University of Baghdad

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Chapter One

NUMERICAL ANALYSIS

Numerical Analysis is the branch of mathematics that provides tools and methods for solving mathematical problems in numerical form.

In numerical analysis we are mainly interested in implementation and analysis of numerical algorithms for finding an approximate solution to a mathematical problem.

هو فرع من فروع الرياضيات الذي يوفر الادوات والاساليب لحل المشكلات الرياضيه وهنا نحن مهتمون بايجاد وتنفيذ الخوارزميات العدديه التي تسمح لنا بايجاد الحلول بأية دقة مطلوبة باستخدام عدد معدود من الخطوات .

NUMERICAL ALGORITHM

A complete set of procedures which gives an approximate solution to a mathematical problem.

وتعرف الخوارزمية بأنها مجموعة من التوجيهات لتنفيذ عمليات حسابية مصممة بشكل يؤدي إلى حل المسألة المعطاة.

STABLE ALGORITHM

Algorithm for which the cumulative effect of errors is limited, so that a useful result is generated is called stable algorithm. Otherwise **Unstable.**

NUMERICAL STABILITY

Numerical stability is about how a numerical scheme propagate error.

NUMERICAL ITERATION METHOD

A mathematical procedure that generates a sequence of improving approximate solution for a class of problems i.e. the process of finding successive approximations.

CONVERGENCE CRITERIA FOR A NUMERICAL COMPUTATION

If the method leads to the value close to the exact solution, then we say that the method is convergent otherwise the method is divergent. **i.e.** $\lim_{n\to\infty} x_n = r$

Why we use numerical iterative methods for solving equations?

As analytic solutions are often either too tiresome or simply do not exist, we need to find an approximate method of solution.

هناك مسائل رياضية عديدة يمكن إيجاد الحلول المضبوطة لها بسهولة مثل إيجاد جذري المعادلة $x^2 - 3x + 2 = 0$ أو إيجاد قيمة التكامل $\int_0^1 x^2 dx$. ولكن في الغالب ليس من السهل إيجاد الحلول المضبوطة للعديد من المسائل مثل المعادلات الجبرية ذات القوى غير الصحيحة أو بعض المعادلات غير الخطية مثل: $x = sinx$ او قيمة التكامل $\int_0^2 e^{-x^2} dx$

RATE OF CONVERGENCE OF AN ITERATIVE METHOD

Suppose that the sequence (x_k) converges to **r** then the sequence (x_k) is said to converge to **r** with order of convergence **a** if there exist a positive constant **P** such that

$$\lim_{k\to\infty} \frac{|x_{k+1-r}|}{|x_{k-r}|^a} = \lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k{}^a} = P$$

Thus if a=1 , **the convergence is linear. If a=2 , the convergence is quadratic and so on.** Where the number a is called convergence factor.

ORDER OF CONVERGENCE OF THE SEQUENCE

Let $(x_0, x_1, x_2, \ldots\ldots)$ be a sequence that converges to a number **r** and set $\epsilon_n = r - x_n$. If there exist a number **a** and a positive constant **c** such that $\lim_{k\to\infty} \frac{|\epsilon_n + 1|}{|\epsilon_n|^a} = c$. Then **a** is called order of convergence of the sequence and **c** **the asymptotic error constant.**

ACCURACY Accuracy means how close are our approximations from exact value.

CONDITION OF A NUMERICAL PROBLEM

A problem is **well conditioned** if small change in the input information causes **small change** in the output. Otherwise it is ill conditioned.

STEP SIZE, STEP COUNT, INTERVAL GAP

The common difference between the points i.e. $h = \frac{b-a}{n} = x_{i+1} - x_i$ is called step size .

ERROR ANALYSIS

ERROR: Error is a term used to denote the amount by which an approximation fails to equal the exact solution.

بما إن الحل العددي لمسألة ما يكون عادة قيمة تقريبية للحل المضبوط لتلك المسألة ، لذا تكون هذه القيمة محملة بأخطاء من المهم قياسها لمعرفة دقة الحل.

SOURCE OF ERRORS: Numerically computed solutions are subject to certain errors. Mainly there are several types of errors

1. INHERENT (EXPERIMENTAL) ERRORS

Errors arise due to assumptions made in the mathematical modeling of problems. Also arise when the data is obtained from certain physical measurements of the parameters of the problem **i.e. errors arising from measurements.**

يظهر هذا النوع من الخطأ في التجارب العملية فعند حساب بيانات المسألة نعتمد على الملاحظة والقياس وان الدقة في هاتين الحالتين تكون محدودة مثال على ذلك عند حساب ضغط الدم أو الفولتية أو درجة الحرارة يظهر هذا النوع من الخطأ نتيجة التعامل مع الأعداد غير النسبية (... ، $\sqrt{2}$ ، π).

2. Residual error or truncation error:

Errors arise when approximations are used to estimate some quantity. These errors corresponding to the facts that a finite (infinite) sequence of computational steps necessary to produce an exact result is "truncated" prematurely after a certain number of steps.

يظهر هذا النوع من الخطأ عندما تكون الدالة *f(x)* على شكل متسلسلة غير منتهية وفي هذه الحالة عند حساب قيمة الدالة عند حد معين وبالتالي يظهر خطأ يكون مساوي لعدد الحدود المحذوفة.

$$f(x) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} / - \frac{x^8}{8!} + \cdots$$

How Truncation error can be removed? Use exact solution.
Error can be reduced by applying the same approximation to a larger number of smaller intervals or by switching to a better approximation.

3. ROUND OFF ERRORS *(Rounding and Chopping)*

Errors arising from the process of rounding off during computations. These are also called chopping i.e. discarding all decimals from some decimals on

وهذه أكثر أنواع الأخطاء شيوعاً فعندما نتعامل مع أعداد غير نسبية (غير منتهية) عندها لابد من بتر العدد بعد عدد معين من المراتب وفي هذه الحالة نلاحظ قيمة العدد على يمين اخر رقم غير مبتور فإذا كان اقل من (٥) فاننا نقطع العدد عند هذا الحد دون إضافة أي رقم إلى العدد الأصلي وهذا ما يعرف بالقطع *(Chopping)* أما إذا كانت قيمة العدد على يمين اخر رقم غير مبتور أكثر أو تساوي (٥) فاننا نضيف (١) إلى اخر رقم غير محذوف وهذا ما يعرف بالتدوير *(Rounding)* مثال على ذلك

$30.6753 \simeq 30.67531...$ (قطع)

$2.4674 \simeq 2.4673915...$ (تدوير)

ملاحظة: الحاسبة الالكترونية لا تدور العدد بل تقطعه.

4. Initial error/Error of the problem:

These are involved in the statement of problem itself. In fact, the statement of a problem generally gives an idealized model and not the exact picture of the actual phenomena. So the value of the parameter (s) involved can only be determined approximately.

5. Accumulated Error

An error whose degree or significance gradually increases in the course of a series of measurements or connected calculations. *Specifically* : an error that is repeated in the same sense or with the same sign

بعض الطرق العددية مثل الحلول العددية للمعادلات التفاضلية تتضمن تكراراً لمجموعة من العمليات الحسابية لخطوات متعاقبة ، فالخطأ في كل خطوة يزداد لاعتماد الحسابات على القيم التقريبية المحسوبة في الخطوات السابقة مما يسبب خطأ يسمى بالخطأ المتراكم.

Errors in Calculations

Absolute error: Let x be a real number and let x^* be an approximation. The **absolute error** in the approximation $x^* \approx x$ is defined as $E = |x - x^*|$.

Relative error is defined as the ratio of the absolute error to the size of x or the approximate value x^*, i.e., $Re = \frac{E}{|x|}$ or $\boldsymbol{Re} = \frac{E}{|x^*|}$, which assumes $x \; and \; x^* \neq 0$

إذا كانت x تمثل القيمة المضبوطة و x^* تمثل القيمة التقريبية فان **الخطأ المطلق** يساوي القيمة المضبوطة – القيمة التقريبية تحت المطلق

ويعرف **الخطأ النسبي** حاصل قسمة الخطأ المطلق على مطلق القيمة المضبوطة وفي الحالات التي تكون فيها القيمة المضبوطة غير معلومة نستخدم بدلاً منها مطلق القيمة التقريبية

Example 1: Let 0.0005 be a real number and let 0.0003 be an approximation. Then find the value of the absolute error and the relative error.

E

$$Re = \frac{\mathrm{E}}{|\mathrm{x}|} = \frac{0.0002}{|0.0005|} = 0.4$$

Example 2: Let x=100000, $x^* = 99950$ then find the value of the absolute error and the relative error.

$$\boldsymbol{E = |x - x^*| = 50} \qquad \textbf{\textit{and}}\ \boldsymbol{Re = \frac{\mathrm{E}}{|\mathrm{x}|} = \frac{50}{100000} = 0.0005}$$

Example 3: Let x=0.000015 , $x^* = 0.000007$ then find the value of the absolute error and the relative error.

$$E = |x - x^*| = |0.000015 \quad - 0.000007 \quad| = 0.000008$$

$$Re = \frac{\mathrm{E}}{|\mathrm{x}|} = \frac{0.000008}{0.000015} = 0.53$$

Some examples:

Ex1/ The infinite Taylor series of

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + \frac{x^{2n}}{n!} \longrightarrow (1)$$

when $\int_0^{1/2} e^{x^2}\,dx = 0.544987104184$

Determine the accuracy of the approximation obtained by replacing the integr and $f(x) = e^{x^2}$ with 1st, 2nd, ---, fifth terms from equation (1).

SoL:- By using one-term

$$\int_0^{1/2} 1\,dx = x\Big]_0^{1/2} = 0.5$$

By using two-terms

$$\int_0^{1/2} (1+x^2)\,dx = \left[x + \frac{x^3}{3}\right]_0^{1/2} = 1/2 + 1/24 = 0.541666666667$$

By using Three-terms

$$\int_0^{1/2} \left(1 + x^2 + \frac{x^4}{2!}\right)dx = \left[x + \frac{x^3}{3} + \frac{x^5}{10}\right]_0^{1/2} = \frac{1}{2} + \frac{1}{24} + \frac{1}{320} =$$

$$= 0.544791666667$$

By using four-terms

$$\int_0^{1/2} \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}\right)dx = \left[x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{7 \times 3!}\right]_0^{1/2} =$$

$$= 0.544977678571$$

By using five-terms

$$\int_0^{1/2}\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}\right)=\left[x+\frac{x^3}{3}+\frac{x^5}{10}+\frac{x^7}{7.3!}+\frac{x^9}{9\times 4!}\right]_0^{1/2}=$$

$$=0.544986720817$$

نلاحظ من القيم اعلاه بزيادة حدود المتسلسل فاننا نقترب من الحل المضبوط ويقل خطأ البتر .

(ill- Conditioned and Well- Conditioned Problems)

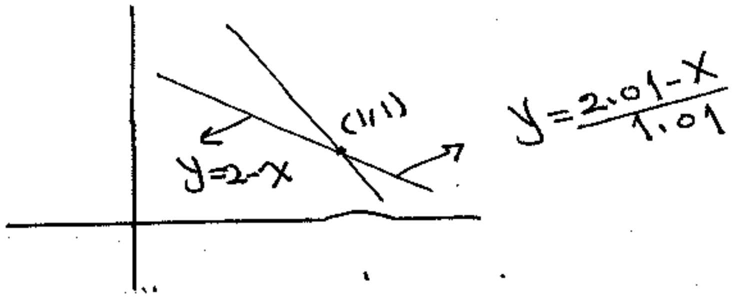
تسمى الطريقة غير مستقرة (unstable) أو (ill-conditioned) اذا كان التغير القليل في البيانات يسبب تغيراً كبيراً في الحل .

اما اذا كان التغير القليل في البيانات يسبب تغيراً صغيراً في الحل فنسمي الطريقة مستقرة (stable) أو (Well-conditioned)

EX2/ suppose that $\boxed{y=\frac{x}{-x+1}}$ with $x=0.93$ and we get $y=13.28$, but when use $x=0.94$ we get $y=15.60$, So we would probably say that this expression is <u>ill- Conditioned</u> when evaluated for x <u>near</u> 0.93.

on the otherhand, if we use $x=-0.93$, we get $y=-0.4818$, and $x=-0.94$, we get $y=-0.4845$ we would say this it is <u>Well-Conditioned</u> for x near <u>-0.93</u>.

EX3:- Consider the linear equations $x+y=2$, $x+1.01y=2.01$ which have solution $y=x=1$. if the number 2.01 is changed to 2.02 then the solution $x=0$ and $y=2$. That means (small) change in the data produces a 100% change in the solution. (this is ill-Conditioned



EX 4:- Consider the following system of two linear equations in two unknowns

$$\begin{bmatrix} 400 & -201 \\ -800 & 401 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ -200 \end{bmatrix}$$

نضرب المعادلة الاولى 2×

$800x_1 - 402x_2 = 400$

$-800x_1 + 401x_2 = -200$

$-x_2 = 200 \rightarrow x_2 = -2$

التعويض بالمعادلة ①

$400x_1 + (201)(200) = 200$

$\longrightarrow x_1 = -100$

The system can be solved and get $x_1 = -100$, $x_2 = -200$

Now, let us make a slight change in a_{11} from 400 to 40

$$\begin{bmatrix} 401 & -201 \\ -800 & 401 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ -200 \end{bmatrix}$$

(الحل بآخر الورقة)

In this case the solution is $x_1 = 40000$ and $x_2 = 79800$ so it is ill-Conditioned problem.

اولاً نضرب المعادلة الاولى X 2 تصبح

$$802X_1 - 402X_2 = 400$$

$$-800X_1 + 401X_2 = -200$$

$$2X_1 - X_2 = 200$$

$$\longrightarrow \boxed{X_2 = 2X_1 - 200}$$

نعوض في المعادلة الاولى الاصلية

$$401X_1 - 201X_2 = 200$$

$$401X_1 - (201)(2X_1 - 200) = 200$$

$$401X_1 - 402X_1 + 40200 = 200$$

$$-X_1 = -40000$$

$$\longrightarrow \boxed{X_1 = 40000}$$

بالتعويض بالمعادلة الاولى

$$401X_1 - 201X_2 = 200$$

$$(401)(40000) - 201X_2 = 200$$

$$\Longrightarrow \boxed{X_2 = 79800}$$

EX5:- let $Q_n = 10^{-n}$ and let $P_n = 10^{-2^n}$ then $\{Q_n\}$ Converges to 0 linearly, and $\{P_n\}$ converges to 0 quadratically.

Sol/ $\{Q_n\}$ Converges to 0 linearly, Since

$$|Q_{n+1} - 0| = 10^{-(n+1)} = 10^{-n} \cdot 10^{-1} = \frac{1}{10} \cdot 10^{-n} = \frac{1}{10} |Q_n - 0|$$

Thus, $\rho = \frac{1}{10}$ and $\alpha = 1$.

Also, $\{P_n\}$ Converge to 0 quadratically, Since

$$|P_{n+1} - 0| = 10^{-2^{n+1}} = 10^{-2^n} \cdot 10^{2^1} = \left(10^{-2^n}\right)^2 = |P_n - 0|^2$$

Thus $\rho = 1$ and $\alpha = 2$.

$$\lim_{k \to \infty} \frac{|x_{k+1} - r|}{|x_k - r|^{\alpha}} = \rho$$

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NUMERICAL ANALYSIS

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Chapter two / SOLUTION OF NON-LINEAR EQUATIONS

ROOTS (SOLUTION) OF AN EQUATION OR ZEROES OF A FUNCTION

Let f(x) continuous function, for any number **r**. The root **r** or also called the zero of an equation is the value which satisfy the equation i.e. **f(r)=0**

ALGEBRAIC EQUATION

متعددة حدود بحتة

The equation f(X) = 0 is called an **algebraic equation** if it is **purely a polynomial in x**. e.g. $x^3+5x^2-6x+3=0$

PROPERTIES OF ALGEBRAIC EQUATIONS

1. Every algebraic equation of degree **n** has **n** and only n roots .e.g. $x^2-1=0$ has distinct roots (1, -1).
2. Complex roots occur in pair. i.e. (a+ib) and (a-ib) are roots of f(x)=0 .
3. If x = a is a root f(x)=0, a polynomial of degree n then (x-a) is factor of f(x)=0 on dividing f(x) by (x-a) we obtain polynomial of degree (n-1).

REMARK

There are two types of methods to find the roots of Algebraic equations.

(1) DIRECT METHODS (2) INDIRECT (ITERATIVE) METHODS

DIRECT METHODS

1. Direct methods give the exact value of the roots in a finite number of steps.
2. These methods determine all the roots at the same time assuming no round off .

INDIRECT (ITERATIVE) METHODS

1. These are based on the concept of successive approximations. The general procedure is to start with one or more approximation to the root and obtain a sequence of iterates x which in the limit converges to the actual or true solution to the root.
2. Indirect Methods determine one or two roots at a time.
3. Rounding error have less effect
4. Easier to program and can be implemented on the computer.

أو تقترب

١. الإجراء العام هو البدء بتقريب واحد أو أكثر للجذر والحصول على سلسلة من التكرارات x والتي تؤدي إلى الحل الفعلي أو الحقيقي للجذر.
٢. تحدد الطرق غير المباشرة جذرًا واحدًا أو اثنين في وقت واحد.
٢. تقريب الخطأ يكون أقل تأثير
٤. أسهل في البرمجة ويمكن تنفيذها على الكمبيوتر.

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REMEMBER: Indirect Methods are further divided into two categories
1. BRACKETING METHODS 2. OPEN METHODS

BRACKETING METHODS

These methods require the limits between which the root lies. e.g. Bisection method, False position method.

OPEN METHODS

These methods require the initial estimation of the solution. e.g. Newton Raphson method.

ADVANTAGES AND DISADVANTAGES OF BRACKETING METHODS

Bracket methods always **converge**. The main **disadvantage** is, if it is not possible to bracket the root, the method cannot applicable.

How to get first approximation?

We can find the approximate value of the root of f(x)=0 by **Graphical method** or by **Analytical method.**

INTERMEDIATE VALUE THEOREM

Suppose **f** is continuous on [a, b] and f(a)≠ f(b) then given a number λ that lies between f(a) and f(b) then there exist a point c such that a< c <b with f(c)= λ

طرق تعيين القيمة التقريبية الاولية (تعيين مواقع الجذور): في الطرق العددية لحلول المعادلات الغير خطية تحتاج عادة الى قيمة تقريبية تخمينية اولية . هناك طريقتان الاولى طريقه الرسم وطريقة المبرمجه هذه الطريقة على تغير اشارة الدالة **f** عند النقاط $x_0 , x_1 , \dots , x_i , x_{i+1} , \dots , x_n$ فاذا كانت اشارة $f(x_i) . f(x_{i+1})$ سالبة فان ذلك يعني وجود جذر في الفترة المغلقة $[x_i, x_{i+1}]$ أما اذا كانت اشارة حاصل الضرب موجبة فهذا يعني لا يوجد جذر للدالة **f** في الفترة المغلقة $[x_i, x_{i+1}]$ وتعتمد هذه الطريقة على مبرهنة القيمة المتوسطة والتي تفترض كون الدالة **f** مستمرة في الفترة المغلقة $[x_i, x_{i+1}]$.

