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A New Mixture Distribution: Theory and Application

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By

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

إِنَّا فَتَحْنَا لَكَ فَتْحًا مُبِينًا (١)لِيَغْفِرَ لَكَ اللَّهُ مَا تَقَدَّمَ
مِنْ ذَنْبِكَ وَمَا تَأَخَّرَ وَيُتِمَّ نِعْمَتَهُ عَلَيْكَ وَيَهْدِيَكَ
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صدق الله العظيم

سورة الفتح - الأيات(١,٢,٣)

الاهداء

الهي لا يطيب الليل الا بشكرك.. ولا يطيب النهار الا بطاعتك .. ولا تطيب اللحظات الا بذكرك .. ولا تطيب الاخرة الا بعفوك.. ولا تطيب الجنة الا برؤيتك..

"الله جل جلاله"

الى من بلغ الرسالة وادى الامانة .. الى نبي الرحمة ونور العالمين ..

"سيدنا محمد(صلى الله عليه وعلى اله وصحبه وسلم)"

الى الغائب الحاضر رحمه الله واسكنه فسيح جناته..

الى من علمني العطاء بدون انتظار ..

الى من احمل اسمه بكل افتخار..

"والدي رحمه الله "

الى من الجنة تحت اقدامها.. واكرمني ربي بوجودها..

"امي الغالية"

الى من سار معي نحو الحلم خطوة بخطوة ... بذرناه معا"..وحصدناه معا" وسنبقى معا"

باذن الله تعالى"زوجي العزيز"

الى سندي في هذه الحياة.. خواتي العزيزات

الى قرة عيني وفرحة عمري..

اولادي الغاليين

"مريم..محمد..موسى"

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Supervisor Certification

I certify that this dissertation was prepared under my supervision at the Department of Mathematics, College of Education for Pure Sciences, Ibn Al-Haitham, University of Baghdad as partial fulfillment of the requirements of the Degree of Doctor of Sciences in Mathematics.

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Date: / / 2019

In view of the available recommendation, I forward this dissertation for debate by the examination committee.

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Date: / / 2019

Committee Certification

We certify that we have read this dissertation entitle **"Nonparametric Estimation of Hazard Function by Using Wavelet"** and as examining committee, we examined the student **(Ali Talib Mohammed)** in its contains and what is connected with it, and that in our opinion it is adequate for the partial fulfillment of the requirement for the degree of Doctor of Science in Mathematics.

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ABSTRACT

The objective of this thesis building the New Mixture distribution by incorporating three distributions which have one parameter and utilizing the survival function of Exponential, Standard Weibull and Rayleigh distributions to define "New Mixture Distribution". The method of mixing contains three main parts: The first part includes the process of mixing the Exponential distribution and Rayleigh distribution depending on the tail in each of them. The second part mixing the exponential distribution and Weibull distribution depending on the tail in each of them as well. Here the exponential distribution plays a big role in the mixing process in both parts. Finally, the third part based on the results of the first and second parts to the blending process, the New Mixture distribution was obtained. Moreover, realizing the mathematical and statistical properties of it as the r^{th} moment about the origin , incomplete moment, the moment generating function , mean , median , obtaining the characterization function and other statistical properties. Furthermore, explaining the shapes of the density function and hazard rate function for the New Mixture distribution. Using the Akaike information criterion, corrected Akaike information criterion and the Bayesian information criterion to compare the new mixture distribution with other related distributions. We illustrate the usefulness of proposed by applications of complete real data.

Later, to estimate the parameters of New Mixture distribution, used classical estimation methods they are (Maximum likelihood estimation method, Ordinary least square method, and the rank set sampling estimation method).

Finally, the Mean Square Error is the measured value to compare the methods above for different sizes of samples generated. Utilizing two methods of

simulation technique to generate different sample sizes. We illustrate the usefulness of proposed by application of complete real data.

LIST OF SYMBOLS

symbol	The meaning
X	Random Variable
x	Value of Random Variable
E	Exponential distribution
W	Weibull distribution
R	Rayleigh distribution
Y	Random Variable of exponential
Z	Random Variable of Weibull
T	Random Variable
U	Random Variable of Rayleigh
V	Random Variable
K	Random Variable
$f_E(y)$	Probability Density Function of Exponential distribution
$f_W(z)$	Probability Density Function of Weibull distribution
$F_E(y)$	Cumulative Function of Exponential distribution
$F_W(z)$	Cumulative Function of Weibull distribution
$S_E(y)$	Survival Function of Exponential distribution
$S_W(y)$	Survival Function of Weibull distribution
$S_{EW}(t)$	Survival Function of Exponential Weibull distribution
$F_{EW}(t)$	Cumulative Function of Exponential Weibull distribution
$f_{EW}(t)$	Probability Density Function of Exponential Weibull distribution
$h_{EW}(t)$	Hazard Function of Exponential Weibull distribution
$f_R(u)$	Probability Density Function of Rayleigh distribution
$F_R(u)$	Cumulative Function of Rayleigh distribution

$S_R(u)$	Survival Function of Rayleigh distribution
$S_{ER}(k)$	Survival Function of Exponential Rayleigh distribution
$F_{ER}(k)$	Cumulative Function of Exponential Rayleigh distribution
$f_{ER}(k)$	Probability Density Function of Exponential Rayleigh distribution
$S_{ER}(t)$	Survival Function of Exponential Rayleigh distribution
$h_{ER}(t)$	Hazard Function of Exponential Rayleigh distribution
$S(x)$	Survival Function of New Mixture distribution
$F(x)$	Cumulative Function of New Mixture distribution
$f(x)$	Probability Density Function of New Mixture distribution
$h(x)$	Hazard Function of New Mixture distribution
Υ	Scale parameter of New Mixture distribution
β	Scale Parameter of New Mixture distribution
α	Shape parameter of New Mixture distribution
$\hat{\Upsilon}$	Estimator Value of Scale Parameter of New Mixture distribution
$\hat{\beta}$	Estimator Value of Scale Parameter of New Mixture distribution
$\hat{\alpha}$	Estimator Value of Shape Parameter of New Mixture distribution
$E(x^r)$	The rth Moment of New Mixture distribution about the Origin
$Var(x)$	The Variance of New Mixture distribution
${}_s\psi_j$	The Wright Generalized Comparing to the Equation of Hypergeometric Function
$G^{m,n}_{s,j}$	The Meijer Function
t	The Value of Failure Time of Random Variable
$M(t)$	The Moment Generating Function
$P_x(t)$	The Factorial Moments Generating function of New Mixture distribution
$\varphi_x(it)$	The Characteristic function of New Mixture distribution
MLE	Maximum likelihood Estimation Method

OLS	Ordinary Least Square Estimation Method
RSS	Rank Set Sampling Estimation Method
$L(\Upsilon, \beta, \alpha, x_i)$	The likelihood function of New Mixture distribution
$g(\Upsilon)$	The First Derivative of Υ of New Mixture distribution
$w(\beta)$	The First Derivative of β of New Mixture distribution
$d(\alpha)$	The First Derivative of α of New Mixture distribution
J	The Jacobean matrix
ε	The Error Value
AIC	The Akaike Information Criterion
AIC_C	The Corrected Akaike Information Criterion
BIC	The Bayesian Information Criterion
$L(\Upsilon, \alpha, x_i)$	The likelihood function of Exponential Weibull distribution
$L(\Upsilon, \beta, x_i)$	The likelihood function of Exponential Rayleigh distribution
$L(\beta, \alpha, x_i)$	The likelihood function of Rayleigh Weibull distribution
$L(\alpha, x_i)$	The likelihood function of Weibull distribution
E_1	The First Experiment
E_2	The Second Experiment
E_3	The Third Experiment
E_4	The Forth Experiment
MSE	The Mean Square Error
L	Number of Repeating

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LIST OF DISSERTATION RELATED PUBLICATIONS

➢ First paper

- **Paper Title:** Study of New Mixture Distribution
- **Authors:** Maysaa Jalil Mohammed and Iden Hasan Hussein
- **Journal Name:** Journal of Engineering and Applied Sciences
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➢ Second paper

- **Paper Title:** Compurgation between New Mixture Distribution and Many Other Distributions [b]
- **Authors:** Maysaa Jalil Mohammed and Iden Hasan Hussein
- **Journal Name:** Journal of Advanced Research in Dynamical and Control Systems - JARDCS
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➢ Third paper

- **Paper Title:** Some Estimation Methods for New Mixture Distribution with Simulation and Application [c]
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INTODUCTION

Statistical distributions play a major role in statistical studies. As it provides us with adequate and important information on how to deal with statistical data and extract important information on how and quality of the work of this data. The process of merging the distributions is not new, but it has undergone many stages of development, depending on how to combine distributions to produce a new distribution, as well as how to choose the distributions to be integrated. All this is done through the information available to the researcher about the quality, characteristics and the nature of the work of each distribution itself. Therefore, we have focused our attention on the creation of a new distribution that is similar in statistical characteristics to the world-famous statistical distributions by using a new method of mixing based on the tail function. Also, study the mathematical and statistical properties of this distribution. It was also possible after studying all the distribution characteristics compared to the distribution of some famous distributions such as Weibull distribution, exponential Weibull distribution, and Rayleigh Weibull distribution. It was compared through the use of some important statistical standards such as Akaike information criterion, corrected Akaike information criterion and the Bayesian information criterion. We present the new distribution with three parameters called "The New Mixture distribution". It is mixed between "exponential Weibull and exponential Rayleigh". The method of the mixture depends on the tail of the exponential Weibull and exponential Rayleigh. exponential distribution has an important role in the mixing process. This thesis consist of five chapters, chapter one included the introduction, aims of work in this thesis, literature review and stricture of the new mixture distribution. Chapter two deals with a study of the mathematical and statistical characteristics of this distribution. Finally, comparison of the new distribution with some statistical distributions such as Weibull distribution, exponential Weibull, exponential Rayleigh, Rayleigh Weibull distribution by using some significant statistical criterions such as Akaike information criterion, corrected Akaike information criterion and the Bayesian information criterion. Chapter three, contains the estimation of distribution parameters was presented in three important classical methods as: (Maximum likelihood estimator method, Ordinary least square method, and the rank set sampling estimator method). It also included the theoretical derivations of the methods. Chapter four, utilizes a simulation study to generate different sizes of samples. Compare the parameters estimator using mean square error for the estimation of the parameters. We applied these methods referred to above on real complete data. Chapter five, it consists of the conclusions, recommendations, and reference of the thesis.

CHAPTER ONE

INTRODUCTION AND METHODOLOGY OF BUILDING THE NEW MIXTURE DISTRIBUTION

1.1 Introduction

In order to keep abreast of developments in the medical and engineering fields and other scientific areas related to data analysis and extract information that will help to take appropriate action based on this information. An important technique in analyzing data, especially in the field of reliability and survival, which was used of statistical distributions and these distributions play important roles in the processing and analysis of data. In view of the developments in modern medical and technological fields, needing to develop statistical methods in processing and extracting data. Weibull distribution is one significant statistical distribution that have a major role in the survival and reliability function analysis. Waloddi Weibull invented the Weibull distribution in 1937 and delivered his hallmark American paper on this subject in 1951. He claimed that his distribution applied to a wide range of problems. Lord Rayleigh (1880) introduced the Rayleigh distribution in connection with a problem in the field of acoustics. Since then, extensive work has taken place related to this distribution in different areas of science and technology. The Rayleigh distribution is a special case of the Weibull distribution when the shape parameter is equal to two. Also, the exponential distribution is the special case of Weibull distribution when the shape parameter of Weibull distribution is equal to one. In this chapter, interesting, the aim of the thesis, literature review and stricture of the new mixture distribution.

1.2 The aim of this work

This work has witnessed many main goals that we can achieve it as follows:

1. Building a new distribution based on a combination of three major distributions: Weibull, exponential, and Rayleigh distributions. New mixture distribution is mixed between the exponential Weibull distribution and exponential Rayleigh distribution; that means employing this method two times for generating. Firstly, mixing between exponential and Weibull distributions to generate exponential Weibull distribution. Secondly, mixing between exponential and Rayleigh distributions to generate exponential Rayleigh distribution. The mixing method depends on the tail function.
2. Proof the statistical and mathematical properties of New Mixture distribution such as rth −moment, incomplete moment, the moment generating function and characteristic function. Obtaining the quantile function and some statistical properties.
3. Comparing the New Mixture distribution with some special distributions as Weibull distribution, exponential Weibull distribution, exponential Rayleigh distribution and Rayleigh Weibull distribution. We illustrate the usefulness of proposed by applications of complete real data.
4. Estimator the three parameters of New Mixture distribution by using the classical estimation methods as (Maximum likelihood estimator method, Ordinary least square method, and the Rank set sampling estimator method).
5. Use two methods to generate data for the New Mixture distribution, the first method depended on the inverse of (CDF) while the second method involves the Newton – Raphson method.
6. Finally, comparing estimator parameters by using the mean square error where MSE=$\sum_{i=1}^{L}\frac{(\vartheta^{\wedge}-\vartheta)^2}{L}$. Repeating the experiments (L=500, 1000), n=10, 30,50,100.

7. Used complete real data for application to illustrate the usefulness of the proposed method.

1.3 literature review

There are many researchers studied the idea of mixed distributions which are as follows:

1. (Fisher, 1934), developed the concept of mixed distributions in order to develop weighted statistical distributions. He pointed out that in the medical, biological and agricultural sciences show random biased samples, leading to the failure of standard statistical distributions [8]. He wanted to overcome the bias that appears in the size and length of the sample, then process and interpret these samples. (Rao, 1985), the development of a general mathematical formula by explained the idea of Fisher through to address such cases and called the statistical distributions weighted [24].

$$f_w(z) = \frac{w(z)f(z)}{E[w(z)]} \qquad (1\text{-}1)$$

Where, $f_w(z)$, probability density function for the weighted distribution.

$f(z)$, Probability functions for standard distribution.

$w(z)$, Weight function for standard distribution and takes two cases:

$w(z) = z$, weight function of the variable length.

$w(z) = z^r$, weight function of the variable size.

$E[w(z)]$, Expectation of the weight function for the standard distribution.

Now, represents some articles which depending on formula (1-1):

(Kilany, 2016), introduced the weighted Lomax distribution [14], in (Ajami and Jahanshahi, 2017), introduced the weighted Rayleigh distribution[2], also, (Saghira, Tazeema and Ahmad, 2017), introduced the weighted exponentiated inverted Weibull distribution [25].

2. (Azzalini, 1985), developed the idea of adding the shape parameter to the normal distribution which does not consist the shape parameter. This distribution was called (Skew normal distribution) [3].
   (Gupta and Kundu, 2009), used the same idea to (Azzalini, 1985) researcher to find a shape parameter for exponential distribution which was containing only the (scale) parameter. This distribution was called (Weighted exponential distribution) [9]. In addition to developing a general mathematical formula to deal with the remediation weighted of statistical distributions as follows:

$$f_w(x) = \frac{1}{pr(x_2 < \theta x_1)} f(x_1)F(\theta x_1) \qquad (1\text{-}2)$$

Where,

$f_w(x)$, probability density function for the weighted distribution, $f(x_1)$, probability function for the standard distribution of the random variable (x_1), and $F(\theta x_1)$, accumulation distribution function in terms of the weighted parameter (θ) for standard distribution.

$pr(x_2 < \theta x_1)$, probability of the random variable$(x_2)$by the random variable (x_1) and the weighted function (θ).

Now, represented some articles which are depending upon formula (1-2) as follows:

(Oguntunde, Owoloko and Balogun, 2016), represented the "On A New Weighted Exponential Distribution: Theory and Application"[20]. Also,

(Oguntunde, Ilori and Okagbue, 2018), introduced the inverted weighted exponential distribution with applications [19]. Finally, (Khongthip, Patummasut and Bodhisuwan, 2018), introduce the discrete weighted exponential distribution [13].

3. (Zografos and Balakrishnan, 2009), studied a broad family of univariate distributions through a particular case of Stacy's generalized gamma distribution [30]. A continuous distribution U with density u and further Stacy's generalized gamma density which is as follows:

$$f(x) = \alpha x^{\alpha\,\theta-1}\, e^{-x^{\alpha}}/\Gamma(\theta) \text{ for } x > 0 \text{ and } \alpha, \theta > 0$$

Based on this density, by replacing x by $-\log[1-G(x)]$ and considering $\alpha = 1$, Zografos and Balakrishnan defined their family with cdf:

$$F(x;\theta) = \alpha\{\theta, -\log[1-G(x)]\},\ x \in X \subseteq R,\ \theta > 0,$$

Where, $\alpha(\theta, r) = \int_0^r s^{\theta-1} e^{-s} ds/\Gamma(\theta)$ denotes the incomplete gamma function and $\Gamma(\cdot)$ is the gamma function. The pdf family is:

$$f(x;\theta) = \frac{1}{\Gamma(\theta)}\,[-\log\{1 - G(x)\}]^{\theta-1}\, g(x).$$

(Bourguignon, Silva and Cordeiro, 2014), depends on (Zografos and Balakrishnan, 2009) ideas, presented general formula to generate Weibull – Ǵ distribution depends on odds $\frac{Ǵ(y)}{1-Ǵ(y)}$ where $\vartheta(y,\vartheta)$ is the (CDF) of random variable such that [5]:

$$F(y,a,b,\vartheta) = \int_0^{\frac{\vartheta(y)}{1-\vartheta(y)}} abz^{b-1}e^{-az^{b}}\,dz = 1 - e^{\left\{-a\left[\frac{\vartheta(y,\vartheta)}{\vartheta(y)}\right]^{b}\right\}} \qquad (1\text{-}3)$$

$$y \in R; a, b > 0.$$

Now, represented some articles which depending on formula (1-3) as follows:

(Merovci and Elbatal, 2015), represented a new distribution called Weibull Rayleigh Distribution. It depends on another method to mix between the distributions [17]. (Bakouch, Bourguignon and Chesneau, 2017), introduced on some properties of the hazard rate function for compound distributions [4].

4. (Cordeiro, Ortega and Lemonte, 2014), introduced a new method to mix the distributions by using a tail of these functions, and represented this new method to mix between exponential of one parameter and Weibull of two parameters called " The exponential –Weibull lifetime distribution"[6]. (Nasiru, 2016), found a new distribution which depends on the Gauss method. This distribution called "Serial Weibull Rayleigh distribution. It was depended on the distribution two parameters of Weibull distribution and Rayleigh distribution one parameter [18].

   Due to the novelty and lack of research used for this method, we have used it more than once to find a new mixed distribution and the full blending method will be presented later in this chapter.

1.4 The New Mixture distribution (Methodology)

Depending on (Cordeiro, Ortega and Lemonte, 2014) method [6], the mathematical structure of this new distribution takes three parts. The first part, mixing between the exponential and Weibull distributions , second part, mixing between exponential and Rayleigh distributions, third part, based on the result of part one and part two to build the New Mixture distribution as follows:

1.4.1 Exponential Weibull distribution

In this part, mixed between exponential and standard Weibull distributions, both have one parameter as follows:

The pdf functions are:

$$f_E(y) = \Upsilon e^{-\Upsilon y} \qquad \Upsilon > 0,\ y>0 \tag{1-4}$$

$$f_W(z) = \alpha z^{\alpha-1} e^{-z^{\alpha}} \qquad \alpha>0,\ z>0 \tag{1-5}$$

The CDF functions are as follows:

$$F_E(y) = 1 - e^{-\Upsilon y} \tag{1-6}$$

$$F_W(z) = 1 - e^{-z^{\alpha}} \tag{1-7}$$

We take the tail (survival) function of exponential distribution and the tail (survival) function of the Weibull distribution as follows:

$$S_E(y) = e^{-\Upsilon y} \tag{1-8}$$

$$S_W(z) = e^{-z^{\alpha}} \tag{1-9}$$

Generalized exponential Standard Weibull distribution with tail (survival) function by multiplying the survival function of exponential distribution and the survival function of standard Weibull distribution as follows:

Let, Y,Z are independent random variables, and T=min(Y,Z).

Then, we get:

$$S_{EW}(t) = \Pr(T > t)$$

$$S_{EW}(t) = \Pr(\min(Y, Z) > t)$$

Since, Y,Z are independent random variables, then we get:

$$S_{EW}(t) = \Pr(Y > t) \cdot \Pr(Z > t)$$

$$S_{EW}(t) = (1 - Pr(Y \leq t)) \cdot ((1 - Pr(Z \leq t))$$

$$S_{EW}(t) = (1 - F_y(t)) \cdot (1 - F_z(t))$$

$$S_{EW}(t) = S_E(t) S_W(t)$$

Since, $S_E(y) = e^{-\Upsilon y}$ and $S_W(z) = e^{-z^{\alpha}}$, its directly we get:

$$S_{EW}(t) = e^{-\Upsilon t} \cdot e^{-t^{\alpha}}$$

$$S_{EW}(t) = e^{-(\Upsilon t + t^{\alpha})} \qquad (1\text{-}10)$$

The survival function of exponential standard Weibull distribution.

The cumulative function of this distribution is:

$$F_{EW}(t) = 1 - e^{-(\Upsilon t + t^{\alpha})} \quad , \quad t>0 \qquad (1\text{-}11)$$

we can write the pdf of the exponential standard Weibull distribution from the derivative of the CDF distribution where $\Upsilon > 0$, is the scale parameter and $\alpha > 0$ is the shape parameter of the exponential standard Weibull distribution as follows:

$$f_{EW}(t) = (\Upsilon + \alpha t^{\alpha - 1})\, e^{-(\Upsilon t + t^{\alpha})} \quad , t > 0 \qquad (1\text{-}12)$$

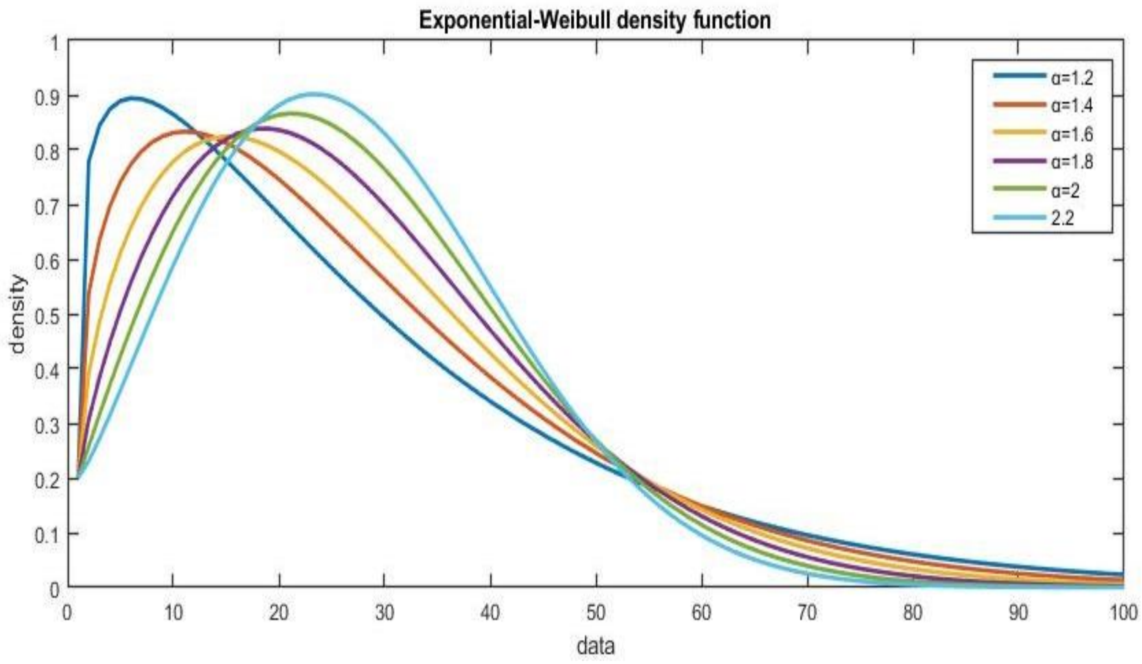


Figure (1): Shape of density function of exponential standard Weibull with scale parameter ($\Upsilon = 0.2$) and different value of shape ($\alpha = 1.2,1.4,1.6,1.8,2,2.2$).

It clear that exponential standard Weibull distribution satisfy the properties that

$$\mathrm{f_{EW}}(\mathrm{t}) > 0 \; and \; \int_0^{\infty} f_{EW}(t)\, dt = 1$$

Also, the hazard rate function is: $h(t) = \frac{f_{EW}(t)}{S_{EW}(t)} = \frac{(\Upsilon+\alpha t^{\alpha-1})\, e^{-(\Upsilon t+t^{\alpha})}}{e^{-(\Upsilon t+t^{\alpha})}}$

$$\mathrm{h_{EW}}(t) = (\Upsilon + \alpha t^{\alpha-1}) \qquad where, t > 0 \tag{1-13}$$

1.4.2 Exponential Rayleigh distribution

In this part, mixed between exponential and Rayleigh distributions, both have one parameter as follows:

The pdf functions are as follows:

Let V,U be two random variables, from equation (1-4):

$$f_E(v) = \Upsilon e^{-\Upsilon v}, \qquad \Upsilon>0, v > 0$$

$$f_R(u) = \beta u e^{-\frac{\beta}{2}u^2}, \ \ \beta > 0, u > 0 \tag{1-14}$$

The CDF functions are as follows:

$$F_E(v) = 1 - e^{-\Upsilon v}, \quad v>0$$

$$\text{and } F_R(u) = 1 - e^{-\frac{\beta}{2}u^2}, \quad u>0 \tag{1-15}$$

We take the tail (survival) function of exponential distribution and the tail (survival) function of the Rayleigh distribution as follows:

$$S_E(v) = e^{-\Upsilon v}, \quad v>0$$

$$S_R(u) = e^{-\frac{\beta}{2}u^2}, \quad u>0 \tag{1-16}$$

We generalized exponential Rayleigh distribution with tail (survival) function by multiplying the survival of exponential distribution and the survival of the Rayleigh distribution as follows:

let V,U are independent random variables and K= min(V,U)

$$S_{ER}(k) = \Pr(K > k)$$

$$S_{ER}(k) = \Pr(\min(V, U) > k)$$

Since, V and U are independent random variables, then we get:

$$S_{ER}(k) = \Pr(V > k) \cdot \Pr(U > k)$$

$$S_{ER}(k) = (1 - \Pr(V \le k)) \cdot ((1 - \Pr(U \le k))$$

$$S_{ER}(k) = \left(1 - F(v)\right) \cdot \left(1 - F(u)\right)$$

Since, $S_E(v) = e^{-\Upsilon v}$ *and* $S_R(u) = e^{-\frac{\beta}{2}u^2}$ then, we get:

$$S_{ER}(k) = S_E(k) \cdot S_R(k)$$

$$S_{ER}(k) = e^{-\Upsilon k} \cdot e^{-\frac{\beta}{2}k^2}$$

$$S_{ER}(k) = e^{-(\Upsilon k+\frac{\beta}{2}k^2)} \quad where, k > 0 \tag{1-17}$$

Then the survival function of exponential Rayleigh distribution is showing in equation (1-5).

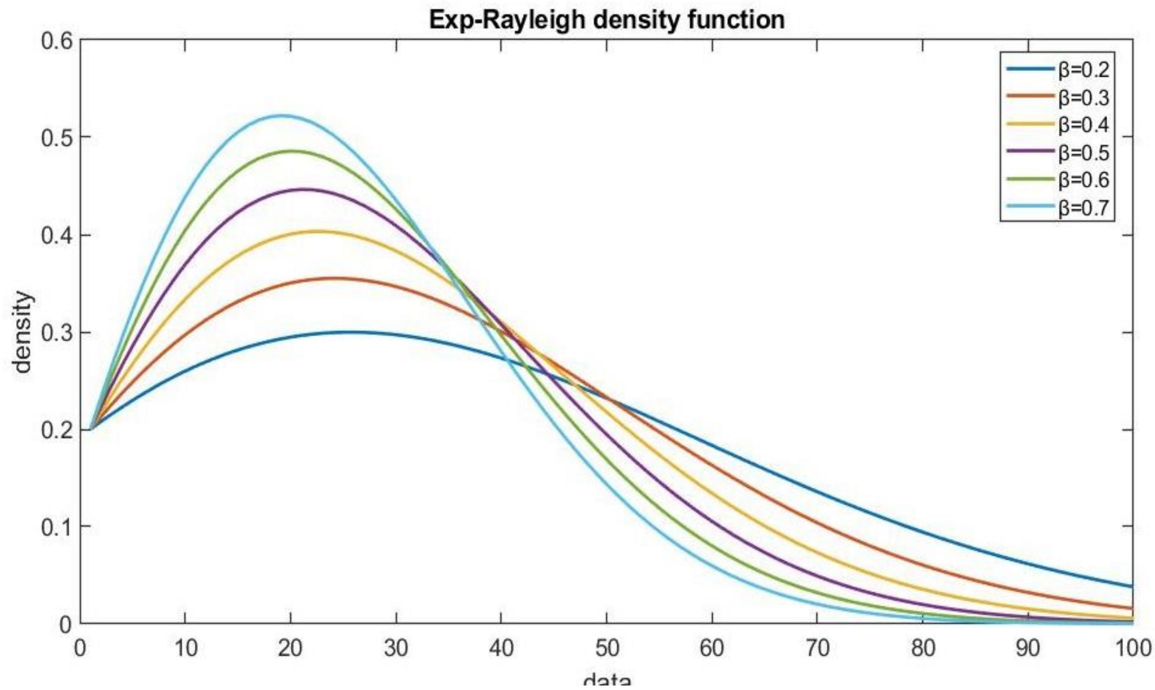
The CDF of this distribution is:

$$F_{ER}(k) = 1 - e^{-(\Upsilon k+\frac{\beta}{2}k^2)} \quad \text{where, } k > 0 \tag{1-15}$$

The pdf of the exponential Rayleigh distribution was obtained from the derivative of the CDF distribution where $\Upsilon > 0$, and $\beta > 0$ which are two scale parameters of the exponential Rayleigh distribution as follows:

$$f_{ER}(k) = (\Upsilon + \beta k)e^{-(\Upsilon k+\frac{\beta}{2}k^2)} \;, k > 0 \tag{1-18}$$

It is clear that $f_{ER}(k) > 0, \int_0^\infty f_{ER}(k)\, dk = 1$.



Figure(2): Shape of density function of exponential Rayleigh with scale parameter $(\Upsilon = 0.2)$ and different values of shape $(\alpha = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7)$.

Also, the hazard rate function is: $h(k) = \frac{f_{ER}(k)}{S_{ER}(k)} = \frac{(\Upsilon+\beta k)e^{-(\Upsilon k+\frac{\beta}{2}k^2)}}{e^{-(\Upsilon k+\frac{\beta}{2}k^2)}}$

$$h_{ER}(k) = (\Upsilon + \beta k) \quad , k > 0 \tag{1-19}$$

1.4.3 <u>Building of New Mixture distribution</u>

Depending (Cordeiro, Ortega and Lemonte, 2014) method [6], finally, we use the results of survival functions from parts one and part two (equ,s (1-10) and (1- 17) to mix between them as the same methods above . Noted, the method of mixing depends on the tail distribution. Then the final result depends of the tail (survival) distribution of exponential standard Weibull distribution and the exponential Rayleigh distribution as the same methods above generates the New Mixture distribution as follows:

Let K,T are independent random variables and X=min(K, T).

Then, $$\mathrm{S(x)} = \Pr(X > x)$$

$$\mathrm{S(x)} = (\Pr(\min(K, T) > x)$$

Since, K,T are independent random variables, then we get:

$$\mathrm{S(x)} = \Pr(\mathrm{K} > \mathrm{x}) \cdot \Pr(\mathrm{T} > \mathrm{x})$$

$$\mathrm{S(x)} = (1 - \Pr(K \leq x)) \cdot (1 - \Pr(T \leq x))$$

$$= \left[1 - \int_0^x \left(1 - e^{-(\Upsilon k + \frac{\beta}{2}k^2)} dk\right)\right] \left[1 - \int_0^x \left(1 - e^{-(\Upsilon t + t^{\alpha})} dt\right)\right]$$

$$\mathrm{S(x)} = \big(1 - \mathrm{F(k)}\big) \cdot \big(1 - \mathrm{F(t)}\big)$$

$$\mathrm{S(x)} = \mathrm{S_{ER}(k)} \cdot \mathrm{S_{Ew}(t)}$$

Since , $\mathrm{S_{ER}(k)} = e^{-(\Upsilon k + \frac{\beta}{2}k^2)}$ $and\ S_{EW}(t) = e^{-(\Upsilon t + t^{\alpha})}$ then, its directly we get:

$$\mathrm{S(x)} = e^{-(\Upsilon x + x^{\alpha})} \cdot e^{-(\Upsilon x + \frac{\beta}{2}x^2)}$$

$$\mathrm{S(x)} = \mathrm{e}^{-(2\Upsilon \mathrm{x} + \frac{\beta}{2}\mathrm{x}^2 + \mathrm{x}^{\alpha})},\ \mathrm{x} > 0 \qquad (1\text{-}20)$$

$\mathrm{S(x)}$, is the survival function of New Mixture distribution. Where $\mathrm{X} = \min\{\mathrm{K}, \mathrm{T}\}$, $\Upsilon > 0$, $\beta > 0$ are two scale parameters and $\alpha > 0$ is the shape parameter of a New Mixture distribution.

