Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Baghdad
College of Education for Pure Science,
Ibn Al-Haitham

Efficient Method for Solving Some Types of Partial Differential Equations

A Thesis
Submitted to the Department of Mathematics, College of Education for Pure Science- Ibn Al-Haitham, University of Baghdad as Partial Fulfillment of Requirements for the Degree of
Master of Science in Mathematics

By

Myasar Obaid Enadi

Supervised By

Prof. Dr. Luma Naji Mohammed Tawfiq

2019 AC 1440 AH

بسم الله الرحمن الرحيم

وَقَالَ رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَى وَالِدَيَّ وَأَنْ
أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَدْخِلْنِي بِرَحْمَتِكَ فِي عِبَادِكَ الصَّالِحِينَ ﴿19﴾

صدق الله العظيم

سورة النمل الآية (19)

Supervisor Certification

I certify that this thesis was prepared under my supervision at the Department of Mathematics, College of Education for pure science / Ibn Al-Haitham, University of Baghdad, as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature: LNM Tawfiq

Name : Prof. Dr. Luma N. M. Tawfiq

Date : 6 / 11 / 2019

In view of the available recommendations, I forward this thesis for debate by examining committee.

Signature: Y. Y. Yousif

Name: Asst. Prof. Dr. Yousif Yaqoub Yousif

Head of the Department of Mathematics,

College of Education for pure science/Ibn Al-Haitham,

University of Baghdad

Date: 7 / 11 / 2019

Committee Certification

We certify that we have read this thesis entitled " **Efficient Method for Solving Some Types of Partial Differential Equations** " and, as an examining Committee, we examined the student **Myasar Obaid Enadi** in its contents and what is connected with it, and that in our opinion it is adequate for the partial fulfillment of the requirement for the degree of master of Science in Mathematics.

Signature:

Name: Prof. Dr. Sadiq A. Mehdi

(Chairman)

Date: 6 / 11/ 2019

Signature:

Name: Asst. Prof. Dr. Maha A. Mohammed

(Member)

Date: 6 / 11/ 2019

Signature:

Name: Lec. Dr. Adel R. Abd Ali

(Member)

Date: 6 / 11/ 2019

Signature: LNM Tawfiq

Name: Prof. Dr. Luma N. M. Tawfiq

(Member and Supervisor)

Date: 6 / 11/ 2019

Approved by the University Committee of graduate studies.

Signature:

Name: Prof. Dr. Hasan Ahmed Hasan

The Dean of College Education for Pure Science

Ibn AL-Haitham University of Baghdad

Date: 2 / 11 / 2019

الإهـداء

الحمد لله على نعمه التي لا تعد ولا تحصى والصلاة والسلام على أشرف المرسلين سيدنا محمد وأله الطيبين الطاهرين

في نهاية رسالتي اهدي ثمرة جهدي المتواضعة

إلى الذين من أجلهم تهون مصاعب الحياة

إلى الذين من أجلهم تستحق الحياة

إلى من سكن حبهم في القلب فكانوا الهواء والنبض " العائلة الكريمة "

إلى من شرفتني بإشرافها على رسالتي الأستاذ الدكتور "لمى ناجي محمد توفيق"

التي لن تكفي الحروف لإيفائها حقها لصبرها الكبير علي ولتوجيهاتها العلمية

التي لا تقدر بثمن و التي ساهمت بشكل كبير في إتمام و انجاز العمل ؛كما أتوجه بخالص شكري و تقديري

إلى بيارق العلم والمعرفة أساتذتي الأفاضل

إلى كل من ساعدني في فترة دراستي

الباحث

Acknowledgements

To “ALLAH” and to his prophet “Mohammed”, my praise and thanks are due for their blessings without which this research would not have been achieved without their blessings. It is pleasure to express my deep appreciation to my supervisor Prof. Dr. Luma Naji Mohammed Tawfiq for her invaluable advice, assistance, cooperation, and support throughout the course of preparing my thesis.

I dedicate this work to my dear parents and my wife for their financial and moral support throughout the study period, and thank them so much.

I would like to thank everyone who helped me and stayed by my side throughout the study period from my family, friends and teachers.

Researcher

Author’s Publications

Journal Papers

1. Enadi, M. O., and Tawfiq, L.N.M., 2019, New Technique for Solving Autonomous Equations, Ibn Al-Haitham Journal for Pure and Applied Science, 32(2), p: 123-130, Doi: 10.30526/32.2.2150

2. Enadi, M. O., and Tawfiq, L.N.M., 2019, Solving Systems of Non-linear Partial differential Equations by Using Coupled Method, International Journal of Modern Mathematical Sciences, 17(1), p: 68-77.

3. Enadi, M. O., and Tawfiq, L.N.M., 2019, New Approach for Solving Three Dimensional Space Partial Differential Equation, Baghdad Science Journal, 16(3), p: 786-792, Doi: http://dx.doi.org/10.21123/bsj.2019.16.3(Suppl.).0780.

4. Enadi, M. O., and Tawfiq, L.N.M., 2019, Using New Coupled Method for Solving System of Nonlinear Wave Equation, Computers and Mathematics with Applications, Elsevier, Under review.

List of Symbols and Abbreviations

Symbol	**Definition**
ADM	Adomian Decomposition Method
A_n	Adomian Polynomials
BCs	Boundary Conditions
BVP	Boundary Value Problem
CuTBs	Cubic Trigonometric Bspline
DTM	Differential Transform Method
Eq.	Equation
Eqs.	Equations
Figs.	Figures
HAM	Homotopy Analysis method
H_n	He Polynomials
HPM	Homotopy Perturbation Method
*i*D-PDEs *i*=1,2,3,4	*i* dimensional Partial differential equations, *i*=1,2,3,4
ICs	Initial Conditions
IVP	Inatial Value Problem
KdV	Korteweg-deVries
LT	Laplace Transformation
MHPM	Modified Homotopy Perturbation Method
Non-Linear	Linear and Nonlinear
NT	New Transformation
NTHPM	New transform homotopy perturbation method

ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
RLW	Real Like Wave
t	Time
VIM	Variational Iteration Method
$\bar{f}$	New transform for the function f
Ω	Omega
$\mathbb{T}$	New Transformation
Γ	Gamma
$\|\ \|$	Norm

Contents

Abstract

In this thesis, a new method based on a combined form of the new transform with homotopy perturbation method is proposed to solve some types of partial differential equations, for finding exact solution in a wider domain. It can be used to solve the problems without any discretization, or resorting to the frequency domain or restrictive assumptions and it is free from round-off errors. This method is called the new transform homotopy perturbation method.

In this thesis, we focuse on some basic concepts of the partial differential equations.The first objective is implement suggested method in order to solve some types of PDEs with initial condition such that: Klein-Gordan equation, wave-like equations, autonomous equation, system of two or three non-linear equations, Burgers' equations, coupled Hirota Satsuma KdV type II, and RLW equation. Finally, The proposed method is used to solve application model which is soil moisture equation where traditional HPM leads to an approximate solution. The second aim which is the convergence of the series solution is studied, the series solution converge to to the exact form is proved.Some examples are provided to illustrate the reliability and capability of the suggested method.The practical results

show that the proposed method is efficient tool for solving those types of partial differential equations.

INTRODUCTION

Introduction

Many phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs). In physics for example, the heat flow and the wave propagation phenomena are well described by PDEs [45,55]. So, it is a useful tool for describing natural phenomena of science and engineering models. Most of engineering problems are nonlinear PDEs, and it is difficult to solve them analytically. The obtaining of the exact solution of nonlinear PDEs in physics and mathematics is still a significant problem that needs new efficient implemented methods to get exact solutions. Various powerful mathematical methods have been proposed for obtaining exact and approximate analytic solutions. Some of the classic analytic methods are perturbation techniques [8] and Hirota's bilinear method [46]. Perturbation techniques were generated useful solutions in describing both quantitative and qualitative properties of the problem, which is an advantage compared to numerical solutions. However, some drawbacks were obvious for complex equations due to either such parameters cause a divergence of solutions as the quantities increase/decrease, or the non-existence of small or large perturbation parameters. In problems where these quantities do not exist, the parameter has to be artificially introduced which may lead to incorrect results [17]. Perturbation techniques are therefore found to be mainly suitable for weakly nonlinear problems.

In recent years, many researchers have paid attention to study the solutions of non-linear PDEs by using various methods. Among these are the Adomian Decomposition Method (ADM) [2,38,39], tanh method, Homotopy Perturbation Method (HPM) [43], Homotopy Analysis Method (HAM) [35], the Differential Transform Method (DTM) [9], Cubic Trigonometric B-Spline Method [29,30], Laplace Decomposition Method [26,43], Variational Iteration Method (VIM) [37, 56], parallel processing [39,40,49] and semi analytic technique [41, 42, 48, 50].

In this thesis we suggested a new method based on combine two efficient methods to get exact solution for some types of PDEs such autonomous equation which describes the appearance of the stripe pattern in two dimensional systems. Moreover, this equation was applied to a number of problems in variety systems, e.g., Rayleigh-Benard convection, Faraday instability, nonlinear optics, chemical reactions and biological systems. The approximate solutions of the autonomous equation were presented by differential transformation method [1], reduce differential transformation [31].

The system of PDEs arises in in many areas of mathematics, engineering and physical sciences. These systems are too complicated to be solved exactly so it is still very difficult to get exact solution for most problems. A vast class of analytical and numerical methods has been proposed to solve such problems. But many systems such as system of high dimensional equations, the required calculations to obtain its solution in some time may be too complicated. Recently, many powerful methods have been presented, such as the coupled method [27,30]. Herein we solved such systems by proposed method and we get exact solution without using computer programming and calculating.

This thesis is organized as follows:

In Chapter one, a brief review of basic definitions and concepts relate to the work is introduced. It includes an overview of PDEs and their types.

Chapter two contains the implementation of proposed method based on coupled two efficient methods such Homotopy Perturbation Method (HPM) and new transform defined by Luma and Alaa in [51], that we will say the New Transform Homotopy Perturbation Method (NTHPM) for obtaining exact solutions to some types of PDEs such as: Klein-Gordan equation, wave-like equations, autonomous equation, 3D-PDEs, and other 3rd order PDEs.

Chapter three contains the implementation of NTHPM for solving some types of system of PDEs. The efficiency of the proposed method is verified by the examples. The convergence of series solution to the exact analytic solution function is proved.

In chapter four the proposed method are successfully implemented to solve 1D, 2D, and 3D soil moisture model equation to determine the moisture content in soil.

Finally, in chapter five the conclusions and future works are given.

Chapter One

Preliminaries

Chapter One
Preliminaries

1.1. Introduction

This chapter includes some basic definitions and concepts related to the problems for this thesis. An overview of differential equations and their types is introduced. In addition, we review some traditional techniques such as HPM for solving partial differential equations, for comparison with the proposed approach NTHPM illustrated by examples.

1.2. Overview of Differential Equations

Differential equations are used in different field of science and engineering. It's a relation involving an unknown function (or functions say dependent variables) of one or several independent variables and their derivatives with respect to those variables. Many real phenomena in various fields such as engineering, physical, biological and chemical are modeled mathematically by using differential equations [47, 48, 50, 60]. Commonly, most real science and engineering processes including more than one independent variable and the corresponding differential equations are called partial differential equations (PDEs). However, the PDEs have been reduced to ordinary differential equations (ODEs) using simplified assumptions.

Where ODE is a differential equation for a function of single independent variable [47].

The order of a PDE is the order of the highest partial derivative that appears in the equation. A PDE s are classified as homogeneous or inhomogeneous. A PDE of any order is called homogeneous if every term of the PDE contains the dependent variable or one of its derivatives; otherwise, it is called an inhomogeneous PDE [55].

In research field, there are several types of PDEs which depends on the application that are used. Each application has its own special governing equations and properties that should considered individually.

A PDE is called linear if the power of the dependent variable and each partial derivative contained in the equation is one and the coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables. However, if any of these conditions is not satisfied, the equation is called nonlinear [55]. Also, it can consider a semi linear, if it is linear in partial derivative only. In addition, it can consider a quasi linear, if it is linear in the first partial derivatives or it is nonlinear in dependent variable [60].

The general form of quasi linear 2nd order inhomogeneous PDE with two independent variables can write as [Hoffman, 2001]:[55]

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad (1.1)$$

Where *A, B, C, D, E, and F* are the coefficients and the inhomogeneous term *G* may depend on *x* and *y*. The above equation (1.1) can be classified to:

1) Elliptic, when $B^2 - 4AC < 0$;

2) Parabolic, when $B^2 - 4AC = 0$;

3) Hyperbolic, when $B^2 - 4AC > 0$.

A solution of a PDE is a function satisfies the equation under discussion and satisfies the given conditions as well. In order to find the solution of PDEs, initial and / or boundary conditions used to solve PDEs, so the PDEs with initial conditions (ICs) is said to be initial value problem (IVPs), the PDEs with boundary conditions (BCs) is said to be boundary value problems (BVPs), but the PDEs with initial and boundary conditions is said to be initial-boundary value problems, the boundary conditions (BCs) which can be classify into three types:[60]

1) *Dirichlet boundary condition*: numerical values of the function are specific of the boundary of the region.

2) *Neumann boundary condition*: specifies the values that the derivative of a solution to take on the boundary of the domain.

3) *Mixed boundary conditions*: defines a BVP in which the solution of the given equation is required to satisfy different boundary conditions on disjoint parts of the boundary of the domain where the condition is stated. In effect, in a mixed BVP, the solution is required to satisfy the Dirichlet or Neumann boundary conditions in a mutually exclusive way on disjoint parts of the boundary.[60]

An elliptic partial differential equation with mixed boundary conditions is called a *Robbins problem* [60].

Some types of differential equation, such as nonlinear differential equation cannot be easy to solve, so we use numerical or approximate solution. In this thesis we suggest an efficient method to solve important types of PDEs to get exact solution.

We are beginning with Homotopy Perturbation Method (HPM).

1.3. Some Basic Concepts of the Homotopy Perturbation Method

The HPM was first proposed by He J. Huan in 1999 [13] for solving differential and integral equations, non-linear and has been successfully applied to solve non-linear differential equations, and other fields for more details see [15]. It is a combine of traditional perturbation method with homotopy method and it suggested to overcome the difficulty arising in calculating Adomian polynomials. This method has many advantages such as it is applied directly to the nonlinear problems without linearizing the problem. In this section, some basic concepts of this method have been explained.

Definition 1.1 [20]

Let X and Y are two topological spaces. Two continuous functions $f: X \to Y$ and $g: X \to Y$ are said to be homotopic, denoted by $\approx g$, if $\exists$ a continuous function $H: X \times [0,1] \to Y$,such that:

$H(x, v) = f(x), \forall x \in X$

$H(x,1) = g(x), \forall x \in X$

In this case, H is said to be a homotopy.

Now, to illustrate definition (1.1), consider the following examples.

Example 1.1[20]

Let $X$and Y be any topological spaces, f be the identity function and g be the zero function, then define $H: X \times [0,1] \to Y$ by:

$H(x,p) = x(1-p), \forall x \in X, \forall\, p \in [0,1]$

Then H is a continuous function and

$H(x,0) = x = f(x), \forall x \in X$

$H(x,1) = 0 = g(x), \forall x \in X$

Therefore $f \approx g$.

Remark

Let $f: R \to R$ and $g: R \to R$ be continuous functions. Define

$H: R \times [0,1] \to R$ by

$H(x,p) = (1-p)f(x) + pg(x), \forall\, x \in R\ , \forall\, p \in [0,1]$

Then, $H(x,0) = f(x), \forall x \in R$

and,$H(x,1) = g(x), \forall x \in R$

Therefore $\approx g$.

Now, to illustrate the basic idea of the HPM, we consider the following nonlinear differential equation:

$$A(u) = f(x),\ x \in \Omega \tag{1.2}$$

where A is differential operator, f is a known function of x. The operator A can generally speaking be divided into two operators L and N, where L is a linear operator, and N is a nonlinear operator. Therefore equation (1.2) can be rewritten as follows:

$$L(u) + N(u) - f(x) = 0$$

According to [16], we can construct a homotopy $\mathscr{H}: \Omega \times [0,1] \longrightarrow \Re$ which satisfies the homotopy equation:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(x)] = 0$$

Or

$$H(v,p) = L(v(x)) - L(u_0(x)) + pL(u_0(x)) + p[N(v(x)) - f(x)] = 0 \tag{1.3}$$

where p $\in[0,1]$, u_0 is an initial approximation solution of equation (1.2).

Obviously, from equation (1.3) we have:

$$H(v, 0) = L(v) - L(u_0) = 0$$

$$H(v,1) = A(v) - f(x) = 0$$

The changing process of p from zero to unity is just that of v(x, p) from $u_0(x)$ to u(x).

Therefore, $L(v) - L(u_0) \cong A(v) - f(x), \quad x \in \Omega$

and $u_0(x) \cong u(x), \quad x \in \Omega$

Assume that the solution of equation (1.2) can be written as a power series in $\boldsymbol{p}$ as follows:

$v(\mathrm{x}, \mathrm{p}) = \sum_{i=0}^{\infty} \mathrm{p}^i \mathrm{v_i}(\mathrm{x}) = v_0 + pv_1 + p^2 v_2 + \cdots$ (1.4)

Setting p =1 in equation (1.4), can get:

$$u(x) = \sum_{i=0}^{\infty} \mathrm{v}_i(\mathrm{x}) = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \quad (1.5)$$

This is the solution of equation (1.2)

To illustrate the efficiency of method, consider the following examples.

Example 1.2[55]

Consider 2nd order linear homogeneous Klein-Gordan equation.

$$u_{tt} = u_{xx} + u_x + 2u \quad , \quad -\infty < x < \infty \quad , \quad t > 0$$

Subject to the IC: $u(x,0) = e^x$, $u_t(x,0) = 0$

Using the HPM we have

$$H(v,p) = (1-p)(v_{tt} - u_{0tt}) + p(v_{tt} - v_{xx} - v_x - 2v)$$

$$= v_{tt} - u_{0tt} + p(u_{0tt} - v_{xx} - v_x - 2v) = 0$$

$p^0 : v_{0tt} - u_{0tt} = 0$

$v_{0t} = u_{0t} \implies v_0 = u_0 = e^x$

$p^1 : v_{1tt} = (v_{0xx} + v_{0x} + 2v_0 - u_{0tt}) = e^x + e^x + 2e^x = 4e^x$

$v_{1t} = 4te^x \Rightarrow v_1 = 4\frac{t^2}{2!}e^x$

$p^2 : v_{2tt} = (v_{1xx} + v_{1x} + 2v_1) = 4\frac{t^2}{2!}e^x + 4\frac{t^2}{2!}e^x + 8\frac{t^2}{2!}e^x = 16\frac{t^2}{2!}e^x$

$v_{2t} = 16\frac{t^3}{3!}e^x \Rightarrow v_2 = 16\frac{t^4}{4!}e^x$

$$p^3: v_{3tt} = (v_{2xx} + v_{2x} + 2v_2) = 16\frac{t^4}{4!}e^x + 16\frac{t^4}{4!}e^x + 32\frac{t^4}{4!}e^x = 64\frac{t^4}{4!}e^x$$

$$v_{3t} = 64\frac{t^5}{5!}e^x \Longrightarrow v_3 = 64\frac{t^6}{6!}e^x$$

And so on, to get

$$u(x,t) = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots$$

$$u(x,t) = e^x + 4\frac{t^2}{2!}e^x + 16\frac{t^4}{4!}e^x + 64\frac{t^6}{6!}e^x + \cdots$$

$$u(x,t) = e^x\left(1 + \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} + \frac{(2t)^6}{6!} + \cdots\right) = e^x\, cosh(2t)$$

Example 1.3[60]

Consider the following 3rd order nonlinear PDE.

$$u_t + \frac{1}{2}u_x^2 = u_{xxt} \quad , \quad -\infty < x < \infty \quad , \quad t > 0$$

Subject to the initial condition (IC): $u(x,0) = x$

Using the HPM we have:

$$H(v,p) = (1-p)(v_t - u_{0t}) + p\left(v_t + \frac{1}{2}(v^2)_x - v_{xxt}\right)$$

$$= v_t - u_{0t} + p\left(u_{0t} + \frac{1}{2}(v^2)_x - v_{xxt}\right) = 0$$

$$p^0: v_{0t} - u_{0t} = 0 \qquad , \quad v_0 = x$$

$$p^1: v_{1t} = -\left(u_{0t} + \frac{1}{2}(v_0^2)_x - v_{0xxt}\right) \qquad , \quad v_1 = -xt$$

$$p^2: v_{2t} = -\left(\frac{1}{2}(2v_0v_1)_x - v_{1xxt}\right) \qquad , \quad v_2 = xt^2$$

$$p^3 : v_{3t} = -\left(\frac{1}{2}(2v_0v_2 + v_1^2)_x - v_{2xxt}\right) \quad , \quad v_3 = -xt^3$$

And so on, to get:

$$u(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) = x\sum_{n=0}^{\infty}(-t)^n = \frac{x}{1+t}$$

This is exact solution

Example 1.4[60]

Consider the following 3rd order linear PDE.

$$u_t + u_x = 2u_{xxt} \quad , \quad -\infty < x < \infty \quad , \quad t > 0$$

Subject to the IC: $u(x,0) = e^{-x}$

Using HPM we have:

$$H(v,p) = (1-p)(v_t - u_{0t}) + p(v_t + v_x - 2v_{xxt})$$

$$= v_t - u_{0t} + p(u_{0t} + v_x - 2v_{xxt}) = 0$$

$$p^0 : v_{0t} - u_{0t} = 0 \quad , \quad v_0 = e^{-x}$$

$$p^1 : v_{1t} = 2v_{0xxt} - v_{0x} - u_{0t} \quad , \quad v_1 = te^{-x} = p_1(t)e^{-x}$$

$$p^2 : v_{2t} = 2v_{1xxt} - v_{1x} \quad , \quad v_2 = \left(2t + \frac{t^2}{2}\right)e^{-x} = p_2(t)e^{-x}$$

$$p^3 : v_{3t} = 2v_{2xxt} - v_{2x} \quad , \quad v_3 = \left(4t + 2t^2 + \frac{t^3}{3!}\right)e^{-x} = p_3(t)e^{-x}$$

And so on, where $p_n(t)$ is a polynomial which has the following form:

$$p_n(t) = 2^{n-1}t + \sum_{j=0}^{n} a_{nj}t^j \; ; \; a_{n2} \, , \; a_{n3}, \dots \dots \dots \; , a_{nn} > 0$$

By induction, we get:

$v_n = p_n(t)e^{-x}$

And

$v_{n+1} = L_t^{-1}[2v_{nxxt} - v_{nx}] = L_t^{-1}[2L_t p_n(t) + p_n(t)]e^{-x}$

$= [2p_n(t) + L_t^{-1}p_n(t)]e^{-x} \equiv p_{n+1}(t)e^{-x}$

It is easy to see that $p_n(t) \to +\infty$ $(n \to +\infty)$ for any $t > 0$. Therefore, the infinite series:

$$\sum_{n=0}^{\infty} v_n = \left[1 + \sum_{n=1}^{\infty} p_n(t)\right] e^{-x}$$

Is divergent.

