Group theory

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تاليف ديفيد بيرتون وترجمه عبد العالي جاسم

Chapter one

Binary Operations

Definition 1.1
Let A be a non empty set. A binary operation on a set A is a function from $A \times A$ into A. (i.e.)

$*: A \times A \to A$ is a binary operation iff

1. $a * b \in A, \forall a, b \in A$ (Closure)
2. If $a, b, c, d \in A$ such that $a = c$ and $b = d$, then $a * b = c * d$ (well-define).

Example 1.2

1) The operations $\{+, -, \times\}$arebinary operations on R, Z, Q, C.
But " $-$ " is not binary operation on N.

2) The operations $\{+, -\}$arenotbinary operations onO **(**odd number).

3) The operation $\div$is abinary operation on $R\backslash\{0\}, Q\backslash\{0\}, C\backslash\{0\}$.

Example 1.3

Let$a * b = a + b + 2, \forall a, b \in Z^+$**.** Is$*$abinary operation on Z^+?

Solution:

1) Closure: let $a, b \in Z^+$, then $a * b = \overbrace{a + b}^{\in Z^+} + 2 \in Z^+$.

2) well-define: $a, b, c, d \in A$ such that $a = c$ and $b = d$, then $a * b = a + b + 2 = c + d + 2 = c * d$
$\Rightarrow *$ is abinary operation on Z^+.

Example 1.4

Let$a * b = a^b$,$a, b \in Z$. Show that$*$is abinary operation on Z.

Solution:

1) Closure: if $a = 3$ and $b = -1$. Then $a * b = 3^{-1} = \frac{1}{3} \notin$ Z$\Rightarrow *$ is not a binary operation on Z.

Remark 1.5: Some time we used the symbols $*$, $\circ$, $\#$, $\odot$, ... to denote abinary operation.

Exercises: which of the following are binary operations?

1) $a * b = a + b, \forall a, b \in R\backslash\{0\}$.

2) $a \odot b = \frac{a}{b}, \forall a, b \in Z$.

3) $a \# b = a + b - 3, \forall a, b \in N$.

4) $a \circ b = a + 2b - 5, \forall a, b \in R$.

5) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \forall \frac{a}{b}, \frac{c}{d} \in Q\backslash\{0\}$.

Definition 1.6 (Commutative)

A binary operation $*$ on a set$A$ is called a Commutative if and only if $a * b = b * a \forall a, b \in A$.

Definition 1.7 (Associative)

A binary operation $*$ on a set A is called an associative if $(a * b) * c = a * (b * c)$ $\forall a, b, c \in A$.

Example 1.8 Let R be a set of real numbers and $*$ be a binary operation on R defined as $a * b = a + b - ab$, then $*$ is commutative and associative.

Solution:

(i) $a * b = a + b - ab = b + a - ba = b * a$

Which implies that $*$ is commutative.

(ii) Let $a, b, c \in R$, then

$$(a * b) * c = (a + b - ab) * c = (a + b - ab) + c - (a + b - ab)c$$

$$= a + b + c - ab - ac - bc + abc \ldots. (1)$$

$$a * (b * c) = a * (b + c - bc)$$

$$= a + (b + c - bc) - a(b + c - bc)$$

$$= a + b + c - bc - ab - ac + abc \ldots (2)$$

$\Rightarrow$ (1)=(2)

$\Rightarrow *$ is associative.

Exercises: which of the following binary operations is a comm., asso.?

(i) $a * b = a - b, \forall a, b \in Z.$

(ii) $a \odot b = 2ab, \forall a, b \in E.$

(iii) $a \# b = a^3 + b^3 \forall a, b \in R.$

Definition 1.9 (**Mathematical System**)

A Mathematical System or (Mathematical Structure) is a non-empty set of elements with one or more binary operations defined on this set.

Example 1.10

$(R, +), (R, .), (R, -), (R\backslash\{0\}, \div), (R, +, .), (N, +), (E, +, \times)$ are Math. System. But $(N, -), (R, \div), (O, +, -)$are not Math. System.

Definition 1.11 (Semi group)

A semi group is a pair $(S, *)$ in which S is an empty set and $*$ is a binary operation on S with associative law.

(i.e.) $(S, *)$ is semi group$\Leftrightarrow$ (1) $S \neq \emptyset$,

(2) $*$ is a binary operation,

(3) $\forall a, b, c \in S$, (a $*$b) $*$c = a $*$(b $*$c).

Example 1.12

(1) $(Z, +), (Z, \times), (N, +), (N, \times), (E, +), (E, \times)$ are semi groups.

