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Analytic and Numerical Solutions for Non-Linear Differential Equations

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

وَلِيَعْلَمَ الَّذِينَ أُوتُوا الْعِلْمَ أَنَّهُ الْحَقُّ مِنْ رَبِّكَ فَيُؤْمِنُوا بِهِ فَتُخْبِتَ لَهُ قُلُوبُهُمْ
وَإِنَّ اللَّهَ لَهَادِ الَّذِينَ آمَنُوا إِلَىٰ صِرَاطٍ مُسْتَقِيمٍ ﴿٥٤﴾

صدق الله العظيم

سورة الحج الآية (٥٤)

الإهــداء

إلى

معلم البشرية ومنبع العلم نبينا محمد (صلى الله عليه وسلم)

إلى الينبوع الذي لا يمل العطاء إلى من حاكت سعادتي بخيوط منسوجة من قلبها إلى

والدتي العزيزة.

إلى من سعى وشقى لأنعم بالراحة والهناء الذي لم يبخل بشيء من اجل دفعي في طريق النجاح

الذي علمني أن ارتقي سلم الحياة بحكمة وصبر إلى

والدي العزيز.

إلى من تقاسموا ا معي عبء الحياة واظهروا لي ما هو أجمل من الحياة إلى

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إلى من علمونا حروفا من ذهب وكلمات من درر إلى من صاغو لنا علمهم حروفا ومن فكرهم منارة تنير لنا مسيرة العلم والنجاح إلى

أساتذتي الكــرام

إلى من مهد لي طريق العلم والمعرفة

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Committee Certification

We certify that we have read this thesis entitled **"Analytic and numerical solutions for non-linear differential equations"** and as examining committee, we examined the student **(Muntasser Ismail Adwan)** in its contains and what is connected with it, and that in our opinion it is adequate for the partial fulfillment of the requirement for the degree of Master of Science in Mathematics.

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ABSTRACT

The main aim of this thesis is to use three iterative methods which are implemented to get the approximate solutions for some important ODEs, PDEs that appeared in physics and engineering.

The first objective of this thesis will be focused on some basic concepts of the differential equations and the existence and uniqueness solution for the ODEs.

The second objective is to implement the three proposed iterative methods, Tamimi-Ansari method, Daftardar-Jafari method and Banach contraction method, which are using to find the approximate solutions to some problems that arise in physics, such as Painlevé I, Painlevé II, Pendulum , and Falkner-skan equations. The obtained results are compared numerically with other numerical methods, such as the fourth order method Runge-Kutta and Euler method. In addition, we have presented several comparisons among these methods, Adomian decomposition method and variational iteration method. Moreover, the convergence of the proposed methods were given are based on the Banach fixed point theorem. The results of the maximal error remainder values show that the present methods are effective and reliable.

The third objective is to use the Tamimi-Ansari method, Daftardar-Jafari method and Banach contraction method to solve the one dimension, two dimension and three dimension non-linear wave equations to get a new approximate solutions approaching to the exact solution. Also, the convergence analysis of the three methods will be presented using the Banach fixed point theorem. Each method does not require any assumption to deal with a nonlinear term. These methods are quite efficient and practically well suited for use in these problems. There are many examples that demonstrate the accuracy and efficiency of this methods.

AUTHOR’S PUBLICATIONS

Journal Papers

1. M. A. Al-Jawary, M. I. Adwan and G. H. Radhi, Three iterative methods for solving second order nonlinear ODEs arising in physics. Journal of King Saud University-Science,(2018),(In press).

2. M. A. Al-Jawary and M. I. Adwan, Reliable iterative methods for solving the Falkner-Skan equation. Gazi University Journal of Science (Under Review).

3. M. A. Al-Jawary and M. I. Adwan, Analytic and numerical solutions for linear and nonlinear 1D, 2D and 3 D wave equations. Computers and Mathematics with Applications (Under Review).

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LIST OF SYMBOLS AND ABBREVIATIONS

$ODEs$: Ordinary Differential Equations

$PDEs$: Partial Differential Equations

TAM: Temimi- Ansari Method

DJM: Daftardar-Jafari method

BCM: Banach Contraction Method

RK4: Runge-Kutta 4 Method

ADM: Adomian Decomposition Method

VIM: Variational Iteration Method

IVP: Initial Value Problem

BVP: Boundary Value Problem

$P(x)$: The Padé approximation

A_n: Adomian Polynomials

ER: Error Remainder

MER: Maximum Error Remainder

λ: Lagrange's multiplayer

$|r_n|$: Absolute Error

vi : The iterative solutions

$\|\ \|$: Norm

Eq. : The equation

Eqs. :The equations

Figs. : The figure

Introduction

Ordinary and partial differential equations have many applications in science and engineering, especially in problems that have the form of non-linear equations. Application of the non-linear ordinary differential equations (ODE) and partial differential equations (PDE) by mathematicians and researchers have become more important and interesting. It has been described different types of phenomenas, such as modeling dynamics, thermal conductivity, diffusion, acoustic waves, transport and many others[89].

In recent years, approximate and analytical methods have been used to solve different types of linear and nonlinear differential equations such as: variational iteration method (VIM) [٩١], Adomian decomposition method (ADM) [٢٤], Homotopy perturbation method (HPM) [٦٥], residual power series method [٥٩], Laplace decomposition method [6٠] etc. In addition, these methods give some useful solutions, but there are some drawbacks like calculating the Lagrange multiplier in the VIM and the Adomian polynomials for the nonlinear problems in ADM, which leads to complex calculations for a number of iterations.

In this thesis, some types of ordinary and partial differential equations that appear in the problems of physics, engineering and other applied sciences will be solved using reliable iterative methods. One of these equations is the Painlevé I and Painlevé II, which are ordinary differential equations of the second-order with initial conditions. The Painlevé equations have appeared in a variety of important physical applications, of which are quantum gravity, quantum field theory general relativity, nonlinear optics [79]. The other problem is the pendulum equation, which is an initial value problem a nonlinear ODEs of the second-order. There are many applications of pendulum equation like clocks, cranes and machinery movement and other.

Moreover, the Falkner-Skan equation, which classified as one of the nonlinear third-order ordinary differential equations. It has modeled as a variety of important physical applications, such as insulating materials, applications of glass and polymer studies[33]. The main problem of numerical methods in solving this equation is how to deal with the infinite boundary conditions.

Furthermore, the other equations that will be solved are the second order 1D, 2D and 3D non-linear wave equations. They have great importance in the field of physics and engineering. The wave equations have taken great interest by the researchers, in addition to solving different types of them [38], as well as, to describe many important phenomena's, such as acoustic problems for the velocity potential, shock waves, chemical exchange processes in chromatography, sediment transport in rivers and waves in plasmas [47]. There are some methods and techniques that have been used by many researchers to solve different types of these problems [59,32,83,87,80].

Recently, Temimi and Ansari (TAM) [35] have suggested a new semi-analytical iterative technique to solve nonlinear problems. The TAM was inspired from the homotopy analysis method (HAM) [79]. The TAM used to solve many ODEs and PDEs, such as PDEs and KdV equations [25], differential algebraic equations (DAEs) [48], nonlinear Burgers advection–diffusion equations [49]. Moreover, this method is effective and reliable and does not require restrictions to deal with non-linear terms.

The other proposed method which is presented for the first in 2006 by Daftardar-Gejji and Jafari, (DJM) [84] to solve nonlinear equations. This method has been used to solve different type of equations, such as fractional differential equations [86], PDEs [75], Volterra integro-differential equations and some applications for the Lane-Emden equations [55] and evolution equations [76]. This method presents a proper solutions, which converges to the exact solution "if such a solution exists" through successive approximations.

The third iterative method depends on the Banach contraction principle (BCP) [85], which considered as the main source of the metric fixed point theory. The Banach contraction principle also known to be Banach's fixed point theorem (BFPT) which has been used to solve various kinds of differential and integral equations [57]. Also, this method does not require complex calculations and requires no restrictions to deal with nonlinear terms.

This thesis has been arranged as follows: in chapter one, the basic concepts of the non-linear differential equations (ordinary and partial) and some analytical and approximate methods will be introduced to solve some scientific applications, such as ADM and VIM. Chapter two, presents the basic ideas of the proposed iterative methods, the convergence of the proposed methods will be given. Furthermore, the proposed iterative methods successfully implemented to solve the Painlevé I, Painlevé II, pendulum and Falkner-skan equation. In chapter three, the proposed methods will be applied to solve the 1D, 2D and 3D linear and nonlinear wave equations and the convergence of the proposed methods will be given. Finally, in chapter four the conclusions and future works are presented.

CHAPTER 1

Basic Concepts

Chapter 1

Basic Concepts

1.1 Introduction

The non-linear differential equations have played an important role in mathematics applications and considered as a tool to interpret many events in engineering, and applied science. Many of these problems are nonlinear equations resulting from a particular application that may be complex and sometimes. However, there are effective and reliable methods to find the approximate, analytic or numerical solutions.

This chapter has been arranged into five sections. In section 2, a number of definitions and theorems will be introduced. In section 3, the existence and uniqueness theory for the solution of boundary value problem will be given. In section 4, some types of ODEs are displayed, such as Painlevé I, Painlevé II, pendulum and Falkner-Skan equations. Also, some analytical and approximate methods will be introduced to solve some scientific applications, such as ADM and VIM. Finally, in section 5, the wave equations in 1D, 2D, and 3D will be presented. Also, the 1D, 2D and 3D wave equations will be solved by ADM and VIM.

1.2 Preliminaries

Definition 1.2.1: [45]

The problem of the initial value for both ordinary and partial differential equation, with a specified value is called the initial condition, at the point in the domain of the solution. The number of initial conditions depends on the order of the differential equation.

Definition 1.2.2: [45]

The problem of the boundary value for the differential equation with a set of boundary conditions, at two or more points is called the boundary conditions. There are four types of boundary conditions which are Dirichlet, Neunmann, Mixed and Robin boundary conditions [5].

Definition 1.2.3: [6,9]

An equation that contains an unknown function y within an integral sign is called an integral equation. A standard form of linear integral equation is:

$$y(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t,y)\, y(t)dt, \tag{1.1}$$

where y is the unknown function that appears in most under inside and outside the integration sign, h and g are two functions which are the limits of integration. The functions f and k are known functions, where k is called the kernel of the integral equation and λ is a constant parameter.

Definition 1.2.4: [23]

Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called contraction on X if there exist a positive real number $k < 1$, such that $\forall\ x, w \in X$ implies $d(T_x, T_w) \leq k\, d(x, w)$.

Definition 1.2.5: [23]

Let $T:[u,w] \longrightarrow [u,w]$ be a mapping, then is said to satisfy a Lipschitz condition if there exist positive a constant L (called the Lipschitz constant) such that for all $x\,,y \in [u\,,w]$, then

$$\left|T_x - T_y\right| \le L\left|x-y\right|.$$

Definition 1.2.6 : [31]

The Padé approximation to y on $[a\,,b]$ is the ratio between two polynomials are $Q_m(x)$ and $R_n(x)$ of degrees m and n, respectively and is given by the following relationship:

$$P(x) = \frac{Q_m(x)}{R_n(x)} = \frac{\sum_{i=0}^{m} a_i\, x^i}{1+\sum_{j=0}^{n} b_j\, x^j}$$
$$= \frac{a_0 + a_1x + a_2x^2 + \cdots + a_mx^m}{1 + b_0 + b_1x + b_2x^2 + \cdots + b_nx^n}\,. \qquad (1.2)$$

Theorem 1.1.7 (Banach Fixed Point Theorem): [23]

Let (X,d) be a metric space , where $X \neq \emptyset$. Consider that X is complete and let $T: X \longrightarrow X$ be a contraction on X , then T admits a unique fixed-point $x^* \in X, \quad T(x^*) = x^*$.

Theorem 1.2.8:[90]

Let F be an operator from a Hilbert space H to H. The finite series solution $y_n(x) = \sum_{i=0}^{n} v_i(x)$ converges if there exists $0<\gamma<1$, such that $\|F[v_0+v_1+\cdots+v_{i+1}]\| \le \gamma\|F[v_0+v_1+\cdots+v_i]\|$ (where $\|v_{i+1}\| \le \gamma\|v_i\|$) $\forall i = 0,1,2,\ldots$.

Theorem(1.2.8) is a special case of Banach's fixed point theorem, which is a sufficient condition to study the convergence.

Theorem 1.2.9: [90]

If the series solution $y(x) = \sum_{i=0}^{\infty} v_i(x)$ is convergent, then this series will represent the exact solution of the current nonlinear problem.

Theorem 1.2.10: [90]

Suppose that the series solution $\sum_{i=0}^{\infty} v_i(x)$ is convergent to the exact solution y. If the truncated series $\sum_{i=0}^{n} v_i(x)$ is used as an approximation to the solution of the current problem, then the maximum error $E_n(x)$ is estimated by

$$E_n(x) \leq \frac{1}{1-\gamma} \gamma^{n+1} \|v_0\|. \tag{1.3}$$

1.3 Existence and Uniqueness of the Solution for the Boundary Value Problems[73]

In this section, we will discuss the existence and uniqueness of the solution for the boundary value problems.

Let us, consider the boundary value problems for equation

$$y^{(n)}(x) = f\left(x, y, y', \dots, y^{(n-1)}\right), \tag{1.4}$$

with one type of the boundary conditions:

$$\alpha_{n-3} y^{(n-3)}(x_1) + \alpha_{n-2} y^{(n-2)}(x_1) = y_1$$

$$\beta_i y^{(i)}(x_2) + \beta_{i+2} y^{(i+2)}(x_2) = y_{i+2}\,, \qquad i = 0, 1, \dots . n-3$$

$$y^{(n-2)}(x_2) = m,$$

or

$$\alpha_{n-3} y^{(n-3)}(x_1) + \alpha_{n-2} y^{(n-2)}(x_1) = y_1$$

$\beta_i y^{(i)}(x_2) + \beta_{i+2} y^{(i+2)}(x_2) = y_{i+2}, \qquad i = 0, 1, \ldots . n-3$

$c_{n-3} y^{(n-3)}(x_3) + c_{n-2} y^{(n-2)}(x_3) = y_n$

Where

$x_1 < x_2 < x_3$, $m, y_k \in R, (k = 1,2, \ldots, n)$ are arbitrary constant and $n > 2$ is a fixed positive integer.

We denote the following conditions by Pi

P_1: $f(x, y_0, y_1, \ldots, y_{n-2}, y_{n-1})$ is continuous on $[x_1, x_3] \times R^n$

P_2: Then, there exists a unique solution of Eq. (1.4) on $[x_1, x_3]$

P_3: $\forall\, x \in (x_1, x_2], f(x, y_0, y_1, \ldots, y_{n-2}, y_{n-1}) < f(x, z_0, z_1, \ldots, z_{n-2}, z_{n-1})$, when $y_j \leq z_{j,}$ $j = 0,1, \ldots, n-3$, $y_{n-2} < z_{n-2}$, $y_{n-1} = z_{n-1}$

P_4: $\forall\, x \in (x_2, x_3], f(x, y_0, y_1, \ldots, y_{n-2}, y_{n-1}) < f(x, z_0, z_1, \ldots, z_{n-2}, z_{n-1})$, *when* $y_j \leq z_{j,}$ $j = 0,1, \ldots, n-3$, $y_{n-2} < z_{n-2}$, $y_{n-1} = z_{n-1}$

we use these results for two and three-points BVPs

$$y'' = f(x, y, y'), \text{with } \; y(x_1) = y_1, \quad y(x_2) = y_2, \tag{1.5}$$

$$y''' = f(x, y, y', y''), \text{with } \; y(x_1) = y_1, \quad y(x_2) = y_2, \qquad y(x_3) = y_3 \tag{1.6}$$

Let us to demonstrate the existence and uniqueness of the solution for two-point value problem by theorem 1.3.5 and three-points boundary value problem by using theorem 1.3.6 .

Theorem 1.3.1: [73]

Suppose that $P_1, P_2 P_3$ are satisfied and $\beta_i\ \beta_{i+2} < 0, i = 0,1, .. n-3$, $\alpha_{n-3} < 0, \alpha_{n-2} > 0$, then for every $m, y_k \in R, k = 1,2, \dots n-1$, then there exists a unique solution two-point boundary value problem on $[x_1, x_2]$.

Theorem 1.3.2: [73]

Suppose that $P_1, P_2 P_3 , P_4$ are satisfied and $\alpha_{n-3} < 0, \alpha_{n-2} > 0, \beta_i\ \beta_{i+2} < 0, i = 0,1, .. n-3$, $c_{n-3} > 0$ then for every $y_k \in R, k = 1,2, \dots n-1$, then there exists a unique solution to the three-points boundary value problem.

1.4 Ordinary Differential Equations

The ODEs are the formula that contains the dependent variable y [61], some of its derivatives and the independent variable x. In other words any equation $F(x, y, y', y'', \dots y^n) = 0$ is called the ordinary differential equation [45]. ODEs are of great importance especially in the interpretation of many physical phenomena, chemical and engineering.

1.4. 1 The Painlevé Equations I, II

The Painlevé equations firstly originated during 1895-1910 through tests conducted by two French mathematicians Paul Painlevé and Bertrand Gambier [20]. It was classified as the differential equations of the second order that have a significant role in many areas of mathematics and physics. Painlevé equations have been used in many applications including nonlinear waves, plasma physics, statistical mechanics, fiber-optic and others [21].

Many methods are successfully used to solve the Painlevé equations. For example, the optimal homotopy asymptotic method (OHAM) [26] to solve the Painlevé equation II, Adomian decomposition method (ADM) [18] and Legendre Tau Method [71]. Hesameddini and Peyrovi [22] have applied the variational iteration method (VIM) and homotopy perturbation method (HPM) to find an approximate solution of Painlevé equation I.

Painlevé equations of the first and second types will be defined by the following formulas [20,13].

PI: $y''(x) = 6y^2(x) + x, \qquad 0 < x < 1 \qquad (1.7)$

with the given initial conditions: $y\,(0) = 0, y'\,(0) = 1$.

PII: $y''(x) = 2y^3(x) + x\,y(x) + \mu, \qquad 0 < x < 1 \qquad (1.8)$

with the initial conditions $y\,(0) = 1,\ y'\,(0) = 0$, where μ is a known parameter.

1.4.2 Analytical methods for solving the Painlevé equations.

In this subsection, the ADM and VIM will be used to solve the nonlinear Painlevé equation I .

1.4.2.1 The ADM

In the late of the 20th century of the 1980s, the ADM was found by George Adomian [28, 29]. This method has an important role in applied mathematics [78], plus easy handling of many types of ODEs, PDEs (linear and nonlinear), integral equations and other equations [72]. The solution resulting from this method is in the form of a series.

Consider the nonlinear functional equation given by following [88]

$$Ly = Ny + f(x), \qquad (1.9)$$

where, L is a linear operator and N is a nonlinear operator, $f(x)$ is a given function.

The solution y is represented as the sum of series

$$y = \sum_{n=0}^{\infty} y_n \;, \tag{1.10}$$

and the nonlinear function $N(y)$ is decomposed as follows:

$$N(y) = \sum_{n=0}^{\infty} A_n \tag{1.11}$$

Where A_n are Adomian polynomials which are computed by

$$A_n = \frac{1}{n!} \frac{d^n}{d\,\lambda^n} \left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0,} \qquad n = 1, 2, 3, \ldots \tag{1.12}$$

Adomian polynomials are organized to have the form

$$A_0 = F(y_0\,),$$

$$A_1 = y_1\, F'\,(y_0),$$

$$A_2 = y_2 F'(y_0) + \frac{{y_1}^2}{2!} F''(y_0),$$

$$A_3 = y_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{{y_1}^3}{3!} F'''(y_0),$$

$$\vdots$$

By substituting Eq. (1.10) and (1.11) in Eq. (1.9), we get:

$$\sum_{n=0}^{\infty} y_n = \sum_{n=0}^{\infty} A_n + f, \tag{1.13}$$

Now, we will create the ADM string terms as follows:

$$\left.\begin{aligned} y_0 &= f, \\ y_n &= A_{n-1}, \quad n = 1, 2, 3, \ldots \end{aligned}\right\} \tag{1.14}$$

Note 1.1:

In order to transform multiple integrations into single, we use the general transformation formula given by [6]

$$\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_{n-1}} y(x_n)dx_n \dots dx_1$$
$$= \frac{1}{(n-1)!}\int_0^x (x-s)^{n-1} y(s)ds. \quad (1.15)$$

1.4.4 Error analysis

Since the exact solutions are unknown for both Painlevé I and II, therefore we used the proper function of the maximal error remainder [42]
to check the accuracy of the approximate solutions.

$$ER_n(x) = L(y) - N(y) - f(x). \quad (1.16)$$

The maximal error remainder is

$$MER_n = \max_{0\le x\le 1} \left| ER_n(x) \right|. \quad (1.17)$$

The ADM will be used to solve the Painlevé differential equations.

Painlevé I equation [82]:

Let us consider the Painlevé equation I (1.7)
with the given initial conditions: $y\,(0) = 0, y'\,(0) = 1.$

Integrating both sides of Eq. (1.7) twice from 0 to x and using the given initial conditions, we can get

$$y(x) = x + \frac{1}{6}x^3 + \int_0^x \int_0^x 6\, A_n \; d\tau\, d\tau \quad (1.18)$$

where A_n are the Adomian polynomials, which represents the nonlinear term $y^2(\tau)$.
By reducing the integration in Eq. (1.18) from double to single [6], we get

$$y(x) = x + \frac{1}{6}x^3 + \int_0^x (x-\tau)(6\, A_n)\, d\tau, \quad (1.19)$$

By applying the ADM, we obtain

$y_0 = x + \frac{1}{6}x^3,$

$$y_1(x) = \int_0^x (x-\tau)(6\,A_0)\,d\tau, \tag{1.20}$$

$$y_1 = \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336},$$

$$y_2(x) = \int_0^x (x-\tau)(6\,A_1)\,d\tau, \tag{1.21}$$

$$y_2 = \frac{x^7}{7} + \frac{x^9}{40} + \frac{71x^{11}}{46200} + \frac{x^{13}}{26208},$$

$$y_3 = \frac{x^{10}}{28} + \frac{23x^{12}}{3080} + \frac{5219x^{14}}{8408400} + \frac{3551x^{16}}{144144000} + \frac{95x^{18}}{224550144},$$

$\vdots$

Then,

$$y_n = \sum_{i=0}^{n} y_i,$$

Continue to get the approximations till $n = 5$, for brevity not listed.

The error remainder function is evaluated:

$$ER_n(x) = y_n''(x) - 6y_n^2(x) - x, \tag{1.22}$$

and the MER_n is:

$$MER_n = \max_{0.01 \le x \le 0.1} |ER_n(x)|, \tag{1.23}$$

Table 1.1 and Figure 1.1 illustrate the convergence of the solution by using the MER_n, the error will be reduced when the number of iterations will be increased. See appendices A**.**

Table 1.1: The maximal error remainder : MER_n by the ADM, where n =1,..,5.

n	MER_n
1	0.0000601952
2	3.22501×10^{-8}
3	1.29031×10^{-11}
4	4.35069×10^{-15}
5	2.08167×10^{-17}

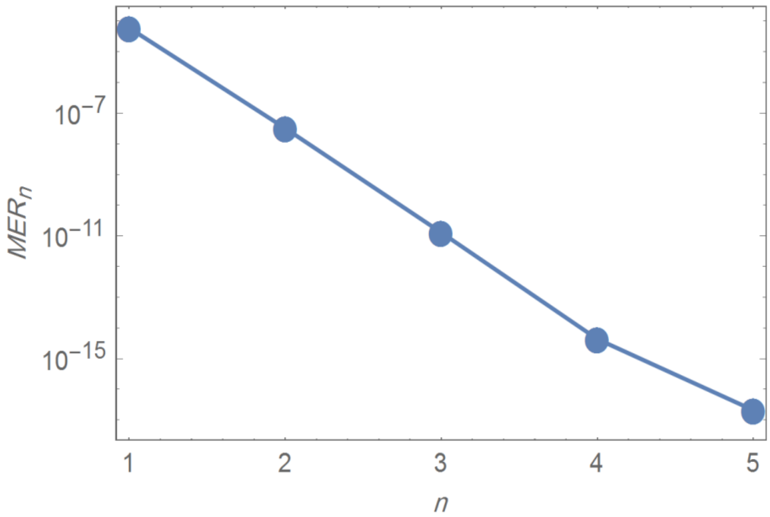


Figure 1.1: Logarithmic plots of MER_n versus n is 1 through 5 by ADM.

Painlevé II equation [18]:

By using Eq.(1.8) with the initial conditions $y\,(0) = 1,\; y'\,(0) = 0$.

Applying the ADM, we get

$$y_0 = 1 + \frac{x^2}{2}$$

$$y_1 = x^2 + \frac{x^3}{6} + \frac{x^4}{4} + \frac{x^5}{40} + \frac{x^6}{20} + \frac{x^8}{224}$$

$$y_2 = \frac{x^4}{2} + \frac{x^5}{10} + \frac{23x^6}{90} + \frac{x^7}{30} + \frac{19x^8}{320} + \frac{x^9}{160} + \frac{131x^{10}}{16800} + \frac{47x^{11}}{123200} + \frac{19x^{12}}{24640} + \frac{3x^{14}}{81536}$$

$\vdots$

$$y_n = \sum_{i=0}^{n} y_i,$$

Continue to get the approximations till $n = 5$, for brevity not listed.

Table 1.2: The maximal error remainder : MER_n by the ADM, where n =1,..,5.

n	MER_n
1	0.0634125
2	0.000950605
3	0.00001041
4	9.77952×10^{-8}
5	8.40299×10^{-10}

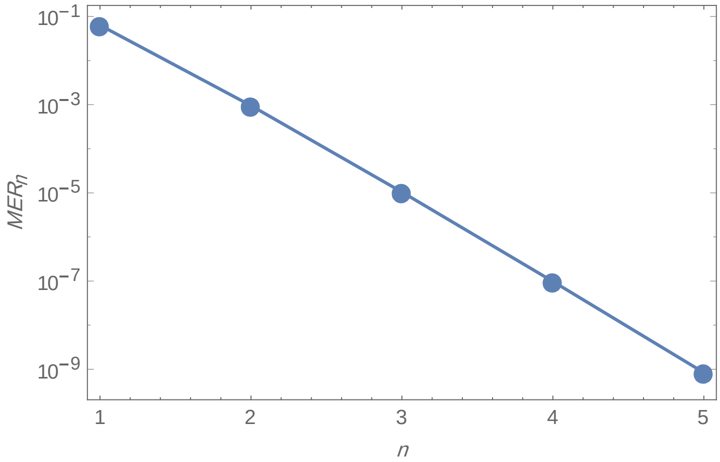


Figure 1.2: Logarithmic plots of MER_n versus n is 1 through 5 by ADM.

1.4.2.2 The Variational Iteration Method (VIM):

The VIM was first suggested in 1999 by Ji-Huan He [43]. It can be applied to many differential and integral equations, linear and nonlinear [54,30]. The

VIM does not require specific treatments for nonlinear problems as in Adomian method [5]. The VIM gives the solution in a series form that replicates to the closed form solution if an exact solution exists [54]. In order to clarify the basic concepts of the VIM, we consider the following

$$Ly(x) + Ny(x) = f(x), \qquad (1.24).$$

The VIM introduces the correction functional for Eq. (1.24) in the form

$$y_{k+1}(x) = y_k(x) + \int_0^x \left[\lambda(t,x)\left(Ly_k(t) + N\tilde{y}_k(t) - f(t)\right)\right] dt, \qquad (1.25)$$

The Lagrang multiplier λ can be calculated for ODEs of n the order in the following form [7, 8]:

$$\lambda = \frac{(-1)^n}{(n-1)!}(t-x)^{n-1}, \qquad n \geq 1. \qquad (1.26)$$

The sequence of solutions are given by: $y(x) = \lim_{n\to\infty} y_n(x)$.

1.4.6 The VIM for solving Painlevé I differential equation

Painlevé I equation [22].

By using Eq.(1.7) with the given initial conditions: $y\,(0) = 0, y'\,(0) = 1$.
The correction functional is:

$$y_{n+1} = y_n(x) + \int_0^x \lambda(t,x)(y_n''(t) - 6{y_n}^2(t) - t)dt, \qquad (1.27)$$

By using the formula in Eq. (1.26) leads to $\lambda(t,x) = t - x$ and substituting this value into the functional Eq.(1.27), to get

$$y_{n+1} = y_n(x) + \int_0^x (t-x)(y_n''(t) - 6{y_n}^2(t) - t)dt, \tag{1.28}$$

Will be the initial approximation as:

$$y_0 = x + \frac{x^3}{6},$$

$$y_1 = x + \frac{x^3}{6} + \int_0^x (t-x)(y_0''(t) - 6{y_0}^2(t) - t)dt, \tag{1.29}$$

$$y_1 = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336},$$

$$y_2 = y_1(x) + \int_0^x (t-x)(y_1''(t) - 6{y_1}^2(t) - t)dt, \tag{1.30}$$

$$y_2 = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^7}{7} + \frac{x^8}{336} + \frac{x^9}{40} + \frac{x^{10}}{60} + \frac{71x^{11}}{46200} + \frac{x^{12}}{330} + \frac{x^{13}}{26208} + \frac{187x^{14}}{764400} + \frac{x^{16}}{100800} + \frac{x^{18}}{5757696}.$$

We continue to find the other approximations till $n = 5$, for brevity they are not listed.

It is possible to calculate the maximal error remainder to see the highest level of precision that we can achieve. Table 1.3 and Fig. 1.3 show the MER_n of the approximate solution obtained by VIM, by increasing the iterations, the errors will be decreasing.

Table 1.3: The maximal error remainder: MER_n by the VIM, where n =1,..,5, and $0.01 \le x \le 0.1$

n	MER_n
1	0.0000601952
2	1.72121×10^{-8}
3	2.29681×10^{-12}
4	1.52656×10^{-16}
5	2.77556×10^{-17}

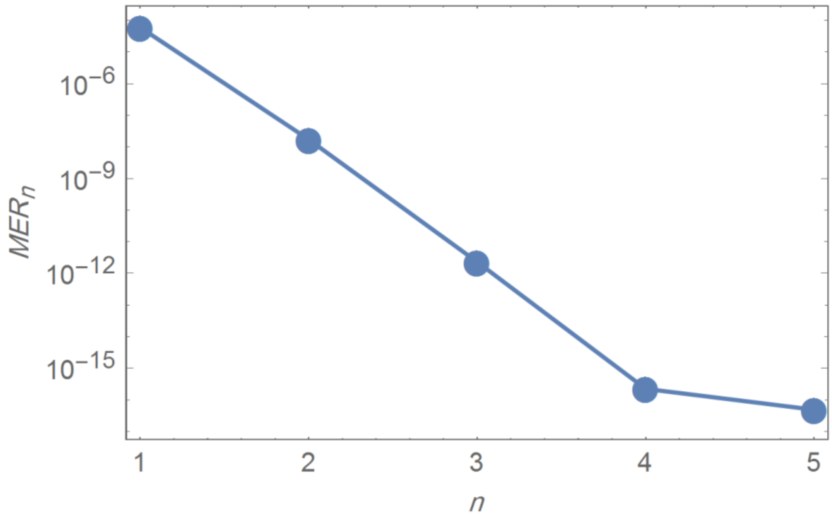


Figure 1.3: Logarithmic plots of MER_nversus n is 1 through 5 by VIM

1.4.7 VIM for solving the Painlevé II differential equation

Painlevé II equation[18]

By using Eq.(1.8) with the initial conditions $y\,(0) = 1,\ y'\,(0) = 0$.