We note that in example 1.3, the PDE is nonlinear and the method gave the exact solution, in example 1.4, the PDE is linear but the method miss fire to get the exact solution. For these reasons we suggest efficient method based on coupled new transformation with HPM and denoted by NTHPM. Now, firstly introduce the new transformation proposed by luma-Alaa [51].

1.4. New Transformation

In this section, a new integral transformation is introduced. The domain of the new transformation (NT) is wider than of the domain of other transformation; therefore, it is more widely used to solve problems.

Definition 1.2 [51]

The new transformation of a function *f*(*t*) is defined by:

$$\bar{f}(u) = \mathbb{T}\{f(t)\} = \int_{0}^{\infty} e^{-t} f\left(\frac{t}{u}\right) dt, \qquad (1.6)$$

Where u is a real number, for those values of u which the improper integral is finite. A list of the NT for common functions is presented in the Table (1.1).

Table 1.1: New transformation for some common functions[51]

$f(t)$	$\bar{f}(u) = \mathbb{T}\{f(t)\}$	$D_{\bar{f}}$
t^n , n=0,1,…	$\frac{n!}{u^n}$	$u \neq 0$
t^a , $a>0$	$\Gamma(a+1)/u^a$	$u \neq 0$
e^{at}	$\frac{u}{u-a}$	$u \in R\backslash[0,a]$ $a \geq 0$ $u \in R\backslash[a,0]$ $a<0$
$\sin(at)$	$\frac{au}{a^2+u^2}$	$u \neq 0$
$\cos(at)$	$\frac{u^2}{a^2+u^2}$	$u \neq 0$
$\sinh(at)$	$\frac{-au}{a^2-u^2}$	$\lvert u\rvert > \lvert a\rvert$
$\cosh(at)$	$\frac{-u^2}{a^2-u^2}$	$\lvert u\rvert > \lvert a\rvert$
$u_a(t)=u(t-a)=H(t-a)$	e^{-au}	$u>0$
$\delta(t-a)$	e^{-au}/u	$u>0$

1.4.1. The General Properties of the New Transformation

If the new transformation $\mathbb{T}\{f\}$ and $\mathbb{T}\{g\}$ of the functions f(t) and $g(t)$ are well-defined and a, b are constants, then the following properties are hold:

1. Linearity property: $\mathbb{T}\{af(t) + bg(t)\} = a\mathbb{T}\{f(t)\} + b\mathbb{T}\{g(t)\}$ (1.7)

2. Convolution property: $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ (1.8)

3. $\mathbb{T}\{t^n\} = \frac{n!}{v^n}$, $v \neq 0$, $n = 0,1,2,3, \ldots$ (1.9)

4. Differentiation property: $\mathbb{T}\{f'\} = v(\mathbb{T}\{f\} - f(0))$ (1.10)

For more details see [51].

1.4.2. The Advantages of the New Transform

The NT has many interesting properties which make it rival to the **Laplace Transform** (LT). Some of these properties are:

1. The domain of the NT is wider than or equal to the domain of LT as illustrate in Table (1.2). This feature makes the NT more widely used for problems.
2. Depending on [51], the NT has the duality with LT, therefore, the NT can be solve all the problems which be solved by LT.
3. The unit step function in the *t*-domain is transformed to unity in the *u*-domain.
4. The differentiation and integration in the *t*-domain are equivalent to multiplication and division of the transformed function F(u) by u in the u-domain.
5. By Linear property (1.7), we have that for any constant $a \in$R, $\mathbb{T}\{a\} = a\mathbb{T}\{1\} = a$, and hence, $\mathbb{T}^{-1}\{a\} = a$, that is, we don't have any problem when we dealing with the constant term(the constant with respect to the parameter u).

Table 1.2: The domain of Laplace and new transformation[51]

f(t)	Laplace		New Transform	
	[$\mathbb{L}$ (f)](s)	Domain	[$\mathbb{T}$ (f)](u)	Domain
t^n n=0,1,2,…	$n!/s^{n+1}$	$s>0$	$n!/u^n$	$u\neq 0$
t^a a>0	$\Gamma(a+1)/s^{a+1}$	$s>0$	$\Gamma(a+1)/u^a$	$u\neq 0$
e^{at}	$1/(s-a)$	$s>a$	$u/(u-a)$	$u\in R/[0,a]$ if $a\geq 0$ $u\in R/[a,0]$ if $a<0$
$\sin(at)$	$a/(s^2+a^2)$	$s>0$	$au/(u^2+a^2)$	$u\neq 0$
$\cos(at)$	$s/(s^2+a^2)$	$s>0$	$u^2/(u^2+a^2)$	$u\neq 0$
$\sinh(at)$	$a/(s^2-a^2)$	$s>\|a\|$	$au/(u^2-a^2)$	$\|u\|>\|a\|$
$\cosh(at)$	$s/(s^2-a^2)$	$s>\|a\|$	$u^2/(u^2-a^2)$	$\|u\|>\|a\|$
$u_a(t)=u(t-a)=H(t-a)$	e^{-as}/s	$s>0$	e^{-au}	$u>0$
$\delta(t-a)$	e^{-as}	$s>0$	e^{-au}/u	$u>0$
$ln(at)$ a>0	$(ln(a/s)-\gamma)/s$	$s>0$	$ln(a/u)-\gamma$	$u>0$
$\gamma = -\int_0^\infty e^{-t} lnt\, dt \cong 0.5772 \ldots$				

In the next chapter, we will use a combination of new transformation (NT) and the HPM to solve types of PDEs and get exact solution without needing computer calculations.

Chapter Two

Couple New Transform with HPM to Solve Some Types of Partial Differential Equations

Chapter Two

Couple New Transform with HPM to Solve Some Types of Partial Differential Equations

2.1. Introduction

In this chapter, we will introduce an approach in obtaining the exact solution of non-linear partial differential equations. The new approach based on combine two methods, new transform with HPM and denoted by **NTHPM**. Then applied it to solve some important model equations. The exact solutions of these equations are compared to the HPM. The comparisons show the efficiency of the proposed NTHPM against the other methods. The method is strongly and powerful to treatment the nonlinear term of nonlinear equations.

2.2. New Transformation–Homotopy Perturbation Method

The **NTHPM** is a new method to solve differential equation; it successfully applied to solve types of PDEs. This method is powerful to obtain the exact solution without using computer calculating. The method suggested firstly by Tawfiq and Jabber in 2018 [18] to solve groundwater equation. The **NTHPM** has many merits and has many advantages over the HPM and ADM. In the present work, the suggested method is used to solve of 1D, 2D, and 3D;

2nd and 3rd order non-linear PDEs equations**,** and comparison has been made to the results obtained by the HPM and NTHPM.

2.3. Solve Linear PDEs by NTHPM

In this section, we will use a combination of new transform (NT) and the HPM to get the new transform that has played an important role because its theoretical interest also in such method that allows to solve in the simplest form; it used have to accelerate the convergence of power series.

To illustrate the ideas of NTHPM to find the exact solution of linear three dimensions 2nd order PDEs of the form:

$$u_{xx} + u_{yy} + u_{zz} = \alpha u_t \; ; \; x, y, z \in R \,\&\, t > 0 \qquad (2.1)$$

with initial condition (IC): $u(x, y, z, 0) = f(x, y, z)$; α is constant.

Firstly, rewrite equation (2.1) as following:

$$L[u(x, y, z, t)] + R[u(x, y, z, t)] = g(x, y, z, t) \qquad (2.2)$$

where L*:* is the linear differential operator ($L = \alpha \frac{\partial}{\partial t}$), R*:* is the remainder of the linear operator, g(x, y, z, t) is the inhomogeneous part.

We construct a Homotopy as:

$$H(u(x, y, z, t), p) = (1 - p)[L(u(x, y, z, t)) - L(u(x, y, z, 0))] +$$

$$p\,[A[u(x, y, z, t)] - g(x, y, z, t)] = 0 \qquad (2.3)$$

Where $p \in [0, 1]$ is an embedding parameter and A defined as A= L+ R.

It is clear that, if p =1, then the homotopy equation (2.3) is converted to the differential equation (2.2).

Substituting equation (2.2) into equation (2.3) and rewrite it as:

$$L(u) - L(f) - pL(u) + pL(f) + pL(u) + pR(u) - p\,g = 0$$

Then

$$L(u) - L(f) + p[L(f) + R(u) - g] = 0 \tag{2.4}$$

Since f(x, y, z) is independent of the variable t and the linear operator L dependent on t so, L(f(x, y, z)) = 0, i.e., the equation (2.4) becomes:

$$L(u) + pR(u) - p\,g = 0 \tag{2.5}$$

According to the classical perturbation technique, the solution of the equation (2.5) can be written as a power series of embedding parameter p, as follows:

$$u(x, y, z, t) = \sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \tag{2.6}$$

The convergence of series (2.6) at p =1 is discussed and proved in [8,25], which satisfies the differential equation (2.2).

The final step is determining the parts u_n (n= 0,1, 2,…) to get the solution $u(x, y, z, t)$.

Here, we couple the NT with HPM as follows:

Taking the NT (with respect to the variable t) for the equation (2.5) to get:

$$\mathbb{T}\{L(u)\} + p\,\mathbb{T}\{R(u)\} - p\,\mathbb{T}\{g\} = 0 \tag{2.7}$$

Now by using the differentiation property of NT (property 4) and equations (2.2), (2.7) becomes:

$$v\alpha\mathbb{T}\{u\} - v\alpha f(x) + p\,\mathbb{T}\{R(u)\} - p\,\mathbb{T}\{g\} = 0 \tag{2.8}$$

Hence:

$$\mathbb{T}\{u\} = f(x,y,z) - p\frac{\mathbb{T}\{R(u)\}}{v\alpha} + p\frac{\mathbb{T}\{g\}}{v\alpha} \tag{2.9}$$

Taking the inverse of the NT on both sides of equation (2.9), to get:

$$u(x,y,z,t) = f(x,y,z) - p\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R(u(x,y,z,t))\}}{v\alpha}\right\} + p\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t)\}}{v\alpha}\right\} \tag{2.10}$$

Then substituting equation (2.6) into equation (2.10) to obtain:

$$\sum_{n=0}^{\infty} p^n u_n = f(x,y,z) - p\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\,\{R(\sum_{n=0}^{\infty} p^n u_n)\}}{v\alpha}\right\} + p\,\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t)\}}{v\alpha}\right\} \tag{2.11}$$

By comparing the coefficient of powers of p in both sides of the equation (2.11), we have:

$$u_0 = f(x,y,z)$$

$$u_1 = -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_0]\}}{v\alpha}\right\} + \mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t)\}}{v\alpha}\right\}$$

$$u_2 = -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_1]\}}{v\alpha}\right\}$$

$$u_3 = -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_2]\}}{v\alpha}\right\} \tag{2.12}$$

$$\vdots$$

$$u_{n+1} = -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_n]\}}{v\alpha}\right\}$$

2.4. Illustrative Application

Here, the NTHPM may be used to solve the 2nd order-PDE with initial condition as following:

Example 2.1[60]

Let us consider the following 3D - PDE

$$u_{xx} + u_{yy} + u_{zz} = \alpha u_t \quad ; \quad all\ x, y, z\ in\ R\ \&\ t > 0$$

Subject to IC: $u(x, y, z, 0) = f(x, y, z) = 5\,sin(ax)\,sin(by)\,sin(cz)$,

where a, b, c and α are constants. According to the equation (2.12) the power series of p get as follows:

$$p^0: u_0(x, y, z, t) = 5\,sin(ax)\,sin(by)\,sin(cz)$$

$$p^1: u_1(x, y, z, t) = -(5\,sin(ax)\,sin(by)\,sin(cz))\left(\frac{t}{\alpha}\right)(a^2 + b^2 + c^2)$$

$$p^2: u_2(x, y, z, t) = (5\,sin(ax)\,sin(by)\,sin(cz))\left(\frac{t^2}{2\ \alpha^2}\right)(a^2 + b^2 + c^2)^2$$

$$p^3: u_3(x, y, z, t) = -(5\,sin(ax)\,sin(by)\,sin(cz))\left(\frac{t^3}{3!\ \alpha^3}\right)(a^2 + b^2 + c^2)^3$$

$$p^4: u_4(x, y, z, t) = (5\,sin(ax)\,sin(by)\,sin(cz))\left(\frac{t^4}{4!\ \alpha^4}\right)(a^2 + b^2 + c^2)^4$$

$$p^5: u_5(x, y, z, t) = -(5\,sin(ax)\,sin(by)\,sin(cz))\left(\frac{t^5}{5!\ \alpha^5}\right)(a^2 + b^2 + c^2)^5$$

$$\vdots$$

$$p^n: u_n(x, y, z, t) = (-1)^n(5\,sin(ax)\,sin(by)\,sin(cz))\left(\frac{t^n}{n!\ \alpha^n}\right)(a^2 + b^2 + c^2)^n$$

Thus, we get the following series form:

$$(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t)$$

$$= \sum_{n=0}^{\infty} (-1)^n 5\, sin(ax)\, sin(by)\, sin(cz) \left(\frac{t^n}{n!\ \alpha^n}\right)(a^2+b^2+c^2)^n$$

So, the closed form of the above series is:

$$u(x,y,z,t) = 5\sin(ax)\sin(by)\sin(cz)\ e^{-\frac{t}{\alpha}(a^2+b^2+c^2)}$$

This gives an exact solution of the problem.

Example 2.2[60]

Consider the following 3D-PDE

$$u_{xx} + u_{yy} + u_{zz} + e^{x+y} = \alpha u_t\ ;\ all\ x,y,z\ in\ R\ \&\ t > 0$$

with IC: $u(x,y,z,0) = f(x,y,z) = d$, where *d* is constants.

From equation (2.12), we get the power series of *p* as follows:

$p^0: u_0(x,y,z,t) = d$

$p^1: u_1(x,y,z,t) = \frac{t}{\alpha}(e^{x+y})$

$p^2: u_2(x,y,z,t) = \frac{t^2}{2\alpha^2}(e^{x+y})$

$p^3: u_3(x,y,z,t) = \frac{t^3}{6\alpha^3}(e^{x+y})$

$p^4: u_4(x,y,z,t) = \frac{t^4}{24\alpha^4}(e^{x+y})$

⋮

$p^n: u_n(x,y,z,t) = \frac{t^n}{n!\alpha^n}(e^{x+y})$

Thus, the following series form is obtained:

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) = d + \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!\ \alpha^{n+1}} (e^{x+y})$$

Therefore, the closed form of the above series is:

$$u(x,y,z,t) = e^{x+y}\left(e^{\frac{t}{\alpha}} - 1\right) + d$$

Example 2.3[55]

Consider another linear homogeneous Klein-Gordan equation

$$u_{tt} = u_{xx} + u_x + 2u \quad , \qquad -\infty < x < \infty \quad , \ \ t > 0$$

Subject to the ICs:

$$u(x,0) = e^x \ , u_t(x,0) = 0$$

Taking new transformation on both sides, subject to the IC, to get:

$$\mathbb{T}[u(x,y,t)] = e^x + \frac{1}{v^2}\mathbb{T}[u_{xx} + u_x + 2u\ ]$$

Taking inverse of new transformation, we get:

$$u(x,y,t) = e^x + \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[u_{xx} + u_x + 2u]\right]$$

by HPM, we get:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$$

Using equation (2.11) in equation (2.10), to get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^x$$
$$+ p\left(\mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}\left[\sum_{n=0}^{\infty} p^n u_{nxx}(x,y,t) + \sum_{n=0}^{\infty} p^n u_{nx}(x,y,t)\right.\right.\right.$$
$$\left.\left.\left. + 2\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right]\right]\right)$$

$p^0: u_0(x,t) = e^x$

$p^1: u_1(x,t) = \frac{(2t)^2}{2!} e^x$

$p^2: u_2(x,t) = \frac{(2t)^4}{4!}\ e^x$

$p^3: u_3(x,t) = \frac{(2t)^6}{6!} e^x$

:

$$u(x,t) = e^x\left(1 + \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} + \frac{(2t)^6}{6!} + \cdots\right) = e^x\, cosh(2t)$$

2.5. Solve Nonlinear PDE by NTHPM

In the NTHPM can be used for solving various types of nonlinear PDEs. To illustrate the basic idea of suggested method, we consider general nonlinear PDEs with the initial conditions of the form:

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t)\,, \qquad (2.13)$$

Subject to ICs: $u(x,0) = h(x)$, $u_t(x,0) = f(x)$.

where L is the 2nd order linear differential operator $L = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than L ; N represents the general nonlinear differential operator and $g(x,t)$is the source term.

Taking the new transformation on both sides of equation (2.13) to get:

$$\mathbb{T}[Lu(x,t)] + \mathbb{T}[Ru(x,t)] + \mathbb{T}[Nu(x,t)] = \mathbb{T}[g(x,t)]\,, \tag{2.14}$$

Using the differentiation property of the new transform, we have:

$$\mathbb{T}[u(x,t)] = h(x) + \frac{f(x)}{v} + \frac{1}{v^2}\mathbb{T}[g(x,t) - Ru(x,t) - Nu(x,t)] \tag{2.15}$$

Operating with the inverse of new transformation on both sides of equation (2.15) gives:

$$u(x,t) = G(x,t) - \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[Ru(x,t) - Nu(x,t)]\right] \tag{2.16}$$

where G(x, t) represents the term arising from the source term and the prescribed initial conditions. Now, we apply the HPM.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{2.17}$$

The nonlinear term can be decomposed as:

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \tag{2.18}$$

for He's polynomials $H_n(u)$ (see [48-49]) that are given by:

$$H_n(u_0, u_1, \dots\dots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right],\ n = 0,1,2,3,\dots \tag{2.19}$$

Substituting equation (2.17), (2.18) and (2.19) in equation (2.16) to get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \mathrm{G}(\mathrm{x},\mathrm{t}) - \mathrm{p}\left(\mathbb{T}^{-1}\left[\frac{1}{\mathrm{v}^2}\mathbb{T}[\mathrm{R}\sum_{\mathrm{n}=0}^{\infty}\mathrm{p}^{\mathrm{n}}\mathrm{u}_{\mathrm{n}}(\mathrm{x},\mathrm{t}) + \mathrm{N}\sum_{\mathrm{n}=0}^{\infty}\mathrm{p}^{\mathrm{n}}\mathrm{H}_{\mathrm{n}}(\mathrm{u})]\right]\right) \quad (2.20)$$

Comparing the coefficient of powers of p, to get:

$p^0: u_0(x,t) = G(x,t)$

$p^1: u_1(x,t) = -\mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[Ru_0(x,t) + H_0(u)]\right],$

$p^2: u_2(x,t) = -\mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[Ru_1(x,t) + H_1(u)]\right],$

$p^3: u_3(x,t) = -\mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[Ru_2(x,t) + H_2(u)]\right],$

and so on.

2.6. Applications

In this section, NTHPM is applied for solving various types of nonlinear wave-like equations with variable coefficients.

Example 2.4[10]

Consider the following 2D- nonlinear wave-like equations with variable coefficients.

$$u_{tt} = \frac{\partial^2}{\partial x \partial y}\left(u_{xx}u_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(xyu_x u_y\right) - u$$

with the ICs: $u(x,y,0) = e^{xy}, \quad u_t(x,y,0) = e^{xy}$

Taking new transformation on both sides, subject to the IC, we get:

$$\mathbb{T}[u(x,y,t)] = e^{xy} + \frac{e^{xy}}{v} + \frac{1}{v^2}\mathbb{T}\left[\frac{\partial^2}{\partial x \partial y}\left(u_{xx}u_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(xyu_x u_y\right) - u\right]$$

Taking inverse of new transform, we have:

$$u(x,y,t) = (1+t)e^{xy} - \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}\left[\frac{\partial^2}{\partial x\partial y}\left(u_{xx}u_{yy}\right) - \frac{\partial^2}{\partial x\partial y}\left(xyu_xu_y\right) - u\right]\right]$$

By HPM, we have:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$$

Substation equation (2.13) in the equation (2.14), to get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = (1+t)e^{xy}$$
$$+ p\left(\mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}\left[\sum_{n=0}^{\infty} p^n H_n(u) - \sum_{n=0}^{\infty} p^n K_n(u) - \sum_{n=0}^{\infty} p^n u_n(x,y,t)\right]\right]\right)$$

Where $H_n(u)$ and $K_n(u)$ are the He's polynomials having the value $H_n(u) = \frac{\partial^2}{\partial x\partial y}\left(u_{xx}u_{yy}\right)$ and $K_n(u) = \frac{\partial^2}{\partial x\partial y}\left(xyu_xu_y\right)$

The first few components of $H_n(u)$ and $K_n(u)$ are given by:

$$H_0(u) = \frac{\partial^2}{\partial x\partial y}\left(u_{0xx}u_{0yy}\right) = \frac{\partial^2}{\partial x\partial y}[(1+t)^2x^2y^2e^{2xy}]$$

$$H_1(u) = \frac{\partial^2}{\partial x\partial y}\left(u_{1xx}u_{0yy} + u_{0xx}u_{1yy}\right)$$
$$= \frac{\partial^2}{\partial x\partial y}\left[-2(1+t)\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)x^2y^2e^{2xy}\right]$$

$$H_2(u) = \frac{\partial^2}{\partial x\partial y}\left(u_{2xx}u_{0yy} + u_{1xx}u_{1yy} + u_{0xx}u_{2yy}\right)$$

And so on

$$K_0(u) = \frac{\partial^2}{\partial x\partial y}\left(xy\left(u_{0x}u_{0y}\right)\right) = \frac{\partial^2}{\partial x\partial y}[(1+t)^2x^2y^2e^{2xy}]$$

$$K_1(u) = \frac{\partial^2}{\partial x\partial y}\left(xy\left(u_{1x}u_{0y} + u_{0x}u_{1y}\right)\right)$$

$$= \frac{\partial^2}{\partial x \partial y}\left[-2(1+t)\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right)x^2y^2e^{2xy}\right]$$

$$K_2(u) = \frac{\partial^2}{\partial x \partial y}\Big(xy\big(u_{2x}u_{0y} + u_{1x}u_{1y} + u_{0x}u_{2y}\big)\Big)$$

:

Comparing the coefficients of various powers of p, to get:

$$p^0: u_0(x,y,t) = ((1+t)e^{xy}$$

$$p^1: u_1(x,y,t) = \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[H_0(u) + K_0(u) - u_0(x,y,t)]\right]$$

$$= -\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right)e^{xy}$$

$$p^2: u_2(x,y,t) = \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[H_1(u) + K_1(u) - u_1(x,y,t)]\right]$$

$$= \left(\frac{t^4}{4!}+\frac{t^5}{5!}\right)e^{xy}$$

$$p^3: u_3(x,y,t) = \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[H_2(u) + K_2(u) - u_2(x,y,t)]\right]$$

$$= -\left(\frac{t^6}{6!}+\frac{t^7}{7!}\right)e^{xy}$$

And so on

Therefore the solution is given by:

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + \cdots$$

$$= e^{xy}\left(1 + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \frac{t^6}{6!} - \frac{t^7}{7!} + \cdots\right)$$

So, $u(x,y,t) = e^{xy}(\cos t + \sin t)$

Example 2.5[10]

Consider the following nonlinear 1D- equation with variable coefficients.

$$u_{tt} = u^2 \frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^2 \frac{\partial^2}{\partial x^2}(u_x^3) - 18u^5 + u$$

with ICs: $u(x,0) = e^x, \qquad u_t(x,0) = e^x$

By applying NTHPM, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = (1+t)e^x$$

$$+ p\left(\mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}\left[\sum_{n=0}^{\infty} p^n H_n(u) - \sum_{n=0}^{\infty} p^n K_n(u) + 18\sum_{n=0}^{\infty} p^n J_n(u) - \sum_{n=0}^{\infty} p^n u_n(x,y,t)\right]\right]\right)$$

Where $H_n(u)$, $K_n(u)$and $J_n(u)$ are He's polynomials. First few components of He's polynomials are given by:

$$H_0(u) = u_0^2 \frac{\partial^2}{\partial x^2}(u_{0x} u_{0xx} u_{0xxx}) = 9(1+t)^5 e^{5x}$$

$$H_1(u) = 2u_0 u_1 \frac{\partial^2}{\partial x^2}(u_{0x} u_{0xx} u_{0xxx}) + u_0^2 \frac{\partial^2}{\partial x \partial y}(u_{1x} u_{0xx} u_{0xxx} + u_{0x} u_{1xx} u_{0xxx} + u_{0x} u_{0xx} u_{1xxx}) = 45(1+t)^4 \left(\frac{t^2}{2!} + \frac{t^3}{3!}\right) e^{5x}$$

:

and

$$K_0(u) = (u_{0x})^2 \frac{\partial^2}{\partial x^2}[(u_{0x})^3] = (1+t)^2 e^{2x} \frac{\partial^2}{\partial x^2}[(1+t)^3 e^{3x}]$$

$$= 9(1+t)^5 e^{5x}$$

$$K_1(u) = 2u_{0x}u_{1x}\frac{\partial^2}{\partial x^2}(u_{0x})^3 + 3(u_{0x})^2\frac{\partial^2}{\partial x^2}[(u_{0x})^2u_{1x}]$$

$$= 45(1+t)^4\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right)e^{5x}$$

:

and

$$J_0(u) = (u_0)^5 = (1+t)^5e^{5x}$$

$$J_1(u) = 5(u_0)^4u_1 = 5(1+t)^4\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right)e^{5x}$$

:

Comparing the coefficients of various powers of p, we get:

$$p^0: u_0(x,t) = (1+t)e^x$$

$$p^1: u_1(x,t) = \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[H_0(u) + K_0(u) - 18J_0(u) + u_0(x,y,t)]\right]$$

$$= \left(\frac{t^2}{2!}+\frac{t^3}{3!}\right)e^x$$

$$p^2: u_2(x,t) = \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[H_1(u) + K_1(u) - 18J_1(u) + u_1(x,y,t)]\right]$$

$$= \left(\frac{t^4}{4!}+\frac{t^5}{5!}\right)e^x$$

and so on, therefore the solution is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$= e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \cdots\right) = e^{x+t}$$

This is the exact solution.