(2) $(O, +), (Z, -), (E, -), (R\backslash\{0\}, \div)$ are not semi groups.

Definition 1.13 (**The identity element**)

Let $(S, *)$ be a Mathematical System and $e \in S$. Then e is called an identity element if $a * e = e * a = a, \forall a \in S.$

Definition 1.14 (The inverse element)

Let $(S,*)$ be a Mathematical System and $a, b \in S$. Then b is called an inverse of a if $a * b = b * a = e$.

Definition 1.15 (The Group)

The pair $(G,*)$ is a group iff $(G,*)$ is a semi group with identity in which each element of G has an inverse.

Definition 1.16 (The Group)

A group $(G,*)$ is a non-empty set G and a binary operation $*$, such that the following axioms are satisfied:

(1) The binary operation $*$ is associative.

(i.e.) $(a * b) * c = a * (b * c), \ \forall a, b, c \in G$

(2) There is an element e in G such that $a * e = e * a = a, \forall a \in G$.

This element e is an identity element for $*$ on G.

(3) for each a in G, there is an element b in G such that $a * b = b * a = e$.

The element b is an inverse of a and denoted by a^{-1}.

Remark 1.17

Every group is a semi group but the converse is not true as in the following example shows.

$(N, +)$ is a semigroup but not group because $\nexists a^{-1} \in N, \forall a \in N$.

Definition 1.18 (Commutative group)

A group $(G,*)$ is called a Commutative group iff $a * b = b * a, \forall a, b \in G$.

Example 1.19

i. $(Z, +), (E, +), (Q, +), (N, \times), (C, +)$ are commutative groups .

ii. $(Z^{+},+)$ is not a group because there is no identity element for $+$ in Z^{+}.

iii. $(Z^{+},\times)$ is not a group because there is an identity element 1 but no inverse of 5.

iv. $(G=\{1,0,-1,2\},+)$ is not group since $+$ is not a binary operation on G, 1+2=3$\notin G$.

v. $(G=\{1,-1\},\times)$ is comm. Group.

vi. $(R\backslash\{0\},\times)$, $(Q\backslash\{0\},\times)$,$(C\backslash\{0\},\times)$ are comm. Groups.

Example 1.20

Let $G=\{a,b,c,d\}$ be a set. Define a binary operation $*$ on G by the following table.

$*$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Is $(G,*)$ a commutative group?

Solution:

(1) Closure is true.

(2) Asso.

(a ∗b) ∗c = a ∗(b ∗c) ?

a∗ d=b ∗ c

d = d

b∗(a∗c)=b∗c=d=(b∗a)∗c

c∗(a∗b)=c∗b=d=(c∗a)∗b

d∗(a∗c)=d∗c=b=(d∗a)∗c ....→

$\Rightarrow *$ is asso.

(3) The identity: To prove $\exists e \in G\ \ s.t. a * e = e * a = a, \forall a \in G$.

$a * a = a,\ b * a = b,\ c * a = c,\ d * a = d$.

$\Rightarrow e = a$ is an identity element of G.

(4) The inverse:

$a * a = a \Rightarrow a^{-1} = a$

$b * d = a \Rightarrow b^{-1} = d$

$c * c = a \Rightarrow c^{-1} = c$

$a * a = a \Rightarrow a^{-1} = a$

$d * b = a \Rightarrow d^{-1} = b$

(5) Comm.

$a * b = b * a$?

$b = b$

$a * c = c * a = c$

$a * d = d * a = d$

$b * c = c * b = d$

$b * d = d * b = a$

$c * d = d * c = b$

$\Rightarrow\ *$ is a comm.

Therefore $(G,*)$ is a comm. Group and called Klein 4-group.

Example 1.21

Let $G = \{1, -1, i, -i\}$ be a set and "." be abinary operation.

Is $(G, .)$ a group ?

Solution:

.	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

1- Closure is true.

2- Asso. Law is true

3- 1 is an identity element.

4- $1^{-1} = 1$, $-1^{-1} = -1, i^{-1} = -i, -i^{-1} = i$

5- Comm .is true

$\therefore$ (G , .) is a comm.group.

Example 1.22

Let G = Z , $a * b = a + b + 2$, show that (G , $*$) is a comm . group.