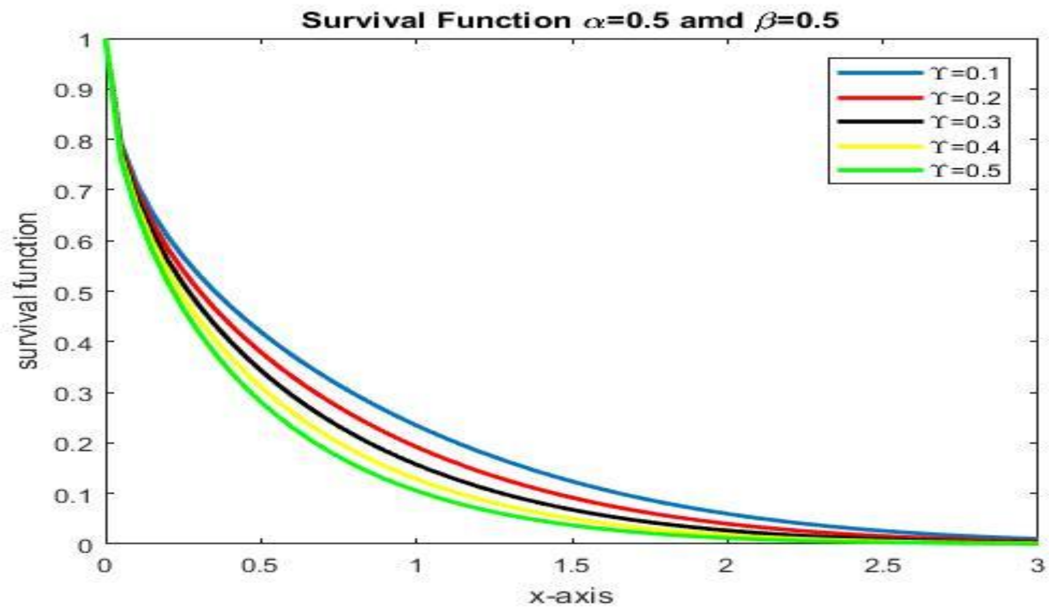


Figure (3): Shape of survival function with constants scale ($\beta = 0.5$), shape ($\alpha = 0.5$),and different scale ($\Upsilon = 0.1,0.2,0.3,0.4,0.5$) parameter values.

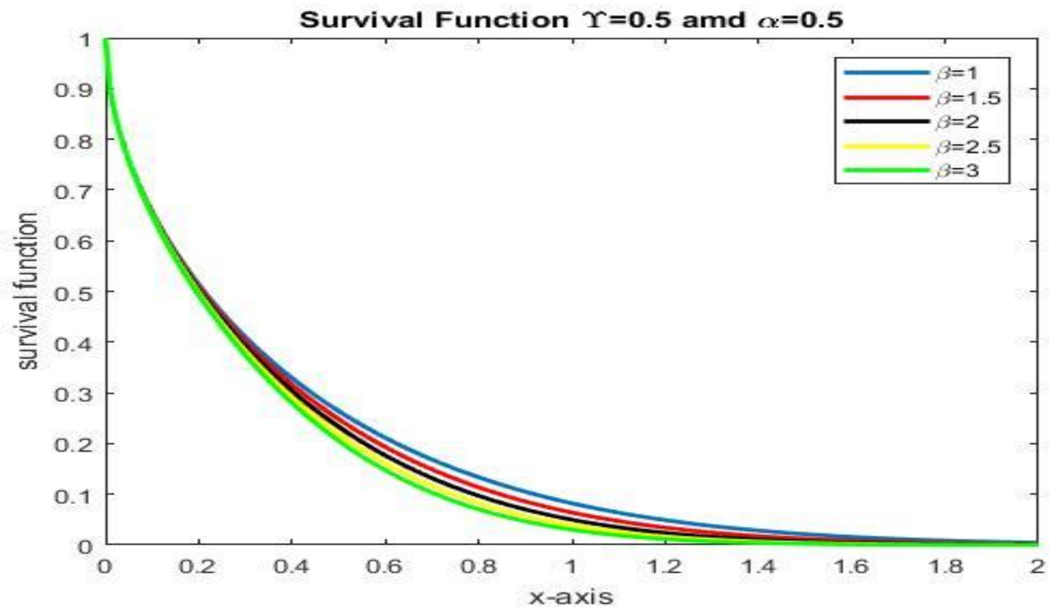


Figure (4): Shape of survival function with constants scale ($\Upsilon = 0.5$), shape ($\alpha = 0.5$), and different scale ($\beta = 1, 1.5, 2, 2.5,3$) parameter values.

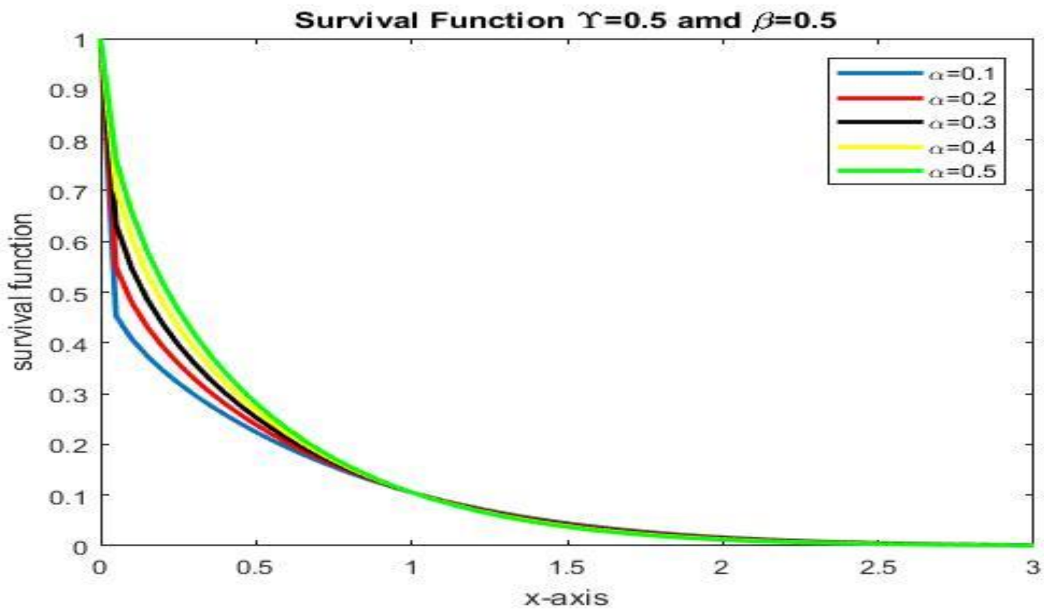


Figure (5): Shape of survival function with constants scales ($\Upsilon = 0.5$), ($\beta = 0.5$), and different shape ($\alpha = 0.1,0.2,0.3,0.4,0.5$) parameter values.

The CDF of New Mixture distribution is:

$$F(x) = 1 - e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)}, \quad \text{where } x > 0 \tag{1-21}$$

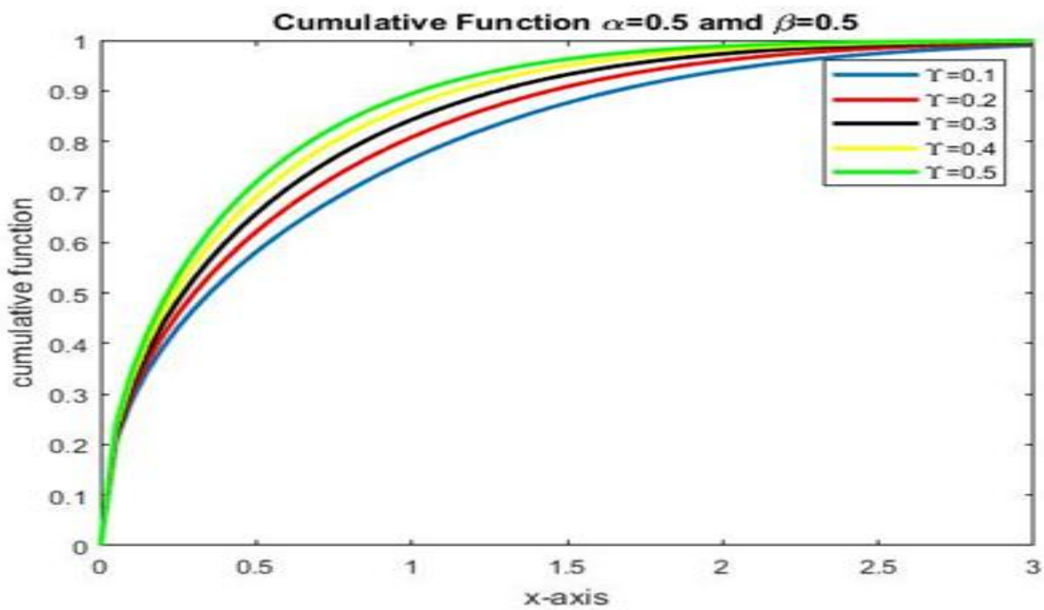


Figure (6): Shape of cumulative function with constants scale ($\beta = 0.5$),shape ($\alpha = 0.5$) and different scale ($\Upsilon = 0.1,0.2,0.3,0.4,0.5$) parameter values.

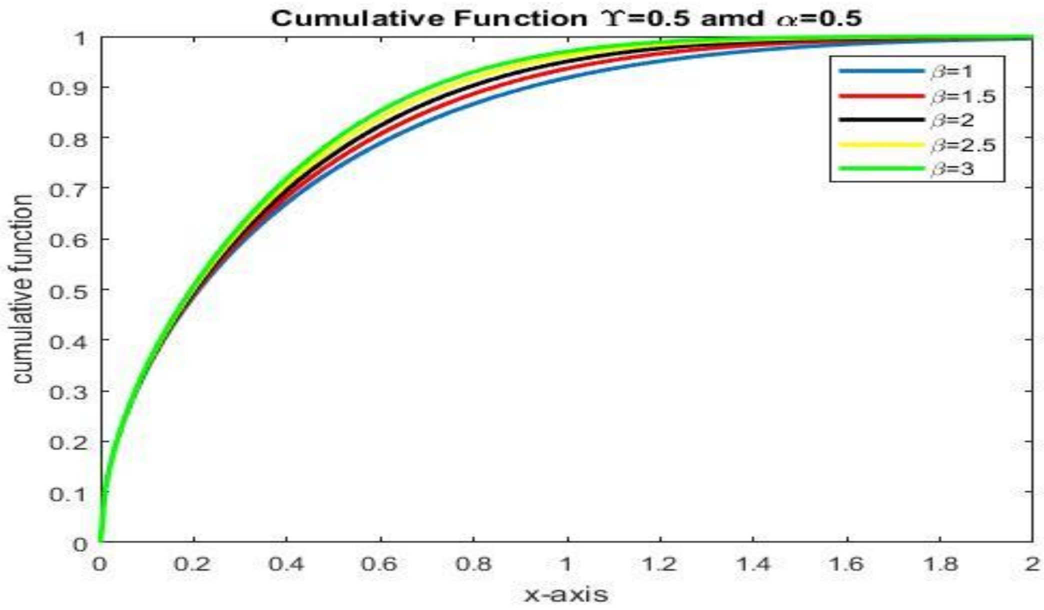


Figure (7): Shape of cumulative function with constants scale ($\Upsilon = 0.5$), shape ($\alpha = 0.5$), and different scale ($\beta = 1{,}1.5, 2, 2.5, 3$) parameter values.

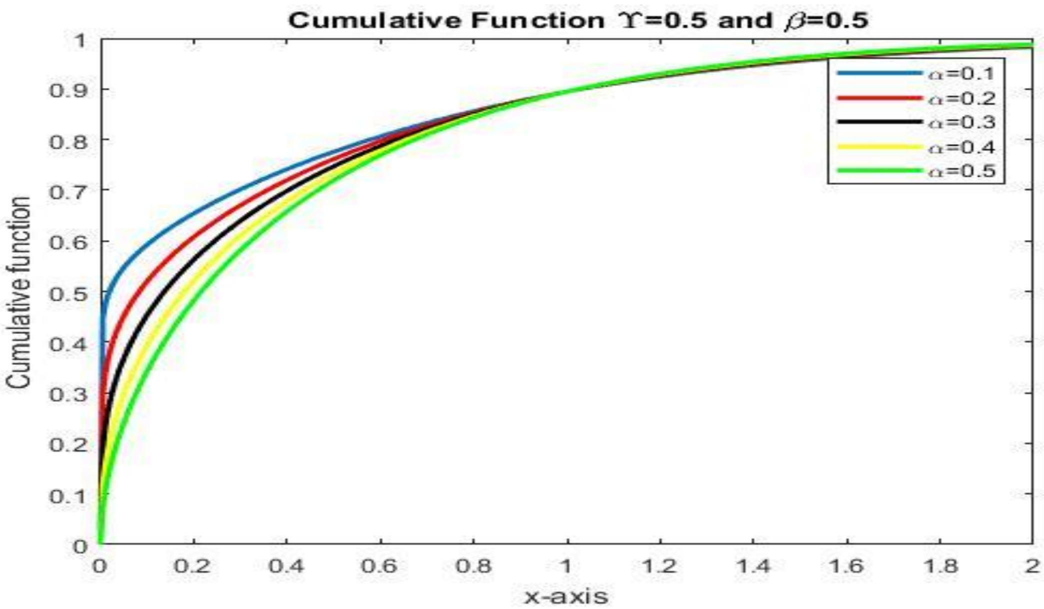


Figure (8): Shape of cumulative function with constants scales ($\Upsilon = 0.5$), ($\beta = 0.5$), and different shape ($\alpha = 0.1{,}0.2{,}0.3{,}0.4{,}0.5$) parameter values.

The pdf function of New Mixture distribution is obtained by derivative the cdf as follows:

$$f(x) = (2\Upsilon + \beta x + \alpha x^{\alpha-1})e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)},\ x > 0 \qquad (1\text{-}22)$$

It is observed that the proposed New Mixture distribution has several interesting properties and it can used effectively to analyze them. Moreover, the pdf of New Mixture distribution satisfy that $f(x) > 0\ and\ \int_0^{\infty} f(x)dx = 1$.

Since, $\int_0^{\infty}(2\Upsilon + \beta x + \alpha x^{\alpha-1})e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)}dx = -\left[e^{-\left(2\beta x + \frac{\mu}{2}x^2 + x^{\alpha}\right)}\right]_0^{\infty}$ =1

The hazard rate function is:

$$h(x) = \frac{f(x)}{S(x)} = \frac{(2\Upsilon + \beta x + \alpha x^{\alpha-1})e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)}}{e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)}}$$

$$h(x) = 2\Upsilon + \beta x + \alpha x^{\alpha-1},\ x > 0 \qquad (1\text{-}23)$$

It is clear that, if $\Upsilon = 0$, the New Mixture pdf transform to Weibull $-$ Rayliegh pdf, if β=0, the New Mixture transform to exponential- Weibull pdf. Furthermore, if $\alpha \neq 0,\ \beta = 0,$ and $\Upsilon = 0$, then the New Mixture distribution reduced to standard Weibull distribution.

1.5 <u>The shapes of New Mixture distribution</u>

We discuss the shapes of the density and the hazard rate functions of New Mixture distribution, the shapes will be introduced with different values of parameters, as follows:

$$\lim_{x\to 0} f(x) = \begin{cases} \infty & \alpha < 1 \\ 2\Upsilon & \alpha > 1 \end{cases}$$

$$\text{and,} \quad \lim_{x\to\infty} f(x) = 0$$

Then, $\ln f(x) = \ln(2\Upsilon + \beta x + \alpha x^{\alpha-1}) - \left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)$

$$\frac{\partial \ln f(x)}{\partial x} = \frac{\beta + \alpha(\alpha-1)x^{\alpha-2}}{2\Upsilon + \beta x + \alpha x^{\alpha-1}} - (2\Upsilon + \beta x + \alpha x^{\alpha-1}) = 0$$

Then, $\beta + \alpha(\alpha-1)x^{\alpha-2} = (2\Upsilon + \beta x + \alpha x^{(\alpha-1)})^2$

There is more than one root of this equation. So, if $x = x_1$is the root of the equation, then, it depending to a local maximum, minimum or a point of inflection which depending on the $J(x_1) < 0$, $J(x_1) > 0$ or $J(x_1) = 0$ where,

$$\frac{\partial \ln f(x)}{\partial x} = (\beta + \alpha(\alpha-1)x^{\alpha-2})\,(2\Upsilon + \beta x + \alpha x^{\alpha-1})^{-1} - (2\Upsilon + \beta x + \alpha x^{\alpha-1}).$$

$$J(x)=\frac{\partial^2 \ln f(x)}{\partial x^2} = -(\beta + \alpha(\alpha-1)x^{\alpha-2})^2\,(2\Upsilon + \beta x + \alpha x^{\alpha-1})^{-2} + (2\Upsilon + \beta x + \alpha x^{\alpha-1})^{-1}\,(\alpha(\alpha-1)(\alpha-2)x^{\alpha-3}) - (\beta + \alpha(\alpha-1)x^{\alpha-2}).$$

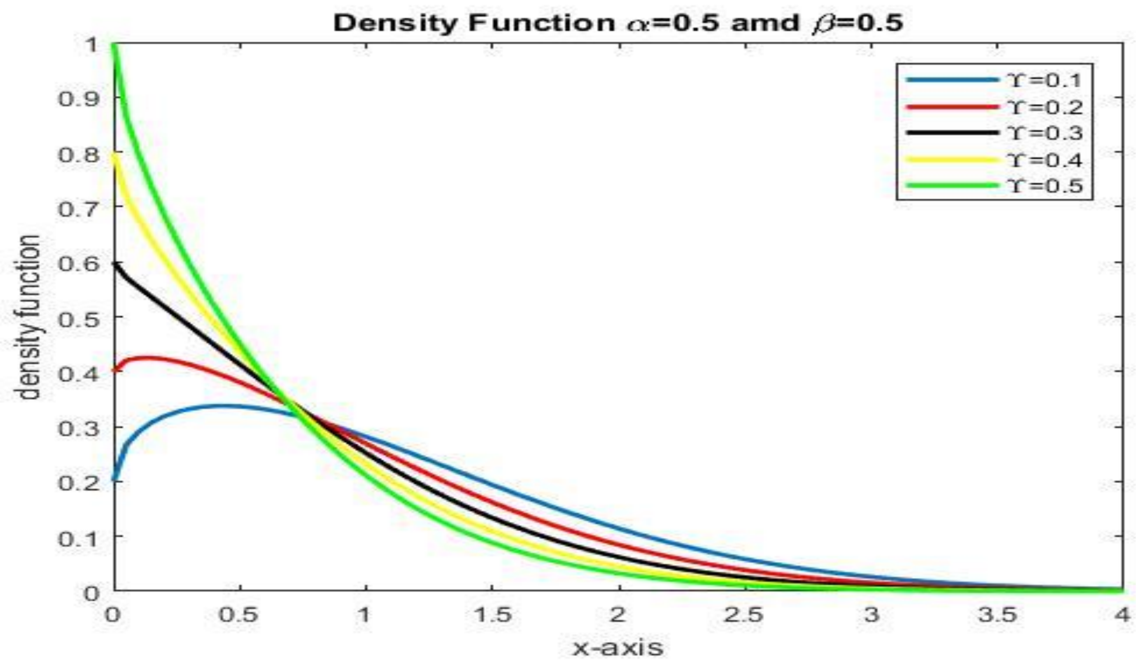


Figure (9): Shape of density function with constants scale ($\beta = 0.5$), shape ($\alpha = 0.5$), and different scale ($\Upsilon = 0.1,0.2,0.3,0.4,0.5$) parameter values.

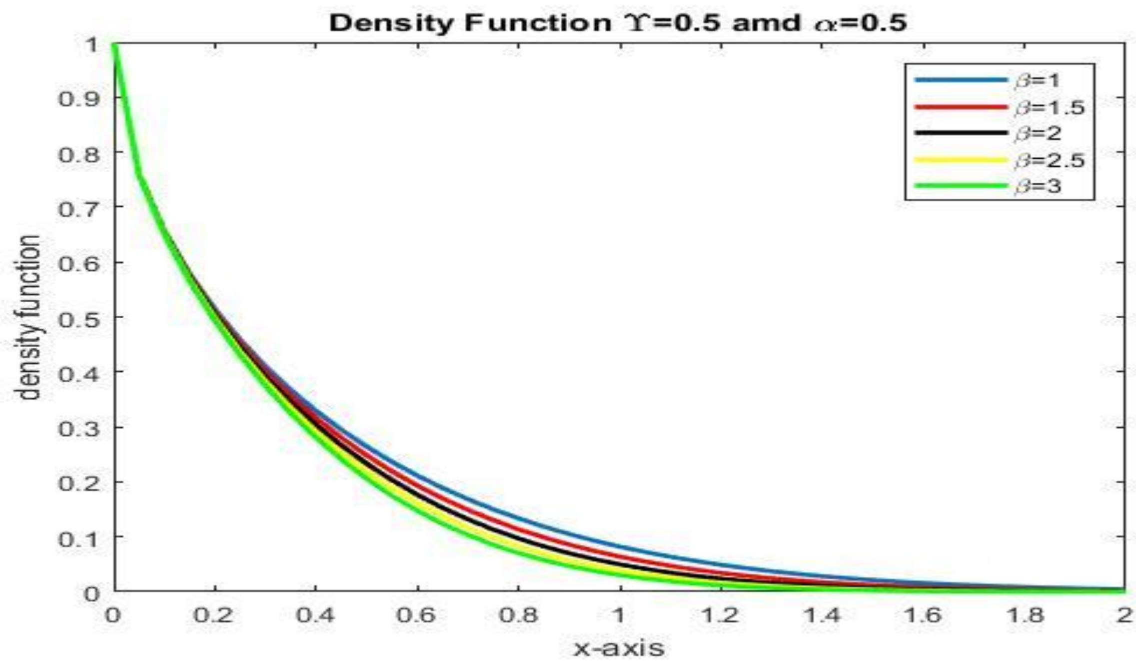


Figure (10): Shape of density function with constants scale ($\Upsilon = 0.5$), shape ($\alpha = 0.5$), and different scale ($\beta = 0.1,0.2,0.3,0.4,0.5$) parameter values.

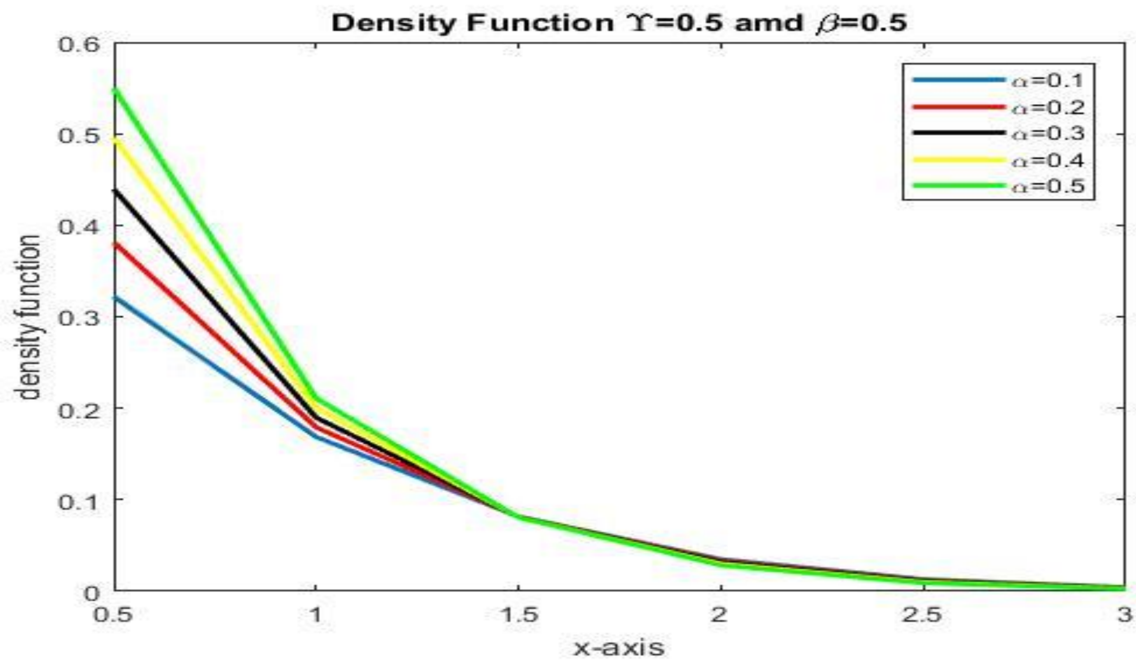


Figure (11): Shape of density function with constants scales ($\Upsilon = 0.5$), ($\beta = 0.5$), and different shape ($\alpha = 0.1,0.2,0.3,0.4,0.5$) parameter values.

Moreover, since the hazard rate function is:

$$h(x) = 2\Upsilon + \beta x + \alpha x^{\alpha-1}$$

$$h'(x) = \beta + \alpha(\alpha - 1)x^{\alpha-2}, \qquad x > 0$$

The shape of hazard function depends on the parameter (α), such that :

If $\alpha<1$, then h(x) will be decreasing function, that is $x>0$ and $h'(x)<0$.

If $\alpha>1$, then h(x) will be increasing function, that is $x>0$ and $h'(x) > 0$.

$$\lim_{x\to 0} h(x) = \begin{cases} \infty & \alpha < 1 \\ 2\Upsilon & \alpha > 1 \end{cases},$$

And,

$$\lim_{x\to \square} h(x) = \infty$$

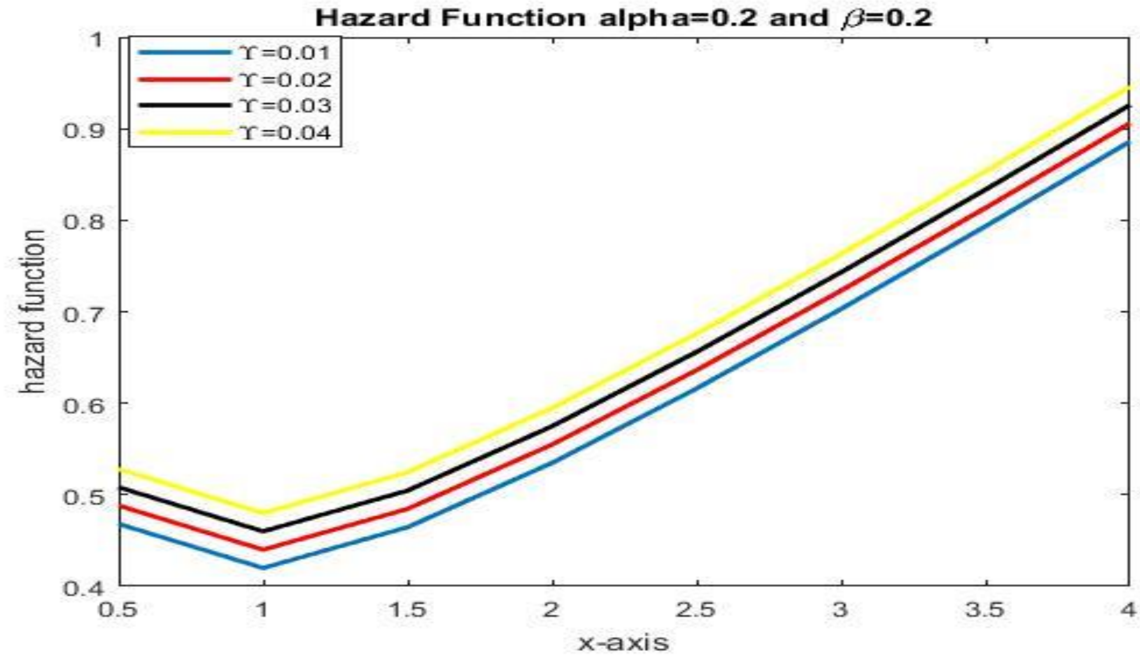


Figure (12): Shape of hazard rate function with constants scale ($\beta = 0.2$), shape ($\alpha = 0.2$), and different scale ($\Upsilon = 0.01, 0.02, 0.03, 0.04$) parameter values.

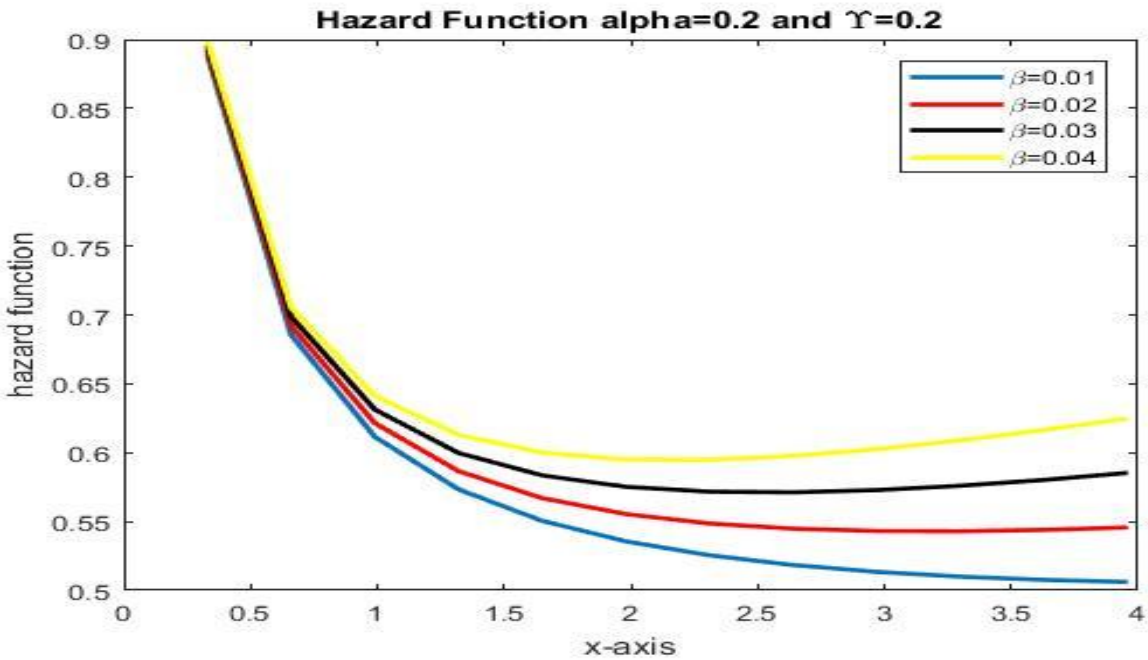


Figure (13): Shape of hazard rate function with constants scale ($\Upsilon = 0.2$), shape ($\alpha = 0.2$), and different scale ($\beta = 0.01, 0.02, 0.03, 0.04$) parameter values.

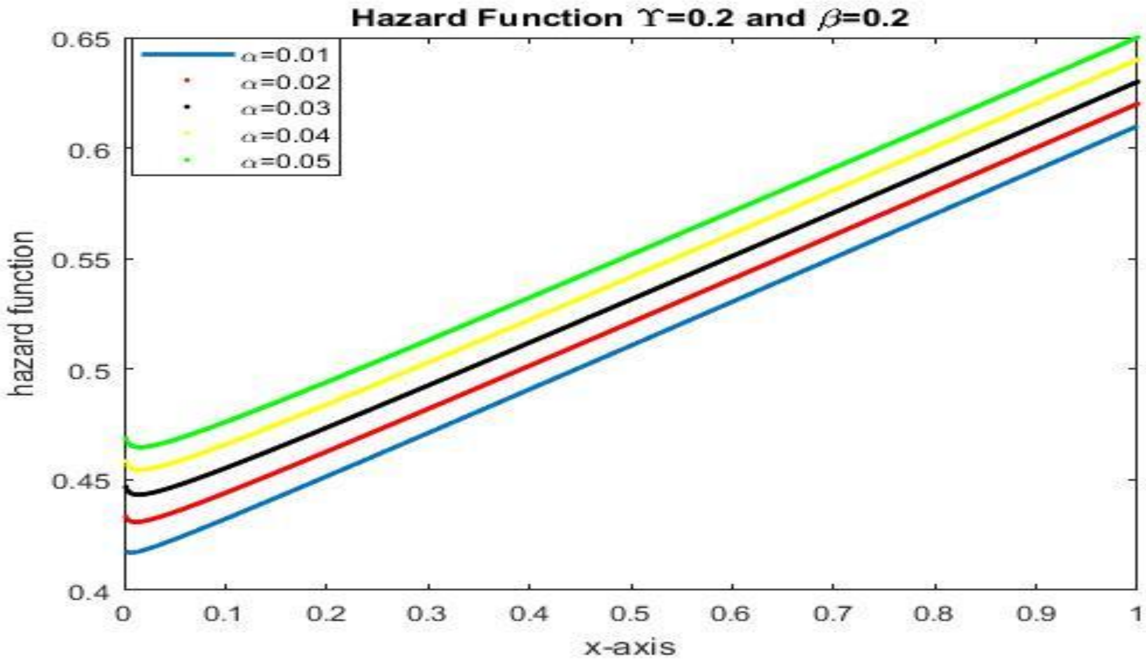


Figure (14): Shape of hazard rate function with constants scales ($\beta = 0.2$), ($\Upsilon = 0.2$), and different scale ($\alpha = 0.01, 0.02, 0.03, 0.04$) parameter values.

