41401×10^-7, 8.93924×10^-7, 1.02777×10^-6,
9.93141×10^-7, 7.86445×10^-7, 4.64662×10^-7, 1.471×10^-7, 2.22045×10^-16 }

r2=Abs[v-v22[x]];

r3=Abs[v-v33[x]];

r4=Abs[v-v44[x]];

r5=Abs[v-v55[x]];

r6=Abs[v-v66[x]];
```

Appendix D: Code of Mathematica for chapter three (the BCM)

****

Code of example (3.4)

****

```
(* v''=3/4*v+v(x/2)-x^2+2, v(0)=0, v'(0)=0  *)

tt=AbsoluteTime[];

zz=SessionTime[];

v0=  x^2-x^4/12;

a0=v0 /. x→x/2;

v1= v0+Integrate[(x-t)*((3/4*v0+a0)/. x→t),{t,0,x}]
```

x^2-13x^6/5760

```
a1=v1 /. x→x/2;
```

```
v2= v0+Integrate[(x-t)*((3/4*v1+a1)/. x→t),{t,0,x}]
```

x^2-91x^8/2949120

```
a2=v2 /. x→x/2;
```

```
v3= v0+Integrate[(x-t)*((3/4*v2+a2)/. x→t),{t,0,x}]
```

x^2-17563x^{10}/67947724800

```
a3=v3 /. x→x/2;
```

```
v4= v0+Integrate[(x-t)*((3/4*v3+a3)/. x→t),{t,0,x}]
```

x^2-13505947x^{12}/9184358065766400

المُستخلص

الهدف الرئيسي من هذه الرسالة هو استخدام اثنتين من طرائق تكرارية شبه تحليلية لإيجاد الحلول التحليلية لبعض المسائل المهمّة في الفيزياء والهندسة.

الهدف الأول هو استخدام الطريقة التكرارية الأولى الذي اقترحه من قبل التميمي والأنصاري ألا وهي الـ (TAM) لإيجاد الحلول التحليلية لمعادلات ريكاتي، بانتوغراف، وتشوّه الذراع المتذبذب. ايضاً، قد تم تطبيق الـ (TAM) على معادلات وانظمة معادلات برجرز الغير خطية الأحادية البعد والثنائية الابعاد والثلاثية الابعاد للحصول على الحلول التحليلية لكلٍ منها. التقارب للـ (TAM) قد تم التحقق منه وإثباته بنجاح لتلك المسائل.

والهدف الثاني هو استخدام تقنية تكرارية على أساس طريقة انقباض بناخ (BCM) التي يسهل تطبيقها في التعامل مع المصطلحات غير الخطية. تم استخدام الـ (BCM) لحل نفس المسائل الفيزيائية والهندسية المذكورة أعلاه. وبالإضافة إلى ذلك، قدمنا العديد من المقارنات بين الـ (TAM) والـ (BCM) والطريقة التغايرية التكرارية (VIM) وطريقة تفكيك أدوميان (ADM) مع بعض الاستنتاجات والأعمال المستقبلية. الطرائق المُقدمة كفوءة وتمتلك دقّة عاليّة. الـ (TAM) والـ (BCM) لا تتطلبان ايّة افتراضات مقيّدة في الحالة غير خطية. بالإضافة الى ذلك، كلا الطريقتين لا تعتمدان على استخدام أي افتراضات مقيّدة اضافية كما هو الحال في الـ (ADM) أوالـ (VIM) أو أية طرائق تكرارية أخرى.

البرنامج المُستخدم للحسابات في هذه الرسالة المُقدّمة هو ماثيماتيكا® ١٠.

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قسم الرياضيات

طرائق تكرارية شبه تحليلية للمعادلات التفاضلية الخطية وغير الخطية

رسالة

مقدمة إلى كلية التربية للعلوم الصرفة / ابن الهيثم ، جامعة بغداد
كجزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

من قبل
مصطفى محمود عزيز
بإشراف
أ. م. د. مجيد أحمد ولي

١٤٣٨ هـ ٢٠١٧ م