Solution:

1- Closure : let a, b $\in$Z , Then

$a * b = a + b + 2 \in Z$ →Closure is true

2- asso. Low : Let a, b, c $\in$Z , then

$a *(b * c) = a * (b + c + 2)= a + (b + c + 2) + 2$

$= a + b + c + 4$ ……..(1)

$(a * b) * c = (a + b + 2) * c= (a + b + 2) + C + 2$

$= a + b + c + 4$ …….(2)

$\therefore$ (1) = (2) $\Rightarrow *$ is asso .

3- Identity : let e $\in$**Z** $\ni a * e = e * a = a$, then

$a * e = a + e + 2 = a \Longrightarrow e = -2$

$e * a = e + a + 2 = a \Longrightarrow e = -2$

$\therefore$ -2 is an identity element of G.

4- Inverse : let $a , b \in Z \ni a * b = b * a = e$

$a * b = a + b + 2 = -2 \Longrightarrow b = -a - 4$

$b * a = b + a + 2 = -2 \Longrightarrow b = -a - 4$

$\therefore a^{-1} = -(a + 4) \in Z$

$\therefore$ (G , $*$) is a group.

5- Comm. To prove $a * b = b * a \ \forall a, b \in Z$

$a * b = a + b + 2 = b + a + 2 = b * a$

$\therefore$ (G, $*$) is a comm. Group.

Example 1.23:

Let $G = \{f_1, f_2, f_3, f_4\}$, where $f_i \ni i = 1,2,3,4$, are mappings on $R\backslash\{0\} \ni \quad f_1(x) = x, f_2(x) = -x , f_3(x) = \frac{1}{x} , f_4(x) = -\frac{1}{x}$.

Show that $(G, \circ)$ is a group.

Solution:

$\circ$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1

1- Closure is true.

For example: $f_1 \circ f_2 (x) = f_1(f_2(x))$

$= f_1(-x)$

$= - x = f_2$

2- Asso. is true. (H .W)

3- The identity : the identity element of G is f_1 , since

$f_1 \circ f_1 = f_1$, $f_2 \circ f_1 = f_2$, $f_3 \circ f_1 = f_3$, $f_4 \circ f_1 = f_4$.

4- The inverse:

$f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$, $f_4^{-1} = f_4$

Is (G , ∗) Comm . ??

Example 1.24

Let **G =** R×R = $\{(a, b) : a, b \in R, a \neq 0\}$ and ∗ be defined by

$(a, b) * (c, d) = (ac, bc + d)$

Prove that (G , ∗) is not comm . group

Solution:

1- Closure : let $(a, b), (c, d) \in G \Rightarrow a \neq 0, \; c \neq 0 \Rightarrow ac \neq 0$

$(a, b) * (c, d) = (ac, bc + d) \in G \;\; ac \neq 0$

2- Asso. : Let $(a, b), (c, d), (e, f) \in G$, we have

$(a, b) * [(c, d) * (e, f)] = (a, b) * (ce, de + f) = (ace, bce + de + f)$ ……(1)

$[(a, b) * (c, d)] * (e, f) = (ac, bc + d) * (e, f)$

$= (ace, (bc + d)e + f)$

$= (ace, bce + de + f)$….. (2)

∴ (1) = (2) , then asso. is true

3-*Identity*: *Let* $(a, b), (x, y) \in G \ni$

$$(a,b) * (x,y) = (x,y) * (a,b) = (a,b)$$

$$(a,b) * (x,y) = (ax, bx+y) = (a,b)$$

$\therefore$ ax = a $\rightarrow x = 1$

$$bx + y = b \rightarrow b + y = b \rightarrow y = 0$$

$$\therefore (x,y) = (1,0)$$

$$(x,y) * (a,b) = (xa, ya+b) = (a,b)$$

$\therefore xa = a \rightarrow x = 1$

$$ya + b = b \rightarrow ya = b - b \rightarrow ya = 0 \rightarrow y = 0$$

$$\therefore (x,y) = (1,0)$$

$$\therefore (1,0) \text{ is an identity element of } G$$

4-inverse: $Let\ (a,b),\ (c,d) \in G\ , a \neq 0, c \neq 0$

$$(a,b) * (c,d) = (c,d) * (a,b) = (1,0)$$

$$(c,d) * (a,b) = (1,0)$$

$$(ac, bc+d) = (1,0) \rightarrow ac = 1 \rightarrow c = \frac{1}{a}$$

$$bc + d = 0 \rightarrow b\frac{1}{a} + d = 0 \rightarrow d = -\frac{b}{a}$$

$$\therefore (c,d) = \left(\frac{1}{a}, \frac{-b}{a}\right) \text{ is an inverse of } G$$

(5) Comm : G is not comm. , since Take (3,5) , (4,6)

$(3,5) * (4,6) = (12,26)$ $\rightarrow$ G is not comm..