CHAPTER TWO

STATISTICAL PROPERTIES OF NEW MIXTURE DISTRIBUTION

2.1 Introduction

Mixed statistical distributions are important distributions in the modeling of many phenomena because these distributions are more flexible than standard distributions. Many researchers have been interested in the study of this type of distributions, whether continuous or intermittent. These distributions are produced by a combination of two distributions or more combinations that combine some common characteristics. Some of the data cannot be represented by standard statistical distributions as required because the nature of these data or phenomena requires the use of complex distributions that are more flexible than the standard distributions in their representation.

The exponential, Rayleigh, Weibull distributions are most commonly used distributions to analyses lifetime data since they have several desirable properties and nice physical interpretations. This work is present a new mixture distribution which is defined by mixed one parameter of three distributions obtained from the tail (survival) of the exponential, Weibull and Rayleigh distributions. The new distribution is quite flexible and can be used effectively in modeling survival data and reliability problems. It includes some well-known lifetime distributions as special sub-models. We provide a comprehensive account of the mathematical properties of the new distribution. A particular, closed-form expressions for the probability density, cumulative distribution and hazard rate functions of this new distribution were given. In this chapter, we introduce some theorems of proofs some special statistical properties for the rth-moments and incomplete moments, respectively. Furthermore, obtaining the mode, median, moment generating function (mgf), expected value, and variance of the new model. Besides that, expansion the quantile functions. Finally, we will discuss the Factorial Moments Generating function, Characteristic function, Kurtosis, and the Skewness.

2.2 Mode

In this section, discussing the mode of the New Mixture distribution by finding the first derivative of the density function for a random variable X is equal to zero as follows:

According to the pdf of the New Mixture distribution (equation (1-22)),

$$f(x) = (2\Upsilon + \beta x + \alpha x^{\alpha-1}) e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)}, \quad x > 0$$

$$f'(x) = -(2\Upsilon + \beta x + \alpha x^{\alpha-1})^2 e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)} + e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)} (\beta + \alpha(\alpha-1)x^{\alpha-2}) = 0$$

$$f'(x) = [-(2\Upsilon + \beta x + \alpha x^{\alpha-1})^2 + (\beta + \alpha(\alpha-1)x^{\alpha-2})] e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)} = 0$$

We can show that

$$f'(x) = \left[-h(x)^2 + h'(x)\right] \cdot s(x) = 0$$

Where, h(x) is the hazard function was defined in equation (1-23), and s(x) is the survival function which is defined in equation (1-20). Then since $s(x) \neq 0$, we can divided the ends of equation on s(x) the result as follows:

$$\beta + \alpha(\alpha-1)x^{\alpha-2} = (2\Upsilon + \beta x + \alpha x^{(\alpha-1)})^2 \qquad (2\text{-}1)$$

There are many roots for the equation (2-1) as we motion in the shape of the density function in chapter one. So if we have $x = x_0$, that is a root of equation (2-1). This root depends on the second derivative of this equation. That mean if the second derivative of the root is less than zero then the point or the root is a local maximum. Or if the root is more than zero then, it is minimum point.

Finally, if the second derivative of the root is equal to zero then the point is inflection.

2.3 Median

Since, $F(x) = 1 - e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)}$, where $x > 0$

Then, to find the median of New Mixture distribution as follows:

Let , $F(x) = \frac{1}{2}$

$$1 - e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)} = \frac{1}{2}$$

$$e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)} = \frac{1}{2}$$

We can take the logarithm to both ends of the equation as follows:

$$-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right) = \ln\left(\frac{1}{2}\right)$$

$$2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha + \ln\left(\frac{1}{2}\right) = 0$$

This equation is polynomial of degree α, therefore the special case of it when $\alpha = 1$. Then,

$$(2\Upsilon + 1)x + \frac{\beta}{2}x^2 + \ln\left(\frac{1}{2}\right) = 0 \qquad (2\text{-}2)$$

Besides that, to compensate for the value of alpha equals one and solves the equation according to the Constitution. As a result, getting the following equation: $x = \frac{-(2\Upsilon+1) \mp \sqrt{(2\Upsilon+1)^2 - 2\beta\ln(1/2)}}{\beta}$

since $x > 0$ then, ignore the negative value of x.

2.4 Moments

Theorem 1:

The rth moment of New Mixture distribution about the origin is:

$$M_r' = E(x^r) = 2\,\Upsilon K(r,\Upsilon,\beta,\alpha) + \beta K(r+1,\Upsilon,\beta,\alpha) + \alpha K(r+\alpha-1,\Upsilon,\beta,\alpha).$$

Where :

$$K(r,\Upsilon,\beta,\alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-2\Upsilon)^n}{n!}\frac{(-\beta)^m}{2^m\, m!}\,\Gamma\left(\frac{r+n+2m+1}{\alpha}\right)$$

Proof:

$$E(x^r) = \int_0^{\infty} x^r f(x)dx = \int_0^{\infty} x^r(2\Upsilon + \beta x + \alpha x^{\alpha-1})e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)}dx$$

Let:

$$K(r,\Upsilon,\beta,\alpha) = \int_0^{\infty} x^r\, e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)}dx \qquad (2\text{-}3)$$

Expanding ($e^{-2\Upsilon x}$)and $e^{-\frac{\beta}{2}x^2}$) asTaylor serieses we get:

$$K(r,\Upsilon,\beta,\alpha) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-2\Upsilon)^n}{n!}\frac{(-\beta)^m}{2^m\, m!}\int_0^{\infty} x^{r+n+2m}\, e^{-x^{\alpha}}dx$$

Then, we will find $\int_0^{\infty} x^{r+n+2m}\, e^{-x^{\alpha}}dx$

Let $u = x^{r+n+2m+1}$,and

$$du = (r+n+2m+1)x^{r+n+2m}dx$$

$$dx = \frac{du}{(r+n+2m+1)\ \ x^{r+n+2m}} \qquad (2\text{-}4)$$

Also since $u = x^{r+n+2m+1}$, then we can get that:

$$u^{\left(\frac{\alpha}{r+n+2m+1}\right)} = x^{\alpha} \qquad (2\text{-}5)$$

From (2-4) and (2-5) we get,

$$\int_0^\infty x^{r+n+2m}\ e^{-x^{\alpha}}dx = \int_0^\infty x^{r+n+2m}\ e^{-u^{\frac{\alpha}{r+n+2m}+1}} \frac{du}{(r+n+2m+1)\ \ x^{r+n+2m}}$$

$$= \frac{1}{r+n+2m+1}\int_0^\infty e^{-u^{\frac{\alpha}{r+n+2m}+1}}\,du$$

$$= \frac{1}{r+n+2m+1}\frac{(r+n+2m+1)\Gamma\left(\frac{r+n+2m+1}{\alpha}, u^{\frac{\alpha}{r+n+2m+1}}\right)}{\alpha}$$

$$= \frac{\Gamma\left(\frac{r+n+2m+1}{\alpha}\right)}{\alpha}$$

This is a special integral Gamma function.

$$K(r,\Upsilon,\beta,\alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-2\Upsilon)^n}{n!}\frac{(-\beta)^m}{2^m\ m!}\ \Gamma\left(\frac{r+n+2m+1}{\alpha}\right)$$

$$K(r,\Upsilon,\beta,\alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\frac{(-2\Upsilon)^n}{n!}\ {}_1\psi_0\left[\left(\frac{r+n+1}{\alpha},\frac{2}{\alpha}\right),\frac{-\beta}{2}\right]$$

Where, ${}_1\psi_0$ is a special case (s=1, j=0) of the Wright generalized comparing to the equation of hypergeometric function which is defined in (Wright, 1935) [29] as :

$$ {}_s\psi_j \left[\begin{matrix}(\alpha_1, F_1), \dots, (\alpha_s, F_s) \\ (\beta_1, D_1), \dots, (\beta_j, D_j)\end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{m=1}^{s} \Gamma(\alpha_m + F_m n)\, x^n}{\prod_{m=1}^{j} \Gamma(\beta_m + D_m n)\, n!}. $$

Finally, the moment equation about origin take the form:

$$ M'_r = E(x^r) = 2\,\Upsilon K(r, \Upsilon, \beta, \alpha) + \beta K(r+1, \Upsilon, \beta, \alpha) + \alpha K(r+\alpha-1, \Upsilon, \beta, \alpha) \quad (2\text{-}6) $$

Where,

$$ K(r, \Upsilon, \beta, \alpha) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-2\Upsilon)^n}{n!} {}_1\psi_0 \left[\left(\frac{r+n+1}{\alpha}, \frac{2}{\alpha}\right), \frac{-\beta}{2}\right] $$

$$ K(r+1, \Upsilon, \beta, \alpha) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-2\Upsilon)^n}{n!} {}_1\psi_0 \left[\left(\frac{r+n+2}{\alpha}, \frac{2}{\alpha}\right), \frac{-\beta}{2}\right] $$

Note that, ignore the denominator of the fracture $\prod_{m=1}^{j} \Gamma(\beta_m + D_m n)\, n!$, since index (j=0), and since (s=1),

$$ \prod_{m=1}^{s} \Gamma(\alpha_m + F_m n)\, x^n = \prod_{m=1}^{1} \Gamma(\alpha_m + F_m n)\, x^n = \Gamma\left(\frac{r+n+2m+1}{\alpha}\right)\left(\frac{-\beta}{2}\right)^m $$

Then,

$$ K(r+\alpha-1, \Upsilon, \beta, \alpha) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-2\Upsilon)^n}{n!} {}_1\psi_0 \left[\left(\frac{r+n+\alpha}{\alpha}, \frac{2}{\alpha}\right), \frac{-\beta}{2}\right] $$

Then, the mean of the New Mixture distribution is:

$$ E(x) = 2\,\Upsilon K(1, \Upsilon, \beta, \alpha) + \beta K(2, \Upsilon, \beta, \alpha) + \alpha K(\alpha, \Upsilon, \beta, \alpha) $$

And, the second moment of the New Mixture distribution is:

$$ E(x^2) = 2\,\Upsilon K(2, \Upsilon, \beta, \alpha) + \beta K(3, \Upsilon, \beta, \alpha) + \alpha K(\alpha+1, \Upsilon, \beta, \alpha) $$

Then, the variance of the New Mixture distribution is:

$$\mathrm{Var}(x) = E(x^2) - [E(x)]^2$$
$$= 2\,\Upsilon K(2,\Upsilon,\beta,\alpha) + \beta K(3,\Upsilon,\beta,\alpha) + \alpha K(\alpha+1,\Upsilon,\beta,\alpha)$$
$$- \left[2\,\Upsilon K(1,\Upsilon,\beta,\alpha) + \beta K(2,\Upsilon,\beta,\alpha) + \alpha K(\alpha,\Upsilon,\beta,\alpha)\right]^2$$

Where,

$$K(2,\Upsilon,\beta,\alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-2\Upsilon)^n}{n!}\frac{(-\beta)^m}{2^m\,m!}\,\Gamma\left(\frac{n+2m+3}{\alpha}\right)$$

$$K(2,\Upsilon,\beta,\alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\frac{(-2\Upsilon)^n}{n!}\,{}_1\psi_0\left[\left(\frac{n+3}{\alpha},\frac{2}{\alpha}\right),\frac{-\beta}{2}\right]$$

Similarly, we can represent ,

$K(3,\Upsilon,\beta,\alpha)$, $K(\alpha+1,\Upsilon,\beta,\alpha)$, $K(\alpha,\Upsilon,\beta,\alpha)$, and $K(1,\Upsilon,\beta,\alpha)$ respectively.

2.5 <u>Incomplete moments</u>

Theorem 2:

In life time model one of the big important things is to know the rth incomplete moments, which is given by $T_r(z) = E(X^r|X \le z)$.

For getting the rth incomplete moments we define:

$$I(r,z) = I(r,z,\beta,\alpha) = \int_0^z x^r\, e^{-x^{\alpha}}\, dx \qquad (2\text{-}7)$$

From equation (1-20) that, $T_r(z) = \int_0^z x^r\, f(x) dx$

We can represent as:

$$T_r(z) = \int_0^z x^r \,(2\Upsilon + \beta x + \alpha x^{\alpha-1}) e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)} \, dx$$

$$T_r(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-2\Upsilon)^n}{n!}\frac{(-\beta)^m}{2^m\, m!} \int_0^z x^{r+n+2m}\, e^{-x^\alpha}\, dx$$

$$T_r(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left(\left(\frac{(-2\Upsilon)^n}{n!}\right)\left(\frac{(-\beta)^m}{2^m\, m!}\right)[2\Upsilon I(r+n+2m, z) + \beta I(r+n+2+1, z)]\right) \tag{2-8}$$

Where, $I(r+n+2m, z) = \int_0^z x^{r+n+2m}\, e^{-x^\alpha}\, dx$

By using the Meijer function $G_{s,j}^{m,n}$ which is define in (Prudnikov, Brychkov and Marichev, 1986) [23], as in equation $(23.24.2.2)[, \mathrm{p.}\, 348$ vols 3],as :

$$\int_0^z x^{\alpha-1}(a-x)^{\beta-1}\, G_{s\,j}^{m\,n}\left(\omega x^{\frac{l}{c}} \mid \begin{matrix} a_s \\ b_j \end{matrix}\right) dx$$
$$= \frac{c^{\mu}\, l^{-\beta}\Gamma(\beta)}{(2\pi)^{w*(c-1)} a^{1-(L\alpha-\beta)}} G_{cs+l\; cj+l}^{cm,c\; n+l}\left(\frac{\omega^c a^l}{c^c (j-s)} \mid \begin{matrix} \Delta(l, 1-\alpha), \Delta(c, a_s) \\ \Delta(c, b_j), \Delta(l, 1-\alpha-\beta) \end{matrix}\right) \tag{2-9}$$

Where, $\mu = (\sum_{k=1}^{j} b_k - \sum_{k=1}^{s} a_k + \frac{s-j}{2}) + 1 \; and \;, a_k - b_k \neq 1,2, \dots$

a_k, b_k are complex parameters, and c, l are co − prime numbers,

$\alpha = {}^{s}/_{j}$, s and j are co − prime numbers $belongs\ to\ N$, $w^* = m + n - \frac{s+j}{2}$.

Here, we can find the value of integration of equation (2-9) , as follows:

$$\exp\left(-g(x)\right) = G_{0,1}^{1,0}\left(g(x)\middle|_{0}^{-}\right) \tag{2-10}$$

from equation (2-3) we get:

$$I(r,z) = \int_{0}^{z} x^{r}\, G_{0,1}^{1,0}\left(x^{s/j}\middle|_{0}^{-}\right) dx \tag{2-11}$$

Where, $\alpha = {}^{s}/_{j}$, s and j are co- prime numbers belongs to N (Jeffrey and Zwillinger, 2007) [12], and $\beta = 1, m = 1,$

$n = 0, s = 0, j = 1.$

Then, $w^* = 1 + 0 - \frac{1}{2} \quad \Rrightarrow \quad w^* = \frac{1}{2}$.

Also, $\alpha - 1 = r \quad \Rrightarrow \ \alpha = r + 1$

Since, $\mu = (\sum_{k=1}^{j} b_k - \sum_{k=1}^{s} a_k + \frac{s-j}{2}) + 1$, then, from equation (2-11) $b_k = 0$.

Then, $\mu = \left(\sum_{k=1}^{1} b_k - \sum_{k=1}^{0} a_k + \frac{0-1}{2}\right) + 1 \quad \Rrightarrow \mu = 1.$

And, $\omega = 1, L = s$, c=j and $\Delta(s,-j) = \frac{-j}{s}, \frac{1-j}{s}, \frac{2-j}{s}, \ldots\ldots.. \frac{(s-1)-j}{s}$.

Then, $\Delta(s,-j-1) = \frac{-j-1}{s}, \frac{1-j-1}{s}, \frac{2-j-1}{s}, \ldots\ldots, \frac{(s-1)-j-1}{s}.$

$$= \frac{-j-1}{s}, \frac{-j}{s}, \frac{1-j}{s}, \ldots\ldots\ldots \frac{s-j-2}{s}.$$

According to equation (2-11),

(2-12)

$$\int_0^z x^r \, G_{0,1}^{1,0}\left(x^{\frac{s}{j}} |_0^-\right) dx$$

$$= \left[\frac{j^1 s^{-1}(1)}{(2\pi)^{(j-1)/2} \, a^{1-s(r+1)-1}} \, G_{cs+l \, cj+l}^{cm,c \, n+l}\left(\frac{\omega^c a^l}{c^c(j-s)} \,\middle|\, \begin{matrix} \Delta(l,1-\alpha), \Delta(c,a_s) \\ \Delta(c,b_j), \Delta(l,1-\alpha-\beta) \end{matrix}\right)\right]$$

So, we can represent $\mathrm{I(r,z)}$ as (Prudnikov, Brychkov and Marichev, 1986) [23].

$$\mathrm{I(r,z)} = \frac{jz^{s(r+1)}}{s(2\pi)^{(j-1)/2}} G_{s,s+j}^{j,s}\left(\frac{z^s}{j^j} \,\middle|\, \begin{matrix} \frac{-r}{s}, \frac{1-r}{s}, \dots, \frac{s-r-1}{s}, - \\ 0, \frac{-r-1}{s}, \frac{r}{s}, \dots, \frac{p-r-2}{s} \end{matrix}\right) \qquad \text{(2-13)}$$

Since, $\Delta(l, 1-\alpha) = \Delta(s, 1-(r+1)) = \Delta(s, -r) = \frac{-r}{s}, \frac{1-r}{s}, \dots, \frac{s-r-1}{s}$,

$\Delta(c, a_s) = \Delta(j, a_0)$, No value of a_0 because the summation of a start from one to s. also, $\Delta(c, b_j) = \Delta(j, b_j) = \Delta(1, b_1) = 0$, since b_1 is equal to zero from comparing the two equations (2-11) with (2-9)

$\Delta(l, 1-\alpha-\beta) = \Delta(s, 1-(r+1)-1) =$

$$\Delta(s, -r-1) = \frac{-r-1}{s}, \frac{1-r-1}{s}, \frac{2-r-1}{s}, \dots\dots., \frac{(s-1)-r-1}{s}.$$

$\Delta(s, -r-1) = \frac{-r-1}{s}, \frac{-r}{s}, \frac{1-r}{s}, \dots\dots., \frac{s-r-2}{s}$.

2.6 Moments generating function

Theorem 3:

The (mgf) of X, $M(t) = E(e^{tx})$, it is given by:

$$M(t) = 2\Upsilon K(2\Upsilon - t, \beta, \alpha) + \beta K(1, 2\Upsilon - t, \beta, \alpha) + \alpha K(\alpha, 2\Upsilon - t, \beta, \alpha).$$

Where :

$$K(2\Upsilon - t, \beta, \alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-(2\Upsilon - t))^n}{n!}\frac{(-\beta)^m}{2^m\, m!}\,\Gamma\left(\frac{n+2m+1}{\alpha}\right),$$

Proof:

$$M(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx$$

$$= \int_0^{\infty} e^{tx}(2\Upsilon + \beta x + \alpha x^{\alpha-1}) e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^{\alpha}\right)} dx$$

$$\int_0^{\infty} (2\Upsilon + \beta x + \alpha x^{\alpha-1})\, e^{-\left[(2\Upsilon - t)x + \frac{\beta}{2}x^2 + x^{\alpha}\right]} dx$$

Let:

$$K(2\Upsilon - t, \beta, \alpha) = \int_0^{\infty} e^{-\left[(2\Upsilon - t)x + \frac{\beta}{2}x^2 + x^{\alpha}\right]}\, dx \qquad (2\text{-}14)$$

$$K(2\Upsilon - t, \beta, \alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-(2\Upsilon - t))^n}{n!}\frac{(-\beta)^m}{2^m\, m!}\,\Gamma\left(\frac{n+2m+1}{\alpha}\right).$$

$$M(t) = 2\Upsilon K(2\Upsilon - t, \beta, \alpha) + \beta K(1, 2\Upsilon - t, \beta, \alpha) + \alpha K(\alpha, 2\Upsilon - t, \beta, \alpha).$$

Where,

$$K(\,2\Upsilon-t,\beta,\alpha)=\frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-(2\Upsilon-t))^{n}}{n!}\frac{(-\beta)^{m}}{2^{m}\,m!}\,\Gamma\left(\frac{n+2m+1}{\alpha}\right),$$

$$K(1,2\Upsilon-t,\beta,\alpha)=\frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-(2\Upsilon-t))^{n}}{n!}\frac{(-\beta)^{m}}{2^{m}\,m!}\,\Gamma\left(\frac{n+2m+2}{\alpha}\right),$$

And,

$$K(\alpha,2\Upsilon-t,\beta,\alpha)=\frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-(2\Upsilon-t))^{n}}{n!}\frac{(-\beta)^{m}}{2^{m}\,m!}\,\Gamma\left(\frac{\alpha+n+2m+1}{\alpha}\right).$$

2.7 Factorial moments generating function

Theorem 4:

The factorial Moments Generating function of X denoted by:

$M_x(t)=2\Upsilon K(2\Upsilon-\ln t,\beta,\alpha)+\beta K(t,2\Upsilon-\ln t,\beta,\alpha)+\alpha K(t+\alpha,2\Upsilon-\ln t,\beta,\alpha)$.

Where:

$$K(2\Upsilon-\ln t\,,\beta,\alpha)=\frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-(2\Upsilon-\ln t))^{n}}{n!}\frac{(-\beta)^{m}}{2^{m}\,m!}\,\Gamma\left(\frac{n+2m+1}{\alpha}\right).$$

Proof:

$$M_x(t)=E(t^{x})=\int_0^{\infty}t^{x}f(x)dx=\int_0^{\infty}e^{\ln t^{x}}f(x)dx=\int_0^{\infty}e^{x\ln t}f(x)dx$$

$$= \int_0^\infty e^{x \ln t}(2\Upsilon + \beta x + \alpha x^{\alpha-1}) e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)} dx$$

$$= \int_0^\infty (2\Upsilon + \beta x + \alpha x^{\alpha-1}) e^{-\left[(2\Upsilon - \ln t)x + \frac{\beta}{2}x^2 + x^\alpha\right]} dx$$

Let:

$$K(2\Upsilon - \ln t, \beta, \alpha) = \int_0^\infty e^{-\left[(2\Upsilon - ln t)x + \frac{\beta}{2}x^2 + x^\alpha\right]} dx$$

$$K(2\Upsilon - ln\, t, \beta, \alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-(2\Upsilon - lnt))^n}{n!} \frac{(-\beta)^m}{2^m\, m!}\, \Gamma\left(\frac{n+2m+1}{\alpha}\right).$$

That yield,

$M_x(t)=2\Upsilon\, K(2\Upsilon - \ln t, \beta,\alpha)+\beta\, K(1, 2\Upsilon - \ln t, \beta,\alpha)+\alpha\, K(\alpha, 2\Upsilon - \ln t, \beta,\alpha)$.

Where,

$$K(2\Upsilon - \ln t, \beta, \alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-(2\Upsilon - lnt))^n}{n!} \frac{(-\beta)^m}{2^m\, m!}\, \Gamma\left(\frac{n+2m+1}{\alpha}\right),$$

$$K(1, 2\Upsilon - \ln t, \beta, \alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-(2\Upsilon - lnt))^n}{n!} \frac{(-\beta)^m}{2^m\, m!}\, \Gamma\left(\frac{n+2m+2}{\alpha}\right),$$

$$K(\alpha, 2\Upsilon - \ln t, \beta, \alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-(2\Upsilon - \ln t))^n}{n!} \frac{(-\beta)^m}{2^m\, m!}\, \Gamma\left(\frac{\alpha+n+2m+1}{\alpha}\right)$$

2.8 Quantile function

The quantile function is denoted by $x = Q(v) = F^{-1}(v)$, as in(Abramowitz and Stegun, 1965) [1] and (Cordeiro, Ortega and Lemonte, 2014) [6]:

$$F(x) = 1 - e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)}$$

$$v = 1 - e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)}$$

$$1 - v = e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)}$$

this yield,

$$\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right) = -\ln(1 - v) \qquad (2\text{-}15)$$

let $z = \left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)$, by Taylar seires represention , we have:

$$x^\alpha = \sum_{i=0}^{\infty} c_i x^i \text{ where,} \quad c_i = \sum_{n=0}^{\infty} \frac{(-1)^{n-i}\binom{n}{i}(\alpha)_n}{n!}$$

and, $(\alpha)_n = \alpha(\alpha - 1)\ldots\ldots(\alpha - n + 1)$is the descending factorial.

Let, $z = \sum_{i=0}^{\infty} \phi_i x^i$, such that:

$$\phi_\circ = c_\circ, \phi_1 = c_1 + 2?, \phi_2 = c_2 + \frac{\beta}{2}, \phi_3 = c_3$$

Therefore, $\phi_i = c_i$ for $i \geq 3$.

If $\phi_1 \neq 0$, we can invert the last series and acquire), as in(Abramowitz and Stegun, 1965) [1], then we get, $x = Q(v) = \sum_{i=1}^{\infty} s_i z^i$, where $s_1 = {\phi_1}^{-1}$, $s_2 = -\phi_2{\phi_1}^{-3}$, $s_3 = (2{\phi_2}^2 - \phi_1\phi_3){\phi_1}^{-5}$, and so on it is inverse interpolation by reversion of series .

Also, in special case if $\alpha = 1$, then from equation(2-15), we compensate for the value of alpha equals one and solve the equation according to the constitution. As a result, getting the following equation:

$$x = \frac{-(2\Upsilon + 1) \mp \sqrt{(2\Upsilon + 1)^2 - 2\beta \ln(1 - v)}}{\beta} \tag{2-16}$$

Because (x) is positive, negative values resulting from this generation are ignored.

2.9 Skewness

The skewness of New Mixture distribution is dependent on moments denoted by:

$$C.S = \frac{M_3}{(M_2)^{\frac{3}{2}}}$$

Where,

$$M_3 = E(x^3) = 2\,\Upsilon K(3, \Upsilon, \beta, \alpha) + \beta K(4, \Upsilon, \beta, \alpha) + \alpha K(\alpha + 2, \Upsilon, \beta, \alpha).$$

$$K(3, \Upsilon, \beta, \alpha) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-2\Upsilon)^n}{n!} \frac{(-\beta)^m}{2^m\, m!}\, \Gamma\left(\frac{4 + n + 2m}{\alpha}\right),$$

$$K(4, \Upsilon, \beta, \alpha) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-2\Upsilon)^n}{n!} \frac{(-\beta)^m}{2^m\, m!}\, \Gamma\left(\frac{5 + n + 2m}{\alpha}\right),$$

$$K(\alpha + 2, \Upsilon, \beta, \alpha) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-2\Upsilon)^n}{n!} \frac{(-\beta)^m}{2^m\, m!}\, \Gamma\left(\frac{\alpha + 3 + n + 2m}{\alpha}\right).$$

Also,

$$(M_2)^{\frac{3}{2}} = [E(x^2)]^{\frac{3}{2}} = [2\,\Upsilon K(2, \Upsilon, \beta, \alpha) + \beta K(3, \Upsilon, \beta, \alpha) + \alpha K(\alpha + 1, \Upsilon, \beta, \alpha)]^{\frac{3}{2}}.$$

Where,

$$K(\alpha+1,\Upsilon,\beta,\alpha)=\frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-2\Upsilon)^n}{n!}\frac{(-\beta)^m}{2^m\,m!}\,\Gamma\left(\frac{\alpha+2+n+2m}{\alpha}\right).$$

2.10 Kurtosis

The kurtosis of New Mixture distribution denoted by:

$$C.K=\frac{M_4}{(M_2)^2}-3$$

Where,

$$M_4=E(x^4)=2\,\Upsilon K(4,\Upsilon,\beta,\alpha)+\beta K(5,\Upsilon,\beta,\alpha)+\alpha K(\alpha+3,\Upsilon,\beta,\alpha).$$

$$[M_2]^2=[E(x^2)]^2=[2\,\Upsilon K(2,\Upsilon,\beta,\alpha)+\beta K(3,\Upsilon,\beta,\alpha)+\alpha K(\alpha+1,\Upsilon,\beta,\alpha)]^2.$$

2.11 Characteristic function

Theorem 5:

The characteristic function of New Mixture distribution is: $\varphi_x(it)=E(e^{itx})=\int_0^\infty e^{itx}f(x)dx$

$$\int_0^\infty e^{itx}(2\Upsilon+\beta x+\alpha x^{\alpha-1})e^{-\left(2\Upsilon x+\frac{\beta}{2}x^2+x^\alpha\right)}\,dx$$

$$\int_0^\infty (2\Upsilon+\beta x+\alpha x^{\alpha-1})\,e^{-\left[(2\Upsilon-it)x+\frac{\beta}{2}x^2+x^\alpha\right]}dx$$

$$K(2\Upsilon-it,\beta,\alpha)=\int_0^\infty e^{-\left[(2\Upsilon-it)x+\frac{\beta}{2}x^2+x^\alpha\right]}dx$$

$$K(2\Upsilon-it,\beta,\alpha)=\frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-(2\Upsilon-it))^n}{n!}\frac{(-\beta)^m}{2^m\,m!}\,\Gamma\left(\frac{n+2m+1}{\alpha}\right).$$

$$\varphi_x(it) = 2\Upsilon K(2\Upsilon - i\,t, \beta, \alpha) + \beta K(1, 2\Upsilon - i\,t, \beta, \alpha) + \alpha K(\alpha, 2\Upsilon - i\,t, \beta, \alpha).$$

Where,

$$K(1, 2\Upsilon - i\,t, \beta, \alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-(2\Upsilon - i\,t))^n}{n!}\frac{(-\beta)^m}{2^m\, m!}\,\Gamma\left(\frac{n + 2m + 2}{\alpha}\right),$$

$$K(\alpha, 2\Upsilon - i\,t, \beta, \alpha) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-(2\Upsilon - i\,t))^n}{n!}\frac{(-\beta)^m}{2^m\, m!}\,\Gamma\left(\frac{\alpha + n + 2m + 1}{\alpha}\right).$$

CHAPTER THREE

CLASSICAL METHODS ESTIMATORES AND INFORMATION CRITERION

3.1 Introduction

This chapter represents the estimation methods of three parameters of New Mixture distribution using the Classical methods which are Maximum Likelihood estimation, Ordinary Least Square estimation, and Rank Sampling set estimation methods. Moreover, use Akaike information criterion, corrected Akaike information criterion and Bayesian information criterion to compare the New Mixture distribution with other distributions. Real data applications are used for strengthening the results.