Example 2.6[10]

Consider the following nonlinear 1D -3rd order wave-like equation with variable coefficients.

$$u_{tt} = x^2 \frac{\partial}{\partial x}(u_x u_{xx}) - x^2(u_{xx}^2) - u \qquad , \qquad 0 < x < 1 \,, t > 0$$

With ICs: $u(x,0) = 0, \qquad u_t(x,y,0) = x^2$

By applying NTHPM, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 t + p\left(\mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}\left[x^2 \sum_{n=0}^{\infty} p^n H_n(u) - x^2 \sum_{n=0}^{\infty} p^n K_n(u) - \sum_{n=0}^{\infty} p^n u_n(x,y,t)\right]\right]\right)$$

Where $H_n(u)$ and $K_n(u)$ are He's polynomials. First few components of He's polynomials are given by:

$$H_0(u) = \frac{\partial}{\partial x}(u_{0x} u_{0xx}) = 4t^2$$

$$H_1(u) = \frac{\partial}{\partial x}(u_{1x} u_{0xx} + u_{0x} u_{1xx}) = -8\frac{t^4}{3!}$$

$$H_2(u) = \frac{\partial^2}{\partial x}(u_{2x} u_{0xx} + u_{1x} u_{1xx} + u_{0x} u_{2xx})$$

:

And

$$K_0(u) = (u_{0xx})^2 = 4t^2$$

$$K_1(u) = 2u_{0xx} u_{1xx} = -8\frac{t^4}{3!}$$

$$K_2(u) = (u_1^2)_{xx} + 2(u_0)_{xx}(u_2)_{xx}$$

:

Comparing the coefficients of various powers of p, to get:

$p^0: u_0(x,t) = x^2 t$

$p^1: u_1(x,t) = \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[x^2 H_0(u) - x^2 K_0(u) - u_0(x,t)]\right] = -x^2\frac{t^3}{3!}$

$p^2: u_2(x,y,t) = \mathbb{T}^{-1}\left[\frac{1}{v^2}\mathbb{T}[x^2 H_1(u) - x^2 K_1(u) - u_1(x,t)]\right] = x^2\frac{t^5}{5!}$

:

Therefore, the solution is given by:

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + \cdots$$

$$= x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) = x^2 \sin t$$

This is the exact solution.

2.7. Solve Autonomous Equation by NTHPM

In this section the proposed method will be used to solve one of the most important of amplitude equations is the autonomous equation which describes the appearance of the stripe pattern in two dimensional systems. Moreover, this equation was applied to a number of problems in variety systems, e.g., Rayleigh-Benard convection, Faraday instability, nonlinear optics, chemical reactions and biological systems [60]. The approximate solutions of the autonomous equation were presented by differential transformation method [18], reduce differential transformation [8]. Here a reliable couple NTHPM is applied for solving autonomous equation.

To illustrate the ideas of NTHPM, firstly rewrite the initial value problem in autonomous equation in the form:

$$u_t(x,t) = cu_{xx}(x,t) + c_1 u(x,t) - c_2 u^n(x,t) \tag{2.20a}$$

With the IC: $u\,(x,t) = g(x)$ $(2.20b)$

where c_1 and c_2 are real numbers and *c* and *n* are positive integers.

Taking new transformation on both sides of the equation (2.20a) and using the linearity property of the new transformation gives:

$$\mathbb{T}\{u_t(x,t)\} = c\mathbb{T}\{u_{xx}(x,t)\} + c_1\mathbb{T}\{u(x,t)\} - c_2\mathbb{T}\{u^n(x,t)\} \tag{2.21}$$

By applying the differentiation property of new transform, we have

$$v\mathbb{T}\{u(x,t)\} - vu(x,0)$$
$$= c\mathbb{T}\{u_{xx}(x,t)\} + c_1\mathbb{T}\{u(x,t)\} - c_2\mathbb{T}\{u^n(x,t)\} \tag{2.22}$$

Thus, we get:

$$(v - c_1)\mathbb{T}\{u(x,t)\} = vg(x) + c\mathbb{T}\{u_{xx}(x,t)\} - c_2\mathbb{T}\{u^n(x,t)\}$$

$$\mathbb{T}\{u(x,t)\} = \frac{vg(x)}{v-c_1} + \frac{c}{v-c_1}\mathbb{T}\{u_{xx}(x,t)\} - \frac{c_2}{v-c_1}\mathbb{T}\{u^n(x,t)\}$$

Taking the inverse of new transformation on equation (2.22), we obtain:

$$u(x,t) = \mathbb{T}^{-1}\left\{\frac{vg(x)}{v-c_1}\right\} + \mathbb{T}^{-1}\left(\frac{c}{v-c_1}\mathbb{T}\{u_{xx}(x,t)\} - \frac{c_2}{v-c_1}\mathbb{T}\{u^n(x,t)\}\right) \tag{2.23}$$

In the HPM, the basic assumption is that the solutions can be written as a power series in *p*:

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t) \tag{2.24}$$

and the nonlinear term *N(u)* = $u^n, n>1$ can be presented by an infinite series as:

$$N(u)=\sum_{n=0}^{\infty}p^n H_n(x,t) \tag{2.25}$$

where $p\in[0,1]$ is an embedding parameter. $H_n(u)$ is He polynomials. Now, substituting (2.24) and (2.25) in (2.23), to get:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=\mathbb{T}^{-1}\left\{\frac{vg(x)}{v-c_1}\right\}$$
$$+p\mathbb{T}^{-1}\left(\frac{c}{v-c_1}\mathbb{T}\left\{\sum_{n=0}^{\infty}p^n\,u_{xx}(x,t)\right\}-\frac{c_2}{v-c_1}\mathbb{T}\left\{\sum_{n=0}^{\infty}p^n H_n\right\}\right)$$

Comparing the coefficient of powers of *p*, the following are obtained.

$$p^0: u_0(x,t)=\mathbb{T}^{-1}\left\{\frac{vg(x)}{v-c_1}\right\}$$

$$p^1: u_1(x,t)=\mathbb{T}^{-1}\left(\frac{c}{v-c_1}\mathbb{T}\{u_{0_{xx}}(x,t)\}-\frac{c_2}{v-c_1}\mathbb{T}\{H_0\}\right)$$

$$p^2: u_2(x,t)=\mathbb{T}^{-1}\left(\frac{c}{v-c_1}\mathbb{T}\{u_{1_{xx}}(x,t)\}-\frac{c_2}{v-c_1}\mathbb{T}\{H_1\}\right)$$

$$p^3: u_3(x,t)=\mathbb{T}^{-1}\left(\frac{c}{v-c_1}\mathbb{T}\{u_{2_{xx}}(x,t)\}-\frac{c_2}{v-c_1}\mathbb{T}\{H_2\}\right)$$

Proceeding in this same manner, the rest of the components $u_n(x, t)$ can be completely obtained and the series solution is thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by truncated series:

$$u(x,t) = \lim_{N\to\infty}\left(\sum_{n=0}^{N} u_n(x,t)\right)$$

The above series solutions generally converge very rapidly.

2.8. Illustrative Examples

In this section, some non-linear autonomous equations with IC are presented to show the advantages of the proposed method.

Example 2.7 [1]

Consider linear 1D, 2nd order autonomous equation.

$$u_t(x,t) = u_{xx}(x,t) - 3u(x,t) \tag{2.26}$$

With IC: $u(x,t) = e^{2x}$

Taking the new transformation on both sides of equation (2.26), we have

$$\mathbb{T}\{u_t(x,t)\} = \mathbb{T}\{u_{xx}(x,t)\} - 3\mathbb{T}\{u(x,t)\}$$

By applying the differentiation property of new transformation, we get:

$$v\mathbb{T}\{u(x,t)\} - vu(x,0) = \mathbb{T}\{u_{xx}(x,t)\} - 3\mathbb{T}\{u(x,t)\}$$

Thus, we have:

$$(v+3)\mathbb{T}\{u(x,t)\} = ve^{2x} + \mathbb{T}\{u_{xx}(x,t)\}$$

$$\mathbb{T}\{u(x,t)\} = \frac{ve^{2x}}{v+3} + \frac{1}{v+3}\mathbb{T}\{u_{xx}(x,t)\} \tag{2.27}$$

Taking the inverse new transformation on equation (2.27), to get:

$$u(x,t) = \mathbb{T}^{-1}\left\{\frac{ve^{2x}}{v+3}\right\} + \mathbb{T}^{-1}\left(\frac{1}{v+3}\mathbb{T}\{u_{xx}(x,t)\}\right)$$

Now, applying the HPM, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{2x-3t} + p\mathbb{T}^{-1}\left(\frac{1}{v+3}\mathbb{T}\left\{\sum_{n=0}^{\infty} p^n\, u_{xx}(x,t)\right\}\right)$$

Comparing the coefficients of powers of p, we have:

$$p^0: u_0(x,t) = \ e^{2x-3t}$$

$$p^1: u_1(x,t) = \mathbb{T}^{-1}\left(\frac{1}{v+3}\mathbb{T}\left\{u_{0_{xx}}(x,t)\right\}\right) = \mathbb{T}^{-1}\left(\frac{1}{v+3}\mathbb{T}\{4e^{2x}e^{-3t}\}\right)$$

$$= 4e^{2x}\mathbb{T}^{-1}\left(\frac{v}{(v+3)^2}\right) = 4te^{2x-3t}$$

$$p^2: u_2(x,t) = \mathbb{T}^{-1}\left(\frac{1}{v+3}\mathbb{T}\left\{u_{1_{xx}}(x,t)\right\}\right) = \mathbb{T}^{-1}\left(\frac{1}{v+3}\mathbb{T}\{16te^{2x}e^{-3t}\}\right)$$

$$= \mathbb{T}^{-1}\left(\frac{1}{v+3} * \frac{16e^{2x}v}{(v+3)^2}\right) = \mathbb{T}^{-1}\left(8e^{2x} * \frac{2!v}{(v+3)^{2+1}}\right) = 8t^2e^{2x-3t}$$

$$p^3: u_3(x,t) = \mathbb{T}^{-1}\left(\frac{1}{v+3}\mathbb{T}\left\{u_{2_{xx}}(x,t)\right\}\right)$$

$$= \mathbb{T}^{-1}\left(\frac{1}{v+3}\mathbb{T}\{32e^{2x}t^2e^{-3t}\}\right) = 32e^{2x}\mathbb{T}^{-1}\left(\frac{1}{v+3} * \frac{2!v}{(v+3)^3}\right)$$

$$= 32e^{2x}\mathbb{T}^{-1}\left(\frac{2!\,v}{(v+3)^4}\right) = \frac{32}{3}e^{2x}\mathbb{T}^{-1}\left(\frac{3!\,v}{(v+3)^{3+1}}\right) = \frac{32}{3}t^3e^{2x-3t}$$

And so on, therefore the solution $u(x, t)$ is given by:

$$u(x,t) = e^{2x-3t}\left(1 + 4t + 8t^2 + \frac{32t^3}{3} + \cdots\right)$$

$$= e^{2x-3t}\left(1 + (4t) + \frac{(4t)^2}{2!} + \frac{(4t)^3}{3!} + \cdots\right) = e^{2x+t}$$

Example 2.8 [1]

Consider nonlinear 1D, 2nd order autonomous equation.

$$u_t(x,t) = 5u_{xx}(x,t) + 2u(x,t) + u^2(x,t) \tag{2.28}$$

With the IC: $u\,(x,t) = \beta$

where β is arbitrary constant. Taking the new transformation on both sides of equation (2.28), we have:

$$\mathbb{T}\{u_t(x,t)\} = 5\mathbb{T}\{u_{xx}(x,t)\} + 2\mathbb{T}\{u(x,t)\} + \mathbb{T}\{u^2(x,t)\}$$

$$v\mathbb{T}\{u(x,t)\} - vu(x,0) = 5\mathbb{T}\{u_{xx}(x,t)\} + 2\mathbb{T}\{u(x,t)\} + \mathbb{T}\{u^2(x,t)\}$$

$$(v-2)\mathbb{T}\{u(x,t)\} = v\beta + 5\mathbb{T}\{u_{xx}(x,t)\} + \mathbb{T}\{u^2(x,t)\}$$

$$\mathbb{T}\{u(x,t)\} = \frac{v\beta}{v-2} + \frac{5}{v-2}\mathbb{T}\{u_{xx}(x,t)\} + \frac{1}{v-2}\mathbb{T}\{u^2(x,t)\} \tag{2.29}$$

Taking the inverse new transformation on equation (2.29), we obtain:

$$u(x,t) = \mathbb{T}^{-1}\left\{\frac{v\beta}{v-2}\right\} + \mathbb{T}^{-1}\left(\frac{5}{v-2}\mathbb{T}\{u_{xx}(x,t)\} + \frac{1}{v-2}\mathbb{T}\{u^2(x,t)\}\right)$$

Now, applying the HPM, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \beta\, e^{2t} + p\mathbb{T}^{-1}\left(\frac{5}{v-2}\mathbb{T}\left\{\sum_{n=0}^{\infty} p^n\, u_{xx}(x,t)\right\} + \frac{1}{v-2}\mathbb{T}\left\{\sum_{n=0}^{\infty} p^n H_n\right\}\right)$$

Comparing the coefficients of powers of p, we have:

$$p^0 : u_0(x,t) = \beta e^{2t}$$

$$p^1 : u_1(x,t) = \mathbb{T}^{-1}\left(\frac{5}{v-2}\mathbb{T}\{u_{0_{xx}}(x,t)\} + \frac{1}{v-2}\mathbb{T}\{H_0(u)\}\right)$$

$$p^2 : u_2(x,t) = \mathbb{T}^{-1}\left(\frac{5}{v-2}\mathbb{T}\{u_{1_{xx}}(x,t)\} + \frac{1}{v-2}\mathbb{T}\{H_1(u)\}\right)$$

$$p^3 : u_3(x,t) = \mathbb{T}^{-1}\left(\frac{5}{v-2}\mathbb{T}\{u_{2_{xx}}(x,t)\} + \frac{1}{v-2}\mathbb{T}\{H_2(u)\}\right)$$

:

First, compute A_n to the nonlinear part $N(u)$, we have:

$$N(u) = N\left(\sum_{n=0}^{\infty} p^n H_n\right)$$

$$= u_0^2 + p\,(2u_0u_1) + p^2(2u_0u_2 + u_1^2) + p^3(2u_0u_3 + 2u_1u_2) + \cdots$$

So,

$$H_0 = u_0^2$$

$$H_1 = 2u_0u_1$$

$$H_2 = 2u_0u_2 + u_1^2$$

and so on.

Moreover, the sequence of the parts u_n is:

$$u_0(x,t) = \beta e^{2t}$$

$$u_1(x,t) = \mathbb{T}^{-1}\left(0 + \frac{1}{v-2}\mathbb{T}\{\beta^2 e^{4t}\}\right)$$

$$= \mathbb{T}^{-1}\left(\frac{1}{v-2} * \frac{\beta^2 v}{v-4}\right) = \beta^2\mathbb{T}^{-1}\left(\frac{v}{(v-2)(v-4)}\right)$$

$$= \beta^2\mathbb{T}^{-1}\left(\frac{v}{2(v-4)} - \frac{v}{2(v-2)}\right) = \beta^2\left[\frac{1}{2}e^{4t} - \frac{1}{2}e^{2t}\right] = \frac{\beta^2}{2}e^{2t}(e^{2t}-1)$$

$$u_2(x,t) = \mathbb{T}^{-1}\left(0 + \frac{1}{v-2}\mathbb{T}\left\{2\beta e^{2t} * \frac{\beta^2}{2}e^{2t}(e^{2t}-1)\right\}\right)$$

$$= \mathbb{T}^{-1}\left(\frac{1}{v-2}\mathbb{T}\{\beta^3(e^{6t}-e^{4t})\}\right) = \mathbb{T}^{-1}\left(\frac{\beta^3}{v-2}\left(\frac{v}{v-6} - \frac{v}{v-4}\right)\right)$$

$$= \beta^3\mathbb{T}^{-1}\left(\frac{v}{(v-2)(v-6)} - \frac{v}{(v-2)(v-4)}\right)$$

$$= \beta^3\mathbb{T}^{-1}\left(\left(\frac{v}{4(v-6)} - \frac{v}{4(v-2)}\right) - \left(\frac{v}{2(v-4)} - \frac{v}{2(v-2)}\right)\right)$$

$$= \beta^3\left[\left(\frac{1}{4}e^{6t} - \frac{1}{4}e^{2t}\right) - \left(\frac{1}{2}e^{4t} - \frac{1}{2}e^{2t}\right)\right] = \beta^3\left[\frac{1}{4}e^{6t} - \frac{1}{2}e^{4t} + \frac{1}{4}e^{2t}\right]$$

$$= \frac{\beta^3}{4}e^{2t}(e^{4t} - 2e^{2t} + 1) = \frac{\beta^3}{4}e^{2t}(e^{2t}-1)^2$$

$$u_3(x,t) = \frac{\beta^4}{8}e^{2t}(e^{2t}-1)^3$$

Therefore the solution $u(x, t)$ is given by:

$$u(x,t) = e^{2t}\left(\beta + \frac{\beta^2}{2}(e^{2t}-1) + \frac{\beta^3}{4}(e^{2t}-1)^2 + \frac{\beta^4}{8}(e^{2t}-1)^3 + \cdots\right)$$

$$= \frac{2\beta e^{2t}}{2 + \beta(1-e^{2t})}$$

2.9. Convergence of the Solution for Linear Case

Now, we need to show the convergence of series form to the exact form for 3D- PDEs.

Lemma 2.1 If f be continues function then

$$\frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = f(t)$$

Proof

Suppose that

$$\int f(x)dx = F(x) + c$$

Assume that $x = t - \tau$ then $dx = -d\tau$ then

$$\frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = -\frac{\partial}{\partial t}\int_t^0 f(x)dx = \frac{\partial}{\partial t}\int_0^t f(x)dx = \frac{\partial}{\partial t}[F(x)|_0^t] =$$

$$\frac{\partial}{\partial t}[F(t) - F(0)] = \frac{\partial}{\partial t}F(t) - \frac{\partial}{\partial t}F(0) = f(t)$$

$$so, \quad \frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = f(t)$$

Lemma 2.2 Let $\mathbb{T}$ is new transformation. Then

$$\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\left\{f(X,t)\right\}\right\}\right) = f(X,t) \text{ ,where } X = (x,y,z)$$

Proof

Using properties 2 and 3 of NT, and lemma (2.1), we have:

$$\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\left\{f(X,t)\right\}\right\}\right) = \frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\left\{1\right\}\mathbb{T}\left\{f(X,t)\right\}\right\}\right) = \frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\mathbb{T}\left\{1 *\right.\right.\right.$$

$$\left.\left.\left.f(X,t)\right\}\right\}\right) = \frac{\partial}{\partial t}(1 * f(X,t)) = \frac{\partial}{\partial t}\left(\int_0^t f(X,t-\tau)d\tau\right) = f(X,t)$$

Theorem 2.1 (Convergence Theorem)

If the series form given in equation (2.6) with $p = 1$, i.e.,

$$u(x,y,z,t)=\sum_{n=0}^{\infty}u_n(x,y,z,t) \tag{2.30}$$

is convergent. Then the limit point converges to the exact solution of equation (2.1), where u_n ($n = 0, 1, \ldots$) are calculated by NTHPM, i.e.,

$$\left.\begin{aligned} u_0(x,y,z,t)+u_1(x,y,z,t) &= \mathbb{T}^{-1}\left\{f+\frac{1}{v\alpha}\,\mathbb{T}\left\{-R[u_0]\right\}\right\} \\ u_n(x,y,z,t) &= -\mathbb{T}^{-1}\left\{\frac{1}{v\alpha}\mathbb{T}\{R[u_{n-1}]\}\right\},n>1 \end{aligned}\right\}$$

Proof

Suppose that equation (2.30) converges to the limit point say as:

$$w(x,y,z,t)=\sum_{n=0}^{\infty}u_n(x,y,z,t)$$

Now, from right hand side of equation (2.1) we have:

$$\alpha\frac{\partial w}{\partial t}=\alpha\,\frac{\partial}{\partial t}\sum_{n=0}^{\infty}u_n(x,y,z,t)\;=\alpha\,\frac{\partial}{\partial t}\left[u_0+u_1+\sum_{n=2}^{\infty}u_n(x,y,z,t)\right]$$

$$=\;\alpha\,\frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\left\{f+\frac{1}{v\alpha}\,\mathbb{T}\left\{-R[u_0]\right\}\right\}-\sum_{n=2}^{\infty}\mathbb{T}^{-1}\left\{\frac{1}{v\alpha}\left[\mathbb{T}\{R[u_{n-1}]\}\right]\right\}\right]$$

$$=\;\alpha\,\frac{\partial f}{\partial t}-R[u_0]-\,\frac{\partial}{\partial t}\,\left(\sum_{n=2}^{\infty}\mathbb{T}^{-1}\left\{\frac{1}{v}\left[\mathbb{T}\{R[u_n]\}\right]\right\}\right)$$

$$=0-R[u_0]-\sum_{n=0}^{\infty}\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{v}\left[\mathbb{T}\{R[u_n]\}\right]\right\}\right) \tag{2.31}$$

By lemma (2.2), the equation (2.31) becomes:

$$\alpha\frac{\partial w}{\partial t} = -\sum_{n=0}^{\infty} R[u_n] = -R\left[\sum_{n=0}^{\infty} u_n\right] = -Rw = w_{xx} + w_{yy} + w_{zz}$$

Then $w(x, y, z, t)$ satisfies equation (2.1). So, it is exact solution.