$(4,6) * (3,5) = (12, 23)$

Example 1.25

Let (G , *) be an arbitrary group .The set of the function from G in to G with the composition (F_G , o) is forms a group , where

$F_G = \{ f_a : a \in G \}$, $f_a : G \to G$ s.t .

$f_a(x) = a * x$, $x \in G$, prove that

Proof :

(1) Closure: let $f_a, f_b \in F_G$, $a,b \in G$

$(f_a \circ f_b)(x) = f_a(f_b(x)) = f_a(b * x)$

$= a * (b * x)$

$= (a * b) * x$, since G is a group .

$= f_{a*b}(x) \in F_G$, since $a*b \in G$

(2) asso : Let $f_a, f_b, f_c \in F_G$, $a,b,c \in G$

$(f_a \circ f_b) \circ f_c = f_{a*b} \circ f_c = f_{(a*b)*c}$

Since * is asso. on G

$= f_{a*(b*c)} = f_a \circ f_{b*c} = f_a \circ (f_a \circ f_c)$

(3) identity : f_e is an identity of FG, since

$f_a \circ f_e = f_{a*e} = f_{e*a} = f_e \circ f_a = f_a$

(4) inverse : The inverse of f_a in F_G is f_a^{-1}, since

$F_a \circ f_a^{-1} = f_{a*a^{-1}} = f_{a^{-1}*a} = f_a^{-1} \circ f_a = f_e$

Also, if G is a Comm . group, then (F_G, o) is a comm. group .

<u>**(Exercises):**</u> Determine the systems (G , ∗) described abelian (comm..) group

1) $G = Z$, $a * b = a+b+3$
2) $G = R \times R = \{ (a, b) : a , b \in R\}$ s.t
   $(a, b) * (c,d) = (a+b, b+d + 2bd)$.
3) ($G = \{f_1 , f_2, f_3, f_4, f_5, f_6 \}$, o) , where
   $f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = 1-x$, $f_4(x) = \frac{x-1}{x}$, $f_5(x) = \frac{x}{x-1}$, $f_6(x) = \frac{1}{1-x}$.

4) $G = \{ (a,b) : a, b \in R , a \neq 0 , b \neq 0 \}$ s.t.
$(a,b) * (c,d) = (ac,bd)$
5) $(G = \{ an : n \in z \} , +)$
6) $G = Q^+ , a * b = \frac{ab}{2}$.

Some properties of Groups:

Theorem (1) : If G is a group with a binary operation $*$, then the left and right cancellation laws hold in G , that is:

1) $a * b = a * c$ implies $b = c$

2) $b * a = c * a$ implies $b = c$

For all $a, b, c \in G$.

Proof :

1) suppose $a*b = a*c$

$\exists\ a^{-1} \in G$ s . t . $a^{-1} * (a * b) = a^{-1} * (a * c)$

$(a^{-1} * a) * b = (a^{-1} * a) * c$

$e * b = e * c$

$\therefore b = c$

2) H.W

Theorem(2): In a group $(G , *)$, there is only one element e in G such that $e * a = a * e = a, \quad \forall a \in G$.

Proof:

Suppose that G has two identity elements e and $e^{/}$ that mean $\forall a \in G$.

$a * e = e * a = a$ and $a * e^{/} = e^{/} * a = a$

Since each e and $e^{/}$ belong to G , so

$e * e^{/} = e^{/} * e = e$ (e عنصر و $e^{/}$ عنصر محايد)
and

$e' * e = e * e'$ (عنصر محايد e عنصر و e')

It follows that $e' = e$.

Theorem(3): In a group (G , ∗), the inverse element of each element in G is unique.

Proof :

Let $a \in G$ and a has two inverse x and x' . Such that $a * x = x * a = e$
$a * x' = x' * a = e$

$$\Longrightarrow x = x * e = x * (a * x') = (x * a) * x' = e * x' = x'$$

$\therefore \quad x = x' \Longrightarrow$ the inverse is an unique element.

Theorem(4): If (G , ∗) is group , then

1- $e^{-1} = e$
2- $(a^{-1})^{-1} = a \quad \forall \ a \in G$
3- $(a * b)^{-1} = b^{-1} * a^{-1} \forall \quad a , b \in G$

Proof :-

1- Let $e^{-1} = x$

e is the identity element of G $\Longrightarrow x * e = e * x = x$ ---- (1)

x is the inverse of $e \Longrightarrow e * x = x * e = e$ ------- (2)

from (1) and(2) $\Longrightarrow x = e \Rightarrow e^{-1} = e$.