3.2 Classical Estimation Methods

Three parameters of New Mixture distribution are estimated by three classical estimation methods which are as follows:

3.2.1 Maximum Likelihood Estimation

The maximum likelihood estimation method is considering as a popular of classical methods which is used to estimate the parameters for any distribution. In one way or another, it copes with different type of samples (uncensored or censored) (White, 1982) [28]. Fisher is the first who apply this method through his many researches. This method aims to make the potential function of random variables as great as possible. Moreover, it is often used to estimate parameters distribution because it has good characteristics as efficient capabilities and less contrast property, as well as a very important feature property which is invariant property. Now, applying this method in case of estimating parameters of the New Mixture distribution as follows:

Let $(x_1, x_2, \dots, x_n)$ be samples of size (n) with a known probability density function $f(x, \Upsilon, \beta, \alpha)$, the likelihood function defined as the common joint probability density function distribution of the data and can be written as follows:

$$L(x_i, \Upsilon, \beta, \alpha) = \prod_{i=1}^{n} f(x_i, \Upsilon, \beta, \alpha) \quad (3\text{-}1)$$

$$L(x_1, x_2, \ldots . x_n, \Upsilon, \beta, \alpha) = \prod_{i=1}^{n} f(x_i, \Upsilon, \beta, \alpha)$$

$$L(\Upsilon, \beta, \alpha, x_i) = \prod_{i=1}^{n} [(2\Upsilon + \beta x_i + \alpha x_i^{\alpha-1})\, e^{-\left(2\Upsilon x_i + \frac{\beta}{2} x_i^2 + x_i^{\alpha}\right)}$$

The log-likelihood function is:

$$\text{InL}(\Upsilon, \beta, \alpha, x_i) = \sum_{i=1}^{n} \ln(2\Upsilon + \beta x_i + \alpha {x_i}^{\alpha-1}) - \sum_{i=1}^{n} \left(2\Upsilon x_i + \frac{\beta}{2} {x_i}^2 + x_i^{\alpha}\right)$$

For the log-likelihood function, the partial derivatives with respect unknown parameters $(\Upsilon, \beta, \alpha)$ as follows:

Assume that,

$$g(\Upsilon) = \frac{\partial InL}{\partial \Upsilon} = (2)[\sum_{i=1}^{n} (2\Upsilon + \beta x_i + \alpha {x_i}^{\alpha-1})^{-1} - \sum_{i=1}^{n} x_i] \quad (3\text{-}2)$$

$$w(\beta) = \frac{\partial InL}{\partial \beta} = \sum_{i=1}^{n} (2\Upsilon + \beta x_i + \alpha {x_i}^{\alpha-1})^{-1} (x_i) - \frac{1}{2} \sum_{i=1}^{n} {x_i}^2 \quad (3\text{-}3)$$

$$d(\alpha) = \frac{\partial InL}{\partial \alpha} = \sum_{i=1}^{n} (2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-1} x_i{}^{\alpha-1}[\alpha \, ln(x_i) + 1] - \sum_{i=1}^{n} x_i{}^{\alpha} \, ln(x_i) \tag{3-4}$$

When $\frac{\partial \mathrm{InL}}{\partial \Upsilon} = \frac{\partial \mathrm{InL}}{\partial \beta} = \frac{\partial \mathrm{InL}}{\partial \alpha} = 0$, there is no closed solution of the equations $(3-2), (3-3), (3-4)$. Therefore, numerical technique (Newton –Raphson method) should be applied to solve these equations as follows:

$$\begin{bmatrix} \hat{\Upsilon}_{i+1} \\ \hat{\beta}_{i+1} \\ \hat{\alpha}_{i+1} \end{bmatrix} = \begin{bmatrix} \hat{\Upsilon}_i \\ \hat{\beta}_i \\ \hat{\alpha} \end{bmatrix} - \mathrm{J}^{-1} \begin{bmatrix} g(\Upsilon) \\ \mathrm{w}(\beta) \\ \mathrm{d}(\alpha) \end{bmatrix} \tag{3-5}$$

Where:

$$\mathrm{J} = \begin{bmatrix} \frac{\partial \, g(\Upsilon)}{\partial \Upsilon} & \frac{\partial \, g(\Upsilon)}{\partial \beta} & \frac{\partial \, g(\Upsilon)}{\partial \alpha} \\ \frac{\partial \mathrm{w}(\beta)}{\partial \Upsilon} & \frac{\partial \mathrm{w}(\beta)}{\partial \beta} & \frac{\partial \mathrm{w}(\beta)}{\partial \alpha} \\ \frac{\partial \mathrm{d}(\alpha)}{\partial \Upsilon} & \frac{\partial \mathrm{d}(\alpha)}{\partial \beta} & \frac{\partial \mathrm{d}(\alpha)}{\partial \alpha} \end{bmatrix}$$

It is the Jacobean matrix.

$$\frac{\partial \, g(\Upsilon)}{\partial \Upsilon} = (-4) \sum_{i=1}^{n} (2\Upsilon + \beta \mathrm{x_i} + \alpha x_i{}^{\alpha-1})^{-2} \tag{3-6}$$

$$\frac{\partial \, g(\Upsilon)}{\partial \beta} = (-2) \sum_{i=1}^{n} (2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-2} (x_i) \tag{3-7}$$

$$\frac{\partial \, g(\Upsilon)}{\partial \alpha} = (-2) \sum_{i=1}^{n} (2\Upsilon + \beta \mathrm{x_i} + \alpha x_i{}^{\alpha-1})^{-2} x_i{}^{\alpha-1} [\alpha \ln(x_i) + 1] \tag{3-8}$$

$$\frac{\partial w(\beta)}{\partial \beta} = (-1)\sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-2}(x_i{}^2) \tag{3-9}$$

$$\frac{\partial w(\beta)}{\partial \alpha} = (-1)\sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-2}\ x_i{}^{\alpha}[\alpha \ln(x_i) + 1] \tag{3-10}$$

$$\frac{\partial d(\alpha)}{\partial \alpha} = \sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-1}\ x_i{}^{\alpha-1}\ln(x_i)[\alpha \ln(x_i) + 2]\,(-\sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-2}\ x_i{}^{2(\alpha-1)}[\alpha \ln(x_i) + 1]^2 - \sum_{i=1}^{n} x_i^{\alpha}\big(\ln(x_i)\big)^2 \tag{3-11}$$

Since the Jacobean matrix must be a non –singular symmetric matrix,

Then, $\frac{\partial g(\Upsilon)}{\partial \beta} = \frac{\partial w(\beta)}{\partial \Upsilon}, \frac{\partial w(\beta)}{\partial \alpha} = \frac{\partial d(\alpha)}{\partial \beta}, \frac{\partial g(\Upsilon)}{\partial \alpha} = \frac{\partial d(\alpha)}{\partial \Upsilon}$.

Windup the Newton- Raphson method, applying the following equation :

$$\begin{bmatrix} \varepsilon_{i+1}(?) \\ \varepsilon_{i+1}(\beta) \\ \varepsilon_{i+1}(\alpha) \end{bmatrix} = \begin{bmatrix} ?_{i+1} \\ \beta_{i+1} \\ \alpha_{i+1} \end{bmatrix} - \begin{bmatrix} ?_{i} \\ \beta_{i} \\ \alpha_{i} \end{bmatrix} \tag{3-12}$$

Where, ε(ϒ), ε(β), ε(α) are the error term, which are assumed and they are very small.

3.2.2 Ordinary Least Square Estimation

The Ordinary Least Square Estimation method is the second estimation procedure which is one of the most popular techniques in estimating the parameters when the equation of the model is linear or non- linear variables. The first decade in the nineteenth century this method was published (Dismuke and Lindrooth, 2006) [7]. It is used in many phenomena's as economics, medical, engineering and mathematical problems. This method contains many properties one of these properties the estimator has linearity property, unbiased property, minimum variance property. Finally, the estimator has consistent property and it is an asymptotically normal distribution. The main goal of this method is to minimize the summation of squared differences between the observed sample values (Y_i) and the expected values $E(Y_i)$ by the formula:

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} [Y_i - E(Y_i)]^2 \tag{3-13}$$

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 x_i - \beta_2 {x_i}^2 - \beta_\alpha {x_i}^\alpha)^2$$

Taking the CDF of three parameters New Mixture distribution:

$$F(x_i) = 1 - e^{-\left(2\Upsilon x_i + \frac{\beta}{2} {x_i}^2 + {x_i}^\alpha\right)}, \ x > 0 \tag{3-14}$$

$$1 - F(x_i) = e^{-\left(2\Upsilon x_i + \frac{\beta}{2} {x_i}^2 + {x_i}^\alpha\right)} \tag{3-15}$$

By taking the logarithm of the two parts of the equation (3-38) as follows:

$$\ln\big(1 - F(x_i)\big) = -2\Upsilon x_i - \frac{\beta}{2} {x_i}^2 - {x_i}^\alpha \tag{3-16}$$

Now, comparing the equation (3-16) to the equation of multiple model getting:

$$Y_i = \ln\big(1 - \mathrm{F}(\mathrm{x_i})\big), \beta_0 = 0, \beta_1 = -2\Upsilon, \beta_2 = -\frac{\beta}{2}, \beta_\alpha = -1$$

So, $$\epsilon_i = (Y_i - \beta_0 - \beta_1 x_i - \beta_2 {x_i}^2 - \beta_\alpha {x_i}^\alpha) \qquad (3\text{-}17)$$

Then, $$\epsilon_i = \ln\big(1 - \mathrm{F}(\mathrm{x_i})\big) + 2\Upsilon \mathrm{x}_i + \tfrac{\beta}{2}{\mathrm{x}_i}^2 + {\mathrm{x}_i}^\alpha \qquad (3\text{-}18)$$

The sum of the equation$(3-18)$, then taking squared it to get:

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left[\ln\big(1 - \mathrm{F}(\mathrm{x_i})\big) + 2\Upsilon \mathrm{x_i} + \frac{\beta}{2}{\mathrm{x_i}}^2 + {\mathrm{x_i}}^\alpha\right]^2 \qquad (3\text{-}19)$$

By deriving the equation $(3-19)$ for the unknown parameters $(\Upsilon, \beta, \alpha)$, we obtained three functions as follows:

$$\frac{\partial \sum_{i=1}^{n} {\epsilon_i}^2}{\partial \Upsilon} = s(\Upsilon) = (4)\sum_{i=1}^{n} [\ln\big(1 - \mathrm{F}(\mathrm{x}_i)\big) + 2\Upsilon \mathrm{x}_i + \frac{\beta}{2}{\mathrm{x}_i}^2 + {\mathrm{x}_i}^\alpha]\,(\mathrm{x}_i) \qquad (3\text{-}20)$$

$$\frac{\partial \sum_{i=1}^{n} {\epsilon_i}^2}{\partial \beta} = q(\beta) = \sum_{i=1}^{n} [\ln\big(1 - \mathrm{F}(\mathrm{x}_i)\big) + 2\Upsilon \mathrm{x}_i + \frac{\beta}{2}{\mathrm{x}_i}^2 + {\mathrm{x}_i}^\alpha]\,({\mathrm{x}_i}^2) \qquad (3\text{-}21)$$

$$\frac{\partial \sum_{i=1}^{n} {\epsilon_i}^2}{\partial \alpha} = j(\alpha)$$
$$= (2)\sum_{i=1}^{n} [\ln\big(1 - \mathrm{F}(\mathrm{x}_i)\big) + 2\Upsilon \mathrm{x}_i + \frac{\beta}{2}{\mathrm{x}_i}^2 + {\mathrm{x}_i}^\alpha]\,({\mathrm{x}_i}^\alpha \ln(\mathrm{x}_i)) \qquad (3\text{-}22)$$

$$= (2)\sum_{i=1}^{n}[\ln\left(1 - \mathrm{F}(\mathrm{x}_i)\right) + 2\Upsilon \mathrm{x}_i + \frac{\beta}{2}\mathrm{x}_i{}^2 + \mathrm{x}_i{}^{\alpha}]\,(\mathrm{x}_i{}^{\alpha}\ln(\mathrm{x}_i))$$

Where, $\mathrm{F}(\mathrm{x}_i)$ is the cumulative function of New Mixture distribution. The functions $(3-20), (3-21) and\ (3-22)$, are system of non-linear equations. It is possible to solve it simultaneously, so, resolving it numerically using the Newton – Raphson method.

The Jacobean matrix is contain the derivative of the first degree for each functions of the functions $\mathrm{s}(\Upsilon), \mathrm{q}(\beta), \mathrm{j}(\alpha)$ with respect to the unknown parameters $(\Upsilon, \beta, \alpha)$.

$$\mathrm{J} = \begin{bmatrix} \frac{\partial\, s(\Upsilon)}{\partial\Upsilon} & \frac{\partial\, s(\Upsilon)}{\partial\beta} & \frac{\partial\, s(\Upsilon)}{\partial\alpha} \\ \frac{\partial \mathrm{q}(\beta)}{\partial\Upsilon} & \frac{\partial \mathrm{q}(\beta)}{\partial\beta} & \frac{\partial \mathrm{q}(\beta)}{\partial\alpha} \\ \frac{\partial \mathrm{j}(\alpha)}{\partial\Upsilon} & \frac{\partial \mathrm{j}(\alpha)}{\partial\beta} & \frac{\partial \mathrm{j}(\alpha)}{\partial\alpha} \end{bmatrix}$$

Where,

$$\frac{\partial\, s(\Upsilon)}{\partial\Upsilon} = (8)\sum_{i=1}^{n} x_i{}^2 \qquad (3\text{-}23)$$

$$\frac{\partial\, s(\Upsilon)}{\partial\beta} = (2)\sum_{i=1}^{n} x_i{}^3 \qquad (3\text{-}24)$$

$$\frac{\partial\, s(\Upsilon)}{\partial\alpha} = (4)\sum_{i=1}^{n} x_i{}^{\alpha+1}\ln(x_i) \qquad (3\text{-}25)$$

$$\frac{\partial\, q(\beta)}{\partial \beta} = \left({}^{1}\!/_{2}\right)\sum_{i=1}^{n} {x_i}^{4} \qquad (3\text{-}26)$$

$$\frac{\partial\, q(\beta)}{\partial \alpha} = \sum_{i=1}^{n} {x_i}^{\alpha+2} \ln(x_i) \qquad (3\text{-}27)$$

$$\begin{aligned}\frac{\partial\, j(\alpha)}{\partial \alpha} = (2)\sum_{i=1}^{n} &[\mathrm{Ln}\big(1 - \mathrm{F}(\mathrm{x}_i)\big)\, \mathrm{x}_i^{\alpha}\, (\ln(\mathrm{x}_i))^2 \\ &+ 2\Upsilon {\mathrm{x}_i}^{\alpha+1} \ln(\mathrm{x}_i) + \frac{\beta}{2}{\mathrm{x}_i}^{\alpha+2} \ln(\mathrm{x}_i) \\ &+ 2{\mathrm{x}_i}^{2\alpha}(\ln(\mathrm{x}_i))^2]\end{aligned} \qquad (3\text{-}28)$$

The Jacobean matrix is symmetric and non- singular matrix, since it is depending upon the first derivatives.

That means: $\frac{\partial s(\Upsilon)}{\partial \beta} = \frac{\partial q(\beta)}{\partial \Upsilon}, \frac{\partial q(\beta)}{\partial \alpha} = \frac{\partial j(\alpha)}{\partial \beta}, \frac{\partial s(\Upsilon)}{\partial \alpha} = \frac{\partial j(\alpha)}{\partial \Upsilon}$.

By applying the equation$(3-28)$, then obtain the estimator of the three parameters of New Mixture distribution.

Windup the Newton- Raphson method, applying the following equation :

$$\begin{bmatrix}\varepsilon_{i+1}(\Upsilon)\\ \varepsilon_{i+1}(\beta)\\ \varepsilon_{i+1}(\alpha)\end{bmatrix} = \begin{bmatrix}\Upsilon_{i+1}\\ \beta_{i+1}\\ \alpha_{i+1}\end{bmatrix} - \begin{bmatrix}\Upsilon_{i}\\ \beta_{i}\\ \alpha_{i}\end{bmatrix} \qquad (3\text{-}29)$$

Where, ε(ϒ), ε(β), ε(α) are the error term, which are assumed and they are very small.

3.2.3 Rank Set Sampling Estimation

One of the important topics in statistics is Estimator parameters. It have taken a large place in statistical studies. One of the most important ways to estimate the parameters is the rank set sampling method(Patil, Sinha and Taillie, 1994)[21]. The development of the rank set sampling method allowed the elements of the samples to be more structured, which led to the use of the method more than others because of the possibility of measuring the actual sample, especially in cases that are difficult to do as (time-consuming tax, costly, and destructive).The rank set sampling method was first introduced to estimate mean pasture by Mcintyre in 1952. Moreover, Halls-dell 1966 proved that rank set sampling compared to simple random sampling is more efficient for estimating the mean of the population. Dell 1972 introduced that although there is a ranking error the rank set sampling in terms of efficiency is better than the simple random sampling in estimating the parameters for logistic distribution. In term of estimation the New Mixture distribution parameters using the rank set sampling method the procedure will be as follows:

The probability density function (p.d.f) of the New Mixture distribution which obtained by increasing ordering random sampling $(X_1, X_2, X_3, \dots \dots \dots X_n)$ is:

$$f(x_i) = \frac{n!}{(i-1)!\,(n-i)!}[F(x_i)]^{i-1}[1-F(x_i)]^{n-i}f(x_i) \qquad (3\text{-}30)$$

$$f(x_i) = \frac{n!}{(i-1)!\,(n-i)!}\left[1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)}\right]^{i-1} \qquad (3\text{-}31)$$

$$\left[1-1+e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^\alpha\right)}\right]^{n-i}\left[(2\Upsilon+\beta x+\alpha x^{\alpha-1})e^{-\left(2\Upsilon x+\frac{\beta}{2}x^2+x^\alpha\right)}\right]$$

Let $M=\frac{n!}{(i-1)!(n-i)!}$ then,

$$f(x_i)=M\left[1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^\alpha\right)}\right]^{i-1}\left[e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^\alpha\right)}\right]^{n-i+1}(2\Upsilon+\beta x+\alpha x^{\alpha-1}) \quad (3\text{-}32)$$

The likelihood function of order samples $(X_{(1)}, X_{(2)}, \dots \dots \dots X_{(n)})$ is:

$$L(\Upsilon,\beta,\alpha,x_i)=M^n\prod_{i=1}^{n}[(2\Upsilon+\beta x_i+\alpha x_i{}^{\alpha-1})\prod_{i=1}^{n}\left[1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^\alpha\right)}\right]^{i-1}\prod_{i=1}^{n}\left[e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^\alpha\right)}\right]^{n-i+1} \quad (3\text{-}33)$$

The log- likelihood function is:

$$\begin{aligned}\ln L = n\ln M &+ \sum_{i=1}^{n}\ln(2\Upsilon+\beta x_i+\alpha x_i{}^{\alpha-1}) \\ &+\sum_{i=1}^{n}(i-1)\ln\left(1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^\alpha\right)}\right) \\ &-\sum_{i=1}^{n}(n-i+1)\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^\alpha\right)\end{aligned} \quad (3\text{-}34)$$

By finding the first derivative of the equation $(3-34)$ with respect to α , β, and ϒ as follows:

$$\frac{\partial \text{InL}}{\partial \Upsilon} = (2)\sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha {x_i}^{\alpha-1})^{-1} + \sum_{i=1}^{n}(i-1)(1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}{x_i}^2+{x_i}^{\alpha}\right)})^{-1}e^{-\left(2\Upsilon x_i+\frac{\beta}{2}{x_i}^2+{x_i}^{\alpha}\right)}(2x_i) - \sum_{i=1}^{n}(n-i+1)(2x_i) \tag{3-35}$$

$$\frac{\partial \text{InL}}{\partial \beta} = \sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha {x_i}^{\alpha-1})^{-1}(x_i) + (i-1)\sum_{i=1}^{n}\left(1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}{x_i}^2+{x_i}^{\alpha}\right)}\right)^{-1}\left(e^{-\left(2\Upsilon x_i+\frac{\beta}{2}{x_i}^2+{x_i}^{\alpha}\right)}\right)\left(\frac{1}{2}{x_i}^2\right) - (n-i+1)\sum_{i=1}^{n}\left(\frac{1}{2}{x_i}^2\right) \tag{3-36}$$

$$\frac{\partial \text{InL}}{\partial \alpha} = \sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha {x_i}^{\alpha-1})^{-1}{x_i}^{\alpha-1}(\alpha \ln(x_i)+1) + \sum_{i=1}^{n}(i-1)(1 - e^{-\left(2\Upsilon x_i+\frac{\beta}{2}{x_i}^2+{x_i}^{\alpha}\right)})^{-1}e^{-\left(2\Upsilon x_i+\frac{\beta}{2}{x_i}^2+{x_i}^{\alpha}\right)}({x_i}^{\alpha}\ln(x_i)) - \sum_{i=1}^{n}(n-i+1)({x_i}^{\alpha}\ln(x_i)) \tag{3-37}$$

Assume that $\frac{\partial \text{InL}}{\partial \Upsilon} = \frac{\partial \text{InL}}{\partial \beta} = \frac{\partial \text{InL}}{\partial \alpha} = 0$, then,

These are non-linear equations. To solve them using the iterative method which is Newton- Raphson method and it steps as follows:

$$\begin{bmatrix} \hat{\Upsilon}_{i+1} \\ \hat{\beta}_{i+1} \\ \hat{\alpha}_{i+1} \end{bmatrix} = \begin{bmatrix} \hat{\Upsilon}_{i} \\ \hat{\beta}_{i} \\ \hat{\alpha}_{i} \end{bmatrix} - \mathrm{J}^{-1} \begin{bmatrix} g(\Upsilon) \\ \mathrm{w}(\beta) \\ \mathrm{d}(\alpha) \end{bmatrix} \qquad (3\text{-}38)$$

We assumed that $\frac{\partial \mathrm{InL}}{\partial \Upsilon} = z(\Upsilon), \frac{\partial \mathrm{InL}}{\partial \beta} = c(\beta)$and $\frac{\partial InL}{\partial \alpha} = r(\alpha)$

To find Jacobean matrix, this is defined by:

$$\mathrm{J} = \begin{bmatrix} \frac{\partial\, z(\Upsilon)}{\partial \Upsilon} & \frac{\partial\, z(\Upsilon)}{\partial \beta} & \frac{\partial\, z(\Upsilon)}{\partial \alpha} \\ \frac{\partial \mathrm{c}(\beta)}{\partial \Upsilon} & \frac{\partial \mathrm{c}(\beta)}{\partial \beta} & \frac{\partial \mathrm{c}(\beta)}{\partial \alpha} \\ \frac{\partial \mathrm{r}(\alpha)}{\partial \Upsilon} & \frac{\partial \mathrm{r}(\alpha)}{\partial \beta} & \frac{\partial \mathrm{r}(\alpha)}{\partial \alpha} \end{bmatrix}$$

Then the first derivative to $z(\Upsilon), c(\beta)\ and\ r(\alpha)$ with respect to α,β, and ϒ as follows:

$$\frac{\partial \mathrm{z}(\Upsilon)}{\partial \Upsilon} = (-4)\sum_{i=1}^{n}(\ 2\Upsilon + \beta \mathrm{x_i} + \alpha {x_i}^{\alpha-1})^{-2}$$
$$-\sum_{i=1}^{n}(i-1)(\,1-\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)})^{-2}\mathrm{e}^{-2\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)}(4{x_i}^2) \qquad (3\text{-}39)$$
$$-\sum_{i=1}^{n}(i-1)(\,1-\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)})^{-1}\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)}(4{x_i}^2)$$

$$\frac{\partial\, z(\Upsilon)}{\partial\beta} = (-2)\sum_{i=1}^{n}(\ 2\Upsilon + \beta \mathrm{x_i} + \alpha {x_i}^{\alpha-1})^{-2}(x_i)$$
$$-\sum_{i=1}^{n}(i-1)(\,1-\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)})^{-2}\mathrm{e}^{-2\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)}({x_i}^3) \qquad (3\text{-}40)$$
$$-\sum_{i=1}^{n}(i-1)(\,1-\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)})^{-1}\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)}({x_i}^3)$$

$$\frac{\partial z(\Upsilon)}{\partial\alpha} = (-2)\sum_{i=1}^{n}(\ 2\Upsilon + \beta \mathrm{x_i} + \alpha {x_i}^{\alpha-1})^{-2}({x_i}^{\alpha-1})(\alpha \ln(x_i)+1)$$
$$-(2)\sum_{i=1}^{n}(i-1)(\,1-\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)})^{-2}\mathrm{e}^{-2\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)}({x_i}^{\alpha+1})(\ln(x_i \qquad (3\text{-}41)$$
$$-(2)\sum_{i=1}^{n}(i-1)(\,1-\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)})^{-1}\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)}({x_i}^{\alpha+1})(\ln(x_i)$$

$$\frac{\partial \mathrm{c}(\beta)}{\partial\beta} = (-1)\sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha {x_i}^{\alpha-1})^{-2}({x_i}^2)$$
$$-\frac{1}{4}\sum_{i=1}^{n}(i-1)(\,1-\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)})^{-2}\mathrm{e}^{-2\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)}({x_i}^4) \qquad (3\text{-}42)$$
$$-\frac{1}{4}\sum_{i=1}^{n}(i-1)(\,1-\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)})^{-1}\mathrm{e}^{-\left(2\Upsilon \mathrm{x}_i+\frac{\beta}{2}\mathrm{x}_i{}^2+\mathrm{x}_i{}^{\alpha}\right)}({x_i}^4)$$

$$\frac{\partial c(\beta)}{\partial \alpha} = (-1)\sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-2}(x_i{}^{\alpha})(\alpha \ln(x_i) + 1)$$
$$-\frac{1}{2}\sum_{i=1}^{n}(i$$
$$-1)(1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)})^{-2}\ e^{-2\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)}(x_i{}^{\alpha+2})(\ln(x_i)) \qquad (3\text{-}43)$$
$$-\frac{1}{2}\sum_{i=1}^{n}(i-1)(\ 1$$
$$-\ e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)})^{-1}e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)}(x_i{}^{\alpha+2})(\ln(x_i))$$

$$\frac{\partial r(\alpha)}{\partial \alpha} = \sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-1}(x_i{}^{\alpha-1}\ln(x_i))(\alpha \ln(x_i) + 2)$$
$$-\sum_{i=1}^{n}(\ 2\Upsilon + \beta x_i + \alpha x_i{}^{\alpha-1})^{-2}\left(x_i{}^{2(\alpha-1)}\right)(\alpha \ln(x_i) + 1)^2$$
$$-\sum_{i=1}^{n}(i-1)\left(1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)}\right)^{-2}\left(e^{-2\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)}\right)(x_i{}^{2\alpha})(\ln(x_i))^2 \qquad (3\text{-}44)$$
$$-\sum_{i=1}^{n}(i-1)\left(1-e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)}\right)^{-1}\left(e^{-\left(2\Upsilon x_i+\frac{\beta}{2}x_i{}^2+x_i{}^{\alpha}\right)}\right)(x_i{}^{2\alpha})(\ln(x_i))^2$$
$$-\sum_{i=1}^{n}(n-i+1)(x_i{}^{\alpha})(\ln(x_i))^2$$

Since, Jacobean is non-singular sympatric matrix.

Then, $\frac{\partial z(\Upsilon)}{\partial \beta} = \frac{\partial c(\beta)}{\partial \Upsilon}, \frac{\partial c(\beta)}{\partial \alpha} = \frac{\partial r(\alpha)}{\partial \beta}, \frac{\partial z(\Upsilon)}{\partial \alpha} = \frac{\partial r(\alpha)}{\partial \Upsilon}.$

Terminated the Newton- Raphson method, applying the following equation:

$$\begin{bmatrix} \varepsilon_{i+1}(\Upsilon) \\ \varepsilon_{i+1}(\beta) \\ \varepsilon_{i+1}(\alpha) \end{bmatrix} = \begin{bmatrix} \Upsilon_{i+1} \\ \beta_{i+1} \\ \alpha_{i+1} \end{bmatrix} - \begin{bmatrix} \Upsilon_{i} \\ \beta_{i} \\ \alpha_{i} \end{bmatrix} \qquad (3\text{-}45)$$

Where, ε(ϒ), ε(β), ε(α) are the error term, which are assumed and they are very small.

3.3 Information criterion

The general idea from information criteria is to know how the good model is at explaining the relationship between variables and when we want to know about how good the model. Choosing the appropriate model for certain data is a difficult and complicated process in terms of composition and mathematical construction. Here comes the role of information criteria in the extraction of information that determines the suitability of models or not. The information criterion measures the difference between the specified model and the actual model. The following are identifying three different types of information criterions.

3.3.1 Akaike information criterion and corrected Akaike information criterion

The Akaike information criterion was first introduced by the Japanese statistical scientist H. Akaike in 1973(Sakamoto, Ishiguro and Kitagawa, 1986) [26]. The Akaki information criterion is inspired from information theory. It can be used to obtain information that can be used in the comparative comparison between the proposed model and several other models for the same data provided for the experiment. The selection process for the optimization model is done by providing the smallest variation of the proposed model with the actual models. In other circumstances, the AIC may have a significant negative bias (Hurvich and Tsai 1989). For such instances, in (1989) Hurvi ch and Tsai were represented the corrected Akaike information $\mathbf{AIC_C}$ (Hurvich and Tsai, 1993) [11]. The advantage of the $\mathbf{AIC_C}$ on the AIC is that the expected difference is estimated with a tendency lower than the **AIC**. The **AIC** feature on $\mathbf{AIC_C}$ is that the **AIC** is more universally applicable because the **AIC** derivation is entirely generic while the $\mathbf{AIC_C}$ derivation depends on the candidate model. The formula of **the AIC** and $\mathbf{AIC_C}$ can be written as follows:

$$\mathbf{AIC} = 2\mathrm{m} - 2\hat{L} \qquad (3\text{-}46)$$

$$\mathbf{AIC_C} = \mathbf{AIC} + \frac{2\mathrm{m}(m+1)}{n-m-1} \qquad (3\text{-}47)$$

Where, n is the sample size of the data applications, m is the number of parameters in the statistical distribution, $\hat{\mathrm{L}}$ is the maximized value of the log-likelihood function under the approved distribution.

3.3.2 Bayesian information criterion

The Bayesian information criterion **(BIC)** is one of the most widely known tools widely used in the selection of statistical models. Their popularity derives from their computational simplicity and effective performance in many modeling frameworks, including Bayesian applications where previous distributions may be elusive(Posada and Buckley, 2004) [22]. The criterion was derived by Schwarz in (1978) to serve as an approximation of the Bayesian probability shift of the candidate model. The formula of the Bayesian information criterion as follows:

$$\mathbf{BIC} = \mathrm{m}\ln(\mathrm{n}) - 2\hat{L} \qquad (3\text{-}48)$$

Where, m, n, and $\hat{\mathrm{L}}$ are define similar as define above in section (3.2.1). When comparing Bayesian information criterion with Akaike information criterion, the penalty for additional parameters will be in the **BIC** than **AIC**. Moreover, the Akaike information criterion generally tries to find an unknown model that has a fact of high dimensions. On the other hand, the Bayesian information criterion comes only through real models. To perform model comparisons can be used the **AIC**, $\mathbf{AIC_C}$, and **BIC**. A lower value of **AIC**, $\mathbf{AIC_C}$,and **BIC** indicates a better fit.

3.4 Maximum Likelihood Estimation for Related distributions

This section presents the maximum likelihood estimation method for some related distributions as exponential Rayleigh, exponential Weibull, Rayleigh Weibull, and Weibull distributions, as a required for information criterions.

3.4.1 Maximum Likelihood Estimation of exponential-Weibull distribution

The probability density function of the exponential – Weibull distribution is:

$$f(x) = (\Upsilon + \alpha x^{\alpha-1})e^{-(\Upsilon x + x^{\alpha})} \text{ , x>0} \tag{3-49}$$

The likelihood function defined as:

$$L(\Upsilon, \alpha, x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f(x_i, \Upsilon, \alpha) \tag{3-50}$$

$$L(\Upsilon, \alpha, x_i) = \prod_{i=1}^{n} \left[\left(\Upsilon + \alpha x_i^{\alpha-1} \right) e^{-(\Upsilon x_i + {x_i}^{\alpha})} \right]$$

$$L(\Upsilon, \alpha, x_i) = \prod_{i=1}^{n} \left[\left(\Upsilon + \alpha x_i^{\alpha-1} \right) e^{-\sum_{i=1}^{n} (\Upsilon x_i + x_i^{\alpha})} \right]$$

The log- likelihood function is:

$$ln\ L(\Upsilon,\alpha,x_i) = \sum_{i=1}^{n} ln\left(\Upsilon + \alpha x_i^{\alpha-1}\right) - \sum_{i=1}^{n}(\Upsilon x_i + x_i^{\alpha}) \qquad (3\text{-}51)$$

$$g(\Upsilon) = \frac{\partial ln\ L}{\partial \Upsilon} = \sum_{i=1}^{n}\left(\Upsilon + \alpha x_i^{\alpha-1}\right)^{-1} - \sum_{i=1}^{n} x_i = 0 \qquad (3\text{-}52)$$

$$w(\alpha) = \frac{\partial ln\ L}{\partial \alpha}$$

$$= \sum_{i=1}^{n}\left(\Upsilon + \alpha x_i^{\alpha-1}\right)^{-1} x_i^{\alpha-1}(\alpha ln(x_i) + 1) - \sum_{i=1}^{n} x_i^{\alpha}\ ln(x_i) = 0 \qquad (3\text{-}53)$$

To solve this non-linear equation using a numerical method which is Newton-Raphson method and the steps as follows:

$$\begin{bmatrix}\hat{\Upsilon}_{i+1} \\ \hat{\alpha}_{i+1}\end{bmatrix} = \begin{bmatrix}\hat{\Upsilon}_{i} \\ \hat{\alpha}_{i}\end{bmatrix} - \mathrm{J}^{-1}\begin{bmatrix}g(\Upsilon) \\ w(\alpha)\end{bmatrix} \qquad (3\text{-}54)$$

Where, $\mathrm{J} = \begin{bmatrix}\frac{\partial g(\Upsilon)}{\partial \Upsilon} & \frac{\partial g(\Upsilon)}{\partial \alpha} \\ \frac{\partial\ w(\alpha)}{\partial \Upsilon} & \frac{\partial\ w(\alpha)}{\partial \alpha}\end{bmatrix}$, is he Jacobean matrix

Where,

$$\frac{\partial g(\Upsilon)}{\partial \Upsilon} = (-1)\sum_{i=1}^{n}\left(\Upsilon + \alpha x_i^{\alpha-1}\right)^{-2} \qquad (3\text{-}55)$$

$$\frac{\partial g(\Upsilon)}{\partial \alpha} = (-1)\sum_{i=1}^{n}\left(\Upsilon + \alpha x_i^{\alpha-1}\right)^{-2} x_i^{\alpha-1}(\alpha ln(x_i) + 1) \qquad (3\text{-}56)$$

$$\frac{\partial\, w(\alpha)}{\partial\alpha} = \sum_{i=1}^{n}(\Upsilon + \alpha x_i^{\alpha-1})^{-1} x_i^{\alpha-1}\, ln(x_i)[2 + \alpha ln(x_i)] - \sum_{i=1}^{n}(\Upsilon + \alpha x_i^{\alpha-1})^{-2} x_i^{2(\alpha-1)}(\alpha ln(x_i) + 1)^2 - \sum_{i=1}^{n} x_i^{\alpha}\left(ln(x_i)\right)^2 \quad (3\text{-}57)$$

The Jacobean matrix is non-singular symmetric matrix. Therefore, $\frac{\partial g(\Upsilon)}{\partial\alpha} = \frac{\partial\, w(\alpha)}{\partial\Upsilon}$.