The correction functional is:

$$y_{n+1} = y_n(x) + \int_0^x \lambda(t,x)(y_n''(t) - 2{y_n}^3(t) - ty(t) - \mu)dt, \qquad (1.31)$$

Then, the iteration formula

$$y_{n+1}=y_n(x)+\int_0^x(t-\ x)(y_n{}''(t)-2y_n{}^3(t)-ty(t)-\mu)dt, \tag{1.32}$$

The initial approximation will be

$$y_0=1+\frac{x^2\mu}{2},$$

$$y_1=y_0(x)+\int_0^x(t-\ x)(y_0{}''(t)-2y_0{}^3(t)-ty_0(t)-\mu)dt, \tag{1.33}$$

$$y_1=1+x^2+\frac{x^3}{6}+\frac{x^2\mu}{2}+\frac{x^4\mu}{4}+\frac{x^5\mu}{40}+\frac{x^6\mu^2}{20}+\frac{x^8\mu^3}{224},$$

$$y_2=y_1(x)+\int_0^x(t-\ x)(y_1{}''(t)-2y_1{}^3(t)-ty_1(t)-\mu)dt, \tag{1.34}$$

$$y_2=1+x^2+\frac{x^3}{6}+\frac{x^4}{2}+\frac{x^5}{10}+\frac{37x^6}{180}+\frac{x^7}{21}+\frac{13x^8}{336}+\frac{x^9}{72}+\frac{x^{10}}{540}+\frac{x^{11}}{11880}+\frac{x^2\mu}{2}+\frac{x^4\mu}{4}+\frac{x^5\mu}{40}+\frac{x^6\mu}{4}+\frac{x^7\mu}{30}+\frac{241x^8\mu}{2240}+\frac{x^9\mu}{40}+\cdots.$$

We continue to obtain the other iterations till n=5 , they are not listed for brevity.

When $\mu=1$ [20].

Table (1.3) and Fig (1.3) show the MER_n of the approximate solution obtained by the VIM, also, by increasing the iterations, the errors will be decreasing.

Table 1.4: The maximal error remainder : MER_n by the VIM, where n =1,..,5, and $0.01 \leq x \leq 0.1$

n	MER_n
1	0.0634125
2	0.000323411
3	6.58637×10^{-7}
4	7.18421×10^{-10}
5	4.8539×10^{-13}

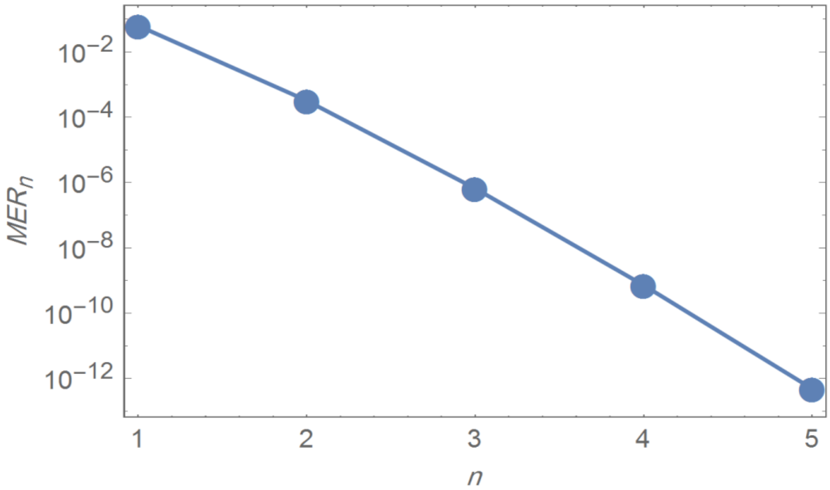


Figure 1.4: Logarithmic plots of MER_n versus n is 1 through 5 by VIM.

1.4.8 Pendulum equation

The pendulum equation is a nonlinear ODEs that arise in physical applications. In the 17th century, the Italian scientist Galileo was the first to study the physical properties of the pendulum [62].The equations of commentators describe many physical phenomenas, including the clock pendulum and others. There are many analytical methods that have been used to solve this type of mathematical model such as ADM [41], VIM[44]and power series method [40]. We consider the general formula of the pendulum equation as follows [1]:

$$y''(t) + \frac{h}{l}\sin y = 0, \tag{1.35}$$

with the initial conditions: $y(0) = y_0$ and $y'(0) = 0$.

Where y is the angle offset, t the time, h is the quickened due to gravity and l is the length of the pendulum.

1.4.9 ADM for solving the nonlinear pendulum equation[41]:

Consider the following pendulum equation.

$$y''(x) + \sin y = 0, \qquad x \in [0,1] \tag{1.36}$$

with the initial conditions: $y(0) = 0$ and $y'(0) = 1$.

We can solve it by using the approximation of $\sin y \approx y - \frac{1}{6}y^3 + \frac{1}{120}y^5$ as it‘s used in [43].

By applying the ADM, we obtain

$$y_0 = x$$

$$y_1 = -\frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$$y_2 = \frac{x^5}{120} - \frac{11x^7}{5040} + \frac{19x^9}{120960} - \frac{x^{11}}{246400} + \frac{x^{13}}{18869760}$$

$\vdots$

Then

$$y_n = \sum_{i=0}^{n} y_i,$$

Continue to get the approximations till $n = 5$, for brevity not listed

Table 1.5: The maximal error remainder : MER_n by the ADM where n =1,..,5.

n	MER_n
1	0.0959857
2	0.0059253
3	0.00115669
4	0.0000367296
5	4.73742×10^{-6}

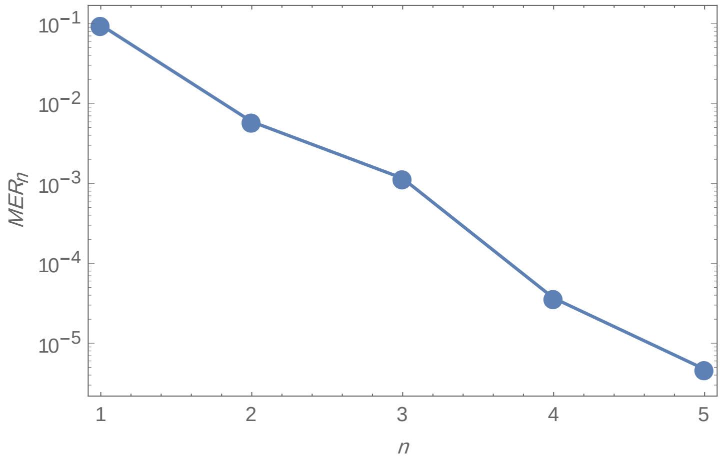


Figure 1.5: Logarithmic plots of MER_n versus n is 1 through 5 by ADM.

1.4.10 VIM for solving the nonlinear pendulum equation[43]:

By using Eq.(1.36) with the initial conditions: $y(0) = 0$ and $y'(0) = 1$. We can solve it by using the approximation of $\sin y \approx y - \frac{1}{6}y^3 + \frac{1}{120}y^5$ as it's used in [43].

The correction functional is:

$$y_{n+1} = y_n(x) + \int_0^x \lambda(t,x)\left(y_n''(t) + y_n(t) - \frac{1}{6}y_n{}^3(t) + \frac{1}{120}y_n{}^5(t)\right)dt, \qquad (1.37)$$

By using the formula in Eq. (1.26) leads to $\lambda = t - x$, to be the iteration formula

$$y_{n+1} = y_n(x) + \int_0^x (t - x)\left(y_n''(t) + y_n(t) - \frac{1}{6}y_n{}^3(t) + \frac{1}{120}y_n{}^5(t)\right)dt, \qquad (1.38)$$

The initial approximation will be

$$y_0 = y(0) + x\,y'(0) = x \qquad (1.39)$$

$$y_1 = y_0(x) + \int_0^x (t - x)\left(y_0''(t) + y_0(t) - \frac{1}{6}y_0{}^3(t) + \frac{1}{120}y_0{}^5(t)\right)dt, \qquad (1.40)$$

$$y_1 = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040},$$

In the same way as before, the second approximation can be written in the form

$$y_2 = y_1(x) + \int_0^x (t - x)\left(y_1''(t) + y_1(t) - \frac{1}{6}y_1{}^3(t) + \frac{1}{120}y_1{}^5(t)\right)dt, \qquad (1.41)$$

$$y_2 = x - \frac{x^3}{6} + \frac{x^5}{60} - \frac{x^7}{420} + \frac{127x^9}{362880} - \frac{893x^{11}}{19958400} + \frac{367x^{13}}{70761600} - \frac{607x^{15}}{1143072000} + \cdots.$$

We continue in this manner to obtain the other iterations till n=5 , they are not listed for brevity, see appendix B

Table 1.4 and Fig. 1.4 illustrate the MER_n of the approximate solution obtained by the VIM, it can be seen that by increasing the iterations, the errors will be decreasing. See appendix B.

Table 1.6: The maximal error remainder : MER_n by the VIM. where n =1,..,5.

n	MER_n
1	0.0959857
2	0.00429285
3	0.0000881802
4	1.05634×10^{-6}
5	8.28868×10^{-9}

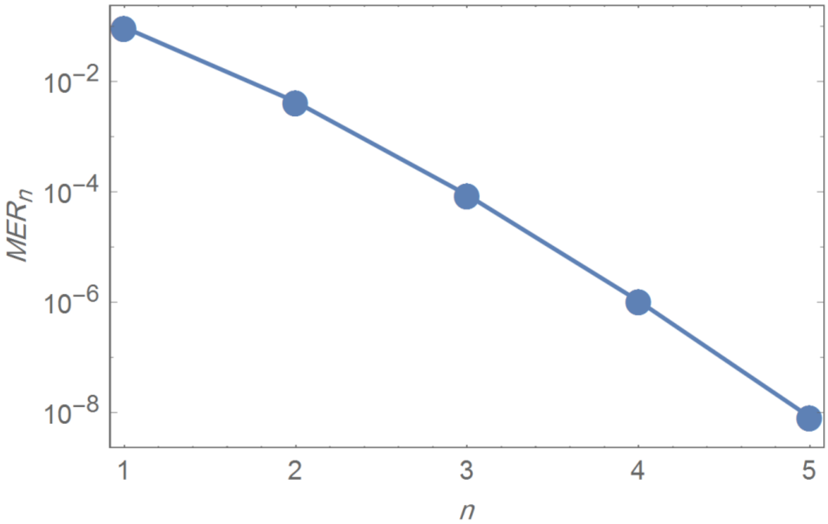


Figure 1.6: Logarithmic plots of MER_n versus n is 1 through 5 by VIM.

1.4.11 Falkner Skan equation

The Falkner-Skan equation was first studied in 1931 by Falkner and Skan [15]. The equation has important applications in several industrial operations, such as cooling of a metallic plate in a cooling bath, an aerodynamic extrusion of plastic sheets, drawing of plastic films, metal spinning, metallic plates and others [33]. There are many methods that have been used to solve this equation, such as an ADM [33], VIM [58,12], OHAM [3]. An iterative finite difference method (IFDM) [34], HPM [67], HAM [14], fourth order Runge-Kutta method (RK4) with shooting techniques[68] and collocation method [64].

The Falkener- Skan problem is defined by the following formula [33]:

$$y'''(x) + y(x)y''(x) + \beta\,[\epsilon^2 - (\,y'(x))^2] = 0, \qquad x \in [0,1] \qquad (1.42)$$

with the boundary conditions:

$$y(0) = 0, \;\; y'(0) = 1 - \epsilon, \;\; y'(\infty) = \epsilon.$$

1.4.12 The ADM for solving the Falkner Skan equation[33]

By using Eq. (1.43) with the boundary conditions $(0) = 0,\ \ y'(0) = 1 - \epsilon,\ \ y'(\infty) = \epsilon.$

Suppose that $y''(0) = a,$ when($\epsilon = 0.1$ and $\beta = 0.5$)[33]

Applying the ADM, we get

$$y_0 = 0.9x + \frac{ax^2}{2} - 0.000833333x^3$$

$$y_1 0.0675x^3 + 0.0000375x^5 + 0.00000694444ax^6 - 4.96032 \times 10^{-9}x^7,$$

$$y_2 = -0.0030375x^5 - 0.0005625ax^6 - 0.00000160714x^7 - 0.00000111607ax^8 + 9.92063 \times 10^{-10}x^9 - 1.37787 \times 10^{-7}a^2x^9 + 2.48016 \times 10^{-10}ax^{10} - 1.127346 \times 10^{-13}x^{11}.$$

Continue in this solution to get the fifth approximation, but for the brevity we can't write all.

By using the Padé approximant in Eq.(1.2), we get

$$P_2^0\left(\frac{dy_5}{dx}\right) = \frac{0.9+(0.9(0.-0.37037a)+a)x+(0.27+(0.-0.37037a)a)x^2}{1+(0.-0.37037a)x+0.075x^2}$$

By taking $\lim\limits_{x\to\infty} P_2^0(\frac{dy_5}{dx})$, we obtain

$$y_5'(x) = 3.6 - 4.93827a^2 = y_5'(\infty)$$

By using the condition value $(y'(\infty) = \epsilon)$ when $\epsilon = 0.1$, we get

$$3.6 - 4.93827a^2 = 0.1 \qquad (1.43)$$

Solving the Eq.(1.43), we obtain

$$a = \pm 0.841873$$

Thus, where dual solution for the Eq.(1.43), we will use the value that achieves better convergence and that is when$(a = 0.841873)$.

Table 1.7: The maximal error remainder : MER_n by the ADM, where n =1,..,5.

n	MER_n
1	0.246021
2	0.0661609
3	0.0101075
4	0.000832153
5	3.40896×10^{-6}

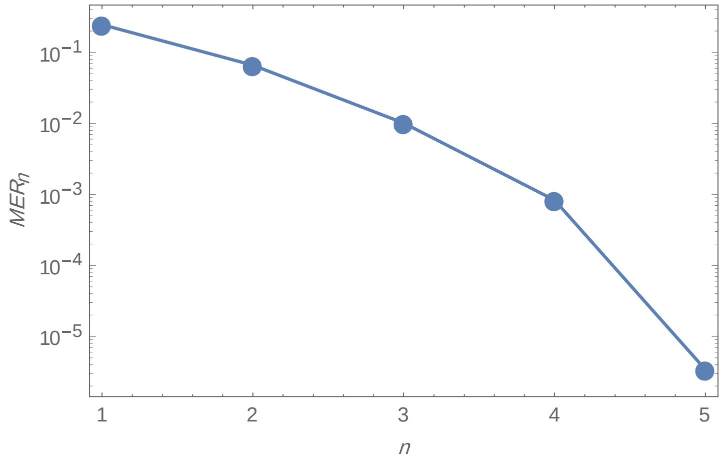


Figure 1.7: Logarithmic plots of MER_nversus n is 1 through 5 by ADM

1.4.13 The VIM for solving the Falkner Skan equation[58]

By using Eq. (1.42)

with the boundary conditions:

$y(0) = 0, \; y'(0) = 1 - \epsilon, \; y'(\infty) = \epsilon.$

Suppose that $y''(0) = a.$

The correction functional for Eq.(1.42) is:

$$y_{n+1} = y_n(x) + \int_0^x \lambda(x,s)\big(y_n'''(s) + y_n(s)y_n''(s)\big) + \beta\,[\epsilon^2 - (y_n'(s))^2]\; ds.$$

The Lagrange multiplier for this problem is $\lambda(x,s) = -\frac{1}{2}(s - x)^2$.

So, we get

$$y_{n+1} = y_n(x) - \frac{1}{2}\int_0^x (s - x)^2 \left(y_n'''(s) + y_n(s)y_n''(s)\right) +$$

$$\beta\, [\epsilon^2 - (\, y_n'(s))^2]\, ds.$$

The first approximation, when($\epsilon = 0.1$ and $\beta = 0.5$)[33] will be the following

$$y_0 = 0.5ax^2 + 0.9x - 0.000833333x^3,$$

$$y_1 = 0.5ax^2 + 0.9x - 0.000833333x^3 - \frac{1}{2}\int_0^x (s - x)^2 \left(y_0'''(s) + y_0(s)y_0''(s)\right) + \beta\, [\epsilon^2 - (\, y_0'(s))^2] ds$$

$$y_1 = 0.9x + 0.5ax^2 + 0.0666667x^3 + 0.0000375x^5 + 0.00000694444\, a\, x^6 - 4.960312 \times 10^{-9}x^7,$$

$$y_2(x) = 0.9x + 0.5ax^2 + 0.0666667x^3 - 0.003x^5 - 0.000555556ax^6 - 0.0000341567x^7 - 0.00000111607ax^8 - 5.42535 \times 10^{-8}x^9 - 1.37787 \times 10^{-7}a^2x^9 - 1.14707 \times 10^{-8}ax^{10} - 1.63465 \times 10^{-12}x^{11} - 3.94571 \times 10^{-12}ax^{12} + 2.92676 \times 10^{-15}x^{13} - 3.3724 \times 10^{-13}a^2x^{13} + 4.73168 \times 10^{-16}ax^{14} - 1.57723 \times 10^{-19}x^{15},$$

and so on till $y_5(x)$, we have

$$y_5(x) = 0.5\mathrm{ax}^2 + 0.9x + 0.0666667x^3 - 0.00833333\mathrm{ax}^2x^4 - 0.003x^5 + \cdots + 1.42488 \times 10^{-163}\mathrm{ax}^2x^{124} + 3.76624 \times 10^{-164}x^{125} - 2.23633 \times 10^{-169}x^{127}$$

By using the Padé approximant in Eq.(1.2), we get

$$a = \pm 0.832666,$$

we will use the value that achieves better convergence and that is when$(a = 0.832666)$.

We can use it to calculate the maximal error remainder to clarify the accuracy level that we achieve. This can be clearly seen in table 1.8 and figure 1.8.

Table 1.8: The maximal error remainder : MER_n by the VIM, where n =1,..,5.

n	MER_n
1	0.246021
2	0.0745229
3	0.0150143
4	0.00226928
5	0.000274414

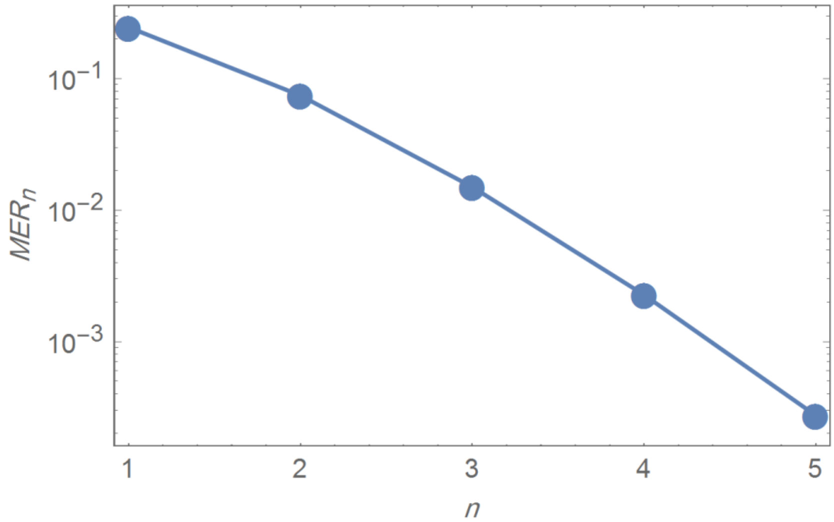


Figure 1.8: Logarithmic plots of MER_nversus n is 1 through 5 by VIM

1.5 Partial Differential Equations

The PDEs is an equation that contains unknown variables or functions and its partial derivatives. In PDEs the function u depends on one or more independent variables $(x, y, \dots)$ with the time variable t. In addition, the PDEs have described many natural phenomenas in the field of physics and geometry. Such as, the heat flow, the wave propagation and other models.

1.5.1 The wave equations

The wave equation was first discovered by Brook Taylor [19].This equation plays an important and significant role in various physical problems, which that requires study in different fields of science and engineering [5]. We consider one dimensional (1D), two dimensional (2D) and three dimensional (3D) non-linear wave equation, which can be given in the following formulae [70]:

$$u_{tt} = u_{xx} + G(u, \dots) + f(x,t)\,, \quad a < x < b\,, \; t > 0, \tag{1.44}$$

with initial conditions

$$u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x).$$

$$u_{tt} = u_{xx} + u_{yy} + G(u, \dots) + f(x,y,t)\,, \quad a < x,y < b\,, \; t > 0, \tag{1.45}$$

with initial conditions

$$u(x,y,0) = f_1(x,y), \quad u_t(x,y,0) = f_2(x,y)$$

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} + G(u, \dots) + f(x,y,z,t)\,, \; a < x,y,z < b\,, \; t > 0, \tag{1.46}$$

with initial conditions

$$u(x,y,z,0) = f_1(x,y,z), \quad u_t(x,y,z,0) = f_2(x,y,z)$$

There are many methods that have been used to solve the wave equations, for example ADM [17,66], HPM [39], VIM [10,38], A new second-order alternating direction implicit method (ADI) [44], Method of difference potentials (MDP) [77], Fixed point iteration method and Newton method [37].

1.5.2 The exact and approximate solutions for non-linear wave equation by ADM

In this subsection, we use the ADM to get the exact and approximate solution for the 1D, 2D, 3D wave equation .

Example 1.1

Let us consider the following 1D linear wave equation[5].

$$u_{tt}(x,t) = u_{xx}(x,t) - 2, \qquad 0 < x < \pi, t > 0 \tag{1.47}$$

with initial conditions: $u(x,0) = x^2$, $u_t(x,0) = sin\ x,$

Integrating both sides of Eq. (1.51) twice from 0 to t and using the given initial conditions, we obtain

$$u(x,t) = -t^2 + x^2 + t\ \text{sin}x + \int_0^t \int_0^t (u_{xx}) dtdt, \tag{1.48}$$

and by reducing the integration in Eq. (1.52) from double to single [6], we get

$$u(x,t) = -t^2 + x^2 + t\ \text{sin}x + \int_0^t (t-s)(u_{xx}) ds \tag{1.49}$$

Then,

$$u_0 = -t^2 + x^2 + t\ \text{sin}x\ , \qquad N(u_{n+1}) = \int_0^t (t-s)(u_{nxx}) ds$$

Applying the ADM, we get

$$u_0 = -t^2 + x^2 + t\ \text{sin}x,$$

$$u_1 = t^2 - \frac{1}{6} t^3 \text{sin}x,$$

$$u_2 = \frac{1}{120} t^5 \sin x ,$$

$$U_n(x,t) = \sum_{i=0}^{n} u_i\,(x,t) \qquad n = 1, 2, \dots \ .$$

$$U_5 \ = \ u_0 + u_1 + \ u_2 + \ u_3 + \ u_4 + \ u_5,$$

$$U_5 = x^2 + t \sin x - \frac{1}{6} t^3 \sin x + \frac{1}{120} t^5 \sin x - \frac{t^7 \sin x}{5040} + \frac{t^9 \sin x}{362880}.$$

The exact solution can be defined by

$$U(x,t) = \sum_{n=0}^{\infty} u_n\,(x,t)$$

$$= x^2 + \sin x \left(t - \frac{1}{6} t^3 + \frac{1}{120} t^5 - \frac{t^7}{5040} + \frac{t^9}{362880} + \cdots \right)$$

$$= x^2 + \sin x \sin t.$$

Which is the same as the exact solution

Example 1.2

Consider the following 1D nonlinear wave equation[10]:

$$u_{tt} = u_{xx} + u + u^2 - xt - x^2 t^2, \qquad 0 \le x \ \le \ 1, \qquad t > 0 \qquad (1.50)$$

with initial condition $u(x,0) = 0,\ u_t(x,0) = x,$

Integrating both sides of Eq. (1.50) twice from 0 to t and using the given initial conditions, we find

$$u(x,t) = \ tx - \frac{t^3 x}{6} - \frac{t^4 x^2}{12} + \int_0^t \int_0^t (u_{ss} + u + u^2)\, ds\, ds. \qquad (1.51)$$

We find the adomian polynomials (A_n) for the nonlinear term u^2 by the relationship in Eq. (1.12).

Reducing the integration in Eq. (1.51) from double to single [6], we get

$$u(x,t) = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12} + \int_0^t (t-s)(u_{ss} + u + A_n)ds \qquad (1.52)$$

By applying the ADM, we get

$$u_0(x,t) = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12},$$

$$A_0 = {u_0}^2$$

$$u_1(x,t) = \int_0^t (t-s)(u_{0ss} + u_0 + A_0)ds,$$

$$= -\frac{t^6}{180} + \frac{t^3x}{6} - \frac{t^5x}{120} + \frac{t^4x^2}{12} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016} - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960},$$

$$A_1 = 2u_0u_1$$

$$u_2(x,t) = \int_0^t (t-s)(u_{1ss} + u_1 + A_1)ds,$$

$$= \frac{t^6}{180} - \frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^5x}{120} - \frac{t^7x}{5040} - \frac{11t^9x}{22680} + \frac{t^{11}x}{26400} + \frac{t^6x^2}{72} - \frac{31t^8x^2}{20160} + \frac{11t^{10}x^2}{302400} + \cdots.$$

⋮

Then :

$$u_5(x,t) = \frac{t^{12}}{1069200} + \frac{1979t^{14}}{4358914560} - \frac{779t^{16}}{5189184000} + \frac{445769t^{18}}{50018544576000} - \frac{10511t^{20}}{62523180720000} - \frac{775837t^{22}}{67548120659712000} + \cdots$$

$$U_5 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = tx - \frac{t^{13}x}{6227020800} - \frac{151t^{14}}{7264857600} - \frac{4673t^{15}x}{16345929600} - \frac{3709t^{16}}{46702656000} - \frac{29357t^{17}x}{694702008000} + \frac{75913t^{18}}{10003708915200} + \cdots$$

The exact solution for Eq.(1.50)

$$u(x,t) = xt.$$

We calculate the absolute error , $|r_n| = |u - U_n|$, to check the accuracy of the approximate solution U_n, where $u(x,t) = xt$ is the exact solution. In table 1.6 the absolute error of ADM with *n* = 1, 3 is provided.

Table 1.9: Results of the absolute errors for ADM, when $t = 1$.

x	$\lvert r_1 \rvert$	$\lvert r_4 \rvert$
0	0.00555556	1.3407×10^{-6}
0. 1	0.00652639	1.9667×10^{-6}
0. 2	0.00778647	2.76817×10^{-6}
0. 3	0.00935701	3.79775×10^{-6}
0. 4	0.011259	5.12235×10^{-6}
0. 5	0.0135134	6.82494×10^{-6}
0. 6	0.0161408	9.00674×10^{-6}
0. 7	0.0191616	0.0000117893
0. 8	0.0225963	0.0000153167
0. 9	0.0264648	0.0000197578
1	0.030787	0.0000253086

In table 1.9, we note that by increasing the iterations, the errors will be decreased.

Example 1.3

Consider the following 2D nonlinear wave equation.

$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) - u(x,y,t)^2 + t^2x^2y^2,$$

$$0 \leq x, y \leq 1, t > 0$$

with initial conditions: $u(x,y,0) = 0$, $u_t(x,y,0) = xy$.

By applying the ADM, we obtain

$$u_0 = txy + \frac{1}{12}t^4x^2y^2$$

$$u_1 = \frac{t^6x^2}{180} + \frac{t^6y^2}{180} - \frac{1}{12}t^4x^2y^2 - \frac{1}{252}t^7x^3y^3 - \frac{t^{10}x^4y^4}{12960}$$

$u_2 = \frac{t^8}{2520} - \frac{t^6x^2}{180} - \frac{11t^9x^3y}{22680} - \frac{t^6y^2}{180} - \frac{t^{12}x^4y^2}{71280} - \frac{11t^9xy^3}{22680} + \frac{1}{252}t^7x^3y^3 - \frac{t^{12}x^2y^4}{71280} + \frac{11t^{10}x^4y^4}{45360} + \frac{37t^{13}x^5y^5}{7076160} + \frac{t^{16}x^6y^6}{18662400}$

$\vdots$

Then ,

$u_4 = -\frac{13t^{16}x^2}{232848000} + \frac{139t^{14}x^4}{170270100} + \frac{t^{11}xy}{16632} + \frac{23t^{17}x^5y}{467026560} - \frac{13t^{16}y^2}{232848000} + \cdots$

This series converges to the exact solution when

$U(x,y,t) = \sum_{n=0}^{\infty} u_n\,(x,y,t) = \ x\,y\,t\,.$

Table 1.10: Results of the absolute errors for ADM, when $y,t = 1$.

x	$\lvert r_1 \rvert$	$\lvert r_4 \rvert$
0	0.00555556	4.36769×10^{-7}
0. 1	0.00560714	4.92375×10^{-7}
0. 2	0.00574591	6.44678×10^{-7}
0. 3	0.00594779	8.85956×10^{-7}
0. 4	0.0061885	1.2063×10^{-6}
0. 5	0.00644359	1.59218×10^{-6}
0. 6	0.00668841	2.02518×10^{-6}
0. 7	0.00689814	2.4808×10^{-6}
0. 8	0.00704776	2.92748×10^{-6}
0. 9	0.00711207	3.32569×10^{-6}
1	0.0070657	3.62723×10^{-6}

Example 1.4.

Consider 3D nonlinear wave equation given in equation

$u_{tt}(x,y,z,t) = u_{xx}(x,y,z,t) + u_{yy}(x,y,z,t) + u_{zz}(x,y,z,t) - u(x,y,z,t)^2 + t^2x^2y^2z^2,\ \ 0 \le x,y,z \le\ 1,\ \ t > 0$

with the initial conditions: $u(x,y,z,0) = 0, u_t(x,y,z,0) = xyz$

By applying the ADM, we obtain

$u_0 = txyz + \frac{1}{12}t^4x^2y^2z^2$

$u_1 =$
$\frac{1}{180}t^6x^2y^2 + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{12}t^4x^2y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960}$

$\vdots$

Then ,

$u_4 = \frac{t^{10}}{37800} - \frac{13t^{16}x^4y^2}{232848000} - \frac{13t^{16}x^2y^4}{232848000} + \frac{139t^{14}x^4y^4}{170270100} - \frac{59t^{13}xyz}{8108100} + \frac{t^{11}x^3yz}{16632} + \frac{t^{11}xy^3z}{16632} + \cdots$

This series converges to the exact solution when

$U(x,y,z,t) = \sum_{n=0}^{\infty} u_n\,(x,y,z,t) = \ xyzt$.

Table 1.11: Results of the absolute errors for ADM, when $y, z, t = 1$.

x	$\lvert r_1 \rvert$	$\lvert r_4 \rvert$
0	0.00555556	3.80939×10^{-7}
0. 1	0.00566269	2.43012×10^{-7}
0. 2	0.00596813	6.6647×10^{-7}
0. 3	0.00644779	8.76941×10^{-7}
0. 4	0.00707739	8.53484×10^{-7}
0. 5	0.00783248	5.69642×10^{-7}
0. 6	0.00868841	3.69911×10^{-9}
0. 7	0.00962036	8.95526×10^{-7}
0. 8	0.01060333	2.132×10^{-6}
0. 9	0.0116121	3.73384×10^{-6}
1	0.0126213	5.71383×10^{-6}

1.5.3 The VIM for solving non-linear wave equation

In this subsection, we will use the VIM to get exact and approximate solution for the wave equation.

Example 1.5

Rewrite the example 1.1 and solve it by VIM.

By using Eq. (1.47), with initial conditions: $u(x,0) = x^2$, $u_t(x,0) = sin\, x$.
The correction functional of the Eq. (1.47) is:

$$u_{n+1} = u_n + \int_0^t \lambda(s)\big((u_{nss}) - (u_{nxx}) + 2\big)ds, \qquad (1.53)$$

By using the formula in Eq. (1.26) leads to $\lambda = s - t$, we have

$$u_{n+1} = u_n + \int_0^t (s - t)\big((u_{nss}) - (u_{nxx}) + 2\big)ds. \qquad (1.54)$$

Then, we get

$$u_0 = -t^2 + x^2 + t\ \mathrm{sin}x, \qquad (1.55)$$

$$u_1 = u_0 + \int_0^t (s - t)\big((u_{0ss}) - (u_{0xx}) + 2\big)ds, \qquad (1.56)$$

$$= x^2 + t\mathrm{sin}x - \frac{1}{6}t^3\mathrm{sin}x\ ,$$

$$u_2 = u_1 + \int_0^t (s - t)\big((u_{1ss}) - (u_{1xx}) + 2\big)ds, \qquad (1.57)$$

$$= x^2 + t\mathrm{sin}x - \frac{1}{6}t^3\mathrm{sin}x + \frac{1}{120}t^5\mathrm{sin}x\ ,$$

continuing in this way till *n* = 5, we find

$$u_5 = x^2 + t\mathrm{sin}x - \frac{1}{6}t^3\mathrm{sin}x + \frac{1}{120}t^5\mathrm{sin}x - \frac{t^7\mathrm{Sin}x}{5040} + \frac{t^9\mathrm{Sin}x}{362880} - \frac{t^{11}\mathrm{Sin}x}{39916800}.$$

The exact solution can be given by,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = x^2 + \sin x \sin t\ .$$

Example 1.6

Solving the example 1.2 by VIM.