2- $(a^{-1})^{-1} = (a^{-1})^{-1} * e$

$= (a^{-1})^{-1} * (a^{-1} * a)$

$= ((a^{-1})^{-1} * a^{-1}) * a$

$= e * a = a.$

3) $(a * b)^{-1} = b^{-1} * a^{-1}, \quad \forall a , b \in G$

Proof :

Since $(a * b) \in G \Longrightarrow (a * b)^{-1} \in G$

$(a * b) * (a * b)^{-1} = (a * b)^{-1} * (a * b) = e$ (def . of inverse)

$(a * b) * (a * b)^{-1} = e$

$a^{-1} * (a * b) * (a * b)^{-1} = a^{-1} * e$

$(a^{-1} * a) * b * (a * b)^{-1} = a^{-1}$

$e * b * (a * b)^{-1} = a^{-1}$

$b^{-1} * b (a * b)^{-1} = b^{-1} * a^{-1}$

$e * (a * b)^{-1} = b^{-1} * a^{-1}$

$\therefore (a * b)^{-1} = b^{-1} * a^{-1}$

Theorem(5) : Let (G , ∗) be a group . Then

i- $(a * b)^{-1} = a^{-1} * b^{-1} \Longleftrightarrow$ G is comm. group.

Proof :

($\Longrightarrow$) Let (G , ∗) be a group and $(a * b)^{-1} = a^{-1} * b^{-1}$. To prove G is comm.

Let a , b ∈ G . To prove $a * b = b * a , \forall a , b \in G$

$a * b =((a * b)^{-1})^{-1}$ (by $(a^{-1}) = a$)

$= (b^{-1} * a^{-1})^{-1}$ (by Th.4)

$= (b^{-1})^{-1} * (a^{-1})^{-1}$

$= b * a$ (by $(a^{-1})^{-1} = a$)

∴ G is comm. gp.

$(\Leftarrow)$ Let $(G, *)$ is a comm . gp. To prove $(a * b)^{-1} = a^{-1} * b^{-1}$

$(a * b)^{-1} = b^{-1} * a^{-1}$ (by Th.4)

$= a^{-1} * b^{-1}$ (by comm.)

ii) if $a = a^{-1}$ then G is comm . gp . (Is the converse true?)

proof :

Let $a = a^{-1}$ T. P. $a * b = b * a$, $\forall a, b \in G$

Let $a, b \in G$ and $a * b \in G \Rightarrow (a * b) = (a * b)^{-1}$

$= b^{-1} * a^{-1}$ (by Th.4)

$= b * a$

$\therefore$ G is comm. Group.

The converse of this part is not true.

(i-e.) if $(G, *)$ is comm . $\not\Rightarrow$ $a = a^{-1}$

For example:

Let $(G = \{1, -1, i, -i\}, .)$ be comm . group,

Let $a = i \Rightarrow a^{-1} = -i$

$\therefore a \neq a^{-1}$

Give another example (H. W.)

Theorem (6): In a group $(G, *)$, the equations $a * x = b$ and $y * a = b$ have a unique solution.

proof : we take

$a * x = b \Rightarrow a^{-1} * (a * x) = a^{-1} * b$

$(a^{-1} * a) * x = a^{-1} * b$

$e * x = a^{-1} * b$

$x = a^{-1} * b$

To show the solution is a unique

Let x' $\in$ G s.t. a $*$ x' = b

$\Longrightarrow$ a $*$ x' = a $*$ x

$\Longrightarrow$ x' = x (by com. law)

By same way, we prove $y * a = b$ has Solution $y = b * a^{-1}$.

Definition.(The integral powers of *a*)

Let (G , $*$) be a group . The integral powers of a, $a \in$ G is defined by :

1- $a^n = \underbrace{a * a \ldots * a}_{n-times}$

2- $a^0 = e$

3- $a^{-n} = (a^{-1})^n, n \in Z^+$

4- $a^{n+1} = a^n * a \quad , n \in Z^+$.

For example :

(1) In (R , +) ,

$3^0 = 0$,

$3^3 = 3 + 3 + 3 = 9$,

$3^{-2} = (3^{-1})^2 = (-3) + (-3)$

$= -6$.