Example 1: find the root of the equation $f(x): x^3 - x - 1 = 0$

0	1	2
-	-	+

Example 2: find the positive root of the equation $f(x): x^4 - x^2 - 3 = 0$

0	1	2
-	-	+

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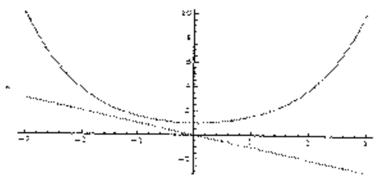
Example 3: find the negative root of the equation $f(x)$: $x^4 - x^2 - 3 = 0$

-2	-1	0
+	-	-

Example 4: find the root of the equation $f(x)$: $x + coshx = 0$

-2	-1	0	1	2
+	+	+	+	+



BISECTION METHOD

Bisection method is one of the bracketing methods. It is based on the **Intermediate value theorem**. The idea behind the method is that if f(x) € C [a, b] and f (a).f(b) < 0 then there exist a root c € (a,b) such that f(c)=0.
This method also known as **BOLZANO METHOD (or) BINARY SECTON METHOD.**

ALGORITHM

For a given continuous function f(x)
1. Find **a, b** such that **f(a).f(b) < 0** .This means there is a root **r€ (a, b)** such that **f(r)=0**
2. Let **c** $=\frac{a+b}{2}$ (mid-point)
3. If f(c)=0; done
4. Else; check if $f(c).f(a) < 0$ or $f(c).f(b) < 0$
5. Pick that interval [a, c] or [c, b] and repeat the procedure until stop criteria satisfied.

STOP CRITERIA

1. Interval small enough.
2. $|f(c_n)|$ almost zero
3. Maximum number of iteration reached
4. Any combination of previous ones

نفرض ان الدالة f لها جذر في الفترة المغلقة [a,b] والمطلوب ايجاد الجذر بطريقة تنصيف الفترات وكما يلي :

١) نأخذ منتصف الفترة [a,b] ولتكن $Q_1 = \frac{a+b}{2}$.

٢) نعوض Q_1 في f ونقارن اشارتها مع اشارة f(a) فاذا كانت مختلفة فهذا يعني ان الجذر يقع بين Q_1 و a. اما اذا كانت متشابهة الاشارات فان ذلك يعني ان الجذر يقع بين Q_1 و b وهكذا نستمر باخذ منتصف الفترة التي يقع فيها الجذر الى نصل الى شرط التوقف حيث ان شرط التوقف هو:

1- $|f(Q_n)| < \epsilon \Rightarrow Q_n$ is approximate root

2- $|Q_n - Q_{n+1}| < \epsilon \Rightarrow Q_{n+1}$ is approximate root

CONVERGENCE CRITERIA **No. of iterations needed in the bisection method to achieve certain accuracy**

Theorem: If the Bisection algorithm is applied to a continues function f on [a,b] and $f(a) \cdot f(b) < 0$, then after **n steps** an approximate root will have been computed with error at most $\frac{\mathbf{b-a}}{\mathbf{2^{n+1}}}$

Note: To get number of iteration or steps , then we have to solve the following inequality for n.

$$\frac{b-a}{2^{n+1}} < \epsilon$$

Sol//

$$\frac{b-a}{2^{n+1}} < \epsilon$$

$$\log(b-a) - n\log(2) - \log(2) < \log(\epsilon)$$
$$\log(b-a) - \log(2\epsilon) < n\log(2)$$

$$\boxed{\frac{\log(b-a) - \log(2\epsilon)}{\log(2)} < n}$$

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MERITS OF BISECTION METHOD

1. The iteration using bisection method always produces a root, since the method brackets the root between two values.
2. As iterations are conducted, the length of the interval gets halved. So one can guarantee the convergence in case of the solution of the equation.
3. Bisection method is simple to program in a computer.

مزايا الطريقه

1 .يؤدي التكرار باستخدام طريقة bisection دائمًا إلى إنتاج الجذر

٢. أثناء إجراء التكرار ، يتم تقليل طول الفاصل إلى النصف. لذلك يمكن للمرء أن يضمن التقارب في حالة حل المعادلة.

3. طريقة Bisection بسيطة في البرمجة في الكمبيوتر

DEMERITS OF BISECTION METHOD

1. The convergence of bisection method is slow as it is simply based on halving the interval.
2. Cannot be applied over an interval where there is discontinuity.
3. Cannot be applied over an interval where the function takes always value of the same sign.
4. Method fails to determine complex roots (give only real roots)
5. If one of the initial guesses a_0 or b_0 is closer to the exact solution, it will take larger number of iterations to reach the root.

عيوب الطريقه

١. تقارب الطريقه بطيئة لأنه يعتمد ببساطة على النصف الفاصل.

٢. لا يمكن تطبيقها على فاصل زمني حيث يكون هناك توقف.

٣. .لا يمكن تطبيقها على فاصل زمني حيث تأخذ الداله دائما قيمة نفس العلامة.

٤. فشل الأسلوب في تحديد جذور معقدة (إعطاء جذور حقيقية فقط)

Example 1: Determined the number of iterations of bisection method necessary to solve *f(x)* on [0,3] with accuracy $\in = 0.001$. a=0 , b=3

$$\frac{\log(b-a)-\log(2\in)}{log(2)} < n$$

$$\frac{\log(3-0)-\log(0.002)}{log(2)} = 10.5 < n$$

n= 11

ما هي التكرارات اللازمة للوصول الى الرقم المطلوب
وما بعدها لا نحتاجها

Therefore, no more than 11 iterations would be needed to achieve the converge to within 0.001

Example 2: Use the Bisection method to find a solution accurate to within $\epsilon = 0.01$ for $f(x): x^2 - \frac{1}{2}$.

x	0	1	0.5	0.75	0.625	0.6875	0.71875	0.7031
$x^2 - 0.5$	-0.5	+0.5	-0.25	+0.0623	-0.1094	-0.0273	+0.0166	-0.0056

0.7031 is the root of function $|-0.0056| < \epsilon = 0.01$

Example 3: Use the Bisection method to find a solution accurate to within $\epsilon = 0.001$ for $f(x): x - 2^{-x}$.

x	0	1	0.5	0.75	0.625	0.6875	0.65625	0.6406
$x - 2^{-x}$	-1	+0.5	-0.2071	+0.1554	-0.0234	+0.0666	+0.0217	-0.0008

0.*6406 is the root of function* $|-0.0008| < \epsilon = 0.001$.

Example 4: Use the Bisection method to find a solution for $f(x): x^3 - 9x + 1$ on [2,4].

X	2	4	3	2.5	2.75	2.875	2.9375	2.9688	2.9532	2.9453
$x^3 - 9x + 1$	-9	+29	1	-5.875	-2.9534	-1.1113	-0.0901	0.4471	0.1772	0.1772

Hence root is 2. 9453 because roots are repeated.

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Ex5/ Use bisection method to find out the roots of the function given by $f(c) = \frac{667.38}{c}\left(1 - \exp(-0.146843c) - 40\right)$ where $c = 12$ to $c = 16$. Perform at least two iterations.

Sol/ given that $f(c) = \frac{667.38}{c}\left[1 - \exp(-0.146843c) - 40\right]$

x	12	13	14	15
$f(x)$	6.670	3.7286	1.5687	-0.4261

Since $f(14) \cdot f(15) < 0$, Therfore root lie between 14 and 15

$x_r = \frac{14+15}{2} = 14.5$, So $f(14.5) = 0.5537$

Again $f(14.5) \cdot f(15) < 0$, Then root lie between 14.5 and 15

$x_r = \frac{14.5+15}{2} = 14.75$, So $f(14.75) = 0.0608$

These are required iterations.

Ex6/ Explain why the equation $e^{-x} = x$ has a solution on the interval $[0,1]$. Use bisection to find the root to 4 decimal places. Can you Prove that there are no other roots?

Sol/ if $f(x) = e^{-x} - x$, Then $f(0) = 1$, $f(1) = 1/e - 1 < 0$

and hence a root is guaranteed by the intermediate value theorem. Using bisection, the value of root $x = 0.567$ $\overline{0.5671}$

Since $f'(x) = -e^{-x} - 1 < 0$, for all x, the function is <u>decreasing</u> and so its graph can only cross the x-axis at a single point which is the root.

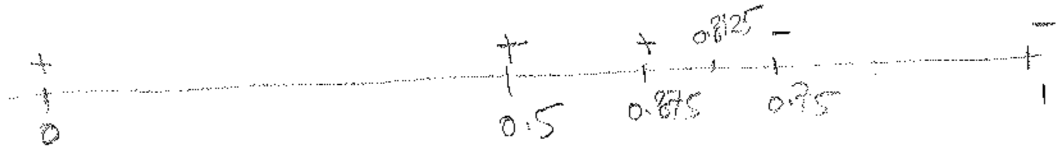
H.W

① Is the function $f(x) = x^3 - 4x^2 - 10$ has a root in $[1,2]$

② let $f(x) = (x+1)^2 e^{(x^2-2)} - 1$ has a root in $[0,1]$ s.t $f(0) < 0$ and $f(1) > 0$. Use Bisection method to find the approximate root with $\epsilon = 0.00001$

③ Determine the number of iterations to solve $f(x) = x^3 + 4x^2 - 10$ with accuracy 10^{-3} using $a = 1$, $b = 2$.

④ Find the approximation to $\sqrt{3}$ correct to within 10^{-4} using the Bisection Algorithm (Hint $f(x) = x^2 - 3$)



$x = 0.5671$

اكبر رقم طالع مرتبة

$e^{-(0.5671)} - 0.5671 = 0.00006$

١

برامج العملي

ايجاد جذر للدالة.

MaTlab built-In Function fzero

The fzero function in MATLAB finds the roots of $f(x) = 0$ for a real function $f(x)$. FZERO Scalar nonlinear zero finding. $X = FZERO(FUN, X_0)$ tries to find a zero of the function FUN near X_0, if X_0 is a scalar.
For example 2.1 use the following Matlab code:

```
clc
clear
fun = @(x) x.^3+4*x.^2-10; % function
x0 = 1; % initial point
x = fzero(fun,x0)
```

the resulte is:
$x = 1.365230013414097$

برنامج طريقة التنصيف

Matlab Code : *. Bisection method*

```
% ****************************************************************
% ************   bisection method   ************
% ****  to find a root of the function f(x) ***
% ****************************************************************
clc
clear
close all
f=@(x) x.^3+4*x.^2-10 ;
% f=@(x) (x+1)^2*exp(x^2-2)-1;
a=1;
b=2;
c=(a+b)/2;
e=0.00001;
k=1;
fprintf('     k         a            b            f(c)      \n');
fprintf('    ___    _________     ___________    __________  \n');

while abs(f(c)) > e
c=(a+b)/2;
if f(c)*f(a)<0
      b=c;
   else
      a=c;
 end
fprintf('%6.f  %10.8f  %10.8f  %10.8f \n', k,a,b,f(c));
k=k+1;
end
fprintf(' The approximated root is c= %10.10f  \n', c);
```

The result as the following table:

```
     k        a            b            f(c)
    ___   __________   __________   __________
     1  1.00000000  1.50000000  2.37500000
     2  1.25000000  1.50000000  -1.79687500
     3  1.25000000  1.37500000  0.16210938
     4  1.31250000  1.37500000  -0.84838867
   ___________________________________________
    18  1.36522675  1.36523056  0.00000903
 The approximated root is c= 1.3652305603
```

طريقة النقطة الصامدة

2 Fixed-Point Iteration

A fixed point for a function is a number at which the value of the function does not change when the function is applied.

Definition . *The number p is a fixed point for a given function g if $g(p) = p$.*

Suppose that the equation $f(x) = 0$ can be rearranged as

$$x = g(x) \tag{2.2}$$

Any solution of this equation is called a fixed point of g. An obvious iteration to try for the calculation of fixed points is

$$x_{n+1} = g(x_n) \qquad n = 0, 1, 2, \cdots \tag{2.3}$$

The value of x_0 is chosen arbitrarily and the hope is that the sequence $x_0, x_1, x_2, \cdots$ converges to a number α which will automatically satisfy equation (2.2).

$\boxed{f(x)=0}$

Moreover, since equation (2.2) is a rearrangement of (2.1), α is guaranteed to be a zero of f.

In general, there are many different ways of rearranging $f(x) = 0$ in the form (2.2). However, only some of these are likely to give rise to successful iterations, as the following example demonstrates.

Example 1 . *Consider the quadratic equation*

$$x^2 - 2x - 8 = 0$$

with roots -2 and 4. Three possible rearrangements of this equation are

(a) $x_{n+1} = \sqrt{2x_n + 8}$

(b) $x_{n+1} = \frac{2x_n+8}{x}$

(c) $x_{n+1} = \frac{x_n^2-8}{2}$

1. $X^2 = 2X+8 \Rightarrow X = \sqrt{2X+8}$
2. $\frac{X^2}{X} = \frac{2X+8}{X} \Rightarrow X = \frac{2X+8}{X}$
3. $X = \frac{X^2-8}{2}$

Numerical results for the corresponding iterations, starting with $x_0 = 5$, are given in Matlab code 2.11 with the Table.

الحل موجود ص 26

Solution:

k	Xa	Xb	Xc
1	4.24264069	3.60000000	8.50000000
2	4.06020706	4.22222222	32.12500000
3	4.01502355	3.89473684	512.0078125
4	4.00375413	4.05405405	131072.0000
5	4.00093842	3.97333333	8589934592.0
6	4.00023460	4.01342282	3.6893e+19

Consider that the sequence converges for (a) and (b), but diverges for (c).
This example highlights the need for a mathematical analysis of the method. Sufficient conditions for the convergence of the fixed point iteration are given in the following (without proof) theorem.
Theorem 1 : *If g' exists on an interval $I = [\alpha - A, \alpha + A]$ containing the starting value x_0 and fixed point α, then x_n converges to α provided*

$$|g'(x)| < 1 \quad on \quad I$$

We can now explain the results of Example 2.5

(a) If $g(x) = (2x+8)^{\frac{1}{2}}$ then $g'(x) = (2x+8)^{-1/2}$ Theorem 2.6 guarantees convergence to the positive root $\alpha = 4$, because $|g'(x)| < 1$ on the interval $I = [3,5] = [\alpha - 1, \alpha + 1]$ containing the starting value $x_0 = 5$. which is in agreement with the results of column Xa in the Table.

(b) If $g(x) = \frac{(2x+8)}{x}$ then $g'(x) = \frac{-8}{x^2}$ Theorem 1 guarantees convergence to the positive root $\alpha = 4$, because $|g'(x)| < 1$ as (a), which is in agreement with the results of column Xb in the Table.

(c) If $g(x) = \frac{(x^2-8)}{2}$ then $g'(x) = x$ Theorem1 cannot be used to guarantee convergence, which is in agreement with the results of column Xc in the Table.

Example 2 *Find the approximate solution for the equation*

$$f(x) = x^4 - x - 10 = 0$$

by fixed point iteration method starting with $x_0 = 1.5$ *with* $|x_n - x_{n-1}| < 0.009$

Solution

The function $f(x)$ has a root in the interval $(1, 2)$, **Why ?**, rearrange the equation as

حلها جذر في الفترة (1,2)

$x_0 = 1.5$

$$x_{n+1} = g(x_n) = \sqrt{x_n + 10}$$

① $x = g(x)$

$\boxed{x = \sqrt[4]{x+10}}$

then

$$g'(x) = \frac{(x+10)^{\frac{-3}{4}}}{4}$$

Achieving the condition

$$|g'(x)| \leq 0.04139 \quad on \quad (1, 2)$$

then we get the solution sequence $\{1.5, 1.8415, 1.85503, 1.8556, \cdots\}$. consider that $|1.85503 - 1.8556| = 0.00057 < 0.009$.

$|X_{n+1} - X_n| < \epsilon$

بتطبيق الشرط (شرط التوقف)

$|g'(x)| = |g'(1.5)| = \left|\frac{1}{4}(1.5+10)^{-3/4}\right| = 0.04 < 1$

لحساب $X_{n+1} = g(x_n)$

x_n	1.5	1.8415	1.855033	1.8556
	1.8415	1.855033	1.8556	

② $\boxed{x = x^4 + 10}$ diverge

③ $x^4 = x + 10 \longrightarrow x = \frac{x+10}{x^3} \longrightarrow |g'(1.5)| > 1 \rightarrow$ diverge

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FIXED POINT ITERATION METHOD

ALGORITHM

1. Consider f(x) =0 and transform it to the form x= φ (x)
2. Choose an arbitrary x_0
3. Do the iterations $x_{k+1}=\varphi(x_k)$; k=0,1,2,3.........

STOPING CRITERIA

Let "ϵ" be the tolerance value

1. $|x_k - x_{k-1}| \le \epsilon$
2. $|x_k - f(x_k)| \le \epsilon$
3. Maximum number of iterations reached.
4. Any combination of above.

CONVERGENCE CRITERIA

Let "x" be exact root such that r=f(x) out iteration is $x_{n+1} = f(x_n)$

Define the error $\epsilon_n = x_n - r$ Then

$$\epsilon_{n+1} = x_{n+1} - r = f(x_n) - r = f(x_n) - f(r) = f'(\S)(x_n - r)$$

(Where $\S \in (x_n, r)$; since f is continuous)

$$\epsilon_{n+1} = f'(\S)\epsilon_n \Rightarrow \epsilon_{n+1} \le |f'(\S)||\epsilon_n|$$

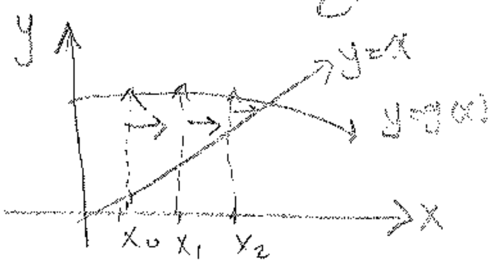
OBSERVATIONS

If $|f'(\S)| < 1$, error decreases, the iteration converges (linear convergence)

If $|f'(\S)| \ge 1$, error increases, the iteration diverges.

REMEMBER: If $|\varphi'(x)| < 1$ in questions then take that point as initial guess.

لايجاد الجذور للمعادلة $f(x)=0$ بالطريقة التكرارية نكتب المعادلة بالشكل $x=g(x)$ فيكون الطرف الايسر خط مستقيم والطرف الايمن دالة مختلفة $g(x)$ ويكون فيه الجذر عند تقاطع هاتين الدالتين كما موضح



Example 3: Find the approximate solution for the equation $f(x): x^2 - x - 3 = 0$ by fixed point iteration method.