We use Eq. (1.50), with initial conditions: $u(x,0) = 0,\ u_t(x,0) = x$.

The correction functional of the Eq. (1.50) is:

$$u_{n+1} = u_n + \int_0^t \lambda(s)\big((u_{nss}) - (u_{nxx}) - u_n - (u_n)^2 + xt + x^2t^2\big)ds. \quad (1.58)$$

By using the formula in Eq. (1.26) leads to $\lambda = s - t$, we have

$$u_{n+1} = u_n + \int_0^t (s - t)((u_{nss}) - (u_{nxx}) - u_n - (u_n)^2 + xt + x^2t^2)ds, \quad (1.59)$$

Then, we get

$$u_0 = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12}, \quad (160)$$

$$u_1 = u_0 + \int_0^t (s - t)((u_{0ss}) - (u_{0xx}) - u_0 - (u_0)^2 + xt + x^2t^2)ds, \quad (1.61)$$

$$= -\frac{t^6}{180} + tx - \frac{t^5x}{120} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016} - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960},$$

$$u_2 = u_1 + \int_0^t (s - t)((u_{1ss}) - (u_{1xx}) - u_1 - (u_1)^2 + xt + x^2t^2)ds, \quad (1.62)$$

$$= -\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7x}{5040} - \frac{11t^9x}{22680} + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} - \frac{11t^8x^2}{20160} + \cdots,$$

continuing in this way till $n = 5$, we find

$$u_5 = -\frac{t^{14}}{335301120} - \frac{179t^{16}}{19813248000} + \frac{9959t^{18}}{72754246656000} + \frac{18763t^{20}}{2303884477440000} + \cdots + tx - \frac{t^{13}x}{6227020800} - \frac{289t^{15}x}{20432412000} + \cdots.$$

This series converges to the exact solution

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = xt.$$

It is possible to calculate absolute error, to show the accuracy of the approximate solution u_n , where $u(x,t) = xt$ is the exact solution. In table 1.12 the absolute error of VIM with $n = 1, 3$ is provided.

Table 1.12: Results of the absolute errors for VIM. When $t = 1$.

x	$\lvert r_1 \rvert$	$\lvert r_4 \rvert$
0	0.00555556	4.61399×10^{-7}
0. 1	0.00652639	5.76688×10^{-7}
0. 2	0.00778647	7.17091×10^{-7}
0. 3	0.00935701	8.8619×10^{-7}
0. 4	0.011259	1.08821×10^{-6}
0. 5	0.0135134	1.32813×10^{-6}
0. 6	0.0161408	1.61177×10^{-6}
0. 7	0.0191616	1.94595×10^{-6}
0. 8	0.0225963	2.33857×10^{-6}
0. 9	0.0264648	2.79877×10^{-6}
1	0.030787	3.33704×10^{-6}

Example 1.7

Solving the example 1.3 by VIM.

By applying the VIM, we get the following iterations

$$u_0 = txy + \frac{1}{12}t^4x^2y^2$$

$$u_1 = \frac{t^6x^2}{180} + txy + \frac{t^6y^2}{180} - \frac{1}{252}t^7x^3y^3 - \frac{t^{10}x^4y^4}{12960}$$

Continuing in this way till n = 4, we find

$$u_4 = -\frac{t^{18}}{6766578000} - \frac{t^{38}}{5309215293981634560000} - \frac{t^{16}x^2}{86486400} + \frac{t^{26}x^2}{166617032400000} - \frac{t^{36}x^2}{1082856018666240000000} + \cdots + txy + \cdots$$

This series converges to the exact solution

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t) = xyt.$$

Table 1.13: Results of the absolute errors for VIM. When $y, t = 1$.

x	$\lvert r_1 \rvert$	$\lvert r_4 \rvert$
0	0.00555556	1.57137×10^{-7}
0. 1	0.00560714	1.84193×10^{-7}
0. 2	0.00574591	2.63572×10^{-7}
0. 3	0.00594779	3.94162×10^{-7}
0. 4	0.0061885	5.74663×10^{-7}
0. 5	0.00644359	8.03297×10^{-7}
0. 6	0.00668841	1.07753×10^{-6}
0. 7	0.00689814	1.39378×10^{-6}
0. 8	0.00704776	1.74719×10^{-6}
0. 9	0.00711207	2.13131×10^{-6}
1	0.0070657	2.53787×10^{-6}

Example 1.8

Solving the example 1.4 by VIM.

By applying the VIM, we find

$$u_0 = txyz + \frac{1}{12}t^4x^2y^2z^2$$

$$u_1 = \frac{1}{180}t^6x^2y^2 + txyz + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960}$$

Continuing in this way till *n* = 4, we find

$$u_4 = xyzt - \frac{59(xyz)t^{13}}{8108100} + \left(\frac{23x^4y^4}{136216080} + \frac{61x^4y^2z^2}{22702680} + \frac{61x^2y^4z^2}{22702680} + \frac{23x^4z^4}{136216080} + \frac{61x^2y^2z^4}{22702680} + \frac{23y^4z^4}{136216080}\right)t^{14} + \left(-\frac{173x^5y^5z^3}{681080400} - \frac{173x^5y^3z^5}{681080400} - \frac{173x^3y^5z^5}{681080400}\right)t^{15} + \cdots$$

This series converges to the exact solution

$$u(x,y,z,t) = \lim_{n\to\infty} u_n(x,y,z,t) = xyzt.$$

Table 1.14: Results of the absolute errors for VIM. When $y, z, t = 1$.

x	$\lvert r_1 \rvert$	$\lvert r_4 \rvert$
0	0.00555556	1.44152×10^{-7}
0. 1	0.00566269	5.3068×10^{-7}
0. 2	0.00596813	1.09838×10^{-6}
0. 3	0.00644779	1.54998×10^{-6}
0. 4	0.00707739	1.87041×10^{-6}
0. 5	0.00783248	2.03909×10^{-6}
0. 6	0.00868841	2.03049×10^{-6}
0. 7	0.00962036	1.81462×10^{-6}
0. 8	0.0106033	1.35764×10^{-6}
0. 9	0.0116121	6.22353×10^{-7}
1	0.0126213	4.31218×10^{-7}

CHAPTER 2

Solving Some Types of Ordinary Differential Equations by Some Iterative Methods

Chapter 2

Solving Some Types of Ordinary Differential Equations by Some Iterative Methods

2.1 Introduction

In this chapter, we implemented three iterative methods to solve several nonlinear ODEs that arise in physics, engineering and other applications. The proposed iterative methods are Tamimi-Ansari method (TAM) [36], which is used to solve many ODEs [36] such as, Duffing equations [50], some chemistry problems [51], thin film flow problem [52] and Fokker-Planck's equations [53]. Daftardar-Jafari method (DJM)[84], is an iterative method used for solving nonlinear equations. Another iterative method is called the Banach Contraction Principle (BCP) which is suggested by Varsha Daftardar-Gejji and Sachin Bhalekar [85].

The approximate solutions resulting upon applying these methods will be compared numerically with other results obtained by applying the Runge-Kutta as well as Euler methods and some analytic methods such as ADM and VIM. The convergence for those presented methods are also discussed. This chapter is organized as follows; In section 2, the basic ideas of the proposed iterative methods will be presented. In section 3, the convergence of the proposed iterative methods will be given. In section 4, some types of nonlinear ODEs equations will be solved by the proposed methods and the convergence will be proved.

2.2 The basic concepts of the proposed iterative methods

Iterative method is a mathematical procedure which generates a sequence of improved approximate solutions for a class of problems. The iterative method leads to an approximate solution which converges to the exact solution if certain conditions and satisfied with some given initial approximations.

Let us introduce the following nonlinear differential equation:

$$L\big(y\,(x)\big) + N\,\big(y(x)\big) + g(x) = 0, \tag{2.1}$$

with the boundary conditions or initial conditions

$$B\left(y, \frac{dy}{dx}\right) = 0, \qquad y(0) = a \text{ and } y^{'}(0) = b \;\; x \in D, \tag{2.2}$$

where x represents the independent variable, $y(x)$ is the unknown function, g is a given known function, $L(\cdot) = \frac{d^n}{dx^n}(\cdot)$ is the linear operator, $N(\cdot)$ is the nonlinear operator, $B(\cdot)$ is a boundary operator and $y(.)$, $y^{'}(.)$ are initial operater. Now, let us begin by introducing the basic ideas of the three iterative methods.

2.2.1 The basic idea of the TAM

We first begin by assuming that $y_0(x)$ is an initial guess to solve the problem and the solution approach begins by solving the following initial value problem [35]:

$$L\big(y_0(x)\big) + g(x) = 0, \text{ and } B\left(y_0, \frac{dy_0}{dx}\right) = 0 \text{ or } y_0(0) = a \text{ and } {y_0}^{'}(0) = b \tag{2.3}$$

The next approximate solutions are obtained by solving the following problems

$$L\big(y_1(x)\big) + g(x) + N\,\big(y_0(x)\big) = 0 \text{ and } B\left(y_1, \frac{dy_1}{dx}\right) = 0 \text{ or } y_1(0) = a \text{ and } {y_1}^{'}(0) = b \tag{2.4}$$

Thus we have a simple iterative procedure, which is the solution of a set of problems i.e.,

$L(y_{n+1}(x)) + g(x) + N(y_n(x)) = 0,$

and $B(y_{n+1}, \frac{dy_{n+1}}{dx}) = 0$ or $y_{n+1}(0) = a \; and \; y_{n+1}{}'(0) = b$ (2.5)

Then, the solution for the problem (2.1) with (2.2) is given by

$$y(x) = \lim_{n\to\infty} y_n(x) \; . \tag{2.6}$$

2.2.2 The basic idea for the DJM

Let us apply the inverse operator $L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\, d\tau\, d\tau$ to the nonlinear problem presented by (2.1) and (2.2). Then, we have

$$y(x) = f(x) + \int_0^x \int_0^x N\big(y(\tau)\big)\, d\tau\, d\tau, \tag{2.7}$$

and by reducing the integration from double to single[6], we get the following form

$$y(x) = f(x) + \int_0^x (x-\tau) N\big(y(\tau)\big)\, d\tau, \tag{2.8}$$

where f is a known analytic function which represents the sum of the available initial conditions and the result of integrating of the function g (if such function is available).

The solution y for Eq. (2.8) can be given by the following series [84]:

$$y = \sum_{i=0}^{\infty} y_i \tag{2.9}$$

Now, the following can be defined

$$G_0 = N(y_0), \tag{2.10}$$

$$G_m = N\left(\sum_{i=0}^{m} y_i\right) - N\left(\sum_{i=0}^{m-1} y_i\right), \qquad m \ge 1$$

So, that $N(y)$ can decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = \underbrace{N(y_0)}_{G_0} + \underbrace{[N(y_0+y_1) - N(y_0)]}_{G_1}$$
$$+ \underbrace{[N(y_0+y_1+y_2) - N(y_0+y_1)]}_{G_2}$$
$$+ \underbrace{[N(y_0+y_1+y_2+y_3) - N(y_0+y_1+y_2)]}_{G_3} + \cdots.$$

Moreover, the relation is defined by recurrence so that

$$y_0 = f, \tag{2.11}$$

$$y_1 = L(y_0) + G_0, \tag{2.12}$$

$$y_{m+1} = L(y_m) + G_m, \qquad m \geq 1 \tag{2.13}$$

Since L represents a linear operator $\sum_{i=0}^{m} L(y_i) = L(\sum_{i=0}^{m} y_i)$, we may write

$$\sum_{i=1}^{m+1} y_i = \sum_{i=0}^{m} L(y_i) + N\left(\sum_{i=0}^{m} y_i\right) = L\left(\sum_{i=0}^{m} y_i\right) + N\left(\sum_{i=0}^{m} y_i\right), \qquad m \geq 1$$

So that,

$$\sum_{i=0}^{\infty} y_i = f + L\left(\sum_{i=0}^{\infty} y_i\right) + N\left(\sum_{i=0}^{\infty} y_i\right)$$

and the approximate solution will be given in

$$y_n = \sum_{i=0}^{n} y_i.$$

2.2.3 The basic idea for the Banach contraction method (BCM)

Consider Eq. (2.8) as a general functional equation. In order to implement the BCM, we define the successive approximations [85]:

$$\begin{aligned} y_0 &= f, \\ y_1 &= y_0 + N(y_0), \\ y_2 &= y_0 + N(y_1), \\ &\vdots \end{aligned} \tag{2.14}$$

and so on, we will get successive approximations for $y_n(x)$ in the following generalized form

$$y_n = y_0 + N(y_{n-1}), \qquad n = 1,2, \ldots. \tag{2.15}$$

Therefore, the solution for the relations (2.1) and (2.2) will be obtained

$$y = \lim_{n \to \infty} y_n . \tag{2.16}$$

2.3 The convergence of the proposed iterative methods

In this section, we will present the fundamental theorems and concepts for the convergence [90] of the considered methods.

The iterations occurred by the DJM are straight used to prove the convergence. However, for the convergence proof of the TAM or BCM, the following procedure should be used for handling Eq. (2.1) with the given conditions (2.2). So, we have the terms

$$\begin{aligned} v_0 &= y_0(x), \\ v_1 &= F[v_0], \\ v_2 &= F[v_0 + v_1], \\ &\vdots \\ v_{n+1} &= F[v_0 + v_1 + \cdots + v_n]. \end{aligned} \tag{2.17}$$

where F represents the following operator

$$F[v_k] = S_k - \sum_{i=0}^{k-1} v_i(x) , \qquad k \geq 1. \tag{2.18}$$

In general, the term S_k is the solution for the problem in the form, for the TAM:

$$L(v_k(x)) + g(x) + N\left(\sum_{i=0}^{k-1} v_i(x)\right) = 0, \;\; k \geq 1. \tag{2.19}$$

For the BCM:

$$v_k = v_0 + N\left(\sum_{i=0}^{k-1} v_i(x)\right), \quad k \geq 1. \tag{2.20}$$

By using the same conditions with the intended iterative technique that will be used. Therefore, we get $y(x) = \lim_{n \to \infty} y_n(x) = \sum_{n=0}^{\infty} v_n$. Hence, by using Eqs. (2.17) and (2.18), the following solution will be obtained in a series form

$$y(x) = \sum_{i=0}^{\infty} v_i(x). \tag{2.21}$$

According to the recursive algorithms of the proposed methods, the sufficient conditions for the convergence of these methods can be given in theorem 1.2.8, 1.2.9 and 1.2.10

Theorems 1.2.8 and 1.2.9 states that the solutions obtained by one of the presented methods, i.e., the relation (2.5) (for the TAM), the relation (2.13) (for the DJM), the relation (2.15) (for the BCM), or (2.17), converges to the exact solution under the condition then there exists $0 < \xi < 1$, such that $\|F[v_0 + v_1 + \cdots + v_{i+1}]\| \leq \xi\|F[v_0 + v_1 + \cdots + v_i]\|$ (that is $\|v_{i+1}\| \leq \xi\|v_i\|$), $\forall i = 0,1,2,\ldots$). In another meaning, for each i, if we define the parameters as below,

$$\beta_i = \begin{cases} \dfrac{\|v_{i+1}\|}{\|v_i\|}, & \|v_i\| \neq 0 \\ 0, & \|v_i\| = 0 \end{cases}, \qquad (2.22)$$

then the series solution $\sum_{i=0}^{\infty} v_i(x)$ for the nonlinear ODE given by (2.1) will be converged to the exact solution $y(x)$, when $0 \leq \beta_i < 1, \forall i = 0,1,2,\ldots$. Also, as in Theorem 1.2.10, the maximum truncation error is estimated to be $\|y(x) - \sum_{i=0}^{n} v_i\| \leq \frac{1}{1-\beta}\beta^{n+1}\|v_0\|$, where $\beta = \max\{\beta_i, i = 0,1,\ldots,n\}$.

2.4 Application of the proposed iterative method with convergence for the nonlinear examples

In this section, some types of non-linear equations which are Painlevé I equation, Painlevé II equation, pendulum equation and Falkner-skan equation will be solved by the suggested methods. In addition, the convergence will be proved.

2.4.1 The proposed iterative methods for solving nonlinear Painlevé I equation

By using Eq.(1.7) with the initial conditions: $y(0) = 0$ and $y'(0) = 1$.

This problem will be solved by using the three proposed iterative methods.

Solving the Painlevé I equation by the TAM:

In order to solve the Eq.(1.7) by the TAM, we have the following form

$L(y) = y''(x),\ N(y) = -6y^2(x)$ and $g(x) = -x$. (2.23)

The initial value problem will be:

$L\big(y_0(x)\big) = x$, with $y_0(0) = 0$ and $y_0'(0) = 1$. (2.24)

We can get the next problems from the following generalized relationship

$L(y_{n+1}(x)) + g(x) + N(y_n(x)) = 0,\ y_{n+1}(0) = 0$ and $y_{n+1}'(0) = 1$. (2.25)

Firstly, to get the zero approximation $y_0(x)$, the following initial problem must be solved:

$y_0''(x) = x,$ (2.26)

By integrating both sides of Eq. (2.26) twice from 0 to x and substituting the initial conditions $y_0(0) = 0$ and $y_0'(0) = 1$, we get

$$y_0(x) = x + \frac{x^3}{6},$$

In a similar manner, the rest of the other iterations can be carried out, the first iteration can be obtained by evaluating

$y_1''(x) = 6y_0^2(x) + x$, with $y_1(0) = 0$ and $y_1'(0) = 1$, (2.27)

Then, the approximate solution for Eq. (2.27) will be then:

$$y_1(x) = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336}$$

The second iteration $y_2(x)$ can be obtained from solving the following

$y_2''(x) = 6y_1^2(x) + x$, with $y_2(0) = 0$ and $y_2'(0) = 1$, (2.28)

Then, by solving Eq. (2.28) approximately, we obtain:

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^7}{7} + \frac{x^8}{336} + \frac{x^9}{40} + \frac{x^{10}}{60} + \frac{71x^{11}}{46200} + \frac{x^{12}}{330} + \frac{x^{13}}{26208} + \frac{187x^{14}}{764400} + \frac{x^{16}}{100800} + \frac{x^{18}}{5757696}$$

Thus, we continue in this process to obtain the approximations till $n = 5$ for $y_n(x)$, but for brevity the terms are not listed.

Solving the Painlevé I by the DJM:

Consider the Eq. (1.7) with initial conditions $\mathrm{y}(0) = 0$ and $\mathrm{y}'\,(0) = 1$.

Integrating both sides of Eq. (1.7) twice from 0 to x and using the given initial conditions, we have

$$y(x) = x + \frac{1}{6}x^3 + \int_0^x \int_0^x 6y^2(\tau)\, d\tau\, d\tau. \tag{2.29}$$

By reducing the integration in Eq. (2.29) from double to single [6], we obtain

$$y(x) = x + \frac{1}{6}x^3 + \int_0^x (x-\tau)(6y^2(\tau))\, d\tau, \tag{2.30}$$

Then, the following relations can be defined:

$y_0 = x + \frac{1}{6}x^3,$

$$N(y_{n+1}) = \int_0^x (x-\tau)(6y_n^2(\tau))\, d\tau, \qquad n \in N \cup \{0\}.$$

By applying the DJM, we get

$$y_0 = \hat{y}_0 = x + \frac{1}{6}x^3,$$

$$\hat{y}_1 = \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336},$$

$$y_1 = \hat{y}_0 + \hat{y}_1 = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336}$$

$$\hat{y}_2 = \frac{x^7}{7} + \frac{x^9}{40} + \frac{x^{10}}{60} + \frac{71x^{11}}{46200} + \frac{x^{12}}{330} + \frac{x^{13}}{26208} + \frac{187x^{14}}{764400} + \frac{x^{16}}{100800} + \frac{x^{18}}{5757696}$$

$$y_2 = \hat{y}_0 + \hat{y}_1 + \hat{y}_2 = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^7}{7} + \frac{x^8}{336} + \frac{x^9}{40} + \frac{x^{10}}{60} + \frac{71x^{11}}{46200} + \frac{x^{12}}{330} + \frac{x^{13}}{26208} + \frac{187x^{14}}{764400} + \frac{x^{16}}{100800} + \frac{x^{18}}{5757696}$$

Therefore, we continue to get approximations till $n = 5$, for $y_n(x)$ but they are not listed.

Solving the Painlevé I by the BCM:

Consider Eq.(1.7), by following the similar procedure as given in the DJM, we get Eq. (2.30). So, let $y_0 = x + \frac{1}{6}x^3$ and $N(y_{n-1}) = \int_0^x (6y_{n-1}^2(\tau))\, d\tau$, $n \in N$.

Applying the BCM, we obtain:

$$y_0 = x + \frac{1}{6}x^3,$$

$$y_1 = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336},$$

$$y_2 = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^7}{7} + \frac{x^8}{336} + \frac{x^9}{40} + \frac{x^{10}}{60} + \frac{71x^{11}}{46200} + \frac{x^{12}}{330} + \frac{x^{13}}{26208} + \frac{187x^{14}}{764400} + \frac{x^{16}}{100800} + \frac{x^{18}}{5757696}$$

We continue to get the approximations till $n = 5$, and for brevity not listed.

It can be seen clearly that the obtained approximate solutions from the three proposed techniques are the same because we got the same series.

In order to access the convergence of the obtained approximate solution for problem 1, the relations given in Eqs. (2.17)-(2.21) will be used. The iterative scheme for Eq. (1.7) can be formulated as

$$v_0(x) = y_0(x) = x + \frac{x^3}{6},$$

By applying the TAM, the operator $F[v_k]$ as defined in Eq. (2.18) with the term S_k which is the solutions for the following problem, will be then

$$v_k''(x) = 6\left(\sum_{i=0}^{k-1} v_i(x)\right)^2 + x, \text{ with } v_k(0) = 0 \text{ and } v_k'(0) = 1,\ k \geq 1. \quad (2.31)$$

Or when applying the BCM, the S_k represents the solution for the following problem,

$$v_k = v_0 + 6\left(\sum_{i=0}^{k-1} v_i(x)\right)^2,\ \ k \geq 1. \quad (2.32)$$

On the other hand, one can use the iterative approximations directly when applying the DJM. Therefore, we have the following terms:

$$v_1 = \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336}$$

$$v_2 = \frac{x^7}{7} + \frac{x^9}{40} + \frac{x^{10}}{60} + \frac{71x^{11}}{46200} + \frac{x^{12}}{330} + \frac{x^{13}}{26208} + \frac{187x^{14}}{764400} + \frac{x^{16}}{100800} + \frac{x^{18}}{5757696}$$

$$v_3 = \frac{2x^{10}}{105} + \frac{41x^{12}}{9240} + \frac{37x^{13}}{5460} + \frac{527x^{14}}{1401400} + \frac{1543x^{15}}{970200} + \frac{105563x^{16}}{112112000} + \frac{91061x^{17}}{571771200} \quad (2.33)$$

As presented in the proof of the convergence of the proposed methods, the terms given by the series $\sum_{i=0}^{\infty} v_i(x)$ in (2.21) satisfy the convergent conditions by evaluating the β_i values in this case, we get

$$\begin{aligned}
\beta_0 &= \frac{\|v_1\|}{\|v_0\|} = 0.488265 < 1 \\
\beta_1 &= \frac{\|v_2\|}{\|v_1\|} = 0.332461 < 1 \\
\beta_2 &= \frac{\|v_3\|}{\|v_2\|} = 0.178092 < 1 \\
\beta_3 &= \frac{\|v_4\|}{\|v_3\|} = 0.102841 < 1 \\
\beta_4 &= \frac{\|v_5\|}{\|v_4\|} = 0.065685 < 1
\end{aligned} \quad (2.34)$$

where, the β_i values for $i \geq 0$ and $0 < x \leq 1$, are less than 1, so the proposed iterative methods satisfy the convergence.

In order to examine the accuracy for the approximate solutions obtained by the proposed methods for Eq. (1.7) and since the exact solution is unknown, the maximal error remainder MER_n will be calculated. The error remainder function for the Painlevé I equation can be defined as

$$ER_n(x) = y_n''(x) - 6y_n^2(x) - x, \tag{2.35}$$

and the MER_n is:

$$MER_n = \max_{0.01 \le x \le 0.1} |ER_n(x)|, \tag{2.36}$$

Table (2.1) and Fig. 2.1 below show the MER_n of the approximate solution obtained by the proposed iterative methods which indicates the efficiency of these methods. It can be seen that by increasing the iterations, the errors will be decreasing.

Table 2.1: The maximal error remainder: MER_n by the proposed methods.

n	MER_n
1	0.0000601952
2	1.72121×10^{-8}
3	2.29681×10^{-12}
4	1.52656×10^{-16}
5	2.77556×10^{-17}

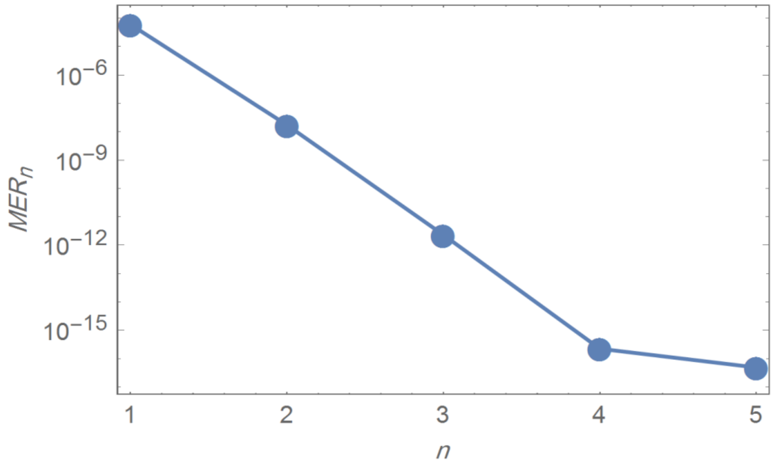


Fig.2.1: Logarithmic plots for the MER_n versus n from 1 to 5, by the proposed methods.

Also, we have made a numerical comparison between the solutions obtained by the proposed methods, the Range-Kutta (RK4) and Euler methods.

The comparison for problem1 is given in Fig. 2.2. It can be seen, a good agreement has achieved.

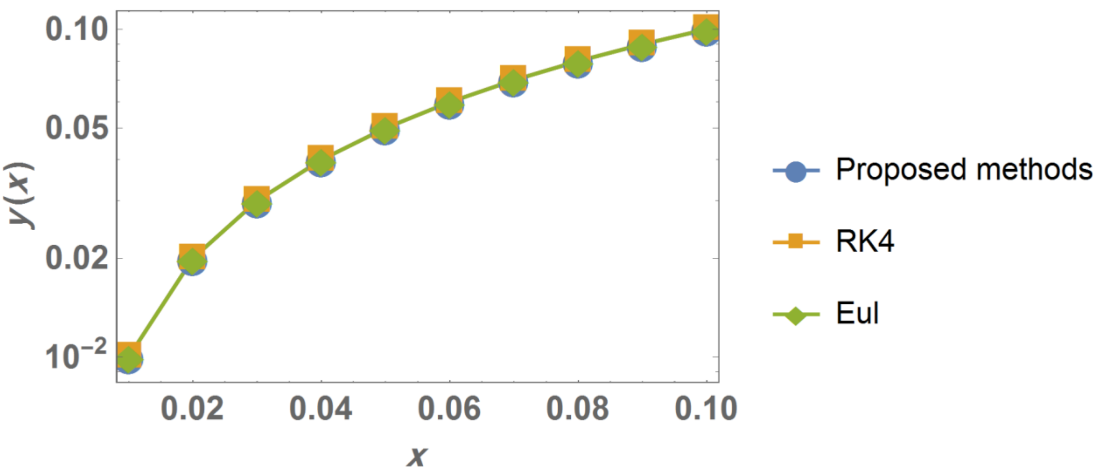


Fig.2.2: The comparison of the numerical solutions for Painlevé I equation.

2.4.2 The proposed iterative methods for solving nonlinear Painlevé II equation

By using Eq.(1.8)

with the initial conditions: $y\ (0) = 1$ and $y'(0) = 0$. Eq.(1.8) which will be solved by the three proposed iterative methods. The parameter μ in this equation will be equal to 1.

Solving the Painlevé II equation by the TAM:

In order to solve Eq.(1.8) by the TAM, we have the following form

$L(y) = y'',\ N(y) = 2y^3 + x\ y$ and $g(x) = 1$. (2.37)

The initial value problem is

$L(y_0(x)) = 1$ with $y_0(0) = 1$ and $y_0'(0) = 0$. (2.38)

The next problems can be found from the generalized iterative formula

$L\big(y_{n+1}(x)\big) + N\big(y_n(x)\big) + g(x) = 0,\ \ y_{n+1}(0) = 1$ and $y_{n+1}'(0) = 0$.

When evaluating the following initial value problem (2.38), one can get the solution

$y_0(x) = 1 + \frac{x^2}{2}.$

The first iteration $y_1(x)$ can be found by solving

$y_1''(x) = 2y_0^3(x) + x\, y_0(x) + 1$ with $y_1(0) = 1, y_1'(0) = 0.$

The solution will be

$$y_1(x) = 1 + \frac{3x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4} + \frac{x^5}{40} + \frac{x^6}{20} + \frac{x^8}{224}$$

Applying the same process for y_2 as follows

$y_2''(x) = 2y_1^3(x) + x\, y_1(x) + 1$, with $y_2(0) = 1$ and $y_2'(0) = 0.$

By solving this problem, we have

$$y_2(x) = 1 + \frac{3x^2}{2} + \frac{x^3}{6} + \frac{3x^4}{4} + \frac{x^5}{8} + \frac{91x^6}{180} + \frac{17x^7}{210} + \frac{1409x^8}{6720} + \frac{13x^9}{288} + \frac{929x^{10}}{16800} + \frac{38593x^{11}}{3326400} + \frac{26683x^{12}}{2217600} + \frac{1483x^{13}}{655200} + \frac{6239x^{14}}{3057600} + \frac{809x^{15}}{2352000} + \frac{4583x^{16}}{16128000} + \frac{2357x^{17}}{60928000} + \frac{31273x^{18}}{959616000} + \frac{37x^{19}}{10944000} + \frac{3499x^{20}}{1191680000} + \frac{109x^{21}}{526848000} + \frac{81x^{22}}{386355200} + \frac{3x^{23}}{507781120} + \frac{x^{24}}{92323840} + \frac{x^{26}}{3652812800}$$

Continuing in this manner to get approximations up to $n = 5$ for $y_n(x)$, but for brevity they are not listed, see appendix C.

Solving the Painlevé II equation by the DJM

Consider the Eq. (1.8) with initial conditions $y(0) = 1$ and $y'(0) = 0$.