(2) In (R , .) ,

$2^0 = 1$,

$2^3 = 2 \times 2 \times 2 = 8$,

$2^{-4} = (2^{-1})^4 = (\frac{1}{2})^4$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{16}$

(3) In (G = { 1, -1 , i , -i}, .) ,

$i^o = 1$, $i^2 = i \times i = -1$, $i^{-2} = (i^{-1})^2 = (-i)^2$

$= -i \times -i$

$= -1$

Theorem:

Let (G , ∗) be agroup and $a \in G$, m , n ∈ Z , then :

1- $a^n * a^m = a^{n+m}$ $\forall n, m \in Z$ (H. W.)
2- $(a^n)^m = a^{nm}$ $\forall n, m \in Z^+$
3- $a^{-n} = (a^n)^{-1}$ $\forall n \in Z^+$
4- $(a * b)^n = a^n * b^n$ $\forall n \in Z \Leftrightarrow$ G is comm . group.

Proof :

2- T.P. $(a^n)^m = a^{nm}$, $\forall n, m \in Z^+$

Let p(m) : (($(a^n)^m = a^{nm}$ $\forall n \in Z^+$))

T.P. is true $\forall m \in Z^+$

If m = 1 ⟹ p(1) : $(a^n)^1 = a^n = a^{n \times 1}$ ⟹ p(1) is true

Suppose that p(k) is true with $k \in Z^+$ and $k \le m$

∴ $(a^n)^k = a^{nk}$

We have to prove that p (k + 1) is true

P (k+ 1) : $(a^n)^{k+1} = a^{n(k+1)}$??

$(a^n)^{k+1} = (a^n)^k * (a^n)^1$ (by define of $a^{n+1} = a^n * a^1$)

$= a^{nk} * a^n$

$= a^{nk+n}$ by (1) above

$= a^{n(k+1)}$

$\therefore$ p (k + 1) is true

By the principle of mathematical induction

$\Longrightarrow$ p (m) is true $\forall$ m $\in Z^+$

$\therefore (a^n)^m = a^{nm}$, $\forall$ n , m $\in Z^+$

3 - T.P. $a^{-n} = (a^{-1})^n = (a^n)^{-1}$, $\forall$ n $\in Z^+$

If n = 1 $\Longrightarrow$ p(1) : $(a^{-1})^1 = a^{-1} = (a^1)^{-1}$

Suppose that if n = k is true $\Longrightarrow$ p (k) = $(a^{-1})^k = (a^k)^{-1}$

We must prove p (k+1) is true

P (k+1) : $(a^{-1})^{k+1} = (a^{k+1})^{-1}$?

$(a^{-1})^{k+1} = (a^{-1})^k * (a^{-1})^1 = (a^k)^{-1} * (a^1)^{-1} = (a^{k+1})^{-1}$

$\therefore$ p (k +1) is true

By the principle of math. ind. $\Longrightarrow$ p (n)is true , $\forall$ n $\in Z^+$.

4-($\Longrightarrow$) If n = 2 $\Longrightarrow$ $(a * b)^2 = a^2 * b^2$, T.P. G is comm. Group.

(a ∗ b) ∗ (a ∗ b) = a ∗ a ∗ b ∗ b (by def . of power int.)

a ∗ (b ∗ a) ∗ b = a ∗ (a ∗ b) ∗ b (by asso .)

(b ∗ a) ∗ b = (a ∗ b) ∗ b (by cancellation law)

b ∗ a = a ∗ b (by cancellation law)

$\therefore$G is comm . group.

($\Longleftarrow$) Let G be comm . group . T.P $(a * b)^n = (a^n * b^n)$, $\forall$ n $\in$ Z.

Let p (n) : $(a * b)^n = a^n * b^n$

I f n = 1 $\Longrightarrow (a * b)^1 = a^1 * b^1$ is true

Suppose that p (k) is true with $k \in Z^+$ and $k \leq n$

s.t . $(a * b)^k = a^k * b^k$

We must prove P (k +1) is true

P(k+1) : $(a * b)^{k+1} = (a * b)^k * (a * b)^1$

$= a^k * b^k * a^1 * b^1$

$= (a^k * b^k) * (b * a)$ since G is comm .

$= a^k * (b^k * b) * a$ (by asso .)

$= a^k * a * b^{k+1}$

$= a^{k+1} * b^{k+1}$

∴ p(k + 1) is true , $\forall n \in Z^+$

Definition: ((**order of a group**))

The number of elements of a group G is called the order of G and is denoted by | G | or o (G).

G is called a finite group if $|G| < \infty$ and infinite group otherwise .

Definition (**the order of an element**)

The order of an element a , $a \in G$ is the least positive integer n such that $a^n = e$, where e is the identity element of G . We denoted to order a by $|a|$ or o (a) .