3.4.2 Maximum Likelihood Estimation of exponential-Rayleigh distribution

The probability density function of the exponential – Rayleigh distribution is:

$$f_{ER}(k) = (\Upsilon + \beta x)e^{-(\Upsilon k + \frac{\beta}{2}x^2)} \;\; , k > 0 \quad (3\text{-}58)$$

The likelihood function defined as:

$$L(\Upsilon, \beta, x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f(x_i, \Upsilon, \beta) \quad (3\text{-}59)$$

$$L(\Upsilon, \beta, x_i) = \prod_{i=1}^{n}\left[(\Upsilon + \beta x_i)\, e^{-\left(\Upsilon x_i + \frac{\beta}{2}x_i^2\right)}\right]$$

$$L(\Upsilon, \beta, x_i) = \prod_{i=1}^{n}\left[(\Upsilon + \beta x_i)\, e^{-\sum_{i=1}^{n}\left(\Upsilon x_i + \frac{\beta}{2}x_i^2\right)}\right]$$

The log- likelihood function is:

$$ln\, L(\Upsilon, \beta, x_i) = \sum_{i=1}^{n} ln(\Upsilon + \beta x_i) - \sum_{i=1}^{n}\left(\Upsilon x_i + \frac{\beta}{2}x_i^2\right) \quad (3\text{-}60)$$

$$g(\Upsilon) = \frac{\partial ln\ L}{\partial \Upsilon} = \sum_{i=1}^{n} (\Upsilon + \beta x_i)^{-1} - \sum_{i=1}^{n} x_i = 0 \qquad (3\text{-}61)$$

$$w(\beta) = \frac{\partial ln\ L}{\partial \beta} = \sum_{i=1}^{n} (\Upsilon + \beta x_i)^{-1}(x_i) - \frac{1}{2}\sum_{i=1}^{n} x_i^2 = 0 \qquad (3\text{-}62)$$

The Newton- Raphson is the numerical method can be used to solve these non-linear equation and it steps as follows:

$$\begin{bmatrix} \hat{\Upsilon}_{i+1} \\ \hat{\beta}_{i+1} \end{bmatrix} = \begin{bmatrix} \hat{\Upsilon}_{i} \\ \hat{\beta}_{i} \end{bmatrix} - \mathrm{J}^{-1} \begin{bmatrix} g(\Upsilon) \\ w(\beta) \end{bmatrix} \qquad (3\text{-}63)$$

Where, $\mathrm{J} = \begin{bmatrix} \frac{\partial g(\Upsilon)}{\partial \Upsilon} & \frac{\partial g(\Upsilon)}{\partial \beta} \\ \frac{\partial\ w(\beta)}{\partial \Upsilon} & \frac{\partial\ w(\beta)}{\partial \beta} \end{bmatrix}$, is he Jacobean matrix

Where,

$$\frac{\partial g(\Upsilon)}{\partial \Upsilon} = (-1)\sum_{i=1}^{n} (\Upsilon + \beta x_i)^{-2} \qquad (3\text{-}64)$$

$$\frac{\partial g(\Upsilon)}{\partial \beta} = (-1)\sum_{i=1}^{n} (\Upsilon + \beta x_i)^{-2}(x_i) \qquad (3\text{-}65)$$

$$\frac{\partial\ w(\beta)}{\partial \beta} = (-1)\sum_{i=1}^{n} \left(\Upsilon + \alpha x_i^{\alpha-1}\right)^{-2}(x_i)^2 \qquad (3\text{-}66)$$

The Jacobean matrix is non-singular symmetric matrix. Therefore, $\frac{\partial g(\Upsilon)}{\partial \beta} = \frac{\partial\ w(\beta)}{\partial \Upsilon}$.

3.4.3 Maximum Likelihood Estimation of Rayleigh-Weibull distribution

The probability density function of the exponential – Rayleigh distribution is:

$$f(x) = (\beta + \alpha x^{\alpha-1})e^{-\left(\frac{\beta}{2}x^2+x^{\alpha}\right)} \text{ , x>0} \tag{3-67}$$

The likelihood function defined as:

$$L(\beta,\alpha,x_1,x_2,\dots,x_n) = \prod_{i=1}^{n} f(x_i,\beta,\alpha) \tag{3-68}$$

$$L(\beta,\alpha,x_i) = \prod_{i=1}^{n}\left[\left(\beta x_i + \alpha x_i^{\alpha-1}\right) e^{-\left(\frac{\beta}{2}x_i^2+{x_i}^{\alpha}\right)}\right]$$

$$L(\beta,\alpha,x_i) = \prod_{i=1}^{n}\left[\left(\beta x_i + \alpha x_i^{\alpha-1}\right) e^{-\sum_{i=1}^{n}\left(\frac{\beta}{2}x_i^2+{x_i}^{\alpha}\right)}\right]$$

The log- likelihood function is:

$$ln\, L(\beta,\alpha,x_i) = \sum_{i=1}^{n} ln\left(\beta x_i + \alpha x_i^{\alpha-1}\right) - \sum_{i=1}^{n}\left(\frac{\beta}{2}x_i^2 + {x_i}^{\alpha}\right) \tag{3-69}$$

$$g(\beta) = \frac{\partial ln\, L}{\partial \beta} = \sum_{i=1}^{n}\left(\beta x_i + \alpha x_i^{\alpha-1}\right)^{-1}(x_i) - \frac{1}{2}\sum_{i=1}^{n}{x_i}^2 = 0 \tag{3-70}$$

$$w(\alpha) = \frac{\partial ln\, L}{\partial \alpha}$$

$$= \sum_{i=1}^{n}\left(\beta x_i + \alpha x_i^{\alpha-1}\right)^{-1} x_i^{\alpha-1}(\alpha ln(x_i) + 1) \tag{3-71}$$

$$- \sum_{i=1}^{n} x_i^{\alpha}\, ln(x_i) = 0$$

To solve this non-linear equation using a numerical method which is Newton-Raphson method and it steps as follows:

$$\begin{bmatrix}\hat{\beta}_{i+1}\\ \hat{\alpha}_{i+1}\end{bmatrix} = \begin{bmatrix}\hat{\beta}_{i}\\ \hat{\alpha}_{i}\end{bmatrix} - \mathrm{J}^{-1}\begin{bmatrix}g(\beta)\\ w(\alpha)\end{bmatrix} \tag{3-72}$$

Where, $\mathrm{J} = \begin{bmatrix}\frac{\partial g(\beta)}{\partial \beta} & \frac{\partial g(\beta)}{\partial \alpha}\\ \frac{\partial\, w(\alpha)}{\partial \beta} & \frac{\partial\, w(\alpha)}{\partial \alpha}\end{bmatrix}$, is he Jacobean matrix

Where,

$$\frac{\partial g(\beta)}{\partial \beta} = (-1)\sum_{i=1}^{n}\left(\beta x_i + \alpha x_i^{\alpha-1}\right)^{-2}({x_i}^2) \tag{3-73}$$

$$\frac{\partial g(\beta)}{\partial \alpha} = (-1)\sum_{i=1}^{n}\left(\beta x_i + \alpha x_i^{\alpha-1}\right)^{-2} x_i^{\alpha}(\alpha ln(x_i) + 1) \tag{3-74}$$

$$\begin{aligned}\frac{\partial\, w(\alpha)}{\partial \alpha} = &\sum_{i=1}^{n}\left(\beta x_i + \alpha x_i^{\alpha-1}\right)^{-1} x_i^{\alpha-1}\, ln(x_i)[2 + \alpha ln(x_i)] \\ &- \sum_{i=1}^{n}\left(\beta x_i + \alpha x_i^{\alpha-1}\right)^{-2} x_i^{2(\alpha-1)}(\alpha ln(x_i) + 1)^2 \\ &- \sum_{i=1}^{n} x_i^{\alpha}\left(ln(x_i)\right)^2\end{aligned} \tag{3-75}$$

The Jacobean matrix is non-singular symmetric matrix. Therefore, $\frac{\partial g(\beta)}{\partial \alpha} = \frac{\partial\, w(\alpha)}{\partial \beta}$.

3.4.4 Maximum Likelihood Estimation of Weibull distribution

The probability density function of the standard Weibull distribution is:

$$f(x) = \alpha x^{\alpha-1} e^{-x^{\alpha}} \quad x>0 \tag{3-76}$$

The likelihood function defined as:

$$L(\alpha, x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f(x_i, \alpha) \tag{3-77}$$

$$L(\alpha, x_i) = \prod_{i=1}^{n} \left[\alpha x_i^{\alpha-1} e^{-x_i^{\alpha}}\right]$$

$$L(\alpha, x_i) = \alpha^n \prod_{i=1}^{n} \left[\, x_i^{\alpha-1} e^{-\sum_{i=1}^{n}(x_i{}^{\alpha})}\right]$$

The log- likelihood function is:

$$ln\, L(\alpha, x_i) = nln\alpha + (\alpha - 1)\sum_{i=1}^{n} ln(x_i) - \sum_{i=1}^{n} x_i{}^{\alpha} \tag{3-78}$$

$$\frac{\partial ln\, L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln(x_i) - \sum_{i=1}^{n} x_i{}^{\alpha} ln(x_i) = 0 \tag{3-79}$$

The numerical method was used to solve this non-linear equation as follows:

$$[\hat{\alpha}_{i+1}] = \hat{\alpha}_i - \mathrm{J}^{-1}\left[\frac{\partial ln\, L}{\partial \alpha}\right] \tag{3-80}$$

Where, $\mathrm{J} = \frac{\partial^2 \ln \mathrm{L}}{\partial^2 \alpha}$ is the Jacobean matrix.

Where,

$$\frac{\partial^2 ln\, L}{\partial^2 \alpha} = -n\alpha^{-2} - \sum_{i=1}^{n} x_i{}^{\alpha} (\ln(x_i))^2 \tag{3-81}$$

3.5 Applications

In this section, we introduce some applications of real complete data to compare between the New Mixture distribution and many others related distributions using AIC, AIC_C and BIC.

Data set(1) (Wang, 2000) [27]: The data represent the times of failure of eighteen electronic devices Wang (2000) are used to show the proposed distribution can be applied in practices following in days: 5, 11, 21, 31, 46, 75, 98, 122, 145,165, 196, 224, 245, 293, 321, 330, 350, 420.

Data set(2) (Linhart and Zucchini, 1986) [15]: (Linhart and Zucchini in 1986 page 69).The following are failure times of air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246 , 21, 42, 20,5,12,120,11,3,14,71,11,14,11,16,90,1,16,52,95.

Data set(3) (Meeker and Escobar, 2014) [16]: The data studied by Meeker and Escobar(1998,pge 388) which gives the time of failure and running times for a sample of devices contain 30 unites are following: 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88 , 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66.

3.5.1 The results and discussion

First: Comparing the New Mixture distribution and many other related distributions and the results tabulated in the table (3-1).

Table (3-1): Electronic devices data of (18) samples in days

	Distribution	Estimation parameters	Standard Error	$-2LL$	AIC	AIC_C	BIC
1	WEIBULL	$\hat{\alpha} = 34.8344$	0.1656	393.8458	395.8458	395.848388	396.736171
2	EXPONENTIAL RAYLIEGH	⍰ = 0.00021709 $\hat{\beta} = 0.00002134$	0.00010708 0.000001034	275.4626	279.4626	280.2626	281.243343
3	RAYLIEGH WEIBULL	$\hat{\beta} = 0.00003745$ $\hat{\alpha} = 0.0020$	0.00011551 0.00188999	263.846	267.8459	268.6459	269.626743
4	EXPONENTIAL WEIBULL	⍰ = 0.0001 $\hat{\alpha}$=0.1682	0.00051999 0.1817999	282.6668	286.6669	287.4669	288.447543
5	NEW MIXTURE DISTRIBUTION	⍰ = 0.00051 $\hat{\beta} = 0.00001760$ $\hat{\alpha} = 0.1007$	0.00031 0.00006314 0.0001	258.966	264.9660	266.680285	267.637115

Where (LL) is the log - likelihood function. Notice, that the New Mixture distribution have the smallest $-2LL, AIC, AIC_C,$ and BIC criterions comparing with exponential Rayleigh, exponential Weibull, Rayleigh Weibull, and Weibull distributions. Indicate the New Mixture distribution is a strong competitor to the rest of existing distributions compared to it.

Second: Represents in the table (3-2) the result of a New Mixture distribution compared with other related distributions with different complete data set (2) as following:

Table (3-2): Data of air conditioning system of an airplane of (30) samples.

	Distribution	Estimation parameters	Standard Error	$-2LL$	AIC	AIC_c	BIC
1	WEIBULL	$\hat{\alpha}$ = 14.8194	0.1806	592.7204	594.7204	594.9346	596.121597
2	EXPONENTIAL RAYLIEGH	⍰ = 0.0002086 $\hat{\beta}$ = 0.0000210	0.00010871 0.0000098871	423.5834	427.5834	428.0278	430.385794
3	RAYLIEGH WEIBULL	$\hat{\beta}$ = 0.0001658 $\hat{\alpha}$ = 0.0003163	0.0000546999 0.0002053	420.3918	424.3918	424.83624	427.194194
4	EXPONENTIAL WEIBULL	⍰ = 0.0001825 $\hat{\alpha}$ = 0.00087059	0.0001713871 0.00911941	561.4449	565.4449	565.88934	568.247294
5	NEW MIXTURE DISTRIBUTION	⍰ = 0.0010 $\hat{\beta}$ = 0.0001 $\hat{\alpha}$ = 0.0016	0.0009 0.000000009 0.0015999	406.2826	412.2826	413.20567	416.486192

Where (LL) is the log - likelihood function. As showing in a table(3-1) the New Mixture distribution is a good fit compared with the list distributions according to -2LL,AIC, AIC_C, and BIC criterions. Moreover, that New Mixture distribution is the smallest results in information criterions.

<u>Third:</u> Comparison of the New Mixture distribution with some distributions by using different complete data set (3) as follows:

Table (3-3): Running times for a sample of devices contains (30) unites.

	Distribution	Estimation parameters	Standard Error	$-2LL$	AIC	AIC_C	BIC
1	WEIBULL	$\hat{\alpha} = 0.5371$	0.236101	182.2498	184.2497	184.392557	185.65049
2	EXPONENTIAL RAYLIEGH	$\hat{▯} = 0.0766$ $\hat{\beta} = 0.0091$	0.035599 0.005	152.414	156.4140	156.858444	159.216394
3	RAYLIEGH WEIBULL	$\hat{\beta} = 0.0894$ $\hat{\alpha} = 0.0234$	0.0404 0.012399	169.2926	173.2925	173.736944	176.09499
4	EXPONENTIAL WEIBULL	$\hat{▯} = 0.5084$ $\hat{\alpha} = 0.0261$	0.00740001 0.01698001	148.2364	152.2363	152.680744	155.03879
5	NEW MIXTURE DISTRIBUTION	$\hat{▯} = 0.0764$ $\hat{\beta} = 0.1855$ $\hat{\alpha} = 0.1018$	0.06709 0.0775 0.189	126.8242	132.8242	133.747276	137.27792

Where (LL) is the log - likelihood function. Noting that, as doing in the table (3-1) and (3-2) analogous as in table (3-3) that the New Mixture distribution is a good fit comparison with the same distributions as above in tables (3-1, 3-2, 3-3) because, it has smallest criterions comparing with Weibull, Rayleigh Weibull, Exponential Weibull, Exponential Rayleigh. Also, using the maximum likelihood estimation method to estimate the parameters of the disruptions and utilizing for calculating -2LL, AIC, AIC_C, and BIC criterions as in tables (3-1, 3-2, 3-3).

CHAPTER FOUR

EMPIRICAL PART

4.1 Introduction

This chapter presents the solution of three parameters estimation $(\Upsilon, \beta, \alpha)$ using the classical methods which are used in chapter three for the New Mixture distribution. That means, finding a numerical results to the classical methods of estimation.

It is commonly accepted that the analytic solution derived from the classical methods as maximum likelihood estimation, ordinary least square estimation, and rank sampling set estimation is direct according to the format of these methods. While the numerical solution which can be founded as a result of substation between the numerical values and the parameters of the model.

4.2 Simulation design

Simulation is defined as a numerical technique for generating different sizes of samples. It is used to create experiments in a computer by a mathematical model to describe the behavior of this model. This chapter introduced two simulation methods to generate different sizes of samples. The first method which depends on the cumulative of New Mixture distribution whiles the second method using the Newton- Raphson formula. The Mean Square Error used to compare between the parameter estimator results that found by maximum likelihood, ordinarily Least Square, and rank set sampling estimation methods. MATLAB (R2018a) program which is used to applied simulation and estimation methods.

4.3 Algorithms of simulation experiments

This section contains two algorithms to generate data experiments and the following will gives the details of these algorithms:

4.3.1 The proposed algorithm of first simulation method

This subsection is include the algorithm of the first method of simulation which is depends of the cumulative function of the New Mixture distribution. There are important steps which are used to build this experiment which is as follows:

- **Step(1)**

Stabilizing the value of α equal to one with two values for β and ϒ as for a New Mixture distribution follows of this table:

Table (4-1): Shows the virtual values for parameters

Experiment		E_1,E_2	E_3,E_4
Parameters	Υ	0.0102	0.03
	β	0.0291	0.01
	α	1	1

The different between E_1,E_2 is that the initial vales of Newton- Raphson which is motion at step4.simlary the different between E_3,E_4.

- **Step(2)**

Determinate different sizes of samples (large, median, small)

n= 10, 30, 50, 100.

- **Step(3)**

Generate random number by using U=Rand as uniform distribution with [0,1].

- **Step(4)**

Assuming that the initial values for Newton-Raphson method as follows:

- **E_1:** $\Upsilon_\circ = 0.0001, \beta_\circ = 0.002, \alpha_\circ = 0.0003$
- **E_2:** $\Upsilon_\circ = 0.0005, \beta_\circ = 0.003333, \alpha_\circ = 0.0006$

- **E$_3$:** $\Upsilon_\circ = 0.00003, \beta_\circ = 0.001, \alpha_\circ = 0.00023$
- **E$_4$:** $\Upsilon_\circ = 0.000019, \beta_\circ = 0.005, \alpha_\circ = 0.00035$

- **Step(5)**

The random variables numbers transform as a uniform distribution to the New Mixture distribution by using the inverse transformation method which based on finding the inverse of cumulative distribution function as follows:

U=F(x), $x = F^{-1}(u)$

To apply this procedure to the New Mixture distribution getting:

$$F(x) = 1 - e^{-(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha)}$$

$$u = 1 - e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)}$$

$$1 - u = e^{-\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)}$$

$$\ln(1-u) = -\left(2\Upsilon x + \frac{\beta}{2}x^2 + x^\alpha\right)$$

Fixing α=1 then getting:

$$\ln(1-u) = -\left((2\Upsilon + 1)x + \frac{\beta}{2}x^2\right)$$

$$\frac{\beta}{2}x^2 + (2\Upsilon + 1)x + ln(1-u) = 0$$

As a result getting the following equation:

$$x = \frac{-(2\Upsilon + 1) \mp \sqrt{(2\Upsilon + 1)^2 - 2\beta \ln(1-u)}}{\beta} \qquad (4\text{-}1)$$

Because (x) is positive, the negative values resulting from this generation are ignored.

- **Step(6)**

Replicate each experiment or sample size 500, 1000 times.

- **Step(7)**

Estimate the three parameters of New Mixture distribution by using all method of this study.

- **Step(8)**

Suppose that the error term in all classical methods(maximum likelihood estimation, least square ordinary estimation, rank sampling set estimation) is ϵ_α =0.01, and ϵ_β, ϵ_Υ =0.001.

- **Step(9)**

Compare between estimation methods by employing the mean square error which is defined as a distance between the estimate value of parameter and the actual value of it. It is given by the equation:

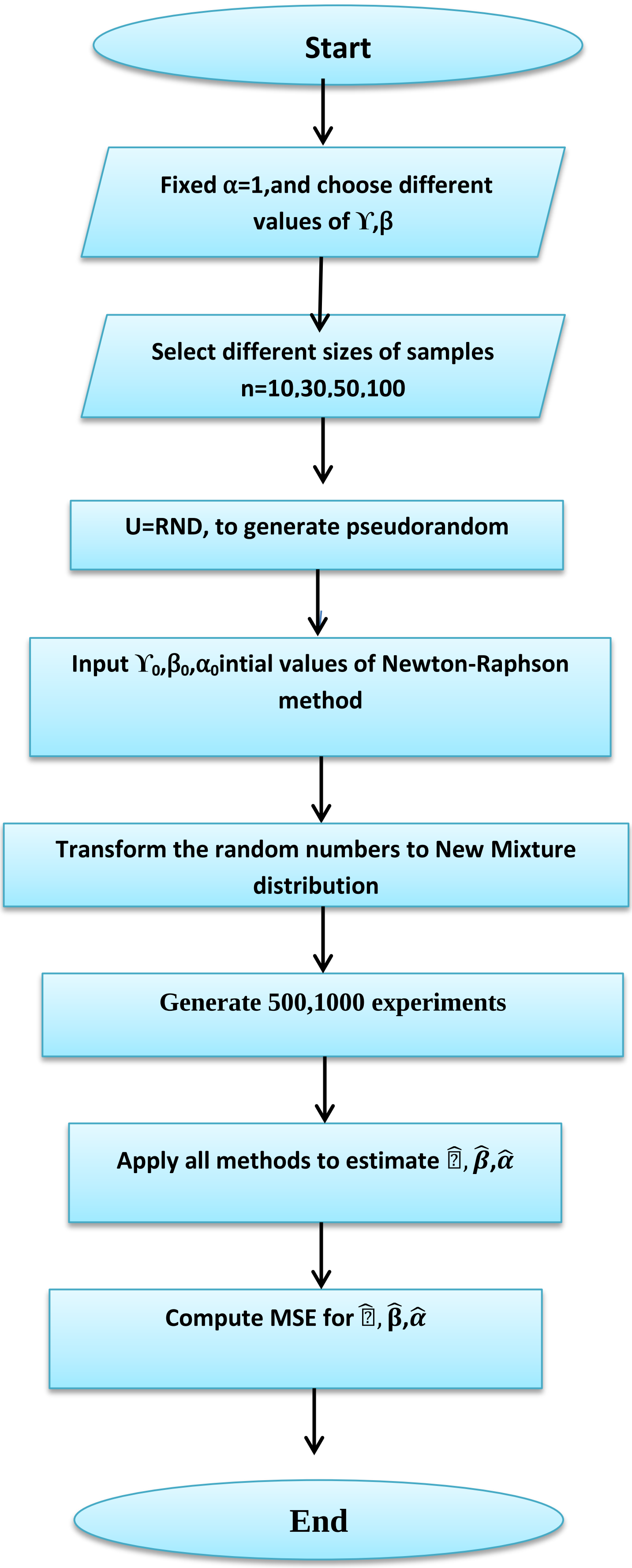
$$\text{MSE}\left(\hat{\Upsilon}\right)=\frac{1}{L}\sum_{i=1}^{L}(\hat{\Upsilon}-\Upsilon)^2 \qquad (4\text{-}2)$$

$$\text{MSE}\left(\hat{\beta}\right)=\frac{1}{L}\sum_{i=1}^{L}(\hat{\beta}-\beta)^2 \qquad (4\text{-}3)$$

$$\text{MSE}\left(\hat{\alpha}\right)=\frac{1}{L}\sum_{i=1}^{L}(\hat{\alpha}-\alpha)^2 \qquad (4\text{-}4)$$

Where, Lis the number of repeating of each experiment and equal to L=500, 1000.

The proposed algorithm of First Simulation Experiment



4.3.1.1 Numerical results to $(\hat{\gamma}, \hat{\beta}, \hat{\alpha})$ for all estimation methods

This subsection includes the estimate values of three parameters $\hat{\Upsilon}, \hat{\beta},$ and $\hat{\alpha}$ using the three estimation methods motioned in chapter 3. Moreover, the tables(4-2, 4-3, 4-4, 4-5) present the numerical results of $\hat{\Upsilon}$, the tables (4-6, 4-7, 4-8 ,4-9) present the numerical results of $\hat{\beta}$ and the tables (4-10, 4-11, 4-12, 4-13) present the numerical results of $\hat{\alpha}$ with repeating L =500. Moreover, the tables (4-14, 4-15, 4-16, 4-17) present the numerical results of $\hat{\Upsilon}$, the tables (4-18, 4-19, 4-20 ,4-21) present the numerical results of $\hat{\beta}$ and the tables (4-22, 4-23, 4-24, 4-25)present the numerical results of $\hat{\alpha}$ with repeating L=1000. Besides that, determined the Mean Square error measure of the estimator $\hat{\Upsilon}, \hat{\beta}$ and $\hat{\alpha}$ for all methods of estimation that are studied in this thesis and showing that for the first experiment the tables(4-26,4-27,4-28) present the MSE of ϒ,β,α for repeating L =500, 1000.Farthoremore, the tables(4-29,4-30,4-31) present the MSE of ϒ,β, α for the second experiment. The tables (4-32,4-33,4-34) present the MSE of α.for the third experiment, the tables(4-35, 4-36, 4-37)present the MSE of ϒ,β,α for the forth experiment.

4.3.1.1.1 Numerical values of estimator ϒ(L=500)

- For n=10, 30 ,50 ,100 present the values of estimator $(\hat{\Upsilon})$in two estimation methods (MLE, OLS) are increasing for increasing true values of parameter(ϒ) except in RSS it is it is oscillation. That means the estimate value of parameter (ϒ) is bigger or smaller than the true value with repeating L=500.
- For n=10, 30, 50, 100 the minimum values of estimator $(\hat{\Upsilon})$for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(ϒ) which are given in the following tables:

Table (4-2): Represents average values of $\widehat{\Upsilon}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	⍰	**0.0001**	**0.0001**	**0.0001**
	⍰̂	**0.000158**	**0.0000999999999997759**	**2.63504E-05**
E2	⍰	**0.0005**	**0.0005**	**0.0005**
	⍰̂	**5.8E-05**	**0.000499999999999941**	**3.76736E-07**
E3	⍰	**0.00003**	**0.00003**	**0.00003**
	⍰̂	**0.000181245**	**0.0000299999999999472**	**3.36E-06**
E4	⍰	**0.000019**	**0.000019**	**0.000019**
	⍰̂	**0.000483181**	**0.0000190000000000003**	**5.97E-06**

Table (4-3): Represents average values of $\widehat{\Upsilon}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	⍰	**0.0001**	**0.0001**	**0.0001**
	⍰̂	**0.000460528**	**0.0000999999999999635**	**0.000125084**
E2	⍰	**0.0005**	**0.0005**	**0.0005**
	⍰̂	**0.000377293**	**0.00049999999999999**	**0.000768331**
E3	⍰	**0.00003**	**0.00003**	**0.00003**
	⍰̂	**0.000600119**	**0.0000300000000000028**	**4.02E-05**
E4	⍰	**0.000019**	**0.000019**	**0.000019**
	⍰̂	**0.001286632**	**0.0000190000000000007**	**1.04862E-05**

Table (4-4): Represents average values of $\widehat{\Upsilon}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	⍰	**0.0001**	**0.0001**	**0.0001**
	⍰̂	**0.000882907**	**0.0000999999999999753**	**1.60E-04**
E2	⍰	**0.0005**	**0.0005**	**0.0005**
	⍰̂	**5.8E-05**	**0.000499999999999941**	**3.76736E-07**
E3	⍰	**0.00003**	**0.00003**	**0.00003**
	⍰̂	**0.000859277**	**0.0000299999999999648**	**1.68E-05**
E4	⍰	**0.000019**	**0.000019**	**0.000019**
	⍰̂	**0.00235443**	**0.000018999999999994**	**3.12307E-05**

Table (4-5): Represents average values of $\hat{\Upsilon}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	Υ	**0.0001**	**0.0001**	**0.0001**
	$\hat{\Upsilon}$	**0.001152784**	**0.000100000000000003**	**0.000173205**
E2	Υ	**0.0005**	**0.0005**	**0.0005**
	$\hat{\Upsilon}$	**0.000267552**	**0.000499999999999994**	**2.52E-05**
E3	Υ	**0.00003**	**0.00003**	**0.00003**
	$\hat{\Upsilon}$	**0.002081312**	**0.000030000000000003**	**1.84E-06**
E4	Υ	**0.000019**	**0.000019**	**0.000019**
	$\hat{\Upsilon}$	**0.004698876**	**0.0000189999999999896**	**8.97E-05**

From tables (4-2, 4-3, 4-4, 4-5) noting that estimate the scale parameter ϒ of New Mixture distribution according to the result which is very closed in the ordinary least square method.

4.3.1.1.2 <u>Numerical values of estimator $\boldsymbol{\beta(L=500)}$</u>

- For n=10, 30 ,50 ,100 present the values of estimator $(\hat{\beta})$in two estimation methods (MLE, OLS) are increasing for increasing true values of parameter(β) except in RSS it is it is oscillation. That means the estimate value of parameter (ϒ) is bigger or smaller than the true value with repeating L=500.
- For n=10, 30, 50, 100 the minimum values of estimator $(\hat{\beta})$for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(β) which are given in the following tables:

Table (4-6): Represents average values of $\hat{\beta}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	β	0.002	0.002	0.002
	$\hat{\beta}$	0.022356667	0.00199999999999996	0.00020366
E2	β	0.003333	0.003333	0.003333
	$\hat{\beta}$	0.043072569	0.00333299999999998	0.000101234
E3	β	0.001	0.001	0.001
	$\hat{\beta}$	0.010660904	0.000999999999999987	6.86101E-05
E4	β	0.005	0.005	0.005
	$\hat{\beta}$	0.053570092	0.00499999999999984	2.27288E-05

Table (4-7): Represents average values of $\hat{\beta}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	β	0.002	0.002	0.002
	$\hat{\beta}$	0.062707271	0.00199999999999995	0.001405363
E2	β	0.003333	0.003333	0.0003333
	$\hat{\beta}$	0.122860722	0.00333299999999989	0.00171472
E3	β	0.001	0.001	0.001
	$\hat{\beta}$	0.029586318	0.000999999999999886	0.000294023
E4	β	0.005	0.005	0.005
	$\hat{\beta}$	0.150495795	0.00499999999999993	0.00246406

Table (4-8): Represents average values of $\hat{\beta}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	β	0.002	0.002	0.002
	$\hat{\beta}$	0.10483256	0.00199999999999999	0.001744593
E2	β	0.003333	0.003333	0.0003333
	$\hat{\beta}$	0.200253497	0.00333299999999986	0.002124367
E3	β	0.001	0.001	0.001
	$\hat{\beta}$	0.049629096	0.000999999999999992	0.001083525
E4	β	0.005	0.005	0.005
	$\hat{\beta}$	0.248900605	0.00499999999999996	0.00338893

Table (4-9): Represents average values of $\hat{\beta}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	β	0.002	0.002	0.002
	$\hat{\beta}$	0.210244689	0.00199999999999997	0.002434
E2	β	0.003333	0.003333	0.0003333
	$\hat{\beta}$	0.397056437	0.00333299999999999	0.0022384
E3	β	0.001	0.001	0.001
	$\hat{\beta}$	0.097973365	0.000999999999999903	0.001095143
E4	β	0.005	0.005	0.005
	$\hat{\beta}$	0.493257205	0.00499999999999997	0.001551137

From tables (4-6, 4-7, 4-8, 4-9) noting that estimate the scale parameter β of New Mixture distribution according to the result which is very closed in the ordinary least square method.