Integrate both sides of Eq. (1.8) twice from 0 to x with using the given initial conditions, we obtain

$$y(x) = 1 + \frac{x^2}{2} + \int_0^x \int_0^x (2y^3(\tau) + \tau y(\tau))\, d\tau\, d\tau, \tag{2.39}$$

and reducing the integration in Eq. (2.39) from double to single [6], we achieve

$$y(x) = 1 + \frac{x^2}{2} + \int_0^x (x-\tau)(2y^3(\tau) + \tau y(\tau))\, d\tau, \tag{2.40}$$

Therefore, we have the following recurrence relation

$$y_0 = 1 + \frac{x^2}{2},$$

$$N(y_{n+1}) = \int_0^x (x-\tau)(2y_n^3(\tau) + \tau y_n(\tau))\, d\tau, \qquad n \in N \cup \{0\}.$$

By applying the DJM, we get

$$y_0 = \hat{y}_0 = 1 + \frac{x^2}{2},$$

$$\hat{y}_1 = x^2 + \frac{x^3}{6} + \frac{x^4}{4} + \frac{x^5}{40} + \frac{x^6}{20} + \frac{x^8}{224}$$

$$y_1 = \hat{y}_0 + \hat{y}_1 = 1 + \frac{3x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4} + \frac{x^5}{40} + \frac{x^6}{20} + \frac{x^8}{224}$$

$$\hat{y}_2 = \frac{x^4}{2} + \frac{x^5}{10} + \frac{41x^6}{90} + \frac{17x^7}{210} + \frac{197x^8}{960} + \frac{13x^9}{288} + \frac{929x^{10}}{16800} + \frac{38593x^{11}}{3326400} + \frac{26683x^{12}}{2217600} + \frac{1483x^{13}}{655200} + \frac{6239x^{14}}{3057600} + \frac{809x^{15}}{2352000} + \frac{4583x^{16}}{16128000} + \frac{2357x^{17}}{60928000} + \frac{31273x^{18}}{959616000} + \frac{37x^{19}}{10944000} + \frac{3499x^{20}}{1191680000} + \frac{109x^{21}}{526848000} + \frac{81x^{22}}{386355200} + \frac{3x^{23}}{507781120} + \frac{x^{24}}{92323840} + \frac{x^{26}}{3652812800}$$

$$y_2 = \hat{y}_0 + \hat{y}_1 + \hat{y}_2 = 1 + \frac{3x^2}{2} + \frac{x^3}{6} + \frac{3x^4}{4} + \frac{x^5}{8} + \frac{91x^6}{180} + \frac{17x^7}{210} + \frac{1409x^8}{6720} + \frac{13x^9}{288} + \frac{929x^{10}}{16800} + \frac{38593x^{11}}{3326400} + \frac{26683x^{12}}{2217600} + \frac{1483x^{13}}{655200} + \frac{6239x^{14}}{3057600} + \frac{809x^{15}}{2352000} + \frac{4583x^{16}}{16128000} + \frac{2357x^{17}}{60928000} + \frac{31273x^{18}}{959616000} + \frac{37x^{19}}{10944000} + \frac{3499x^{20}}{1191680000} + \frac{109x^{21}}{526848000} + \frac{81x^{22}}{386355200} + \frac{3x^{23}}{507781120} + \frac{x^{24}}{92323840} + \frac{x^{26}}{3652812800}$$

Therefore, we continue to get approximations till $n = 5$, for $y_n(x)$ but for brevity the terms are not listed.

Solving the Painlevé II equation by the BCM

Consider Eq.(1.8), by following the similar procedure in the DJM, we get the Eq. (2.40). So, if we let $y_0 = 1 + \frac{x^2}{2}$ and

$N(y_{n-1}) = \int_0^x (x-\tau)(2y_{n-1}^3(\tau) + \tau y_{n-1}(\tau))\, d\tau.$

Applying the BCM, we obtain:

$$y_0 = 1 + \frac{x^2}{2},$$

$$y_1 = 1 + \frac{3x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4} + \frac{x^5}{40} + \frac{x^6}{20} + \frac{x^8}{224},$$

$$\begin{aligned} y_2 = 1 &+ \frac{3x^2}{2} + \frac{x^3}{6} + \frac{3x^4}{4} + \frac{x^5}{8} + \frac{91x^6}{180} + \frac{17x^7}{210} + \frac{1409x^8}{6720} + \frac{13x^9}{288} \\ &+ \frac{929x^{10}}{16800} + \frac{38593x^{11}}{3326400} + \frac{26683x^{12}}{2217600} + \frac{1483x^{13}}{655200} + \frac{6239x^{14}}{3057600} \\ &+ \frac{809x^{15}}{2352000} + \frac{4583x^{16}}{16128000} + \frac{2357x^{17}}{60928000} + \frac{31273x^{18}}{959616000} \\ &+ \frac{37x^{19}}{10944000} + \frac{3499x^{20}}{1191680000} + \frac{109x^{21}}{526848000} + \frac{81x^{22}}{386355200} \\ &+ \frac{3x^{23}}{507781120} + \frac{x^{24}}{92323840} + \frac{x^{26}}{3652812800} \end{aligned}$$

We continue to get the approximations till $n = 5$, for brevity they are not listed.

The obtained solutions from the three proposed methods are the same because we got the same series. Hence, as presented in the proof of the convergence for these methods in the previous section and by following similar procedure that presented for Painlevé II equation, the terms given by the series $\sum_{i=0}^{\infty} v_i(x)$ in Eq. (2.21) satisfy the convergent conditions by evaluating the β_i values for each iterative methods, we get

$$\begin{aligned} \beta_0 &= \frac{\|v_1\|}{\|v_0\|} = 0.997421 < 1 \\ \beta_1 &= \frac{\|v_2\|}{\|v_1\|} = 0.983067 < 1 \\ \beta_2 &= \frac{\|v_3\|}{\|v_2\|} = 0.736222 < 1 \\ \beta_3 &= \frac{\|v_4\|}{\|v_3\|} = 0.500802 < 1 \\ \beta_4 &= \frac{\|v_5\|}{\|v_4\|} = 0.32419 < 1 \end{aligned} \tag{2.41}$$

where, the β_i values for $i \geq 0$ and $0 < x \leq 1$, are less than 1, so the proposed iterative methods are convergent.

Further, the investigation can be done, in order to examine the accuracy for the approximate solution for this problem, the error remainder function is evaluated:

$$ER_n(x) = y_n''(x) - 2y_n^3(x) - xy_n(x) - 1, \tag{2.42}$$

and the MER_n is:

$$MER_n = \max_{0.01 \leq x \leq 0.1} |ER_n(x)|, \tag{2.43}$$

Table 2.2 and Fig. 2.2 shows the MER_n of the approximate solutions obtained by the proposed iterative methods which indicates the efficiency of these methods. Also, by increasing the iterations, the errors will be decreasing.

Table 2.2: The maximal error remainder: MER_n by the proposed methods.

n	MER_n
1	0.0634125
2	0.000323411
3	6.58637×10^{-7}
4	7.18421×10^{-10}
5	4.8539×10^{-13}

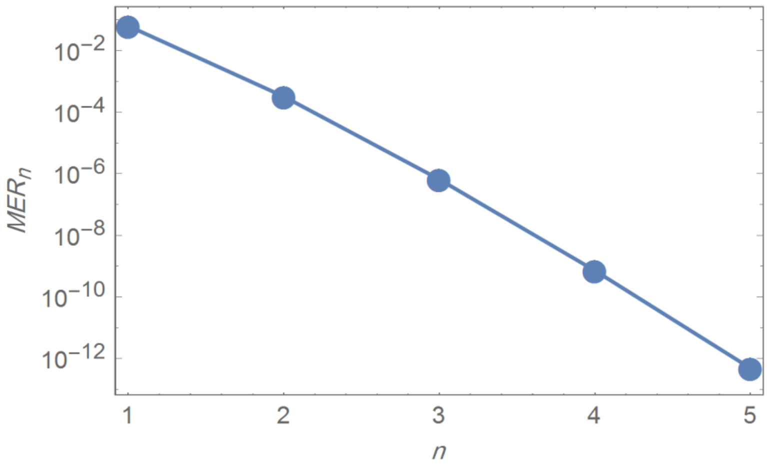


Fig.2.3: Logarithmic plots for the MER_n versus *n* from 1 to 5, by the proposed methods.

The numerical comparison between the solutions obtained by the proposed methods, the Range-Kutta (RK4) and Euler methods for Painlevé II equation are given in Fig. 2.4 below, and good agreement was clearly obtained.

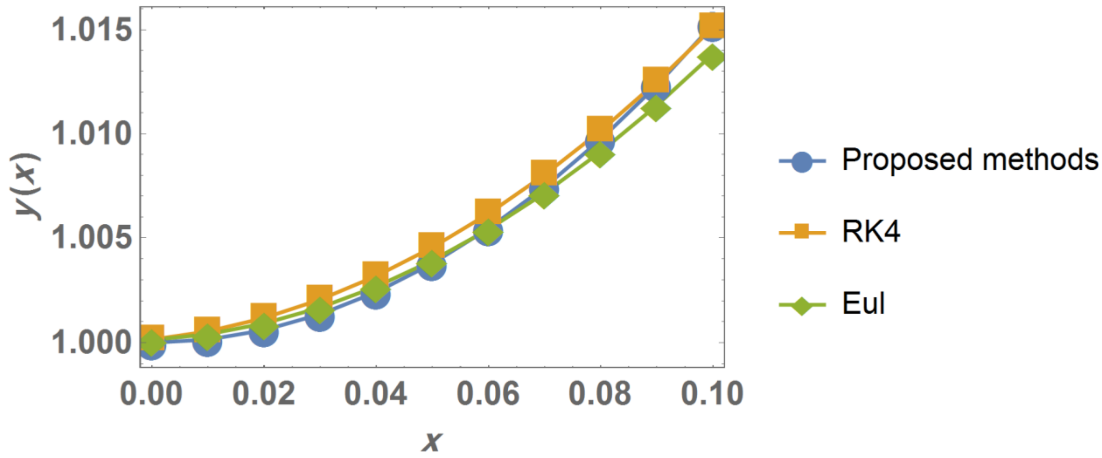


Fig.2.4**:** The comparison of the numerical solutions for Painlevé II equation.

2.4.3 The proposed iterative methods for solving nonlinear pendulum equation.

The pendulum equation presented by the form [41]:

$$y''(x) + \sin y = 0, \tag{2.44}$$

with the given initial conditions: $y(0) = 0$ and $y'(0) = 1$, can be solved by using the approximation of $\sin y \approx y - \frac{1}{6}y^3 + \frac{1}{120}y^5$ as it used in [43]. Hence, the pendulum equation (2.44) may be written as in the following second order nonlinear ODE

$$y''(x) + y - \frac{1}{6}y^3 + \frac{1}{120}y^5 = 0. \tag{2.45}$$

The exact solution for Eq. (2.44) is expressed by the following Jacobi elliptic function $y = 2\arcsin(\frac{1}{2}\mathrm{sn}(x, \frac{1}{4}))$.

Solving the pendulum equation by the TAM:

In order to solve the pendulum equation given in Eq. (2.45) with the given conditions by the TAM, we have:

$$L(y) = y'', \qquad N(y) = y - \frac{1}{6}y^3 + \frac{1}{120}y^5, \tag{2.46}$$

The initial value problem is

$$L(y_0(x)) = 0 \text{ with } y_0(0) = 0 \text{ and } y_0'(0) = 1. \tag{2.47}$$

The next problems can be found from the generalized iterative formula

$L(y_{n+1}(x)) + N(y_n(x)\ ) = 0,\ \ y_{n+1}(0) = 0$ and $y_{n+1}'(0) = 1.$

By solving the initial problem (2.47), one get

$y_0(x) = x.$

The first iteration $y_1(x)$ can be found by solving

$y_1''(x) = -(y_0 - \frac{1}{6}y_0^3 + \frac{1}{120}y_0^5)$ with $y_1(0) = 0$ and $y_1'(0) = 1.$

The solution will be as below

$$y_1(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}.$$

Applying the same process for y_2, we have to solve

$y_2''(x) = -(y_1 - \frac{1}{6}y_1^3 + \frac{1}{120}y_1^5)$ with $y_2(0) = 0$ and $y_2'(0) = 1.$

By solve this problem, we get

$$y_2(t) = x - \frac{x^3}{6} + \frac{x^5}{60} - \frac{x^7}{420} + \frac{127x^9}{362880} - \frac{893x^{11}}{19958400} + \frac{367x^{13}}{70761600} - \frac{607x^{15}}{1143072000} + \frac{56881x^{17}}{1243662336000} - \frac{2521x^{19}}{781861248000} + \frac{17x^{21}}{92177326080} - \frac{22129x^{23}}{2591207055360000} + \frac{17651x^{25}}{55306395648000000} - \frac{61787x^{27}}{6470848290816000000} + \frac{2021x^{29}}{8981758653235200000} - \frac{73x^{31}}{18002231783424000000} + \frac{13x^{33}}{245294925978009600000} - \frac{x^{35}}{221137092358963200000} + \frac{x^{37}}{5198022476861276160000}.$$

Continuing in this manner to get approximations up to $n = 5$ for $y_n(x)$, but for brevity they are not listed.

Solving the pendulum equation by the DJM:

Consider the Pendulum equation given in Eq. (2.45) with the given conditions $y(0) = 0$ and $y'(0) = 1$.

Integrating both sides of Eq. (2.45) twice from 0 to x, we get

$$y(t) = x - \int_0^x \int_0^x \left(y - \frac{1}{6}y^3 + \frac{1}{120}y^5\right) d\tau\, d\tau, \qquad (2.48)$$

and reducing the integration in Eq. (2.48) from double to single [6], we obtain

$$y(t) = x - \int_0^x (x-\tau)\left(y - \frac{1}{6}y^3 + \frac{1}{120}y^5\right) d\tau, \qquad (2.49)$$

Therefore, we have the following recurrence relation

$$y_0 = x,$$

$$N(y_{n+1}) = -\int_0^x (x-\tau)\left(y_n - \frac{1}{6}y_n^3 + \frac{1}{120}y_n^3\right) d\tau, \qquad n = 0, 1, 2, \ldots$$

By applying the DJM, we get

$$y_0 = \hat{y}_0 = x,$$

$$\hat{y}_1 = -\frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040},$$

$$y_1 = \hat{y}_0 + \hat{y}_1 = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$$\hat{y}_2 =$$

$$\frac{x^5}{120} - \frac{11x^7}{5040} + \frac{127x^9}{362880} - \frac{893x^{11}}{19958400} + \frac{367x^{13}}{70761600} - \frac{607x^{15}}{1143072000} + \frac{56881x^{17}}{1243662336000} - \frac{2521x^{19}}{781861248000} + \frac{17x^{21}}{92177326080} - \frac{22129x^{23}}{2591207055360000} + \frac{17651x^{25}}{55306395648000000} - \frac{61787x^{27}}{6470848290816000000} + \frac{2021x^{29}}{8981758653235200000} - \frac{73x^{31}}{18002231783424000000} + \frac{13x^{33}}{245294925978009600000} - \frac{x^{35}}{2211370923589632000000} + \frac{x^{37}}{5198022476861276160000000}.$$

$y_2 = \hat{y}_0 + \hat{y}_1 + \hat{y}_2 = x - \frac{x^3}{6} + \frac{x^5}{60} - \frac{x^7}{420} + \frac{127x^9}{362880} - \frac{893x^{11}}{19958400} + \frac{367x^{13}}{70761600} - \frac{607x^{15}}{1143072000} + \frac{56881x^{17}}{1243662336000} - \frac{2521x^{19}}{781861248000} + \frac{17x^{21}}{92177326080} - \frac{22129x^{23}}{2591207055360000} + \frac{17651x^{25}}{55306395648000000} - \frac{61787x^{27}}{6470848290816000000} + \frac{2021x^{29}}{8981758653235200000} - \frac{73x^{31}}{18002231783424000000} + \frac{13x^{33}}{245294925978009600000} - \frac{x^{35}}{2211370923589632000000} + \frac{x^{37}}{519802247686127616000000}$.

Therefore, we continue to get the other iterations till $n = 5$, for $y_n(x)$ but for brevity the terms are not listed.

Solving the pendulum equation by the BCM:

Consider Eq.(2.45), by applying the same way as in the DJM, we get the Eq. (2.49). So, let $y_0 = x$ and $N(y_{n-1}) = -\int_0^x (x-\tau)(y_{n-1} - \frac{1}{6}y_{n-1}^3 + \frac{1}{120}y_{n-1}^3)\, d\tau,\ n \in N.$

By applying the BCM, we obtain:

$y_0 = x$,

$$y_1 = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040},$$

$y_2 = x - \frac{x^3}{6} + \frac{x^5}{60} - \frac{x^7}{420} + \frac{127x^9}{362880} - \frac{893x^{11}}{19958400} + \frac{367x^{13}}{70761600} - \frac{607x^{15}}{1143072000} + \frac{56881x^{17}}{1243662336000} - \frac{2521x^{19}}{781861248000} + \frac{17x^{21}}{92177326080} - \frac{22129x^{23}}{2591207055360000} + \frac{17651x^{25}}{55306395648000000} - \frac{61787x^{27}}{6470848290816000000} + \frac{2021x^{29}}{8981758653235200000} - \frac{73x^{31}}{18002231783424000000} + \frac{13x^{33}}{245294925978009600000} - \frac{x^{35}}{2211370923589632000000} + \frac{x^{37}}{519802247686127616000000}$

We continue in the manner to get the other approximations till $n = 5$, for brevity they are not listed.

The obtained solutions by the three proposed methods are equal to each other because we got the same series.

Hence, as presented in the proof of the convergence in the previous section, the terms given by the series $\sum_{i=0}^{\infty} v_i(x)$ in Eq. (2.21) satisfy the convergent conditions by evaluating the β_i values for each iterative methods, we get

$$\begin{aligned}
\beta_0 &= \frac{\|v_1\|}{\|v_0\|} = 0.114522 < 1 \\
\beta_1 &= \frac{\|v_2\|}{\|v_1\|} = 0.0358901 < 1 \\
\beta_2 &= \frac{\|v_3\|}{\|v_2\|} = 0.019054 < 1 \\
\beta_3 &= \frac{\|v_4\|}{\|v_3\|} = 0.0114183 < 1 \\
\beta_4 &= \frac{\|v_5\|}{\|v_4\|} = 0.00760755 < 1
\end{aligned} \tag{2.50}$$

where, the β_i values for $i \geq 0$ and $0 < x \leq 1$, are less than 1, so the proposed iterative methods are convergent.

To examine the accuracy of the obtained approximate solution for this problem, the error remainder function is evaluated

$$ER_n(x) = y_n''(x) + y_n(x) - \frac{1}{6} y_n^3\,(x) + \frac{1}{120} y_n^5\,(x), \tag{2.51}$$

and the MER_n is:

$$MER_n = \max_{0 \leq x \leq 1} |ER_n(x)|, \tag{2.52}$$

Table 2.3 and Fig. 2.3 below, shows the MER_n of the approximate solution obtained by the proposed iterative methods which indicates the efficiency of these methods. Moreover, by increasing the iterations, the errors will be decreasing.

Table 2.3: The maximal error remainder: MER_n by the proposed methods.

n	MER_n
1	0.0959857
2	0.00429285
3	0.0000881802
4	1.05634×10^{-6}
5	8.28868×10^{-9}

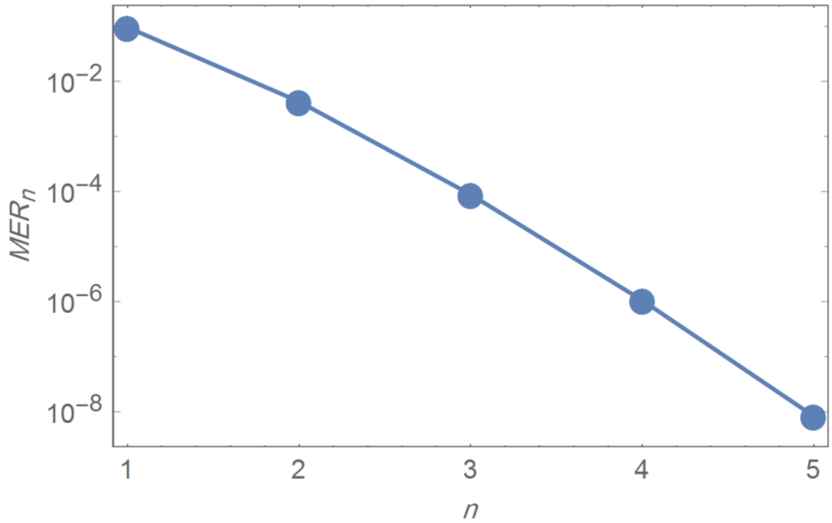


Fig.2.5: Logarithmic plots for the MER_n versus n from 1 to 5, by the proposed methods.

In addition, the numerical comparison between the solutions obtained by the proposed methods, exact solution, the Range-Kutta (RK4) and Euler methods for pendulum equation are presented in Fig. 2.6. The agreement between the solutions can be clearly seen.

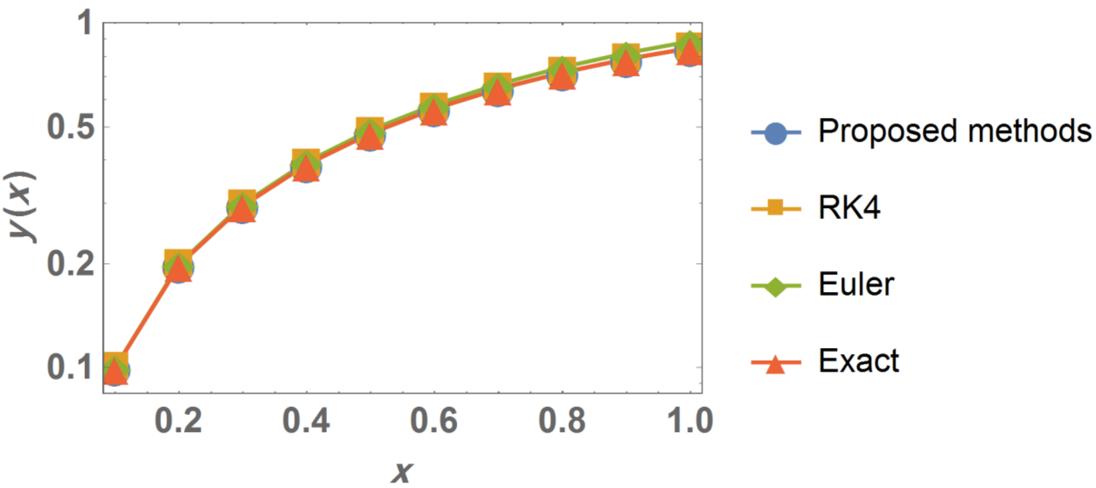


Fig.2.6: The comparison of the numerical solutions for pendulum equation.

2.4.4 The proposed iterative methods for solving nonlinear Falkner skan equation [33].

The Falkner skan equation is given by the following form:

$$y'''(x) + y(x)y''(x) + \beta\,[\epsilon^2 - (\,y'(x))^2] = 0, \tag{2.53}$$

with the following boundary conditions:

$y(0) = 0,\ \ y'(0) = 1 - \epsilon,\ \ y'(\infty) = \epsilon.$

Where, β and ϵ are parameters will look at their numerical values later.

Solving Falkner skan equation by the TAM:

In order to solve the Eq. (2.53) with the boundary conditions by the TAM, we have the following form

$$L(y) = y'''(x),\ N(y) = -y(x)y''(x) + \beta(\,y'(x))^2,\ g(x) = -\beta\epsilon^2. \tag{2.54}$$

According to the steps previously presented, we begin to solve the problem

$$y_0'''(0) = -\beta\epsilon^2,\ \ y_0(0) = 0,\quad y_0'(0) = 1 - \epsilon,\quad y_0''(0) = a. \tag{2.55}$$

Therefore, the initial problem will be:

$$L\big(y_0(x)\big) = -\beta\epsilon^2,\quad y_0(0) = 0,\ \ y_0'(0) = 1 - \epsilon,,\quad y_0''(0) = a. \tag{2.56}$$

The following problems can be obtained by the general relationship

$L\big(y_{n+1}(x)\big)+\ g(x)+N\ \big(y_n(x)\big)=0,\ y_{n+1}(0)=0,\ y_{n+1}{}'(0)=1-\ \epsilon,$
$y_{n+1}{}''(0)=\ a.$ (2.57)

Then,

$$y_0{}'''(x)=-\beta\epsilon^2\ . \tag{2.58}$$

By integration Eq. (2.58) from 0 to x three times and using the initial conditions $y_0(0)=\ 0,\ \ y_0{}'(0)=1-\epsilon,\ \ y_0{}''(0)=a,$ we get

$$y_0(x)=x+\frac{ax^2}{2}-x\epsilon-\frac{1}{6}x^3\beta\epsilon^2. \tag{2.59}$$

First iteration can be carried and is given as

$y_1'''(x)=\ -y_0(x)y_0{}''(x)-\beta\ \epsilon^2+\beta(\ y_0{}'(x))^2$ with $y_1(0)=$
$0,\ \ y_0{}'(0)=1-\epsilon$ and $y_1{}''(0)=a.$ (2.60)

Then, the solution for Eq. (2.60) will be:

$$y_1(x)=x+\frac{ax^2}{2}-\frac{ax^4}{24}-\frac{a^2x^5}{120}+\frac{x^3\beta}{6}+\frac{1}{12}ax^4\beta+\frac{1}{60}a^2x^5\beta-x\epsilon+\frac{1}{24}ax^4\epsilon-\frac{1}{3}x^3\beta\epsilon-\frac{1}{12}ax^4\beta\epsilon+\frac{1}{60}x^5\beta\epsilon^2+\frac{1}{180}ax^6\beta\epsilon^2-\frac{1}{60}x^5\beta^2\epsilon^2-\frac{1}{120}ax^6\beta^2\epsilon^2-\frac{1}{60}x^5\beta\epsilon^3+\frac{1}{60}x^5\beta^2\epsilon^3-\frac{x^7\beta^2\epsilon^4}{1260}+\frac{1}{840}x^7\beta^3\epsilon^4.$$

The second iteration $y_2(x)$ can be obtained similarly by solving the following

$y_2'''(x)=\ -y_1(x)y_1{}''(x)-\beta\ \epsilon^2+\beta(\ y_1{}'(x))^2$ with $y_2(0)=0,\ \ y_2{}'(0)=$
$1-\epsilon$ and $y_2{}''(0)=a$ (2.61)

Then, by solving the Eq. (2.61), we get:

$$y_2(x) = x + \frac{ax^2}{2} - \frac{ax^4}{24} - \frac{a^2x^5}{120} + \frac{ax^6}{240} + \frac{11a^2x^7}{5040} + \frac{11a^3x^8}{40320} - \frac{a^2x^9}{24192} - \frac{a^3x^{10}}{64800} - \frac{a^4x^{11}}{712800} + \frac{x^3\beta}{6} + \frac{1}{12}ax^4\beta - \frac{x^5\beta}{60} + \frac{1}{60}a^2x^5\beta - \frac{1}{60}ax^6\beta - \frac{2}{315}a^2x^7\beta + \frac{ax^8\beta}{2688} + \cdots + \frac{x^{15}\beta^5\epsilon^8}{24766560} - \frac{23x^{15}\beta^6\epsilon^8}{412776000} + \frac{x^{15}\beta^7\epsilon^8}{39312000}.$$

We continue to get the other approximations till $n = 5$, for $y_n(x)$ but for brevity the terms are not listed.

After finding the approximate solution of the series $y_5(x)$ which contains the value of the missing condition $y_5''(0) = a$, then to find the numerical value of a, by using the Padé approximation that was defined in Eq.(1.2) Now, by applying the Padé approximation for $y'_5(x)$ (since the third used boundary condition $y'(\infty) = \epsilon$), with $(\beta = 0.5,\ \epsilon = 0.1)$[33] , we get

$$P_2^2\left(y'_5(x)\right) = \frac{0.9 + (0.9(0. - 0.375a) + a)x + (0.2675 + (0. - 0.375a)a)x^2}{1 + (0. - 0.375a)x + 0.075x^2}$$

By taking $\lim_{x\to\infty} P_2^2(x)$, we obtain

$P_2^2\,(x) = 3.56667 - 5a^2,$

By applying the condition value $(y'(\infty) = \epsilon = 0.1)$, we get

$3.56667 - 5\mathrm{a}^2 = 0.1,$

Then, the value of a will be $(a = \pm 0.832666)$ this means a dual solution of equation. Thus, we will use the value that achieves better convergence, when $(a = 0.832666)$, i.e. $y''(0) = 0.832666$.

Solving Falkner skan equation by the DJM:

To solve the problem of the Falkner-Skan given in Eq. (2.53) by the DJM, with boundary conditions:

$y(0) = 0,\ \ y'(0) = 1 - \epsilon,\ \ y'(\infty) = \epsilon.$

The following steps will be used:

$$y'''(x) = -\,y(x)y''(x) - \beta\,\epsilon^2 + \beta(\,y'(x))^2. \qquad (2.62)$$

By integration both sides of Eq. (2.62) three times from 0 to x and using the initial conditions when $y''(0) = a$, we get :

$$\mathrm{y}(x) = x + \frac{ax^2}{2} - x\epsilon - \frac{1}{6}x^3\beta\epsilon^2 + \int_0^x\int_0^x\int_0^x (-\,y(t)y''(t) + \beta(y'(t))^2)dt\,dtdt \qquad (2.63)$$

According to the reducing of multiple integrals [6], then the functional Eq. (2.63) become as:

$$\mathrm{y}(x) = x + \frac{ax^2}{2} - x\epsilon - \frac{1}{6}x^3\beta\epsilon^2 + \frac{1}{2}\int_0^x (x-t)^2(-\,y(t)y''(t) + \beta\,(y'(t))^2\,)dt. \qquad (2.64)$$

Then, $y_0(x) = x + \frac{ax^2}{2} - x\epsilon - \frac{1}{6}x^3\beta\epsilon^2$,

$$N(y_{n+1}) = \frac{1}{2}\int_0^x (x-t)^2(-\,y_n(t){y_n}''(t) + \beta\,({y_n}'(t))^2\,)dt, \quad n = 0, 1, 2, \ldots\ .$$

By applying the DJM, we get

$y_0 = \hat{y}_0 = x + \frac{ax^2}{2} - x\epsilon - \frac{1}{6}x^3\beta\epsilon^2,$

$\hat{y}_1 = \frac{ax^4}{24} - \frac{a^2x^5}{120} + \frac{x^3\beta}{6} + \frac{1}{12}ax^4\beta + \frac{1}{60}a^2x^5\beta + \frac{1}{24}ax^4\epsilon - \frac{1}{3}x^3\beta\epsilon - \frac{1}{12}ax^4\beta\epsilon + \frac{1}{6}x^3\beta\epsilon^2 + \frac{1}{60}x^5\beta\epsilon^2 + \frac{1}{180}ax^6\beta\epsilon^2 - \frac{1}{60}x^5\beta^2\epsilon^2 - \frac{1}{120}ax^6\beta^2\epsilon^2 - \frac{1}{60}x^5\beta\epsilon^3 + \frac{1}{60}x^5\beta^2\epsilon^3 - \frac{x^7\beta^2\epsilon^4}{1260} + \frac{1}{840}x^7\beta^3\epsilon^4,$

$$y_1 = \hat{y}_0 + \hat{y}_1 = x + \frac{ax^2}{2} - \frac{ax^4}{24} - \frac{a^2x^5}{120} + \frac{x^3\beta}{6} + \frac{1}{12}ax^4\beta + \frac{1}{60}a^2x^5\beta - x\epsilon + \frac{1}{24}ax^4\epsilon - \frac{1}{3}x^3\beta\epsilon - \frac{1}{12}ax^4\beta\epsilon + \frac{1}{60}x^5\beta\epsilon^2 + \frac{1}{180}ax^6\beta\epsilon^2 - \frac{1}{60}x^5\beta^2\epsilon^2 - \frac{1}{120}ax^6\beta^2\epsilon^2 - \frac{1}{60}x^5\beta\epsilon^3 + \frac{1}{60}x^5\beta^2\epsilon^3 - \frac{x^7\beta^2\epsilon^4}{1260} + \frac{1}{840}x^7\beta^3\epsilon^4$$

$$\hat{y}_2 = \frac{ax^6}{240} + \frac{11a^2x^7}{5040} + \frac{11a^3x^8}{40320} - \frac{a^2x^9}{24192} - \frac{a^3x^{10}}{64800} - \frac{a^4x^{11}}{712800} - \frac{x^5\beta}{60} - \frac{1}{60}ax^6\beta - \frac{2}{315}a^2x^7\beta + \frac{ax^8\beta}{2688} - \frac{a^3x^8\beta}{1260} + \frac{53a^2x^9\beta}{181440} + \cdots + \frac{x^{13}\beta^6\epsilon^7}{1235520} - \frac{x^{15}\beta^4\epsilon^8}{103194000} + \frac{x^{15}\beta^5\epsilon^8}{24766560} - \frac{23x^{15}\beta^6\epsilon^8}{412776000} + \frac{x^{15}\beta^7\epsilon^8}{39312000}.$$

$$y_1 = \hat{y}_0 + \hat{y}_1 + \hat{y}_2 = x + \frac{ax^2}{2} - \frac{ax^4}{24} - \frac{a^2x^5}{120} + \frac{ax^6}{240} + \frac{11a^2x^7}{5040} + \frac{11a^3x^8}{40320} - \frac{a^2x^9}{24192} - \frac{a^3x^{10}}{64800} - \frac{a^4x^{11}}{712800} + \frac{x^3\beta}{6} + \frac{1}{12}ax^4\beta - \frac{x^5\beta}{60} + \frac{1}{60}a^2x^5\beta - \frac{1}{60}ax^6\beta - \frac{2}{315}a^2x^7\beta + \frac{ax^8\beta}{2688} + \cdots + \frac{x^{15}\beta^5\epsilon^8}{24766560} - \frac{23x^{15}\beta^6\epsilon^8}{412776000} + \frac{x^{15}\beta^7\epsilon^8}{39312000}.$$

Since, $y_n = \sum_{i=0}^{n} \hat{y}_i$ for all $n = 1, 2, 3, \ldots$.