Solution :

يوجد جذراً للمعادلة في الفترة $\left(\overset{-}{2}, \overset{+}{3}\right) \Leftarrow x_0 = 2.5$

نكتب المعادلة على شكل $x = g(x)$

هناك اكثر من طريقة واسلوب لكتابة المعادلة بالصيغة $x = g(x)$

a) $x = 1 + \frac{3}{x} = g_1(x)$

b) $x = x^2 - 3 = g_2(x)$

c) $x = \frac{9x - x^2 + 3}{8} = g_3(x)$

d) $x = \frac{x^2 + 3}{2x - 1} = g_4(x)$

x	$x_{n+1} = g_1(x_n)$	$x_{n+1} = g_2(x_n)$	$x_{n+1} = g_3(x_n)$	$x_{n+1} = g_4(x_n)$
x_0	2.5	2.5	2.5	2.5
x_1	2.2	3.25	2.40625	2.3125
x_2	2.36364	7.5625	2.35828	2.302802
x_3	2.26923	54.1914	2.33288	2.302776
x_4	2.32203	2933.71	2.31920	2.302776
x_5	2.29197	8606642.63	2.31176	2.302776
x_6	2.30892	7.41×10^{13}	2.30770	2.302776

هنا التقارب ثابت للجذر غير قريب

كيف نختار g ؟ ان الشرط الكافي لتقارب الصيغة التكرارية $x_{n+1} = g(x_n)$ هو $|g'(x)| < 1$ بحسب المبرهنة التالية

Fixed Point Theorem:

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in [a, b]. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| < k \quad \forall x \in (a, b).$$

Then, for any number p_0 in [a, b], the sequence defined by $p_{n+1} = g(p_n)$ $n = 0,1,2, \ldots$ converges to the unique fixed point p in [a, b].

For the above example we note

$g_1'(x) = \frac{-3}{x^2} \Rightarrow |g_1'(2.5)| = 0.48 < 1$

$g_2'(x) = 2x \Rightarrow |g_2'(2.5)| = 5 > 1$

$g_3'(x) = \frac{9}{8} - \frac{x}{4} \Rightarrow |g_3'(2.5)| = 0.5 < 1$

$g_4'(x) = \frac{(2x-1)2x - (x^2+3)2}{(2x-1)^2} \Rightarrow |g_4'(2.5)| = 0.0938 < 1$

افضل الرموز

EXAMPLE 4

Find the root of equation $2x = cosx + 3$ correct to three decimal points using fixed point iteration method.

SOLUTION

Given that $f(x) = 2x - cosx - 3 = 0$

X	0	1	2
F(X)	-4	-1.5403	1.4161

Root lies between "1" and "2"

Now $2x - cosx - 3 = 0 \Rightarrow x = \frac{cosx+3}{2} = \varphi(x)$

$\Rightarrow \varphi'(x) = \frac{1}{2}(-sinx) \Rightarrow |\varphi'(x)| = |\frac{1}{2}(-sinx)|$

Now $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{1}{2}(cosx_n + 3)$

Here we will take "x_0" as mid-point. So

If by putting 1 we get $|\varphi'(x)| < 1$ then take it as "x_0" if not then check for 2 rather take their mid-point

$x_0 = \frac{1+2}{2} = 1.5$

$x_1 = \frac{1}{2}(\cos x_0 + 3) = 1.9998$ $\qquad f(x_1) = 0.0002$

$x_2 = \frac{1}{2}(\cos x_1 + 3) = 1.9997$ $\qquad f(x_2) = 0.000008$

$x_3 = \frac{1}{2}(\cos x_2 + 3) = 1.9997$ $\qquad f(x_3) = 0.000008$

Hence the real root is 1.9997

هنا التقارب للجذر هو ثابت غير متذبذب

EXAMPLE 5

Find the root of equation $e^{-x} = 10x$ correct to four decimal points using fixed point iteration method.

SOLUTION

Given that

$f(x) = e^{-x} - 10x = 0$

X	0	1
F(X)	1	-9.6321

Root lies between "0" and "1"

Now $e^{-x} - 10x = 0 \Rightarrow x = \frac{e^{-x}}{10} = \varphi(x)$

$\Rightarrow \varphi'(x) = -\frac{e^{-x}}{10}$

Now since $|\varphi'(0)| = 0.1$ is less than "1" therefore $x_0 = 0$

Now $x_{n+1} = \varphi(x_n) \Rightarrow x_{n+1} = \frac{e^{-x_n}}{10}$

$X_1 = \frac{e^{-x_0}}{10} = \frac{e^{-0}}{10} = 0.1000$	$F(x_1) = -0.0952$
$X_2 = 0.0905$	$F(x_2) = 0.0085$
$X_3 = 0.0913$	$F(x_3) = -0.0003$
$x_4 = 0.0913$	$F(x_4) = -0.0003$

Hence the real root is 0.0913

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EX/ Find the root of $f(x) = x^2 - 2x - 8 = 0$ using Fixed Point iteration method.

Sol/ The roots for the equation are -2 and 4

$x_0 = 3.5$ نجد جذر بين الفترة $[3,4]$. نبدأ بنقطة ابتدائية

هناك أكثر من اسلوب لكتابة $f(x)=0$ بشكل $x = g(x)$

(1) if $g(x) = (2x+8)^{1/2}$ then $g'(x) = (2x+8)^{-1/2}$

$|g'(x_0)| < 1 \rightarrow |0.258| < 1$ Converge

(2) $g(x) = \dfrac{2x+8}{x}$ then $g'(x) = \dfrac{-8}{x^2}$ Converge

$|g'(3.5)| < 1 \longrightarrow |0.65| < 1$

(3) $g(x) = \dfrac{x^2-8}{2}$ Then $g'(x) = x$, diverge

ثم نكمل الحل ...
(نتوقف بعد خمس تكرارات)

H.W

(1) use fixed point iteration to find the root of

(1) $x - \cos x = 0$ (2) $x^2 + \ln x = 0$

with $x_0 = 1$, $\epsilon = 0.5 \times 10^{-2}$

(2) Find the first nine terms generated by $x_{n+1} = e^{-x_n}$

with $x_0 = 1$

	g_1	g_2
$\boxed{x_0 = 3.5}$	3.8729	4.2857
$x_2 =$	3.9918	3.8666
$x_3 =$	3.9979	4.0693
$x_4 =$	3.9994	10.1039
$x_5 =$	3.9998	9.7918

② برنامج العملي / النقطة الصامدة

Matlab Code *Fixed Point Iteration*

```
clc
clear
close all

xa =5; % Initial value of root
xb =5;
xc =5;
fprintf('     k          Xa               Xb                Xc      \n');
fprintf('    ___      ________         ________         ________    \n');

for k=1:1:6
 xa=sqrt(2*xa+8);
 xb =(2*xb +8)/xb;
 xc =(xc^2-8)/2;
fprintf('%6.f   %10.8f   %10.8f   %10.8f \n', k, xa , xb , xc
    );
end
```

The result as the following table:

k	Xa	Xb	Xc
1	4.24264069	3.60000000	8.50000000
2	4.06020706	4.22222222	32.12500000
3	4.01502355	3.89473684	512.0078125
4	4.00375413	4.05405405	131072.0000
5	4.00093842	3.97333333	8589934592.0
6	4.00023460	4.01342282	3.6893e+19

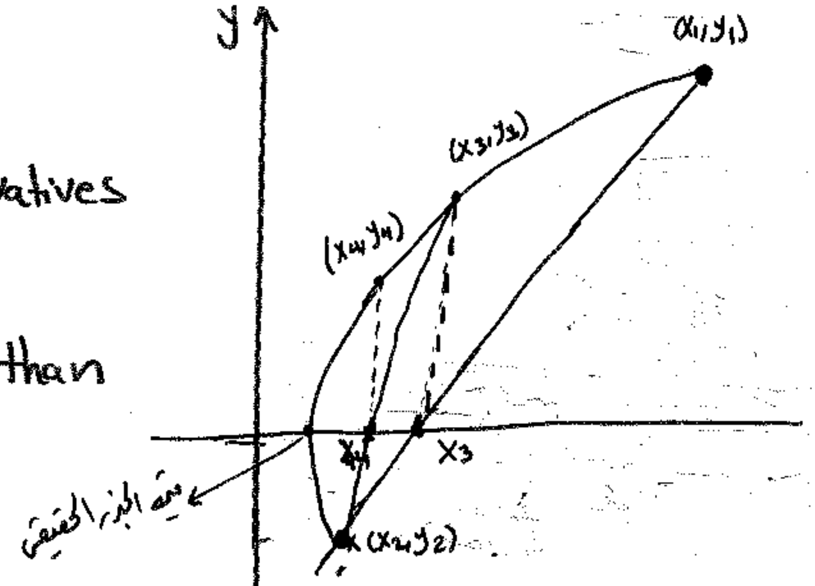
Secant Method

The Secant Method is a simple variant of the method of false position which it is no longer required that the function (f) has opposite signs at the end points of each interval generated, not even the intial interval.

In the other words, one starts with two arbitrary initial approximations $x_0 \neq x_1$ and continues with

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] \quad n = 1, 2, 3, 4 \ldots \quad (1)$$

This method also known as Quasi Newton's Method



Advantages

1. No computations of derivatives
2. one $f(x)$ computation
3. Also rapid convergence than falsi method.

في هذه الطريقة نبدأ بنقطتين اوليتين للحل ليس من الضروري ان يكونا على جهتين مختلفتين من الجذر، ثم نصل بين النقطتين بخط مستقيم فيقطع محور السينات في نقطة ثالثة x_3 فتكون اقرب الجذر الجذر من x_1 ، x_2 ثم نأخذ المستقيم المار بالنقطتين x_2 ، x_3 فيقطع محور السينات في النقطة x_4 فيكون اقرب للجذر من x_2 ، x_3 وهكذا نستمر الى ان نصل الى شرط التوقف.

Ex1:- Use secant method to find the root of the function $f(x) = x^3 - 4$ to 5 decimal places.

Sol/ Since the secant method is given using the iterative equation in (1). Starting with an initial value $x_0 = 1$ and $x_1 = 1.5$, using (1) we can compute

$$x_2 = 1.5 - (0.625)\left[\frac{1.5 - 1}{-0.625 - (-3)}\right] = 1.63158$$

The continued iterations can be computed as shown in table (1) which shows a stop at iteration no. 5 since the error is $x_5 - x_4 < 10^{-5}$ resulting in a root of

$x^* = 1.58740$.

iteration	x_{n-1}	x_n	x_{n+1} using 1	$f(x_{n+1})$	$x_{n+1} - x_n$
1	$x_0 = 1$	$x_1 = 1.5$	1.63158	0.34335	0.13158
2	1.5	1.63158	1.58493	-0.01865	-0.04665
3	1.63158	1.58493	1.58733	-0.00054	0.0024
4	1.58493	1.58733	1.58740	-7.95238×10^{-6}	0.00007
5	1.58733	1.58740	1.58740	-7.95238×10^{-6}	$< 10^{-5}$

Stop criteria:- $|f(x_{n+1})| < \epsilon$ or $|x_{n+1} - x_n| < \epsilon$

ملاحظة/ نختار نقطتين بحيث ان المشتقة لاتساوي صفر ولا قريب من الصفر اي ميل
المستقيم المار بهما لا يساوي صفر ولا قريب منه لانه في هذه الحالة يكون التقارب بطيء
او لا يوجد تقارب اصلاً

$$x_{n+2} = x_{n+1} - \frac{y_{n+1}}{\underbrace{\frac{y_{n+1} - y_n}{x_{n+1} - x_n}}_{m}}$$

EX2/ Solve the equation $e^{-x} = 3\log(x)$ to $\underline{5}$ decimal Places using Secant method, assuming initial guess $x_0 = 1$ and $x_1 = 2$

Sol/ let $f(x) = e^{-x} - 3\log(x)$ to solve the given, it is now equivalent to find the root of $f(x)$. using (1) we can compute

$$x_2 = 2 - (-0.76775)\left[\frac{2-1}{-0.76775-(0.36788)}\right] = 1.32394.$$

The continued iterations can be computed as shown in blew table which shown a stop at iteration no.5 since the error is $x_5 - x_4 < 10^{-5}$ resulting in a root of

$x^* = 1.24682.$

iteration	x_{n-1}	x_n	x_{n+1} using 1	$f(x_{n+1})$	$x_{n+1} - x_n$
1	$x_0 = 1$	$x_1 = 2$	1.32394	-0.09952	-0.67606
2	2	1.32394	1.22325	0.03173	-0.10069
3	1.32394	1.22325	1.24759	-1.01955×10^{-3}	0.02434
4	1.22325	1.24759	1.24683	-7.27178×10^{-6}	-0.00076
5	1.24759	1.24683	1.246823	6.65199×10^{-6}	$< 10^{-5}$

-0.0000069

$f(1) = 0.367\cancel{88}944$

باستخدام التقريب ٥ مراتب

$f(1) = 0.36788$

$f(2) =$

Example: Find a solution for the equation $f(x): x^3-2x^2-5=0$ by secant method if you know that $\in = 0.03$.

Solution: We begin with two points (2,-5) and (1,-6)

$$m=\frac{-5-(-6)}{2-1}=1\neq 0$$

$$x_3=2-\frac{-5}{1}=7\Rightarrow y_3=240 \Longrightarrow (x_3,y_3)=(7{,}240)$$

$$x_4=7-\frac{240(7-2)}{(240+5)}=2.1020\Rightarrow y_3=-4.5491$$

$$\Longrightarrow (x_3,y_3)=(2.1020,-4.5491)$$

x_n	1	2	7	2.1020	2.1931	2.9693	2.6080	2.6788	2.6696
y_n	-6	-5	240	-4.5491	-4.0712	3.5449	-0.8646	0.1290	-0.2279

$|2.6696-2.6788|=0.0092<\in \Longrightarrow x_n=2.6696$

While if we begin with another points (2,-5) and (3,4)

$$m=\frac{4+5}{3-2}=9\neq 0$$

x_n	2	3	2.5556	2.6691	2.6923
y_n	-5	4	-1.3722	-0.2330	0.0185

$|0.0185|<\in \Longrightarrow x_n=2.6923$

Example: Find a solution for the equation $f(x): x-\cos(x)=0$ by secant method if you know that $\in = 0.004$.

$m=\frac{2.5714}{1.5714}=1.6364\neq 0 \Longleftarrow \quad (0,-1),\ (\frac{\pi}{2},1.57079)$

x_n	0	$\frac{\pi}{2}$	0.61101	0.72329	0.7395
y_n	-1	1.57079	-0.2081	-0.0263	0.0007

$|0.0007|<\in \Longrightarrow x_n=0.7395$

Note

ان طريقة القاطع اسرع بالوصول للجذر التقريبي (اقل عدد من التكرارات) والدقة تكون اعلى بالمقارنة مع طريقة التنصيف

Ex:- Find the roote of $f(x) = x^2 - \frac{1}{2}$ with $\epsilon = 0.001$ using Secant Method and Bisection method, Compare the results.

Ex:– Find a root of $f(x)=x^2-3=0$ with $\epsilon=0.001$ using Secant method.

هنا المعادلة اكثر من جذر ولكن المطلوب ايجاد جذر واحد فقط

Sol/

يوجد جذر في الفترة [1,2]

$x_1=1 \to f(x_1)=-2$

$x_2=2 \to f(x_2)=1$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$=\frac{1-(-2)}{2-1}=\frac{3}{1}=3\neq 0$$

نحسب

$$x_3=x_2-\frac{y_2}{\frac{y_2-y_1}{x_2-x_1}}=2-1/3=1.6667$$

$$f(x_3)=y_3=-0.221$$

$$x_4=x_3-\frac{y_3}{\frac{y_3-y_2}{x_3-x_2}}=1.7273 \rightsquigarrow f(x_4)=y_4=-0.01653$$

$$x_5=x_4-\frac{y_4}{\frac{y_4-y_3}{x_4-x_3}}=1.7321$$

$$f(x_5)=0.000318$$

$$|f(x_5)|=|f(1.7341)|<\epsilon=0.001$$

$\Rightarrow$ 1.7321 is approximet root.

Newton-Raphson method

Newton-Raphson method is one of the most popular techniques for finding roots of non-linear equations.

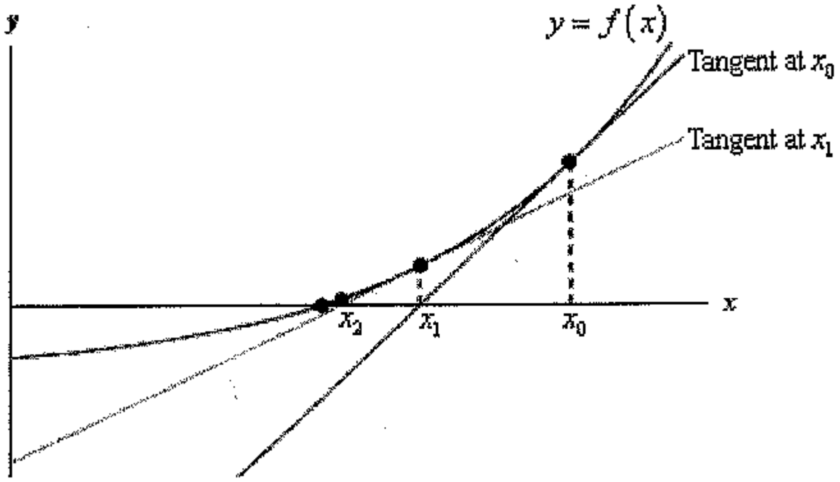


Figure 2.2: sketch of the Newton Raphson method

Newton-Raphson Formula:

Now Suppose that x_0 is a known approximation to a root of the function $y = f(x)$, as shown in Fig. 2.2.
The next approximation, x_2 is taken to be the point where tangent graph of $y = f(x)$ at $x = x_0$ intersects the x—axis.
From Taylor series we have

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + f''(x_0)\frac{(x_1 - x_0)^2}{2!} + f'''(x_0)\frac{(x_1 - x_0)^3}{3!}$$
$$+ \cdots + f^{(n)}(a)\frac{(x_1 - x_0)^n}{n!} + \cdots$$

هذه الفكرة تعود الى تيلور ثم أخذ المشتقة الأولى

نستخدم هذه الطريقة عندما تكون مشتقة الدالة بسيطة من السهل حسابها

والفكرة هي $f \in C^2[a,b]$ وتملك جذر في الفترة $[a,b]$

① نبدأ بقيمة x_0 تقريبية $x_0 = \frac{a+b}{2}$

$f'(x_0) \neq 0$, $|x - x_0|$ صغير جداً

consider x_1 as a root and take only the first two terms as an approximation:

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$
$$(x_1 - x_0) = -\frac{f(x_0)}{f'(x_0)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

So, we can find the new approximation x_1. Now we can repeat the whole process to find an even better approximation.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

we will arrive at the following formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \cdots \tag{2.4}$$

Note that when $f'(x_n) = 0$ the calculation of x_{n+1} fails. This is because the tangent at x_n is horizontal.

Example *Newton's method for calculating the zeros of*

$$f(x) = e^x - x - 2$$

is given by

$$x_{n+1} = x_n - \frac{e^{x_n} - x_n - 2}{e^{x_n} - 1}$$
$$= \frac{e^{x_n}(x_n - 1) + 2}{e^{x_n} - 1}$$

The graph of f, sketched in Fig. 2.3, shows that it has two zeros. It is clear from this graph that x_n converges to the negative root if $x_0 < 0$ and to the positive root if $x_0 > 0$, and that it breaks down if $x_0 = 0$. The results obtained with $x_0 = -10$ and $x_0 = 10$ are listed in next table.