4.3.1.1.3 Numerical values of estimator α(L=500)

- For n=10, 30, 50, 100 present the values of estimator ($\hat{\alpha}$)in two estimation methods (MLE, OLS) are increasing for increasing true values of parameter(α) except in RSS it is it is oscillation. That means the estimate value of parameter (α) is bigger or smaller than the true value with repeating L=500.
- For n=10, 30, 50, 100 the minimum values of estimator ($\hat{\alpha}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(α) which are given in the following tables:

Table (4-10): Represents average values of $\hat{\alpha}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	α	0.0003	0.0003	0.0003
	$\hat{\alpha}$	0.003679243	0.000299999999999659	0.000295134
E2	α	0.0006	0.0006	0.0006
	$\hat{\alpha}$	0.007963821	0.00059999999999997	0.000556402
E3	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.002724949	0.000229999999999685	0.000215051
E4	α	0.00035	0.00035	0.00035
	$\hat{\alpha}$	0.00399701	0.000349999999999989	0.000319873

Table (4-11): Represents average values of $\hat{\alpha}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	α	0.0003	0.0003	0.0003
	$\hat{\alpha}$	0.010826236	0.000299999999999763	7.76483E-05
E2	α	0.0006	0.0006	0.0006
	$\hat{\alpha}$	0.024325356	0.000599999999999979	0.000589899
E3	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.008042246	0.000230000000000005	3.77274E-06
E4	α	0.00035	0.00035	0.00035
	$\hat{\alpha}$	0.011470908	0.000350000000000004	0.000103614

Table (4-12): Represents average values of $\hat{\alpha}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	α	0.0003	0.0003	0.0003
	$\hat{\alpha}$	0.017852048	0.000299999999999825	0.000300002
E2	α	0.0006	0.0006	0.0006
	$\hat{\alpha}$	0.039566233	0.000599999999999937	0.000606985
E3	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.013285258	0.000229999999999952	0.000231763
E4	α	0.00035	0.00035	0.00035
	$\hat{\alpha}$	0.018804473	0.000349999999999976	0.000339659

Table (4-13): Represents average values of $\hat{\alpha}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	α	0.0003	0.0003	0.0003
	$\hat{\alpha}$	0.035310401	0.000299999999999997	0.000301298
E2	α	0.0006	0.0006	0.0006
	$\hat{\alpha}$	0.080963196	0.000599999999999974	0.000601449
E3	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.026273891	0.00023000000000004	0.000233726
E4	α	0.00035	0.00035	0.00035
	$\hat{\alpha}$	0.037440488	0.000349999999999944	0.000346402

From tables (4-10, 4-11, 4-12, 4-13) noting that estimate the shape parameter α of New Mixture distribution according to the result which is very closed in the ordinary least square method.

. 4.3.1.1.4 Numerical values of estimator ϒ(L=1000)

- The results in tables(4-14, 4-15, 4-16, 4-17) present the values of estimator ($\hat{\Upsilon}$)in two estimation methods(MLE,OLS) are increasing for increasing true values of parameter(ϒ) except in RSS it is oscillation. That means the estimate value of parameter (ϒ) is bigger or smaller than the true value for n=10, 30, 50, 100 repeating L=1000.
- The minimum values of estimator ($\hat{\Upsilon}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(ϒ) for n=10, 30, 50, 100 which are given in the following tables:

Table (4-14): Represents average values of □̂ where n=10

experiment	parameter	MLE	OLS	RSS
E1	□	0.0001	0.0001	0.0001
	□̂	0.000110337	0.00010000000000000003	6.72736E-05
E2	□	0.0005	0.0005	0.0005
	□̂	0.000429963	0.000499999999999753	0.000119417
E3	□	0.00003	0.00003	0.00003
	□̂	0.000201831	0.000029999999999968	5.43E-06
E4	□	0.000019	0.000019	0.000019
	□̂	0.000452433	0.000018999999999976	7.99E-06

Table (4-15): Represents average values of □̂ where n=30

experiment	parameter	MLE	OLS	RSS
E1	□	0.0001	0.0001	0.0001
	□̂	0.00054409	0.0000999999999999292	0.000219716
E2	□	0.0005	0.0005	0.0005
	□̂	4.70E-05	0.000499999999999965	0.000148971
E3	□	0.00003	0.00003	0.00003
	□̂	0.000556248	0.0000299999999999561	4.11E-05
E4	□	0.000019	0.000019	0.000019
	□̂	0.001357835	0.0000189999999999984	5.56558E-05

Table (4-16): Represents average values of □̂ where n=50

experiment	parameter	MLE	OLS	RSS
E1	□	0.0001	0.0001	0.0001
	□̂	0.000923625	0.0000999999999999638	1.63E-04
E2	□	0.0005	0.0005	0.0005
	□̂	1.14E-05	0.000499999999999994	0.000449013
E3	□	0.00003	0.00003	0.00003
	□̂	0.00096016	0.0000299999999999919	3.48E-05
E4	□	0.000019	0.000019	0.000019
	□̂	0.002132136	0.0000189999999999807	0.000116719

Table (4-17): Represents average values of $\hat{\Upsilon}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	Υ	**0.0001**	**0.0001**	**0.0001**
	$\hat{\Upsilon}$	**0.001641408**	**0.0000999999999999914**	**0.000129609**
E2	Υ	**0.0005**	**0.0005**	**0.0005**
	$\hat{\Upsilon}$	**0.000329351**	**0.000499999999999995**	**0.000189746**
E3	Υ	**0.00003**	**0.00003**	**0.00003**
	$\hat{\Upsilon}$	**0.002087276**	**0.0000299999999999982**	**2.99E-05**
E4	Υ	**0.000019**	**0.000019**	**0.000019**
	$\hat{\Upsilon}$	**0.004695927**	**0.0000189999999999864**	**9.87E-05**

From tables (4-14, 4-15, 4-16, 4-17) with repeating L=1000 according to the result which we motion above, noting that the estimate the scale parameter ϒ of New Mixture distribution which is very closed in the ordinary least square method.

4.3.1.1.5 <u>Numerical values of estimator β(L=1000)</u>

- The results in tables(4-18, 4-19, 4-20, 4-21)present the values of estimator ($\hat{\beta}$)in two estimation methods(MLE, OLS) are increasing for increasing true values of parameter(β) except in RSS it is oscillation. That means the estimate value of parameter (β) is bigger or smaller than the true value for n=10, 30, 50, 100 repeating L=1000.
- The minimum values of estimator ($\hat{\beta}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(β) for n=10, 30, 50, 100 which are given in the following tables:

Table (4-18): Represents average values of $\hat{\beta}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	β	**0.002**	**0.002**	**0.002**
	$\hat{\beta}$	**0.022634746**	**0.0019999999999998**	**7.54264E-05**
E2	β	**0.003333**	**0.003333**	**0.0003333**
	$\hat{\beta}$	**0.041166613**	**0.00333299999999998**	**3.10194E-05**
E3	β	**0.001**	**0.001**	**0.001**
	$\hat{\beta}$	**0.010464704**	**0.000999999999999706**	**1.99085E-06**
E4	β	**0.005**	**0.005**	**0.005**
	$\hat{\beta}$	**0.053733839**	**0.00499999999999978**	**0.000266914**

Table (4-19): Represents average values of $\hat{\beta}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	β	**0.002**	**0.002**	**0.002**
	$\hat{\beta}$	**0.064029948**	**0.00199999999999997**	**9.0437E-05**
E2	β	**0.003333**	**0.003333**	**0.0003333**
	$\hat{\beta}$	**0.118007106**	**0.00333299999999999**	**0.002831399**
E3	β	**0.001**	**0.001**	**0.001**
	$\hat{\beta}$	**0.03011417**	**0.000999999999999947**	**0.000195644**
E4	β	**0.005**	**0.005**	**0.005**
	$\hat{\beta}$	**0.150503331**	**0.00499999999999996**	**0.00278809**

Table (4-20): Represents average values of $\hat{\beta}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	β	**0.002**	**0.002**	**0.002**
	$\hat{\beta}$	**0.105692771**	**0.002000000000000002**	**0.001332158**
E2	β	**0.003333**	**0.003333**	**0.003333**
	$\hat{\beta}$	**0.19248939**	**0.00333299999999995**	**0.001362862**
E3	β	**0.001**	**0.001**	**0.001**
	$\hat{\beta}$	**0.048349036**	**0.000999999999999843**	**0.000554854**
E4	β	**0.005**	**0.005**	**0.005**
	$\hat{\beta}$	**0.249032692**	**0.00499999999999997**	**0.003314065**

Table (4-21): Represents average values of $\hat{\beta}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	β	0.002	0.002	0.002
	$\hat{\beta}$	0.001641408	0.00199999999999999	0.001896108
E2	β	0.003333	0.003333	0.003333
	$\hat{\beta}$	0.388610019	0.00333299999999999	0.002144883
E3	β	0.001	0.001	0.001
	$\hat{\beta}$	0.097754666	0.000999999999999991	0.000955043
E4	β	0.005	0.005	0.005
	$\hat{\beta}$	0.492778618	0.00499999999999995	0.001911629

From tables (4-18, 4-19, 4-20, 4-21) with repeating L=1000, according to the result that we motion above, noting that the best classical method that used to estimate the scale parameter β of New Mixture distribution which is the ordinary least square method.

4.3.1.1.6 Numerical values of estimator $\hat{\alpha}$ (L=1000)

- The results in tables(4-22, 4-23, 4-24, 4-25)present the values of estimator ($\hat{\alpha}$) in two estimation methods(MLE, OLS) are increasing for increasing true values of parameter(α) except in RSS it is oscillation. That means the estimate value of parameter (α) is bigger or smaller than the true value for n=10, 30, 50, 100 repeating L=1000.
- The minimum values of estimator ($\hat{\alpha}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(α) for n=10, 30, 50, 100 which are given in the following tables:

Table (4-22): Represents average values of $\hat{\alpha}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	α	0.0003	0.0003	0.0003
	$\hat{\alpha}$	0.003664633	0.000299999999999918	0.000247082
E2	α	0.0006	0.0006	0.0006
	$\hat{\alpha}$	0.008430446	0.000599999999999865	8.83972E-05
E3	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.002780255	0.000229999999999919	0.000174649
E4	α	0.00035	0.00035	0.00035
	$\hat{\alpha}$	0.004032716	0.000349999999999961	4.80264E-05

Table (4-23): Represents average values of $\hat{\alpha}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	α	0.0003	0.0003	0.0003
	$\hat{\alpha}$	0.010791337	0.000299999999999817	0.000115705
E2	α	0.0006	0.0006	0.0006
	$\hat{\alpha}$	0.024533793	0.000599999999999974	0.000603627
E3	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.007887591	0.000229999999999973	0.000229004
E4	α	0.00035	0.00035	0.00035
	$\hat{\alpha}$	0.011314863	0.000349999999999976	0.000230941

Table (4-24): Represents average values of $\hat{\alpha}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	α	0.0003	0.0003	0.0003
	$\hat{\alpha}$	0.017448438	0.000299999999999905	0.000288506
E2	α	0.0006	0.0006	0.0006
	$\hat{\alpha}$	0.040697573	0.000599999999999907	0.000574815
E3	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.013180136	0.000229999999999966	0.000229993
E4	α	0.00035	0.00035	0.00035
	$\hat{\alpha}$	0.018882194	0.000349999999999999	0.000331446

Table (4-25): Represents average values of $\hat{\alpha}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	α	0.0003	0.0003	0.0003
	$\hat{\alpha}$	0.03538451	0.000299999999999769	0.000300014
E2	α	0.0006	0.0006	0.0006
	$\hat{\alpha}$	0.081112354	0.000599999999999956	0.000600842
E3	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.026392533	0.000229999999999826	0.000230447
E4	α	0.00035	0.00035	0.00035
	$\hat{\alpha}$	0.037452916	0.00034999999999992	0.000349213

From tables (4-22, 4-23, 4-24, 4-25) with repeating L=1000, according to the result that we motion above, noting that the best classical method that used to estimate the shape parameter α of New Mixture distribution which is the ordinary least square method.

Now the following tables introduce the mean square error of the three parameters of New Mixture distribution of all classical methods of this study include different sizes of samples n=10, 30, 50, 100 with repeating L=500, 1000.

Table (4-26): Represents the MSE of ⍰ for the E_1 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000000027866	1.4926E-34	0.00000096557	OLS
	30	0.00000024403	1.1442E-34	0.00000064516	OLS
	50	0.00000072405	2.4156E-35	0.00000075774	OLS
	100	0.0000028512	2.240E-35	0.00000035098	OLS
1000	10	0.000000029059	5.0137E-35	0.000001044	OLS
	30	0.00000023517	1.5237E-35	0.0000016759	OLS
	50	0.00000070817	8.7501E-36	0.00000087165	OLS
	100	0.0000026167	1.2301E-35	0.00000075154	OLS

Table (4-27): Represents the MSE of β for the E_1 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.00043536	3.4426E-34	0.0000020175	OLS
	30	0.0039	4.4767E-34	0.0000015805	OLS
	50	0.0109	6.7647E-35	0.00000099255	OLS
	100	0.0436	7.2146E-35	0.000001909	OLS
1000	10	0.00043405	8.5989E-35	0.0000017581	OLS
	30	0.00390	5.0045E-35	0.00000028676	OLS
	50	0.0109	1.7291E-35	0.00000075795	OLS
	100	0.0435	2.6098E-35	0.00000074618	OLS

Table (4-28): Represents the MSE of α for the E_1 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000012432	7.563E-34	0.00000000057195	OLS
	30	0.00011133	2.2386E-33	0.0000000011446	OLS
	50	0.00031027	9.0362E-34	0.00000000022375	OLS
	100	0.0012	5.657E-34	0.00000000024307	OLS
1000	10	0.000012454	2.4816E-34	0.00000000064134	OLS
	30	0.00011145	6.0281E-34	0.00000000044959	OLS
	50	0.00031221	1.2903E-34	0.00000000058221	OLS
	100	0.00123	3.1527E-34	0.00000000011676	OLS

Table (4-29): Represents the MSE of □ for the E_2 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.0000000031257	2.3097E-35	0.000003621	OLS
	30	0.0000000033285	9.7552E-35	0.0000036458	OLS
	50	0.000000012459	2.8607E-35	0.0000031874	OLS
	100	0.00000001926	2.732E-35	0.0000033321	OLS
1000	10	0.0000000022466	6.9431E-35	0.0000063446	OLS
	30	0.0000000069127	2.6313E-35	0.0000014406	OLS
	50	0.00000004528	6.0573E-35	0.0000046884	OLS
	100	0.0000000084142	2.3237E-35	0.0000020565	OLS

Table (4-30): Represents the MSE of β for the E_2 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.00161	2.4032E-35	0.0000076628	OLS
	30	0.0144	2.5728E-34	0.0000030141	OLS
	50	0.0397	7.1127E-35	0.0000026675	OLS
	100	0.1587	5.3807E-35	0.0000017343	OLS
1000	10	0.00167	3.2366E-34	0.0000036113	OLS
	30	0.0143	7.9492E-35	0.0000086552	OLS
	50	0.0398	1.4812E-34	0.0000011745	OLS
	100	0.1587	6.8359E-35	0.0000038636	OLS

Table (4-31): Represents the MSE of α for the E_2 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000064593	2.3452E-34	0.000000010564	OLS
	30	0.00058686	5.2573E-34	0.0000000088097	OLS
	50	0.00166	2.6478E-34	0.000000018149	OLS
	100	0.0065	2.4085E-34	0.0000000016646	OLS
1000	10	0.000066984	3.1709E-34	0.000000018462	OLS
	30	0.00058836	5.4602E-34	0.000000011192	OLS
	50	0.001644	4.7076E-34	0.000000017234	OLS
	100	0.0065	4.7548E-34	0.00000000089309	OLS

Table (4-32): Represents the MSE of ϒ for the E_3 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000000041591	6.6653E-34	0.00000026655	OLS
	30	0.00000037785	4.0618E-35	0.00000035396	OLS
	50	0.0000010981	1.5564E-35	0.00000012564	OLS
	100	0.0000042857	2.8892E-35	0.00000017207	OLS
1000	10	0.000000045557	4.1228E-35	0.0000004001	OLS
	30	0.00000038628	1.0844E-35	0.00000032766	OLS
	50	0.000001074	2.8705E-35	0.1106	OLS
	100	0.0000042873	2.5786E-35	0.00000017752	OLS

Table (4-33): Represents the MSE of β for the E_3 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000095733	5.1842E-33	0.00000055382	OLS
	30	0.00086019	1.0108E-34	0.00000018573	OLS
	50	0.0024	4.0218E-35	0.00000056003	OLS
	100	0.0095	1.2089E-34	0.0000003255	OLS
1000	10	0.000094557	3.3828E-34	0.00000036038	OLS
	30	0.00085812	5.0379E-35	0.00000020064	OLS
	50	0.0024	1.0978E-34	0.0231	OLS
	100	0.0095	5.8559E-35	0.00000032936	OLS

Table (4-34): Represents the MSE of α for the E_3 of the third experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.0000068853	8.3771E-34	0.00000000017941	OLS
	30	0.000062027	9.9644E-34	0.00000000031559	OLS
	50	0.0001737	4.8967E-34	0.00000000020676	OLS
	100	0.00069157	4.6667E-34	0.00000000028427	OLS
1000	10	0.0000069863	2.9986E-34	0.00000000052148	OLS
	30	0.00006208	5.0638E-34	0.000000000062119	OLS
	50	0.00017276	8.6535E-34	0.00000000018086	OLS
	100	0.00069128	4.5132E-34	0.00000000013999	OLS

Table (4-35): Represents the MSE of ϒ for the E_4 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.00000022511	6.0895E-35	0.000006518	OLS
	30	0.0000020382	7.1003E-35	0.0000026489	OLS
	50	0.0000056924	1.9033E-35	0.0000010893	OLS
	100	0.000022791	1.8548E-35	0.0000010506	OLS
1000	10	0.00000022301	4.9014E-35	0.0000035825	OLS
	30	0.0000020669	4.7652E-35	0.5143	OLS
	50	0.0000056479	2.6645E-35	0.0000015244	OLS
	100	0.000022633	2.562E-35	0.0000016763	OLS

Table (4-36): Represents the MSE of β for the E_4 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.0024	8.2467E-34	0.0000044907	OLS
	30	0.0215	2.0396E-34	0.0000059536	OLS
	50	0.0598	4.3961E-35	0.000010252	OLS
	100	0.2392	6.7973E-35	0.000007895	OLS
1000	10	0.0024	8.3318E-35	0.0000083002	OLS
	30	0.0216	1.1874E-34	2.3163	OLS
	50	0.0598	5.5777E-35	0.0000079894	OLS
	100	0.2392	6.322E-35	0.0000049711	OLS

Table (4-37): Represents the MSE of α for the E_4 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000013862	9.7576E-34	0.0000000046022	OLS
	30	0.00012406	6.7493E-34	0.0000000082266	OLS
	50	0.00034436	4.7142E-34	0.0000000013072	OLS
	100	0.0014	7.1198E-34	0.0000000044947	OLS
1000	10	0.000013823	2.8655E-34	0.0000000033028	OLS
	30	0.00012381	5.6882E-34	0.000022266	OLS
	50	0.00034486	3.1739E-34	0.00000000077438	OLS
	100	0.0014	3.8149E-34	0.00000000020041	OLS

From tables (4-26, 4-27, 4-28, 4-29,4-30, 4-31,4-32,4-33, 4-34, 4-35, 4-36, 4-37) according to the result that we motion above, noting that the best classical method that used to estimate the parameters (Υ,β, α) of New Mixture distribution which is the ordinary least square method.

4.3.2 The Modified algorithm of second simulation method

This subsection is include the algorithm of the second method of simulation which is depends of the Newton- Raphson formula. In 2015[16], Faton Merovci and Ibrahim Elbatal using this formula to generate data of Rayleigh Weibull distribution for size of sample n=30 without repeating. Modifying that algorithm to generate different sizes of samples. There are important steps which are used to build this algorithm which is as follows:

- **Step(1)**

Select different two values for α, β and Υ as for a New Mixture distribution follows of this table:

Table (4-38): Shows the virtual values for Parameters

Experiment		E_1,E_2	E_3,E_4
Parameters	Υ	0.0102	0.03
	β	0.0291	0.01
	α	1.7	1.8

The different between E_1, E2 is that the initial vales of Newton- Raphson which is motion at the step4.simlary the different between E3, E4.

- **Step(2)**

Determinate different sizes of samples (large, median, small)

n= 10, 30, 50, 100.

- **Step(3)**

Generate random number by using U=Rand as uniform distribution with [0,1].

➢ **Step(4)**

Generate x^0 as a random number of uniform distribution and generate $x^*$by using the Newton- Raphson formula which is:

$$x^* = x^0 - \frac{F(x)-U(x)}{f(x)}$$

Where, f(x) is the density function of New Mixture distribution.

F(x) is the cumulative function of New Mixture distribution.

U(x) is rand as the uniform distribution with [0,1].

➢ **Step(5)**

Assuming that the initial values for Newton-Raphson method as follows:

- **E_1:** $\Upsilon_\circ = 0.0002, \beta_\circ = 0.000137, \alpha_\circ = 0.00013$
- **E_2:** $\Upsilon_\circ = 0.0003, \beta_\circ = 0.000156, \alpha_\circ = 0.00023$
- **E_3:** $\Upsilon_\circ = 0.00025, \beta_\circ = 0.000447, \alpha_\circ = 0.00032$
- **E_4:** $\Upsilon_\circ = 0.0005, \beta_\circ = 0.000147, \alpha_\circ = 0.00023$

➢ **Step(6)**

Replicate each experiment or sample size 500, 1000 times.

➢ **Step(7)**

Estimate the three parameters of New Mixture distribution by using all methods of this study.

➢ **Step(8)**

Suppose that the error term in all classical methods (maximum likelihood estimation, least square ordinary estimation is ϵ_α =0.01, and ϵ_β, ϵ_Υ =0.0001) and rank sampling set estimation is ϵ_α =0.0001, and ϵ_β, ϵ_Υ =0.000001.

- **Step(9)**

Compare between estimation methods by employing the mean square error which is defined as a distance between the estimate value of parameter and the actual value of it. It is given by the equation:

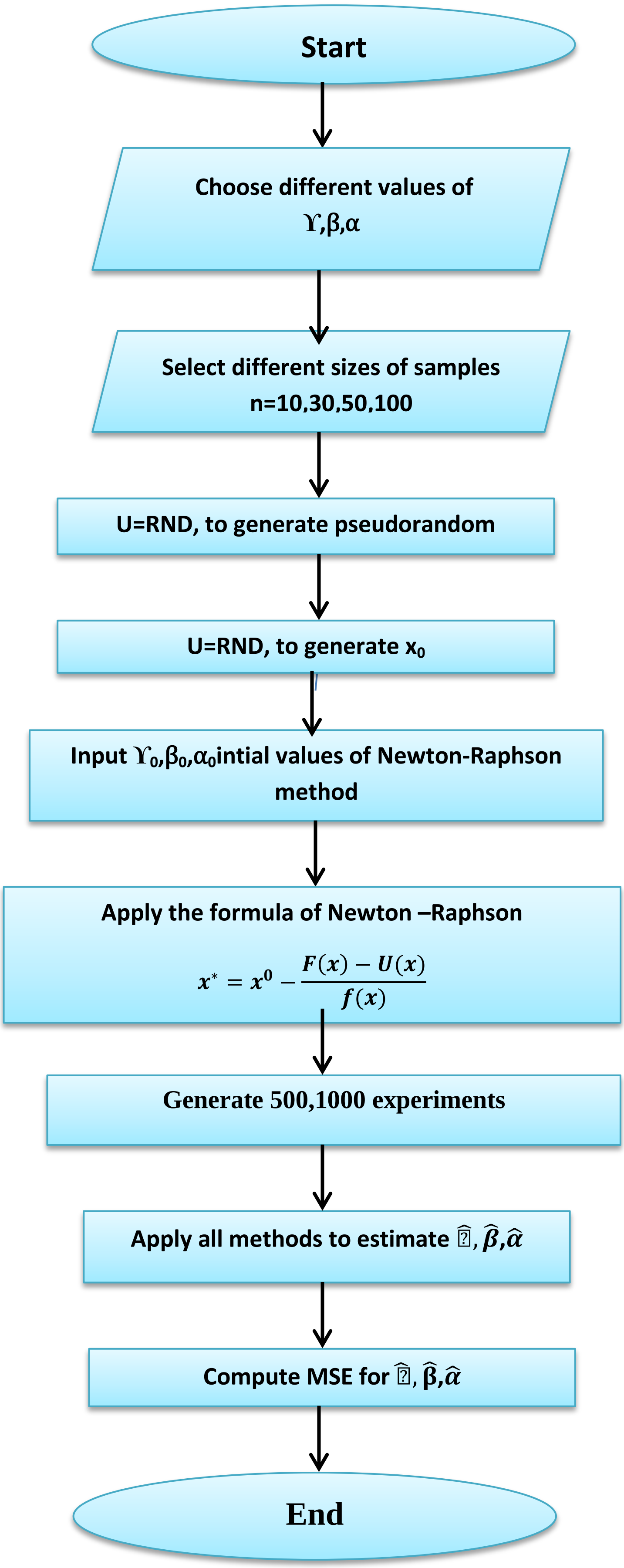
$$\text{MSE}\left(\hat{\Upsilon}\right) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\Upsilon} - \Upsilon)^2 \qquad (4\text{-}5)$$

$$\text{MSE}\left(\hat{\beta}\right) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\beta} - \beta)^2 \qquad (4\text{-}6)$$

$$\text{MSE}\,(\hat{\alpha}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\alpha} - \alpha)^2 \qquad (4\text{-}7)$$

Where Lis the number of repeating in each experiment and equal to L=500, 1000.

The modified algorithm of Second Simulation Experiment



4.3.2.1 Numerical results to $(\hat{\gamma}, \hat{\beta}, \hat{\alpha})$ for all estimation methods

This subsection includes the estimate values of three parameters $\hat{\Upsilon}, \hat{\beta}$, and $\hat{\alpha}$ using the three estimation methods motioned in chapter 3. Moreover, the tables(4-39, 4-40, 4-41, 4-42) present the numerical results of $\hat{\Upsilon}$, the tables (4-43, 4-44, 4-45 ,4-46) present the numerical results of $\hat{\beta}$ and the tables (4-47, 4-48, 4-49, 4-50) present the numerical results of $\hat{\alpha}$ with repeating L =500. Moreover, the tables (4-51, 4-52, 4-53, 4-54) present the numerical results of $\hat{\Upsilon}$, the tables (4-55, 4-56, 4-57 ,4-58) present the numerical results of $\hat{\beta}$ and the tables (4-59, 4-60, 4-61, 4-62) present the numerical results of $\hat{\alpha}$ with repeating L=1000. Besides that, determined the Mean Square error measure of the estimator $\hat{\Upsilon}, \hat{\beta}$ and $\hat{\alpha}$ for all methods of estimation that are studied in this thesis and showing that for the first experiment the tables(4-63,4-64,4-65) present the MSE of ϒ,β,α for repeating L =500, 1000.Farthoremore, the tables(4-66,4-67,4-68) present the MSE of ϒ,β, α for the second experiment. The tables (4-69,4-70,4-71) present the MSE of α.for the third experiment, the tables(4-72, 4-73, 4-74)present the MSE of ϒ,β,α for the forth experiment.

4.3.2.1.1 Numerical values of estimator ϒ(L=500)

- For n=10, 30, 50, 100 present the values of estimator $(\hat{\Upsilon})$for all estimation methods are increasing for increasing true values of parameter (ϒ) repeating L=500.
- For n=10, 30, 50, 100 the minimum values of estimator $(\hat{\Upsilon})$for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(ϒ) which are given in the following tables:

Table (4-39): Represents average values of $\hat{\square}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	$\square$	**0.0002**	**0.0002**	**0.0002**
	$\hat{\square}$	**0.000564117**	**0.000199999999999991**	**0.00037727**
E2	$\square$	**0.0003**	**0.0003**	**0.0003**
	$\hat{\square}$	**0.000671652**	**0.000299999999999949**	**0.000585486**
E3	$\square$	**0.00025**	**0.00025**	**0.00025**
	$\hat{\square}$	**0.001224201**	**0.000249999999999997**	**0.000418805**
E4	$\square$	**0.0005**	**0.0005**	**0.0005**
	$\hat{\square}$	**0.000942011**	**0.000499999999999939**	**0.000933892**

Table (4-40): Represents average values of $\hat{\square}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	$\square$	**0.0002**	**0.0002**	**0.0002**
	$\hat{\square}$	**0.001062428**	**0.000199999999999997**	**0.000369607**
E2	$\square$	**0.0003**	**0.0003**	**0.0003**
	$\hat{\square}$	**0.001250046**	**0.000300000000000001**	**0.000521055**
E3	$\square$	**0.00025**	**0.00025**	**0.00025**
	$\hat{\square}$	**0.002193213**	**0.00025000000000002**	**0.000416282**
E4	$\square$	**0.0005**	**0.0005**	**0.0005**
	$\hat{\square}$	**0.001861138**	**0.000499999999999975**	**0.000918127**

Table (4-41): Represents average values of $\hat{\square}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	$\square$	**0.0002**	**0.0002**	**0.0002**
	$\hat{\square}$	**0.001665917**	**0.000199999999999981**	**0.000180251**
E2	$\square$	**0.0003**	**0.0003**	**0.0003**
	$\hat{\square}$	**0.000225374**	**0.000299999999999992**	**0.000498443**
E3	$\square$	**0.00025**	**0.00025**	**0.00025**
	$\hat{\square}$	**0.005769982**	**0.000249999999999944**	**0.0002883**
E4	$\square$	**0.0005**	**0.0005**	**0.0005**
	$\hat{\square}$	**0.004706899**	**0.000499999999999936**	**0.000184433**

Table (4-42): Represents average values of ⍰̂ where n=100

experiment	parameter	MLE	OLS	RSS
E1	⍰	0.0002	0.0002	0.0002
	⍰̂	0.002001206	0.00019999999999998	0.000314079
E2	⍰	0.0003	0.0003	0.0003
	⍰̂	0.004236219	0.00030000000000006	0.000420754
E3	⍰	0.00025	0.00025	0.00025
	⍰̂	1.09E-02	0.000249999999999995	1.52E-05
E4	⍰	0.0005	0.0005	0.0005
	⍰̂	0.015486943	0.000500000000000004	2.25745E-05

From tables (4-39, 4-40, 4-41, 4-42) according to the result which we motion above, noting that the estimate the scale parameter ϒ of New Mixture distribution which is very closed in the ordinary least square method.

4.3.2.1.2 Numerical values of estimatorβ(L=500)

- For n=10, 30, 50, 100 present the values of estimator ($\hat{\beta}$)for all estimation methods are increasing for increasing true values of parameter (β).
- For n=10, 30, 50, 100 the minimum values of estimator ($\hat{\beta}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(β) which are given in the following tables:

Table (4-43): Represents average values of $\hat{\beta}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	β	0.000137	0.000137	0.000137
	$\hat{\beta}$	1.27E-03	0.000136999999999662	0.000215863
E2	β	0.000156	0.000156	0.000156
	$\hat{\beta}$	0.000683016	0.000155999999999895	0.000323294
E3	β	0.000447	0.000447	0.000447
	$\hat{\beta}$	0.001894013	0.000446999999999775	0.000979441
E4	β	0.000147	0.000147	0.000147
	$\hat{\beta}$	0.001281661	0.000146999999999997	0.00038403

Table (4-44): Represents average values of $\hat{\beta}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	β	0.000137	0.000137	0.000137
	$\hat{\beta}$	0.004585143	0.000136999999999873	0.000255977
E2	β	0.000156	0.000156	0.000156
	$\hat{\beta}$	0.002262211	0.000155999999999685	0.000420235
E3	β	0.000447	0.000447	0.000447
	$\hat{\beta}$	0.006795929	0.000446999999999913	0.000976149
E4	β	0.000147	0.000147	0.000147
	$\hat{\beta}$	0.005017763	0.000146999999999984	0.000335994

Table (4-45): Represents average values of $\hat{\beta}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	β	0.000137	0.000137	0.000137
	$\hat{\beta}$	0.00483477	0.000136999999999996	0.000289522
E2	β	0.000156	0.000156	0.000156
	$\hat{\beta}$	0.006395646	0.000155999999999999	7.18428E-05
E3	β	0.000447	0.000447	0.000447
	$\hat{\beta}$	0.010642324	0.00044699999999995	0.001037969
E4	β	0.000147	0.000147	0.000147
	$\hat{\beta}$	0.004984129	0.000146999999999993	0.001405705

Table (4-46): Represents average values of $\hat{\beta}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	β	0.000137	0.000137	0.000137
	$\hat{\beta}$	0.016335924	0.000136999999999702	0.000395757
E2	β	0.000156	0.000156	0.000156
	$\hat{\beta}$	0.00801536	0.000155999999999418	0.00041557
E3	β	0.000447	0.000447	0.000447
	$\hat{\beta}$	0.000567945	0.000446999999999917	0.001191444
E4	β	0.000147	0.000147	0.000147
	$\hat{\beta}$	0.010453787	0.000146999999999989	0.000381901

From tables (4-43, 4-44, 4-45, 4-46) according to the result which we motion above, noting that the estimate the scale parameter β of New Mixture distribution which is very closed in the ordinary least square method.