Continue to find other approximations up to $n = 5$, for $y_n(x)$ but for brevity the terms are not listed.

We got the same value of a as in TAM, because the approximate solutions are the same .

Solving Falkner skan equation by the BCM:

Consider Eq.(2.53), by applying the same procedure as in the DJM, we get the Eq.(2.64). So, let

$y_0(x) = 0.9x + 0.5ax^2 - 0.000833333x^3,$

$N(y_{n-1}) = \frac{1}{2}\int_0^x (x-t)^2(-y_{n-1}(t)y_{n-1}''(t) + \beta\,(y_{n-1}'(t))^2)dt, \quad n \in N.$

$$y_0 = x + \frac{ax^2}{2} - x\epsilon - \frac{1}{6}x^3\beta\epsilon^2,$$

$$y_1(x) = x + \frac{ax^2}{2} - \frac{ax^4}{24} - \frac{a^2x^5}{120} + \frac{x^3\beta}{6} + \frac{1}{12}ax^4\beta + \frac{1}{60}a^2x^5\beta - x\epsilon + \frac{1}{24}ax^4\epsilon - \frac{1}{3}x^3\beta\epsilon - \frac{1}{12}ax^4\beta\epsilon + \frac{1}{60}x^5\beta\epsilon^2 + \frac{1}{180}ax^6\beta\epsilon^2 - \frac{1}{60}x^5\beta^2\epsilon^2 - \frac{1}{120}ax^6\beta^2\epsilon^2 - \frac{1}{60}x^5\beta\epsilon^3 + \frac{1}{60}x^5\beta^2\epsilon^3 - \frac{x^7\beta^2\epsilon^4}{1260} + \frac{1}{840}x^7\beta^3\epsilon^4,$$

$$y_2(x) = x + \frac{ax^2}{2} - \frac{ax^4}{24} - \frac{a^2x^5}{120} + \frac{ax^6}{240} + \frac{11a^2x^7}{5040} + \frac{11a^3x^8}{40320} - \frac{a^2x^9}{24192} - \frac{a^3x^{10}}{64800} - \frac{a^4x^{11}}{712800} + \frac{x^3\beta}{6} + \frac{1}{12}ax^4\beta - \frac{x^5\beta}{60} + \frac{1}{60}a^2x^5\beta - \frac{1}{60}ax^6\beta - \frac{2}{315}a^2x^7\beta + \frac{ax^8\beta}{2688} + \cdots + \frac{x^{15}\beta^5\epsilon^8}{24766560} - \frac{23x^{15}\beta^6\epsilon^8}{412776000} + \frac{x^{15}\beta^7\epsilon^8}{39312000}.$$

We continue to get other approximations till $n = 5$, for $y_n(x)$ but all are not listed for brevity, we got the same value of a as for the previous methods. TAM and DJM, since the approximate solutions are the same because we got the same series.

Now, we can find the β_i values in order to prove the convergence condition. Hence, the terms of the series $\sum_{i=0}^{\infty} v_i(x)$ given in Eq. (2.21), where $\beta = 0.5$, $\epsilon = 0.1$ we get

$$\beta_0 = \frac{\|v_1\|}{\|v_0\|} = 0.0854743 < 1,$$

$$\beta_1 = \frac{\|v_2\|}{\|v_1\|} = 0.0818763 < 1,$$

$$\beta_2 = \frac{\|v_3\|}{\|v_2\|} = 0.125112 < 1, \tag{2.65}$$

$$\beta_3 = \frac{||v_4||}{||v_3||} = 0.124664 < 1,$$

$$\beta_4 = \frac{||v_5||}{||v_4||} = 0.115635 < 1,$$

where, the β_i values for $i \geq 0$ and $0 < x \leq 1$, are less than 1, so the solution for the TAM is convergent. Since, the approximate solutions for the all three proposed methods are equal, therefore, the values of β_i are the same, then DJM and BCM approaches are convergent.

To examine the accuracy for the approximate solution for this problem; the error remainder function is

$$ER_n(\,x) = y_n'''(x) + y_n(x)y_n''(x) + \beta\,[\epsilon^2 - (y_n'(x))^2] \tag{2.66}$$

and the MER_n is:

$$MER_n = \max_{0\leq x\leq 1}|ER_n(x)|, \tag{2.67}$$

Table 2.4 and Fig. 2.4 shows the MER_n of the approximate solution obtained by the proposed iterative methods, it can be seen that by increasing the iterations, the errors will be decreasing.

Table 2.4: The maximal error remainder by the proposed methods. when $\beta = 0.5$ and $\epsilon = 0.1$.

n	MER_n
1	0.246021
2	0.0745229
3	0.0150143
4	0.00226928
5	0.000274414

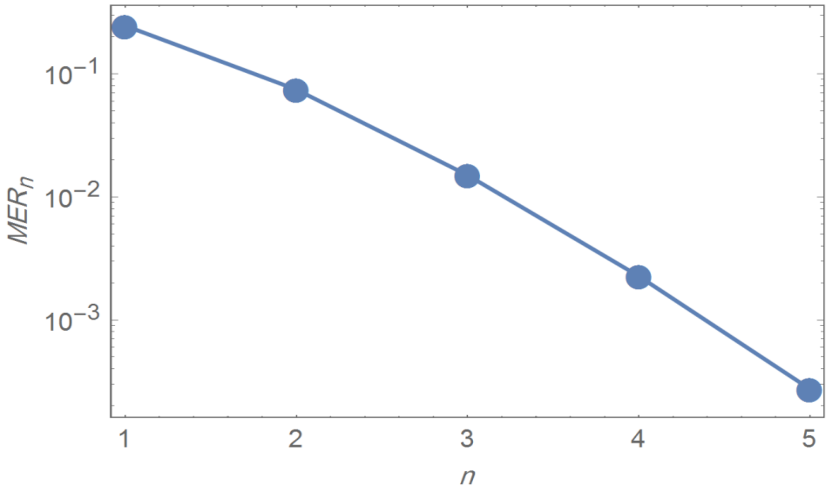


Fig.2.7: Logarithmic plots for the MER_n versus n from 1 to 5 when $\beta = 0.5$ and $\epsilon = 0.1$, by the proposed methods.

Moreover, the numerical comparison between the solutions obtained by the proposed methods, the Range-Kutta (RK4) and Euler methods for the Falkner skan equation is presented in Fig.2.8 and a good agreement can be clearly seen.

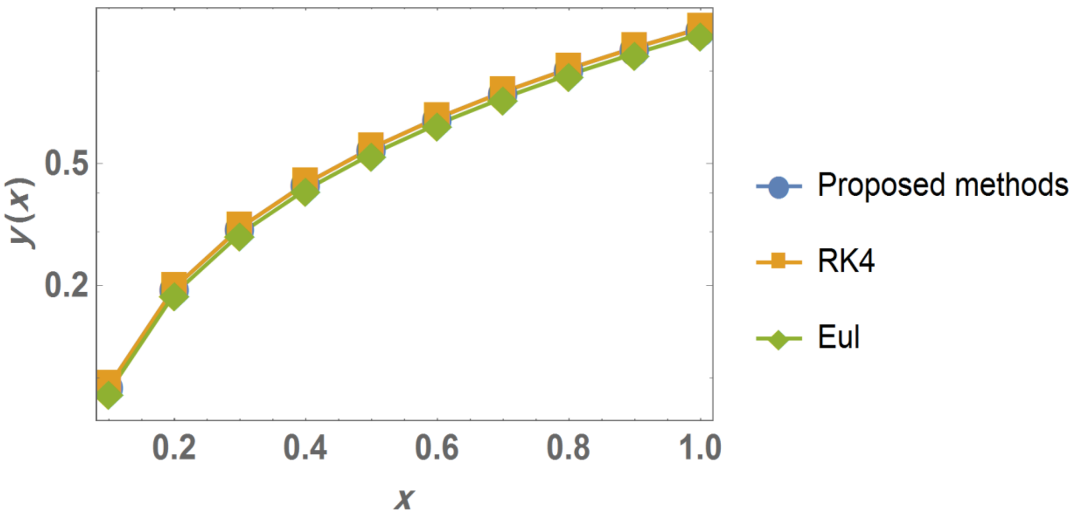


Fig.2.8: The comparison of the numerical solutions for Falkner skan equation.

CHAPTER 3

Solving 1D, 2D and 3D Wave Equations by TAM, DJM, and BCM

Chapter 3

Solving 1D, 2D and 3D Wave Equations by TAM, DJM, and BCM.

3.1 Introduction

The PDEs, have a great importance in engineering and physical applications. The wave equation is partial differential equation for the time variable t, with one or more variable spatial variables $(x, y, \ldots)$ and scalar function $u = u(x, y, \ldots)$[5]. This equation described many phenomena including the vibration of a beam membrane or an elastic rod, a vibrating string, shallow water waves, transmission of electric signals along a cable and other applications[89].

The main goal of this chapter is to implement the three iterative methods TAM, DJM and BCM to find the approximate solutions for the wave equation. This chapter has been organized as follows: In section two, the proposed methods will be used to solve linear 1D, 2D and 3D wave equations. In section three, the convergent of the three iterative methods will be given. In section four, 1D, 2D and 3D nonlinear wave equations will be solved by the TAM, DJM and BCM and the convergence will be provided.

3.2 Application of the proposed methods for solving linear 1D, 2D and 3D wave equations.

In this section, the proposed methods will be used to solve linear 1D, 2D and 3D wave equations.

Example 3.1.

Let us recall example 1.1

$$u_{tt}(x,t) = u_{xx}(x,t) - 2, \tag{3.1}$$

with initial conditions: $u(x,0) = x^2$, $u_t(x,0) = sin\, x$. Thus Eq.(3.1) will be solved by applying the three proposed iterative methods.

Example 3.1 by using the TAM.

We first begin by solving the following initial value problem as follows:

$$L\big(u(x,t)\big) = u_{tt}(x,t), N\left(u(x,t)\right) = u_{xx}(x,t),\ g(x,t) = -2. \tag{3.2}$$

The primary problem can be written as

$$L\big(u_0(x,t)\big) = -2, \text{ with } u_0(x,0) = x^2, \quad u_{0t}(x,0) = sin\, x. \tag{3.3}$$

We get the following problems from the general relationship

$$L\big(u_{n+1}(x,t)\big) = g(x,t) + N\left(u_n(x,t)\right) = 0,\ u_{n+1}(x,0) = x^2,\ u_{(n+1)t}(x,0) = sin\, x \tag{3.4}$$

$$u_{0tt}(x,t) = -2, \tag{3.5}$$

By integrating both sides of Eq. (3.5) twice from 0 to t, with $u_0(x,0) = x^2$, $u_{0t}(x,0) = sin\, x$, we obtain

$$u_0(x,t) = -t^2 + x^2 + t\,\text{sin}x,$$

By following this procedure, the rest of the other iterations can be obtained, the first iteration can be obtained by calculating.

$$u_{1tt}(x,t) = u_{0xx}(x,t) - 2 \text{ with } u_1(x,0) = x^2, u_{1t}(x,0) = sin\, x \qquad (3.6)$$

Then, the solution for Eq. (3.6) will be then:

$$u_1(x,t) = x^2 + t\,\sin x - \frac{1}{6}t^3 \sin x,$$

The second iteration $u_2(x,t)$ can be obtain from solving the following

$$u_{2tt}(x,t) = u_{1xx}(x,t) - 2 \text{ with } u_2(x,0) = x^2\,, u_{2t}(x,0) = sin\, x \qquad (3.7)$$

Then, by solving Eq. (3.7), we get

$$u_2(x,t) = x^2 + t\sin x - \frac{1}{6}t^3 \sin x + \frac{1}{120}t^5 \sin x$$

Also, third Iteration $u_3(x,t)$ can be obtain by solving the following

$$u_{3tt}(x,t) = u_{2xx}(x,t) - 2 \text{ with } u_3(x,0) = x^2\,, u_{3t}(x,0) = sin\, x \qquad (3.8)$$

By solving Eq. (3.8), we obtain

$$u_3(x,t) = x^2 + t\sin x - \frac{1}{6}t^3 \sin x + \frac{1}{120}t^5 \sin x - \frac{t^7 \sin x}{5040},$$

In a similar way, we obtain

$$u_4 = x^2 + t \sin x - \frac{1}{6}t^3 \sin x + \frac{1}{120}t^5 \sin x - \frac{t^7 \sin x}{5040} + \frac{t^9 \sin x}{362880},$$

and,

$$u_5(x,t)$$
$$= x^2 + t \sin x - \frac{1}{6}t^3 \sin x + \frac{1}{120}t^5 \sin x - \frac{t^7 \sin x}{5040} + \frac{t^9 \sin x}{362880}$$
$$- \frac{t^{11} \sin x}{39916800}, \qquad (3.9)$$

$u(x,t) = \lim_{n\to\infty} u_n$

$$u(x,t) = x^2 + \sin x\left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}\sin x}{39916800} + \cdots\right),$$

$= x^2 + \sin x \sin t$, which is the exact solution.

Solving the Example 3.1 by the DJM:

Consider the Eq. (3.1) with initial conditions: $u(x,0) = x^2$, $u_t(x,0) = sin\ x$.

Integrate both sides of Eq. (3.1) twice from 0 to t with using the given initial conditions, we get

$$u(x,t) = -t^2 + x^2 + t \sin x + \int_0^t \int_0^t (u_{xx}(x,s))dsds. \tag{3.10}$$

By reducing the integration in Eq. (3.10) from double to single [6], we obtain

$$u(x,t) = -t^2 + x^2 + t \sin x + \int_0^t (t-s)(u_{xx}(x,s))ds. \tag{3.11}$$

Then,

$u_0 = -t^2 + x^2 + t \sin x$, $N(u_{n+1}) = \int_0^t (t-s)(u_{nxx}(x,s))ds$

Applying the DJM, we get

$u_0(x,t) = \hat{u}_0(x,t) = -t^2 + x^2 + t \sin x,$

$$\hat{u}_1(x,t) = \int_0^t (t-s)(\hat{u}_{0xx}(x,s))ds = t^2 - \frac{1}{6}t^3 \sin x, \tag{3.12}$$

$u_1(x,t) = \hat{u}_0 + \hat{u}_1 = x^2 + t \sin x - \frac{1}{6}t^3 \sin x$

$\hat{u}_2(x,t) =$

$$\int_0^t (t-s)((\hat{u}_{0xx+}(x,s)+\hat{u}_{1xx}(x,s))ds \;-\; \hat{u}_1 \;=\; \frac{1}{120}t^5 \sin x \qquad (3.13)$$

$$u_2(x,t) = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 = x^2 + t\sin x - \frac{1}{6}t^3 \sin x + \frac{1}{120}t^5 \sin x$$

⋮

and so on, we get

$$\hat{u}_5(x,t) = -\frac{t^{11}\mathrm{Sin}\, x}{39916800}$$

Then , we have

$$u_n = \sum_{i=0}^{n} \hat{u}_i \qquad n = 1, 2, \dots .$$

$u_5 = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4 + \hat{u}_5$.

Then, we get

$u_5 = x^2 + t\sin x - \frac{1}{6}t^3 \sin x + \frac{1}{120}t^5 \sin x - \frac{t^7 \sin x}{5040} + \frac{t^9 \sin x}{362880} - \frac{t^{11} \sin x}{39916800}$, is the same of the solution in Eq. (3.9) because we got the same series.

Solving the Example3.1 by the BCM:

Also, consider the Eq. (3.1) with initial conditions: $u(x,0) = x^2$, $u_t(x,0) = sin\ x$

Integrate both sides of Eq. (3.1) twice from 0 to t with using the given initial conditions, we get

$$u(x,t) = -t^2 + x^2 + t\sin x + \int_0^t \int_0^t (u_{xx}) dx dx. \qquad (3.14)$$

Reducing the integration in Eq. (3.14) from double to single [6], we find

$u(x,t) =$

$$-t^2 + x^2 + t\,\sin x + \int_0^t (t-s)(u_{xx}(x,s))ds \tag{3.15}$$

Let $u_0 = -t^2 + x^2 + t\,\sin x$ and $N(u_{n-1}) =$

$$\int_0^t (t-s)(u_{n-1xx}(x,s))ds, \tag{3.16}$$

Applying the BCM, we obtain:

$u_0 = -t^2 + x^2 + t\,\sin x$,

$u_1(x,t) =$

$$u_0 + \int_0^t (t-s)(u_{0xx})ds = x^2 + t\,\sin x - \frac{1}{6}t^3 \sin x, \tag{3.17}$$

$$u_2(x,t) = u_0 + \int_0^t (t-s)(u_{1xx})ds = x^2 + t\sin x - \frac{1}{6}t^3 \sin x + \frac{1}{120}t^5 \sin x, \tag{3.18}$$

⋮

and so on till u_5, then

$u_5(x,t) = x^2 + t\,\sin x - \frac{1}{6}t^3 \sin x + \frac{1}{120}t^5 \sin x - \frac{t^7 \sin x}{5040} + \frac{t^9 \sin x}{362880} - \frac{t^{11} \sin x}{39916800}$, is the same of the solution in Eq. (3.9).

Example3.2. Consider the two-dimensional linear wave equation given in equation[44]

$$u_{tt} - (u_{xx} + u_{yy}) = 0, \qquad 0 \le x,y \le 1, \qquad t > 0 \tag{3.19}$$

with the initial conditions : $u(x,y,0) = e^{x+y}$, $u_t(x,y,0) = -\sqrt{2}\,e^{x+y}$

Eq. (3.19) will be solved by the three proposed iterative methods

Solving the Example 3.2 by the TAM:

By applying the TAM, we obtain the following iterations

$u_0(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t,$

$$u_1(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3\,,$$

$$u_2(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}}\,,$$

⋮

and so on, then

$$u_5(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}} + \frac{1}{90}e^{x+y}t^6 - \frac{e^{x+y}t^7}{315\sqrt{2}} + \frac{e^{x+y}t^8}{2520} - \frac{e^{x+y}t^9}{11340\sqrt{2}} + \frac{e^{x+y}t^{10}}{113400} - \frac{e^{x+y}t^{11}}{623700\sqrt{2}}, \quad (3.20)$$

$$u(x,y,t) = \lim_{n\to\infty} u_n = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \cdots.$$

This series converges to the exact solution[44]

$$u(x,y,t) = e^{x+y-\sqrt{2}t} = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\left(\sqrt{2}e^{x+y}\right)t^3 + \cdots.$$

Solving the Example 3.2 by the DJM:

Consider the Eq. (3.19) with initial conditions:

$u(x,y,0) = e^{x+y}$, $u_t(x,y,0) = -\sqrt{2}e^{x+y}$.

By applying the DJM, we get

$$u_0(x,y,t) = \hat{u}_0(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t,$$

$$\hat{u}_1(x,y,t) = e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3$$

$$u_1(x,y,t) = \hat{u}_0 + \hat{u}_1 = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3$$

$$\hat{u}_2(x,y,t) = \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}}\,,$$

$u_2(x,y,t) = \hat{u}_0 + \hat{u}_1 + \hat{u}_1 = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}}$

$\vdots$

$$\hat{u}_5 = \frac{e^{x+y}t^{10}}{113400} - \frac{e^{x+y}t^{11}}{623700\sqrt{2}}.$$

$$u_n = \sum_{i=0}^{n} \hat{u}_i \qquad n = 1, 2, \ldots .$$

$u_5 = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4 + \hat{u}_5 = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}} + \frac{1}{90}e^{x+y}t^6 - \frac{e^{x+y}t^7}{315\sqrt{2}} + \frac{e^{x+y}t^8}{2520} - \frac{e^{x+y}t^9}{11340\sqrt{2}} + \frac{e^{x+y}t^{10}}{113400} - \frac{e^{x+y}t^{11}}{623700\sqrt{2}}.$

$$u(x,y,t) = \sum_{i=0}^{\infty} \hat{u}_i = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \cdots$$

The series converges to the closed form of the exact solution

$$u(x,y,t) = e^{x+y-\sqrt{2}t} = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\left(\sqrt{2}e^{x+y}\right)t^3 + \cdots$$

Solving the Example 3.2 by the BCM:

Consider Eq.(3.19) with initial conditions $u(x,y,0) = e^{x+y}$, $u_t(x,y,0) = -\sqrt{2}e^{x+y}$.

Applying the BCM, we obtain:

$$u_0(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t,$$

$$u_1(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 ,$$

$$u_2(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}},$$

$\vdots$

$$u_5(x,y,t) = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\sqrt{2}e^{x+y}t^3 + \frac{1}{6}e^{x+y}t^4 - \frac{e^{x+y}t^5}{15\sqrt{2}} + \frac{1}{90}e^{x+y}t^6 - \frac{e^{x+y}t^7}{315\sqrt{2}} + \frac{e^{x+y}t^8}{2520} - \frac{e^{x+y}t^9}{11340\sqrt{2}} + \frac{e^{x+y}t^{10}}{113400} - \frac{e^{x+y}t^{11}}{623700\sqrt{2}},$$

$$u(x,y,t) = \lim_{n\to\infty} u_n = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \tfrac{1}{3}\sqrt{2}e^{x+y}t^3 + \cdots$$

This series converges to the exact solution

$$u(x,y,t) = e^{x+y-\sqrt{2}t} = e^{x+y} - \sqrt{2}e^{x+y}t + e^{x+y}t^2 - \frac{1}{3}\left(\sqrt{2}e^{x+y}\right)t^3 + \cdots$$

Example 3.3. Let us consider the following 3D linear wave equation as[5]

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} + \sin x + \sin y, \qquad 0 < x,y,z < \pi \tag{3.21}$$

with initial conditions: $u(x,y,z,0) = \sin x + \sin y,\ u_t(x,y,z,0) = \sin z$

Eq. (3.21) will be solved by the three proposed iterative methods

Example 3.3 by using the TAM:

By applying the TAM, we obtain:

$$u_0 = \sin x + \frac{1}{2}t^2\sin x + \sin y + \frac{1}{2}t^2\sin y + t\sin z.$$

$$u_1 = \sin x - \frac{1}{24}t^4\sin x + \sin y - \frac{1}{24}t^4\sin y + t\sin z - \frac{1}{6}t^3\sin z,$$

$$u_2 = \sin x + \frac{1}{720}t^6\sin x + \sin y + \frac{1}{720}t^6\sin y + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z\,,$$

⋮

$$u_5 = \sin x - \frac{t^{12}\sin x}{479001600} + \sin y - \frac{t^{12}\sin y}{479001600} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800}. \quad (3.22)$$

$$u(x,y,z,t) = \lim_{n\to\infty} u_n = \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \cdots.$$

This series converges to the exact solution

$$u(x,y,z,t) = \sin x + \sin y + \sin z\,\sin t = \; = \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \cdots.$$

Solving Example 3.3 by the DJM:

By applying the DJM, we obtain:

$$u_0 = \hat{u}_0 = \sin x + \tfrac{1}{2}t^2\sin x + \sin y + \tfrac{1}{2}t^2\sin y + t\sin z,$$

$$\hat{u}_1 = -\tfrac{1}{2}t^2\sin x - \tfrac{1}{24}t^4\sin x - \tfrac{1}{2}t^2\sin y - \tfrac{1}{24}t^4\sin y - \tfrac{1}{6}t^3\sin z,$$

$$u_1 = \hat{u}_0 + \hat{u}_1 = \sin x - \frac{1}{24}t^4\sin x + \sin y - \frac{1}{24}t^4\sin y + t\sin z - \frac{1}{6}t^3\sin z$$

$$\hat{u}_2 = \frac{1}{24}t^4\sin x + \frac{1}{720}t^6\sin x + \frac{1}{24}t^4\sin y + \frac{1}{720}t^6\sin y + \frac{1}{120}t^5\sin z\,,$$

$u_2 = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 = \sin x + \frac{1}{720}t^6\sin x + \sin y + \frac{1}{720}t^6\sin y + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z$

⋮

$$\hat{u}_5 = -\frac{t^{10}\sin x}{3628800} - \frac{t^{12}\sin x}{479001600} - \frac{t^{10}\sin y}{3628800} - \frac{t^{12}\sin y}{479001600} - \frac{t^{11}\sin z}{39916800}.$$

$$u_n = \sum_{i=0}^{n} \hat{u}_i \qquad n = 1,2,3,4,\ldots$$

$$u_5 = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4 + \hat{u}_5.$$

$$u_5 = \sin x - \frac{t^{12}\sin x}{479001600} + \sin y - \frac{t^{12}\sin y}{479001600} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800},$$

$u = \sum_{i=0}^{\infty} \hat{u}_i = \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \frac{t^{13}\sin z}{6227020800} - \frac{t^{15}\sin z}{1307674368000} + \cdots$

This series converges to the exact solution

$u(x,y,z,t) = \sin x + \sin y + \sin z\sin t = \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \frac{t^{13}\sin z}{6227020800} - \frac{t^{15}\sin z}{1307674368000} + \cdots$

Solving the Example 3.3 by the BCM:

By applying the BCM, we obtain:

$$u_0 = \sin x + \frac{1}{2}t^2\sin x + \sin x + \frac{1}{2}t^2\sin x + t\sin x,$$

$$u_1 = \sin x - \frac{1}{24}t^4\sin x + \sin y - \frac{1}{24}t^4\sin y + t\sin z - \frac{1}{6}t^3\sin z,$$

$$u_2 = \sin x + \frac{1}{720}t^6\sin x + \sin y + \frac{1}{720}t^6\sin y + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z\,,$$

$\vdots$

and so on, thus, we have

$$u_5 = \sin x - \frac{t^{12}\sin x}{479001600} + \sin y - \frac{t^{12}\sin y}{479001600} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800}.$$

$$u(x,y,z,t) = \lim_{n\to\infty} u_n = \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \cdots$$

This series converges to the exact solution

$$u(x,y,z,t) = \sin x + \sin y + \sin z\,\sin t =$$

$$= \sin x - \frac{t^{16}\sin x}{20922789888000} + \sin y - \frac{t^{16}\sin y}{20922789888000} + t\sin z - \frac{1}{6}t^3\sin z + \frac{1}{120}t^5\sin z - \frac{t^7\sin z}{5040} + \frac{t^9\sin z}{362880} - \frac{t^{11}\sin z}{39916800} + \cdots$$

3.3 The convergent of the proposed methods

To prove the convergence of the proposed methods for the linear and nonlinear 1D wave equations [91], we define the following new iterations:

$$
\begin{aligned}
v_0 &= u_0(x,t), \\
v_1 &= F[v_0], \\
v_2 &= F[v_0 + v_1], \\
&\;\;\vdots \\
v_{n+1} &= F[v_0 + v_1 + \cdots + v_n].
\end{aligned}
\tag{3.23}
$$

where F is an operator which can be defined as follows:

$$F[v_k] = S_k - \sum_{i=0}^{k-1} v_i(x,t)\,, k = 1,2, \ldots. \tag{3.24}$$

The term S_k represents the solution of the following problem

$$L(v_k(x,t)) + g(x,t) + N\left(\sum_{i=0}^{k-1} v_i(x,t)\right) = 0, \qquad k = 1, 2, \ldots, \tag{3.25}$$

using the same given conditions of the problem, in this way, we have $u(x,t) = \lim_{n\to\infty} u_n(x,t) = \sum_{n=0}^{\infty} v_n\,(x,t)$. So, the solution of the problem represented, we can access it by using (3.23) ,(3.24) in the resulted series

$$u(x,t) = \sum_{i=0}^{\infty} v_i(x,t). \tag{3.26}$$

According to this procedure for the proposed methods, the sufficient conditions for convergence of these methods can be given in theorems 1.2.8, 1.2.9and 1.2.10.

The procedure can be generalized for 2D, 3D as a similar way.

3.4 Application of the proposed methods for solving nonlinear 1D, 2D and 3D wave equations with convergence.

In this section, the proposed methods will be applied to solve the nonlinear 1D, 2D and 3D wave equations. Moreover, the convergence will be proved.

Example 3.4: Let us recall example 1.2

$$u_{tt} = u_{xx} + u + u^2 - xt - x^2t^2, \tag{3.27}$$

with initial condition : $u(x,0) = 0,\ u_t(x,0) = x,$ Eq. (3.27) will be solved by the proposed iterative methods

Solving the Example 3.4 by the TAM:

In order to solve the Eq. (3.27) by TAM with the given initial conditions, we have the following form

$$L(u) = u_{tt}(x,t),\ N\ (u) = u_{xx}(x,t) + u(x,t) + u(x,t)^2, \quad g(x,t) = -xt - x^2t^2 \tag{3.28}$$

The initial problem is

$$L(u_0) = -xt - \ x^2t^2 \text{ with } \ u_0(x,0) = 0,\ u_{0_t}(x,0) = x. \tag{3.29}$$

We will be found the next problems from the generalized iterative formula

$$L(u_{n+1}) + N(u_n) = g(x,t), \qquad u_{n+1}(x,0) = 0, \ \ u_{n+1_t}(x,0) = x$$

By solving the Eq. (3.29), we get

$$u_0 = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12}$$

The first iteration $u_1(x,t)$ can be obtained by solving

$u_{1tt} = u_{0_{xx}}(x,t) + u_0(x,t) + u_0(x,t)^2 - xt - x^2t^2$, with $u_1(x,0) = 0$, $u_{1_t}(x,0) = x$.