2.10. Convergence of the Solution for Nonlinear Case

Now, we must prove the convergence of solution of equation (2.13) to the exact solution when we used the NTHPM, the solution is given in equation (2.32), where u_n,, (n= 0, 1, …), are calculated by new transformation, i.e.,

$$u(x, y, z, t) = u_0(x, y, z, t) + u_1(x, y, z, t) + \cdots = \sum_{n=0}^{\infty} u_n(x, y, z, t) \qquad (2.32)$$

$$\left.\begin{aligned} u_0 &= f(x, y, z) \\ u_n &= -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_{n-1}] + A_{n-1}\}}{v}\right\} \;, n \geq 1 \end{aligned}\right\} \qquad (2.33)$$

and A_n, (n = 0, 1, …), are defined as

$$A_n = u_n\frac{\partial u_0}{\partial z} + u_{n-1}\frac{\partial u_1}{\partial z} + \cdots + u_0\frac{\partial u_n}{\partial z} = \sum_{k=0}^{n} u_k\,\frac{\partial u_{n-k}}{\partial z} \qquad (2.34)$$

Now we proof the convergence in the following theorem.

Theorem 2.2 (Convergence Theorem)

If the series (2.32) which was calculated by NTHPM, is convergent then the limit point converges to the exact solution for the equation (2.13). Suppose that the limit point is:

$$w(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, y, z, t)$$

Now, from left hand side of equation (2.13), we have:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\sum_{n=0}^{\infty} u_n(x,y,z,t) = \frac{\partial}{\partial t}\left[u_0(x,y,z,t) + \sum_{n=1}^{\infty} u_n(x,y,z,t)\right]$$

$$= \frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\{f\} - \sum_{n=1}^{\infty} \mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_{n-1}] + A_{n-1}\}}{v}\right\}\right]$$

$$= \frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\{f\} - \sum_{n=0}^{\infty} \mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_n]\}}{v}\right\} - \sum_{n=0}^{\infty} \mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{A_n\}}{v}\right\}\right]$$

$$= \frac{\partial f}{\partial t} - \frac{\partial}{\partial t}\sum_{n=0}^{\infty} \mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_n]\}}{v}\right\} - \frac{\partial}{\partial t}\sum_{n=0}^{\infty} \mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{A_n\}}{v}\right\}$$

$$= \frac{\partial f}{\partial t} - \sum_{n=0}^{\infty} \frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_n]\}}{v}\right\}\right] - \sum_{n=0}^{\infty} \frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{A_n\}}{v}\right\}\right] \qquad (2.35)$$

By lemma (2.2) and equation (2.35), we get:

$$\frac{\partial w}{\partial t} = 0 - \sum_{n=0}^{\infty} R[u_n] - \sum_{n=0}^{\infty} A_n \qquad (2.36)$$

However, from equation (2.34), we have:

$$\sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} u_k \frac{\partial u_{n-k}}{\partial z}$$

$$= u_0\frac{\partial u_0}{\partial z} + u_0\frac{\partial u_1}{\partial z} + u_1\frac{\partial u_0}{\partial z} + u_0\frac{\partial u_2}{\partial z} + u_1\frac{\partial u_1}{\partial z} + u_2\frac{\partial u_0}{\partial z}$$

$$+ u_0\frac{\partial u_3}{\partial z} + u_1\frac{\partial u_2}{\partial z} + u_2\frac{\partial u_1}{\partial z} + u_3\frac{\partial u_0}{\partial z} + \cdots$$

$$= u_0\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) + u_1\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) +$$

$$u_2\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) + u_3\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) + \cdots$$

$$= (u_0 + u_1 + u_2 + u_3 + \cdots)\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) =$$

$$\left(\sum_{n=0}^{\infty} u_n\right)\left(\sum_{n=0}^{\infty} \frac{\partial u_n}{\partial z}\right) = \left(\sum_{n=0}^{\infty} u_n\right)\left(\frac{\partial}{\partial z}\sum_{n=0}^{\infty} u_n\right) \qquad (2.37)$$

Then substitute equation (2.37) in equation (2.36) to obtain:

$$\frac{\partial w}{\partial t} = -R\left[\sum_{n=0}^{\infty} u_n\right] - \left(\sum_{n=0}^{\infty} u_n\right)\left(\frac{\partial}{\partial z}\sum_{n=0}^{\infty} u_n\right) = -R[w] - w\frac{\partial w}{\partial z}$$

$$then \quad \frac{\partial w}{\partial t} = \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] - w\frac{\partial w}{\partial z}$$

Then $w(x, y, z, t)$ is satisfy equation (2.13). So, its exact solution.

Chapter Three

Solving System of Partial Differential Equations by New Couple Methods

Chapter Three

Solving System of Partial Differential Equations by New Couple Method

3.1. Introduction

The system of PDEs arises in many areas of mathematics, engineering and physical sciences. These systems are too complicated to be solved exactly so it is still very difficult to get closed-form solutions for most problems. A vast class of analytical and numerical methods has been proposed to solve such problems. Such as the ADM [24, 57], VIM [6, 54], HPM [3, 7, 8, 59], HAM [5] and DTM [28]. But many systems such that system of high dimensional equations, the required calculations to obtain its solution in some time may be too complicated. Recently, many powerful methods have been presented, such as the coupled method [10, 32, 52].

In this chapter, new coupled method based on HPM and new transform NTHPM, is presented to solve systems of PDEs. The efficiency of the NTHPM is verified by some examples.

3.2. Solving System of Nonlinear PDEs by NTHPM

This section consist the procedure of the NTHPM to solve system of nonlinear PDEs. Firstly, writes the system of nonlinear PDEs as follows:

$$L[u(x,y,t)] + R[u(x,y,t)] + N[u(x,y,t)] = g_1(x,y,t)$$

$$L[w(x,y,t)] + R[w(x,y,t)] + N[w(x,y,t)] = g_2(x,y,t) \qquad (3.1a)$$

With IC:

$$u(x,y,0) = f(x,y)$$

$$w(x,y,0) = g(x,y) \qquad (3.1b)$$

where all $x, y\ in\ R$, L is a linear differential operator ($L = \frac{\partial}{\partial t}$), R is a remained of the linear operator, N is a nonlinear differential operator and $g_1(x,y,t), g_2(x,y,t)$ are the nonhomogeneous part.

We construct a homotopy u(x, p): $R^n\times$ [0, 1] → R, using the homotopy perturbation technique which satisfies

$$H(u(x,y,t),p) = (1-p)\big[L(u(x,y,t)) - L\big(u(x,y,0)\big)\big] + p[A(u(x,y,t)) - g_1(x,y,t)] = 0$$

$$H(w(x,y,t),p) = (1-p)\big[L(w(x,y,t)) - L\big(w(x,y,0)\big)\big] + p[A(v(x,y,t)) - g_1(x,y,t)] = 0 \qquad (3.2)$$

Where $p \in [0,1]$ is an embedding parameter and the operator A defined as:

$A = L + R + N$.

Obviously, if p = 0, the system (3.2) becomes:

$L(u(x,y,t)) = L\big(u(x,y,0)\big)$, and $L(w(x,y,t)) = L\big(w(x,y,0)\big)$.

It is clear that, if $p = 1$ then the homotopy system (3.2) convert to the main system (3.1). In topology, this deformation is called homotopic.

Substitute equation (3.1b) in system (3.2) and rewrite it as:

$$L\big(u(x,y,t)\big) - L\big(f(x,y)\big) - pL\big(u(x,y,t)\big) + pL\big(f(x,y)\big) +$$

$$pL\big(u(x,y,t)\big) + pR\big(u(x,y,t)\big) + pN\big(u(x,y,t)\big) - pg_1(x,y,t) = 0$$

$$L(w(x,y,t)) - L\big(g(x,y)\big) - pL\big(w(x,y,t)\big) + pL\big(g(x,y)\big) +$$

$$pL\big(w(x,y,t)\big) + pR\big(w(x,y,t)\big) + pN\big(w(x,y,t)\big) - pg_2(x,y,t) = 0$$

Then

$$L(u(x,y,t)) - L\big(f(x,y)\big) + p\big[L\big(f(x,y)\big) + R(u(x,y,t)) +$$

$$N(u(x,y,t)) - g_1(x,y,t)\big] = 0$$

$$L(w(x,y,t)) - L\big(g(x,y)\big) + p\big[L\big(g(x,y)\big) + R\big(w(x,y,t)\big) +$$

$$N\big(w(x,y,t)\big) - g_2(x,y,t)\big] = 0 \qquad (3.3)$$

Since $f(x,y)$ and $g(x,y)$ are independent of the variable t and the linear operator L dependent on t so, $L\big(f(x,y)\big) = 0, L\big(g(x,y)\big) = 0$, i.e., system (3.3) becomes:

$$L(u(x,y,t)) + p[R(u(x,y,t)) + N(u(x,y,t)) - g_1(x,y,t)] = 0$$

$$L\big(w(x,y,t)\big) + p\big[R\big(w(x,y,t)\big) + N\big(w(x,y,t)\big) - g_2(x,y,t)\big] = 0 \qquad (3.4)$$

According to the classical perturbation technique, the solution of system (3.4) can be written as a power series of embedding parameter p, in the form:

$$u(x,y,t) = \sum_{n=0}^{\infty} p^n u_n(x,y,t)$$

$$w(x,y,t) = \sum_{n=0}^{\infty} p^n w_n(x,y,t) \tag{3.5}$$

For most cases, the series form in (3.5) is convergent and the convergent rate depends on the nonlinear operator $N(u(x,y,t))$ and $N(w(x,y,t))$.

Taking the NT (with respect to the variable t) for the system (3.4) to get:

$$\mathbb{T}\{L(u)\} + p\,\mathbb{T}\{R(u) + N(u) - g_1\} = 0$$

$$\mathbb{T}\{L(w)\} + p\,\mathbb{T}\{R(w) + N(w) - g_2\} = 0 \tag{3.6}$$

Now by using the differentiation property of NT and IC in (3.1b), so system (3.6) becomes:

$$v\mathbb{T}\{u\} - vf(x,y) + p\,\mathbb{T}\{R(u) + N(u) - g_1\} = 0$$

$$v\mathbb{T}\{w\} - vg(x,y) + p\,\mathbb{T}\{R(w) + N(w) - g_2\} = 0$$

Hence:

$$\mathbb{T}\{u\} = f(x,y) + p\frac{1}{v}\mathbb{T}\{g_1 - R(u) - N(u)\}$$

$$\mathbb{T}\{w\} = g(x,y) + p\frac{1}{v}\mathbb{T}\{g_2 - R(w) - N(w)\} \tag{3.7}$$

By taking the inverse of NT on both sides of system (3.7), to get:

$$u(x,y,t) = f(x,y) + p\,\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_1(x,y,t) - R\big(u(x,y,t)\big) - N\big(u(x,y,t)\big)\right\}\right\}$$

$$w(x,y,t) = g(x,y) + p\,\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_2(x,y,t) - R\big(w(x,y,t)\big) - N\big(w(x,y,t)\big)\right\}\right\} \tag{3.8}$$

Then, substitute system (3.5) in system (3.8) to get:

$$\sum_{n=0}^{\infty} p^n u_n = f(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_1(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n u_n\right) - N\left(\sum_{n=0}^{\infty} p^n u_n\right)\right\}\right\}$$

$$\sum_{n=0}^{\infty} p^n w_n = g(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_2(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n w_n\right) - N\left(\sum_{n=0}^{\infty} p^n w_n\right)\right\}\right\} \qquad (3.9)$$

The nonlinear part can be decomposed, as will be explained later, by substituting system (3.5) in it as:

$$N(u) = N\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right) = \sum_{n=0}^{\infty} p^n H_n$$

$$N(w) = N\left(\sum_{n=0}^{\infty} p^n w_n(x,y,t)\right) = \sum_{n=0}^{\infty} p^n K_n$$

Then system (3.9) becomes:

$$\sum_{n=0}^{\infty} p^n u_n = f(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_1(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n u_n\right) - \sum_{n=0}^{\infty} p^n H_n\right\}\right\}$$

$$\sum_{n=0}^{\infty} p^n w_n = g(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_2(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n w_n\right) - \sum_{n=0}^{\infty} p^n K_n\right\}\right\} \qquad (3.10)$$

By comparing the coefficient with the same power of p, in both sides of the system (3.10) we have:

$$u_0 = f(x,y), \qquad w_0 = g(x)$$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{g_1(x,y,t) - R(u_0) - H_0\}\right\},\ w_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{g_2(x,y,t) - R(w_0) - K_0\}\right\}$$

$$u_2 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(u_1) + H_1\}\right\}, \qquad w_2 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(w_1) + K_1\}\right\}$$

$$u_3 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(u_2) + H_2\}\right\}, \qquad w_3 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(w_2) + K_2\}\right\}$$

$\vdots$

$u_{n+1} = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(u_n) + H_n\}\right\}, \quad w_{n+1} = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(w_n) + K_n\}\right\}$

According to the series solution in system (3.5), when at p=1 we can get

$u(x, y, t) = u_0(x, y, t) + u_1(x, y, t) + \cdots = \sum_{n=0}^{\infty} u_n(x, y, t)$

$w(x, y, t) = w_0(x, y, t) + w_1(x, y, t) + \cdots = \sum_{n=0}^{\infty} w_n(x, y, t)$

3.3. Illustrative Examples for System of 1D-PDEs

In this section, the **NTHPM** can be used to solve system of 1D, nonlinear PDEs.

Example 3.1 [7]

Consider the following system of 1D, nonhomogeneous nonlinear PDEs.

$$\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial t} + u\frac{\partial w}{\partial t} = -1 + e^x \sin(t)$$

$$\frac{\partial w}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial u}{\partial x} = -1 - e^{-x}\cos(t)$$

Subject to IC: $u(0, t) = \sin(t)$, $w(0, t) = \cos(t)$

$u_0 = \sin(t)$, $w_0 = \cos(t)$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{w_0\frac{\partial u_0}{\partial t} - u_0\frac{\partial w_0}{\partial t} - 1 + e^x \sin(t)\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{e^x \sin(t)\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v} * \frac{v\sin(t)}{v-1}\right\} = \sin(t)\mathbb{T}^{-1}\left\{\frac{v}{v-1} - 1\right\} = e^x \sin(t) - \sin(t)$$

$$w_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-\frac{\partial u_0}{\partial t}\frac{\partial w_0}{\partial x} - \frac{\partial w_0}{\partial t}\frac{\partial u_0}{\partial x} - 1 - e^{-x}\cos(t)\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-1 - e^{-x}\cos(t)\}\right\}$$

$$= \mathbb{T}^{-1}\left\{-\frac{1}{v} - \frac{v\cos(t)}{v(v+1)}\right\} = -x + \cos(t)\,\mathbb{T}^{-1}\left\{\frac{-v}{v+1} + 1\right\}$$

$$= -x + e^{-x}\cos(t) - \cos(t)$$

$$u_2 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(w_1\frac{\partial u_0}{\partial t} + w_0\frac{\partial u_1}{\partial t}\right) - (u_1\frac{\partial w_0}{\partial t} + u_0\frac{\partial w_1}{\partial t})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-2 + e^{x} + e^{-x} - x\cos(t)\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{-2}{v} + \frac{1}{v-1} + \frac{1}{v+1} - \frac{\cos(t)}{v^2}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{-2}{v} + \frac{v}{v-1} - 1 - \frac{v}{v+1} + 1 - \frac{\cos(t)}{v^2}\right\}$$

$$= -2t + e^{x} - e^{-x} - \frac{x^2}{2!}\cos(t)$$

$$w_2 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{(\frac{\partial u_1}{\partial t}\frac{\partial w_0}{\partial x} + \frac{\partial u_0}{\partial t}\frac{\partial w_1}{\partial x}) + (\frac{\partial w_1}{\partial t}\frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial t}\frac{\partial u_1}{\partial x})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-\cos(t) - e^{-x}\cos^2(t) - e^{x}\sin^2(t)\}\right\}$$

$$= -\mathbb{T}^{-1}\left\{-\frac{\cos(t)}{v} - \frac{v\cos^2(t)}{v(v+1)} - \frac{v\sin^2(t)}{v(v-1)}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{\cos(t)}{v} + \cos^2(t)\left(\frac{-v}{(v+1)} + 1\right) + \sin^2(t)\,(\frac{v}{(v-1)} - 1)\right\}$$

$$= x\cos(t) - e^{-x}\cos^2(t) + \cos^2(t) + e^{x}\sin^2(t) - \sin^2(t)$$

$$= x\cos(t) - e^{-x}\cos^2(t) + e^{x}\sin^2(t) + \cos(2t)$$

and so on, therefore

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sin(t) + e^{x}\sin(t) - \sin(t) + \cdots = e^{x}\sin(t)$$

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = \cos(t) + e^{-x}\cos(t) - \cos(t) - x + \cdots =$$
$$e^{-x}\cos(t)$$

This is the exact solution.

Example 3.2 [3]

Consider a system of 1D, 3rd order nonlinear KdV equations (type1).

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} - w\frac{\partial w}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial w}{\partial x} = 0$$

Subject to IC: $u(x,0) = 3 - 6\tanh^2\left(\frac{x}{2}\right)$, $w(x,0) = -3i\sqrt{2}\tanh^2\left(\frac{x}{2}\right)$

We have the following terms:

$p^0{:}\, u_0(x,t) = 3 - 6\tanh^2\left(\frac{x}{2}\right)$, $p^0{:}\, w_0(x,t) = -3i\sqrt{2}\tanh^2\left(\frac{x}{2}\right)$

$$p^1{:}\, u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial^3 u_0}{\partial x^3} + u_0\frac{\partial u_0}{\partial x} + w_0\frac{\partial w_0}{\partial x}\right\}\right\}$$

$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(-6\,tanh^3(\frac{x}{2})sech^2(\frac{x}{2}) + 12\,tanh(\frac{x}{2})\,sech^4(\frac{x}{2})\right) + \left(-18\tanh(\frac{x}{2})\,sech^2(\frac{x}{2}) + \right.\right.\right.$

$\left.\left.\left. 36\,tanh^3(\frac{x}{2})sech^2(\frac{x}{2})\right) + \left(-18\,tanh^3\left(\frac{x}{2}\right)sech^2(\frac{x}{2})\right)\right\}\right\}$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{12\tanh^3(\frac{x}{2})\operatorname{sech}^2(\frac{x}{2}) + 12\tanh(\frac{x}{2})\operatorname{sech}^4(\frac{x}{2}) - 18\tanh(\frac{x}{2})\operatorname{sech}^2(\frac{x}{2})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{12\tanh(\frac{x}{2})\operatorname{sech}^2(\frac{x}{2}) - 18\tanh(\frac{x}{2})\operatorname{sech}^2(\frac{x}{2})\right\}\right\}$$

$$= \left(-6\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right)t$$

$$p^1: w_1 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial^3 u_0}{\partial x^3} + u_0\frac{\partial w_0}{\partial x}\right\}\right\}$$

$w_1 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(-6i\sqrt{2}\,tanh^3(\frac{x}{2})sech^2(\frac{x}{2}) + 12i\sqrt{2}\,tanh(\frac{x}{2})\,sech^4(\frac{x}{2})\right) + \right.\right.$

$\left.\left.\left(-9i\sqrt{2}\,tanh(\frac{x}{2})\,sech^2(\frac{x}{2}) + 18i\sqrt{2}\,tanh^3(\frac{x}{2})sech^2(\frac{x}{2})\right)\right\}\right\}$

$w_1 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{12i\sqrt{2}\tanh^3(\frac{x}{2})\operatorname{sech}^2(\frac{x}{2}) + 12i\sqrt{2}\tanh(\frac{x}{2})\operatorname{sech}^4(\frac{x}{2}) - \right.\right.$

$\left.\left. 9i\sqrt{2}\tanh(\frac{x}{2})\operatorname{sech}^2(\frac{x}{2})\right\}\right\}$

$$= -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{12i\sqrt{2}\tanh(\frac{x}{2})\operatorname{sech}^2(\frac{x}{2}) - 9i\sqrt{2}\tanh(\frac{x}{2})\operatorname{sech}^2(\frac{x}{2})\right\}\right\}$$

$$= \left(-3i\sqrt{2}\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right)t$$

and so on, therefore

$$u = 3 - 6\tanh^2\left(\frac{x}{2}\right) - \left(6\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right)t + \ldots\ldots\ldots$$

$$w = -3i\sqrt{2}\tanh^2\left(\frac{x}{2}\right) - \left(3i\sqrt{3}\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right)t + \ldots\ldots\ldots$$

The above series closed to exact solution:

$$u = 3 - 6\tanh^2\left(\frac{x+t}{2}\right) \quad , \quad w = -3i\sqrt{2}\tanh^2\left(\frac{x+t}{2}\right)$$

Example 3.3 [3]

Consider a system of 1D, 3rd order nonlinear KdV equations (type 2).