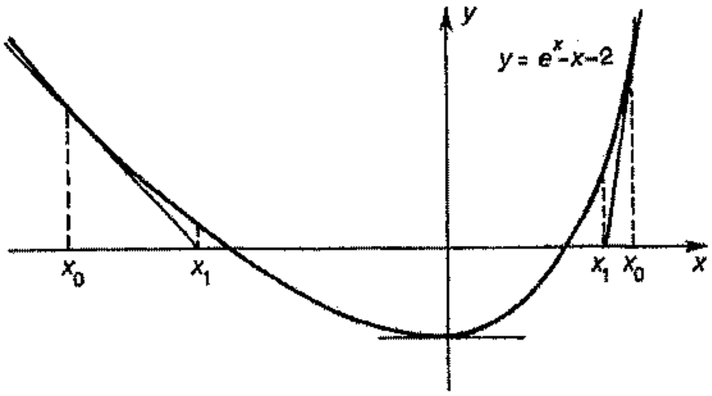


Figure 2.3: sketch of the Newton Raphson method

Sufficient conditions for the convergence of Newton's method are given in the following theorem.

Theorem *If f'' is continuous on an interval $[\alpha - A, \alpha + A]$, then x_n converges to α provided $f'(\alpha) \neq 0$ and x_0 is sufficiently close to α.*

Proof. Comparison of equation

$$x_{n+1} = g(x_n) \qquad n = 0, 1, 2, \cdots$$

and the equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

shows that Newton's method is a fixed point iteration with

$$g(x) = x - \frac{f(x)}{f'(x)}$$

By the quotient rule,

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

let $x = \alpha$ then

$g'(\alpha) < 1$

$$g'(\alpha) = \frac{f(\alpha)f''(\alpha)}{(f'(\alpha))^2}$$

$$\frac{|f(\alpha)f''(\alpha)|}{|(f'(\alpha))^2|} < 1$$

$$\Rightarrow \boxed{|f(\alpha)f''(\alpha)| < |(f'(\alpha))^2|}$$

This implies that $g'(\alpha) = 0$, because $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Hence by the continuity of f'', there exists an interval $I = [\alpha - \delta, \alpha + \delta]$, for some $\delta > 0$, on which $|g'(x)| < 1$. Theorem 2.6 then guarantees convergence provided $x_0 \in I$, i.e. provided x_0 is sufficiently close to α. $\square$

شروط التوقف

STOP CRITERIA

1. $|f(x_{n+1})| < \in.$ $\Rightarrow$ x_{n+1} is approximate root

   Or

2. $|x_{n+1} - x_n| < \in.$ $\Rightarrow$ x_{n+1} is approximate root

ملاحظة: من ملاحظة قيمة $h = |p - p_0|$ نجد ان قيمة المشتقة كلما كانت كبيرة كلما صغرت قيمة h وهذا يعني ان التقارب يكون سريعاً وفعالاً عندما يكون ميل المماس للمنحني في النقاط القريبة من p_0 شاقولياً تقريباً والعكس عندما تكون قيمة المشتقة صغيرة هذا يعني ان h كبيرة وبالتالي يكون التقارب بطيئاً ولا يوجد تقارب لذا لا يستحسن استخدام طريقة نيوتن- رافسون عندما يكون منحني الدالة موازياً لمحور السينات في النقاط القريبة من الجذر وانما نستخدم طريقة القاطع أو تنصيف الفترات.

Example : Find a solution for the equation $f(x): x - e^{-x} = 0$ by Newton- Raphson method if you know that $\in = 9 \times 10^{-4}$.

$x_0 = 0.5$ $\Leftarrow$ *There is a root in the interval (0,1)*

x	0.5	0.5663	0.5671
$f(x): x - e^{-x}$	-0.1065	-0.0013	-0.00005
$f'(x): 1 + e^{-x}$	1.6065	1.5676	

Since $|-.00005| < \in = 0.0009$ $\Rightarrow$ 0.5671 *is approximate root.*

Example: Find the value of $\sqrt{8}$ by Newton- Raphson method if you know that $\in = 8 \times 10^{-4}$.

$x = \sqrt{8} \Rightarrow x^2 = 8 \Rightarrow x^2 - 8 = 0 \Rightarrow f(x): x^2 - 8 = 0$

$x_0 = 2.5$

x	2.5	2.85	2.8285
$f(x): x^2 - 8$	-1.75	0.1225	0.0004
$f'(x): 2x$	5	5.7	

Since $|0.0004| < \in = 0.0008$ $\Rightarrow$ 2.8285 *is approximate root.*

H.W: Find a solution for the equation $f(x): x^3 - x - 1 = 0$ by Newton- Raphson method if you know that $\in = 0.0003$.

EXAMPLE

Apply Newton's Raphson method for $cosx = xe^x$ at $x_0 = 1$ correct to three decimal places.

SOLUTION

$f(x) = cosx - xe^x$

$f'(x) = -sinx - e^x - xe^x$

Using formula $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

at $x_0 = 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.653 \quad (after\ solving)$

$f(x_1) = -0.460\ ;\ f'(x_1) = -3.783$

Similarly

n	x_n	$f(x_n)$	$f'(x_n)$
2	0.531	-0.041	-3.110
3	0.518	-0.001	-3.043
4	0.518	-0.001	-3.043

Hence root is "0.518"

REMARK

1. **If two are more roots are nearly equal, then method is not fastly convergent.**
2. **If root is very near to maximum or minimum value of the function at the point, NR-method fails.**

NEWTON SCHEME OF ITERATION FOR FINDING THE SQUARE ROOT OF POSITION NUMBER

The square root of "N" can be carried out as a root of the equation

$x=\sqrt{N} \Rightarrow x^2 = N \Rightarrow x^2 - N = 0$

Here $f(x) = x^2 - N$; $f(x_n) = x_n^2 - N$

$f'(x) = 2x$; $f'(x_n) = 2x_n$

Using Newton Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\Rightarrow x_{n+1} = x_n - \frac{(x_n^2 - N)}{2x_n}$

$\Rightarrow x_{n+1} = \frac{1}{2}[x_n + \frac{N}{x_n}]$ This is required formula.

QUESTION

Evaluate $\sqrt{12}$ by Newton Raphson formula.

SOLUTION

Let $x=\sqrt{12} \Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$

Here $f(x) = x^2 - 12$; $f'(x) = 2x$; $f''(x) = 2$

X	0	1	2	3	4
F(x)	-12	-11	-8	-3	4

Root lies between 3 and 4 and $x_0 = 4$

Now using formula $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right] \Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{12}{x_n}\right]$ ……………………..(1)

For n=0 $x_1 = \frac{1}{2}\left[x_0 + \frac{12}{x_0}\right] \Rightarrow x_1 = \frac{1}{2}\left[4 + \frac{12}{4}\right]$=3.5

For n=1 $x_2 = \frac{1}{2}\left[x_1 + \frac{12}{x_1}\right] \Rightarrow x_2 = \frac{1}{2}\left[3.5 + \frac{12}{3.5}\right] = 3.4643$

Similarly $x_3 = 3.4641$ *and* $x_4 = 3.4641$

Hence $\sqrt{12} = 3.4641$

NEWTON SCHEME OF ITERATION FOR FINDING THE "pth" ROOT OF POSITION NUMBER "N"

Consider $x = N^{\frac{1}{p}} \quad \Rightarrow \quad x^p = N \Rightarrow x^p - N = 0$

Here $\quad f(x) = x^p - N \quad ; \quad f(x_n) = x_n^p - N$

$\quad f'(x) = px^{p-1} \quad ; \quad f'(x_n) = px_n^{p-1}$

Since by Newton Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \Rightarrow x_{n+1} = x_n - \frac{(x_n^p - N)}{\left(px_n^{p-1}\right)} \quad \Rightarrow x_{n+1} = \frac{1}{px_n^{p-1}}\left[px_n^{p-1+1} - x_n^p + N\right]$$

$$x_{n+1} = \frac{1}{px_n^{p-1}}\left[(p-1)x_n^p + N\right] \Rightarrow x_{n+1} = \frac{1}{p}\left[\frac{(p-1)x_n^p + N}{x_n^{p-1}}\right]$$ Required formula for pth root.

QUESTION

Obtain the cube root of 12 using Newton Raphson iteration.

SOLUTION

Consider $x = 12^{\frac{1}{3}} \quad \Rightarrow \quad x^3 = 12 \Rightarrow x^3 - 12 = 0$

Here $\quad f(x) = x^3 - 12$ and $f'(x) = 3x^2 \quad ; \quad f''(x) = 6x$

For interval

X	0	1	2	3
F(x)	-12	-11	-4	15

Root lies between 2 and 3 and $x_0=3$

Since by Newton Raphson formula for pth root.

$$x_{n+1} = \frac{1}{p}\left[\frac{(p-1)x_n^p + N}{x_n^{p-1}}\right] \Rightarrow x_{n+1} = \frac{1}{3}\left[\frac{(3-1)x_n^3 + 12}{x_n^{3-1}}\right] = \frac{1}{3}\left[\frac{2x_n^3 + 12}{x_n^2}\right]$$

Put n=0 $\quad x_1 = \frac{1}{3}\left[\frac{2x_0^3 + 12}{x_0^2}\right] = \frac{1}{3}\left[\frac{2(3)^3 + 12}{(3)^2}\right] = 2.4444$

Similarly $\quad n = 1$

$x_2 = 2.2990$, $x_3 = 2.2895$, $x_4 = 2.2894 \quad x_5 = 2.2894$

Hence $\sqrt[3]{12} = 2.2894$

2.1 EXERCISE

1. Use the Bisection method and fixed point method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

2. Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1)$ Use the Bisection method and fixed point method on the intervals $[-2, 1.5]$ and $[-1.25, 2.5]$ to find p_3.

3. Use the Bisection method on the solutions accurate to within 10^{-2} for $f(x) = x^3 - 7x^2 + 14x - 6 = 0$ on each intervals: $[0, 1]$, $[1, 3.2]$ and $[3.2, 4]$.

4. Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the Bisection Algorithm. Hint: Consider $f(x) = x^2 - 3$.

5. Use an appropriate fixed point iteration to find the root of

   (a) $x - \cos x = 0$

   (b) $x^2 + \ln x = 0$

   starting in each case with $x_0 = 1$. Stop when $|x_{n+1} - x_n| < 0.5 \times 10^{-2}$.

6. Find the first nine terms of the sequence generated by $x_{n+1} = e^{-x_n}$ starting with $x_0 = 1$.

7. Use Newton's method to find the roots of

   (a) $x - \cos x = 0$

   (b) $x^2 + \ln x = 0$

   starting in each case with $x_0 = 1$. Stop when $|x_{n+1} - x_n| < 10^{-6}$.

8. Find the roots of $x^2 - 3x - 7$ using Newton's method with $\epsilon = 10^{-4}$ or maximum 20 iterations.

Matlab Code 2.12. *Newton Raphson method*

```
% ********  Newton Raphson method  ************
% ****  to find a root of the function f(x) ***
clc
clear
close all
f=@(x) exp(x)-x-2 ; % the function f(x)
fp=@(x) exp(x)-1 ; % the derivative f'(x) of f(x)
xa=-10; % Initial value of first root
xb=10;  % Initial value of second root
r = 'failure';
fprintf('     k          Xa              Xb   \n');
fprintf('    ___      ________        ________  \n');
fprintf('%6.f    %10.8f    %10.8f \n', 0, xa , xb );
for k=1:1:14
     if fp(xa)==0; r
     return
     elseif fp(xb)==0; r
     return
     end
 xa=xa-f(xa)/fp(xa);
 xb=xb-f(xb)/fp(xb);
 fprintf('%6.f    %10.8f    %10.8f \n', k, xa , xb );
end
```

The result as the following table:

```
     k          Xa             Xb
    ___     ___________    __________
     1      -1.99959138    9.00049942
     2      -1.84347236    8.00173312
     3      -1.84140606    7.00474864
    ___     ___________    __________
    13      -1.84140566    1.14619325
    14      -1.84140566    1.14619322
>>
```

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Newton- Raphson for polynomials (Synthetic Division method)

This method is appropriate to find the roots of equation f (x) = 0 when f is polynomial. Let x^* be an approximate value of f and let f be a polynomial of degree m.

تعتبر طريقة القسمة التركيبية ملائمة لايجاد جذور المعادلة $f(x) = 0$ عندما تكون f متعددة حدود .لتكن x^* هي قيمة تقريبية للدالة f ولتكن f متعددة حدود من الدرجة m

Since we often need to evaluate the value of f(x) and sometimes also of $f'(x)$ as in **Newton- Raphson** method

$$f(x) = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots + a_{m-1}x + a_m \qquad \ldots(1)$$

$$\frac{f(x)}{x - x^*} = b_0x^{m-1} + b_1x^{m-2} + \cdots + b_{m-1} + \frac{b_m}{x - x^*}$$

وبصيغة اخرى فان :

$$f(x) = (x - x^*)(b_0x^{m-1} + b_1x^{m-2} + \cdots + b_{m-1}) + b_m \qquad \ldots(2)$$

وبتعويض x^* في المعادلة (2) نحصل على

$$f(x^*) = b_m$$

To find this value and other values of b_k in equations (1) and (2), we get

ولايجاد هذه القيمة والقيم الاخرى لل b_k نقارن بين معاملات القوى المتساوية الى x في المعادلتين (1) و (2) نحصل على

$$a_0 = b_0$$

$$a_k = b_k - x^*b_{k-1} \qquad ,k = 1,2,\ldots,m$$

To find b_m

$$b_0 = a_0$$

$$b_k = a_k + x^*b_{k-1} \qquad ,k = 1,2,\ldots,m$$

وهكذا نحصل على قيم المعاملات بالتعاقب واخرها b_m وهي قيمة متعددة الحدود في النقطة x^*.

Now to find the $f'(x^*)$ which is also a polynomial of the form

ولحساب قيمة $f'(x^*)$ نكتب المعادلة (2) بالشكل

$$f(x) = (x - x^*)g(x) + b_m$$

$$f'(x) = (x - x^*)g'(x) + g(x)$$

عند تعويض x^* ينتج

$$f'(x^*) = g(x^*)$$

g متعددة حدود من الدرجة m-1 لذا نستخدم نفس الاسلوب في حساب $g(x^*)$ بايجاد قيمة c_{m-1} حيث :

$$c_0 = b_0$$

$$c_k = b_k + x^* c_{k-1} \qquad , k = 1,2,\dots,m-1$$

Synthetic division proceeds as follows:

لتطبيق طريقة نيوتن – رافسون لمتعددات الحدود نتبع ما يلي :

١- نحدد الفترة التي يقع بها الجذر ونختار القيمة التقريبية الاولية منتصف هذه الفترة.

٢- نحسب القيمة التقريبية الافضل للجذر من علاقة نيوتن – رافسون

$$x_1 = x_0 - \frac{b_m}{c_{m-1}}$$

حيث ان

$$b_m = f(x_0) \text{ و } c_{m-1} = f'(x_0)$$

٣- نحسب قيمة b_m و c_{m-1} من القوانين

$$c_0 = b_0 = a_0$$

$$b_k = a_k + x_0 b_{k-1} \qquad , k = 1,2,\dots,m$$

$$c_k = a_k + x_0 c_{k-1} \qquad , k = 1,2,\dots,m-1$$

٤- نكرر الخطوة ٢ و٣ الى ان نصل الى شرط التوقف

$$|x_{n+1} - x_n| < \epsilon \qquad or \qquad |b_m| < \epsilon$$

	a_0	a_1	a_2	...	a_{m-1}	a_m
x_0		$x_0 b_0$	$x_0 b_1$		$x_0 b_{m-2}$	$x_0 b_{m-1}$
	b_0	b_1	b_2	...	b_{m-1}	$\boldsymbol{b_m = f(x_0)}$
x_0		$x_0 c_0$	$x_0 c_1$		$x_0 c_{m-2}$	
	c_0	c_1	c_2	...	$\boldsymbol{c_{m-1} = f'(x_0)}$	

Example1: Find the root of the equation $f(x): x^4 - x^2 - 10 = 0$ by using synthetic division method if you know that $\in = 0.07$.

$x_0 = 1.5 \Leftarrow There\ is\ a\ root\ in\ interval\ (1,2)$

	1	0	−1	0	−10
1.5		1.5	2.25	1.875	2.8125
	1	1.5	1.25	1.875	$\boxed{-7.1875}$ $b_m = \text{f}(1.5)$
1.5		1.5	4.5	8.625	
	1	3	5.75	$\boxed{10.5}$ $c_{m-1} = f'(1.5)$	

$$x_1 = x_0 - \frac{b_m}{c_{m-1}}$$

$$x_1 = 1.5 - \frac{-7.1875}{10.5} = 2.1845$$

	1	0	−1	0	−10
2.1845		2.1845	4.7720	8.24002	18.0003
	1	2.1845	3.7720	8.24002	$\boxed{8.0003}$ $b_m = f(2.1845)$
2.1845		2.1845	9.5441	29.0891	
	1	4.369	13.3161	$\boxed{37.3291}$ $c_{m-1} = f'(2.1845)$	

$$x_2 = 2.1845 - \frac{8.0003}{37.3291} = 1.9702$$

	1	0	−1	0	−10
1.9702		1.9702	3.8817	5.6775	11.1858
	1	1.9702	2.8817	5.6775	$\boxed{1.1858}$ $b_m = f(1.9702)$

1.9702		1.9702	7.7634	20.9729
	1	3.9404	10.6451	$\boxed{26.6504}$ $c_{m-1} = f'(1.9702)$

$$x_3 = 1.9702 - \frac{1.1858}{26.6504} = 1.9257$$

$$|x_3 - x_2| = 0.0445 < \in$$

Example 2: Find $f(2), f'(2), f''(2)$ for the equation $f(x): 5x^4 - 2x^3 + x - 1 = 0$ by using synthetic division method.

Sol: $b_m = f(2)$ $\quad\quad$ $c_{m-1} = f'(2)$

	5	-2	0	1	-1
2		10	16	32	66
	5	8	16	33	$\boxed{65}$ $b_m = f(2)$
2		10	36	104	
	5	18	52	$\boxed{137}$ $c_{m-1} = f'(2)$	

$f'(x) = 20x^3 - 6x^2 + 1$ $\quad\quad$ ولحساب $f''(2)$ نجد المشتقة للدالة

ونطبق القسمة التركيبية على المشتقة

	20	-6	0	1
2		40	68	136
	20	34	68	$\boxed{137}$ $f'(2)$
2		40	148	104
	20	74	$\boxed{216}$ $c_{m-1} = f''(2)$	

Exercise: Find the root of the equation $f(x): x^3 - x^2 + 2x + 5 = 0$ by using synthetic division method if you know that $\epsilon = 0.003$, start the solution from $x_0 = -1$.

Exercise: Find the root of the equation $f(x): x^3 - 11x^2 + 39x - 45 = 0$ by using synthetic division method if you know that $\epsilon = 0.001$, start the solution from $x_0 = 3$.

Exercise: Find the root of the equation $f(x): x^4 - 10x^3 + 32x^2 - 38x + 15 = 0$ by using synthetic division method, start the solution from $x_0 = 0$.

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Chapter 3 / Numerical Solutions for System of Non-Linear Equations

If we have several equations and several unknowns and must find those values of unknowns which satisfy all the equations at the same time.