4.3.2.1.3 Numerical values of estimator α (L=500)

- For n=10, 30, 50, 100 present the values of estimator ($\hat{\alpha}$)for all estimation methods are increasing for increasing true values of parameter (α).
- For n=10, 30, 50, 100 the minimum values of estimator ($\hat{\alpha}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(α) which are given in the following tables:

Table (4-47): Represents average values of $\hat{\alpha}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	α	0.00013	0.00013	0.00013
	$\hat{\alpha}$	0.000325	0.000129999999999965	2.49E-04
E2	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.000518	0.000229999999999922	0.000467991
E3	α	0.00032	0.00032	0.00032
	$\hat{\alpha}$	0.006950834	0.000319999999999959	0.000324004
E4	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.005651516	0.000229999999999974	0.000480429

Table (4-48): Represents average values of $\hat{\alpha}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	α	0.00013	0.00013	0.00013
	$\hat{\alpha}$	0.010221528	0.00012999999999995	0.000276443
E2	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.01326005	0.00023000000000005	0.00049109
E3	α	0.00032	0.00032	0.00032
	$\hat{\alpha}$	0.019775703	0.000319999999999958	0.000613644
E4	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.022455149	0.0002300000000004	0.00050191

Table (4-49): Represents average values of $\hat{\alpha}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	α	0.00013	0.00013	0.00013
	$\hat{\alpha}$	0.00185	0.000129999999999982	0.000275527
E2	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.023748436	0.000229999999999984	0.000306553
E3	α	0.00032	0.00032	0.00032
	$\hat{\alpha}$	0.038983722	0.000319999999999962	4.49736E-05
E4	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.039994548	0.000229999999999945	0.000906176

Table (4-50): Represents average values of $\hat{\alpha}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	α	0.00013	0.00013	0.00013
	$\hat{\alpha}$	0.027335652	0.000129999999999963	0.000294395
E2	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.05142268	0.000230000000077777	5.37E-04
E3	α	0.00032	0.00032	0.00032
	$\hat{\alpha}$	0.078454261	0.000320000000007777	6.62E-04
E4	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.098358833	0.000229999999999987	0.000606512

From tables (4-47, 4-48, 4-49, 4-50) with repeating L=1000, according to the result which we motion above, noting that the estimate the shape parameter α of New Mixture distribution which is very closed in the ordinary least square method.

4.3.2.1.4 <u>Numerical values of estimator ϒ(L=1000)</u>

- The results in tables(4-14, 4-15, 4-16, 4-17)present the values of estimator ($\hat{\Upsilon}$)for all estimation methods are increasing for increasing true values of parameter(ϒ) for n=10, 30 ,50 ,100 repeating L=1000.
- the minimum values of estimator ($\hat{\Upsilon}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(ϒ) for n=10, 30, 50, 100 which are given in the following tables:

Table (4-51): Represents average values of ϒˆwhere n=10

experiment	parameter	MLE	OLS	RSS
E1	□	**0.0002**	**0.0002**	**0.0002**
	□̂	**0.00038158**	**0.000199999999999913**	**0.000356273**
E2	□	**0.0003**	**0.0003**	**0.0003**
	□̂	**0.000571655**	**0.000299999999999912**	**0.000559347**
E3	□	**0.00025**	**0.00025**	**0.00025**
	□̂	**0.000939965**	**0.000249999999999985**	**0.000413799**
E4	□	**0.0005**	**0.0005**	**0.0005**
	□̂	**0.001067404**	**0.000499999999999995**	**0.000893372**

Table (4-52): Represents average values of □̂ where n=30

experiment	parameter	MLE	OLS	RSS
E1	□	**0.0002**	**0.0002**	**0.0002**
	□̂	**0.000848711**	**0.000199999999999991**	**0.000358776**
E2	□	**0.0003**	**0.0003**	**0.0003**
	□̂	**0.002869857**	**0.00029999999999998**	**0.000549242**
E3	□	**0.00025**	**0.00025**	**0.00025**
	□̂	**0.002141921**	**0.000249999999999916**	**0.000418468**
E4	□	**0.0005**	**0.0005**	**0.0005**
	□̂	**0.003539951**	**0.00049999999999999**	**0.000657736**

Table (4-53): Represents average values of □̂ where n=50

experiment	parameter	MLE	OLS	RSS
E1	□	**0.0002**	**0.0002**	**0.0002**
	□̂	**0.000756835**	**0.000199999999999928**	**0.000365115**
E2	□	**0.0003**	**0.0003**	**0.0003**
	□̂	**0.004262595**	**0.000299999999999995**	**0.00037115**
E3	□	**0.00025**	**0.00025**	**0.00025**
	□̂	**0.005589419**	**0.000249999999999994**	**3.99E-05**
E4	□	**0.0005**	**0.0005**	**0.0005**
	□̂	**0.000809752**	**0.000499999999999992**	**0.000871802**

Table (4-54): Represents average values of ⍰̂ where n=100

experiment	parameter	MLE	OLS	RSS
E1	⍰	**0.0002**	**0.0002**	**0.0002**
	⍰̂	**0.004362411**	**0.00020000000000001**	**0.000290022**
E2	⍰	**0.0003**	**0.0003**	**0.0003**
	⍰̂	**0.00630609**	**0.000300000000000007**	**0.000305359**
E3	⍰	**0.00025**	**0.00025**	**0.00025**
	⍰̂	**0.01367093**	**0.000249999999999982**	**3.44E-05**
E4	⍰	**0.0005**	**0.0005**	**0.0005**
	⍰̂	**0.014985962**	**0.00050000000000003**	**0.000688938**

From tables (4-51, 4-52, 4-53, 4-54) with repeating L=1000, according to the result which we motion above, noting that the estimate the scale parameter ϒ of New Mixture distribution which is very closed in the ordinary least square method.

4.3.2.1.5 Numerical values of estimator $\boldsymbol{\beta}(\boldsymbol{L}=\mathbf{1000})$

- For n=10, 30, 50, 100 present the values of estimator ($\hat{\beta}$)for all estimation methods are increasing for increasing true values of parameter (β).
- For n=10, 30, 50, 100 the minimum values of estimator ($\hat{\beta}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(β) which are given in the following tables:

Table (4-55): Represents average values of $\hat{\beta}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	β	0.000137	0.000137	0.000137
	$\hat{\beta}$	0.001586077	0.000136999999999993	0.000343454
E2	β	0.000156	0.000156	0.000156
	$\hat{\beta}$	0.001321738	0.000155999999999698	0.000275842
E3	β	0.000447	0.000447	0.000447
	$\hat{\beta}$	0.001206676	0.000446999999999979	0.00075569
E4	β	0.000147	0.000147	0.000147
	$\hat{\beta}$	0.001887596	0.000146999999999254	0.000436496

Table (4-56): Represents average values of $\hat{\beta}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	β	0.000137	0.000137	0.000137
	$\hat{\beta}$	0.002874711	0.000136999999999671	0.000332377
E2	β	0.000156	0.000156	0.000156
	$\hat{\beta}$	0.000240374	0.000155999999999968	0.000354787
E3	β	0.000447	0.000447	0.000447
	$\hat{\beta}$	0.006747182	0.000446999999999998	0.000471455
E4	β	0.000147	0.000147	0.000147
	$\hat{\beta}$	0.002925721	0.000146999999999971	0.000443028

Table (4-57): Represents average values of $\hat{\beta}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	β	0.000137	0.000137	0.000137
	$\hat{\beta}$	0.008345166	0.000136999999999986	0.000324249
E2	β	0.000156	0.000156	0.000156
	$\hat{\beta}$	0.001160273	0.000155999999999871	0.000170744
E3	β	0.000447	0.000447	0.000447
	$\hat{\beta}$	0.000142032	0.000446999999999779	0.001307354
E4	β	0.000147	0.000147	0.000147
	$\hat{\beta}$	0.004888114	0.000146999999999984	0.000342793

Table (4-58): Represents average values of $\hat{\beta}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	β	0.000137	0.000137	0.000137
	$\hat{\beta}$	0.01193716	0.000136999999999947	0.000303149
E2	β	0.000156	0.000156	0.000156
	$\hat{\beta}$	0.002263163	0.000155999999999879	0.000228196
E3	β	0.000447	0.000447	0.000447
	$\hat{\beta}$	0.006848347	0.000446999999999998	0.001276205
E4	β	0.000147	0.000147	0.000147
	$\hat{\beta}$	0.012049378	0.000146999999999975	0.000427006

From tables (4-55, 4-56, 4-57, 4-58) with repeating L=1000, according to the result which we motion above, noting that the estimate the scale parameter β of New Mixture distribution which is very closed in the ordinary least square method.

4.3.2.1.6 Numerical values of estimator $\hat{\alpha}$ (L=1000)

- The results in tables(4-22, 4-23, 4-24, 4-25)present the values of estimator ($\hat{\alpha}$)for all estimation methods are increasing for increasing true values of parameter(α) for n=10, 30 ,50 ,100 repeating L=1000.
- the minimum values of estimator ($\hat{\alpha}$)for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter(α) for n=10, 30, 50, 100 which are given in the following tables:

Table (4-59): Represents average values of $\hat{\alpha}$ where n=10

experiment	parameter	MLE	OLS	RSS
E1	α	**0.00013**	**0.00013**	**0.00013**
	$\hat{\alpha}$	**0.003032984**	**0.000129999999999951**	**0.000281648**
E2	α	**0.00023**	**0.00023**	**0.00023**
	$\hat{\alpha}$	**0.005340238**	**0.00023000000568712**	**0.000468886**
E3	α	**0.00032**	**0.00032**	**0.00032**
	$\hat{\alpha}$	**0.00667624**	**0.000319999999999919**	**1.3729E-05**
E4	α	**0.00023**	**0.00023**	**0.00023**
	$\hat{\alpha}$	**0.005797184**	**0.000230000000000006**	**0.000509607**

Table (4-60): Represents average values of $\hat{\alpha}$ where n=30

experiment	parameter	MLE	OLS	RSS
E1	α	**0.00013**	**0.00013**	**0.00013**
	$\hat{\alpha}$	**0.008229825**	**0.000129999999999987**	**0.00027673**
E2	α	**0.00023**	**0.00023**	**0.00023**
	$\hat{\alpha}$	**0.01988364**	**0.000229999999999961**	**0.00048997**
E3	α	**0.00032**	**0.00032**	**0.00032**
	$\hat{\alpha}$	**0.019104957**	**0.000319999999999779**	**7.76052E-06**
E4	α	**0.00023**	**0.00023**	**0.00023**
	$\hat{\alpha}$	**0.02526177**	**0.000229999999999878**	**0.000653818**

Table (4-61): Represents average values of $\hat{\alpha}$ where n=50

experiment	parameter	MLE	OLS	RSS
E1	α	**0.00013**	**0.00013**	**0.00013**
	$\hat{\alpha}$	**0.012794253**	**0.000129999999999916**	**0.000272226**
E2	α	**0.00023**	**0.00023**	**0.00023**
	$\hat{\alpha}$	**0.031882553**	**0.000229999999999931**	**0.000288377**
E3	α	**0.00032**	**0.00032**	**0.00032**
	$\hat{\alpha}$	**0.040784979**	**0.000319999999999939**	**0.000900716**
E4	α	**0.00023**	**0.00023**	**0.00023**
	$\hat{\alpha}$	**0.038844907**	**0.000229999999999976**	**0.000509321**

Table (4-62): Represents average values of $\hat{\alpha}$ where n=100

experiment	parameter	MLE	OLS	RSS
E1	α	0.00013	0.00013	0.00013
	$\hat{\alpha}$	0.038467604	0.000129999999999982	0.000292506
E2	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.059319026	0.00022999999999999	0.000272653
E3	α	0.00032	0.00032	0.00032
	$\hat{\alpha}$	0.08342274	0.000319999999999906	0.000388816
E4	α	0.00023	0.00023	0.00023
	$\hat{\alpha}$	0.097087378	0.00023000000000002	0.000632103

From tables (4-59, 4-60, 4-61, 4-62) with repeating L=1000, according to the result which we motion above, noting that the estimate the shape parameter α of New Mixture distribution which is very closed in the ordinary least square method.

Now the following tables introduce the mean square error of the three parameters of New Mixture distribution of all classical methods of this study include different sizes of samples n=10, 30, 50, 100 with repeating L=500, 1000.

Table (4-63): Represents the MSE of ⍰ for the E_1 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.00000036694	3.1661E-35	0.00490	OLS
	30	0.0000022527	1.864E-35	0.0026	OLS
	50	0.0000058363	7.2008E-36	0.0002518	OLS
	100	0.000022792	3.6964E-35	0.00590	OLS
1000	10	0.00000024117	2.4222E-35	0.0132	OLS
	30	0.0000021444	2.2967E-35	0.0193	OLS
	50	0.0000057182	1.4031E-35	0.0208	OLS
	100	0.000023375	1.1023E-35	0.0017	OLS

Table (4-64): Represents the MSE of β for the E_1 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.0000016459	2.4673E-34	0.00072359	OLS
	30	0.000023055	3.8979E-35	0.0022	OLS
	50	0.000068348	1.8527E-35	0.0014	OLS
	100	0.00028596	3.1759E-34	0.0018	OLS
1000	10	0.0000026663	2.4243E-34	0.00390	OLS
	30	0.000023606	2.1432E-34	0.0066	OLS
	50	0.000072725	1.6479E-34	0.008	OLS
	100	0.00027563	2.6748E-35	0.002	OLS

Table (4-65): Represents the MSE of α for the E_1 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000018551	2.9652E-34	0.000052633	OLS
	30	0.00014717	1.4389E-34	0.00031098	OLS
	50	0.00040088	8.8051E-35	0.000041055	OLS
	100	0.0016	1.5123E-34	0.000027275	OLS
1000	10	0.000015795	3.2238E-34	0.00028761	OLS
	30	0.00014292	9.8248E-35	0.00013256	OLS
	50	0.00039478	2.9083E-34	0.000034274	OLS
	100	0.0016	4.2076E-34	0.00010771	OLS

Table (4-66): Represents the MSE of □ for the E_2 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.0000015819	3.9245E-35	0.00580	OLS
	30	0.0000093269	6.3841E-35	0.0036	OLS
	50	0.000024128	1.8639E-36	0.0126	OLS
	100	0.1041	1.0187E-34	0.0133	OLS
1000	10	0.0000010968	2.5808E-35	0.0836	OLS
	30	0.000020984	8.8624E-36	0.0875	OLS
	50	0.000051189	6.8427E-36	0.075	OLS
	100	0.00011124	4.7433E-36	0.0777	OLS

Table (4-67): Represents the MSE of β for the E_2 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.00000083494	1.9259E-34	0.0061	OLS
	30	0.000011098	6.2725E-34	0.0098	OLS
	50	0.000054059	6.7375E-36	0.0032	OLS
	100	0.0666	1.1911E-33	0.0025	OLS
1000	10	0.0000019116	1.7021E-34	0.00032403	OLS
	30	0.0000022678	1.4221E-35	0.00025287	OLS
	50	0.000013551	7.8262E-35	0.0023	OLS
	100	0.00020023	7.0526E-35	0.00063636	OLS

Table (4-68): Represents the MSE of α for the E_2 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000059651	4.6033E-34	0.00004524	OLS
	30	0.00047438	3.1974E-34	0.000067033	OLS
	50	0.0012	9.1289E-35	0.000022839	OLS
	100	5.6925	5.3839E-34	0.000050329	OLS
1000	10	0.000051243	1.5414E-33	0.00003171	OLS
	30	0.00060054	4.6232E-35	0.00016859	OLS
	50	0.0016	1.0408E-34	0.0008517	OLS
	100	0.0051	1.2156E-34	0.00043572	OLS

Table (4-69): Represents the MSE of □ for the E_3 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000002242	1.533E-35	0.04	OLS
	30	0.000020088	2.2026E-35	0.0125	OLS
	50	0.000051148	9.3857E-36	0.0236	OLS
	100	0.0002167	4.8546E-36	0.0342	OLS
1000	10	0.0000022769	1.0287E-35	0.1631	OLS
	30	0.000019084	1.3669E-35	0.0618	OLS
	50	0.000050979	5.5551E-36	0.1494	OLS
	100	0.00020789	4.0511E-36	0.0487	OLS

Table (4-70): Represents the MSE of β for the E_3 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.0000056575	2.4537E-34	0.0062	OLS
	30	0.000051542	3.3027E-35	0.0204	OLS
	50	0.00015268	7.3695E-35	0.0071	OLS
	100	0.00061956	1.5893E-35	0.00490	OLS
1000	10	0.0000036045	1.3745E-35	0.0266	OLS
	30	0.000052092	1.1966E-34	0.0407	OLS
	50	0.000147	5.8327E-35	0.011	OLS
	100	0.00070951	1.8575E-35	0.0712	OLS

Table (4-71): Represents the MSE of $\boldsymbol{\alpha}$ for the E_3 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.000073631	1.5578E-34	0.000040197	OLS
	30	0.0006677	1.7322E-34	0.000059738	OLS
	50	0.0018	1.0577E-34	0.00010888	OLS
	100	0.00725	1.499E-34	0.00016092	OLS
1000	10	0.00007863	1.4793E-34	0.0000065982	OLS
	30	0.00066032	1.5038E-34	0.000017942	OLS
	50	0.0018	1.6515E-34	0.00082004	OLS
	100	0.0071	9.1043E-35	0.0034	OLS

Table (4-72): Represents the MSE of □ for the E_4 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.0000029811	1.8408E-35	0.0083	OLS
	30	0.000021048	3.6077E-36	0.0187	OLS
	50	0.000064161	1.1774E-35	0.0226	OLS
	100	0.00025022	4.9422E-36	0.000026496	OLS
1000	10	0.0000019142	6.9029E-35	0.0143	OLS
	30	0.000019367	4.8306E-36	0.1398	OLS
	50	0.000058241	1.1857E-36	0.0417	OLS
	100	0.00024171	4.0029E-36	0.0133	OLS

Table (4-73): Represents the MSE of β for the E_4experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.0000020359	2.3105E-34	0.0247	OLS
	30	0.000028554	5.3439E-36	0.0118	OLS
	50	0.00005109	1.641E-34	0.0052	OLS
	100	0.00028896	4.277E-36	0.0327	OLS
1000	10	0.0000042342	6.1845E-34	0.1702	OLS
	30	0.000029651	2.4641E-35	0.0025	OLS
	50	0.000077224	2.2387E-36	0.0811	OLS
	100	0.00028176	3.772E-36	0.0652	OLS

Table (4-74): Represents the MSE of α for the E_4 experiment

L	n	MLE	OLS	RSS	BEST
500	10	0.00010437	3.3484E-35	0.00017865	OLS
	30	0.00088283	1.4195E-34	0.00029966	OLS
	50	0.0026	4.6172E-35	0.00081742	OLS
	100	0.01	7.118E-35	0.0011	OLS
1000	10	0.000092168	2.204E-34	0.0012	OLS
	30	0.0008544	1.2284E-34	0.0016	OLS
	50	0.002534	8.8528E-35	0.0001643	OLS
	100	0.013	6.5771E-35	0.0031	OLS

From tables (4-63, 4-64, 4-65, 4-66,4-67, 4-68,4-69,4-70, 4-71, 4-72, 4-73, 4-74) according to the result that we motion above, noting that the best classical method that used to estimate the parameters (ϒ,β, α) of New Mixture distribution which is the ordinary least square method.

4.4 Combination between the two methods of simulation

In this subsection, introduce simple combination between the first and second method of simulation. Besides, represents the Similarities and differences between them as the following table:

Table (4-75): Represents the combination between the first and second method of simulation

n	First method	Second method
1	The method depends on the inverse of the cumulative function of the New Mixture distribution.	The method depends on the Newton-Raphson formula.
2	Using uniform distribution one time for U=F(x)	Using uniform distribution two times for the formula one time of the fraction's numerator and the second to generate x_0.
3	The size of the sample is N=10,30,50,100 with repeating the experiments 500, 1000 times.	The size of the sample is N=10,30,50,100 with repeating the experiments 500, 1000 times.
4	Fixed α=1 for all experiments to generate positive values of x	Choose different value of the α, in order to use it to generate positive values of x
5	One of the positives of this method is the singularity of error for all classical methods.	The singularity of error for two classical MLE,OLS methods but the deference for the rank sampling set method.
6	The best method with minimum square error is ordinary least square estimation method	The best method with minimum square error is ordinary least square estimation method
7	For all estimation methods with n=10, 30 , 50, 100 the value of $(\hat{\Upsilon},\hat{\beta},\hat{\alpha})$ are increasing true values of parameters (ϒ,β,α)for MLE and OLS but it is oscillation in RSS. That means the estimate values of parameters are bigger or smaller than the true value.	For all estimation methods with n=10, 30 , 50, 100 the value of $(\hat{\Upsilon},\hat{\beta},\hat{\alpha})$ are increasing for increasing true values of parameters (ϒ,β,α).
8	The second order of the best estimation method according to the minimum square error is rank sampling set method	The second order of the best estimation method according to the minimum square error is rank sampling set method
9	The minimum values of estimators$(\hat{\Upsilon},\hat{\beta},\hat{\alpha})$ for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter for n=10, 30, 50, 100.	The minimum values of estimators$(\hat{\Upsilon},\hat{\beta},\hat{\alpha})$ for all estimation methods which are very close to the true values especially in ordinary least square method which is converged to the true value of the parameter for n=10, 30, 50, 100.

4.5 <u>Practical application (1)</u>

The Kut textile factory of the General Company for Textile Industries consists of two parts:

Yarn section and textile section. The textile department has been selected to collect the samples required for the following application because the section includes various industrial products such as (jerseys, underwear Stockings, textiles of all kinds). This data is collect by researcher Haider R., Talib[9]. The data on the machines of the spinning department were collected if the failure times were taken until the sample was 100In the table:

Table (4-76): Represents data of the actual sample stopping times for one month:

10	**9**	**12.5**	**3**	**9.5**	**18.5**	**8.5**	**13**	**7.5**	**4**
9.5	17	1.5	9	11.5	18	21	13.5	15.5	16
8	22	20	2	15	14.5	7.5	4	2	4
6.5	26.5	7	7	8	3	5	7	14.5	18.5
10	14	12	14	9.5	14	9.5	15.5	17.5	6.5
12	4.5	18	21.5	13	3	12	8	4.5	20
11	8	12	18	8.5	5	12	3.5	3	6.5
18.5	9	15	14	20	12	22.5	1	27	8.5
8	5	5.5	8	21.5	7	20.5	4	10	15
12	12	2	1	19	16.5	17.5	13	16	14

To show this data distribute as the data of New Mixture distribution, follow this hypothesis test:

H_0: Suppose θ_0 distribute as a data for New Mixture distribution.

H_1: Suppose θ_0 not distribute as a data for New Mixture distribution.

Using the test standard is likelihood ratio test as formula:

LR=$-2\ln[\frac{\ell(\theta_0)}{\ell(\widehat{\theta})}]$

Suppose that θ_0=(ϒ= 0.0078, β = 0.0073, α = 0.0006)

Then, -2 $ln\ell(\theta_0) = 843.4663$, using maximum likelihood estimation method to estimate the θ_0 and find $\hat{\theta} = (? = 0.0109, \beta = 0.0088, \alpha = 0.0003)$

Getting, 2 $ln\ell(\widehat{\theta})$=-838.4730, denote the value of LR is λ.

$$\lambda = -2(ln\ell(\theta_0) - ln\ell(\widehat{\theta})) = \left(2ln\ell\widehat{\theta}) - 2ln\ell(\theta_0)\right)$$

$$= -838.4730 + 843.4663 = 4.9933$$

Then $\lambda = 4.9933$ *is* the likelihood ratio test take c is the critical region 0≤c≤1.

if the value of λ>c then do not reject H_0.

If the value of $\lambda < c$ then reject H_0.

As the result, since $\lambda = 4.9933 \; > c \; then \; do \; not \; reject \; H_0$.

Then this data has distributed as a new Mixture distribution.

4.5.1 The Information creation of real data

Comparing the New Mixture distribution with many related distributions as Weibull, exponential Rayleigh, exponential Weibull, and Rayleigh Weibull distributions. Using some information creations noting that, the New Mixture distribution is good fitting comparing with the distributions motion above according to the result of the following table:

Table (4-77): Represents the Information Criterion for data of actual sample stopping times for one month

	Distribution	Estimation parameters	Standard Error	$-2LL$	AIC	AIC_C	BIC
1	WEIBULL	$\hat{\alpha}$ = 3.1603	0.3397	897.5384	899.5384	899.579216	902.1435702
2	EXPONENTIAL RAYLEIGH	⍰ = 0.0156 $\hat{\beta}$ = 0.000152	0.01557350. 0086480	849.5994	853.5994	855.313685	858.9079404
3	RAYLIEGH WEIBULL	$\hat{\beta}$ = 0.0056 $\hat{\alpha}$ = 0.0308	0.0023 0.01087	864.8482	868.8482	868.971912	874.0585404
4	EXPONENTIAL WEIBULL	⍰ = 0.01 $\hat{\alpha}$= 0.0000397	0.0005 0.0036603	1143.1	1147.1	1147.22372	1152.310341
5	NEW MIXTURE DISTRIBUTION	⍰ = 0.01031744 $\hat{\beta}$ = 0.00914199 $\hat{\alpha}$ = 0.000065394	0.00301744 0.0012494 0.00053461	838.0050	844.0050	844.255	851.820510

4.5.2 The result of application

This subsection introduced the mean square of all estimation methods(Maximum Likelihood estimation, Ordinary Least Square estimation, and Rank Sampling set) which is used to estimate the three parameters (ϒ,β,α) of New Mixture distribution as follows:

Table(4-78): Represents the parameters estimator of New Mixture distribution

n	MLE $(\widehat{\square},\hat{\beta},\hat{\alpha})$	OLS $(\widehat{\square},\hat{\beta},\hat{\alpha})$	RSS $(\widehat{\square},\hat{\beta},\hat{\alpha})$
100	0.000670521	0.000949999999999996	0.0011328
	0.000191996	0.00000950000000000102	0.000157766
	0.000171061	0.000370000000000003	0.000272019

The initial value of Newton-Raphson method which is used of this application is ϒ= 0.00095, β = 0.0000097, α = 0.00037. The mean square error is the measure to compare between all of the classical methods which are used of this study. The result of mean square error which is motion of this table:

Table(4-79):the MSE of all estimation method

n	MLE $(\widehat{\square},\hat{\beta},\hat{\alpha})$	OLS $(\widehat{\square},\hat{\beta},\hat{\alpha})$	RSS $(\widehat{\square},\hat{\beta},\hat{\alpha})$
100	7.81084E-08	1.60925E-35	3.34158E-08
	3.32318E-08	1.04005E-36	2.19236E-08
	3.95769E-08	8.88968E-36	9.60034E-09

Noting that the ordinary least square method is the best method comparing with the all classical methods which are used of this study because the mean square error of it is less than the mean square error of the Maximum Likelihood estimation and Rank Sampling set estimation method. The second order of the best is the Rank Sampling estimation and finally is the Maximum Likelihood estimation method.

4.6 Practical application (2)

These data were collected from the Ministry of Health, especially from the Yarmouk Teaching Hospital for patients who died of Prostate and bladder cancer. It starts from 24/7/2010 to 28/7/2016. The sample included ten deceased:

Table (4-80): Represents the number of days the patient has lived from the moment she enters the hospital until his death.

N	1	2	3	4	5	6	7	8	9	10
patient death	2	16	36	54	59	159	197	210	364	533

To show this data distribute as the data of New Mixture distribution, follow this hypothesis test:

H_0: Suppose θ_0 distribute as a data for New Mixture distribution.

H_1: Suppose θ_0 not distribute as a data for New Mixture distribution.

Using the test standard is likelihood ratio test as formula:

LR=$-2\ln[\frac{\ell(\theta_0)}{\ell(\hat{\theta})}]$

Suppose that θ_0=(ϒ= 0.00192, β = 0.0000001, α = 0.0017)

Then, -2$ln\ \ell(\theta_0) = 146.8552$, using ranking sampling set method to estimate the θ_0 and find $\hat{\theta} = (□ = 0.000334, \beta = 0.0000134, \alpha = 0.0099)$

Getting , 2 $ln\ell(\widehat{\theta})$= $-$ 145.0383, denote the value of LR is λ.

$$\lambda = -2(ln\ell(\theta_0) - ln\ell(\widehat{\theta})) = \left(2ln\ell\widehat{\theta}) - 2ln\ell(\theta_0)\right) =$$

$$-145.0383 + 146.8552 = 1.8169$$

Then $\lambda = 1.8169$the likelihood ratio test take c is the critical region 0≤c≤1.

if the value of λ>c then do not reject H_0.

If the value of $\lambda<c$ then reject H_0.

As the result, since $\lambda = 1.8169 > c$ then, do not reject H_0.

That means this data has distributed as a New Mixture distribution.

4.6.1 The information criterion of real data

Using some information criterion as the Akaike information criterion, corrected Akaike information criterion and the Bayesian information criterion to compare the New Mixture distribution with other distributions. According to the result of the following table, observing that the New Mixture distribution is good fitting than other related distributions. The following table includes the New Mixture distribution comparing with other distribution as follows:

Table (4-81): Represents the Information Criterion of the number of days the patient has lived from the moment she or he enters the hospital until his death.

	Distribution	Estimation parameters	Standard Error	$-2LL$	AIC	AIC_C	BIC
1	WEIBULL	$\hat{\alpha} =$ 35.893756	0.15933	174.3939	176.3939	176.8939	176.6964850
2	EXPONENTIAL RAYLIEGH	$\hat{⍰}$ = 0.00020355 $\hat{\beta}$ = 0.00000355	0.000201650 0.00009645	150. 9496	154.9496	156.663886	155.5547701
3	RAYLIEGH WEIBULL	$\hat{\beta}$ = 0.00002475 $\hat{\alpha}$ = 0.0046	0.00000825 0.002607	152.6202	156.6202	158.334485	157.2253701
4	EXPONENTIAL WEIBULL	$\hat{⍰}$ = 0.0009499 $\hat{\alpha}$= 0.00006531	0.000009910 0.000034689	162. 1965	166.1965	167.910786	166.8016701
5	NEW MIXTURE DISTRIBUTION	$\hat{⍰}$ = 0.0011 $\hat{\beta}$ = 0.00000581 $\hat{\alpha}$ = 0.0179	0.000820 0.00000419 0.0001	143.3172	149.3172	153.3172	150.224955

4.6.2 The result of application

This subsection introduced the mean square of all estimation methods(Maximum Likelihood estimation, Ordinary Least Square estimation, and Rank Sampling set) which is used to estimate the three parameters (ϒ,β,α) of New Mixture distribution as follows:

Table (4-82): Represents the parameters estimator of New Mixture distribution

n	MLE $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$	OLS $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$	RSS $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$
10	0.001261151	0.00192000000000003	0.000334289
	8.49E-06	9.99999999999859E-08	1.33E-05
	0.001483083	0.00169999999999999	0.009869731

The initial value of Newton-Raphson method which is used of this application is ϒ=0.00192, β=0.0000001, α=0.0017. The mean square error is the measure to compare between all of the classical methods which is used of this study. The result of mean square error which is motion of this table:

Table (4-83):the MSE of all estimation method

n	MLE $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$	OLS $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$	RSS $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$
10	4.34082E-07	8.95445E-34	2.51448E-06
	7.04607E-11	1.98673E-40	1.73204E-10
	4.70532E-08	9.94938E-35	6.67445E-05

Noting that the ordinary least square method is the best method comparing with the all classical methods which is used of this study because the mean square error of it is less than the mean square error of the maximum likelihood estimation and rank sampling set estimation method. The second order of the best is maximum likelihood estimation for estimate the parameters ϒ, β, α and finally is the rank sampling estimation method.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5-1 Introduction

This chapter consists many survey conclusions and recommendations for this thesis. It is discussed with the theoretical part about a New Mixture distribution according to the contributions and the results of the simulation study that can be used to evaluate the behavior of the estimators for unknown shape parameter(α) and scale parameters(ϒ,β) of New Mixture distribution.

5-2 Conclusion

There are many significant conclusions of this study which are as the following:

1. In the first method of simulation study for MLE, OLS estimation methods, noting that the estimator of scales and shape parameters $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$are increasing for increasing true value of parameter ϒ,β,α.
2. In the first method of simulation study observing that, for RSS estimation method that the estimator of scales and shape parameters $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$ are oscillation. That means the estimate values of parameters are bigger or smaller than the true value of parameter ϒ,β,α.
3. In the second method of simulation study noting that, for all estimation methods the scales and shape parameters $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$ are increasing for increasing the true value of parameter ϒ,β,α.
4. In OLS estimation method, note that the estimators of scales and shape parameters $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$ are very closed to the true value of parameter ϒ for all sample size which is used of this study.
5. Noting that, for all estimation methods, the mean squares error for $(\hat{\Upsilon}, \hat{\beta}, \hat{\alpha})$ are oscillation for increasing true values of parameter (ϒ,β,α).