The solution will be

$$u_1 = -\frac{t^6}{180} + tx - \frac{t^5x}{120} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016} - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960},$$

Applying the same process for u_2, we have

$u_{2tt} = u_{1_{xx}}(x,t) + u_1(x,t) + u_1(x,t)^2 - xt - x^2t^2$ with $u_2(x,0) = 0$, $u_{2_t}(x,0) = x$

By solving this problem, we get

$$u_2 = -\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7x}{5040} - \frac{11t^9x}{22680} + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} - \frac{11t^8x^2}{20160} + \frac{t^{10}x^2}{181440} + \cdots$$

⋮

and so on. Then, we get

$$u_5 = xt - \frac{xt^{13}}{6227020800} + \left(-\frac{1}{335301120} - \frac{41x^2}{43589145600}\right)t^{14} + \left(-\frac{289x}{20432412000} - \frac{29x^3}{13076743680}\right)t^{15} + \cdots \quad (3.30)$$

This series converges to the exact solution when

$u(x,t) = xt.$

Solving Example 3.4 by the DJM:

Consider the Eq. (3.28) with the initial conditions $u(x,0) = 0$, $u_t(x,0) = x$. Integrating both sides of Eq. (3.27) twice from 0 to t, we get

$$u(x,t) = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12} + \int_0^t \int_0^t u_{xx} + u + u^2 \, dx \, dx, \quad (3.31)$$

and reducing the integration in Eq. (3.31) from double to single [6], we find

$$u(x,t) = t^2 - \frac{t^4}{6} + x^2 - t^2x^2 + \int_0^t (t-s)(u_{xx} + u + u^2)ds \qquad (3.32)$$

Therefore, we have the following recurrence relation

$u_0 = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12}$,

$N(u_{n+1}) = \int_0^t (t-s)(u_{nxx} + u_n + {u_n}^2)ds.$

By applying the DJM, we get

$u_0 = \hat{u}_0 = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12}$

$\hat{u}_1 = -\frac{t^6}{180} + \frac{t^3x}{6} - \frac{t^5x}{120} + \frac{t^4x^2}{12} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016} - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960}$,

$u_1 = \hat{u}_0 + \hat{u}_1 = -\frac{t^6}{180} + tx - \frac{t^5x}{120} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016} - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960}$

$\hat{u}_2 = \frac{t^6}{180} - \frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + \frac{t^5x}{120} - \frac{t^7x}{5040} - \frac{11t^9x}{22680} + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} + \frac{t^6x^2}{72} - \frac{t^8x^2}{960} + \frac{t^{10}x^2}{181440} + \cdots$

$u_1 = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 = -\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7x}{5040} - \frac{11t^9x}{22680} + \frac{t^{11}x}{47520} + \frac{t^{13}x}{1684800} - \frac{11t^8x^2}{20160} + \frac{t^{10}x^2}{181440} + \cdots$

$\vdots$

and so on. Thus, we have

$\hat{u}_5 = \frac{t^{12}}{3326400} + \frac{239t^{14}}{1452971520} - \frac{6619t^{16}}{435891456000} - \frac{29839t^{18}}{72754246656000} + \frac{60709t^{20}}{6911653432320000} - \frac{68107471t^{22}}{46833363657400320000} + \cdots$

$u_n = \sum_{i=0}^{n} \hat{u}_i \qquad n = 1, 2, 3, \dots .$

$u_5 = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4 + \hat{u}_5$,

$u_5 = xt - \frac{xt^{13}}{6227020800} + (-\frac{1}{335301120} - \frac{41x^2}{43589145600})t^{14} + (-\frac{289x}{20432412000} - \frac{29x^3}{13076743680})t^{15} + \cdots$

The series converges to the exact solution when

$u(x,t) = xt,$ see appendix D.

Solving Example 3.4 by the BCM:

Consider Eq.(3.27), by followed the same procedure as in the DJM, we get the Eq. (3.32).

So , let $u_0 = tx - \frac{t^3 x}{6} - \frac{t^4 x^2}{12}$,

$N(u_{n-1}) = \int_0^t (t-s)(\partial_{xx} u_{n-1} + u_{n-1} + {u_{n-1}}^2) ds$

By applying the BCM, we obtain:

$$u_0 = tx - \frac{t^3 x}{6} - \frac{t^4 x^2}{12},$$

$$u_1 = -\frac{t^6}{180} + tx - \frac{t^5 x}{120} - \frac{t^6 x^2}{72} + \frac{t^8 x^2}{2016} - \frac{t^7 x^3}{252} + \frac{t^9 x^3}{2592} + \frac{t^{10} x^4}{12960},$$

$$u_2 = -\frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + tx - \frac{t^7 x}{5040} - \frac{11 t^9 x}{22680} + \frac{t^{11} x}{47520} + \frac{t^{13} x}{1684800} - \frac{11 t^8 x^2}{20160} + \frac{t^{10} x^2}{181440} + \cdots.$$

⋮

and so on. Then, we have

$$u_5 = xt - \frac{xt^{13}}{6227020800} + \left(-\frac{1}{335301120} - \frac{41x^2}{43589145600}\right)t^{14} + \left(-\frac{289x}{20432412000} - \frac{29x^3}{13076743680}\right)t^{15} + \cdots,$$

is the same of the approximate solution in Eq.(3.30).

We see that the approximate solutions obtained from the three proposed methods are the same because we got the same series.

To prove the convergence analysis for the proposed methods, we will be applied the process as given in Eqs. (3.23)-(3.26). The iterative scheme for Eq. (3.27) can be formulated as

$v_0(x,t) = u_0(x,t) = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12}$.

Applying the TAM, the operator $F[v_k]$ as defined in Eq. (3.27) with the term S_k which is the solution for the following problem, it will be then

$v_{ktt}(x,t) = \left(\sum_{i=0}^{k-1} v_{ixx}(x,t)\right) + \left(\sum_{i=0}^{k-1} v_i(x,t)\right) + \left(\sum_{i=0}^{k-1} v_i(x,t)\right)^2 - xt - x^2t^2$,

with $v_k(x,0) = 0, v_{kt}(x,0) = x \qquad k \geq 1.$

Also, when applying the BCM, the S_k represents the solution for the following problem,

$$v_k = v_0 + \left(\sum_{i=0}^{k-1} v_{ixx}(x,t)\right) + \left(\sum_{i=0}^{k-1} v_i(x,t)\right) + \left(\sum_{i=0}^{k-1} v_i(x,t)\right)^2, \ k \geq 1.$$

Can be used the iterative approximations directly when applying the DJM. Therefore, we have the following terms

$$v_1 = -\frac{t^6}{180} + \frac{t^3x}{6} - \frac{t^5x}{120} + \frac{t^4x^2}{12} - \frac{t^6x^2}{72} + \frac{t^8x^2}{2016} - \frac{t^7x^3}{252} + \frac{t^9x^3}{2592} + \frac{t^{10}x^4}{12960},$$

$$v_2 = \frac{t^6}{180} - \frac{t^8}{1680} + \frac{t^{10}}{90720} + \frac{t^{14}}{5896800} + \frac{t^5x}{120} - \frac{t^7x}{5040} - \frac{11t^9x}{22680} + \cdots,$$

$$v_3 = \frac{t^8}{1680} - \frac{t^{10}}{33600} + \frac{t^{12}}{5987520} - \frac{t^{14}}{11531520} + \frac{11t^{16}}{1415232000} + \frac{47t^{18}}{46637337600} - \frac{t^{20}}{28957824000} + \cdots.$$

We use the above duplicates in computing the values of βi for the equation as in (3.27), we obtain

$$\beta_0 = \frac{\|v_1\|}{\|v_0\|} = 0.265249 < 1$$

$$\beta_1 = \frac{\|v_2\|}{\|v_1\|} = 0.132357 < 1$$

$$\beta_2 = \frac{\|v_3\|}{\|v_2\|} = 0.0800386 < 1$$

$$\beta_3 = \frac{\|v_4\|}{\|v_3\|} = 0.045514 < 1$$

$$\beta_4 = \frac{\|v_5\|}{\|v_4\|} = 0.0301515 < 1,$$

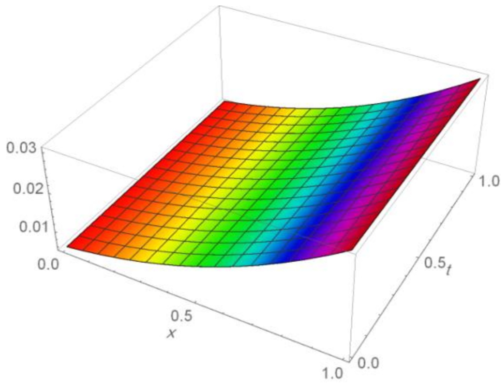
where, the β_i values for $i \geq 0$ and $0 < x \leq 1$ are less than 1 when $t = 1$, so the proposed iterative methods satisfy the convergence.

We calculate the absolute error $|r_n|$, to check the accuracy of the approximate solution u_n , where $u = xt$ is the exact solution.

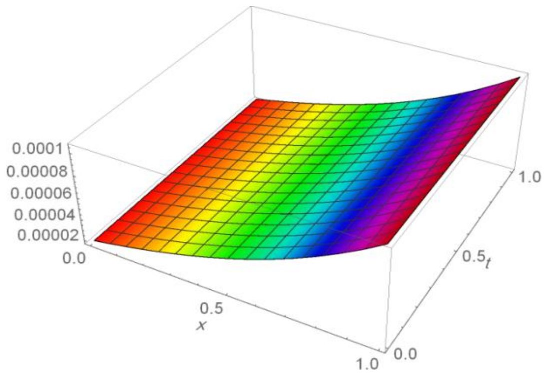
Table3.1and Fig3. $1(a, b, c)$below shows the 3D plotted graph of the $|r_n|$ for approximate solution obtained by the suggested iterative methods. Also, by increasing the iterations, the errors are decreasing and the accuracy of the approximate solution increases.

Table3.1: Results of the absolute errors by the proposed methods where $t = 1$.

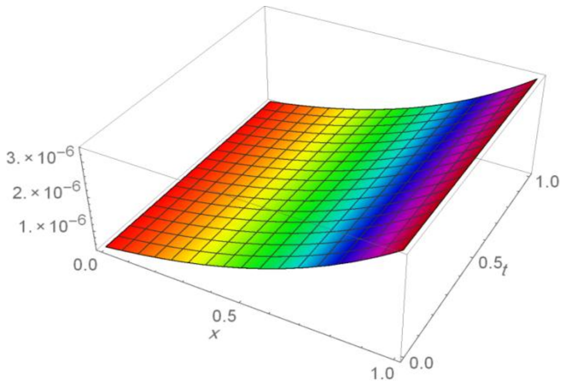
$\boldsymbol{x}$	$\lvert \boldsymbol{r_1} \rvert$	$\lvert \boldsymbol{r_3} \rvert$	$\lvert \boldsymbol{r_4} \rvert$
0	0.00555556	0.0000184804	4.61399×10^{-7}
0. 1	0.00652639	0.0000228675	5.76688×10^{-7}
0. 2	0.00778647	0.0000278468	7.17091×10^{-7}
0. 3	0.00935701	0.0000335267	8.8619×10^{-7}
0. 4	0.011259	0.0000400357	1.08821×10^{-6}
0. 5	0.0135134	0.000047523	1.32813×10^{-6}
0. 6	0.0161408	0.0000561599	1.61177×10^{-6}
0. 7	0.0191616	0.0000661411	1.94595×10^{-6}
0. 8	0.0225963	0.0000776859	2.33857×10^{-6}
0. 9	0.0264648	0.0000910392	2.79877×10^{-6}
1	0.030787	0.000106473	3.33704×10^{-6}



$a = u_1(x,t)$



$b = u_3(x,t)$



$\mathrm{c} = u_4(x,t)$

Figure3.1$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$**:** The plots of the absolute error $|r_n|$ at $n = 1, 3$ and 4

Example 3.5. We take the following 2D nonlinear wave equation

$$u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) - u(x,y,t)^2 + t^2x^2y^2 \qquad (3.33)$$

with initial conditions:

$u(x,y,0) = 0, \quad u_t(x,y,0) = xy.$

The Eq. (3.33) will be solved by the three iterative methods with the initial conditions.

Solving Example 3.5 by the TAM:

By applying the TAM

$$u_0 = txy + \frac{1}{12}t^4x^2y^2\,,$$

$$u_1=\frac{t^6x^2}{180}+txy+\frac{t^6y^2}{180}-\frac{1}{252}t^7x^3y^3-\frac{t^{10}x^4y^4}{12960},$$

$$u_2=\frac{t^8}{2520}-\frac{t^{14}x^4}{5896800}+txy-\frac{11t^9x^3y}{22680}-\frac{t^{14}x^2y^2}{2948400}-\frac{t^{12}x^4y^2}{142560}-\frac{11t^9xy^3}{22680}+\frac{t^{15}x^5y^3}{4762800}-\frac{t^{14}y^4}{5896800}-\frac{t^{12}x^2y^4}{142560}+\cdots,$$

continuing in this way till n = 4, we find

$$u_4=xyt+\left(\frac{23x^4}{136216080}+\frac{61x^2y^2}{22702680}+\frac{23y^4}{136216080}\right)t^{14}+\left(-\frac{173x^5y^3}{681080400}-\frac{173x^3y^5}{681080400}\right)t^{15}+\cdots \tag{3.34}$$

This series converges to the exact solution when

$$u(x,y,t)=\lim_{n\to\infty}u_n(x,y,t)=\ x\ y\ t\ .$$

Solving Example 3.5 by the DJM:

By applying the DJM

$$u_0=\hat{u}_0=txy+\frac{1}{12}t^4x^2y^2\ ,$$

$$\hat{u}_1=\frac{t^6x^2}{180}+\frac{t^6y^2}{180}-\frac{1}{12}t^4x^2y^2-\frac{1}{252}t^7x^3y^3-\frac{t^{10}x^4y^4}{12960},$$

$$u_1=\hat{u}_0+\hat{u}_1=\frac{t^6x^2}{180}+txy+\frac{t^6y^2}{180}-\frac{1}{252}t^7x^3y^3-\frac{t^{10}x^4y^4}{12960}$$

$$\hat{u}_2=\frac{t^8}{2520}-\frac{t^6x^2}{180}-\frac{t^{14}x^4}{5896800}-\frac{11t^9x^3y}{22680}-\frac{t^6y^2}{180}-\frac{t^{14}x^2y^2}{2948400}-\frac{t^{12}x^4y^2}{142560}-\frac{11t^9xy^3}{22680}+\frac{1}{252}t^7x^3y^3+\cdots$$

$$u_2=\hat{u}_0+\hat{u}_1+\hat{u}_2=\frac{t^8}{2520}-\frac{t^{14}x^4}{5896800}+txy-\frac{11t^9x^3y}{22680}-\frac{t^{14}x^2y^2}{2948400}-\frac{t^{12}x^4y^2}{142560}-\frac{11t^9xy^3}{22680}+\frac{t^{15}x^5y^3}{4762800}-\frac{t^{14}y^4}{5896800}-\frac{t^{12}x^2y^4}{142560}+\cdots,$$

Continue to till $n=4$

$$\hat{u}_4 = \frac{139t^{18}}{378928368000} - \frac{t^{38}}{530921529398163456000} - \frac{t^{16}x^2}{3891888000} + \frac{t^{26}x^2}{166617032400000} + \cdots$$

$u_n \;=\; \sum_{i=0}^{n} \hat{u}_i \qquad n = 1, 2, \ldots\ .$

$u_4 \;= \hat{u}_0 + \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4\ .$

$u_4 \;= xyt + (\frac{23x^4}{136216080} + \frac{61x^2y^2}{22702680} + \frac{23y^4}{136216080})t^{14} + (-\frac{173x^5y^3}{681080400} - \frac{173x^3y^5}{681080400})t^{15} + \cdots$

Which is the same of the approximate solution in Eq. (3.34), and converges to the exact solution .

$u = \sum_{i=0}^{\infty} \hat{u}_i \;= xyt$

Solving the Example 3.5 by the BCM:

By applying the BCM

$$u_0 = txy + \frac{1}{12}t^4x^2y^2\ ,$$

$$u_1 = \frac{t^6x^2}{180} + txy + \frac{t^6y^2}{180} - \frac{1}{252}t^7x^3y^3 - \frac{t^{10}x^4y^4}{12960}\ ,$$

$$u_2 = \frac{t^8}{2520} - \frac{t^{14}x^4}{5896800} + txy - \frac{11t^9x^3y}{22680} - \frac{t^{14}x^2y^2}{2948400} - \frac{t^{12}x^4y^2}{142560} - \frac{11t^9xy^3}{22680} + \frac{t^{15}x^5y^3}{4762800} - \frac{t^{14}y^4}{5896800} - \frac{t^{12}x^2y^4}{142560} + \cdots,$$

$\vdots$

Continue to till $n = 4$

$$u_4 = xyt + (\frac{23x^4}{136216080} + \frac{61x^2y^2}{22702680} + \frac{23y^4}{136216080})t^{14} + (-\frac{173x^5y^3}{681080400} - \frac{173x^3y^5}{681080400})t^{15} + \cdots.$$

is the same approximate solution as given in (3.34), we see that the approximate solutions obtained from the three proposed methods are the same because we got the same series.

To prove the convergence analysis for the proposed methods, we can find the β_i values in the problem as in Eq. (3.34). Hence, the terms of the series $\sum_{i=0}^{\infty} v_i(x,y,t)$ given in Eq. (3.26), we have

$$\beta_0 = \frac{\|v_1\|}{\|v_0\|} = 0.0564883 < 1$$

$$\beta_1 = \frac{\|v_2\|}{\|v_1\|} = 0.142938 < 1$$

$$\beta_2 = \frac{\|v_3\|}{\|v_2\|} = 0.0770897 < 1$$

$$\beta_3 = \frac{\|v_4\|}{\|v_3\|} = 0.066341 < 1$$

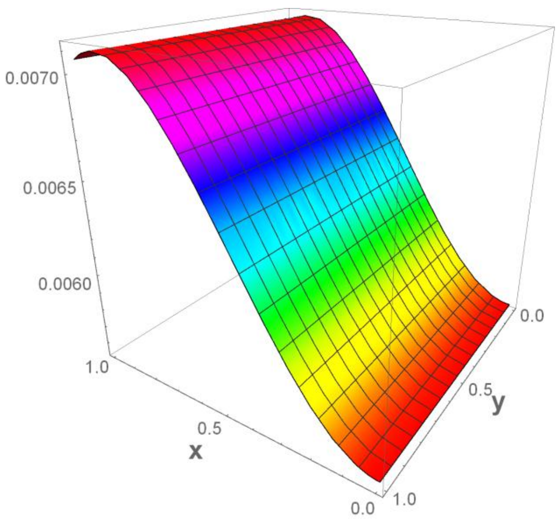
where, the β_i values for $i \geq 0$ and $0 < x, y \leq 1$ are less than 1 when $t = 1$, so the proposed iterative methods satisfy the convergence.

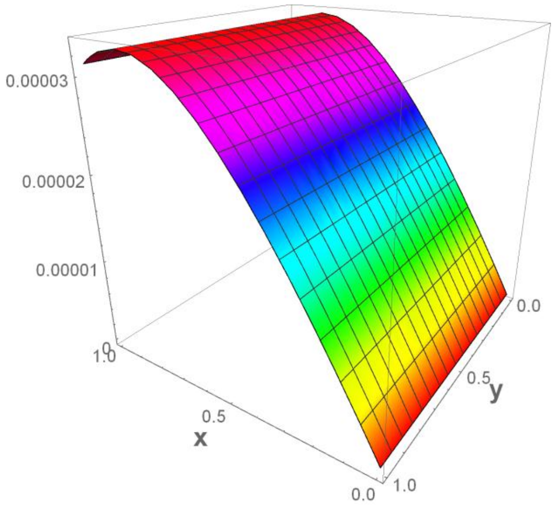
In order to test the accuracy of the approximate solution, we calculate the $|r_n|$ where $u = x\ y\ t$ is the exact solution.

Table 3.2 and Fig. $3.2(a,b,c)$ shows the 3D plotted graph of the $|r_n|$ for approximate solution obtained by the suggested iterative methods. It can be seen clearly, by increasing the number of iterations the error will be reduced and the solution becomes more accurate.

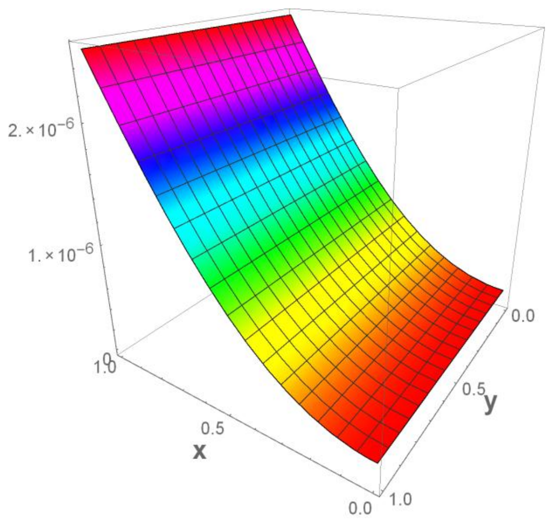
Table3.2: Results of the absolute errors by the proposed methods, where $y, t = 1$.

x	$\lvert r_1 \rvert$	$\lvert r_3 \rvert$	$\lvert r_4 \rvert$
0	0.00555556	8.89033×10^{-8}	1.57137×10^{-7}
0. 1	0.00560714	5.95471×10^{-6}	1.84193×10^{-7}
0. 2	0.00574591	0.0000115097	2.63572×10^{-7}
0. 3	0.00594779	0.0000166991	3.94162×10^{-7}
0. 4	0.0061885	0.0000214338	5.74663×10^{-7}
0. 5	0.00644359	0.0000255918	8.03297×10^{-7}
0. 6	0.00668841	0.0000290187	1.07753×10^{-6}
0. 7	0.00689814	0.0000315298	1.39378×10^{-6}
0. 8	0.00704776	0.0000329106	1.74719×10^{-6}
0. 9	0.00711207	0.0000329184	2.13131×10^{-6}
1	0.0070657	0.0000312834	2.53787×10^{-6}





$a = u_1(x, y, t)$ $b = u_3(x, y, t)$



$c = u_4(x, y, t)$

Figure3.2$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$: The plots of the absolute error $|r_n|$ at $n = 1, 3$ *and* 4 with $t = 1$.

Example 3.6. Consider 3D nonlinear wave equation given in equation

$$u_{tt}(x, y, z, t) = u_{xx}(x, y, z, t) + u_{yy}(x, y, z, t) + u_{zz}(x, y, z, t) - u(x, y, z, t)^2 + t^2x^2y^2z^2, \quad (3.35)$$

with the initial conditions: $u(x, y, z, 0) = 0, u_t(x, y, z, 0) = xyz$

Solving the Example 3.6 by the TAM:

By applying the TAM

$$u_0 = txyz + \frac{1}{12}t^4x^2y^2z^2$$

$$u_1 = \frac{1}{180}t^6x^2y^2 + txyz + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960},$$

$$u_2 = \frac{t^8x^2}{2520} + \frac{t^8y^2}{2520} - \frac{t^{14}x^4y^4}{5896800} + txyz - \frac{11t^9x^3y^3z}{22680} + \frac{t^8z^2}{2520} - \frac{t^{14}x^4y^2z^2}{2948400} - \frac{t^{14}x^2y^4z^2}{2948400} - \frac{t^{12}x^4y^4z^2}{142560} - \frac{11t^9x^3yz^3}{22680} - \frac{11t^9xy^3z^3}{22680} + \cdots.$$

$\vdots$

and so on. Then, we have

$$u_4 = xyzt - \frac{59(xyz)t^{13}}{8108100} + \left(\frac{23x^4y^4}{136216080} + \frac{61x^4y^2z^2}{22702680} + \frac{61x^2y^4z^2}{22702680} + \frac{23x^4z^4}{136216080} + \frac{61x^2y^2z^4}{22702680} + \frac{23y^4z^4}{136216080}\right)t^{14} + \left(-\frac{173x^5y^5z^3}{681080400} - \frac{173x^5y^3z^5}{681080400} - \frac{173x^3y^5z^5}{681080400}\right)t^{15} + \cdots \qquad (3.36)$$

Is the approximate solution, which converges to the exact solution when

$$u(x,y,z,t) = \lim_{n\to\infty} u_n = xyzt.$$

Solving Example3.6 by the DJM:

By applying the DJM

$$u_0 = \hat{u}_0 = txyz + \frac{1}{12}t^4x^2y^2z^2,$$

$$\hat{u}_1 = \frac{1}{180}t^6x^2y^2 + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{12}t^4x^2y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960},$$

$$u_1 = \hat{u}_0 + \hat{u}_1 = \frac{1}{180}t^6x^2y^2 + txyz + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960}$$

$$\hat{u}_2 = \frac{t^8x^2}{2520} + \frac{t^8y^2}{2520} - \frac{1}{180}t^6x^2y^2 - \frac{t^{14}x^4y^4}{5896800} - \frac{11t^9x^3y^3z}{22680} + \frac{t^8z^2}{2520} - \frac{1}{180}t^6x^2z^2 + \cdots$$

$$u_2 = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 = \frac{t^8x^2}{2520} + \frac{t^8y^2}{2520} - \frac{t^{14}x^4y^4}{5896800} + txyz - \frac{11t^9x^3y^3z}{22680} + \frac{t^8z^2}{2520} - \frac{t^{14}x^4y^2z^2}{2948400} - \frac{t^{14}x^2y^4z^2}{2948400} - \frac{t^{12}x^4y^4z^2}{142560} - \frac{11t^9x^3yz^3}{22680} - \frac{11t^9xy^3z^3}{22680} + \cdots$$

⋮

Continue to till $n = 4$

$$\hat{u}_4 = -\frac{t^{10}}{37800} - \frac{t^{22}}{660124080000} - \frac{t^{20}x^2}{36921225600} + \frac{139t^{18}x^4}{378928368000} + \frac{t^{30}x^4}{31952405923200000} - \frac{t^{38}x^8}{530921529398163456 0000} - \frac{t^{20}y^2}{36921225600} - \frac{t^{18}x^2y^2}{5262894000} \cdots$$

$$u_n = \sum_{i=0}^{n} \hat{u}_i \qquad n = 1, 2, 3, \dots .$$

$$u_4 = \hat{u}_0 + \hat{u}_1 + \hat{u}_2 + \hat{u}_3 + \hat{u}_4 = xyzt - \frac{59(xyz)t^{13}}{8108100} + \left(\frac{23x^4y^4}{136216080} + \frac{61x^4y^2z^2}{22702680} + \frac{61x^2y^4z^2}{22702680} + \frac{23x^4z^4}{136216080} + \frac{61x^2y^2z^4}{22702680} + \frac{23y^4z^4}{136216080}\right)t^{14} + \left(-\frac{173x^5y^5z^3}{681080400} - \frac{173x^5y^3z^5}{681080400} - \frac{173x^3y^5z^5}{681080400}\right)t^{15} + \cdots$$

Which is the same of the approximate solution in Eq. (3.36), and converges to the exact solution.

Solving the Example3.6 by the BCM:

By applying the BCM

$$u_0 = txyz + \frac{1}{12}t^4x^2y^2z^2$$

$$u_1 = \frac{1}{180}t^6x^2y^2 + txyz + \frac{1}{180}t^6x^2z^2 + \frac{1}{180}t^6y^2z^2 - \frac{1}{252}t^7x^3y^3z^3 - \frac{t^{10}x^4y^4z^4}{12960},$$

$$u_2 = \frac{t^8x^2}{2520} + \frac{t^8y^2}{2520} - \frac{t^{14}x^4y^4}{5896800} + txyz - \frac{11t^9x^3y^3z}{22680} + \frac{t^8z^2}{2520} - \frac{t^{14}x^4y^2z^2}{2948400} - \frac{t^{14}x^2y^4z^2}{2948400} - \frac{t^{12}x^4y^4z^2}{142560} - \frac{11t^9x^3yz^3}{22680} - \frac{11t^9xy^3z^3}{22680} + \cdots.$$

⋮

and so on. Then, we have

$$u_4 = xyzt - \frac{59(xyz)t^{13}}{8108100} + \left(\frac{23x^4y^4}{136216080} + \frac{61x^4y^2z^2}{22702680} + \frac{61x^2y^4z^2}{22702680} + \frac{23x^4z^4}{136216080} + \frac{61x^2y^2z^4}{22702680} + \frac{23y^4z^4}{136216080}\right)t^{14} + \left(-\frac{173x^5y^5z^3}{681080400} - \frac{173x^5y^3z^5}{681080400} - \frac{173x^3y^5z^5}{681080400}\right)t^{15} + \cdots$$

Is the same of the approximate solution in Eq. (3.36), and converges to the exact solution, we see that the approximate solutions obtained from the three proposed methods are the same because we got the same series

To prove the state of convergence we find values of β_i for the problem as in (3.35). Hence, the terms of the series $\sum_{i=0}^{\infty} v_i(x, y, z, t)$ given in Eq. (3.26), we get

$$\beta_0 = \frac{\|v_1\|}{\|v_0\|} = 0.053744 < 1$$

$$\beta_1 = \frac{\|v_2\|}{\|v_1\|} = 0.213378 < 1$$

$$\beta_2 = \frac{\|v_3\|}{\|v_2\|} = 0.0530355 < 1$$

$$\beta_3 = \frac{\|v_4\|}{\|v_3\|} = 0.157401 < 1,$$

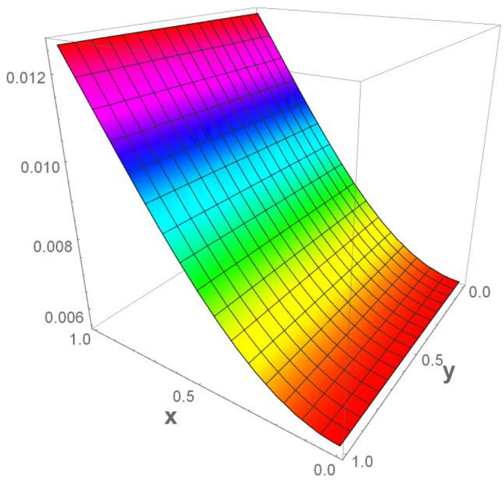
where, the β_i values for $i \geq 0$ and $0 < x, y, z \leq 1$ are less than 1 when $t = 1$, so the proposed iterative methods satisfy the convergence.

To examine the accuracy of the approximate solutions for this example, we have to calculate the absolute error of the approximate solution, where $u = t\ x\ y\ z$ is the exact solution .

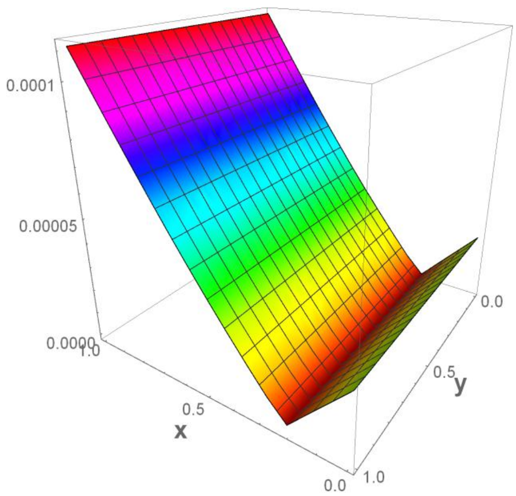
Table 3.3 Fig3. $3(a,b,c)$ show the 3D plotted graph of the $|r_n|$ for approximate solution obtained by the proposed iterative methods, we note that by increasing the number of iterations, errors will decreasing and the accuracy of the approximate solutions is increasing

Table 3.3: Results of the absolute errors by the proposed methods, where $y,z,t = 1$.

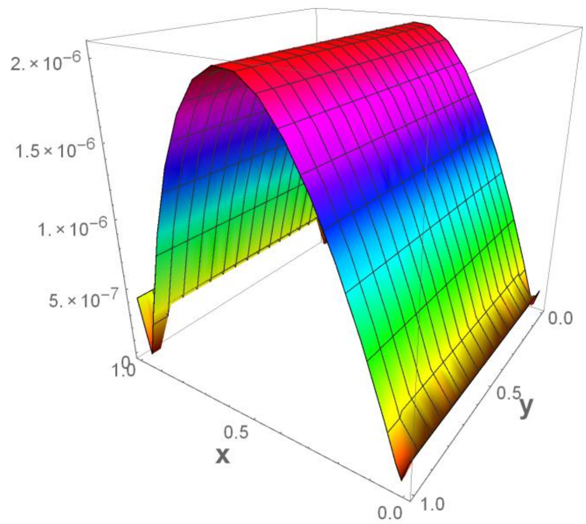
x	$\|r_1\|$	$\|r_3\|$	$\|r_4\|$
0	0.00555556	0.0000263533	1.44152×10^{-7}
0. 1	0.00566269	0.0000144065	5.3068×10^{-7}
0. 2	0.00596813	2.41046×10^{-6}	1.09838×10^{-6}
0. 3	0.00644779	9.88883×10^{-6}	1.54998×10^{-6}
0. 4	0.00707739	0.000022676	1.87041×10^{-6}
0. 5	0.00783248	0.0000360674	2.03909×10^{-6}
0. 6	0.00868841	0.0000501122	2.03049×10^{-6}
0. 7	0.00962036	0.0000647933	1.81462×10^{-6}
0. 8	0.0106033	0.0000800281	1.35764×10^{-6}
0. 9	0.0116121	0.0000956701	6.22353×10^{-7}
1	0.0126213	0.000111509	4.31218×10^{-7}



$a = u_1(x,y,z,t)$



$b = u_3(x,y,z,t)$



$c = u_4(x, y, z, t)$

Figure3. 3(a, b, c): The plots of the absolute error $|r_n|$ at $n = 1, 3$ and 4, with $t, z = 1$.