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^3} - 2w\frac{\partial u}{\partial x} - u\frac{\partial w}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} - u\frac{\partial u}{\partial x} = 0$$

Subject to ICs: $u(x,0) = -\tanh\left(\frac{x}{\sqrt{3}}\right)$, $w(x,0) = -\frac{1}{6} - \frac{1}{2}\tanh^2\left(\frac{x}{\sqrt{3}}\right)$

Now, we solve the system by using our suggested method to get:

$$p^0 : u_0(x,t) = -\tanh\left(\frac{x}{\sqrt{3}}\right) \ , \ p^0 : w_0(x,t) = -\frac{1}{6} - \frac{1}{2}\tanh^2\left(\frac{x}{\sqrt{3}}\right)$$

$$p^1 : u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial^3 u_0}{\partial x^3} + 2w_0\frac{\partial u_0}{\partial x} + u_0\frac{\partial w_0}{\partial x}\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(-\frac{4}{3\sqrt{3}}\tanh^2(\tfrac{x}{\sqrt{3}})\operatorname{sech}^2(\tfrac{x}{\sqrt{3}}) + \frac{2}{3\sqrt{3}}\operatorname{sech}^4(\tfrac{x}{\sqrt{3}})\right) + \left(\frac{1}{6\sqrt{3}}\operatorname{sech}^2(\tfrac{x}{\sqrt{3}}) + \frac{1}{2\sqrt{3}}\tanh^2(\tfrac{x}{\sqrt{3}})\operatorname{sech}^2(\tfrac{x}{\sqrt{3}})\right) + \left(\frac{1}{\sqrt{3}}\tanh^2\left(\frac{x}{\sqrt{3}}\right)\operatorname{sech}^2(\tfrac{x}{\sqrt{3}})\right)\right\}\right\}$$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{2}{3\sqrt{3}}\tanh^2(\frac{x}{\sqrt{3}})\,\mathrm{sech}^2(\frac{x}{\sqrt{3}}) + \frac{2}{3\sqrt{3}}\mathrm{sech}^4(\frac{x}{\sqrt{3}})\right.\right.$$

$$\left.\left. + \frac{1}{3\sqrt{3}}\mathrm{sech}^2(\frac{x}{\sqrt{3}})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{2}{3\sqrt{3}}\mathrm{sech}^2(\frac{x}{\sqrt{3}}) + \frac{1}{3\sqrt{3}}\mathrm{sech}^2(\frac{x}{\sqrt{3}})\right\}\right\}$$

$$= \left(\frac{1}{\sqrt{3}}\mathrm{sech}^2\left(\frac{x}{\sqrt{3}}\right)\right)t$$

$$p^1 : w_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{u_0\frac{\partial u_0}{\partial x}\right\}\right\}$$

$$w_1 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(\frac{1}{\sqrt{3}}\tanh\left(\frac{x}{\sqrt{3}}\right)\mathrm{sech}^2(\frac{x}{\sqrt{3}})\right)\right\}\right\}$$

$$= \left(\frac{1}{\sqrt{3}}\tanh(\frac{x}{\sqrt{3}})\,\mathrm{sech}^2(\frac{x}{\sqrt{3}})\right)t$$

and so on, therefore

$$u = -\tanh\left(\frac{x}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\mathrm{sech}^2\left(\frac{x}{\sqrt{3}}\right)\right)t + \ \ldots\ldots\ldots$$

$$w = -\frac{1}{6} - \frac{1}{2}\tanh^2\left(\frac{x}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\tanh(\frac{x}{\sqrt{3}})\,\mathrm{sech}^2(\frac{x}{\sqrt{3}})\right)t + \ \ldots\ldots\ldots$$

The above series closed to exact solution:

$$u = -\tanh\left(\frac{-t+x}{\sqrt{3}}\right) \quad , \quad w = -\frac{1}{6} - \frac{1}{2}\tanh^2\left(\frac{-t+x}{\sqrt{3}}\right)$$

3.4. Illustrative Examples for System of 2D-PDEs

In this section, the **NTHPM** will be used to solve system of 2D, nonlinear PDEs.

Example 3.4 [7]

Consider the following system of 2D, nonhomogeneous nonlinear PDEs.

$$\frac{\partial u}{\partial t} - w\frac{\partial u}{\partial x} - \frac{\partial w}{\partial t}\frac{\partial u}{\partial y} = 1 - x + y + t$$

$$\frac{\partial w}{\partial t} - u\frac{\partial w}{\partial x} - \frac{\partial u}{\partial t}\frac{\partial w}{\partial y} = 1 - x - y - t$$

Subject to ICs:

$$f(x,y) = u(x,y,0) = x + y - 1\ , \quad g(x,y) = w(x,y,0) = x - y + 1$$

$L[u(x,y,t)] = \frac{\partial u(x,y,t)}{\partial t}$, $R[u(x,y,t)] = 0$,

$N[u(x,y,t)] = -w\frac{\partial u}{\partial x} - \frac{\partial w}{\partial t}\frac{\partial u}{\partial y}$, $g(x,y,t) = 1 - x + y + t$

$N[w(x,y,t)] = -u\frac{\partial w}{\partial x} - \frac{\partial u}{\partial t}\frac{\partial w}{\partial y}$, $g(x,y,t) = 1 - x - y - t$

First, compute the nonlinear of $N(u)$ to get:

$$w\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} H_{(1,n)}(u,w) \quad , \quad \frac{\partial w}{\partial t}\frac{\partial u}{\partial y} = \sum_{n=0}^{\infty} H_{(2,n)}(u,w)$$

Also, compute the nonlinear of $N(w)$ to get:

$u\frac{\partial w}{\partial x} = \sum_{n=0}^{\infty} K_{(1,n)}(u,w)$, $\frac{\partial u}{\partial t}\frac{\partial w}{\partial y} = \sum_{n=0}^{\infty} K_{(2,n)}(u,w)$

$H_{(1,0)} = w_0\frac{\partial u_0}{\partial x}$, $H_{(2,0)} = \frac{\partial w_0}{\partial t}\frac{\partial u_0}{\partial y}$

$$H_{(1,1)} = w_1 \frac{\partial u_0}{\partial x} + w_0 \frac{\partial u_1}{\partial x} \quad , \ H_{(2,1)} = \frac{\partial w_1}{\partial t}\frac{\partial u_0}{\partial y} + \frac{\partial w_0}{\partial t}\frac{\partial u_1}{\partial y}$$

And so on

$$K_{(1,0)} = u_0 \frac{\partial w_0}{\partial x} \qquad , \qquad K_{(2,0)} = \frac{\partial u_0}{\partial t}\frac{\partial w_0}{\partial y}$$

$$K_{(1,1)} = u_1 \frac{\partial w_0}{\partial x} + u_0 \frac{\partial w_1}{\partial x} \quad , \quad K_{(2,1)} = \frac{\partial u_1}{\partial t}\frac{\partial w_0}{\partial y} + \frac{\partial u_0}{\partial t}\frac{\partial w_1}{\partial y}$$

And so on.

Moreover, we have the sequence of u_n , w_n as:

$u_0 = x + y - 1 \quad , \quad w_0 = x - y + 1$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{w_0 \frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial t}\frac{\partial u_0}{\partial y} + 1 - x + y + t\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{2 + t\}\right\} = \mathbb{T}^{-1}\left\{\frac{2}{v} + \frac{1}{v^2}\right\} = 2t + \frac{t^2}{2!}$$

$$w_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{u_0 \frac{\partial w_0}{\partial x} + \frac{\partial u_0}{\partial t}\frac{\partial w_0}{\partial y} + 1 - x - y - t\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-t\}\right\} = \mathbb{T}^{-1}\left\{-\frac{1}{v^2}\right\} = -\frac{t^2}{2!}$$

$$u_2 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{(w_1 \frac{\partial u_0}{\partial x} + w_0 \frac{\partial u_1}{\partial x}) + (\frac{\partial w_1}{\partial t}\frac{\partial u_0}{\partial y} + \frac{\partial w_0}{\partial t}\frac{\partial u_1}{\partial y})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-\frac{t^2}{2!} - t\right\}\right\} = \mathbb{T}^{-1}\left\{-\frac{1}{v^3} - \frac{1}{v^2}\right\} = -\frac{t^3}{3!} - \frac{t^2}{2!}$$

$$w_2 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{(u_1\frac{\partial w_0}{\partial x} + u_0\frac{\partial w_1}{\partial x}) + (\frac{\partial u_1}{\partial t}\frac{\partial w_0}{\partial y} + \frac{\partial u_0}{\partial t}\frac{\partial w_1}{\partial y})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-2 + t + \frac{t^2}{2!}\right\}\right\} = \mathbb{T}^{-1}\left\{-\frac{2}{v} + \frac{1}{v^2} + \frac{1}{v^3}\right\} = -2\mathrm{t} + \frac{t^2}{2!} + \frac{t^3}{3!}$$

$$u_3 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{(w_2\frac{\partial u_0}{\partial x} + w_1\frac{\partial u_1}{\partial x} + w_0\frac{\partial u_2}{\partial x}) + (\frac{\partial w_2}{\partial t}\frac{\partial u_0}{\partial y} + \frac{\partial w_1}{\partial t}\frac{\partial u_1}{\partial y} + \frac{\partial w_0}{\partial t}\frac{\partial u_2}{\partial y})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-2 - t + t^2 + \frac{t^3}{3!}\right\}\right\} = \mathbb{T}^{-1}\left\{-\frac{2}{v} - \frac{1}{v^2} + \frac{2}{v^3} + \frac{1}{v^4}\right\}$$

$$= -2t - \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}$$

$$w_3 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{(u_2\frac{\partial w_0}{\partial x} + u_1\frac{\partial w_1}{\partial x} + u_0\frac{\partial w_2}{\partial x}) + (\frac{\partial u_2}{\partial t}\frac{\partial w_0}{\partial y} + \frac{\partial u_1}{\partial t}\frac{\partial w_1}{\partial y} + \frac{\partial u_0}{\partial t}\frac{\partial w_2}{\partial y})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{t - \frac{t^3}{3!}\right\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v^2} - \frac{1}{v^4}\right\} = \frac{t^2}{2!} - \frac{t^4}{4!}$$

$$u_4 = \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} - \frac{t^5}{5!}$$

$$w_4 = 2t - \frac{t^2}{2!} - \frac{t^3}{2} + \frac{t^4}{4!} + \frac{t^5}{5!}$$

$$u_5 = 2\mathrm{t} + \frac{t^2}{2!} - \frac{2t^3}{3} - \frac{t^4}{12} + \frac{t^5}{60} + \frac{t^6}{6!}$$

$$w_5 = -\frac{t^2}{2!} + \frac{t^4}{12} - \frac{t^6}{6!}$$

$$u_6 = -\frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{12} + \frac{t^5}{60} - \frac{t^6}{6!} - \frac{t^7}{7!}$$

$$w_6 = -2t + \frac{t^2}{2!} + \frac{5t^3}{3!} - \frac{t^4}{12} - \frac{t^5}{30} + \frac{t^6}{6!} + \frac{t^7}{7!}$$

And so on. Therefore

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = x + y - 1 + 2t + \frac{t^2}{2!} - \frac{t^2}{2!} - \frac{t^3}{3!} - 2t - \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots = x + y + t - 1$$

$$w(x,y,t) = \sum_{n=0}^{\infty} w_n(x,y,t) = x - y + 1 - \frac{t^2}{2!} - 2t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^2}{2!} - \frac{t^4}{4!} + 2t - \frac{t^2}{2!} - \frac{t^3}{2} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots = x - y - t + 1$$

This is the exact solution.

Example 3.5 [22]

Consider the following 2D, nonlinear system of Burgers' equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3.11}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y} = \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right),$$

Subject to the ICs:

$$u(x,y,0) = x + y \ , \ w(x,y,0) = x - y \quad ; (x,y,t) \in R^2 \times \left[0\,,\frac{1}{\sqrt{2}}\right).$$

To solve equation (3.11) by using NTHPM, we construct the following homotopy

$$H(u,p) = (1-p)\left(\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right) = 0$$

$$H(w,p) = (1-p)\left(\frac{\partial w}{\partial t} - \frac{\partial w_0}{\partial t}\right) + p\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right) = 0$$

Or equivalent:

$$\frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} - p\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u_0}{\partial t}\right)$$

$$\frac{\partial w}{\partial t} = \frac{\partial w_0}{\partial t} - p\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial w_0}{\partial t}\right)$$

Applying NT on both sides of above system, to get:

$$\mathbb{T}\left\{\frac{\partial u}{\partial t}\right\} = \mathbb{T}\left\{\frac{\partial u_0}{\partial t} - p\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u_0}{\partial t}\right)\right\}$$

$$\mathbb{T}\left\{\frac{\partial w}{\partial t}\right\} = \mathbb{T}\left\{\frac{\partial w_0}{\partial t} - p\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial w_0}{\partial t}\right)\right\}$$

Now by using the differentiation property of NT we obtain:

$$v\mathbb{T}\{u\} = vu(x,y,0) + \mathbb{T}\left\{\frac{\partial u_0}{\partial t} - p\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u_0}{\partial t}\right)\right\}$$

$$v\mathbb{T}\{w\} = vw(x,y,0) + \mathbb{T}\left\{\frac{\partial w_0}{\partial t} - p\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y} - \frac{1}{R}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial w_0}{\partial t}\right)\right\}$$

By applying inverse of NT we have:

$$u(x,y,t) = u(x,y,0) + \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial u_0}{\partial t} - p\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u_0}{\partial t}\right)\right\}\right\}$$

$$w(x,y,t) = w(x,y,0) + \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial w_0}{\partial t} - p\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y} - \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial w_0}{\partial t}\right)\right\}\right\} \quad (3.12)$$

Suppose the series solution has the form:

$$u = u_0 + pu_1 + p^2u_2 + \cdots$$

$$w = w_0 + pw_1 + p^2w_2 + \cdots \tag{3.13}$$

Substituting the system (3.12) into the system (3.13) and comparing coefficients of the terms with the identical powers of p, this lead to

$$p^0 = \begin{cases} u_0(x,y,t) = u(x,y,0) + \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial u_0}{\partial t}\right\}\right\} \\ w_0(x,y,t) = w(x,y,0) + \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial w_0}{\partial t}\right\}\right\} \end{cases}$$

$$p^1 = \begin{cases} u_1(x,y,t) = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-\frac{\partial u_0}{\partial t} - u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y} + \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right)\right\}\right\} \\ w_1(x,y,t) = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-\frac{\partial w_0}{\partial t} - u_0\frac{\partial w_0}{\partial x} - w_0\frac{\partial w_0}{\partial y} + \left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right)\right\}\right\} \end{cases}$$

$$p^2 = \begin{cases} u_2(x,y,t) = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-u_0\frac{\partial u_1}{\partial x} - u_1\frac{\partial u_0}{\partial x} - w_0\frac{\partial u_1}{\partial y} - v_1\frac{\partial u_0}{\partial y} + \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right)\right\}\right\} \\ w_2(x,y,t) = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-u_0\frac{\partial w_1}{\partial x} - u_1\frac{\partial w_0}{\partial x} - w_0\frac{\partial w_1}{\partial y} - w_1\frac{\partial w_0}{\partial y} + \left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right)\right\}\right\} \end{cases}$$

And so on, we have

$$p^j = \begin{cases} u_j(x,y,t) = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\sum_{k=0}^{j-1}\left(-u_k\frac{\partial u_{j-k-1}}{\partial x} - w_k\frac{\partial u_{j-k-1}}{\partial y}\right) + \left(\frac{\partial^2 u_{j-1}}{\partial x^2} + \frac{\partial^2 u_{j-1}}{\partial y^2}\right)\right\}\right\}, j = 1,2,\ldots \\ w_j(x,y,t) = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\sum_{k=0}^{j-1}\left(-u_k\frac{\partial w_{j-k-1}}{\partial x} - w_k\frac{\partial w_{j-k-1}}{\partial y}\right) + \left(\frac{\partial^2 w_{j-1}}{\partial x^2} + \frac{\partial^2 w_{j-1}}{\partial y^2}\right)\right\}\right\}, j = 1,2,\ldots \end{cases}$$

From above system, we obtain the following recurrent relations:

$$u_j(x,y,t)=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\sum_{k=0}^{j-1}\left(-u_k\frac{\partial u_{j-k-1}}{\partial x}-w_k\frac{\partial u_{j-k-1}}{\partial y}\right)+\left(\frac{\partial^2 u_{j-1}}{\partial x^2}+\frac{\partial^2 u_{j-1}}{\partial y^2}\right)\right\}\right\}$$

$$w_j(x,y,t)=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\sum_{k=0}^{j-1}\left(-u_k\frac{\partial w_{j-k-1}}{\partial x}-w_k\frac{\partial w_{j-k-1}}{\partial y}\right)+\left(\frac{\partial^2 w_{j-1}}{\partial x^2}+\frac{\partial^2 w_{j-1}}{\partial y^2}\right)\right\}\right\}$$

Starting with

$$u_0=x+y \qquad , \quad w_0=x-y$$

Then we get the following results:

$$u_1(x,y,t)=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-\frac{\partial u_0}{\partial t}-u_0\frac{\partial u_0}{\partial x}-w_0\frac{\partial u_0}{\partial y}+\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right)\right\}\right\}$$

$$=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{0-(x+y)(1)-(x-y)(1)+0\}\right\}=\mathbb{T}^{-1}\left\{-\frac{2x}{v}\right\}=-2xt$$

$$w_1(x,y,t)=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-\frac{\partial w_0}{\partial t}-u_0\frac{\partial w_0}{\partial x}-w_0\frac{\partial w_0}{\partial y}+\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right)\right\}\right\}$$

$$w_1=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{0-(x+y)(1)-(x-y)(-1)+0\}\right\}=\mathbb{T}^{-1}\left\{-\frac{2y}{v}\right\}=-2yt$$

$$u_2(x,y,t)=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-u_0\frac{\partial u_1}{\partial x}-u_1\frac{\partial u_0}{\partial x}-w_0\frac{\partial u_1}{\partial y}-w_1\frac{\partial u_0}{\partial y}+\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}\right)\right\}\right\}$$

$$u_2=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-(x+y)(-2t)-(-2xt)(1)-(x-y)(0)-(-2yt)(1)+0\}\right\}$$

$$=\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{2xt+2yt+2xt+2yt\}\right\}=\mathbb{T}^{-1}\left\{\frac{4x}{v^2}+\frac{4y}{v^2}\right\}$$

$$=2xt^2+2yt^2$$

$$w_2(x,y,t) = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-u_0\frac{\partial w_1}{\partial x} - u_1\frac{\partial w_0}{\partial x} - w_0\frac{\partial w_1}{\partial y} - w_1\frac{\partial w_0}{\partial y} + \left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right)\right\}\right\}$$

$$w_2 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-(x+y)(0) - (-2xt)(1) - (x-y)(-2t) - (-2yt)(-1) + 0\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{2xt + 2xt - 2yt - 2yt\}\right\} = \mathbb{T}^{-1}\left\{\frac{4x}{v^2} - \frac{4y}{v^2}\right\}$$

$$= 2xt^2 - 2yt^2$$

$$u_3 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-(x+y)(2t^2) - (-2xt)(-2t) - (2xt^2 + 2yt^2)(1) - (x-y)(2t^2)\right.$$

$$\left. - (-2yt)(0) - (2xt^2 - 2yt^2)(1) + 0\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-2xt^2 - 2yt^2 - 4xt^2 - 2xt^2 - 2yt^2 - 2xt^2 + 2yt^2 - 2xt^2 + 2yt^2\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-12xt^2\}\right\} = \mathbb{T}^{-1}\left\{\frac{-24x}{v^3}\right\} = -4xt^3$$

$$w_3 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-(x+y)(2t^2) - (-2xt)(0) - (2xt^2 + 2yt^2)(1) - (x-y)(-2t^2)\right.$$

$$\left. - (-2yt)(-2t) - (2xt^2 - 2yt^2)(-1) + 0\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-2xt^2 - 2yt^2 - 2xt^2 - 2yt^2 + 2xt^2 - 2yt^2 - 4yt^2 + 2xt^2 - 2yt^2\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-12yt^2\}\right\} = \mathbb{T}^{-1}\left\{\frac{-24y}{v^3}\right\} = -4yt^3$$

$u_4 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-(x+y)(-4t^3) - (-2xt)(2t^2) - (2xt^2 + 2yt^2)(-2t) -\right.$

$\left.(-4xt^3)(1) - (x-y)(0) - (-2yt)(2t^2) - (2xt^2 - 2yt^2)(0) - (-2yt^3)(1)\}\right\} =$

$\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{4xt^3 + 4yt^3 + 4xt^3 + 4xt^3 + 4yt^3 + 4xt^3 + 4yt^3 + 4yt^3\}\right\} =$

$\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{16xt^3 + 16yt^3\}\right\} = \mathbb{T}^{-1}\left\{\frac{96x}{v^4} + \frac{96y}{v^4}\right\} = 4xt^4 + 4yt^4$

$w_4 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-(x+y)(0) - (-2xt)(2t^2) - (2xt^2 + 2yt^2)(0) - (-4xt^3)(1) -\right.$

$\left.(x-y)(-4t^3) - (-2yt)(-2t^2) - (2xt^2 - 2yt^2)(-2t) - (-4yt^3)(-1)\}\right\}$

$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{4xt^3 + 4xt^3 + 4xt^3 - 4yt^3 - 4yt^3 + 4xt^3 - 4yt^3 - 4yt^3\}\right\} =$

$\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{16xt^3 - 16yt^3\}\right\} = \mathbb{T}^{-1}\left\{\frac{96x}{v^4} - \frac{96y}{v^4}\right\} = 4xt^4 - 4yt^4$

$u_5 = -8xt^5$, $\quad w_5 = -8yt^5$

$u_6 = 8xt^6 + 8yt^6$, $\quad w_6 = 8xt^6 - 8yt^6$

And so on, the solution of the system (3.11) can be obtained by setting $p = 1$, i.e.,

$$u(x,y,t) = \lim_{p\to 1} u_n(x,y,t) = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + \cdots$$

$$u(x,y,t) = x + y - 2xt + 2xt^2 + 2yt^2 - 4xt^3 + 4xt^4 + 4yt^4 - 8xt^5 + 8xt^6 + 8yt^6$$

$$u(x,y,t) = (x+y)(1 + 2t^2 + 4t^4 + 8t^6 + \cdots) - 2xt(1 + 2t^2 + 4t^4 + 8t^6 + \cdots)$$

$$u(x,y,t) = \frac{x+y}{1-2t^2} - \frac{2xt}{1-2t^2} = \frac{x+y-2xt}{1-2t^2}$$

Also by similar way, we get:

$$w(x,y,t)=\lim_{p\to 1} v_n(x,y,t)=w_0+w_1+w_2+w_3+w_4+w_5+w_6+\cdots$$

$$w(x,y,t)=x-y-2yt+2xt^2-2yt^2-4yt^3+4xt^4-4yt^4-8yt^5+8xt^6-8yt^6$$

$$w(x,y,t)=(x-y)(1+2t^2+4t^4+8t^6+\cdots)-2yt(1+2t^2+4t^4+8t^6+\cdots)$$

$$w(x,y,t)=\frac{x-y}{1-2t^2}-\frac{2yt}{1-2t^2}=\frac{x-y-2yt}{1-2t^2}$$

3.5. Solving System for 3Equations Nonlinear 1D-PDEs

In this section, the procedure of **NTHPM** will be used to solve system of 1D, for 3 equations nonlinear PDEs.