(i.e.) $|a| = n$ if $a^n = e$, $n \in Z^+$

Example (1): (Z , +) is an infinite group

Example (2): the trivial group G = { 0 }

| G | = 1 , G is the only group of order 1.

Example (3): find the order of G and the order of each element of (G, .)

. Such that G = { 1, -1 , i, -i}.

Ans.

$|G| = 4$ and

$|a| : a = 1$, then $|a| = |1| = 1$ (since $e = 1$)

If $a = -1$, then $|-1| : (-1)^2 = 1 \Longrightarrow |-1| = 2$

If $a = i$, then $|i| : i^2 = -1$, $i^4 = 1 \Longrightarrow |i| = 4$

If $a = -i$, then $|-i| : -i^2 = -1$, $-i^3 = i$, $-i^4 = 1$

$\therefore |-i| = 4$

((The group of integers modulo *n*))

Definition: Let $a , b \in$, Z , $n > 0$. Then a is congruent to b modulo n if $a - b = nk$, $k \in Z$ and denoted by $a \equiv b$ or $a \equiv b \pmod{n}$

1- $17 \equiv 5 \pmod{6}$, sine $17 - 5 = 12 = (6)(2)$
2- $8 \equiv 4 \pmod{2}$, since $8 - 4 = 4 = (2)(2)$
3- $-12 \equiv 3 \pmod{3}$, since $-12 - 3 = -15 = (3)(-5)$
4- $5 \not\equiv 2$ (mode2), since $5 - 2 = 3 \neq (2)(k)$, $\forall\ k \in Z$

Theorem: The congruence module n is an equivalence relation on the set of integers.

Proof:

Let $a , b , c \in Z$, $n > 0$

1- $a - a = 0 = (n)(0)$

$\therefore a \equiv a \pmod{n}$ reflexive is true

2- if $a \equiv b \pmod{n}$, T. P. $b \equiv a \pmod{n}$

$\therefore a \equiv b \pmod{n} \Longrightarrow a - b = nk$, $k \in Z$ so , $b - a = -nk = (n)(-k)$, $-k \in Z$

$\therefore b \equiv a \pmod{n} \Longrightarrow$ symmetric is true

3- If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$

T. P. $a \equiv c \pmod{n}$

Since $a \equiv b \pmod{n}$, then $a\text{-}b = nk$

And $b \equiv c \pmod{n}$, then $b - c = nk^{/}$

By adding these two eqs . $\Longrightarrow$ $a - c = n (k + k^{/})$, $k + k^{/} \in Z$

$\therefore a \equiv c \pmod{n}$

$\Longrightarrow$ Transitive is true

$\therefore$ The congruence modulo n is an equivalence relation .

Definition:

Let $a \in Z$, $n > o$. The congruence class of a modulo n, denoted by [a] is the set of all integers that are congruent to a modulo n .

(i.e.)

$[a] = \{ z \in Z : z \equiv a \pmod{n} \}$

$= \{ z \in Z : z = a + kn , k \in Z \}$

Example(1):

If n = 2 , find [0] , [1]

$[0] = \{ z \in Z : z \equiv 0 \pmod{2} \}$

$= \{ z \in Z : z = 0 + 2K , K \in Z \}$

$= \{ 0 , \mp 2 , \mp 4 , \ldots\ldots\ldots. \}$

$[1] = \{ z \in Z : z \equiv 1 \pmod{2} \}$

$= \{ z \in Z : z = 1 + 2k , k \in \}$

$= \{ \mp 1 , \mp 3 , \mp 5 , \ldots. \}$.

Example(2):

If n = 3 , find [1] , [7]

[1] = { z ∈ Z : z ≡ 1 (mod 3) }

= { 1 , 1 ∓ 3, 1 ∓ 6 …. }

= { 1 , -2 , 4 , 7 , -5 , ….}.

[7] (H. W.)

Definition:

The set of all congruence classes modulo *n* is denoted by Z_n (which is read Z mod *n*). Thus

Z_n = { [0] , [1] , [2] , …… , [n -1] }, or

$Z_n = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}\}$

Z_n has *n* elements.

Example:

$Z_1 = \{\bar{0}\}$

$Z_2 = \{ \bar{0}, \bar{1} \}$

$Z_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

Now, we define addition on Z_n (write $+_n$) by the following :

For any [a] , [b] ∈ Z_n [a] $+_n$ [b] = [a $+_n$ b]

Similarly, we define multiplication on Z_n (write "$\cdot_n$ " by the following :

[a] $\cdot_n$ [b] = [a $\cdot_n$ b] , ∀ [a] , [b] ∈ Z_n

It is easy to see that (Z_n , $+_n$) is an abelian group with identity [0] and for every [a] ∈Z_n , $[a]^{-1}$ = [*n* – a] . This group is called the Additive Group of integers modulo *n* .