CHAPTER 4

Conclusions and Future Works

Chapter 4

Conclusions and Future Works

4.1 Introduction

The main objective of this work has been achieved by solving some non-linear ordinary and partial differential equation by various iterative methods. This purpose was obtained by applying three iterative methods, which are TAM, DJM and BCM. Also, the comparison between these suggested methods and other methods such as ADM and VIM.

4.2 Conclusions

1.The three iterative methods are reliable and effective to find the approximate solutions for Painlevé I, II, Pendulum and Falkner –skan equations, and the linear and nonlinear wave equations in 1D, 2D and 3D. The accuracy and efficiency of those proposed methods have been demonstrated through the study of convergence.

2. The three proposed methods do not require any restrictive assumptions to deal with non-linear terms unlike other iterative methods. Such as the ADM that need, to calculate the adomain Polynomial and VIM required to evaluate the Lagrange Multiplier.

3.The convergence of the proposed methods is given based on the Banach fixed point theorem. The results of the maximal error remainder values show that the present methods are effective and reliable.

4. We solved these problems by numerical methods which are the Rang-Kutta (RK4) and Euler methods. We compared the numerical results with approximate solutions and were in good agreement.

5. For comparison, we take examples for nonlinear painlevé I equation and Painlevé II equation.

In comparing results for by suggested methods with some existing methods such as ADM and VIM. We note that the approximate solutions for painlevé equation I obtained by the suggested methods the same as VIM and better than ADM without adomain Polynomial and Lagrange Multiplier. That can be clarified in the table 4.1 , figure 4.1.

Table 4.1: Comparative results of the maximal error remainder: MER_n for proposed methods, VIM and ADM, for Painlevé I equation, where n=1,...,5 .

n	MER_n by the Proposed methods	MER_n by the ADM	MER_n by the VIM
1	0.0000601952	0.0000601952	0.0000601952
2	1.72121×10^{-8}	3.22501×10^{-8}	1.72121×10^{-8}
3	2.29681×10^{-12}	1.29031×10^{-11}	2.29681×10^{-12}
4	1.52656×10^{-16}	4.35069×10^{-15}	1.52656×10^{-16}
5	2.77556×10^{-17}	2.08167×10^{-17}	2.77556×10^{-17}

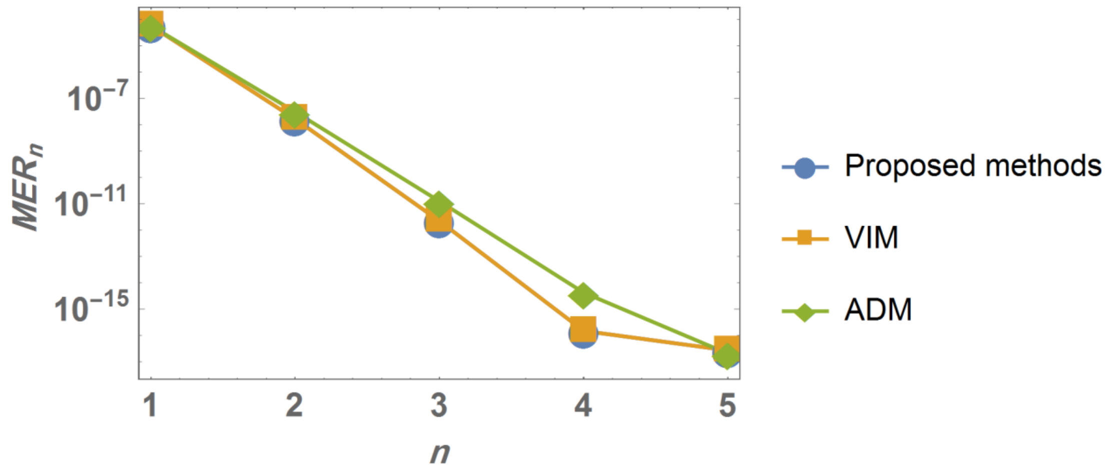


Figure 4.1: Comparison of the maximal error remainder for the proposed methods, VIM and ADM.

Also, we note that the approximate solutions for Painlevé equation II obtained by suggested methods the same as VIM without adomain Polynomial and Lagrange Multiplier. That can be clarified in the table 4.2 figure, 4.2.

Table 4.2: Comparative results of the maximal error remainder: MER_n for the proposed methods, VIM and ADM, for Painlevé II equation where n =1,...,5 .

n	MER_n by the Proposed methods	MER_n by the ADM	MER_n by the VIM
1	0.0634125	0.0634125	0.0634125
2	0.000323411	0.000950605	0.000323411
3	6.58637×10^{-7}	0.00001041	6.58637×10^{-7}
4	7.18421×10^{-10}	9.77952×10^{-8}	7.18421×10^{-10}
5	4.8539×10^{-13}	8.40299×10^{-10}	4.8539×10^{-13}

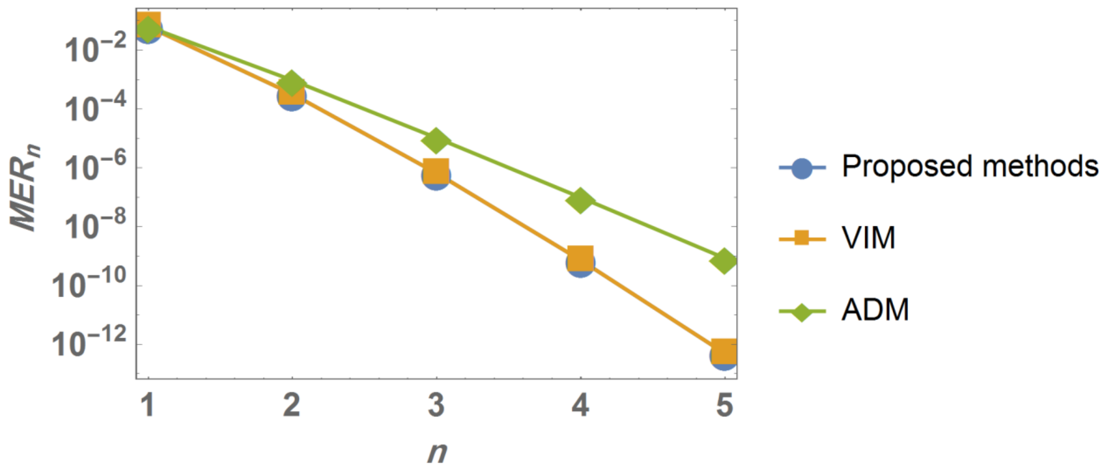


Figure 4.2: Comparison of the maximal error remainder for proposed methods, VIM and ADM.

6. We take example for nonlinear Pendulum equation. In comparison the results by the proposed methods with the some existing methods such as ADM and VIM. We note that the approximate solutions obtained by suggested methods the same as VIM without adomain Polynomial and

Lagrange Multiplier. It can be seen that by increasing the iterations, the errors will be decreasing, that can be clarified in the table 4.3 and figure 4.3.

Table 4.3: Comparison results of the maximal error remainder: MER_n for the proposed methods, VIM and ADM, where n =1,...,5 .

n	MER_n by the Proposed methods	MER_n by the ADM	MER_n by the VIM
1	0.0959857	0.0959857	0.0959857
2	0.00429285	0.0059253	0.00429285
3	0.0000881802	0.00115669	0.0000881802
4	1.05634×10^{-6}	0.0000367296	1.05634×10^{-6}
5	8.28868×10^{-9}	4.73742×10^{-6}	8.28868×10^{-9}

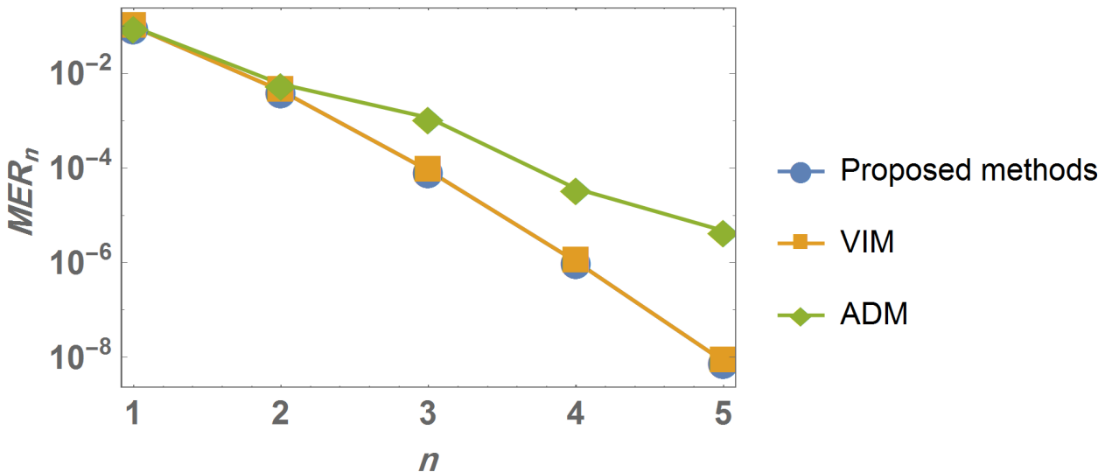


Figure 4.3: Comparison of the maximal error remainder for proposed methods, VIM and ADM.

7. We take example for nonlinear Falkner- skan equation, when comparing the results of the proposed methods with those of the ADM and VIM. We note that the approximate solutions obtained by the proposed methods the same as VIM without adomain Polynomial and Lagrange Multiplier.

Note that when increasing the number of iterations, The maximal error remainders will be decreased, that can be illustrated by a table 4.4 and figure 4.4

Table 4.4: Comparative results of the maximal error remainder: MER_n for the proposed methods, VIM and ADM, where $n = 1, \ldots, 5$.

n	MER_n by the Proposed methods	MER_n by the ADM	MER_n by the VIM
1	0.246021	0.246021	0.246021
2	0.0745229	0.0661609	0.0745229
3	0.0150143	0.0101075	0.0150143
4	0.00226928	0.000832153	0.00226928
5	0.000274414	3.40896×10^{-6}	0.000274414

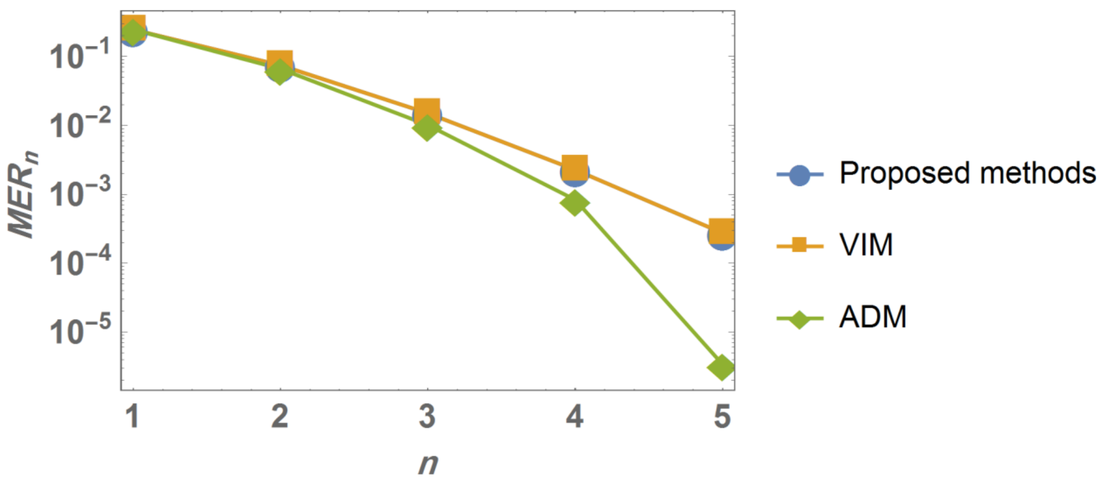


Figure 4.4: Comparison of the maximal error remainder for the Falkner-skan equation between the proposed methods, VIM and ADM.

8. In chapter three, we used the proposed methods TAM, DJM and BCM to find exact solution for linear equation and approximate solutions for nonlinear 1 D, 2D and 3D wave equations. We compared the approximate solutions with the exact solutions for the nonlinear equations, by the absolute error, then we found that increasing the iterations it would reduce the errors and increase the accuracy of the approximate solutions.

9. We take examples for nonlinear 1D, 2D and 3D wave equation. In comparison the results by the proposed methods with some methods such as ADM and VIM. In addition, it is noted that the approximate solutions obtained by the proposed methods converge quicker without any constrained assumptions. As given in tables 4.5, 4.6 and 4.9, we compare the absolute error for the methods used, with $n = 4$ together with the VIM, and ADM.

Table 4.5: Comparative results for the absolute errors of the proposed methods , ADM and VIM for nonlinear 1D, where $t = 1$

x	$\lvert r_4 \rvert$ **by the Proposed methods**	$\lvert r_4 \rvert$ **by ADM**	$\lvert r_4 \rvert$ **by VIM**
0	4.61399×10^{-7}	1.3407×10^{-6}	4.61399×10^{-7}
0. 1	5.76688×10^{-7}	1.9667×10^{-6}	5.76688×10^{-7}
0. 2	7.17091×10^{-7}	2.76817×10^{-6}	7.17091×10^{-7}
0. 3	8.8619×10^{-7}	3.79775×10^{-6}	8.8619×10^{-7}
0. 4	1.08821×10^{-6}	5.12235×10^{-6}	1.08821×10^{-6}
0. 5	1.32813×10^{-6}	6.82494×10^{-6}	1.32813×10^{-6}
0. 6	1.61177×10^{-6}	9.00674×10^{-6}	1.61177×10^{-6}
0. 7	1.94595×10^{-6}	0.0000117893	1.94595×10^{-6}
0. 8	2.33857×10^{-6}	0.0000153167	2.33857×10^{-6}
0. 9	2.79877×10^{-6}	0.0000197578	2.79877×10^{-6}
1	3.33704×10^{-6}	0.0000253086	3.33704×10^{-6}

It can be observed clearly from table 4.5, the absolute error for the proposed methods and VIM are lower than ADM. In addition, we note that the approximate solutions obtained by the proposed methods and VIM are the same.

Table 4.6: Comparative results for the absolute errors of the proposed methods , ADM and VIM, for nonlinear 2D, where $y, t = 1$.

$\boldsymbol{x}$	$\lvert r_4 \rvert$ by the Proposed methods	$\lvert r_4 \rvert$ by ADM	$\lvert r_4 \rvert$ by VIM
0	1.57137×10^{-7}	4.36769×10^{-7}	1.57137×10^{-7}
0. 1	1.84193×10^{-7}	4.92375×10^{-7}	1.84193×10^{-7}
0. 2	2.63572×10^{-7}	6.44678×10^{-7}	2.63572×10^{-7}
0. 3	3.94162×10^{-7}	8.85956×10^{-7}	3.94162×10^{-7}
0. 4	5.74663×10^{-7}	1.2063×10^{-6}	5.74663×10^{-7}
0. 5	8.03297×10^{-7}	1.59218×10^{-6}	8.03297×10^{-7}
0. 6	1.07753×10^{-6}	2.02518×10^{-6}	1.07753×10^{-6}
0. 7	1.39378×10^{-6}	2.4808×10^{-6}	1.39378×10^{-6}
0. 8	1.74719×10^{-6}	2.92748×10^{-6}	1.74719×10^{-6}
0. 9	2.1313×10^{-6}	3.32569×10^{-6}	2.13131×10^{-6}
1	2.53787×10^{-6}	3.62723×10^{-6}	2.53787×10^{-6}

Table 4.7: Comparative results for the absolute errors of the proposed methods , ADM and VIM, for nonlinear 3D, $y, z, t = 1$.

$\boldsymbol{x}$	$\lvert r_4 \rvert$ by the Proposed methods	$\lvert r_4 \rvert$ by ADM	$\lvert r_4 \rvert$ by VIM
0	1.44152×10^{-7}	3.80939×10^{-7}	1.44152×10^{-7}
0. 1	5.3068×10^{-7}	2.43012×10^{-7}	5.3068×10^{-7}
0. 2	1.09838×10^{-6}	6.6647×10^{-7}	1.09838×10^{-6}
0. 3	1.54998×10^{-6}	8.76941×10^{-7}	1.54998×10^{-6}
0. 4	1.87041×10^{-6}	8.53484×10^{-7}	1.87041×10^{-6}
0. 5	2.03909×10^{-6}	5.69642×10^{-7}	2.03909×10^{-6}
0. 6	2.03049×10^{-6}	3.69911×10^{-9}	2.03049×10^{-6}
0. 7	1.81462×10^{-6}	8.95526×10^{-7}	1.81462×10^{-6}
0. 8	1.35764×10^{-6}	2.132×10^{-6}	1.35764×10^{-6}
0. 9	6.22353×10^{-7}	3.73384×10^{-6}	6.22353×10^{-7}
1	4.31218×10^{-7}	5.71383×10^{-6}	4.31218×10^{-7}

It can be observed clearly from table 4.6 and 4.7, the absolute error for the proposed methods and VIM are lower than ADM. In addition, we note that the approximate solutions obtained by the proposed methods and VIM are the same.

4.3-Future works

In this section some of future works will be suggested

1- Solving the beam-type actuators equation[27], by using TAM

$$\frac{d^4v}{dx^4} = \frac{\alpha_k}{(1-v(x))^k} + \frac{\beta}{(1-v(x))^2} + 0.65\frac{g}{w}\frac{\beta}{(1-v(x))^2}, \qquad k = 3,4,$$

with boundary conditions:

$$v(0) = v'(0) = v''(1) = v'''(1) = 0$$

2- Using BCM for solving the Blasius equation[46]

$$v'''(x) + \frac{1}{2}v(x)v''(x) = 0, \qquad 0 < x < \infty,$$

With initial-boundary conditions:

$$v(0) = v'(0) = 0 \text{ and } v'(\infty) = 1.$$

3-Applying the Homotopy-Pad´e method [74], for solving Falkner skan equation .

4- Using the DJM for solving the sixth-order boundary-value problems[63]

$$y^{(6)}(x) = (1+c)y^{(4)}(x) - cy^{(2)}(x) + cx, \qquad 0 \le x \le 1,$$

with the boundary conditions:

$$y(0) = 1,\ y^{(1)}(0) = 1,\ y^{(2)}(0) = 0,$$

$$y(1) = \frac{7}{6} + \sinh(1),\ y^{(1)}(1) = 1 + \cosh(1), y^{(2)}(0) = 1 + \sinh(1)$$

The exact solution : $y(x) = \left(1 + \frac{x}{6}\right) + \sin(x)$.

5- Using the TAM for solving Lane-Emden equation[4]

$y''(x) + \frac{2}{x} y'(x) + g(x,y) = f(x),$

with the initial conditions:

$y(0) = a,\ y'(0) = b.$

6-Using the DJM for solving advection-diffusion-reaction equation (ADRE)(1D) [81]

$a(x)\frac{d^2v(x)}{dx^2} + b(x)\frac{dv(x)}{dx} + c(x)v(x) = f(x), \quad 0 \le x \le 1,$

with boundary conditions:

$v(0) = v_0, \quad v(1) = v_1,\ v_0 + v_1 > 0.$

7-Solving the Benjamin-Bona-Mahoney-Burgers (BBMB) equations [16] by using the TAM

$v_t - v_{xxt} + v_x + \left(\frac{u^2}{2}\right)_x = 0$

With initial conditions:

$v(x,0) = sech^2\left(\frac{x}{4}\right).$

8-Using the harmonic balance method (HBM)[56]for solving Pendulum equation.

9-Solving the class of boundary value problems with polynomial coefficients [2] by the TAM .

10- Solving the Painlevé equation I by the Modification of homotopy perturbation method[11].

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Appendices

Appendices

A: Code of Mathematica for chapter one (the ADM)

……………………………………………………………………………………………

Code of Painlevé equation I

$(*u'' = 6u^2 + x, \qquad u(0) = 0, \qquad u'(0) = 1*)$

$$(*u = x + \frac{x^3}{6} + 6\int_0^x ((\int_0^x (u^2/.x \to t)\, dt)/.x \to t)\, dt *)$$

$$\text{u0} = x + \frac{x^3}{6}$$

$$x + \frac{x^3}{6}$$

$\text{A0} = \text{u0}^2;$

$\text{u1} = 6\int_0^x ((\int_0^x (\text{A0}/.x \to t)\, dt)/.x \to t)\, dt$

$$6(\frac{x^4}{12} + \frac{x^6}{90} + \frac{x^8}{2016})$$

Simplify[u1]

$$\frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336}$$

$\text{A1} = 2\text{u0u1};$

$\text{u2} = 6\int_0^x ((\int_0^x (\text{A1}/.x \to t)\, dt)/.x \to t)\, dt$U1=u0+u1;

$$6(\frac{x^7}{42} + \frac{x^9}{240} + \frac{71x^{11}}{277200} + \frac{x^{13}}{157248})$$

Simplify[u2]

$$\frac{x^7}{7} + \frac{x^9}{40} + \frac{71x^{11}}{46200} + \frac{x^{13}}{26208}$$

$\text{A2} = \text{u1}^2 + 2\text{u0u2};$

$$u3 = 6\int_0^x((\int_0^x(A2/.x \to t)\,dt)/.x \to t)\,dt$$

$$\frac{x^{10}}{28} + \frac{23x^{12}}{3080} + \frac{5219x^{14}}{8408400} + \frac{3551x^{16}}{144144000} + \frac{95x^{18}}{224550144}$$

$$A3 = 2(u1u2 + u0u3);$$

$$u4 = 6\int_0^x((\int_0^x(A3/.x \to t)\,dt)/.x \to t)\,dt\,;$$

$$A4 = u2^2 + 2u1u3 + 2u0u4;$$

$$u5 = 6\int_0^x((\int_0^x(A4/.x \to t)\,dt)/.x \to t)\,dt\,;$$

U1=u0+u1;

U2=u0+u1+u2;

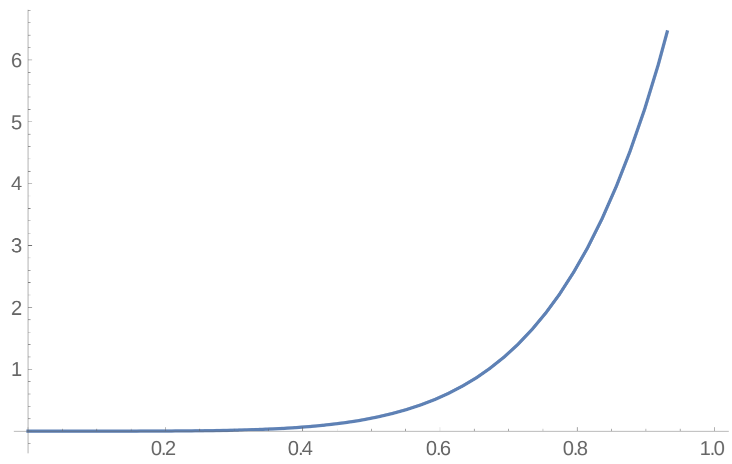
U3=u0+u1+u2+u3;

U4=u0+u1+u2+u3+u4;

U5=u0+u1+u2+u3+u4+u5;

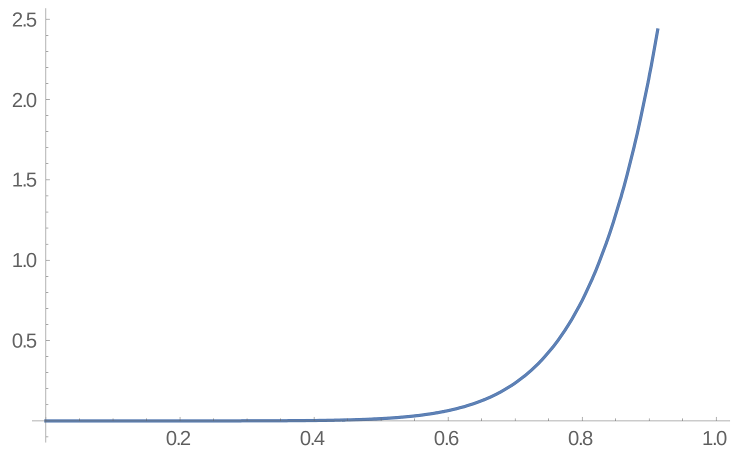
$$r1 = \mathrm{Abs}[D[U1,\{x,2\}] - 6U1^2 - x];$$

Plot[r1, $\{x, 0,1\}$]



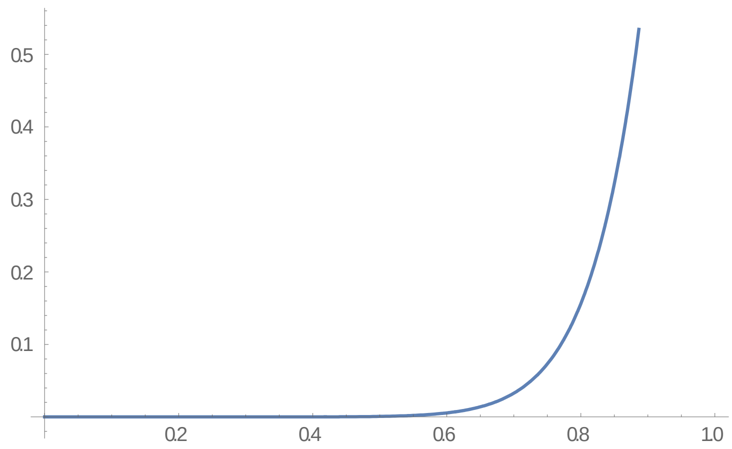
$$r2 = \mathrm{Abs}[D[U2,\{x,2\}] - 6U2^2 - x];$$

Plot[r2, $\{x, 0,1\}$]



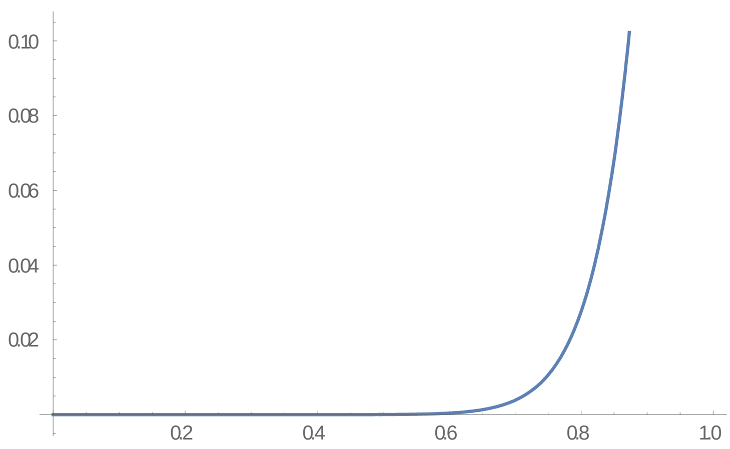
$\mathrm{r3} = \mathrm{Abs}[D[\mathrm{U3}, \{x, 2\}] - 6\mathrm{U3}^2 - x];$

$\mathrm{Plot}[\mathrm{r3}, \{x, 0,1\}]$



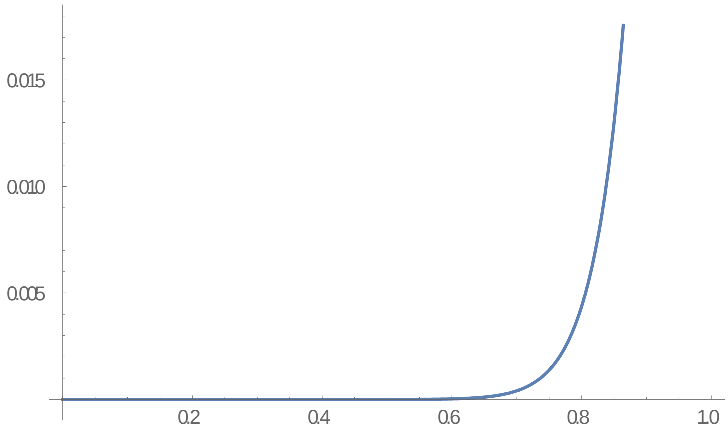
$\mathrm{r4} = \mathrm{Abs}[D[\mathrm{U4}, \{x, 2\}] - 6\mathrm{U4}^2 - x];$

$\mathrm{Plot}[\mathrm{r4}, \{x, 0,1\}]$



$r5 = \mathrm{Abs}[D[\mathrm{U5}, \{x, 2\}] - 6\mathrm{U5}^2 - x];$

$\mathrm{Plot}[r5, \{x, 0{,}1\}]$



y1=Max[r1]

0.0000601952

y2=Max[r2]

$3.22501*10^{-8}$

y3=Max[r3]

$1.29031*10^{-11}$

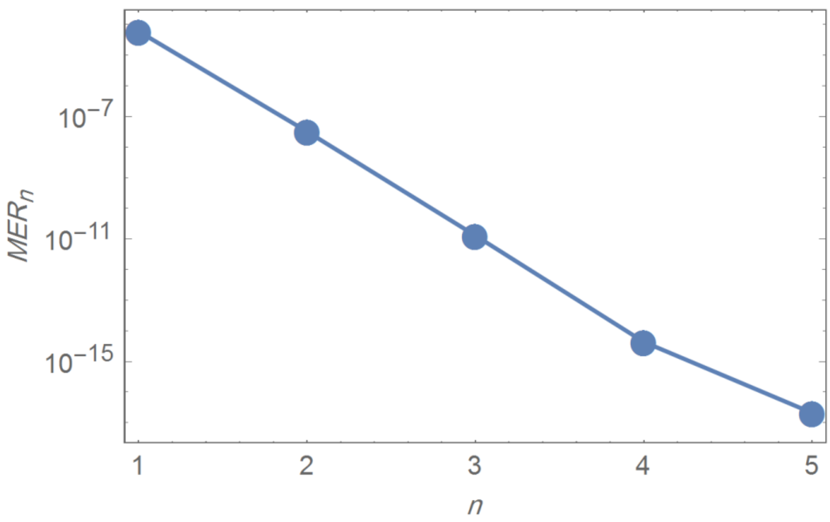
y4=Max[r4]

$4.36456*10^{-15}$

y5=Max[r5]

$1.38778*10^{-17}$

ListLogPlot[{{1, y1}, {2, y2}, {3, y3}, {4, y4}, {5, y5}}, Joined → True, PlotRange → All, Frame → True, Axes → True, FrameLabel → {Row[{Style["n", FontSlant → Italic]}], Row[{Style[MER_n, FontSlant → Italic]}]}, PlotMarkers → {Automatic, 15}]



B: Code of Mathematica for chapter one (the VIM)

…………………………………………………………………………………………………

Code of pendulum equation

$$(*\mathrm{u}'' + \gamma\mathrm{Sin}[u] = 0, u[0] = 0, u'[0] = 1, \text{the exact solutionisexpressedinJacobiellipticfunction} u = 2 * \mathrm{ArcSin}[k * \mathrm{sn}(\sqrt{\gamma} * t, k^2)]*)$$

$$(*u(t) = t - \gamma \int_0^t \int_0^t \mathrm{Sin}[u]\mathrm{dsds}*)$$

$$(*\mathrm{Sin}[u] = u - \frac{u^3}{6} + \frac{u^5}{120} + O[u]^7*)$$

ClearAll["Global` * "]

$\mathrm{u0} = x$

x

$$\mathrm{u1} = \mathrm{u0} + \int_0^x (t - x) * ((D[\mathrm{u0}, \{x, 2\}] + \mathrm{u0} - \frac{\mathrm{u0}^3}{6} + \frac{\mathrm{u0}^5}{120} /. x \to t))\, dt$$

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$$\mathrm{u2} = \mathrm{u1} + \int_0^x (t - x) * ((D[\mathrm{u1}, \{x, 2\}] + \mathrm{u1} - \frac{\mathrm{u1}^3}{6} + \frac{\mathrm{u1}^5}{120} /. x \to t))\, dt$$

$$x-\frac{x^3}{6}+\frac{x^5}{60}-\frac{x^7}{420}+\frac{127x^9}{362880}-\frac{893x^{11}}{19958400}+\frac{367x^{13}}{70761600}-\frac{607x^{15}}{1143072000}+\frac{56881x^{17}}{1243662336000}-\frac{2521x^{19}}{781861248000}+\frac{17x^{21}}{92177326080}-\frac{22129x^{23}}{2591207055360000}+\frac{17651x^{25}}{55306395648000000}-\frac{61787x^{27}}{6470848290816000000}+\frac{2021x^{29}}{8981758653235200000}-\frac{73x^{31}}{18002231783424000000}+\frac{13x^{33}}{245294925978009600000}-\frac{x^{35}}{2211370923589632000000}+\frac{x^{37}}{519802247686127616000000}$$