اذا كان لدينا اكثر من معادله لاكثر من متغير والمطلوب ايجاد الجذور فسوف نستخدم الطرق السابقه بعد تعميمها لحل منظومه معادلات انيه غير خطيه . سنقتصر على حل منومه تتكون من معادلتين بمتغيرين فقط

(Fixed Point Iteration Method)

Let $f(x,y)=0$
$g(x,y)=0$

ثم نجد الفترة التي يقع بها الجذر بالنسبة لل x وفترة بالنسبة لل y ونبدأ الحل بنقطة تقريبية اولية تمثل منتصف الفترات السابقة ولتكن $(x_0\,,y_0)$ ثم نعيد كتابة المعادلات بالصيغة التالية

$x=F(x,y)$
$y=G(x,y)$

ثم نختبر شرط التقارب حيث ان شرط التقارب

$$\left|\frac{\partial F}{\partial \mathrm{x}}\right|_{(\mathrm{x_0,y_0})}+\left|\frac{\partial G}{\partial \mathrm{x}}\right|_{(\mathrm{x_0,y_0})}<1 \qquad and \qquad \left|\frac{\partial F}{\partial \mathrm{y}}\right|_{(\mathrm{x_0,y_0})}+\left|\frac{\partial G}{\partial \mathrm{y}}\right|_{(\mathrm{x_0,y_0})}<1$$

وبعد ان يتحقق شرط التقارب نبدأ الحل بالصيغة التكرارية وكما يلي:

$$x_{n+1}=F(x_n,y_n)$$
$$y_{n+1}=G(x_{n+1},y_n) \qquad n=0,1,2,\dots$$

ملاحظة : عوضنا في الدالة G النقطة (x_{n+1},y_n) لتعجيل التقارب.

وهكذا نستمر الى نصل الى شرط التوقف حيث ان شرط التوقف هو :

$$|x_{n+1}-x_n|<\in \qquad and \qquad |y_{n+1}-y_n|<\in$$

وبنفس الطريقة اذا كانت لدينا منظومة معادلات مكونة من m من المعادلات وm من المتغيرات.

Example: Find the solution for the system of equations by fixed point iteration method

$\cosh(x+y)-4x=0$

$e^{y-x}-2y=0$

if you know that $\in=0.001$.

Solution : $(0,0)$, $(1,1)$ يوجد للمنظومة جذر بين النقطتين

$$f: \quad + \quad -$$

$$g: \quad + \quad -$$

$\therefore\ (x_0, y_0) = (0.5, 0.5)$

$$x = F(x,y)$$
$$y = G(x,y)$$

$$x = \frac{\cosh(x+y)}{4} = F(x,y)$$

$$y = \frac{e^{y-x}}{2} = G(x,y)$$

الان نحقق شرط التقارب

$$\left|\frac{\partial F}{\partial x}\right|_{(0.5,0.5)} + \left|\frac{\partial G}{\partial x}\right|_{(0.5,0.5)} < 1 \quad and \quad \left|\frac{\partial F}{\partial y}\right|_{(0.5,0.5)} + \left|\frac{\partial G}{\partial y}\right|_{(0.5,0.5)} < 1$$

$$\left|\frac{\partial F}{\partial x}\right|_{(0.5,0.5)} + \left|\frac{\partial G}{\partial x}\right|_{(0.5,0.5)} = \left|\frac{\sinh(x+y)}{4}\right|_{(0.5,0.5)} + \left|\frac{-e^{y-x}}{2}\right|_{(0.5,0.5)}$$

$$= 0.2938+0.5=0.7938 <1$$

And

$$\left|\frac{\partial F}{\partial y}\right|_{(0.5,0.5)} + \left|\frac{\partial G}{\partial y}\right|_{(0.5,0.5)} = \left|\frac{\sinh(x+y)}{4}\right|_{(0.5,0.5)} + \left|\frac{e^{y-x}}{2}\right|_{(0.5,0.5)}$$

$$= 0.2938+0.5=0.7938 <1$$

Let $x_{n+1} = F(x_n, y_n)$

$y_{n+1} = G(x_{n+1}, y_n) \qquad n = 0,1,2,\ldots$

n	$x_{n+1} = \frac{\cosh(x_n + y_n)}{4}$	$y_{n+1} = \frac{e^{y_n - x_{n+1}}}{2}$
0	0.5	0.5
1	0.3858	0.5605
2	0.3705	0.6046
3	0.3786	0.6268
4	0.3874	0.6353
5	0.3925	0.6373
6	0.3947	0.63726
7	0.3954	0.6368

$|x_7 - x_6| = 0.0007 < \in$ *and* $|y_7 - y_6| = 0.0004 < \in$

$\therefore (x_7, y_7)$ is approximate root

Exercise 1: Find the solution for the system of equations by fixed point iteration method

$x^2 + y^2 - x = 0$

$x^2 - y^2 - y = 0$

if you know that $\in = 0.001$, starting with $(x_0, y_0) = (0.1, 0.1)$

Exercise 2: Find the solution for the system of equations
by fixed point iteration method

$x^2 + y^2 - 25 = 0$

$x^2 - y^2 - 7 = 0$

if you know that $\in = 0.007$, starting with $(x_0, y_0) = (3,7)$

Ex/ Solve the system of nonlinear equation using fixed Point iteration method.

$$x+3\log_{10} x - y^2 = 0 \rightarrow (1) \quad \text{with } \epsilon = 0.009$$

$$2x^2 - xy - 5x + 1 = 0 \rightarrow (2)$$

	(3,2)	(4,3)
f	−	+
g	−	+

هنا الدوال تختلف الاشارة

$\log_a x = y$

$a^y = x$

Sol/ let $f(3,2)$, $f(4,3)$

$$x_0 = \frac{3+4}{2} = 3.5$$

$$y_0 = \frac{2+3}{2} = 2.5$$

let $x = F(x,y)$ and $y = G(x,y)$

From equation (2)

$$2x^2 - xy - 5x + 1 = 0$$

$$x^2 = \frac{xy+5x-1}{2} = \frac{x(y+5)-1}{2}$$

$$\Rightarrow \boxed{x = \sqrt{\frac{x(y+5)-1}{2}}}$$

From equation (1)

$$y^2 = x + 3\log_{10} x \Rightarrow y = (x+3\log_{10} x)^{1/2}$$

شرط التقارب

$$\left|\frac{dF}{dx}\right|_{(3.5,2.5)} + \left|\frac{dG}{dx}\right| < 1 \quad \text{and}$$

$$\left|\frac{dF}{dy}\right|_{(3.5,2.5)} + \left|\frac{dG}{dy}\right|_{(3.5,2.5)} < 1$$

$\log_{10} x = \log x$

$\log_e x = \ln x$

$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$

$\log_a x^r = r \log_a x$

$$\left|\frac{dF}{dx}\right|_{(3.5,2.5)} = \frac{1}{2\sqrt{\frac{x(y+5)-1}{2}}} \cdot \frac{y+5}{2}$$

$$\left|\frac{dF}{dx}\right|_{(3.5,2.5)} = \left|\frac{y+5}{4*\sqrt{\frac{x(y+5)-1}{2}}}\right| = \frac{7.5}{4*\sqrt{\frac{3.5*(7.5)-1}{2}}}$$

$$= 0.5277$$

$$\left|\frac{dG}{dx}\right|_{(3.5,2.5)} = \left|\frac{1+\frac{3}{x*\ln(10)}}{2*\sqrt{x+3\log_{10} x}}\right|_{(3.5,2.5)} =$$

$$\frac{1+\frac{3}{3.5*\ln(10)}}{2*\sqrt{3.5+3*\log_{10}(3.5)}} = 0.3029$$

$$0.5277+0.3029 = 0.8306 < 1$$

$$\left|\frac{dF}{dy}\right| + \left|\frac{dG}{dy}\right| < 1$$ (واجب)

n	$x_{n+1} = \left(\frac{x_n(y_n+5)-1}{2}\right)^{1/2}$	$y_{n+1} = \left(x_{n+1}+3\log_{10} x_{n+1}\right)^{1/2}$
0	3.5	2.5
1	3.5532	2.2815
2	3.5265	2.2734
3	3.5107	2.2687
4	3.5103	2.2658

علي

The following Matlab code is to solve System of Two Non Linear Equations By Fixed Point Method:

```
% ****************************************************
% *******   find a root of a System ***********
% **   of Two nonlinear  equations f and g    **
% ******          By Fixed Point Method        **
% ****************************************************
clc
clear
close all
% Define the functions f and g
% and their partial derivative
f=@(x,y) (x^3+y^3+3)/6 ; % the function f(x,y)
g=@(x,y) (x^3-y^3+2)/6 ;   % the function g(x,y)
fx=@(x,y) x*x*0.5;       % partial derivative of f to x
fy=@(x,y) y*y*0.5;       % partial derivative of f to y
gx=@(x,y) x*x*0.5 ;        % partial derivative of f to x
gy=@(x,y) -y*y*0.5;      % partial derivative of f to y
a=0.5; b=0.5;  % Initial root value
 fprintf(' n         Xn             Yn   \n')
 fprintf('%2.0f      %2.8f         %2.8f \n', 0 ,a,b)
for k=1:1:8
     w1=abs(fx(a,b)+fy(a,b));
     w2=abs(gx(a,b)+gy(a,b));
      if w1 > 1  ; break  ; end
      if w2 > 1  ; break  ; end
     a=f(a,b);
     b=g(a,b)  ;
   fprintf('%2.0f      %2.8f          %2.8f \n', k ,a,b)
end
```

The result as the following table:

```
n        Xn              Yn
0     0.50000000      0.50000000
1     0.54166667      0.33898775
2     0.53298008      0.35207474
3     0.53250741      0.35122633
4     0.53238788      0.35126185
=============================
8     0.53237038      0.35125745
```

Numerical Solutions for System of Non-Linear Equations

(Newton- Raphson Method)

$$f(x,y) = 0$$
$$g(x,y) = 0$$

نبدأ الحل بنقطة تقريبية اولية ولتكن (x_0 , y_0)

ليكن λ الجذر المضبوط لـ x و μ هو الجذر المضبوط لـ y فان هذا يعني $\lambda = x_0 + h$ و $\mu = y_0 + k$ اي انه

$f(\lambda,\mu) = 0$, $g(\lambda,\mu) = 0$ ثم نستخدم توسيع تايلر لكل من f و g حول النقطة الابتدائية (x_0 , y_0)

$$0 = f(x_0 ,y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + k\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2) + O(k^2)$$

$$0 = g(x_0 ,y_0) + h\left(\frac{\partial g}{\partial x}\right)_0 + k\left(\frac{\partial g}{\partial y}\right)_0 + O(h^2) + O(k^2)$$

ونحل المعادلتين اعلاه انياً نحصل على قيم h و k كما يلي:

$$\begin{bmatrix} h \\ \mathrm{k} \end{bmatrix} = -\begin{bmatrix} \left(\frac{\partial f}{\partial x}\right)_0 & \left(\frac{\partial f}{\partial y}\right)_0 \\ \left(\frac{\partial g}{\partial x}\right)_0 & \left(\frac{\partial g}{\partial y}\right)_0 \end{bmatrix}^{-1} \begin{bmatrix} f(x_0 ,y_0) \\ g(x_0 ,y_0) \end{bmatrix}$$

$$x_1 = x_0 + h \quad , \quad y_1 = y_0 + h$$

ثم نكرر العملية على النقطة (x_1 , y_1) وهكذا نستمر الى ان نصل الى شرط التوقف حيث ان شرط التوقف هو:

$$|x_{n+1} - x_n| < \in \qquad and \qquad |y_{n+1} - y_n| < \in$$

ملاحظة: اذا كان المحدد يساوي صفر اي ان النظام ليس له معكوس اذاً لا يحل بطريقة نيوتن رافسون وانما بالطريقة التكرارية.

Example: Find the solution for the system of equations by Newton – Raphson method

$$3x^2 - y^2 = 0$$

$$3xy^2 - x^3 - 1 = 0$$

if you know that $\in = 0.009$, starting with $(x_0, y_0) = (0.5, 0.75)$.

$f(0.5 ,0.75) = 0.1875 \qquad g(0.5 ,0.75) = -0.2813$

$$\left(\frac{\partial f}{\partial x}\right)_0 = 6x = 3 \qquad \left(\frac{\partial f}{\partial y}\right)_0 = -2y = -1.5$$

$$\left(\frac{\partial g}{\partial x}\right)_0 = 3y^2 - 3x^2 = 0.9375 \qquad \left(\frac{\partial g}{\partial y}\right)_0 = 6xy = 2.25$$

$$\begin{bmatrix} h \\ \mathrm{k} \end{bmatrix} = -\begin{bmatrix} 3 & -1.5 \\ 0.9375 & 2.25 \end{bmatrix}^{-1} \begin{bmatrix} 0.1875 \\ -0.2813 \end{bmatrix}$$

$$= \frac{-1}{6.75 + 1.4063} \begin{bmatrix} 2.25 & 1.5 \\ -0.9375 & 3 \end{bmatrix} \begin{bmatrix} 0.1875 \\ -0.2813 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2759 & -0.1839 \\ 0.11494 & -0.3678 \end{bmatrix} \begin{bmatrix} 0.1875 \\ -0.2813 \end{bmatrix} = \begin{bmatrix} -0.00000018 \\ 0.12501 \end{bmatrix}$$

$$x_1 = x_0 + h = 0.5 + 1 \times 10^{-7} = 0.499998$$

$$y_1 = y_0 + k = 0.75 + 0.12501 = 0.87501$$

f(x1,y1)

$$\begin{bmatrix} h \\ \mathrm{k} \end{bmatrix} = -\begin{bmatrix} 2.99994 & -1.75 \\ 1.54695 & 2.62497 \end{bmatrix}^{-1} \begin{bmatrix} -0.0157 \\ 0.0234 \end{bmatrix} = \begin{bmatrix} 0.00002 \\ -0.00892 \end{bmatrix}$$

g(x1,y1)

$$x_2 = x_1 + h = 0.4999982 + 0.00002 = 0.50001$$

$$y_2 = y_1 + k = 0.87501 - 0.00892 = 0.8661$$

$$|x_2 - x_1| = 0.000012 < \epsilon \quad \text{and} \quad |y_2 - y_1| = 0.008 < \epsilon$$

$$\therefore\ (x_2, y_2) = (0.50001, 0.8661) \quad \textit{is approximate root}$$

Remark:- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ex1/ Solve the system of nonlinear equations using Newton Raphson method.

$$x + 3\log_{10} x - y^2 = 0$$
$$2x^2 - xy - 5x + 1 = 0$$

$(4,3)$, $(3,2)$ الجذر موجود في الفترة

Sol/ $X_0 = (3.5, 2.5)$

$$F(x,y) = x + 3\log_{10} x - y^2$$

$$G(x,y) = 2x^2 - xy - 5x + 1$$

$$\left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} = 1 + \frac{3}{x\ln 10} = 1 + \frac{3}{3.5 \cdot \ln 10} = \boxed{1.3723}$$

$$\left.\frac{\partial F}{\partial y}\right|_{x_0,y_0} = -2y\Big|_{(3.5,2.5)} = -5$$

$$\left.\frac{\partial G}{\partial x}\right|_{(x_0,y_0)} = 4x - y - 5 = 6.5$$

$$\left.\frac{\partial G}{\partial y}\right|_{(x_0,y_0)} = -x\Big|_{(3.5,2.5)} = -3.5$$

$$F(3.5, 2.5) = -1.1178$$

$$G(3.5, 2.5) = -0.75$$

$$\begin{bmatrix} h_1 \\ k_1 \end{bmatrix} = -\begin{bmatrix} 1.3723 & -5 \\ 6.5 & -3.5 \end{bmatrix}^{-1}\begin{bmatrix} -1.1178 \\ -0.75 \end{bmatrix}$$

$h_1 = -0.0059$

$k_1 = -0.2252$

$$(x_1, y_1) = (x_0 + h_1, y_0 + k_1) = (3.5 - 0.0059, 2.5 - 0.2252)$$
$$= (3.4941, 2.2748)$$

$$(x_2, y_2) = (x_1 + h_2, y_1 + k_2) = (3.49 + h_2, 2.27 + k_2)$$

$$\begin{bmatrix} h_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} -0.1360 & 0.1771 \\ -0.2608 & 0.0534 \end{bmatrix} \begin{bmatrix} F(x_1, y_1) \\ G(x_1, y_1) \end{bmatrix}$$

$F(x_1, y_1) = -0.0507$ $\quad (h_2, k_2) = (-0.0067, -0.0132)$

$G(x_1, y_1) = -0.0013$ $\quad (x_2, y_2) = (3.4875, 2.2617)$

Also, $(x_3, y_3) = (x_2 + h_3, y_2 + k_3) = (3.4875 + h_3, 2.2617 + k_3)$

$$\begin{bmatrix} h_3 \\ k_3 \end{bmatrix} = - \begin{bmatrix} -0.1370 & 0.1776 \\ -0.2627 & 0.0539 \end{bmatrix} \begin{bmatrix} F(x_2, y_2) \\ G(x_2, y_2) \end{bmatrix}$$

$F(x_2, y_2) = -1.755e-004$

$G(x_2, y_2) = 1.3000e-006$

$$\begin{bmatrix} h_3 \\ k_3 \end{bmatrix} = 1.0e-004 * \begin{bmatrix} -0.2426 \\ -0.4617 \end{bmatrix}$$

$(x_3, y_3) = (3.4874, 2.2616)$ ملاحظات

$|x_3 - x_2| = |3.4874 - 3.4875| = \underline{0.0001}$

$|y_3 - y_2| = |2.2616 - 2.2617| = \underline{0.0001}$

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Ex2/ Solve the systems of nonlinear equations using Newton-Raphson method.

$$f_1(x,y) = x^2+y^2-50=0$$

$$f_2(x,y) = xy-25=0 \qquad (x_0,y_0)=(2,1),\ \epsilon=0.02$$

Sol/ $\left.\dfrac{\partial f_1}{\partial x}\right|_{(2,1)} = 2x = 4 \qquad \dfrac{\partial f_1}{\partial y} = \left. 2y\right|_{(2,1)} = 2$

$\left.\dfrac{\partial f_2}{\partial x}\right|_{(2,1)} = y| = 1 \qquad \dfrac{\partial f_2}{\partial y} = \left. x\right|_{(2,1)} = 2$

$f_1(x_0,y_0) = -45$

$f_2(x_0,y_0) = -23$

$$\begin{bmatrix} h_1 \\ k_1 \end{bmatrix} = -\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -45 \\ -23 \end{bmatrix}$$

$h_1 = 7.33$, $k_1 = 7.83 \rightarrow x_1 = 9.33$, $y_1 = 8.83$

$\Rightarrow (x_2,y_2) = (x_1+h_2, y_1+k_2) = (9.33+h_2, 8.83+k_2)$

$$\begin{bmatrix} h_2 \\ k_2 \end{bmatrix} = -\begin{bmatrix} 18.66 & 17.66 \\ 8.83 & 9.33 \end{bmatrix} \begin{bmatrix} 115.1389 \\ 57.444 \end{bmatrix}$$

$h_2 = -3.2905$, $k_2 = -3.0405$

$(x_2,y_2) = (6.0428, 5.7928)$

وبنفس الاسلوب نجد

$(x_3,y_3) = (5.1337, 5.0087)$

$(x_4,y_4) = (5.0317, 4.9692)$

$(x_5,y_5) = (5.0156, 4.984)$

$$|x_5 - x_4| = |5.0156 - 5.0317| = 0.016 < 0.02 = \epsilon$$

and

$$|y_5 - y_4| = |4.9844 - 4.9692| = 0.0152 < 0.02 = \epsilon$$

Ex/ Solve the system of nonlinear equations using Newton-Raphson method and fixed Point method.