Firstly the system is written as follows:

$$L[u(x,y,t)]+R[u(x,y,t)]+N[u(x,y,t)]=g_1(x,y,t)$$

$$L[w(x,y,t)]+R[w(x,y,t)]+N[w(x,y,t)]=g_2(x,y,t) \qquad (3.14)$$

$$L[z(x,y,t)]+R[z(x,y,t)]+N[z(x,y,t)]=g_3(x,y,t)$$

Subject to IC:

$$u(x,y,0)=f(x,y)$$

$$w(x,y,0)=g(x,y)$$

$$z(x,y,0)=h(x,y)$$

Where all $x,y \in R$, L is a linear differential operator ($L=\frac{\partial}{\partial t}$), R is a remained of the linear operator; N is a nonlinear differential operator and g_1, g_2, g_3 is the nonhomogeneous part.

We construct a homotopy as: u(x, p): $R^n\times$ [0, 1] $\rightarrow R$,

$H_1(u(x,y,t),p)=(1-p)\big[L(u(x,y,t))-L\big(u(x,y,0)\big)\big]+p[A(u(x,y,t))-g_1(x,y,t)]=0$

$H_2(w(x,y,t),p)=(1-p)\big[L(w(x,y,t))-L\big(w(x,y,0)\big)\big]+p[A(w(x,y,t))-g_2(x,y,t)]=0$

$H_3(z(x,y,t),p)=(1-p)\big[L(z(x,y,t))-L\big(z(x,y,0)\big)\big]+p[A(z(x,y,t))-g_3(x,y,t)]=0$

Where $p\in[0,1]$ is an embedding parameter and the operator A defined as:

$A=L+R+N.$

Obviously, if p = 0, then the above system becomes

$L(u(x,y,t))=L\big(u(x,y,0)\big)$, $L(w(x,y,t))=L\big(w(x,y,0)\big)$ and $L(z(x,y,t))=L\big(z(x,y,0)\big)$.

Substitute ICs in above system and rewrite it as:

$L\big(u(x,y,t)\big)-L\big(f(x,y)\big)-pL\big(u(x,y,t)\big)+pL\big(f(x,y)\big)+pL\big(u(x,y,t)\big)+pR\big(u(x,y,t)\big)+pN\big(u(x,y,t)\big)-pg_1(x,y,t)=0$

$L(w(x,y,t))-L\big(g(x,y)\big)-pL\big(w(x,y,t)\big)+pL\big(g(x,y)\big)+pL\big(w(x,y,t)\big)+pR\big(w(x,y,t)\big)+pN\big(w(x,y,t)\big)-pg_2(x,y,t)=0$

$L\big(z(x,y,t)\big)-L\big(h(x,y)\big)-pL\big(z(x,y,t)\big)+pL\big(h(x,y)\big)+pL\big(z(x,y,t)\big)+pR\big(z(x,y,t)\big)+pN\big(z(x,y,t)\big)-pg_3(x,y,t)=0$

Then

$L(u(x,y,t))-L\big(f(x,y)\big)+p\big[L\big(f(x,y)\big)+R(u(x,y,t))+N(u(x,y,t))-g_1(x,y,t)\big]=0$

$$L(w(x,y,t))-L\big(g(x,y)\big)+p\big[L\big(g(x,y)\big)+R\big(w(x,y,t)\big)+N\big(w(x,y,t)\big)-g_2(x,y,t)\big]=0 \quad (3.15)$$

$L(z(x,y,t)) - L\big(h(x,y)\big) + p\big[L\big(h(x,y)\big) + R(z(x,y,t)) + N(z(x,y,t)) -$

$g_3(x,y,t)\big] = 0$

Since $f(x,y), g(x,y)$ and $h(x,y)$ are independent of the variable t and the linear operator L dependent on t so, $L\big(f(x,y)\big) = 0, L\big(g(x,y)\big) = 0$, and $L\big(h(x,y)\big) = 0$; i.e., the system (3.15) becomes:

$$L(u(x,y,t)) + p[R(u(x,y,t)) + N(u(x,y,t)) - g_1(x,y,t)] = 0$$

$$L(w(x,y,t)) + p[R(w(x,y,t)) + N(w(x,y,t)) - g_2(x,y,t)] = 0$$

$$L(z(x,y,t)) + p[R(z(x,y,t)) + N(z(x,y,t)) - g_3(x,y,t)] = 0 \qquad (3.16)$$

According to the classical perturbation technique, the solution of the above system can be written as a power series of embedding parameter p, as the form

$$u(x,y,t) = \sum_{n=0}^{\infty} p^n u_n(x,y,t)$$

$$w(x,y,t) = \sum_{n=0}^{\infty} p^n w_n(x,y,t) \qquad (3.17)$$

$$z(x,y,t) = \sum_{n=0}^{\infty} p^n z_n(x,y,t)$$

For most cases, the series form in (3.17) is convergent and the convergent rate depends on the nonlinear operator.

Taking the NT (with respect to the variable t) for the system (3.16) to get:

$$\mathbb{T}\{L(u)\} + p\ \mathbb{T}\{R(u) + N(u) - g\} = 0$$

$$\mathbb{T}\{L(w)\} + p\ \mathbb{T}\{R(w) + N(w) - g\} = 0 \qquad (3.18)$$

$\mathbb{T}\{L(z)\} + p\,\mathbb{T}\{R(z) + N(z) - g\} = 0$

Now by using the differentiation property of NT and IC, the above system becomes:

$v\mathbb{T}\{u\} - vf(x,y) + p\,\mathbb{T}\{R(u) + N(u) - g_1\} = 0$

$v\mathbb{T}\{w\} - vg(x,y) + p\,\mathbb{T}\{R(w) + N(w) - g_2\} = 0$

$v\mathbb{T}\{z\} - vh(x,y) + p\,\mathbb{T}\{R(z) + N(z) - g_3\} = 0$

Hence:

$\mathbb{T}\{u\} = f(x,y) + p\frac{1}{v}\mathbb{T}\{g_1 - R(u) - N(u)\}$

$\mathbb{T}\{w\} = g(x,y) + p\frac{1}{v}\mathbb{T}\{g_2 - R(w) - N(w)\}$

$\mathbb{T}\{z\} = h(x,y) + p\frac{1}{v}\mathbb{T}\{g_3 - R(z) - N(z)\}$

By taking the inverse of new transform on both sides of the above system, to get:

$$u(x,y,t) = f(x,y) + p\,\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_1(x,y,t) - R\big(u(x,y,t)\big) - N\big(u(x,y,t)\big)\right\}\right\}$$

$$w(x,y,t) = g(x,y) + p\,\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_2(x,y,t) - R\big(w(x,y,t)\big) - N\big(w(x,y,t)\big)\right\}\right\} \quad (3.19)$$

$$z(x,y,t) = h(x,y) + p\,\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_3(x,y,t) - R\big(z(x,y,t)\big) - N\big(z(x,y,t)\big)\right\}\right\}$$

Then, substitute system (3.17) in the system (3.19) to get:

$$\sum_{n=0}^{\infty} p^n u_n = f(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_1(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n u_n\right) - N\left(\sum_{n=0}^{\infty} p^n u_n\right)\right\}\right\}$$

$$\sum_{n=0}^{\infty} p^n w_n = g(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_2(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n w_n\right) - N\left(\sum_{n=0}^{\infty} p^n w_n\right)\right\}\right\} \quad (3.20)$$

$$\sum_{n=0}^{\infty} p^n z_n = h(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_3(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n z_n\right) - N\left(\sum_{n=0}^{\infty} p^n z_n\right)\right\}\right\}$$

The nonlinear part can be decomposed, as will be explained later, by substituting system (3.17) in it as:

$$N(u) = N\left(\sum_{n=0}^{\infty} p^n u_n(x,y,t)\right) = \sum_{n=0}^{\infty} p^n H_n$$

$$N(w) = N\left(\sum_{n=0}^{\infty} p^n w_n(x,y,t)\right) = \sum_{n=0}^{\infty} p^n K_n$$

$$N(z) = N\left(\sum_{n=0}^{\infty} p^n z_n(x,y,t)\right) = \sum_{n=0}^{\infty} p^n J_n$$

Then the system (3.20) becomes:

$$\sum_{n=0}^{\infty} p^n u_n = f(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_1(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n u_n\right) - \sum_{n=0}^{\infty} p^n H_n\right\}\right\}$$

$$\sum_{n=0}^{\infty} p^n w_n = g(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_2(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n w_n\right) - \sum_{n=0}^{\infty} p^n K_n\right\}\right\} \quad (3.21)$$

$$\sum_{n=0}^{\infty} p^n z_n = h(x,y) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g_3(x,y,t) - R\left(\sum_{n=0}^{\infty} p^n z_n\right) - \sum_{n=0}^{\infty} p^n J_n\right\}\right\}$$

By comparing the coefficient with the same power of p, in both sides of the system (3.21) we have:

$$u_0 = f(x,y), \qquad w_0 = g(x,y), \qquad z_0 = h(x,y)$$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{g_1(x,y,t) - R(u_0) - H_0\}\right\},$$

$$w_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{g_2(x,y,t) - R(w_0) - K_0\}\right\}$$

$$z_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{g_3(x,y,t) - R(z_0) - J_0\}\right\}$$

$$u_2 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(u_1) + H_1\}\right\},$$

$$w_2 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(w_1) + K_1\}\right\}$$

$$z_2 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(z_1) + J_1\}\right\}$$

$$u_3 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(u_2) + H_2\}\right\},$$

$$w_3 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(w_2) + K_2\}\right\}$$

$$z_3 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(z_2) + J_2\}\right\}$$

And so on, therefor

$$u_{n+1} = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(u_n) + H_n\}\right\}$$

$$w_{n+1} = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(w_n) + K_n\}\right\}$$

$$z_{n+1} = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(z_n) + J_n\}\right\}$$

According to the series solution in system (3.17), when at p=1, we can get:

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + \cdots = \sum_{n=0}^{\infty} u_n(x,y,t)$$

$$w(x,y,t) = w_0(x,y,t) + w_1(x,y,t) + \cdots = \sum_{n=0}^{\infty} w_n(x,y,t)$$

$$z(x,y,t) = z_0(x,y,t) + z_1(x,y,t) + \cdots = \sum_{n=0}^{\infty} z_n(x,y,t)$$

Example 3.6 [7]

Consider the following nonlinear system of inhomogeneous PDEs.

$$\frac{\partial u}{\partial t} - \frac{\partial z}{\partial x}\frac{\partial w}{\partial t} - \frac{1}{2}\frac{\partial z}{\partial t}\frac{\partial^2 u}{\partial x^2} = -4xt$$

$$\frac{\partial w}{\partial t} - \frac{\partial z}{\partial t}\frac{\partial^2 u}{\partial x^2} = 6t$$

$$\frac{\partial z}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial w}{\partial x}\frac{\partial z}{\partial t} = 4xt - 2t - 2$$

Subject to the ICs:

$$u(x,0) = x^2 + 1 \quad , \quad w(x,0) = x^2 - 1 \quad , \quad z(x,0) = x^2 - 1$$

$$u_0(x,0) = x^2 + 1 \quad , \quad w_0(x,0) = x^2 - 1 \quad , \quad z_0(x,0) = x^2 - 1$$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial z_0}{\partial x}\frac{\partial w_0}{\partial t} + \frac{1}{2}\frac{\partial z_0}{\partial t}\frac{\partial^2 u_0}{\partial x^2} - 4xt\right\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-4xt\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{-4x}{v^2}\right\} = -2xt^2$$

$$w_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(\frac{\partial z_0}{\partial t}\frac{\partial^2 u_0}{\partial x^2} + 6\mathrm{t}\right)\right\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{6t\}\right\} = \mathbb{T}^{-1}\left\{\frac{6}{v^2}\right\} = 3t^2$$

$$z_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x}\frac{\partial z_0}{\partial t} + 4xt - 2t - 2\right)\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{4xt - 2t\}\right\} = \mathbb{T}^{-1}\left\{\frac{4x}{v^2} - \frac{2}{v^2}\right\} = 2xt^2 - t^2$$

$$u_2 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{(\frac{\partial z_1}{\partial x}\frac{\partial w_0}{\partial t} + \frac{\partial z_0}{\partial x}\frac{\partial w_1}{\partial t}) + \frac{1}{2}(\frac{\partial z_1}{\partial t}\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial z_0}{\partial t}\frac{\partial^2 u_1}{\partial x^2})\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{16xt - 2t\}\right\} = \mathbb{T}^{-1}\left\{\frac{16x}{v^2} - \frac{2}{v^2}\right\} = 8xt^2 - t^2$$

$$w_2 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{\partial z_1}{\partial t}\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial z_0}{\partial t}\frac{\partial^2 u_1}{\partial x^2}\right\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{8xt - 4t\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{8x}{v^2} - \frac{4}{v^2}\right\} = 4xt^2 - 2t^2$$

$$z_2 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(\frac{\partial^2 u_1}{\partial x^2} + (\frac{\partial w_1}{\partial x}\frac{\partial z_0}{\partial t} + \frac{\partial w_0}{\partial x}\frac{\partial z_1}{\partial t}\right)\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{8x^2 t - 4xt\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{8x^2}{v^2} - \frac{4x}{v^2}\right\} = 4x^2t^2 - 2xt^2$$

$$u_3 = 3t^4 - 6xt^2 + 12x^2t^2$$

$$w_3 = 8x^2t^2 - 4xt^2$$

$$z_3 = 8x^3t^2 - 4x^2t^2$$

$$u_4 = -5t^4 + 16xt^4 + 24x^3t^2 - 12x^2t^2$$

$w_4 = 16x^3t^2 - 8x^2t^2$

$z_4 = 8t^3 + 4xt^4 - 2t^4 + 16x^4t^2 - 8x^3t^2$

$u_5 = 60x^2t^4 - 10t^4 + 48x^4t^2 - 24x^3t^2 + 8t^3$

$w_5 = 16t^3 + 32xt^4 - 16t^4 + 32x^4t^2 - 16x^3t^2$

$z_5 = 64xt^3 - 8t^3 - 32x^2t^4 - 16xt^4 + 2t^4 + 32x^5t^2 - 16x^4t^2$

And so on, thus

$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 + 1 - 2xt^2 + 8xt^2 - t^2 + 3t^4 - 6xt^2 + 12x^2t^2 - 5t^4 + \cdots = x^2 - t^2 + 1$

$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = x^2 - 1 + 3t^2 + 4xt^2 - 2t^2 + 8x^2t^2 - 4xt^2 + \cdots = x^2 + t^2 - 1$

$z(x,t) = \sum_{n=0}^{\infty} z_n(x,t) = x^2 - 1 + 2xt^2 - t^2 + 4x^2t^2 - 2xt^2 + 8x^3t^2 - 4x^2t^2 + \cdots = x^2 - t^2 - 1$

Example 3.7 [3]

Consider the generalized coupled Hirota Satsuma KdV type II.

$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^3 u}{\partial x^3} + 3u\frac{\partial u}{\partial x} - 3\frac{\partial}{x}(wz) = 0$$

$$\frac{\partial w}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 3u\frac{\partial w}{\partial x} = 0$$

$$\frac{\partial z}{\partial t} + \frac{\partial^3 z}{\partial x^3} - 3u\frac{\partial z}{\partial x} = 0$$

Subject to IC:

$$u(x,0) = -\frac{1}{3} + 2\tanh^2(x)\,,\;\; w(x,0) = \tanh(x)\,,\;\; z(x,0) = \frac{8}{3}\tanh(x)$$

$$p^1 : u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{1}{2}\frac{\partial^3 u_0}{\partial x^3} - 3u_0\frac{\partial u_0}{\partial x} + 3\frac{\partial}{\partial x}(w_0 z_0)\right\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{(8\tanh^3(x)\operatorname{sech}^2(x) - 16\tanh(x)\operatorname{sech}^4(x)) + (4\tanh(x)\operatorname{sech}^2(x) -\right.\right.$$

$$\left.\left. 24\tanh^3(x)\operatorname{sech}^2(x)) + \left(16\tanh(x)\operatorname{sech}^2(\tfrac{x}{2})\right)\right\}\right\}$$

$$u_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-16\tanh^3(x)\operatorname{sech}^2(x) + 20\tanh(x)\operatorname{sech}^2(x)\right.$$

$$\left. - 16\tanh(x)\operatorname{sech}^4(x)\}\right\}$$

$$= \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{-16\tanh(x)\operatorname{sech}^2(x) + 20\tanh(x)\operatorname{sech}^2(x)\}\right\}$$

$$= (4\tanh(x)\operatorname{sech}^2(x))t$$

$$p^1 : w_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-\frac{\partial^3 u_0}{\partial x^3} + 3u_0\frac{\partial w_0}{\partial x}\right\}\right\}$$

$$= -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{(-4\tanh^2(x)\operatorname{sech}^2(x) + 2\operatorname{sech}^4(x)) + (-\operatorname{sech}^2(x) +\right.$$

$$\left. 6\tanh^2(x)\operatorname{sech}^2(x))\}\right\}$$

$$w_1 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{2\tanh^2(x)\operatorname{sech}^2(x) + 2\operatorname{sech}^4(x) - \operatorname{sech}^2(x)\}\right\}$$

$$= -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{2\,\mathrm{sech}^2(x) - \mathrm{sech}^2(x)\}\right\} = (\mathrm{sech}^2(x))t$$

$$p^1: z_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{-\frac{\partial^3 z_0}{\partial x^3} + 3u_0\frac{\partial z_0}{\partial x}\right\}\right\}$$

$$z_1 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\left(-\frac{32}{3}\tanh^2(x)\mathrm{sech}^2(x) + \frac{16}{3}\mathrm{sech}^4(x)\right) + \right.\right.$$

$$\left.\left.\left(-\frac{8}{9}\mathrm{sech}^2(x) + \frac{16}{3}\tanh^2(x)\mathrm{sech}^2(x)\right)\right\}\right\}$$

$$z_1 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{16}{3}\tanh^2(x)\,\mathrm{sech}^2(x) + \frac{16}{3}\mathrm{sech}^4(x) - \frac{8}{3}\mathrm{sech}^2(x)\right\}\right\}$$

$$= -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{\frac{16}{3}\mathrm{sech}^2(x) - \frac{8}{3}\mathrm{sech}^2(x)\right\}\right\} = \left(\frac{8}{3}\mathrm{sech}^2(x)\right)t$$

And so on, so

$$u = -\frac{1}{3} + 2\tanh^2(x) + (4\tanh(x)\,\mathrm{sech}^2(x))t + \ \dots\dots\dots$$

$$w = \tanh(x) + (\mathrm{sech}^2(x))t + \ \dots\dots\dots$$

$$z = \frac{8}{3}\tanh(x) + \left(\frac{8}{3}\mathrm{sech}^2(x)\right)t + \ \dots\dots\dots$$

The above series closed to exact solution as

$$u = -\tanh(t+x)\,,\ w = -\frac{1}{6} - \frac{1}{2}\tanh^2(t+x)\,,\ z = \frac{8}{3}\tanh(t+x)$$

3.6. Convergence for the Series Solution

In this section the convergence of NTHPM for systems of nonlinear PDEs is presented. The sufficient condition for convergence of the method is addressed. Since mathematical modeling of numerous scientific and engineering experiments lead to system of equations, it is worth trying new methods to solve these systems. Here we show the series solution for systems of previous sections closed to exact solution.

Definition 3.1 [33]

A Banach space is a complete, normed, vector space.

All norms on a finite-dimensional vector space are equivalent. Every finite-dimensional normed space over **R** or **C** is a Banach space [50].

Definition 3.2 [34]

Let X is a set and let $f: X \to X$ be a function that maps X into itself. Such a function is often called an operator. A fixed point of f is an element $x \in X$, for which $f(x) = x$.

Definition 3.3 [34]

Let (X, d) be a metric space. A mapping $T : X \to X$ is a contraction mapping, or contraction, if there exists a constant c, with $0 \leq c < 1$, such that $d\big(T(x), T(y)\big) \leq c d(x, y), \ \forall \ x, y \in X$.

Theorem 3.1 [46]

A contractive function T on a Banach space S has a unique fixed point x^* in R^2. see [46]

Theorem 3.2 (Sufficient Condition of Convergence)

If X and Y are Banach spaces and $N : X \to Y$ is a contractive nonlinear mapping, that is

$$\forall\, w\,, w^* \in X\,;\ \|N(w) - N(w^*)\| \leq \gamma\|w - w^*\|\,, 0 < \gamma < 1\,.$$

Then according to Banach's fixed-point theorem, N has a unique fixed-point u, that is $N(u) = u$.

Assume that the sequence generated by NTHPM can be written as:

$$w_n = N(w_{n-1}), \qquad w_{n-1} = \sum_{i=0}^{n-1} w_i\ , \quad n = 1,2,3, \dots \dots$$

And suppose that $W_0 = w_0 \in B_r(w)\ where\ B_r(w) = \{w^* \in X|\ \|w^* - w\| < r\}$, then we have

i. $w_n \in B_r(w)$,

ii. $\lim_{n\to\infty} W_n = w$

Proof

(i) By inductive approach, for $n = 1$, we have

$$\|W_1 - w\| = \|N(W_0) - N(w)\| \leq \gamma\|w_0 - w\|$$

Assume that $\|W_{n-1} - w\| \leq \gamma \|w_{n-1} - w\|$

$$\leq \gamma^2 \|w_{n-2} - w\|$$

$$\leq \gamma^3 \|w_{n-3} - w\|$$

$$\leq \gamma^{n-1} \|w_0 - w\|$$

As induction hypothesis, then

$$\|W_n - w\| = \|N(W_{n-1}) - N(w)\| \leq \gamma \|w_{n-1} - w\| \leq \gamma^n \|w_0 - w\|$$

Using (i), we have

$$\|W_n - w\| \leq \gamma^n \|w_0 - w\| \leq \gamma^n r < r \Rightarrow W_n \in B_r(w)$$

Because of $\|W_n - w\| \leq \gamma^n \|w_0 - w\|$ and

$\lim_{n\to\infty} \gamma^n = 0$, $\lim_{n\to\infty} \|W_n - w\| = 0$,

that is

$$\lim_{n\to\infty} W_n = w$$

Chapter Four

Application Model

Chapter Four

Application Model

4.1. Introduction

To illustrate the importance of suggested method, we use it to solve soil moisture equation that is equation describe rate of moisture content in soil. The term moisture content is used in hydrogeology, soil sciences and soil Mechanics. Here a model of moisture content is derived and used to build up the one, two and three dimensional space 2^{nd} order nonlinear homogenous PDE. Then NTHPM is used to solve this equation. The derivation of formulation model is illustrated in section 4.2. Basic idea for suggested method that be used to solve the model equation is introduced in section 4.3. Illustrating applicability is presented in section 4.4. While a convergence of the solution is proved in section 4.5.

4.2. Formulation Mathematical Model [22]

Moisture content is the quantity of water contained in a soil called soil moisture. The saturated zone is one in which the space is occupied by water. In the unsaturated zone only part of the space is occupied by water.