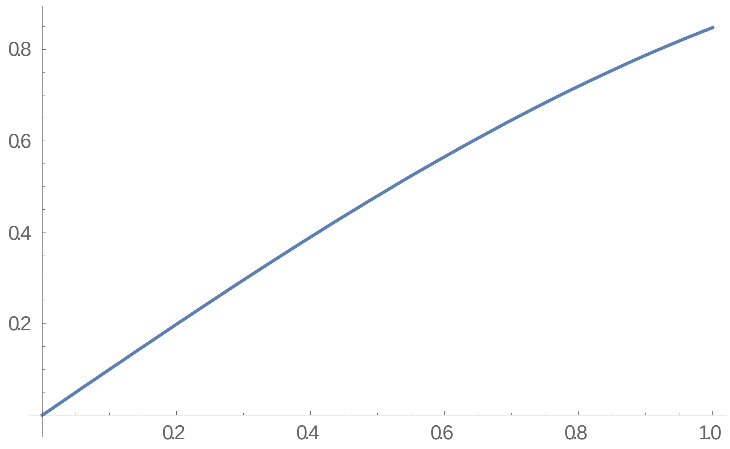
$$\mathrm{u3}=\mathrm{u2}+\int_0^x (t-x)*((D[\mathrm{u2},\{x,2\}]+\mathrm{u2}-\frac{\mathrm{u2}^3}{6}+\frac{\mathrm{u2}^5}{120}/.x\to t))\,dt;$$

$$\mathrm{u4}=\mathrm{u3}+\int_0^x (t-x)*((D[\mathrm{u3},\{x,2\}]+\mathrm{u3}-\frac{\mathrm{u3}^3}{6}+\frac{\mathrm{u3}^5}{120}/.x\to t))\,dt;$$

$$\mathrm{ux}=2*\mathrm{ArcSin}[\frac{1}{2}*\mathrm{JacobiSN}[x,\frac{1}{4}]]$$

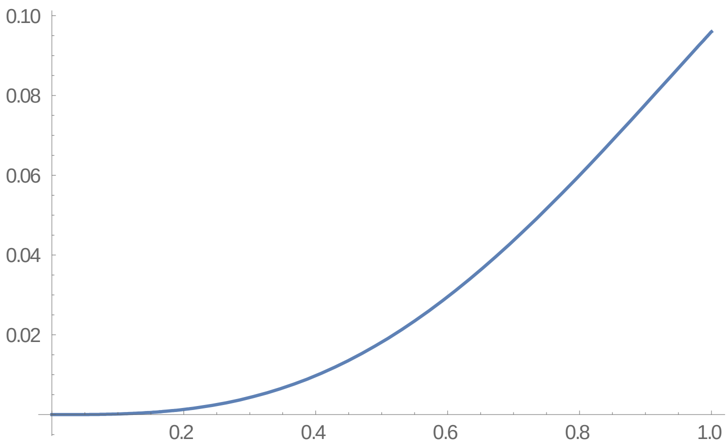
$$2\mathrm{ArcSin}[\tfrac{1}{2}\mathrm{JacobiSN}[x,\tfrac{1}{4}]]$$

Plot[ux, {x, 0,1}]



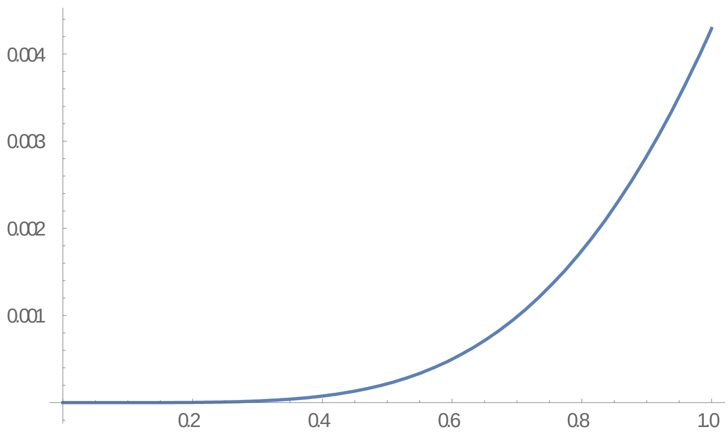
$$\mathrm{r1}=\mathrm{Abs}[D[\mathrm{u1},\{x,2\}+(\mathrm{u1}-\frac{\mathrm{u1}^3}{6}+\frac{\mathrm{u1}^5}{120})];$$

Plot[r1, {x, 0,1}]



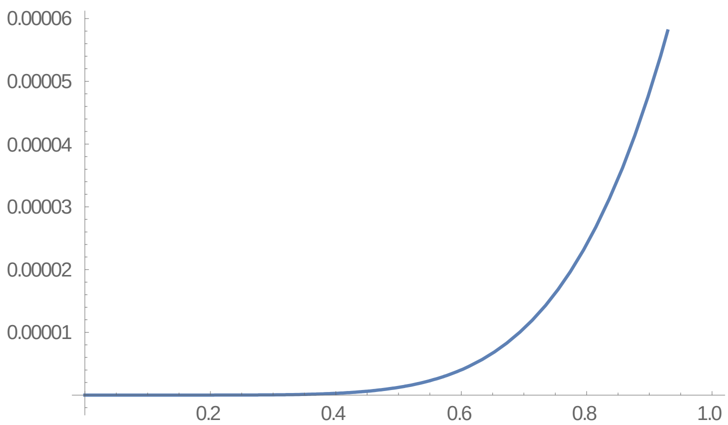
$$r2 = \mathrm{Abs}[D[u2, \{x, 2\}] + (u2 - \frac{u2^3}{6} + \frac{u2^5}{120})];$$

$$\mathrm{Plot}[r2, \{x, 0,1\}]$$



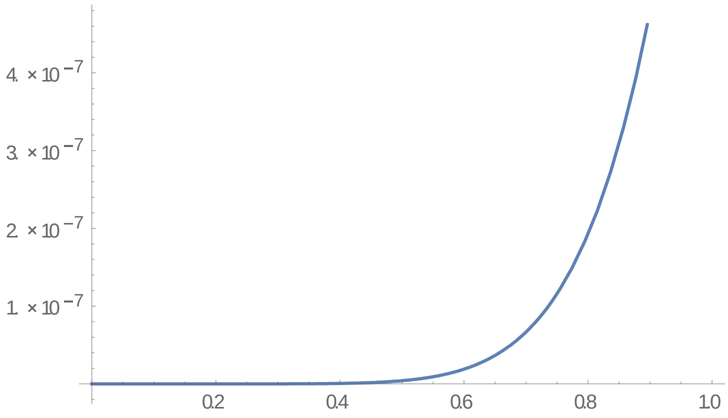
$$r3 = \mathrm{Abs}[D[u3, \{x, 2\}] + (u3 - \frac{u3^3}{6} + \frac{u3^5}{120})];$$

$$\mathrm{Plot}[r3, \{x, 0,1\}]$$



$$r4 = \mathrm{Abs}[D[\mathrm{u4}, \{x, 2\}] + (\mathrm{u4} - \frac{\mathrm{u4}^3}{6} + \frac{\mathrm{u4}^5}{120})];$$

Plot[r4, {t, 0,1}]



y1=N[Max[r1]]

0.0959857

y2=N[Max[r2]]

0.00429285

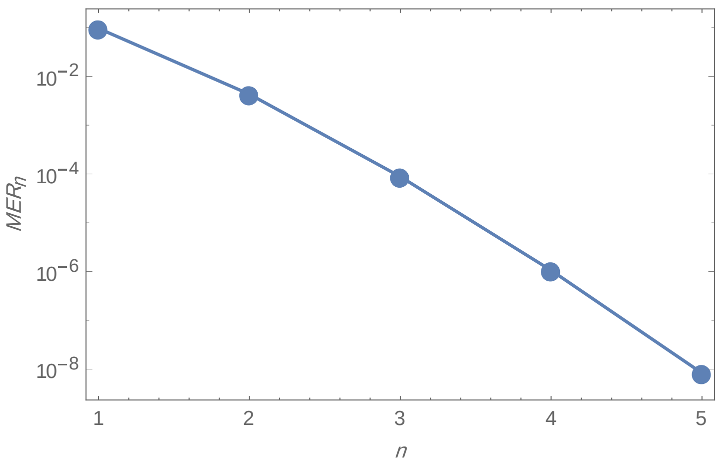
y3=N[Max[r3]]

0.0000881802

y4=N[Max[r4]]

1.05634*10^{-6}

g10=ListLogPlot[{{1,y1},{2,y2},{3,y3},{4,y4}},Joined->True,PlotRange->All,Frame->True, Axes->True,FrameLabel->{Row[{Style["n",FontSlant->Italic]}],Row[{Style["MER_n",FontSlant->Italic]}]},PlotMarkers->{Automatic,15}]



C: Code of Mathematica for chapter two (the TAM)

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Code of Painlevé equation II

(*Theequation(u``$(x) = 2 * u(x)$^3 $+ x * u(x) + \mu, u(0) = 1$, u`(0) $= 0$*)

(*$L(u) =$ u``, $N(u) = -2 * u(x)$^3 $- x * u(x) - \mu$and$g(x) = x$*)

Clear All["Global`*"]

tt=AbsoluteTime;

zz=SessionTime;

$\mu = 1$

1

u01[x_] = u0[x]/. First@DSolve[{u0''[x] == μ, u0′[0] == 0, u0[0] == 1}, u0[x], x]

$$\frac{1}{2}(2 + x^2)$$

u11[x_] = u1[x]/. First@DSolve[{u1''[x] == 2 * u01[x]^3 + x * u01[x] + μ, u1′[0] == 0, u1[0] == 1}, u1[x], x]

$$\frac{3360 + 5040x^2 + 560x^3 + 840x^4 + 84x^5 + 168x^6 + 15x^8}{3360}$$

u22[x_] = u2[x]/. First@DSolve[{u2''[x] == 2 * u11[x]^3 + x * u11[x] + μ, u2′[0] == 0, u2[0] == 1}, u2[x], x];

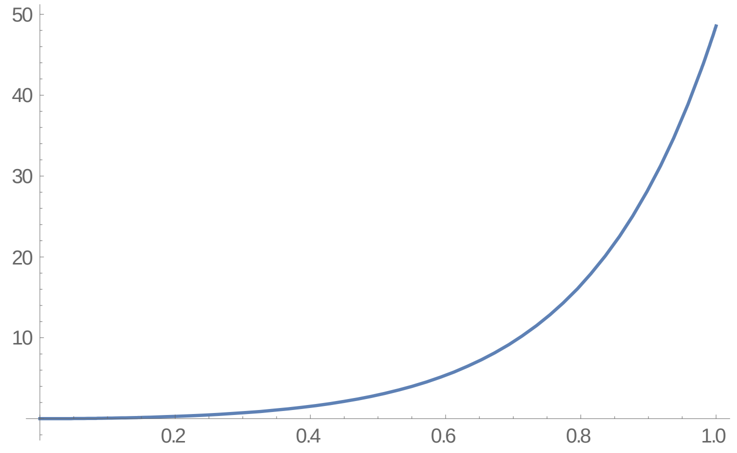
u33[x_]=u3[x]/.First@DSolve[{u3''[x]==2*u22[x]^3+x*u22[x]+μ,u3'[0]==0, u3[0]==1},u3[x],x];

u44[x_]=u4[x]/.First@DSolve[{u4''[x]==2*u33[x]^3+x*u33[x]+μ,u4'[0]==0, u4[0]==1},u4[x],x];

u55[x_]=u5[x]/.First@DSolve[{u5''[x]==2*u44[x]^3+x*u44[x]+μ,u5'[0]==0,
u5[0]==1},u5[x],x];

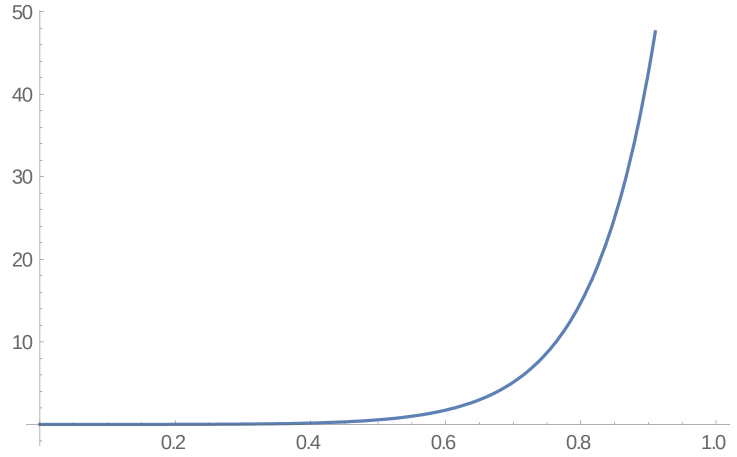
$r1 = Abs[((D[u11[x],\{x,2\}]) - (2 * u11[x]^\wedge 3) - x * u11[x] - \mu)];$

$Plot[r1,\{x,0,1\}]$



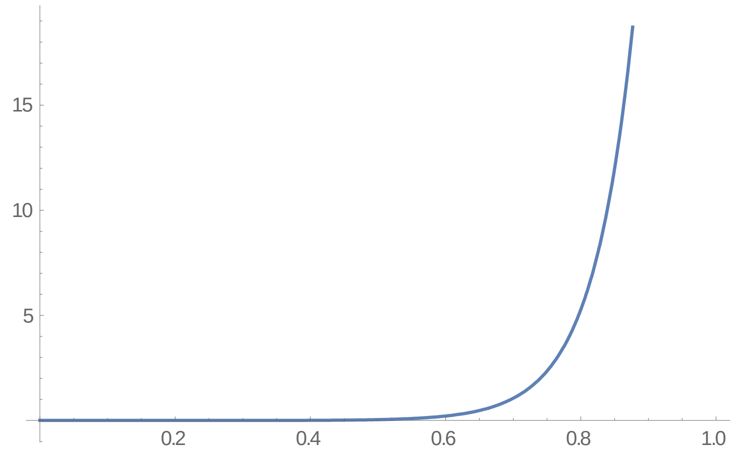
$r2 = Abs[((D[u22[x],\{x,2\}]) - (2 * u22[x]^\wedge 3) - x * u22[x] - \mu)];$

$Plot[r2,\{x,0,1\}]$



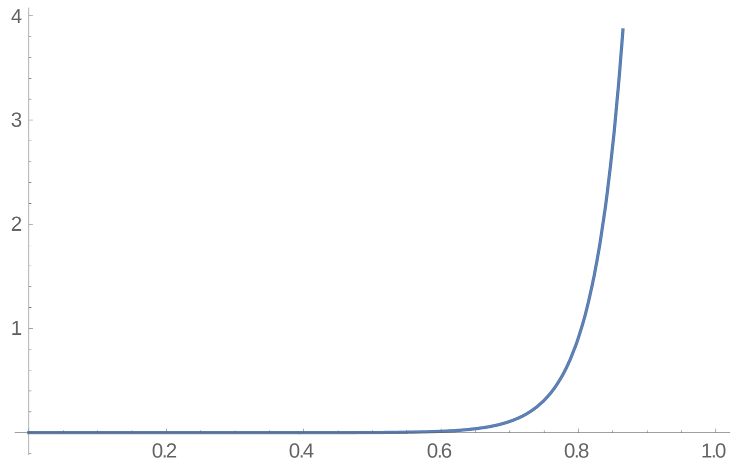
$r3 = Abs[((D[u33[x],\{x,2\}]) - (2 * u33[x]^\wedge 3) - x * u33[x] - \mu)];$

$Plot[r3,\{x,0,1\}]$



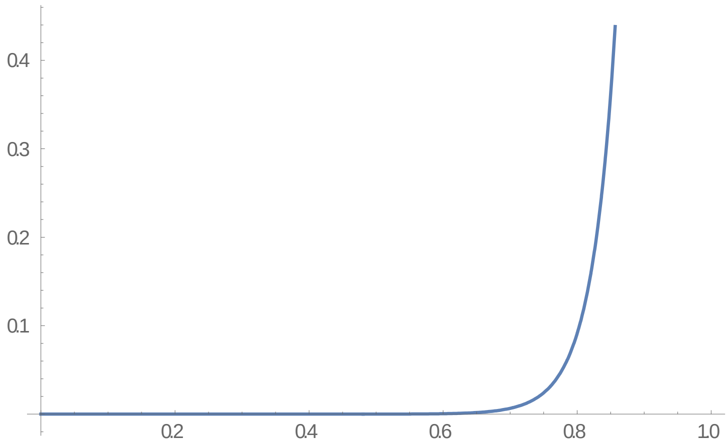
$r4 = \mathrm{Abs}[((D[\mathrm{u}44[x],\{x,2\}]) - (2 * \mathrm{u}44[x]\text{^}3) - x * \mathrm{u}44[x] - \mu)];$

$\mathrm{Plot}[\mathrm{r}4,\{x,0{,}1\}]$



$r5 = \mathrm{Abs}[((D[\mathrm{u}55[x],\{x,2\}]) - (2 * \mathrm{u}55[x]\text{^}3) - x * \mathrm{u}55[x] - \mu)];$

$\mathrm{Plot}[\mathrm{r}5,\{x,0{,}1\}]$



z1=Max[r1]

0.0634125

z2=Max[r2]

0.000323411

z3=Max[r3]

$6.58637*10^{-7}$

z4=Max[r4]

$7.18423*10^{-10}$

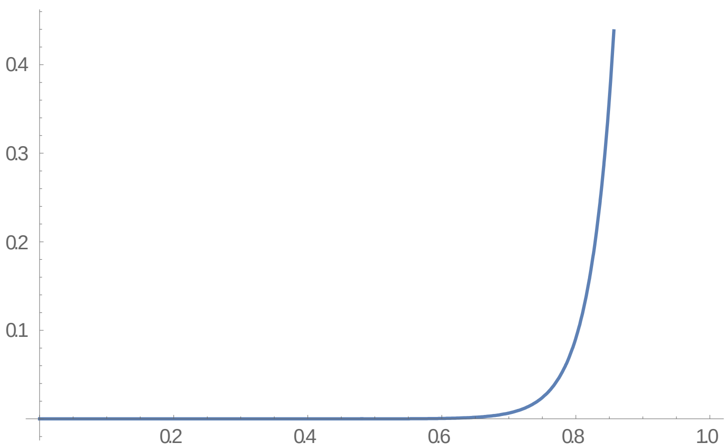
z5=Max[r5]

$4.876*10^{-13}$

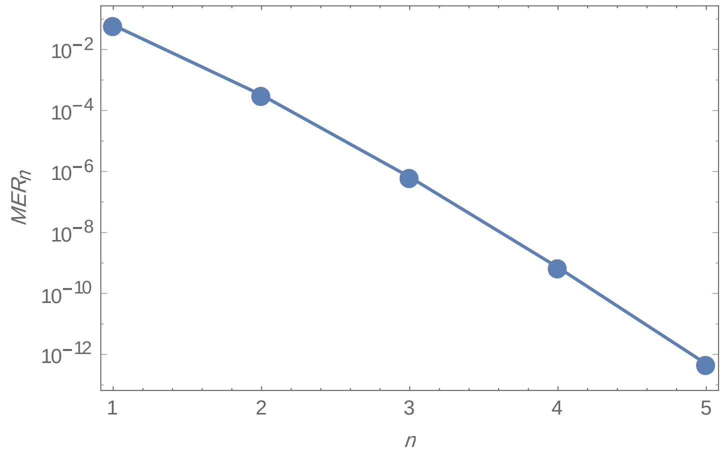
Log[z4/z3]/Log[z3/z2]

1.1007606329593649

$z = \{z1, z2, z3, z4, z5\};$



$$\text{ListLogPlot}[\{\{1,\text{z1}\},\{2,\text{z2}\},\{3,\text{z3}\},\{4,\text{z4}\},\{5,\text{z5}\}\},\text{Joined} \to \text{True},\text{PlotRange} \to \text{All},\text{Frame} \to \text{True},\text{Axes} \to \text{True},\text{FrameLabel} \to \{\text{Row}[\{\text{Style}[\text{"n"},\text{FontSlant} \to \text{Italic}]\}],\text{Row}[\{\text{Style}[\text{MER}_\text{n},\text{FontSlant} \to \text{Italic}]\}]\},\text{PlotMarkers} \to \{\text{Automatic},15\}]$$



D: Code of Mathematica for chapter three (the DJM)

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Code of example (3.4)

$$(*\partial_{t,t}u = \partial_{x,x}u + u + u^2 - xt - x^2t^2, \text{initialconditions}: u(x,0) = 0, \partial_t u(x,0) = x*)$$

$$(*u = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12} + \int_0^t \left(\int_0^t ((\partial_{x,x}u + u + u^2)/.t \to s)\, ds\, /.t \to s\right) ds\, *)$$

ClearAll["Global` * "]

$$\text{u0} = tx - \frac{t^3x}{6} - \frac{t^4x^2}{12}$$

$$tx - \frac{t^3x}{6} - \frac{t^4x^2}{12}$$

$$\text{u1} = \int_0^t \left(\int_0^t ((\partial_{x,x}\text{u0} + \text{u0} + \text{u0}^2)/.t \to s)\, ds\, /.t \to s\right) ds$$

$$-\frac{t^6}{180}+\frac{t^3x}{6}-\frac{t^5x}{120}+\frac{t^4x^2}{12}-\frac{t^6x^2}{72}+\frac{t^8x^2}{2016}-\frac{t^7x^3}{252}+\frac{t^9x^3}{2592}+\frac{t^{10}x^4}{12960}$$

$$\mathrm{u2}=\int_0^t(\int_0^t((\partial_{x,x}(\mathrm{u0}+\mathrm{u1})+(\mathrm{u0}+\mathrm{u1})+(\mathrm{u0}+\mathrm{u1})^2)/.\,t\to s)\,ds\,/.\,t\to s)\,ds-\mathrm{u1};$$

$$\mathrm{u3}=\int_0^t(\int_0^t((\partial_{x,x}(\mathrm{u0}+\mathrm{u1}+\mathrm{u2})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2})^2)/.\,t\to s)\,ds\,/.\,t\to s)\,ds-\int_0^t(\int_0^t((\partial_{x,x}(\mathrm{u0}+\mathrm{u1})+(\mathrm{u0}+\mathrm{u1})+(\mathrm{u0}+\mathrm{u1})^2)/.\,t\to s)\,ds\,/.\,t\to s)\,ds\,;$$

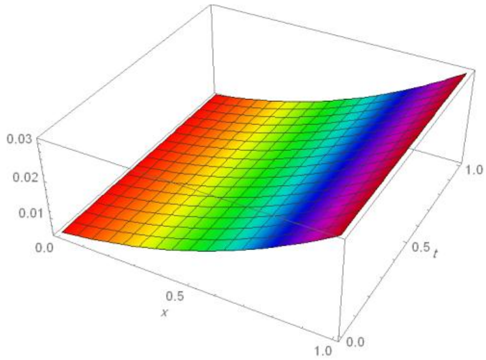
$$\mathrm{u4}=\int_0^t(\int_0^t((\partial_{x,x}(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3})^2)/.\,t\to s)\,ds\,/.\,t\to s)\,ds-\int_0^t(\int_0^t((\partial_{x,x}(\mathrm{u0}+\mathrm{u1}+\mathrm{u2})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2})^2)/.\,t\to s)\,ds\,/.\,t\to s)\,ds\,;$$

$$\mathrm{u5}=\int_0^t(\int_0^t((\partial_{x,x}(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3}+\mathrm{u4})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3}+\mathrm{u4})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3}+\mathrm{u4})^2)/.\,t\to s)\,ds\,/.\,t\to s)\,ds-\int_0^t(\int_0^t((\partial_{x,x}(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3})+(\mathrm{u0}+\mathrm{u1}+\mathrm{u2}+\mathrm{u3})^2)/.\,t\to s)\,ds\,/.\,t\to s)\,ds\,;$$

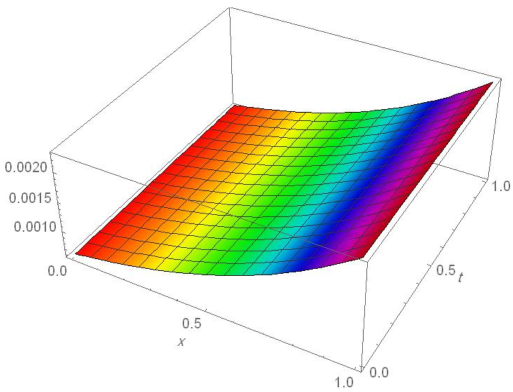
```
U1=u0+u1;
U2=u0+u1+u2;
U3=u0+u1+u2+u3;
U4=u0+u1+u2+u3+u4;

TableForm[Table[Abs[U1-uex],{x,0,1,0.1},{func,{u}}],TableHeadings->{Range[0,1,0.1],(ToString[#]<>"[x,t]")&/@{"Absolute error1"}}]
{
 {x, Absolute error1[x,t]},
 {0., 0.00555556},
 {0.1, 0.00652639},
```

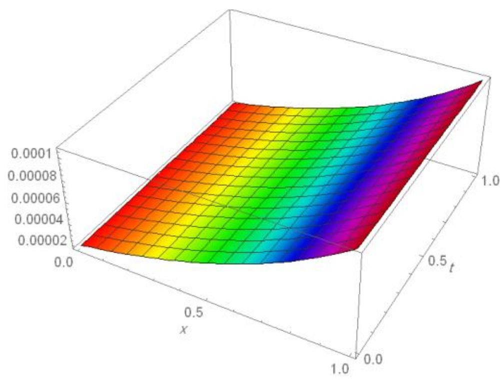
{0.2, 0.00778647},
{0.3, 0.00935701},
{0.4, 0.011259},
{0.5, 0.0135134},
{0.6, 0.0161408},
{0.7, 0.0191616},
{0.8, 0.0225963},
{0.9, 0.0264648},
{1., 0.030787}
absr1=N[Abs[uex-U1]];
Plot3D[absr1,{x,0,1},{t,0,1}]



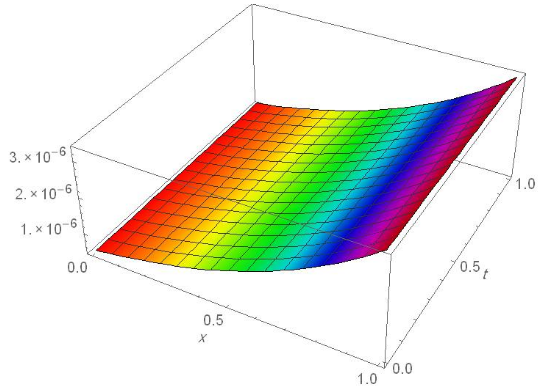
TableForm[Table[Abs[U2-uex],{x,0,1,0.1},{func,{u}}],TableHeadings->{Range[0,1,0.1],(ToString[#]< >"[x,t]")&/@{"Absolute error2"}}]
{
{x, Absolute error2[x,t]},
{0., 0.000584046},
{0.1, 0.000655976},
{0.2, 0.000741215},
{0.3, 0.000842609},
{0.4, 0.000963196},
{0.5, 0.0011062},
{0.6, 0.00127503},
{0.7, 0.00147329},
{0.8, 0.00170474},
{0.9, 0.00197335},
{1., 0.00228325}
}
absr2=N[Abs[uex-U2]];
Plot3D[absr2,{x,0,1},{t,0,1}]



```
TableForm[Table[Abs[U3-uex],{x,0,1,0.1},{func,{u}}],TableHeadings-
>{Range[0,1,0.1],( To String[#]< >"[x,t]")&/@{"Absolute error3"}}]
{
 {x, Absolute error3[x,t]},
 {0., 0.0000184804},
 {0.1, 0.0000228675},
 {0.2, 0.0000278468},
 {0.3, 0.0000335267},
 {0.4, 0.0000400357},
 {0.5, 0.000047523},
 {0.6, 0.0000561599},
 {0.7, 0.0000661411},
 {0.8, 0.0000776859},
 {0.9, 0.0000910392},
 {1., 0.000106473}
}
absr3=N[Abs[uex-U3]];
Plot3D[absr3,{x,0,1},{t,0,1}]
```




```
TableForm[Table[Abs[U4-uex],{x,0,1,0.1},{func,{u}}],TableHeadings-
>{Range[0,1,0.1],(ToString[#]< >"[x,t]")&/@{"Absolute error4"}}]
{
 {x, Absolute error4[x,t]},
 {0., 4.61399*10^-7},
 {0.1, 5.76688*10^-7},
 {0.2, 7.17091*10^-7},
 {0.3, 8.8619*10^-7},
 {0.4, 1.08821*10^-6},
 {0.5, 1.32813*10^-6},
 {0.6, 1.61177*10^-6},
 {0.7, 1.94595*10^-6},
 {0.8, 2.33857*10^-6},
 {0.9, 2.79877*10^-6},
 {1., 3.33704*10^-6}
absr4=N[Abs[uex-U4]];
Plot3D[absr4,{x,0,1},{t,0,1}]
```


3. × 10^-6
2. × 10^-6
1. × 10^-6
0.0
0.5
x
1.0
0.0
0.5
t
1.0

المستخلص

الهدف الرئيسي لهذه الرسالة هو استخدام ثلاث طرائق تكراريه تم تنفيذها للحصول على الحلول التقريبية لبعض المشاكل الهامه في مجال الفيزياء والهندسة.

سيكون الهدف الأول من هذه الرسالة هو التركيز علي بعض المفاهيم الأساسية للمعادلات التفاضلية اضافة الى شروط وجود ووحدانية الحل .

الهدف الثاني هو استخدام ثلاث طرائق تكراريه مقترحه هي طريقه التميمي- الأ نصاري (TAM) ، وطريقه الانكماش في دافتردار- جئفري (DJM)، وأسلوب انقباض بناخ (BCM) . لإيجاد الحلول التقريبية لبعض المشاكل التي تنشا في الفيزياء مثل معادلات بينليف الأولى ، بينليف الثانية ، البندول ، وفالكنر- سكان. يتم مقارنه النتائج التي تم الحصول عليها عدديا مع الأساليب العددية الأخرى مثل أساليب رنك كوتا الرابعة (Runge-kutta 4) وأويلر وبعض الطرق التحليلية مثل طريقة أدوميان للتجزئة(ADM) والطريقة التغايرية التكرارية (VIM) بالإضافة إلى ذلك ، يتم التقارب بين الطرق المقترحة استنادا إلى نظرية بناخ النقطة الصامدة. وتظهر نتائج القيم القصوى للخطأ المتبقي أن الأساليب الحالية فعالة وموثوقة.

والهدف الثالث هو استخدام نفس الطرائق المقترحة الثلاثة لحل معادلة الموجة الخطية وغير الخطية ذات البعد الواحد والبعدين والثلاثة أبعاد. أيضا ، ولتحليل التقارب للأساليب الثلاثة سوف نستخدم نظرية النقطة الصامدة. كل طريقه لا تتطلب اي افتراض للتعامل مع مصطلح غير الخطية. هذه الأساليب هي كفوءة جدا وهي عمليا مناسبه تماما للاستخدام في هذه المشاكل. هناك العديد من الأمثلة التي توضح دقه وكفاءه هذه الأساليب.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بغداد
كلية التربية للعلوم الصرفة/ ابن الهيثم
قسم الرياضيات

الحلول التحليلية والعددية للمعادلات التفاضلية الخطية وغير الخطية

رسالة

مقدمة إلى كلية التربية للعلوم الصرفة / ابن الهيثم ، جامعة بغداد
كجزء من متطلبات نيل درجة ماجستير في علوم الرياضيات

من قبل
منتصر إسماعيل عدوان

بإشراف
أ. م. د. مجيد احمد ولي

١٤٤٠هـ **٢٠١٨م**