(1) $f(x,y) = x^3 + y^3 - 6x + 3 = 0$
$g(x,y) = x^3 - y^3 - 6y + 2 = 0$ $\quad (x_0, y_0) = (0,0)$

(2) $x = \sin y$
$y = \cos x$

using newton-Raphson method and fixed Point method with $(x_0, y_0) = (1,1)$

(3) $f(x,y) = x^2 + y^2 - 4 = 0$
$g(x,y) = 2x - y^2 = 0$

with $(x_0, y_0) = (1,1)$ using newton-Raphson method

The following Matlab code is for System of two Non Linear Equations: Matlab

```
% ***********************************************
% *******  find a root of a System ***********
% **   of Two nonlinear  equations f and g    **
% ***********************************************
clc
clear
close all
% Define the functions f and g
% and their partial derivative
f=@(x,y) x^2+y^2-4 ; % the function f(x,y)
g=@(x,y) 2*x-y^2 ;   % the function g(x,y)
fx=@(x,y) 2*x;       % partial derivative of f to x
fy=@(x,y) 2*y;       % partial derivative of f to y
gx=@(x,y) 2 ;        % partial derivative of f to x
gy=@(x,y) -2*y;      % partial derivative of f to y
a=1; b=1;  % Initial root value
 fprintf(' n        Xn              Yn   \n')
for k=1:1:5
 X=[a;b];
 xn(k)=a;yn(k)=b;
 F=[f(a,b);g(a,b)];
 J=[fx(a,b),fy(a,b);gx(a,b),gy(a,b)]; % the Jacobian matrix
 X=X-inv(J)*F;
 a=X(1);
 b=X(2);
 fprintf('%2.0f     %2.6f        %2.6f  \n', k  ,a,b)
end
```

Matlab The result as the following table:

```
 n       Xn          Yn
 1    1.250000    1.750000
 2    1.236111    1.581349
 3    1.236068    1.572329
 4    1.236068    1.572303
 5    1.236068    1.572303
>>
```

Chapter four

Dr. Huda and Dradil

THE SOLUTION OF LINEAR SYSTEM OF EQUATIONS

A system of "m" linear equations in "n" unknowns "$x_1, x_2, x_3, \dots\dots\dots, x_n$" is a set of the equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots\dots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots\dots a_{2n}x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots\dots\dots a_{mn}x_n = b_m$$

Where the coefficients "a_{ik}" and "b_i" are given numbers.

The system is said to be homogeneous if all the "b_i" are zero. Otherwise it is said to be non-homogeneous.

SOLUTION OF LINEAR SYSTEM EQUATIONS

A solution of system is a set of numbers "$x_1, x_2, x_3, \dots\dots\dots, x_n$" which satisfy all the "m" equations.

PIVOTING: Changing the order of equations is called pivoting.

We are interested in following types of Pivoting

1. PARTIAL PIVOTING
2. TOTAL PIVOTING

PARTIAL PIVOTING

In partial pivoting we interchange rows where pivotal element is zero.

In Partial Pivoting if the pivotal coefficient "a_{ii}" happens to be zero or near to zero, the i^{th} column elements are searched for the numerically largest element. Let the j^{th} row ($j>i$) contains this element, then we interchange the "i^{th}" equation with the "j^{th}" equation and proceed for elimination. This process is continued whenever pivotal coefficients become zero during elimination.

TOTAL PIVOTING

In Full (complete, total) pivoting we interchange rows as well as column.

In Total Pivoting we look for an absolutely largest coefficient in the entire system and start the elimination with the corresponding variable, using this coefficient as the pivotal coefficient (may change row and column). Similarly, in the further steps. It is more complicated than Partial Pivoting. Partial Pivoting is preferred for hand calculation.

Why is Pivoting important?

Because Pivoting made the difference between non-sense and a perfect result.

PIVOTAL COEFFICIENT

For elimination methods (Guass's Elimination, Guass's Jordan) the coefficient of the first unknown in the first equation is called Pivotal Coefficient.

BACK SUBSTITUTION

The analogous algorithm for upper triangular system "Ax=b" of the form

$$\begin{pmatrix} a_{11} & a_{12} \dots\dots\dots a_{1n} \\ 0 & a_{22} \dots\dots\dots a_{2n} \\ \vdots & \vdots \qquad\qquad \vdots \\ 0 & 0 \quad \cdots\cdots \quad a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$ **is called Back Substitution.**

The solution "x_i" is computed by $x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$ $\quad ; i = 1, 2, 3, \dots\dots\dots n$

FORWARD SUBSTITUTION

The analogous algorithm for lower triangular system "Lx=b" of the form

$$\begin{pmatrix} l_{11} & 0 \quad \dots\dots 0 \\ l_{21} & l_{22} \dots\dots\dots 0 \\ \vdots & \vdots \qquad\qquad \vdots \\ l_{n1} & l_{n2} \cdots\cdots \quad l_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$ **Is called Forward Substitution.**

The solution "x_i" is computed by $x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} x_j}{l_{ii}}$ $\quad ; i = 1, 2, 3, \dots\dots\dots n$

Chapter Four

THINGS TO REMEMBER

Let the system $AX = B$ is given

- If $B \neq 0$ then system is called non homogenous system of linear equation.
- If $B = 0$ then $AX = 0$ then system is called homogenous system of linear equation.
- If the system $AX = B$ has solution then this system is called consistent.
- If the system $AX = B$ has no solution then this system is called inconsistent.

RANK OF A MATRIX رتبة المصفوفة تساوي عدد الصفوف غير الصفرية في السلّمي

The rank of a matrix 'A' is equal to the number of non – zero rows in its echelon form or the order of I_r in the conical form of A.

رتبه ترتيب المصفوفة

KEEP IN MIND

- TYPE I: when number of equations is equal to the number of variables and the system $AX = B$ is non – homogeneous then unique solution of the system exists if matrix 'A' is non- singular after applying row operation.
- TYPE II: when number of equations is not equal (may be equal) to the number of variables and the system $AX = B$ is non – homogeneous then system has a solution if $rank\ A = rankA_b$
- TYPE III: a system of 'm' homogeneous linear equations $AX = 0$ in 'n' unknown has a non- trivial solution if $rank\ A < n$ where 'n' is number of columns of A.
- TYPE IV: if $rank\ A = rankA_b < number\ of\ unknown$ then infinite solution exists
- TYPE V: if $rank\ A \neq rankA_b$ then no solution exists

ملاحظات/

① اذا كان عدد المعادلات m اقل من عدد المجاهيل n (m<n) فان المنظومة سيكون لها حل ولكنه ليس وحيد (مالانهاية من الحلول)

② اذا كان عدد المعادلات اكثر من عدد المجاهيل n (m>n) فان المنظومة قد لايكون لها حل على الاطلاق

③ اذا كان عدد المعادلات m مساوي لعدد المجاهيل (m=n) فان المنظومة سيكون لها حل وحيد $\Leftrightarrow$ اذا كان المصفوفة A معكوس اي $|A| \neq 0$ اي ان المصفوفة غير منفردة.

chapter four

Linear Algebraic Equations

Many important problems in science and engineering require the solution of systems of simultaneous linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{4.1}$$

Where the coefficients a_{ij} and the right hand sides b_i are given numbers, and the quantities x_i are the unknowns which need to be determined. In matrix notation this system can be written as

$$AX = b \tag{4.2}$$

where $A = (a_{ij})$, $b = (b_i)$ and $x = (x_i)$. We shall assume that the $n \times n$ matrix A is non-singular (i.e. that the determinant of A is non-zero) so that equation (4.2) has a unique solution.

There are two classes of method for solving systems of this type:

- **Direct methods** find the solution in a finite number of steps.
- **iterative methods** start with an arbitrary first approximation to x and then improve this estimate in an infinite but convergent sequence of steps.

Gauss elimination:

Gauss elimination is used to solve a system of linear equations by transforming it to an upper triangular system (i.e. one in which all of the coefficients below the leading diagonal are zero) using elementary row operations. The solution

of the upper triangular system is then found using back substitution.
We shall describe the method in detail for the general example of 3×3 system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

In matrix notation this system can be written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

STEP 1

The first step is eliminates the variable x_1 from the second and third equations. This can be done by subtracting multiples $m_{21} = \frac{a_{21}}{a_{11}}$ and $m_{31} = \frac{a_{31}}{a_{11}}$ of row 1 from rows 2 and 3, respectively, producing the equivalent system

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^{(2)} \\ b_3^{(2)} \end{pmatrix}$$

where $a_{ij}^{(2)} = a_{ij} - m_{ij}a_{1j}$ and $b^{(2)} = b_i - m_{i1}b_1$ $(i, j = 2, 3)$.

STEP 2

The second step eliminates the variable x_2 from the third equation. This can be done by subtracting a multiple $m_{32} =$

$\frac{a_{32}^{(2)}}{a_{22}^{(2)}}$ from row 2 and 3, producing the equivalent upper triangular system

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^{(2)} \\ b_3^{(3)} \end{pmatrix}$$

where $a_{33}^{(3)} = a_{33}^{(2)} - m_{32}a_{23}^{(2)}$ and $b_3^{(2)} = b_3^{(2)} - m_{32}b_2^{(2)}$.
Since these row operations are reversible, the original system and the upper triangular system have the same solution. The upper triangular system is solved using back substitution. The last equation implies that

$$x_3 = \frac{b_3^{(3)}}{a_{33}^{(3)}}$$

This number can then be substituted into the second equation and the value of x_2 obtained from

$$x_2 = \frac{b_2^{(2)} - a_{23}^{(2)}x_3}{a_{22}^{(2)}}$$

Finally, the known values of x_2 and x_3 can be substituted into the first equation and the value of x_1 obtained from

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

It is clear from previous equations that the algorithm fails if any of the quantities $a_{jj}^{(j)}$ are zero, since these numbers are used as the denominators both in the multipliers m_{ij} and in the back substitution equations. These numbers are usually referred to as pivots. Elimination also produces poor results if any of the multipliers are greater than one in modulus. It is possible to prevent these difficulties by

using row interchanges. At step j, the elements in column j which are on or below the diagonal are scanned. The row containing the element of largest modulus is called the pivotal row. Row j is then interchanged (if necessary) with the pivotal row.

It can, of course, happen that all of the numbers $a_{jj}^{(j)}$, $a_{j+1,j}^{(j)}$, $\cdots$, $a_{nj}^{(j)}$ are exactly zero, in which case the coefficient matrix does not have full rank and the system fails to possess a unique solution.

Example: To illustrate the effect of partial pivoting, consider the solution of

$$\begin{pmatrix} 0.61 & 1.23 & 1.72 \\ 1.02 & 2.15 & -5.51 \\ -4.34 & 11.2 & -4.25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.792 \\ 12 \\ 16.3 \end{pmatrix}$$

using three significant figure arithmetic with rounding. This models the more realistic case of solving a large system of equations on a computer capable of working to, say, ten significant figure accuracy.

Without partial pivoting we proceed as follows:
Step 1: The multipliers are $m_{21} = \frac{1.02}{0.61} = 1.67$ and $m_{31} = \frac{-4.34}{0.61} = -7.11$, which give

$$\begin{pmatrix} 0.61 & 1.23 & 1.72 \\ 0 & 0.10 & -8.38 \\ 0 & 20.0 & 7.95 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.792 \\ 10.7 \\ 21.9 \end{pmatrix}$$

Step 2 The multiplier is $m_{32} = \frac{20}{0.1} = 200$, which gives

$$\begin{pmatrix} 0.61 & 1.23 & 1.72 \\ 0 & 0.10 & -8.38 \\ 0 & 0 & 1690 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.792 \\ 10.7 \\ -2120 \end{pmatrix}$$

Solving by back substitution, we obtain
$x_3 = -1.25 \qquad x_2 = 2 \qquad x_1 = 0.790$

With partial pivoting we proceed as follows:
Step 1: Since $|-4.34| > |0.610|$ and $|1.02|$, rows 1 and 3 are interchanged to get

$$\begin{pmatrix} -4.34 & 11.2 & -4.25 \\ 1.02 & 2.15 & -5.51 \\ 0.61 & 1.23 & 1.72 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16.3 \\ 12 \\ 0.792 \end{pmatrix}$$

The multiplier is $m_{21} = \frac{1.02}{-4.34} = -0.235$ and $m_{31} = \frac{0.610}{-4.34} = -0.141$ which gives

$$\begin{pmatrix} -4.34 & 11.2 & -4.25 \\ 0 & 4.78 & -6.51 \\ 0 & 2.81 & 1.12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16.3 \\ 15.8 \\ 3.09 \end{pmatrix}$$

Step 2 Since $|4.78| > |2.81|$, no further interchanged are needed and $m_{32} = \frac{2.81}{4.78} = 5.88$, which gives

$$\begin{pmatrix} -4.34 & 11.2 & -4.25 \\ 0 & 4.78 & -6.51 \\ 0 & 0 & 4.95 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16.3 \\ 15.8 \\ -6.20 \end{pmatrix}$$

Solving by back substitution, we obtain
$x_3 = -1.25 \qquad x_2 = 1.60 \qquad x_1 = 0.1.59$
By substituting these values into the original system of equations it is easy to verify that the result obtained with partial pivoting is a reasonably accurate solution. (In fact, the exact solution, rounded to three significant figures, is given by $x_3 = -1.26$, $x_2 = 1.60$ and $x_1 = 1.61$) However, the values obtained without partial pivoting are totally unacceptable; the value of x_1 is not even correct to one significant figure.

GUASS ELIMINATION METHOD

ALGORITHM

- In the first stage, the given system of equations is reduced to an equivalent upper triangular form using elementary transformation.
- In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order "$x_n, x_{n-1}, \dots \dots \dots x_2, x_1$"

REMARK

Guass's Elimination method fails if any one of the Pivotal coefficient become zero. In such a situation, we rewrite the equation in a different order to avoid zero Pivotal coefficients.

QUESTION Solve the following system of equations using Elimination Method.

$$2x + 3y - z = 5$$

$$4x + 4y - 3z = 3$$

$$-2x + 3y - z = 1$$

SOLUTION We can solve it by elimination of variables by making coefficients same.

$$2x + 3y - z = 5 \quad \dots\dots\dots\dots\dots\dots\dots (i)$$

$$4x + 4y - 3z = 3 \quad \dots\dots\dots\dots\dots\dots\dots (ii)$$

$$-2x + 3y - z = 1 \quad \dots\dots\dots\dots\dots\dots\dots (iii)$$

$r_2 = 2r_1 - r_2$
$r_3 = r_1 - r_3$

Multiply (i) by 2 and subtracted by (ii) $\quad 2y + z = 7 \quad \dots\dots\dots\dots\dots\dots\dots (iv)$

Adding (i) and (iii) $\quad 6y - 2z = 6 \quad \dots\dots\dots\dots\dots\dots\dots (v)$

$$\begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & 2 & 1 & 7 \\ 0 & 6 & -2 & 6 \end{bmatrix}$$

Now eliminating "y" Multiply (iv) by 3 then subtract from (v) $\quad z = 3$

Using "z" in (iv) we get $\quad y = 2 \quad$ and Using "y", "z" in (i) we get $\quad x = 1$

Hence solution is $\quad x = 1, y = 2, z = 3$

$r_3 = r_2 - \frac{r_3}{3}$

$$\begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & 2 & 1 & 7 \\ 0 & 0 & \frac{5}{3} & 5 \end{bmatrix}$$

QUESTION

Solve the following system of equations by Guass's Elimination method with partial pivoting.

$$x + y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

SOLUTION

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 7 \\ 16 \end{bmatrix} \quad \sim R_{12} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 16 \end{bmatrix} \quad \sim \frac{1}{3}R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 16 \end{bmatrix} \quad \sim R_2 - R_1 \Rightarrow \begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 0 \end{bmatrix} \quad \sim R_3 - 2R_1$$

$r_2 = r_2 - r_1$ $\qquad$ $r_3 = r_3 - 2r_1$

2^{nd} row cannot be used as pivot row as $a_{22} = 0$, So interchanging the 2^{nd} and 3^{rd} row we get

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \qquad \sim R_{23}$$

Using back substitution

$$-\frac{1}{3}z = -1 \quad \Rightarrow z = 3$$

$$-y + \frac{1}{3}z = 0 \quad \Rightarrow y = \cancel{3}\ 1 \qquad \therefore z = 3$$

$$x + y + \frac{4}{3}z = 8 \quad \Rightarrow x = 3 \qquad \therefore\ y = \cancel{3}\ 1\ , \ z = 3$$

QUESTION

Solve the following system of equations using Guass's Elimination Method with partial pivoting.

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$

$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$

$$4x_1 + 5x_2 + 65x_3 + 2x_4 = 26$$

$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

SOLUTION

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 26 \\ 21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix} \quad \sim R_{14}$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 65 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix} \quad \sim \frac{1}{9} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 4 & 2 & 8 \\ 4 & 5 & 65 & 2 \\ 4 & 10 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 24 \\ 26 \\ 32 \end{bmatrix} \quad \sim R_{24}$$

$r_2 = r_2 - 4r_1$

$r_3 = r_3 - 4r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 4 & 2 & 8 \\ 0 & \frac{29}{9} & \frac{85}{18} & 2 \\ 0 & \frac{74}{9} & \frac{29}{9} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 24 \\ 16.6668 \\ 22.668 \end{bmatrix} \quad \sim R_3 - 4R_1 \text{ and } \sim R_4 - 4R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 29/9 & 85/18 & 2 \\ 0 & 74/9 & 29/9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ 16.6668 \\ 22.668 \end{bmatrix} \quad \sim \frac{1}{4} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & ½ & 2 \\ 0 & 0 & 3.111 & -4.444 \\ 0 & 0 & -0.889 & 16.444 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ -2.6665 \\ -26.665 \end{bmatrix} \qquad \sim R_3 - \frac{29}{9}R_2 \; and \sim R_4 - \frac{74}{9}R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 4/9 & 4/9 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 1 & -1.428 \\ 0 & 0 & 0 & 15.175 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 6 \\ -0.857 \\ -27.427 \end{bmatrix} \qquad \sim \frac{R_3}{3.111} \; and \sim R_4 + 0.889R_3$$

$\Rightarrow \quad 15.175x_4 = -27.427$

$\Rightarrow x_4 = -1.8074$

$\Rightarrow x_3 - 1.428x_4 = -0.857$

$\Rightarrow x_3 = -3.438 \quad \therefore x_4 = -1.8074$

$\Rightarrow x_2 + \frac{1}{2}x_3 + 2x_4 = 6$

$\Rightarrow x_2 = 11.3338 \quad \therefore x_4 = -1.8074 \; , \; x_3 = -3.438$

$\Rightarrow x_1 + \frac{4}{9}x_2 + \frac{4}{9}x_3 = 2.333$

$\Rightarrow x_1 = -1.1762 \quad \therefore x_2 = 11.3338 \; , \; x_3 = -3.438$

Hence required solutions are

$x_1 = -1.1762 \; , \; x_2 = 11.3338 \; , \; x_3 = -3.438 \; , \; x_4 = -1.8074$

Example: Solve the system of linear equations by Gaussian elimination method

$$2x_1 + 3x_2 - x_3 = 5$$
$$4x_1 + 4x_2 - 3x_3 = 3$$
$$-2x_1 + 3x_2 - x_3 = 1$$

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix} \quad , b = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \quad , X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right] \xrightarrow[r_1(1)+r_3]{r_1(-2)+r_2} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 6 & -2 & 6 \end{array}\right]$$

$$\xrightarrow[r_2(3)+r_3]{} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

وبالتعويض المتراجع نحصل على قيم X كالاتي

$$-5x_3 = -15 \Rightarrow x_3 = 3$$
$$-2x_2 - x_3 = -7 \Rightarrow -2x_2 - 3 = -7 \Rightarrow x_2 = 2$$
$$2x_1 + 3x_2 - x_3 = 5 \Rightarrow 2x_1 + 3(2) - 3 = 5 \Rightarrow x_1 = 1$$
$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$$

ملاحظة: يمكن ايجاد محدد المصفوفة A وذلك بحساب حاصل ضرب عناصر القطر الرئيسي بعد تحويلها لمصفوفة مثلثية عليا .