There is no moisture in the dry soil, so the value of moisture content is 1 when the porous medium is fully saturated by water and its value is 0 in the unsaturated porous medium. So, the range of moisture content is [0, 1]. The region of the unsaturated soil is called as unsaturated zone. In typical soil profiles some distance separates the earth's surface from the water table, which is the upper limit of completely water-saturated soil. In this inverting zone the water saturation varies between 0 and 1 the rest of the pore space normally being occupied by air. Water flow in this unsaturated zone is complicated by the fact that the soil's permeability to water depends on its water saturation. In many practical situations the flow of water through soil is unsteady because the moisture content changes as a function of time and it is slightly saturated because all the spaces are not completely filed with flowing liquid.

In the formulation model, we assume that the diffusivity coefficient will be small enough constant and regarded as a perturbation parameter because of they are equivalent to their average value over the whole range of moisture content. Additionally, the permeability of the medium is varied directly to the square of the moisture content.

Darcy's law gives the motion of water in the isotropic homogeneous medium as:

$$V = -K \, \nabla \emptyset \tag{4.1}$$

Where V is volume flux of moisture content, K is coefficients of aqueous the conductivity, ϕ is a total moisture potential and ∇ is the gradient, i.e., $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

In any unsaturated porous media, continuity equation governs the motion of water flow given as (depending on [23]):

$$\frac{\partial(\rho s\theta)}{\partial t} = -\nabla . M \tag{4.2}$$

where ρ_S : bulk density for medium on dry weight basis, θ : the moisture content in any position (x, y, z) on a dry weight basis, and M : mass of flux for moisture in any $t \geq 0$.

From (4.1) and (4.2) we get:

$$\frac{\partial(\rho s\theta)}{\partial t} = -\nabla . M = -\nabla .(\rho . V) = \nabla .(\rho . K . \nabla\phi) \tag{4.3}$$

Where ρ indicate the flux density of the medium.

Now, depending on [21] we have $\phi = \psi - gz$. Hence,

$$\frac{\partial(\rho s\theta)}{\partial t} = \nabla .(\rho . K . \nabla(\psi - gz)) \tag{4.4}$$

Where ψ indicates the pressure (capillary) potential, g is the gravitation constant, z is an elevation of the unit of mass for water above a consistent datum (which is the level of saturation) and the positive direction of the z-axis is the same as that of gravity. We find

$$\rho s \frac{\partial \theta}{\partial t} = \nabla.(\rho.K.\nabla\psi) - \nabla.\left(\rho.K.\nabla(gz)\right) \qquad (4.5)$$

Since $\nabla(gz) = \left(\frac{\partial(gz)}{\partial x}, \frac{\partial(gz)}{\partial y}, \frac{\partial(gz)}{\partial z}\right) = (0,0,g)$ and divided both sides of equation (4.5) by $\rho S,$ we get:

$$\frac{\partial \theta}{\partial t} = \nabla.\left(\frac{\rho}{\rho s}.K.\nabla\psi\right) - \nabla.\left(\frac{\rho}{\rho s}.K.(0,0,g)\right) \qquad (4.6)$$

Consider ψ and θ are single valued function, so equation (4.6) can be written as

$$\frac{\partial \theta}{\partial t} = \nabla.\left(\frac{\rho}{\rho s}.K.\frac{\partial \psi}{\partial \theta}\nabla\theta\right) - \nabla.\left(\frac{\rho}{\rho s}.K.(0,0,g)\right) \qquad (4.7)$$

According to [21], we have $D = \frac{\rho}{\rho s}.K.\frac{\partial \psi}{\partial \theta}$ is called the diffusivity coefficient, which is constant as we assumed, so we get:

$$\frac{\partial \theta}{\partial t} = \nabla.(D.\nabla\theta) - \nabla.\left(\frac{\rho}{\rho s}.K.(0,0,g)\right) \qquad (4.8)$$

Let D_a is the average value of D over the whole range. According to [22] we have $K \propto \theta^2$, i.e., $K \propto K_0.\theta^2$,where K_0 is a constant. Hence, equation (4.8) becomes

$$\frac{\partial \theta}{\partial t} = \nabla.(D_a.\nabla\theta) - \nabla.\left(\frac{\rho}{\rho s}.K_0.\theta^2.(0,0,g)\right) \qquad (4.9)$$

So, equation (4.9) get:

$$\frac{\partial\theta}{\partial t}+\frac{\rho}{\rho s}2g.K_0.\theta\frac{\partial\theta}{\partial z}=D_a(\nabla.\nabla\theta) \tag{4.10}$$

Suppose $K_1=\frac{\rho}{\rho s}.2g.K_0$ and substituting it in equation (4.10) we get:

$$\frac{\partial\theta}{\partial t}+K_1.\theta.\frac{\partial\theta}{\partial z}=D_a(\nabla.\nabla\theta) \tag{4.11}$$

Hence, the final form of the equation is

$$\frac{\partial\theta}{\partial t}+K_1.\theta.\frac{\partial\theta}{\partial z}=D_a\left[\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right] \tag{4.12}$$

We can simplify equation (4.12), by supposing:

$$\bar{x}=\frac{x}{k_1}\ ,\bar{y}=\frac{y}{k_1}\ ,\bar{z}=\frac{z}{k_1} \tag{4.13}$$

It is clear that:

$$\begin{aligned}&\frac{\partial\theta}{\partial x}=\frac{1}{k_1}\frac{\partial\theta}{\partial\bar{x}}\ ,\quad \frac{\partial\theta}{\partial y}=\frac{1}{k_1}\frac{\partial\theta}{\partial\bar{y}}\ ,\quad \frac{\partial\theta}{\partial z}=\frac{1}{k_1}\frac{\partial\theta}{\partial\bar{z}}\\ &\frac{\partial^2\theta}{\partial x^2}=\frac{1}{k_1^2}\frac{\partial^2\theta}{\partial\bar{x}^2}\ ,\quad \frac{\partial^2\theta}{\partial y^2}=\frac{1}{k_1^2}\frac{\partial^2\theta}{\partial\bar{y}^2}\ ,\quad \frac{\partial^2\theta}{\partial z^2}=\frac{1}{k_1^2}\frac{\partial^2\theta}{\partial\bar{z}^2}\end{aligned} \tag{4.14}$$

Substituting equation (4.14) into equation (4.12), we get:

$$\frac{\partial\theta}{\partial t}+\theta\frac{\partial\theta}{\partial\bar{z}}=\frac{D_a}{k_1^2}\left[\frac{\partial^2\theta}{\partial\bar{x}^2}+\frac{\partial^2\theta}{\partial\bar{y}^2}+\frac{\partial^2\theta}{\partial\bar{z}^2}\right] \tag{4.15}$$

Let $\alpha=\frac{D_a}{k_1^2}$, then equation (4.15) will be:

$$\frac{\partial\theta}{\partial t}+\theta\frac{\partial\theta}{\partial\bar{z}}=\alpha\left[\frac{\partial^2\theta}{\partial\bar{x}^2}+\frac{\partial^2\theta}{\partial\bar{y}^2}+\frac{\partial^2\theta}{\partial\bar{z}^2}\right] \tag{4.16}$$

For simplification, the original symbols will be used instead of the symbols in equation (4.16), i.e.,

$$\frac{\partial\theta}{\partial t} + \theta\frac{\partial\theta}{\partial z} = \alpha\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right] \qquad (4.17)$$

4.3. Solving Model Equation by the NTHPM

The main purpose of this section is discuss the new implementation of the combine NT algorithm with the HPM to solve the suggested model equation (4.17).

To explain the NTHPM, firstly writes the nonlinear PDE (4.17) as follow:

$$L[\theta(X,t)] + R[\theta(X,t)] + N[\theta(X,t)] = g(X,t) \qquad (4.18)$$

Subject to IC: $\theta(X,0) = f(X)$

Where $X \in R^n$, L is a linear differential operator ($L = \frac{\partial}{\partial t}$) , R is a remained of the linear operator, N is a nonlinear differential operator and $g(X,t)$ is the nonhomogeneous part.

We construct a Homotopy as: (X,p): $R^n \times [0,1] \rightarrow R$, which satisfies

$$H(\theta(X,t),p) = (1-p) * \left[L\big(\theta(X,t)\big) - L\big(\theta(X,0)\big)\right] +$$

$$p\left[A\left(\big(\theta(X,t)\big)\right) - g(X,t)\right] = 0 \qquad (4.19)$$

Where $p \in [0,1]$ is an embedding parameter and the operator A defined as $A = L + R + N$.

Obviously, if $p = 0$, equation (4.19) becomes $L(\theta(X,t)) - L(\theta(X,0))$.

It is clear that, if $p = 1$ then the homotopy equation (4.19) convert to the main differential equation (4.18). Substitute IC in equation (4.19) and rewrite it as:

$$L(\theta(X,t)) - L(f(X)) - pL(\theta(X,t)) + pL(f(X)) + pL(\theta(X,t)) + pR(\theta(X,t)) + pN(\theta(X,t)) - pg(X,t) = 0$$

Then

$$L(\theta(X,t)) - L\big(f(X)\big) + p\Big[L\big(f(X)\big) + R\big(\theta(X,t)\big) + N\big(\theta(X,t)\big) - g(X,t)\Big] = 0 \tag{4.20}$$

Since $f(X)$ is independent of the variable t and the linear operator L dependent on t so, $L(f(X)) = 0$, i.e., (4.20) becomes:

$$L(\theta(X,t)) + p[R(\theta(X,t)) + N(\theta(X,t)) - g(X,t)] = 0 \tag{4.21}$$

According to the classical perturbation technique, the solution of equation (4.21) can be written as a power series of embedding parameter p, in the form

$$\theta(X,t) = \sum_{n=0}^{\infty} p^n \theta_n(X,t) \tag{4.22}$$

For most cases, the series form (4.22) is convergent and the convergent rate depends on the nonlinear operator $N(\theta(X,t))$.

Taking the NT (with respect to the variable t) for the equation (4.21), we have:

$$\mathbb{T}\{L(\theta)\} + p\mathbb{T}\{R(\theta) + N(\theta) - g\} = 0 \tag{4.23}$$

Now by using the differentiation property of LT, so equation (4.23), becomes:

$$v\mathbb{T}\{\theta\} - f(X) + p\mathbb{T}\{R(\theta) + N(\theta) - g\} = 0$$

Hence:

$$\mathbb{T}\{\theta\} = f(X) + p\frac{1}{v}\mathbb{T}\{g - R(\theta) - N(\theta)\} \tag{4.24}$$

By taking the inverse of new transformation on both sides of equation (4.24), to get:

$$\theta(X,t) = f(X) + p\,\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\big\{g(X,t) - R\big(\theta(X,t)\big) - N\big(\theta(X,t)\big)\big\}\right\} \tag{4.25}$$

Then substitute equation (4.22) in equation (4.25) to obtain:

$$\sum_{n=0}^{\infty} p^n\theta_n = f(X) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g(X,t) - R\left(\sum_{n=0}^{\infty} p^n\theta_n\right) - N\left(\sum_{n=0}^{\infty} p^n\theta_n\right)\right\}\right\} \tag{4.26}$$

The nonlinear part can be decomposed, as will be explained later, by substituting equation (4.22) in it as:

$$N(\theta) = N\left(\sum_{n=0}^{\infty} p^n\theta_n(X,t)\right) = \sum_{n=0}^{\infty} p^n H_n$$

Then equation (4.26) becomes:

$$\sum_{n=0}^{\infty} p^n \theta_n = f(X) + p\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\left\{g(X,t) - R\left(\sum_{n=0}^{\infty} p^n \theta_n\right) - \sum_{n=0}^{\infty} p^n H_n\right\}\right\} \tag{4.27}$$

By comparing the coefficient with the same power of p, in both sides of the equation (4.27) we have:

$$\theta_0 = f(X)$$

$$\theta_1 = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{g(X,t) - R(\theta_0) - H_0\}\right\}$$

$$\theta_2 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(\theta_1) + H_1\}\right\}$$

$$\theta_3 = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(\theta_2) + H_2\}\right\}$$

$$\vdots$$

$$\theta_{n+1} = -\mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{R(\theta_n) + H_n\}\right\}$$

According to the series solution in equation (4.22), and p converges to 1, we get:

$$\theta(X,t) = \theta_0(X,t) + \theta_1(X,t) + \cdots = \sum_{n=0}^{\infty} \theta_n(X,t) \tag{4.28}$$

4.4. Experiment Application

In this section, the suggested method will be used to solve the one; two and three-dimensions model equation, with appropriate initial condition:

Problem 4.1

Here suggested method that can used to solve the 2D model equation (1D space) i.e., equation governs the motion of water flow in 1D-horizontal:

$$\frac{\partial\theta}{\partial t}+\theta\frac{\partial\theta}{\partial x}=\alpha\left[\frac{\partial^2\theta}{\partial x^2}\right]$$

$$f(x)=\theta(x,0)=\frac{\gamma+\beta+(\beta-\gamma)\mathrm{e}^{\mu}}{e^{\mu}+1}$$

Where $\mu=\frac{\gamma}{\alpha}(x-\lambda)$, α, β and λ are parameters dimension

We have

$$L[\theta(X,t)]=\frac{\partial\theta(x,t)}{\partial t} \quad , \quad R[\theta(X,t)]=-\alpha\left[\frac{\partial^2\theta}{\partial x^2}\right],$$

$$N[\theta(X,t)]=\theta\frac{\partial\theta}{\partial x} \quad \text{and} \quad g(X,t)$$

First, compute H_n to the nonlinear part $N(\theta)$ we get:

$$N(\theta)=N\left(\sum_{n=0}^{\infty}p^n\theta_n\right)=\left(\sum_{n=0}^{\infty}p^n\theta_n\right)\left(\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty}p^n\theta_n\right]\right)=\left(\sum_{n=0}^{\infty}p^n\theta_n\right)\left(\sum_{n=0}^{\infty}p^n\frac{\partial\theta_n}{\partial x}\right)$$

$$=\theta_0\frac{\partial\theta_0}{\partial x}+p\left(\theta_0\frac{\partial\theta_1}{\partial x}+\theta_1\frac{\partial\theta_0}{\partial x}\right)+p^2\left(\theta_0\frac{\partial\theta_2}{\partial x}+\theta_1\frac{\partial\theta_1}{\partial x}+\theta_2\frac{\partial\theta_0}{\partial x}\right)$$

$$+p^3\left(\theta_0\frac{\partial\theta_3}{\partial x}+\theta_1\frac{\partial\theta_2}{\partial x}+\theta_2\frac{\partial\theta_1}{\partial x}+\theta_3\frac{\partial\theta_0}{\partial x}\right)+\cdots$$

So,

$$H_0 = \theta_0 \frac{\partial \theta_0}{\partial x}$$

$$H_1 = \theta_1 \frac{\partial \theta_0}{\partial x} + \theta_0 \frac{\partial \theta_1}{\partial x}$$

$$H_2 = \theta_2 \frac{\partial \theta_0}{\partial x} + \theta_1 \frac{\partial \theta_1}{\partial x} + \theta_0 \frac{\partial \theta_2}{\partial x}$$

$$H_3 = \theta_3 \frac{\partial \theta_0}{\partial x} + \theta_2 \frac{\partial \theta_1}{\partial x} + \theta_1 \frac{\partial \theta_2}{\partial x} + \theta_0 \frac{\partial \theta_3}{\partial x}$$

And so on.

From IC, we can get

$$H_0 = -\frac{2\,\gamma^2\, e^{\mu}\,(\gamma+\beta+(\beta-\gamma)e^{\mu})}{\alpha\,(e^{\mu}+1)^3}$$

$$H_1 = t\,\frac{2\,\gamma^3\beta\, e^{\mu}\,(\gamma+\beta-4\gamma\, e^{\mu}+(\gamma-\beta)e^{2\mu})}{\alpha^2\,(e^{\mu}+1)^4}$$

$$H_2 = -t^2\,\frac{\gamma^4\beta^2\, e^{\mu}\,(\gamma+\beta-(11\gamma+3\beta)e^{\mu}+(11\gamma-3\beta)e^{2\mu}+(\beta-\gamma)e^{3\mu})}{\alpha^3\,(e^{\mu}+1)^5}$$

$$H_3 = t^3\,\frac{\gamma^5\beta^3\, e^{\mu}\,(\gamma+\beta-(26\gamma+10\beta)e^{\mu}+66\gamma e^{2\mu}-(26\gamma-10\beta)e^{3\mu}+(\gamma-\beta)e^{4\mu})}{3\,\alpha^4\,(e^{\mu}+1)^6}$$

And so on.

Moreover, the sequence of parts θ_n is:

$$\theta_0 = \frac{\gamma+\beta+(\beta-\gamma)e^{\mu}}{(e^{\mu}+1)}$$

$$\theta_1 = t\,\frac{2\,\gamma^2\beta\, e^{\mu}}{\alpha(e^{\mu}+1)^2}$$

$$\theta_2 = -t^2\,\frac{\gamma^3\beta^2\, e^{\mu}\,(1-e^{\mu})}{\alpha^2\,(e^{\mu}+1)^3}$$

$\theta_3 = t^3 \frac{\gamma^4 \beta^3 \ e^{\mu} \ (1-4 \ e^{\mu}+e^{2\mu})}{3 \ \alpha^3 \ (e^{\mu}+1)^4}$

$\theta_4 = -t^4 \frac{\gamma^5 \beta^4 \ e^{\mu} \ (1-11 \ e^{\mu}+11 e^{2\mu}-e^{3\mu})}{12 \ \alpha^4 \ (e^{\mu}+1)^5}$

And so on.

Substituting the above values in series form (4.28), hence the solution of the problem is close to the form:

$\theta(x,t) = \frac{\gamma+\beta+(\beta-\gamma)\mathrm{e}^{\mu-\frac{\gamma\beta}{\alpha}t}}{\mathrm{e}^{\mu-\frac{\gamma\beta}{\alpha}t}+1}$

Problem 4.2

Here suggested method will be used to solve the 3D model equation (2D space) i.e., equation governs the motion of water flow in horizontal and vertical:

$$\frac{\partial \theta}{\partial t} + \theta \frac{\partial \theta}{\partial x} = \alpha \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right]$$

$f(x,y) = \theta(x,y,0) = \frac{\gamma+\beta+(\beta-\gamma)\mathrm{e}^{\mu}}{e^{\mu}+1}$

Where $\mu = \frac{\gamma}{\alpha}(x+y-\lambda)$, α, β and λ are parameters dimension

We have

$L[\theta(X,t)] = \frac{\partial \theta(x,y,t)}{\partial t}$, $R[\theta(X,t)] = -\alpha \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right]$,

$N[\theta(X,t)] = \theta \frac{\partial\theta}{\partial x}$ and $g(X,t) = 0$

First, compute H_n to the nonlinear part $N(\theta)$ we get:

$$N(\theta) = N\left(\sum_{n=0}^{\infty} p^n \theta_n\right) = \left(\sum_{n=0}^{\infty} p^n \theta_n\right)\left(\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} p^n \theta_n\right]\right) = \left(\sum_{n=0}^{\infty} p^n \theta_n\right)\left(\sum_{n=0}^{\infty} p^n \frac{\partial \theta_n}{\partial x}\right)$$

$$= \theta_0 \frac{\partial\theta_0}{\partial x} + p\left(\theta_0 \frac{\partial\theta_1}{\partial x} + \theta_1 \frac{\partial\theta_0}{\partial x}\right) + p^2\left(\theta_0 \frac{\partial\theta_2}{\partial x} + \theta_1 \frac{\partial\theta_1}{\partial x} + \theta_2 \frac{\partial\theta_0}{\partial x}\right)$$

$$+ p^3\left(\theta_0 \frac{\partial\theta_3}{\partial x} + \theta_1 \frac{\partial\theta_2}{\partial x} + \theta_2 \frac{\partial\theta_1}{\partial x} + \theta_3 \frac{\partial\theta_0}{\partial x}\right) + \cdots$$

So,

$$H_0 = \theta_0 \frac{\partial\theta_0}{\partial x}$$

$$H_1 = \theta_1 \frac{\partial\theta_0}{\partial x} + \theta_0 \frac{\partial\theta_1}{\partial x}$$

$$H_2 = \theta_2 \frac{\partial\theta_0}{\partial x} + \theta_1 \frac{\partial\theta_1}{\partial x} + \theta_0 \frac{\partial\theta_2}{\partial x}$$

$$H_3 = \theta_3 \frac{\partial\theta_0}{\partial x} + \theta_2 \frac{\partial\theta_1}{\partial x} + \theta_1 \frac{\partial\theta_2}{\partial x} + \theta_0 \frac{\partial\theta_3}{\partial x}$$

And so on.

From IC, we can get:

$$H_0 = -\frac{\gamma^2 e^{\mu} (\gamma+\beta+(\beta-\gamma)e^{\mu})}{\alpha (e^{\mu}+1)^3}$$

$$H_1 = t\,\frac{\gamma^3 \beta\, e^{\mu} (\gamma+\beta-4\gamma\, e^{\mu}+(\gamma-\beta)e^{2\mu})}{2\,\alpha^2 (e^{\mu}+1)^4}$$

$$H_2 = -t^2\,\frac{\gamma^4 \beta^2 e^{\mu} (\gamma+\beta-(11\gamma+3\beta)e^{\mu}+(11\gamma-3\beta)e^{2\mu}+(\beta-\gamma)e^{3\mu})}{8\,\alpha^3 (e^{\mu}+1)^5}$$

$$H_3 = t^3 \frac{\gamma^5\beta^3\, e^{\mu}\, (\gamma+\beta-(26\gamma+10\beta)e^{\mu}+66\gamma e^{2\mu}-(26\gamma-10\beta)e^{3\mu}+(\gamma-\beta)e^{4\mu})}{48\,\alpha^4\,(e^{\mu}+1)^6}$$

And so on.

Moreover, the sequence of parts θ_n is:

$$\theta_0 = \frac{\gamma+\beta+(\beta-\gamma)e^{\mu}}{(e^{\mu}+1)}$$

$$\theta_1 = t\,\frac{\gamma^2\beta\, e^{\mu}}{\alpha(e^{\mu}+1)^2}$$

$$\theta_2 = -t^2\,\frac{\gamma^3\beta^2\, e^{\mu}\,(1-e^{\mu})}{4\alpha^2\,(e^{\mu}+1)^3}$$

$$\theta_3 = t^3\,\frac{\gamma^4\beta^3\, e^{\mu}\,(1-4\,e^{\mu}+e^{2\mu})}{24\,\alpha^3\,(e^{\mu}+1)^4}$$

$$\theta_4 = -t^4\,\frac{\gamma^5\beta^4\, e^{\mu}\,(1-11\,e^{\mu}+11e^{2\mu}-e^{3\mu})}{192\,\alpha^4\,(e^{\mu}+1)^5}$$

And so on.