6. For all estimation methods, noting that the mean squares error of parameter $(\hat{\gamma}, \hat{\beta}, \hat{\alpha})$ are increasing for increasing the samples sizes especially in the first method of simulation study.
7. The first order of the best estimation method in the two methods of simulation study is the least square estimation method.
8. The second order of the best estimation method in the two methods of the simulation study is the rank set sampling method and finally is the maximum likelihood estimation method.
9. For real data applications, the best estimation method is the ordinary least square method depends on the mean square error as a measure of comparison between the classical estimation methods.
10. The second order of the best estimation method for real data applications is the maximum likelihood estimation method and finally is the ranking set sampling estimation method.
11. For real data applications, the scale parameter $(\hat{\gamma})$ are increasing for increasing the true value of parameter (Υ).
12. For real data applications, the scale parameter $(\hat{\beta})$ are oscillation for increasing the true value of parameter (β).
13. For real data applications, the shape parameter $(\hat{\alpha})$ are oscillation for increasing the true value of parameter (α).
14. The last conclusion is that the mean squares error of real data applications are increasing for increasing the samples sizes.

5-3 Recommendations

There are many important notes which are used for fracture study as follows:

1. Using the concurring data for the application of the New Mixture distribution.
2. Find other estimation methods as Bayesian estimation methods to estimate the three parameters of New Mixture distribution.
3. Comparing between the Bayesian estimation method and the classical estimation methods.
4. Using other classical methods to estimate the three parameters of New Mixture distribution.
5. A study of the generalization of this method used for mixing different distributions.

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APPENDIX

SOME PROGGRAMING OF THE NEW MIXTURE DISTRIBUTION

MLE for New Mixture Distribution

```
n=size of sample;
a(1,1)= ……;a(2,1)= ……..;a(3,1)=……..;
q=0;w=0;s=0;r=0;t=0;y=0;u=0;v=0;
qq=0;ww=0;ss=0;rr=0;ty=0;yt=0;qw=0;wq=0;sr=0;rs=0;tt=0;yy=0;uu=0;vv
=0;
for i=1:n
   sd(i)=(2*a(1,1)+(a(2,1)*x(i))+(a(3,1)*(x(i)^(a(3,1)-1))^(-2)));
   q=q+1;
   qq=qq+sd(i);

f1=(-4)*qq;

   sf(i)=((2*a(1,1)+(a(2,1)*x(i))+(a(3,1)*(x(i)^(a(3,1)-1))))^(-2))*(x(i));
   w=w+1;
   ww=ww+sf(i);

f2=(-2)*ww;

   sg1(i)=((2*a(1,1)+(a(2,1)*x(i))+(a(3,1)*(x(i)^(a(3,1)-1))))^(-
2))*(x(i)^(a(3,1)-1));
   sg2(i)=a(3,1)*log(x(i))+1;
   s=s+1;
   ss=ss+(sg1(i)*sg2(i));

f3=(-2)*ss;

   sh(i)=((2*a(1,1)+(a(2,1)*x(i))+(a(3,1)*(x(i)^(a(3,1)-1))))^(-2))*(x(i))^2;
   r=r+1;
   rr=rr+sh(i);

f4=(-1)*rr;

   sk1(i)=((2*a(1,1)+(a(2,1)*x(i))+(a(3,1)*(x(i)^(a(3,1)-1))))^(-
2))*(x(i)^a(3,1));
   sk2(i)=(a(3,1)*log(x(i)))+1;
   t=t+1;
   tt=tt+(sk1(i)*sk2(i));

f5=(-1)*tt;
```

```
    sv1(i)=((2*a(1,1)+(a(2,1)*x(i))+(a(3,1)*(x(i)^(a(3,1)-1))))^(-
1))*(x(i)^(a(3,1)-1)*log(x(i)));
    sv2(i)=(a(3,1)*log(x(i)))+2;
    t=t+1;
    tt=tt+(sv1(i)*sv2(i));


    sv3(i)=((2*a(1,1)+(a(2,1)*x(i))+(a(3,1)*(x(i)^(a(3,1)-1))))^(-
2))*(x(i)^(2*a(3,1)-2));
    sv4(i)=(a(3,1)*log(x(i))+1)^2;
    u=u+1;
    uu=uu+(sv3(i)*sv4(i));


    sx(i)=(x(i)^a(3,1))*((log(x(i)))^2);
    v=v+1;
    vv=vv+sx(i);

f6=tt-uu-vv;
end
J=[f1 f2 f3 ; f2 f4 f5 ; f3 f5 f6];
J1=inv(J);

 for j=1:n
    sd1(j)=(2*a(1,1)+a(2,1)*x(j)+(a(3,1)*(x(j)^(a(3,1)-1))))^(-1);

    qw=qw+sd1(j);

   wq=wq+(x(j));

F=2*(qw)-2*(wq);

    sd2(j)=((2*a(1,1)+a(2,1)*x(j)+a(3,1)*(x(j)^(a(3,1)-1)))^(-1))*(x(j));

    sr=sr+sd2(j);

  rs=rs+(x(j).^2);

G=sr-(rs/2);

    sd3(j)=((2*a(1,1)+(a(2,1)*x(j))+(a(3,1)*(x(j)^(a(3,1)-1))))^(-
1))*(x(j)^(a(3,1)-1));
    sd4(j)=(a(3,1)*log(x(j)))+1;
    sd5(j)=sd3(j)*sd4(j);
```

```
    ty=ty+sd5(j);

    sd6(j)=(x(j)^a(3,1))*(log(x(j)));

    yt=yt+sd6(j);
end
K=ty-yt;
a=[a(1,1);a(2,1);a(3,1)]-J1*([F;G;K])

 while abs(a-a(1,1))>0.001
     a=a(1,1);
    while abs((a-a(2,1)))>0.001
      a=a(2,1);
       while abs(a-a(3,1))>0.001
      a=a(3,1);
      end
    end
 end
%------------------------------------------------------------------------------------------
```

Calculated the AIC, AIC_c ,BIC

```
n=size of sample;
p=a(1,1);m=a(2,1);h=a(3,1);
q=0;s=0;
qq=0;ss=0;
for i=1:n
    sd1(i)=log((2*p)+(m*x(i))+(h*((x(i))^(h-1))));


    q=q+1;
    qq=qq+sd1(i);
end
for j=1:n
    sd2(j)=(2*p*x(j))+((m/2)*(x(j).^2))+(x(j).^h);
    s=s+1;
    ss=ss+sd2(j);
end

b=6-2*(qq-ss);
bb=3*ln(n)- 2*(qq-ss);
bs=2*k*(k+1)/(n-k-1);
bk=b+bs;
%--------------------------------------------------------------------------------------
```

MLE for Rayleigh Weibull Distribution

```
n=size of sample;
m=…….;h=………;
q=0;w=0;s=0;t=0;y=0;
qq=0;ww=0;ss=0;tt=0;yy=0;
for i=1:n
    q=q+1;
        a(i)=(m*x(i))+(h*(x(i)^(h-1)));
    qq=qq+((1/(a(i))^2)*(x(i)^2));
end
f1=(-1)*qq
for j=1:n
  w=w+1;
    b(j)=(m*x(j))+(h*(x(j)^(h-1)));
    c(j)=(1/(b(j))^2)*(x(j)^h)*(h*log(x(j))+1);
    ww=ww+c(j);
end
f2=(-1)*ww
for k=1:n
    s=s+1;
    l(k)=(m*x(k))+(h*(x(k)^(h-1)));
    p(k)=(1/(l(k)))*(x(k)^(h-1))*(2+(h*(log(x(k)))));
    ss=ss+p(k);
end
for z=1:n
    t=t+1;
    ll(z)=(m*x(z))+(h*(x(z)^(h-1)));
    pp(z)=(1/(ll(z))^2)*(x(z)^(2*(h-1)))*((h*log(x(z))+1)^2);
    tt=tt+pp(z);
end
for r=1:n
    y=y+1;
    e(r)=((x(r))^h)*((log(x(r)))^2);
    yy=yy+e(r);
end
f3=ss-tt-yy
J=[f1 f2 ; f2 f3]
J1=inv(J);
%------------------------------------------------------------------------------------------
```

Calculated the AIC, AIC_c, BIC

```
n=size of sample;
m=……..;h=………;
q=0;w=0;s=0;t=0;y=0;v=0;
qq=0;ww=0;ss=0;tt=0;yy=0;vv=0;
for i=1:n
    q=q+1;
    a(i)=(m*x(i))+(h*(x(i)^(h-1)));
    qq=qq+((1/(a(i)))*(x(i)));
end
for u=1:n
    y=y+1;
    yy=yy+(x(u)^2);
end
f=qq-(0.5*yy);
for j=1:n
    w=w+1;
    b(j)=(m*x(j))+(h*(x(j)^(h-1)));
    c(j)=(1/(b(j)))*(x(j)^(h-1))*(h*log(x(j))+1);
    ww=ww+c(j);
end

for k=1:n
    s=s+1;
    l(k)=(x(k)^h)*log(x(k));
    ss=ss+l(k);
end
g=ww-ss;
a=[m;h]-(J1*[f;g])
while abs(a-m)>0.001
     a=m;
    while abs((a-h))>0.001
      a=h
end
for p=1:n
    v=v+1;
    vv=vv+log(((a(1)*x(p))+(a(2)*(x(p))^(a(2)-
1)))*(1/exp(((a(1)/2)*x(p)^2)+(x(p)^a(2)))));

ff=4-(2*vv);
```

```
bb=2*ln(n)- 2*(vv);
bs=2*k*(k+1)/(n-k-1);
bk=ff+bs;
```

MLE for Exponential Rayleigh distribution

```
n=size of sample;
m=……..;p= ………;
q=0;w=0;s=0;
qq=0;ww=0;ss=0;
for i=1:n
    a(i)=p+(m*x(i));
    q=q+1;
    qq=qq+(1/(a(i)^2));
end
f1=(-1)*qq
for j=1:n
    b(j)=p+(m*x(j));
    c(j)=x(j)/(b(j)^2);
    w=w+1;
    ww=ww+c(j);
end
f2=(-1)*ww
for k=1:n
    l(k)=p+(m*x(k));
    r(k)=(x(k)^2)/(l(k)^2);
    s=s+1;
    ss=ss+r(k);
end
f3=(-1)*ss

J=[f1 f2 ; f2 f3]
J1=inv(J);
%-----------------------------------------------------------------------------------------
```

Calculated the AIC, AIC_c ,BIC

```
n=size of sample;
m=…….;p= ……..;
q=0;w=0;s=0;t=0;v=0;
qq=0;ww=0;ss=0;tt=0;vv=0;
for i=1:n
    a(i)=p+(m*x(i));
    q=q+1;
    qq=qq+(1/(a(i)));
```

```
end
for u=1:n
    s=s+1;
    ss=ss+x(u);
end
f=qq-ss
for j=1:n
    b(j)=p+(m*x(j));
    c(j)=x(j)/b(j);
    w=w+1;
    ww=ww+c(j);
end
for z=1:n
    t=t+1;
    tt=tt+((x(z))^2);
end
g=ww-(tt/2)
a=[p;m]-(J1*[f;g])
while abs(a-m)>0.001
     a=m;
    while abs((a-p))>0.001
      a=p
end
end

for k=1:n
    bb(k)=a(1)+(a(2)*x(k));
    aa(k)=exp(a(1)*x(k)+(0.5*a(2)*x(k)^2));
    v=v+1;
    vv=vv+log(bb(k)/aa(k));
end
ff=4-(2*vv);
bf=2*ln(n)- 2*(vv);
bs=2*k*(k+1)/(n-k-1);
bk=ff+bs;

%-------------------------------------------------------------------------------------------
```

MLE for Exponential Weibull Distribution

```
n=size of sample;
m=…….;h=……..;
q=0;w=0;s=0;t=0;y=0;
```

```
qq=0;ww=0;ss=0;tt=0;yy=0;
for i=1:n

    a(i)=m+(h*((x(i))^(h-1)));
    q=q+1;
    qq=qq+(1/(a(i))^2);
end
f1=(-1)*qq;
for j=1:n

    b(j)=m+(h*((x(j))^(h-1)));
    c(j)=((x(j))^(h-1))*(h*log(x(j))+1);
    w=w+1;
    ww=ww+(c(j)/(b(j))^2);
end
f2=(-1)*ww
for k=1:n

    L(k)=m+(h*((x(k))^(h-1)));
    p(k)=((x(k))^(h-1))*(log(x(k)))*(2+h*log(x(k)));
    s=s+1;
    ss=ss+(p(k)/L(k));
end
for z=1:n

    ll(z)=m+(h*(x(z))^(h-1));
    pp(z)=((x(z))^(2*(h-1)))*((h*log(x(z))+1)^2);
    t=t+1;
    tt=tt+(pp(z)/(ll(z))^2);
end
for r=1:n
    y=y+1;
    e(r)=((x(r))^h)*((log(x(r)))^2);
    yy=yy+e(r);
end
f3=ss-tt-yy
J=[f1 f2 ; f2 f3]
J1=inv(J);
%------------------------------------------------------------------------------------------
```

Calculated the AIC, AIC_c ,BIC

```
n=size of sample;
m=……;h=………;
```

```
r=0;q=0;w=0;s=0;v=0;u=0;
rr=0;qq=0;ww=0;ss=0;vv=0;uu=0;
for i=1:n
    q=q+1;
    qq=qq+(1/(m+(h*(x(i)^(h-1)))));
end
for k=1:n
    r=r+1;
    rr=rr+x(k);
end
f=qq-rr;
for j=1:n
    sd1(j)=1/(m+(h*(x(j)^(h-1))));
    sd2(j)=(x(j)^(h-1))*(h*(log(x(j)))+1);
    s=s+1;
    ss=ss+(sd2(j)/sd1(j));
end
for z=1:n
    w=w+1;
    ww=ww+((x(z)^h)*(log(x(z))));
end
g=ss-ww;
a= [m;h]-(J1*[f;g])
while abs(a-m)>0.001
     a=m;
    while abs((a-h))>0.001
      a=h;
    end
end

for t=1:n
    sa1(t)=log(a(1)+(a(2)*(x(t)^(a(2)-1))));
    v=v+1;
    vv=vv+sa1(t);
end
for e=1:n
    sa2(e)=(a(1)*x(e))+(x(e)^a(2));
    u=u+1;
    uu=uu+sa2(e);
end
ff=4-(2*(vv-uu));
bf=2*ln(n) -(2*(vv-uu));
bs=2*k*(k+1)/(n-k-1);
```

```
bk=ff+bs;

%-----------------------------------------------------------------------------------------
```

MLE for Weibull Distribution and Calculated the AIC, AIC_c BIC

```
n=size of sample;
h=……..;
r=0;s=0;q=0;u=0;rr=0;ss=0;qq=0;t=0;tt=0;uu=0;
for i=1:n
   r=r+1;
   v(i)=(log(x(i)))^2;
   rr=rr+((x(i)^h)*v(i));
end
a=(-1)*(n/(h^2))-rr;
for j=1:n
   s=s+1;
   ss=ss+log(x(j));
end
b=ss;
for k=1:n
   q=q+1;
   z(k)=log(x(k));
   qq=qq+(((x(k))^h)*z(k));
end
c=qq;
f=(n/h)+b-c;
d=h-(f/a)
while abs((d-h))>0.001
     d=h;
end

for L=1:n
   y(L)=log(x(L));
   t=t+1;
   tt=tt+y(L);
end
for e=1:n
   u=u+1;
   uu=uu+x(e);
end

w=(n*log(h))+((h-1)*tt)-uu;
```

```
m=2-(2*w);
ff=ln(n) -(2*w);
bs=2*k*(k+1)/(n-k-1);
bk=ff+m;
%-------------------------------------------------------------------------------------------
```

Generalized Data By the First Method of Simulation

```
a=[…….;……;1];
u=rand(n,L);
for j=1:L
for i=1:n
 CDF(i,j)=log(1-u(i,j));
     r(i,j)=sqrt(((2*a(1,1)+1)^2)-(2*a(2,1)*CDF(i,j)));
     x(i,j)=(1/a(2,1))*((-2*a(1,1)-1)+r(i,j));
     k(i,j)=(1/a(2,1))*((-2*a(1,1)-1)-r(i,j));

end
end
%-------------------------------------------------------------------------------------------
```

Generalized Data By the Second Method of Simulation

```
a1=……;b=……..;c=…..;
y=rand(n,L);
for j=1:L
    for i=1:n
     x(i,j)=y(1,j)-((1-exp((-2*a1*y(1,j))-((b/2)*y(1,j)^2)-(y(1,j)^c))-rand(i-(i-
1))).((((2*a1+b*y(1,j))+(c*(y(1,j)^(c-1))))*(exp((-2*a1*y(1,j))-
((b/2)*y(1,j)^2)-(y(1,j)^c)))));

    end
end
%-------------------------------------------------------------------------------------------
```

MLE for New Mixture Distribution for Simulation Samples

```
n=size of sample;
m1=L;
p=…..;m=……;h=…….;
q=0;w=0;s=0;r=0;t=0;y=0;u=0;v=0;
qq=0;ww=0;ss=0;rr=0;ty=0;yt=0;qw=0;wq=0;sr=0;rs=0;tt=0;yy=0;uu=0;vv
=0;
for k=1:m1
for i=1:n
```

```
    sd(i)=(2*p+(m*x(i,k))+(h*(x(i,k)^(h-1))^(-2)));
    q=q+1;
    qq=qq+sd(i);

f1=(-4)*qq;

    sf(i)=((2*p+(m*x(i,k))+(h*(x(i,k)^(h-1))))^(-2))*(x(i,k));
    w=w+1;
    ww=ww+sf(i);

f2=(-2)*ww;

    sg1(i)=((2*p+(m*x(i,k))+(h*(x(i,k)^(h-1))))^(-2))*(x(i,k)^(h-1));
    sg2(i)=h*log(x(i,k))+1;
    s=s+1;
    ss=ss+(sg1(i)*sg2(i));

f3=(-2)*ss;

    sh(i)=((2*p+(m*x(i,k))+(h*(x(i,k)^(h-1))))^(-2))*(x(i,k))^2;
    r=r+1;
    rr=rr+sh(i);

f4=(-1)*rr;

    sk1(i)=((2*p+(m*x(i,k))+(h*(x(i,k)^(h-1))))^(-2))*(x(i,k)^h);
    sk2(i)=(h*log(x(i,k)))+1;
    t=t+1;
    tt=tt+(sk1(i)*sk2(i));

f5=(-1)*tt;

    sv1(i)=((2*p+(m*x(i,k))+(h*(x(i,k)^(h-1))))^(-1))*(x(i,k)^(h-1)*log(x(i,k)));
    sv2(i)=(h*log(x(i,k)))+2;
    t=t+1;
    tt=tt+(sv1(i)*sv2(i));


    sv3(i)=((2*p+(m*x(i,k))+(h*(x(i,k)^(h-1))))^(-2))*(x(i,k)^(2*h-2));
    sv4(i)=(h*log(x(i,k))+1)^2;
    u=u+1;
    uu=uu+(sv3(i)*sv4(i));


    sx(i)=(x(i,k)^h)*((log(x(i,k)))^2);
```

```
    v=v+1;
    vv=vv+sx(i);

f6=tt-uu-vv;
J=[f1 f2 f3 ; f2 f4 f5 ; f3 f5 f6];
J1=inv(J);
for j=1:n
    sd1(j)=(2*p+m*x(j,k)+(h*(x(j,k)^(h-1))))^(-1);

    qw=qw+sd1(j);


    wq=wq+(x(j,k));

F=2*(qw)-2*(wq);

    sd2(j)=((2*p+m*x(j,k)+h*(x(j,k)^(h-1)))^(-1))*(x(j,k));

    sr=sr+sd2(j);



    rs=rs+(x(j,k)^2);

G=sr-(rs/2);

    sd3(j)=((2*p+(m*x(j,k))+(h*(x(j,k)^(h-1))))^(-1))*(x(j,k)^(h-1));
    sd4(j)=(h*log(x(j,k)))+1;
    sd5(j)=sd3(j)*sd4(j);

    ty=ty+sd5(j);

    sd6(j)=(x(j,k)^h)*(log(x(j,k)));

    yt=yt+sd6(j);

K=ty-yt;
  end
end
a1=J1*[F;G;K];
 a(:,k)=[p;m;h]-a1;
   while abs((a(1,k))-p)>0.01
     p=a(1,k);
    while abs((a(2,k))-m)>0.001
      m=a(2,k);
```

```
      while abs((a(3,k))-h)>0.001
      h=a(3,k);
      end
    end
   end
  end
%-------------------------------------------------------------------------------------
```

OLS for New Mixture Distribution for Simulation Samples

```
n=size of sample
m=L;
para_esti=zeros(3,m);A=zeros(3,1);CDF=zeros(n,m);
lemda=……;beta=…….;alpha=……..;
aa=0;bb=0;cc=0;dd=0;ee=0;ff=0;gg=0;hh=0;ii=0;

for k1=1:m
    for i=1:n
CDF(i,k1)=1-
(1/exp((2*lemda*x(i,k1))+(0.5*beta*(x(i,k1)^2))+(x(i,k1)^alpha)));
end
end
for k=1:m
    for i1=1:n
        a4(i1)=x(i1,k)^2;
        dd=dd+a4(i1);
f1=8*dd;
    end

for i2=1:n
        a5(i2)=x(i2,k)^3;
        ee=ee+a5(i2);
f2=2*ee;
end

for i3=1:n
        a6(i3)=(x(i3,k)^(alpha+1))*log(x(i3,k));
        ff=ff+a6(i3);
f3=4*ff;
end
for i4=1:n
        a7(i4)=x(i4,k)^4;
        gg=gg+a7(i4);
```

```
f4=0.5*gg;
end

for i5=1:n
        a8(i5)=(x(i5,k)^(alpha+2))*log(x(i5,k));
        hh=hh+a8(i5);
f5=hh;
end

for i6=1:n
a9(i)=((x(i6,k)^alpha)*log(x(i6,k)))*((log(x(i6,k))*log(1-
CDF(i6,k)))+(2*lemda*x(i6,k)^(alpha+1))*((log(x(i6,k))+(0.5*beta*x(i6,k)^(
alpha+2))+(2*(x(i6,k)^(2*alpha))*log(x(i6,k)^2)))));
ii=ii+a9(i6);
f6=2*ii;
end

 for j1=1:n
a1(j1)=x(j1,k)*((log(1-
CDF(j1,k)))+(2*lemda*x(j1,k))+((beta/2)*x(j1,k)^2)+(x(j1,k)^(alpha)));
aa=aa+a1(j1);
F_lemda=4*aa;
 end

 for j2=1:n
a2(j2)=(x(j2,k)^2)*((log(1-
CDF(j2,k)))+(2*lemda*x(j2,k))+((beta/2)*x(j2,k)^2)+(x(j2,k)^(alpha)));
bb=bb+a2(j2);
G_beta=bb;
 end

 for j3=1:n
a3(j3)=((x(j3,k)^(alpha))*log(x(j3,k)))*((log(1-
CDF(j3,k)))+(2*lemda*x(j3,k))+((beta/2)*x(j3,k)^2)+(x(j3,k)^(alpha)));
cc=cc+a3(j3);
K_alpha=2*cc;
 end

     J1=[f1 f2 f3; f2 f4 f5; f3 f5 f6];
     J=inv(J1);
     A1=[F_lemda;G_beta;K_alpha];
     A(:,k)=J*A1;

     para_esti(:,k)=[lemda;beta;alpha]- A(:,k);
```

```
while abs((para_esti(1,k))-lemda)>0.001
     lemda=para_esti(1,k);
while abs((para_esti(2,k))-beta)>0.001
      beta=para_esti(2,k)
while abs((para_esti(3,k))-alpha)>0.001
      alpha=para_esti(3,k)
end
end
end
end
%------------------------------------------------------------------------------------------
```

RSS for New Mixture Distribution for Simulation Samples

```
n=size of sample;
u1=L; x=sort(x);
lemda=……;beta= …….;alpha =………;
for k=1:u1
    for i=1:n
        F(i,k)=exp((-2*lemda*x(i,k))-((beta/2)*x(i,k)^2)-(x(i,k)^(alpha)));
        f(i,k)=  ((2*lemda)+(beta*x(i,k))+(alpha*x(i,k)^(alpha-1)));

    end
end
for k1=1:u1
    for i1=1:n
        a1(i1,k1)=-4*f(i1,k1)^(-2);
        a2(i1,k1)=4*(i1-1)*(x(i1,k1)^2)*(F(i1,k1)^2)*((1-F(i1,k1))^(-2));
        a3(i1,k1)=4*(i1-1)*(x(i1,k1)^2)*(F(i1,k1))*((1-F(i1,k1))^(-1));

    end
end
    f1=sum(a1)-sum(a2)-sum(a3);
for k2=1:u1
    for i2=1:n
        b1(i2,k2)=-2*x(i2,k2)*(f(i2,k2)^(-2));
        b2(i2,k2)=(i2-1)*(x(i2,k2)^3)*(F(i2,k2)^2)*((1-F(i2,k2))^(-2));
        b3(i2,k2)=(i2-1)*(x(i2,k2)^3)*(F(i2,k2))*((1-F(i2,k2))^(-1));

    end
end
    f2=sum(b1)-sum(b2)-sum(b3);
    for k3=1:u1
    for i3=1:n
```

```
        c1(i3,k3)=(-2)*(x(i3,k3)^(alpha-1))*(f(i3,k3)^(-
2))*(alpha*log(x(i3,k3))+1);
        c2(i3,k3)=2*(i3-1)*(x(i3,k3)^(alpha+1))*(F(i3,k3)^2)*((1-F(i3,k3))^(-
2))*(log(x(i3,k3)));
        c3(i3,k3)=2*(i3-1)*(x(i3,k3)^(alpha+1))*(F(i3,k3))*((1-F(i3,k3))^(-
1))*(log(x(i3,k3)));
    end
    end
  f3=sum(c1)-sum(c2)-sum(c3);
  for k4=1:u1
    for i4=1:n
        d1(i4,k4)=(-1)*(x(i4,k4)^2)*f(i4,k4)^(-2);
        d2(i4,k4)=(1/4)*(i4-1)*(x(i4,k4)^4)*(F(i4,k4)^2)*((1-F(i4,k4))^(-2));
        d3(i4,k4)=(1/4)*(i4-1)*(x(i4,k4)^4)*(F(i4,k4))*((1-F(i4,k4))^(-1));

    end
  end
    f4=sum(d1)-sum(d2)-sum(d3);
    for k5=1:u1
    for i5=1:n
        e1(i5,k5)=(-1)*(x(i5,k5)^alpha)*(alpha*log(x(i5,k5))+1)*(f(i5,k5)^(-2));
        e2(i5,k5)=(1/2)*(i5-1)*(x(i5,k5)^(alpha+2))*(F(i5,k5)^2)*((1-
F(i5,k5))^(-2))*log(x(i5,k5));
        e3(i5,k5)=(1/2)*(i5-1)*(x(i5,k5)^(alpha+2))*(F(i5,k5))*((1-F(i5,k5))^(-
1))*log(x(i5,k5));

    end
    end
    f5=sum(e1)-sum(e2)-sum(e3);
    for k6=1:u1
    for i6=1:n
        g1(i6,k6)=(x(i6,k6)^(alpha-1))*(f(i6,k6)^(-
1))*log(x(i6,k6))*(alpha*log(x(i6,k6))+2);
        g2(i6,k6)=(x(i6)^(2*alpha-2))*(f(i6,k6)^(-
2))*(alpha*log(x(i6,k6))+1)^2;
        g3(i6,k6)=(i6-1)*(x(i6,k6)^(2*alpha))*(F(i6,k6)^2)*((1-F(i6,k6))^(-
2))*((log(x(i6)))^2);
        g4(i6,k6)=(i6-1)*(x(i6,k6)^(2*alpha))*(F(i6,k6))*((1-F(i6,k6))^(-
1))*((log(x(i6,k6)))^2);
        g5(i6,k6)=(n-i6+1)*(x(i6,k6)^(alpha))*((log(x(i6,k6)))^2);
    end
    end
    f6=sum(g1)-sum(g2)-sum(g3)-sum(g4)-sum(g5);
```

```
   for k7=1:u1
 for j=1:n
     h1(j,k7)=2*f(j)^(-1);
     h2(j,k7)=(j-1)*(2*x(j,k7))*(F(j,k7))*((1-F(j,k7))^(-1));
     h3(j,k7)=(n-j+1)*(2*x(j,k7));
 end
   end
 f_lemda=sum(h1)+sum(h2)-sum(h3);
 for k8=1:u1
 for j1=1:n
     k1(j1,k8)=(x(j1,k8))*(f(j1,k8)^(-1));
     k2(j1,k8)=0.5*(j1-1)*(x(j1,k8)^2)*(F(j1,k8))*((1-F(j1,k8))^(-1));
     k3(j1,k8)=0.5*(n-j1+1)*(x(j1,k8)^2);
 end
 end
 g_beta=sum(k1)+sum(k2)-sum(k3);
 for k9=1:u1
 for j2=1:n
     m1(j2,k9)=(x(j2,k9)^(alpha-1))*(f(j2,k9)^(-1))*(alpha*log(x(j2,k9))+1);
     m2(j2,k9)=(j2-1)*(x(j2,k9)^(alpha))*(log(x(j2,k9)))*(F(j2,k9))*((1-
F(j2,k9))^(-1));
     m3(j2,k9)=(n-j2+1)*(x(j2,k9)^(alpha)*log(x(j2,k9)));
 end
 end

 k_alpha=sum(m1)+sum(m2)-sum(m3);
 for v=1:u1
 J1=[f1(v) f2(v) f3(v); f2(v) f4(v) f5(v); f3(v) f5(v) f6(v)];
 J=inv(J1);
 A=J*[f_lemda(v); g_beta(v); k_alpha(v)];
a(:,v)=[lemda; beta; alpha]-A;
  while abs((a(1,v))-lemda)>0.001
     lemda=a(1,v);
    while abs((a(2,v))-beta)>0.001
        beta=a(2,v);
      while abs((a(3,v))-alpha)>0.001
      alpha=a(3,v);
      end
    end
   end
 end
%------------------------------------------------------------------------------------
```

Mean Square Error for all Estimation Methods

```
m1=L;
p1=zeros(1,m1);p2=zeros(1,m1);p3=zeros(1,m1);
p=…….;m=……..;h=………;
for j=1:m1
p1(j)=((a(1,j)-p)^2);
p2(j)=((a(2,j)-m)^2);
p3(j)=((a(3,j)-h)^2);
end
ms1=sum(p1);
ms2=sum(p2);
ms3=sum(p3);
mse1=ms1/m1
 mse2=ms2/m1
 mse3=ms3/m1
```

المستخلص

الهدف من هذه الرسالة بناء توزيع مختلط جديد من خلال دمج ثلاثة توزيعات لها معلمة واحدة واستخدام دالة البقاء لكل من التوزيع الأسي وتوزيع ويبل وتوزيع رايلي لتحديد "توزيع المختلط الجديد". تستند طريقة الدمج على ثلاثة أجزاء رئيسية: الجزء الأول يشمل عملية الدمج بين التوزيع الأسي وتوزيع رايلي اعتمادًا على الذيل في كل منها. الجزء الثاني دمج التوزيع الأسي وتوزيع ويبل اعتمادا على الذيل في كل منها ايضا. هنا التوزيع الأسي يلعب دورا كبيرا في عملية الدمج في كلا الجزأين. أخيرًا ، الجزء الثالث يكون استنادًا إلى نتائج الجزأين الأول والثاني لعملية الدمج ، تم الحصول على التوزيع المختلط الجديد. بالاضافة الى تحقق الخواص الاحصائية والرياضية مثل ايجاد العزوم حول الصفروالعزوم الغير كاملة،ايجاد الدالة المولدة للعزوم،الوسط ، المنوال، الحصول على الدالة المميزة وغيرها من الخواص الاحصائية. علاوة على ذلك التعبير عن اشكال كل من دالة الكثافة ودالة المخاطرة بالنسبة للتوزيع الجديد. باستخدام معيار الاكاكي ،معيار اكاكي المصحح والمعيار البيزي للمعلومات لمقارنة توزيع الخليط الجديد مع توزيعات احصائية اخرى ذات صلة. وقد تم توضيح الفائدة من التوزيع الجديد من خلال استخدام بيانات حقيقية كاملة.

في وقت لاحق ، لتقدير معلمات توزيع المزيج الجديد ، تستخدم طرق التقدير الكلاسيكية المستخدمة وهي: (طريقة تقدير الاحتمالية القصوى ، طريقة المربعات الصغرى العادية ، وطريقة تقدير عينات مجموعة الرتب).

أخيرًا ، "Mean Square Error" المعيارالمستخدم لمقارنة الطرق أعلاه لأحجام مختلفة من العينات التي تم إنشاؤها. الاستفادة من طريقتين لتقنية المحاكاة لإنشاء أحجام عينات مختلفة. وقد تم تطبيق كل ما سبق من خلال بيانات حقيقية كاملة.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة بغداد
كلية التربية للعلوم الصرفة (ابن الهيثم)
قسم الرياضيات

توزيع مختلط جديد: نظرية و تطبيق

أطروحة

مقدمة الى كلية التربية للعلوم الصرفة (ابن الهيثم)

جامعة بغداد

وهي جزء من متطلبات نيل شهادة الدكتوراه في فلسفة الرياضيات

من قبل

ميساء جليل محمد

بأشراف

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١٤٤٠ ٢٠١٩