في المثال اعلاه اذا كان المطلوب ايضا ايجاد المحدد لمصفوفة المعاملات A سيكون

$$|A| = (2)(-2)(-5) = 20$$

Example Solve the system of linear equations by Gaussian elimination method and find the determinant of the coefficient matrix A.

$$2x_1 + x_2 = 7$$
$$3x_1 - 4x_2 = 5$$

$$[A|b] = \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 3 & -4 & 5 \end{array}\right] \xrightarrow{r_1\left(\frac{-3}{2}\right)+r_2} \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 0 & \frac{-11}{2} & \frac{-11}{2} \end{array}\right]$$

$$\frac{-11}{2} x_2 = \frac{-11}{2} \Rightarrow x_2 = 1$$
$$2x_1 + 1 = 7 \Rightarrow x_1 = 3$$
$$X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$|A| = (2)\left(\frac{-11}{2}\right) = -11$$

Exercise : Solve the system of linear equations by Gaussian elimination method and find the determinant of the coefficient matrix A.

$$x_1 - x_2 = 0$$
$$2x_1 + x_3 = 4$$
$$-2x_1 + x_2 - 2x_3 = 1$$

EX3/ Use Gaussian elimination method to solve the following system of linear equations.

$$3x_1 - x_2 + 2x_3 = 12$$
$$x_1 + 2x_2 + 3x_3 = 11$$
$$2x_1 - 2x_2 - x_3 = 2$$

Sol/ $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 12 \\ 11 \\ 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

نقوم بتحويل المصفوفة A الى مثلث علوي

$$[A|b] = \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 1 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{array}\right] \xrightarrow[-2r_1+3r_3]{-r_1+3r_2} \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 7 & 7 & 21 \\ 0 & -4 & -7 & -18 \end{array}\right] \xrightarrow{4r_2+7r_3}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 7 & 7 & 21 \\ 0 & 0 & -21 & -42 \end{array}\right]$$

وبالتعويض التراجعي نحصل على

$$-21x_3 = -42 \rightarrow x_3 = 2$$

$$7x_2 + 7x_3 = 21 \rightarrow 7x_2 + 14 = 21 \rightarrow x_2 = 1$$

and

$$3x_1 - x_2 + 2x_3 = 12$$
$$3x_1 - 1 + 4 = 12 \rightarrow 3x_1 = 9 \rightarrow x_1 = 3$$

$$X = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Ex:-/ Use gaussian elimination to solve:

$$0.0002x_1 + 1.54x_2 = 1.560 \longrightarrow (1)$$

$$0.6540x_1 - 2.440x_2 = 1.01 \longrightarrow (2)$$

Sol/ في حالة اختيار المعادلة (1) لتكون معادلة الارتكاز نلاحظ

$$A = \begin{bmatrix} 0.0002 & 1.540 \\ 0.6540 & -2.440 \end{bmatrix} \quad b = \begin{bmatrix} 1.560 \\ 1.01 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

نلاحظ ان $a_{11} = 0.0002$ رقم صغير جداً

$$[A|b] = \left[\begin{array}{cc|c} 0.0002 & 1.540 & 1.560 \\ 0.6540 & -2.440 & 1.01 \end{array}\right] \xrightarrow{r_1 * \left(\frac{-0.6540}{0.0002}\right) + r_2}$$

$$\longrightarrow \left[\begin{array}{cc|c} 0.0002 & 1.540 & 1.560 \\ 0 & -5038 & -5100 \end{array}\right]$$

بالتعويض المتراجع نجد ان

$$x_2 = \frac{-5100}{-5038} = 1.012$$

$$\text{and} \quad x_1 = \frac{1.560 - 1.540 * 1.012}{0.0002} = 10$$

بينما الحل المضبوط هو $x_1 = 5.32$ اي ان الحل يحيد عن الحل المضبوط

$$x_2 = 1.012$$

الان نغير ترتيب المعادلات بحيث تكون المعادلة 2 معادلة الارتكاز

$$\left[\begin{array}{cc|c} 0.654 & -2.440 & 1.01 \\ 0.0002 & 1.540 & 1.56 \end{array}\right] \xrightarrow{r_1 * \left(\frac{-0.0002}{0.6540}\right) + r_2}$$

$$\left[\begin{array}{cc|c} 0.654 & -2.440 & 1.01 \\ 0 & 1.5407 & 1.5597 \end{array}\right]$$

بالتعويض نجد أن

$x_2 = 1.0125$

$x_1 = 5.3219$ وهي قيمة تقريبية جيدة

① متى Gauss نفشل عندما يكون هناك سطر كله اصفار ؟

② هل دائماً كاوس أو جوردن مع المصفوفة المربعة ... ؟ يمكن ان يستخدم مع المصفوفات غير المربعة لعمل الحل التقريبي

③ كاوس لها مصفوفة مثلثية علوية فقط

④ اذا كان يريد استخراج المحدد هنا الافضل كاوس لان عدد الخطوات اقل

⑤ $|A^{-1}| = \frac{1}{|A|}$

⑥ يجب ان تكون عناصر القطر الرئيسي $a_{ii} \neq 0$ اذا صفر نبدل الاسطر

⑦ محدد المصفوفة الناتجة = محدد المصفوفة الاصلية = حاصل ضرب عناصر القطر الرئيسي
يمكن كاوس تعطي المحدد في نفس الوقت

مع الارتكاز ① هنا نحدد عنصر الارتكاز من اول عمود الاول ← نصفر الصف الثاني والثالث ...
من العمود الثاني a_{22} ← نصفر الصف الثالث والرابع ... العمود الاخير
الثالث a_{33} ← نصفر الصف الرابع ... العمود الاخير

② حالة تساوي العدد في الصفوف نأخذ الصف الاول بالترتيب

Chapter Four

MATRIX INVERTION

A "$n \times n$" matrix "M" is said to be non-singular (or Invertible) if a "$n \times n$" matrix "M^{-1}" exists with "MM^{-1}" = "$M^{-1}M$" $= I$ then matrix "M^{-1}" is called the inverse of "M". A matrix without an inverse is called Singular (or Non-invertible)

MATRIX INVERSION THROUGH GUASS ELIMINATION

1. Place an identity matrix, whose order is same as given matrix.
2. Convert matrix in upper triangular form.
3. Take largest value as Pivot.
4. Using back substitution get the result.

NOTE: In order to increase the accuracy of the result, it is essential to employ Partial Pivoting. In the first column use absolutely largest coefficient as the pivotal coefficient (for this we have to interchange rows if necessary). Similarly, for the second column and vice versa.

MATRIX INVERSION THROUGH GUASS JORDAN ELIMINATION

1. Place an identity matrix, whose order is same as given matrix.
2. Convert matrix in upper triangular form.
3. No need to take largest value as Pivot.
4. Using back substitution get the result.

With Pivoting

QUESTION : Find inverse using Guass Elimination Method $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$

ANSWER

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_{12} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \quad \frac{1}{4}R_1$$

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{4} & \frac{-1}{4} & & 0 & \frac{1}{4} & 0\\ 0 & \frac{1}{4} & \frac{5}{4} & \vdots & 1 & -\frac{1}{4} & 0\\ 3 & 5 & 3 & & 0 & 0 & 1\end{bmatrix} R_2 - R_1 \Rightarrow \begin{bmatrix}1 & \frac{3}{4} & \frac{-1}{4} & & 0 & \frac{1}{4} & 0\\ 0 & \frac{1}{4} & \frac{5}{4} & \vdots & 1 & -\frac{1}{4} & 0\\ 0 & \frac{11}{4} & \frac{15}{4} & & 0 & -\frac{3}{4} & 1\end{bmatrix} \quad R_3 - 3R_1$$

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{4} & \frac{-1}{4} & & 0 & \frac{1}{4} & 0\\ 0 & \frac{11}{4} & \frac{15}{4} & \vdots & 0 & -\frac{3}{4} & 1\\ 0 & \frac{1}{4} & \frac{5}{4} & & 1 & -\frac{1}{4} & 0\end{bmatrix} R_{23} \Rightarrow \begin{bmatrix}1 & \frac{3}{4} & \frac{-1}{4} & & 0 & \frac{1}{4} & 0\\ 0 & 1 & \frac{15}{11} & \vdots & 0 & -\frac{3}{11} & \frac{4}{11}\\ 0 & \frac{1}{4} & \frac{5}{4} & & 1 & -\frac{1}{4} & 0\end{bmatrix} \quad \frac{4}{11}R_2$$

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{4} & \frac{-1}{4} & & 0 & \frac{1}{4} & 0\\ 0 & 1 & \frac{15}{11} & \vdots & 0 & -\frac{3}{11} & \frac{4}{11}\\ 0 & 0 & \frac{10}{11} & & 1 & -\frac{2}{4} & -\frac{1}{11}\end{bmatrix} R_3 - \frac{1}{4}R_2 \Rightarrow \begin{bmatrix}1 & \frac{3}{4} & \frac{-1}{4} & & 0 & \frac{1}{4} & 0\\ 0 & 1 & \frac{15}{11} & \vdots & 0 & -\frac{3}{11} & \frac{4}{11}\\ 0 & 0 & 1 & & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \quad \frac{11}{10}R_3$$

$$\Rightarrow\begin{bmatrix}1 & \frac{3}{4} & 0 & & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40}\\ 0 & 1 & 0 & \vdots & -\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \quad R_1 + \frac{1}{4}R_3 \text{ and } R_2 - \frac{15}{11}R_3$$

$$\Rightarrow\begin{bmatrix}1 & 0 & 0 & & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5}\\ 0 & 1 & 0 & \vdots & -\frac{3}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix} \quad R_1 - \frac{3}{4}R_2$$

Hence

$$A^{-1} = \begin{bmatrix}\frac{7}{5} & \frac{1}{5} & -\frac{2}{5}\\ -\frac{3}{2} & 0 & \frac{1}{2}\\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10}\end{bmatrix}$$

Inverse of the Coefficient Matrix by Gaussian Elimination Method:

نأخذ المصفوفة الموسعة $[A|I]$ حيث ان I هي مصفوفة الوحدة رتبتها مساوية الى رتبة A ثم نجري العمليات السابقة في طريقة كاوس للحذف وبشكل متسلسل لتحويلها الى مصفوفة مثلثية عليا (المصفوفة A تتحول الى مصفوفة مثلثية عليا) ثم نحصل على اعمدة المعكوس من خلال التعويض المتراجع n من المرات في كل مرة نحصل على عمود من المعكوس.

الارتكاز الجزئي:(The Pivot Process)

المعادلة التي تستعمل لحذف احد المجاهيل من المعادلات الباقية تسمى معادلة الارتكاز ومعامل ذلك المجهول في تلك المعادلة يسمى عنصر الارتكاز.

ان طريقة كاوس للحذف تعتمد على عنصر الارتكاز والذي يجب ان لا يكون صفر او قريب جداً من الصفر وان كان صفر او قريب من الصفر فلا يمكن ان تستعمل كمعادلة ارتكاز ولذلك نعيد ترتيب المعادلات ونختار معادلة اخرى لتكون معادلة ارتكاز.

ملاحظات:

١- لاختيار عنصر الارتكاز حيث انه يجب ان يكون اكبر عنصر في العمود الذي يحتوي عليه (دون مراعاة الاشارة).

٢- عندما نبدل الصفوف في الارتكاز الجزئي فان قيمة المحدد في كل تغيير تضرب في سالب لذلك يجب ان ناخذ بعين الاعتبار عدد التغيرات في الصفوف عند حساب المحدد.

Example: Solve the system of linear equations by Gaussian elimination method and find the determinant and the inverse of the coefficient matrix A with The pivot process .

$x_2 + x_3 = 1$

$x_1 + x_3 = 1$

$x_1 + x_2 = 1$

$$[A|b] = \left[\begin{array}{ccc|c|ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_1 \Leftrightarrow r_2} \left[\begin{array}{ccc|c|ccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_1(-1) + r_3} \left[\begin{array}{ccc|c|ccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{r_2(-1) + r_3} \left[\begin{array}{ccc|c|ccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & -1 & 1 \end{array}\right]$$

وبالتعويض المتراجع نحصل على قيم X كالاتي

$$-2x_3 = -1 \;\Rightarrow\; x_3 = \frac{1}{2}$$

$$x_2 + x_3 = 1 \;\Rightarrow\; x_2 + \frac{1}{2} = 1 \;\Rightarrow x_2 = \frac{1}{2}$$

$$x_1 + x_3 = 1 \Rightarrow x_1 + \frac{1}{2} = 1 \Rightarrow x_1 = \frac{1}{2}$$

$$X = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$|A| = -(1)(1)(-2) = 2$$

نجد المعكوس بالتعويض المتراجع ٣ مرات (n=3)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$-2x_3 = -1 \;\Rightarrow\; x_3 = \frac{1}{2}$$

$$x_2 + x_3 = 1 \;\Rightarrow\; x_2 + \frac{1}{2} = 1 \;\Rightarrow x_2 = \frac{1}{2}$$

$$x_1 + x_3 = 0 \Rightarrow x_1 + \frac{1}{2} = 0 \Rightarrow x_1 = -\frac{1}{2}$$

$$X_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$ العمود الاول للمعكوس

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$-2x_3 = -1 \;\Rightarrow\; x_3 = \frac{1}{2}$$

$$x_2 + x_3 = 0 \;\Rightarrow\; x_2 + \frac{1}{2} = 0 \;\Rightarrow x_2 = -\frac{1}{2}$$

$$x_1 + x_3 = 1 \Rightarrow x_1 + \frac{1}{2} = 1 \Rightarrow x_1 = \frac{1}{2}$$

(80)

$$X_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$ العمود الثاني للمعكوس

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$-2x_3 = 1 \;\Rightarrow\; x_3 = -\frac{1}{2}$$

$$x_2 + x_3 = 0 \;\Rightarrow\; x_2 - \frac{1}{2} = 0 \;\Rightarrow x_2 = \frac{1}{2}$$

$$x_1 + x_3 = 0 \Rightarrow x_1 - \frac{1}{2} = 0 \Rightarrow x_1 = \frac{1}{2}$$

$$X_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$ العمود الثالث للمعكوس

$$A^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

Ex:- Solve the systems of linear equations using Gauss elimination (1) without Pivoting (2) with Partial Pivoting

(1)
$$0.005x_1 + x_2 + x_3 = 2$$
$$x_1 + 2x_2 + x_3 = 4$$
$$-3x_1 - x_2 + 6x_3 = 2$$

(2)
$$x_1 - x_2 + 2x_3 = 5$$
$$2x_1 - 2x_2 + x_3 = 1$$
$$30x_1 - 2x_2 + 7x_3 = 20$$

Gauss Jordan Method:
The following **row operations** produce an equivalent system, i.e., a system with the same solution as the original one. The row operations are:

1. Interchange any two rows.
2. Multiply each element of a row by a nonzero constant.
3. Replace a row by the sum of itself and a constant multiple of another row of the matrix.

Convention: For these row operation, we will use the following notations:

- $R_i \longleftrightarrow Rj$ means: Interchange row i and row j.
- $R_i = \alpha R_i$ means: Replace row i with α times row i.
- $R_i = R_i + \alpha R_j$ means: Replace row i with the sum of row i and α times row j.

The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.

1. Write the extended matrix of the system.
2. Use row operations to transform the extended matrix to have following properties:
   (a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.
   (b) In each row that does not consist entirely of zeros, the left nonzero element is 1 (called a leading 1 or a pivot).
   (c) Each column that contains a leading 1 has zeros in all other entries.

(d) The leading 1 in any row is to the left of any leading 1's in the rows below it.

3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Example: Solve the following system of equations using the Gauss Jordan elimination method.

$$\begin{aligned} x+y+z &= 5 \\ 2x+3y+5z &= 8 \\ 4x+5z &= 2 \end{aligned}$$

Solution: The extended matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

use the row operations as following:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow[R_3 = \frac{1}{13}R_3]{R_3 = R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - R_2]{\substack{R_2 = R_2 - 3R_3 \\ R_1 = R_1 - 3R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$x = 3, \qquad y = 4, \qquad z = -2$$

Example: Solve the following system of equations using the Gauss Jordan elimination method.

$$\begin{aligned} x + 2y - 3z &= 2 \\ 6x + 3y - 9z &= 6 \\ 7x + 14y - 21z &= 13 \end{aligned}$$

Solution: The extended matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

use the row operations as following:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow[R_3 = R_3 - 7R_1]{R_2 = R_2 - 6R_1} \left[\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent, i.e., it has no solutions.

Example: Solve the following system of equations using the Gauss Jordan elimination method.

$$\begin{aligned} 4y + z &= 2 \\ 2x + 6y - 2z &= 3 \\ 4x + 8y - 5z &= 4 \end{aligned}$$

Solution: The extended matrix of the system is the follow-

ing.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

use the row operations as following:

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow[R_3 = R_3 - 2R_1]{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 = \frac{1}{2}R_1]{\substack{R_2 = \frac{1}{4}R_2 \\ R_1 = R_1 - 6R_2}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{-7}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We can stop This because the form of the last matrix. It corresponds to the following system.

$$x - \frac{7}{4}z = 0$$
$$y + \frac{1}{4}z = \frac{1}{2}$$

We can express the solutions of this system as

$$x = \frac{7}{4}z \qquad y = -\frac{1}{4}z + \frac{1}{2}$$

Since there is no specific value for z, it can be chosen arbitrarily. This means that there are **infinitely many** solutions for this system. We can represent all the solutions by using a parameter t as follows.

$$x = \frac{7}{4}t \qquad y = -\frac{1}{4}t + \frac{1}{2} \qquad z = t$$

Any value of the parameter t gives us a solution of the system. For example:

$t = 4$ gives the solution $(x, y, z) = (7, \frac{-1}{2}, 4)$

$t = -2$ gives the solution $(x, y, z) = (\frac{-7}{2}, 1, -2)$

Chapter Four

GUASS JORDAN ELIMINATION METHOD

The method is based on the idea of reducing the given system of equations "$Ax = b$" to a diagonal system of equations "$Ix = b$" where "I" is the identity matrix, using row operation. It is the verification of Gauss's Elimination Method.