Substituting the above values in series form (4.28), hence the solution of the problem is close to the form:

$$\theta(x,y,t) = \frac{\gamma+\beta+(\beta-\gamma)\mathrm{e}^{\mu-\frac{\gamma\beta}{2\alpha}t}}{\mathrm{e}^{\mu-\frac{\gamma\beta}{2\alpha}t}+1}$$

Problem 4.3

Here suggested method will be used to solve the 4D model equation (3D space):

$$\frac{\partial\theta}{\partial t} + \theta\,\frac{\partial\theta}{\partial x} = \alpha\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + + \frac{\partial^2\theta}{\partial z^2}\right]$$

$$f(x,y,z) = \theta(x,y,z,0) = \frac{\gamma+\beta+(\beta-\gamma)\mathrm{e}^{\mu}}{e^{\mu}+1}$$

Where $\mu = \frac{\gamma}{\alpha}(x+y+z-\lambda), \alpha, \beta$ and λ are parameters dimension

We have

$$L[\theta(X,t)] = \frac{\partial\theta(x,y,z,t)}{\partial t} \quad , \quad R[\theta(X,t)] = -\,\alpha\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right],$$

$$N[\theta(X,t)] = \theta\frac{\partial\theta}{\partial x} \quad \text{and} \quad g(X,t) = 0$$

First, compute H_n to the nonlinear part $N(\theta)$ we get:

$$N(\theta) = N\left(\sum_{n=0}^{\infty} p^n\theta_n\right) = \left(\sum_{n=0}^{\infty} p^n\theta_n\right)\left(\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} p^n\theta_n\right]\right) = \left(\sum_{n=0}^{\infty} p^n\theta_n\right)\left(\sum_{n=0}^{\infty} p^n\frac{\partial\theta_n}{\partial x}\right)$$

$$= \theta_0\frac{\partial\theta_0}{\partial x} + p\left(\theta_0\frac{\partial\theta_1}{\partial x} + \theta_1\frac{\partial\theta_0}{\partial x}\right) + p^2\left(\theta_0\frac{\partial\theta_2}{\partial x} + \theta_1\frac{\partial\theta_1}{\partial x} + \theta_2\frac{\partial\theta_0}{\partial x}\right)$$

$$+ p^3\left(\theta_0\frac{\partial\theta_3}{\partial x} + \theta_1\frac{\partial\theta_2}{\partial x} + \theta_2\frac{\partial\theta_1}{\partial x} + \theta_3\frac{\partial\theta_0}{\partial x}\right) + \cdots$$

So,

$$H_0 = \theta_0\frac{\partial\theta_0}{\partial z}$$

$$H_1 = \theta_1\frac{\partial\theta_0}{\partial z} + \theta_0\frac{\partial\theta_1}{\partial z}$$

$$H_2 = \theta_2\frac{\partial\theta_0}{\partial z} + \theta_1\frac{\partial\theta_1}{\partial z} + \theta_0\frac{\partial\theta_2}{\partial z}$$

And so on.

From IC, we can get

$$H_0 = -\frac{2\,\gamma^2\, e^{\mu}\,(\gamma+\beta+(\beta-\gamma)e^{\mu})}{3\,\alpha\,(e^{\mu}+1)^3}$$

$$H_1 = t\,\frac{2\,\gamma^3\beta\, e^{\mu}\,(\gamma+\beta-4\gamma\, e^{\mu}+(\gamma-\beta)e^{2\mu})}{9\,\alpha^2\,(e^{\mu}+1)^4}$$

$$H_2 = -t^2\,\frac{\gamma^4\beta^2\, e^{\mu}\,(\gamma+\beta-(11\gamma+3\beta)e^{\mu}+(11\gamma-3\beta)e^{2\mu}+(\beta-\gamma)e^{3\mu})}{27\,\alpha^3\,(e^{\mu}+1)^5}$$

$$H_3 = t^3\,\frac{\gamma^5\beta^3\, e^{\mu}\,(\gamma+\beta-(26\gamma+10\beta)e^{\mu}+66\gamma\, e^{2\mu}-(26\gamma-10\beta)e^{3\mu}+(\beta-\gamma)e^{4\mu})}{243\,\alpha^4\,(e^{\mu}+1)^6}$$

And so on.

Moreover, the sequence of parts θ_n is:

$$\theta_0 = \frac{\gamma+\beta+(\beta-\gamma)e^{\mu}}{(e^{\mu}+1)}$$

$$\theta_1 = t\,\frac{2\,\gamma^2\beta\, e^{\mu}}{3\alpha(e^{\mu}+1)^2}$$

$$\theta_2 = -t^2\,\frac{\gamma^3\beta^2\, e^{\mu}\,(1-e^{\mu})}{9\alpha^2\,(e^{\mu}+1)^3}$$

$$\theta_3 = t^3\,\frac{\gamma^4\beta^3\, e^{\mu}\,(1-4\, e^{\mu}+e^{2\mu})}{81\,\alpha^3\,(e^{\mu}+1)^4}$$

$$\theta_4 = -t^4\,\frac{\gamma^5\beta^4\, e^{\mu}\,(1-11e^{\mu}+11e^{2\mu}-e^{3\mu})}{972\,\alpha^4\,(e^{\mu}+1)^5}$$

And so on.

Substituting the above values in series form (4.28), hence the solution of the problem is close to the form:

$$\theta(x,y,z,t) = \frac{\gamma+\beta+(\beta-\gamma)\mathrm{e}^{\mu-\frac{\gamma\beta}{3\alpha}t}}{\mathrm{e}^{\mu-\frac{\gamma\beta}{3\alpha}t}+1}$$

From problems 1, 2, 3, we can see that the proposed NTHPM is applied to find the exact solution of the nonlinear 2^{nd} order (2, 3, 4 – D) model equations does not require any restrictive assumptions to deal with nonlinear terms.

Chapter Five

CONCLUSIONS
& FUTURE WORKS

Chapter Five

Conclusions and Future Work

5.1. Conclusions

In this thesis, the combination of new transform suggested by Luma and Alaa with HPM method is proposed to get exact solution of some types of non-linear, non-homogenous PDEs; 1D, 2D, and 3D; 2^{nd} or 3^{rd} order such as Klein-Gordan equation, wave-like equations, autonomous equation, system of two or three non-linear equations, 2D-Burgers' equations, coupled Hirota Satsuma KdV type II, and RLW equation. Finally, the suggested method is used to solve application model such soil moisture equation where traditional HPM leads to an approximate solution. So, the results reveal that NTHPM is a powerful method for solving those types of PDEs with initial conditions. The basic idea described in this thesis is strong enough to be employed to solve other types of equations. The advantage of suggested method is capability of combining two powerful methods for obtaining exact solutions for those types of PDEs, where the HPM was disability to get the exact solution for the same problems and solved its approximately.

Moreover, the research finding how the solution of PDE by MTHPM provide the agreement with real life problems. The method attacks the problem in a direct way and in a straightforward fashion without using linearization, or any other restrictive assumption that may change the physical behavior of the model under discussion.

The experimental results show that the NTHPM is computationally efficient for solving those types of problems and can easily be implemented without computer.

The suggested method is free of unnecessary mathematical complexities.

The fast convergence and simple applicability of NTHPM provid excellent foundation for using these functions in analytical solution of variety problems.

The obtained results show that our proposed method has several advantages such like being free of using Adomian polynomials when dealing with the nonlinear terms like in the ADM and being free of using the Lagrange multiplier as in the VIM.

The results reveal that the presented methods are reliable, effective, very accurate and applicable to solve other nonlinear problems.

The advantage of NTHPM is its capability of combining two powerful methods for obtaining exact.

5.2. Future Works

Based on the results of the proposed method and its illustrative, the following future works may suggest:

1- Using NTHPM for solving high dimensions PDEs.

2- Use NTHPM to solve nonlocal problems.

3- Use NTHPM to solve integral and integro-differential equations.

4- Use NTHPM to solve system of integral or integro-differential equations.

5- Use NTHPM to solve differential equations with fractional orders.

REFERENCES

References

[1] Aasaraai, A., 2011, Analytic Solution for Newell-Whitehead-Segel Equation by Differential Transform Method, Middle East J. Sci. Res. Vol.10, No.2, p: 270–273.

[2] Adomian, G., 1994, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, MA.

[3] Alquran, M., and Mohammad, M., 2011, Approximate Solutions to System of Nonlinear Partial Differential Equations Using Homotopy Perturbation Method, International Journal of Nonlinear Science, Vol. 12, No. 4, p: 485-497.

[4] Ayaz, F., 2003, On the Two-Dimensional Differential Transform Method, Applied Mathematics and Computation, Vol. 143, No. 2-3, p: 361–374.

[5] Bataineh, A. S., Noorani, M. S. M., and Hashim, I., 2008, Approximation Analytical Solution of System of PDEs by Homotopy Analysis Method, Computers and Mathematics with Applications, Vol. 55, p: 2913-2923.

[6] Batiha, B., Noorani M. S. M., and Hashim, I., 2008, Numerical Simulations of Systems of PDEs by Variational Iteration Method, Phys. Lett. A., Vol. 372, p: 822–829.

[7] Biazar, J., and Eslami, M., 2011, A New Homotopy Perturbation Method for Solving Systems of Partial Differential Equations, Computers and Mathematics with Applications, Vol. 62, p: 225–234.

[8] Biazar, J., Eslami, M., and Ghazvini, H., 2007, Homotopy Perturbation Method for System of Partial Differential Equations, International Journal of Nonlinear Sciences and Numerical Simulations, Vol. 8, No. 3, p: 411-416.

[9] Biazar, J., and Eslami, M., 2010, Analytic Solution for Telegraph Equation by Differential Transform Method, Physics Letters, Vol. 374, p: 2904–2906.

[10] Fadaei, J., 2011, Application of Laplace−Adomian Decomposition Method on Linear and Nonlinear System of PDEs, Applied Mathematical Sciences, Vol. 5, No. 27, p: 1307–1315.

[11] Ghazi, F. F., 2016, Employ Mathematical Model and Neural Networks for Determining Rate of the Environmental Contamination, MSc. Thesis, College of Education for Pure Science Ibn Al–Haitham, University of Baghdad.

[12] He, J.H., and Wu, X.H., 2007, Variational Iteration Method: New Development and Applications, Computers and Mathematics with Applications, Vol. 54, p: 881–894.

[13] He, J. H., 1999, Homotopy Perturbation Technique, Computer Methods in Applied Mechanics and Engineering, Vol. 178, No. (3-4), p: 257-262. doi:10.1016/s0045-7825(99)00018-3.

[14] He, J. H., 2000, A Coupling Method of Homotopy Technique and Perturbation Technique for Nonlinear Problems. Int. J. Non- linear Mech., Vol. 35, p: 37-43.

[15] He, J. H., 2004, The Homotopy Perturbation Method for Solving Non-Linear Oscillators with Discontinuities, Appl. Math. Comp., Vol. 151, p: 287-292.

[16] He, J. H., Wu, G.C. and Austin, F., 2009, The Variational Iteration Method Which Should be Followed, Nonlinear Science Letters A1. , p: 1–30.

[17] Holmes, M. H., 1995, Introduction to Perturbation Methods, 1st edition, Springer-Verlag, New York.

[18] Jabber, A. K., 2018, Design Mathematical Model for Groundwater and its Application in Iraq, PhD Thesis, Department of Mathematics, College of Education for Pure Science Ibn Al-Haitham, University of Bagdad, Bagdad, Iraq.

[19] Khan, Y., 2009, An Effective Modification of the Laplace Decomposition Method for Nonlinear Equations, Journal of Nonlinear Sciences and Numer. Simulation, Vol. 10, p: 1373–1376.

[20] Massey, W. S., 1967, Algebric Topology, Harcourt, Brace Worl. Inc.

[21] Mehta, M. N., 1975, A Singular Perturbation Solution of One-Dimensional flow in Unsaturated Porous Media with Small Diffusivity Coefficient, Proceeding of the National Conference on Fluid Mechanics and Fluid Power (FMEP ’75), E1-E4.

[22] Mehta, M. N., and Patel, T., 2006, A Solution of Burger’s Equation Type One-Dimensional Groundwater Recharge by Spreading in Porous Media, The Journal of the Indian Academy of Mathematics, Vol. 28, No. 1, p: 25-32.

[23] Moench, A. F., 1997, Flow to a Well of Finite Diameter in a Homogeneous, Anisotropic Water Table Aquifer, Water Resources Research, Vol. 33, No. 6, p: 1397-1407. doi:10.1029/97wr00651.

[24] Mohammed Rabie, E. A., and Elzaki, T. M., 2014, Systems of Linear Partial Differential Equation by Using Adomian and Modified Decomposition Methods, African Journal of Mathematics and Computer Science Research, Vol. 7, No. 6, p: 61-67.

[25] Rahimi, E., Rahimifar, A., Mohammadyari, R., Rahimipetroudi, I. and Rahimi, EM., 2016, Analytical Approach for Solving Two-Dimensional Laminar Viscous Flow Between Slowly Expanding and Contracting Walls, ASEJ. , Vol. 7, No. 4, p: 1089-1097. doi:10.1016/j.asej.2015.07.013.

[26] Rajnee, T. and Hradyesh, K. M., 2016, Homotopy Perturbation Method with Laplace Transform (LT‑HPM) for Solving Lane–Emden Type Differential Equations (LETDEs). Tripathi and Mishra Springer Plus, Vol. 5, No. 4, p: 1-21, DOI 10.1186/s40064-016-3487-4.

[27] Rana, M. A., Siddiqui, A. M. and Ghori, Q. K., 2007, Application of He's Homotopy Perturbation Method to Sumudu Transform. International Journal of Nonlinear Sciences and Numerical Simulation, Vol. 8, No. 2, p: 185–190.

[28] Ravi Kanth, A. S. V. and Aruna, K., 2008, Differential Transform Method for Solving Linear and Non-linear Systems of Partial Differential Equations, Physics Letters A., Vol. 372, p: 6896–6898.

[29] Salih, H., Tawfiq, L. N. M. , Yahya, Z. R. and Zin, S. M., 2018, Numerical Solution of the Equation Modified Equal Width Equation by Using Cubic Trigonometric B-Spline Method, International Journal of Engineering and Technology, Vol. 7, No. 3.7, p: 340-344.

[30] Salih, H., Tawfiq, L. N. M., Yahya, Z. R. I. and Zin, S. M., 2018, Solving Modified Regularized Long Wave Equation Using Collocation Method, Journal of Physics: Conference Series, Vol. 1003, No. 012062, p: 1-10. doi :10.1088/1742-6596/1003/1/012062.

[31] Saravanan, A., and Magesh, N., 2013, A Comparison Between the Reduced Differential Transform Method and the Adomian Decomposition Method for the Newell-Whitehead-Segel Equation, Egyptian Math. Soc., Vol. 21, p: 259–265.

[32] Sedeeg, K. A. H. and Mahgoub, M. A. M., 2016, Aboodh Transform Homotopy Perturbation Method for Solving System of Nonlinear Partial Differential Equations, Mathematical Theory and Modeling, Vol. 6, No. 8.

[33] Steven, H. W., 2011, Advanced Linear Algebra, The Mathematical Association of America, USA.

[34] Suhas, P., and Uttam, D., 2015, Random Fixed Point Theorems for Contraction Mappings in Metric Space, International Journal of Science and Research (IJSR), Vol. 23, No. 19, p: 64–70.

[35] Sunil, K., Jagdev S., Devendra, K. and Saurabh, K., 2014, New Homotopy Analysis Transform Algorithm to Solve Volterra Integral Equation, ASEJ., Vol. 5, No. 2, p: 243–246.

[36] Sweilam, N. H. and Khader, M. M., 2009, Exact Solutions of Some Coupled Nonlinear Partial Differential Equations Using the Homotopy Perturbation Method. Computers and Mathematics with Applications, Vol. 58, p: 2134–2141.

[37] Tamer, A. A., Magdy, A. E. T. and El-Zoheiry, H., 2007, Modified Variational Iteration Method for Boussinesq Equation, Computers and Mathematics with Applications. , Elsevier Publisher, Vol. 54, p: 955–965.

[38] Tawfiq, L. N. M. , Jasim, K. A. and Abdulhmeed, E. O., 2016, Numerical Model for Estimation the Concentration of Heavy Metals in Soil and its Application in Iraq, Global Journal of Engineering science and Researches, Vol. 3, No. 3, p: 75-81**.**

[39] Tawfiq, L. N. M. and Hassan, M.A., 2018, Estimate the Effect of Rainwaters in Contaminated Soil by Using Simulink Technique. In Journal of Physics: Conference Series, Vol. 1003, No. 012057, p: 1-7.

[40] Tawfiq, L. N. M. and Hussein, A. A. T., 2013, Design Feed Forward Neural Network to Solve Singular Boundary Value Problems, ISRN Applied Mathematics, p: 1-8, http://dx.doi.org/10.1155/2013/650467.

[41] Tawfiq, L. N. M. and Hussein, R. W., 2013, On Solution of Regular Singular Initial Value Problems. Ibn Al-Haitham Journal for Pure and Applied Sciences, Vol. 26, No. 1, p: 257- 264.

[42] Tawfiq, L. N. M. and Hussein, R. W., 2014, Efficient Semi-Analytic Technique for Solving Nonlinear Singular Initial Value Problems, Ibn Al-Haitham Journal for Pure and Applied Sciences, Vol. 27, No. 3, p: 513-522.

[43] Tawfiq, L. N. M. and Jabber, A. K., 2016, Mathematical Modeling of Groundwater Flow, GJESR. , Vol.3, No.10, p: 15-22. doi: 10.5281/zenodo.160914.

[44] Tawfiq, L. N. M. and Jabber, A. K., 2017, "Solve the Groundwater Model Equation Using Fourier Transforms Method", Int. J. Adv. Appl. Math. and Mech., Vol. 5, No. 1, p: 75 – 80.

[45] Tawfiq, L. N. M. and Jabber, A. K., 2018, Steady State Radial Flow in Anisotropic and Homogenous in Confined Aquifers, Journal of Physics: Conference Series, Vol. 1003, No. 012056, p: 1-12, IOP Publishing.

[46] Tawfiq, L. N. M. and Rasheed, H. W., 2013, On Solution of Non Linear Singular Boundary Value Problem, Ibn Al-Haitham Journal for Pure and Applied Sciences, Vol. 26, No. 3, p: 320- 328.

[47] Tawfiq, L. N. M. and Rasheed, H. W., 2013, On Solution of Regular Singular Ordinary Boundary Value Problem, Ibn Al-Haitham Journal for Pure and Applied Sciences, Vol. 26, No. 2, p: 249-256.

[48] Tawfiq, L. N. M. and Yassien, S. M., 2013, Solution of High Order Ordinary Boundary Value Problems Using Semi-Analytic Technique, Ibn Al-Haitham Journal for Pure and Applied Sciences, Vol. 26, No. 1, p: 281-291.

[49] Tawfiq, L. N. M., 2016, Using Collocation Neural Network to Solve Eigenvalue Problems, MJ Journal on Applied Mathematics, Vol. 1, No. 1, p: 1-8.

[50] Tawfiq, L. N. M., and Hilal, M. M., 2014, Solution of 2nd Order Nonlinear Three-Point Boundary Value Problems By Semi-Analytic Technique, Ibn Al-Haitham Journal for Pure and Applied Sciences, Vol. 27, No. 3, p: 501-512.

[51] Tawfiq, L. N. M., and Jabber, A. K., 2018, New Transform Fundamental Properties and its Applications, Ibn Al-Haitham Journal for Pure and Applied Science, Vol. 31, No. 1, p: 151-163, doi: http://dx.doi.org/10.30526/31.2.1954.

[52] Vahidi, J., 2016, The Combined Laplace-Homotopy Analysis Method for Partial Differential Equations, Journal of Mathematics and Computer Science, Vol. 1, p: 95-112.

[53] Wazwaz, A. M., 2006, The Modified Decomposition Method for Analytic Treatment of Differential Equations, Appl. Math. Comput, Vol. 173, No. 1, p: 165–176.

[54] Wazwaz, A. M., 2007, The Variational Iteration Method for Solving Linear and Nonlinear System of PDEs, Comput, Math, Appl., Vol. 54, p: 895 - 902.

[55] Wazwaz, A. M., 2009, Partial Differential Equations and Solitary Waves Theory, 1st edition, Beijing and Berlin: Springer, ISBN 978-3-642-00250-2, e-ISBN 978-3-642-00251-9.

[56] Wazwaz, A. M., 2017, Dual Solutions for Nonlinear Boundary Value Problems by the Variational Iteration Method, International Journal of Numerical Methods for Heat & Fluid Flow, Vol. 27, No. 1, p: 210-220, DOI 10.1108/HFF-10-2015-0442.

[57] Wazwaz, A. M., 2000, The Decomposition Method Applied to Systems of Partial Differential Equations and to the Reaction_Diffusion Brusselator Model,Applied Mathematics and Computation, Vol. 110, No. 2-3, p: 251-264.

[58] Yusufoglu, E., 2006, Numerical Solution of Diffusing Equation by the Laplace Decomposition Algorithm, Applied Mathematics and Computation, Vol. 177, p: 572–580.

[59] Yusufoglu, E., 2009, An Improvment to Homotopy Perturbation Method for Solving System of Linear Equations, Computers and Mathematic with Applications, Vol. 58, p: 2231-2235.

[60] Zwillinger, D., 1997 Handbook of Differential Equations, 3rd edition, Academic Press.

المستَخلَص

في هذه الرسالة اقترحنا طريقة جديدة تستند على اقتران صيغة تحويل جديد مع طريقة الاضطراب الهوموتوبي لحل بعض انواع المعادلات التفاضلية الجزئية لإيجاد الحل المضبوط في مجال اوسع. ويمكن استخدام الطريقة لحل المسائل دون أي تقسيم او اعادة ترتيب تردد المجال او التقييد بفرضيات و هو حر من اخطاء التدوير. هذه الطريقة تسمى طريقة التحويل الجديد للاضطراب الهوموتوبي.

الهدف الاول في الرسالة سيكون التركيز على بعض المفاهيم الاساسية للمعادلات التفاضلية الجزئية.

الهدف الثاني هو تطبيق الطريقة المقترحة بالإضافة الى حل بعض انواع من المعادلات التفاضلية الجزئية ذات الشروط الابتدائية مثل

Klein-Gordan equation, wave-like equations, autonomous equation, system of two or three non-linear equations, Burgers' equations, coupled Hirota Satsuma KdV type II, and RLW equation.

اخيرا استخدمت لحل نموذج تطبيقي مثل معادلة رطوبة التربة حيث ان حلها بطريقة الاضطراب الهوموتوبي التقليدية يؤدي الى الحل التقريبي. ايضا اثبتنا تقارب متسلسلة الحل الى الصيغة المضبوطة . ايضا تضمنت بعض الامثلة التي توضح موثوقية و قابلية الطريقة المقترحة.

النتائج العملية اثبتت ان الطريقة المقترحة اداة كفوءة لحل تلك الانواع من المعادلات التفاضلية الجزئية.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

جامعة بغداد

كلية التربية للعلوم الصرفة /ابن الهيثم

طريقة كفوءة لحل بعض أنواع المعادلات التفاضلية الجزئية

رسالة

مقدمة إلى كلية التربية للعلوم الصرفة – ابن الهيثم، جامعة بغداد

وهي جزء من متطلبات نيل شهادة ماجستير علوم

في الرياضيات

من قبل

ميسر عبيد عنادي

بإشراف

أ. د. لمى ناجي محمد توفيق

1440 هـ   2019 م