ALGORITHM

1) Make the elements below the first pivot in the augmented matrix as zeros, using the elementary row transformation.
2) Secondly make the elements below and above the pivot as zeros using elementary row transformation.
3) Lastly divide each row by its pivot so that the final matrix is of the form

$$\begin{bmatrix} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & d_3 \end{bmatrix}$$

Then it is easy to get the solution of the system as $x_1 = d_1, x_2 = d_2, x_3 = d_3$

Partial Pivoting can also be used in the solution. We may also make the pivot as "1" before performing the elimination.

ADVANTAGE/DISADVANTAGE

The Guass's Jordan method looks very elegant as the solution is obtained directly. However, it is computationally more expensive than Guass's Elimination. Hence we do not normally use this method for the solution of the system of equations.

The most important application of this method is to find inverse of a non-singular matrix.

What is Gauss Jordan variation?

In this method Zeroes are generated both below and above each pivot, by further subtractions. The final matrix is thus diagonal rather than triangular and back substitution is eliminated. The idea is attractive but it involves more computing than the original algorithm, so it is little used.

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Guass Jourdan

QUESTION

Solve the system of equations using Elimination method

$$x + 2y + z = 8$$

$$2x + 3y + 4z = 20$$

$$4x + 3y + 2z = 16$$

ANSWER

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 2 & 3 & 4 & \vdots & 20 \\ 4 & 3 & 2 & & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & -1 & +2 & \vdots & 4 \\ 0 & -5 & -2 & & -16 \end{bmatrix} \quad R_2 - 2R_1 \; and \; R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 1 & +2/5 & & 16/5 \end{bmatrix} \quad (-1)R_2 \; and \; (-1/5)R_3$$

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 0 & 12/5 & & 36/5 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & & 8 \\ 0 & 1 & -2 & \vdots & -4 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad (5/12)R_3$$

$$\begin{bmatrix} 1 & 2 & 0 & & 5 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad R_1 - R_3 \; and \; R_2 + 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & & 1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & & 3 \end{bmatrix} \quad R_1 - 2R_2$$

Hence solutions are $x = 1, y = 2, z = 3$

QUESTION

Find inverse using Guass's Jordan Elimination Method $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$

ANSWER

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \quad R_2 - 4R_1 \text{ and } R_3 - 3R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \quad -1R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \quad R_3 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \quad -\frac{1}{10}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{-1}{10} & \frac{1}{5} & \frac{1}{10} \\ 0 & 1 & 0 & \frac{-3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \quad R_1 - R_3 \text{ and } R_2 - 5R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \quad R_1 - R_2$$

Hence

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

QUESTION: Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

SOLUTION: we first find the co-factor of the elements of A

$a_{11} = (-1)^2(2+1) = 3$

$a_{12} = (-1)^3(-1) = 1$

$a_{13} = (-1)^4(0-2) = -2$

$a_{21} = (-1)^3(0+2) = -2$

$a_{22} = (-1)^4(1-2) = -1$

$a_{23} = (-1)^5(-1-0) = 1$

$a_{31} = (-1)^4(0-4) = -4$

$a_{32} = (-1)^5(1-0) = -1$

$a_{33} = (-1)^6(2-0) = 2$

Thus $[A_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$

$adjA = [A'_{ij}]_{3\times3} = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ $\quad and \quad |A| = -1$

So $A^{-1} = \frac{1}{|A|}\ adjA = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$ after putting the values.

Gauss- Jordan Elimination Method

ان طريقة كاوس – جوردن لحل منظومة المعادلات الخطية مشابهة الى طريقة كاوس للحذف ماعدا ان المصفوفة A تختزل الى مصفوفة قطرية في كاوس – جوردن بينما في طريقة كاوس للحذف تختزل الى مصفوفة مثلثية عليا.

Example 1: Solve the system of linear equations by Gauss- Jordan elimination method

$$2x_1+3x_2-x_3=5$$
$$4x_1+4x_2-3x_3=3$$
$$-2x_1+3x_2-x_3=1$$

$$A=\begin{bmatrix}2&3&-1\\4&4&-3\\-2&3&-1\end{bmatrix}\quad ,b=\begin{bmatrix}5\\3\\1\end{bmatrix}\quad ,X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$[A|b]=\left[\begin{array}{ccc|c}2&3&-1&5\\4&4&-3&3\\-2&3&-1&1\end{array}\right]\xrightarrow[r_1(1)+r_3]{r_1(-2)+r_2}\left[\begin{array}{ccc|c}2&3&-1&5\\0&-2&-1&-7\\0&6&-2&6\end{array}\right]$$

$$\xrightarrow[r_2(3)+r_3]{r_2\left(\frac{3}{2}\right)+r_1}\left[\begin{array}{ccc|c}2&0&\frac{-5}{2}&\frac{-11}{2}\\0&-2&-1&-7\\0&0&-5&-15\end{array}\right]\xrightarrow[r_3\left(\frac{-1}{5}\right)+r_2]{r_3\left(\frac{-1}{2}\right)+r_1}\left[\begin{array}{ccc|c}2&0&0&2\\0&-2&0&-4\\0&0&-5&-15\end{array}\right]$$

نحصل على قيم X كالاتي

$$-5x_3=-15\ \Rightarrow\ x_3=3$$
$$-2x_2=-4\ \Rightarrow\ x_2=2$$
$$2x_1=2\ \Rightarrow x_1=1$$
$$X=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

ملاحظة: يمكن ايجاد محدد المصفوفة A وذلك بحساب حاصل ضرب عناصر القطر الرئيسي بعد تحويلها لمصفوفة قطرية .

Inverse of the Coefficient Matrix by Gauss- Jordan Elimination Method:

نأخذ المصفوفة الموسعة $[A|I]$ حيث ان I هي مصفوفة الوحدة رتبتها مساوية الى رتبة A ثم نجري العمليات السابقة في طريقة كاوس - جوردن وبشكل متسلسل لتحويلها الى مصفوفة قطرية ثم نحصل مصفوفة المعكوس بعد اختزال المصفوفة A الى مصفوفة وحدة.

ملاحظة: فكرة الارتكاز الجزئي الذي تم تطبيقها في طريقة كاوس للحذف هي نفسها التي سوف يتم تطبيقها بطريقة كاوس – جوردن.

Example: Solve the system of linear equations by Gauss- Jordan Elimination Method and find the determinant and the inverse of the coefficient matrix A with The pivot process .

$x_1 + x_2 - x_3 = 1$

$2x_1 + x_2 = -2$

$-4x_1 + x_3 = 0$

$$[A|b|I] = \left[\begin{array}{ccc|c|ccc} 1 & 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & -2 & 0 & 1 & 0 \\ -4 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xRightarrow{r_1 \Leftrightarrow r_3} \left[\begin{array}{ccc|c|ccc} -4 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & -2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xRightarrow[r_1\left(\frac{1}{4}\right) + r_3]{r_1\left(\frac{1}{2}\right)+r_2} \left[\begin{array}{ccc|c|ccc} -4 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & \frac{1}{2} & -2 & 0 & 1 & \frac{1}{2} \\ 1 & 1 & \frac{-3}{4} & 1 & 1 & 0 & \frac{1}{4} \end{array}\right]$$

$$\xRightarrow{r_2(1)+r_3} \left[\begin{array}{ccc|c|ccc} -4 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & \frac{1}{2} & -2 & 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{-1}{4} & -1 & 1 & 1 & \frac{3}{4} \end{array}\right]$$

$$\xRightarrow[r_3(2) + r_2]{r_3(4)+r_1} \left[\begin{array}{ccc|c|ccc} -4 & 0 & 0 & -4 & 4 & 4 & 4 \\ 0 & -1 & 0 & -4 & 2 & 3 & 2 \\ 0 & 0 & \frac{-1}{4} & \frac{-1}{4} & 1 & 1 & \frac{3}{4} \end{array}\right]$$

$$\xRightarrow[\substack{r_2(-1) \\ r_3(-4)}]{r_1\left(\frac{-1}{4}\right)} \left[\begin{array}{ccc|c|ccc} 1 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 4 & -2 & -3 & -2 \\ 0 & 0 & 1 & 4 & -4 & -4 & -3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ -2 & -3 & -2 \\ -4 & -4 & -3 \end{bmatrix} \quad , |A| = -(-4)(-1)\left(\frac{-1}{4}\right) = 1$$

Ex:- Use Gauss-Jordan method to solve

$$2x+y=7$$
$$3x-4y=5$$

Sol/ $A=\begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$ $\quad X=\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad b=\begin{bmatrix} 7 \\ 5 \end{bmatrix}$

تحول المصفوفة A الى مصفوفة قطرية

$$[A|b]=\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 3 & -4 & 5 \end{array}\right] \xrightarrow{-3r_1+2r_2} \left[\begin{array}{cc|c} 2 & 1 & 7 \\ 0 & -11 & -11 \end{array}\right] \xrightarrow{\frac{1}{11}r_3+r_1}$$

$$\left[\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & -11 & -11 \end{array}\right]$$

$$-11y=-11 \rightarrow y=1 \quad , \quad 2x_1=6 \rightarrow \boxed{x_1=3}$$

Ex/ Use Gauss-Jordan method to solve

$$2x_1+3x_2-x_3=5$$
$$4x_1+4x_2-3x_3=5$$
$$-2x_1+3x_2-x_3=1$$

Sol/ $A=\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix}$ $\quad X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad b=\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$

تحول المصفوفة A الى قطرية

$$[A|b]=\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right] \xrightarrow[r_1+r_3]{2r_1+r_2} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 6 & -2 & 6 \end{array}\right]$$

$$\xrightarrow[r_2+r_3]{r_2+r_1}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -5/2 & -1/2 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & -5 & -15 \end{array}\right] \xrightarrow[-1/5 r_3 + r_2]{\frac{-1}{2} r_3 + r_1}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & -5 & -15 \end{array}\right] \Longrightarrow$$

$2x_1 = 2 \longrightarrow x_1 = 1$

$-2x_2 = -4 \longrightarrow x_2 = 2$

$-5x_3 = -15 \longrightarrow x_3 = 3$

Ex:-/ use gauss-Jordan method to solve

$x + y + z = 5$
$2x + 3y + 5z = 8$
$4x + 5z = 2$

Ex:- use gauss-Jordan elimination method to solve

$x + 2y - 3z = 2$
$6x + 3y - 9z = 6$
$7x + 14y - 21z = 13$

Ex:- Solve the systems by gauss-Jordan

$4y + z = 2$
$2x + 6y - 2z = 3$
$4x + 8y - 5z = 4$

Ex:- Find inverse of $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ using gauss-Jordan method.

Cramer's Rule:

Cramer's rule begins with the clever observation

$$\begin{vmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{vmatrix} = x_1$$

That is to say, if you replace the first column of the identity matrix with the vector $\mathbf{x} = \left(x_1, x_2, x_3 \right)^T$, the determinant is x_1. Now, we've illustrated this for the 3×3 case and for column one. In general, if you replace the ith column of an $n \times n$ identity matrix with a vector $\mathbf{x}$, the determinant of the matrix you get will be x_i, the ith component of $\mathbf{x}$.

Note that if $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

then

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}.$$

Take determinants of both sides then we get

$$det(A)x_1 = det(B_1)$$

where B_1 is the matrix we get when we replace column 1 of A by the vector $\mathbf{b}$. So,

$$x_1 = \frac{det(B_1)}{det(A)}.$$

In general

$$x_i = \frac{det(B_i)}{det(A)},$$

where B_i is the matrix we get by replacing column i of A with **b**.

Example: Use Cramer's rule to solve for the the linear system:

$$
\begin{aligned}
2x_1 + x_2 - 5x_3 + x_4 &= 8\\
x_1 - 3x_2 - 6x_4 &= 9\\
2x_2 - x_3 + 2x_4 &= -5\\
x_1 + 4x_2 - 7x_3 + x_4 &= 0
\end{aligned}
$$

Solution: write the system in matrix notation $AX = b$, then we have

$$A = \begin{pmatrix} 2 & 1 & -5 & 1\\ 1 & -3 & 0 & -6\\ 0 & 2 & -1 & 2\\ 1 & 4 & -7 & 6 \end{pmatrix} \text{ and } \quad b = \begin{pmatrix} 8\\ 9\\ -5\\ 0 \end{pmatrix}.$$

Now we need to calculate $det(A), det(B_1), det(B_2), det(B_3), det(B_4)$:

$$A = \begin{pmatrix} 2 & 1 & -5 & 1\\ 1 & -3 & 0 & -6\\ 0 & 2 & -1 & 2\\ 1 & 4 & -7 & 6 \end{pmatrix} \xRightarrow{then} det(A) = 27 \neq 0$$

$$B_1 = \begin{pmatrix} 8 & 1 & -5 & 1\\ 9 & -3 & 0 & -6\\ -5 & 2 & -1 & 2\\ 0 & 4 & -7 & 6 \end{pmatrix} \xRightarrow{then} det(B_1) = 81$$

$$B_2 = \begin{pmatrix} 2 & 8 & -5 & 1\\ 1 & 9 & 0 & -6\\ 0 & -5 & -1 & 2\\ 1 & 0 & -7 & 6 \end{pmatrix} \xRightarrow{then} det(B_2) = -108$$

$$B_3 = \begin{pmatrix} 2 & 1 & 8 & 1 \\ 1 & -3 & 9 & -6 \\ 0 & 2 & -5 & 2 \\ 1 & 4 & 0 & 6 \end{pmatrix} \xRightarrow{then} det(B_3) = -27$$

$$B_4 = \begin{pmatrix} 2 & 1 & -5 & 8 \\ 1 & -3 & 0 & 9 \\ 0 & 2 & -1 & -5 \\ 1 & 4 & -7 & 0 \end{pmatrix} \xRightarrow{then} det(B_4) = 27$$

This lead to:

$$x_1 = \frac{det(B_1)}{det(A)} = \frac{81}{27} = 3$$

$$x_2 = \frac{det(B_2)}{det(A)} = \frac{-108}{27} = -4$$

$$x_3 = \frac{det(B_3)}{det(A)} = \frac{-27}{27} = -1$$

$$x_4 = \frac{det(B_4)}{det(A)} = \frac{27}{27} = 1$$

Cramer's Rule

يمكن صياغة طريقة كرامر لايجاد عمود المجاهيل X في منظومة المعادلات الخطية بالشكل التالي:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

حيث ان D تمثل محدد مصفوفة المعاملات

وان D_{x_n} هو نفس المحدد D ولكن باستبدال العمود n بعمود النتائج b حيث

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_m & a_{m2} & \cdots & a_{mn} \end{vmatrix}, \quad D_{x_2} = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & b_m & \cdots & a_{mn} \end{vmatrix}$$

و

$$D_{x_n} = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & b_m \end{vmatrix}$$

وبالتالي فان قيم عمود المجاهيل X يتم حسابها بالشكل التالي:

$$x_1 = \frac{D_{x_1}}{D} \quad , x_2 = \frac{D_{x_2}}{D} , \ldots , x_n = \frac{D_{x_n}}{D}$$

Example 1: Solve the system of linear equations by Cramer's rule method.

$2x_1 + x_2 = 7$

$3x_1 - 4x_2 = 5$

$$D = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -8 - 3 = -11$$

$$D_{x_1} = \begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix} = -28 - 5 = -33$$

$$D_{x_2} = \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix} = 10 - 21 = -11$$

$$x_1 = \frac{D_{x_1}}{D} = \frac{-33}{-11} = 3$$

$$x_2 = \frac{D_{x_2}}{D} = \frac{-11}{-11} = 1$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Example 2: Solve the system of linear equations by Cramer's rule method.

$2x_1 + x_2 - x_3 = 3$

$x_1 + x_2 + x_3 = 1$

$x_1 - 2x_2 - 3x_3 = 4$

$$D=\begin{vmatrix}2 & 1 & -1\\ 1 & 1 & 1\\ 1 & -2 & -3\end{vmatrix}=2*\begin{vmatrix}1 & 1\\ -2 & -3\end{vmatrix}-1*\begin{vmatrix}1 & 1\\ 1 & -3\end{vmatrix}-1*\begin{vmatrix}1 & 1\\ 1 & -2\end{vmatrix}$$

$$=2(-1)-1(-4)-1(-3)=-2+4+3=5$$

$$D_{x_1}=\begin{vmatrix}3 & 1 & -1\\ 1 & 1 & 1\\ 4 & -2 & -3\end{vmatrix}=3*\begin{vmatrix}1 & 1\\ -2 & -3\end{vmatrix}-1*\begin{vmatrix}1 & 1\\ 4 & -3\end{vmatrix}-1*\begin{vmatrix}1 & 1\\ 4 & -2\end{vmatrix}$$

$$=3(-1)-1(-7)-1(-6)=10$$

$$D_{x_2}=\begin{vmatrix}2 & 3 & -1\\ 1 & 1 & 1\\ 1 & 4 & -3\end{vmatrix}=2*\begin{vmatrix}1 & 1\\ 4 & -3\end{vmatrix}-3*\begin{vmatrix}1 & 1\\ 1 & -3\end{vmatrix}-1*\begin{vmatrix}1 & 1\\ 1 & 4\end{vmatrix}$$

$$=2(-7)-3(-4)-1(3)=-5$$

$$D_{x_3}=\begin{vmatrix}2 & 1 & 3\\ 1 & 1 & 1\\ 1 & -2 & 4\end{vmatrix}=2*\begin{vmatrix}1 & 1\\ -2 & 4\end{vmatrix}-1*\begin{vmatrix}1 & 1\\ 1 & 4\end{vmatrix}+3*\begin{vmatrix}1 & 1\\ 1 & -2\end{vmatrix}$$

$$=2(6)-1(3)+3(-3)=0$$

$$x_1=\frac{D_{x_1}}{D}=\frac{10}{5}=2$$

$$x_2=\frac{D_{x_2}}{D}=\frac{-5}{5}=-1$$

$$x_3=\frac{D_{x_3}}{D}=\frac{0}{5}=0 \qquad \therefore X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 0\end{bmatrix}$$

Exercise : Solve the system of linear equations by Cramer's rule method.

1) $x_1-2x_2+x_3=3$

$2x_1+x_2-x_3=5$

$3x_1-x_2+2x_3=12$

(2) $2x_1+x_2-5x_3+x_4=8$

$x_1-3x_2+6x_4=9$

$2x_2-x_3+2x_4=-5$

$x_1+4x_2-7x_3+x_4=0$

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Ex3: Use Gramers' rule to solve

$x+2y=1$

$3x+4y=3$

Sol/ let $D=\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}=4-6=-2$

$Dx=\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}=-2$

$Dy=\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}=3-3=0$

$x=\dfrac{Dx}{D}=\dfrac{-2}{-2}=1 \quad , \quad y=\dfrac{Dy}{D}=0$

في هذا المثال نلاحظ ان معاملات x و b متساوية لذلك
نتيجه x=1 و y=0

Ex4/ Use Gramers' rule to solve

$2x+5y=5$

$4x+6y=6$

Sol/ $D=\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix}=12-20=-8$

$Dx=\begin{vmatrix} 5 & 5 \\ 6 & 6 \end{vmatrix}=3-3=0$

$Dy=\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix}=12-20=-8$

$x=\dfrac{Dx}{D}=\dfrac{0}{-8}=0 \qquad y=\dfrac{Dy}{D}=\dfrac{-8}{-8}=1$

في هذا المثال نلاحظ ان معاملات y و b متساوية لذلك
y=1 و x=0

يتبع

Ex/ use Cramer's rule to solve the vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{pmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$