Also, (Z_n , $\cdot_n$) is abelian semi group with identity [1] . It is called the multiplicative semi group of integers modulo *n*.

Example (1): $(Z_4, +_4)$

$+_4$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
1	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

$Z_4 = \{ \bar{0} , \bar{1} , \bar{2} , \bar{3} \}$

(1) Closure is true

(2) Asso. is true

(3) $\bar{0}$ is an identity element

(4) Inverse:

$\bar{1}^{-1} = \bar{4} - \bar{1} = \bar{3}$

$\bar{2}^{-1} = \bar{4} - \bar{2} = \bar{2}$

$\bar{3}^{-1} = \bar{4} - \bar{3} = \bar{1}$

(5) Comm : $\bar{1} + \bar{2} = \bar{3} = \bar{2} + \bar{1}$

$\bar{1} + \bar{3} = \bar{0} = \bar{3} + \bar{1}$

↓

$\therefore (Z_4, +_4)$ is a Comm.group.

Example (2): $(Z_4, \cdot_4)$

$\cdot_4$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
1	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

It is clear that we cannot have a group. Since the number $\bar{1}$ is identity but the numbers $\bar{0}$ and $\bar{2}$ have no inverse. It follows that $(Z_4, \cdot_4)$ is not a group, but it is semi group.

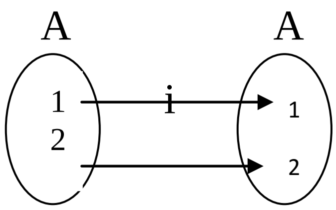
The Permutations : (التباديل)

Definition: A Permutation or symmetric of a set A is a function from A in to A that is both one to one and on to.

$$f : A \xrightarrow{1-1, onto} A$$

Symm (A) = $\{f \mid f : A \xrightarrow{1-1, onto} A\}$ the set of all permutation on A .
If A is the finite set {1,2,…, n}, then the set of all permutation of A is denoted by S_n or P_n and $o(S_n)$ = n! , where n! = n (n-1) … (3)(2) (1)

Example (1): Let A= {1,2} . Write all permutation on A.



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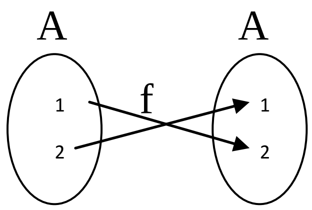


Symm(A) = {i , f } = $\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \}$.

Example (2): Let $A = \{1, 2, 3\}$. Write all Perm. on A.

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

$P_3 = Symm(A) = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

o(P_3)= 3 ! = (3)(2) = 6

<u>**Theorem :**</u> If $A \neq \varphi$, then the set of all permutation on A Forms agroup with composition of Mapps.

(i.e.) Let $\neq \varphi$, then (Symm(A) , o) is a group.

Proof :

Symm (A) = $\{f \mid f: A \xrightarrow{1-1, onto} A \text{ } is\ a\ mapp.\}$,

T.P. (Symm(A) ,o) is a group.

since $\exists\ i_A: A \xrightarrow{1-1, onto} A$ a perm. on A

$\therefore i_A \in$ Symm(A) $\Rightarrow$ Symm(A) $\neq \varphi$.

(1) Closure : Let f , $g \in$ symm(A) , it follows that

$$f: A \xrightarrow{1-1, onto} A \ , g: A \xrightarrow{1-1, onto} A$$

$\Rightarrow fog: A \xrightarrow{1-1, onto} A \Rightarrow fog \in$ Symm(A)

(2) Asso. : True since the composition of maps is an asso.

(3) The identity : since $i_A \in$ symm(A) and $i_A of = foi_A = f$

for all f in symm(A) $\Rightarrow i_A$ is an idenetity element

(4) The inverse : $\forall f: A \xrightarrow{1-1, onto} A, \exists f^{-1}: A \xrightarrow{1-1, onto} A$

$\therefore f^{-1} \in$ Symm(A) and $fof^{-1} = f^{-1}of = i_A$

$\therefore$ (Symm(A), o) is a group.

Is (Symm(A), o) comm. group ? (H.W.)

Example: Let $A = \{1,2,3\}$, then

$S_3 = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ and (S_3, o) is a group.

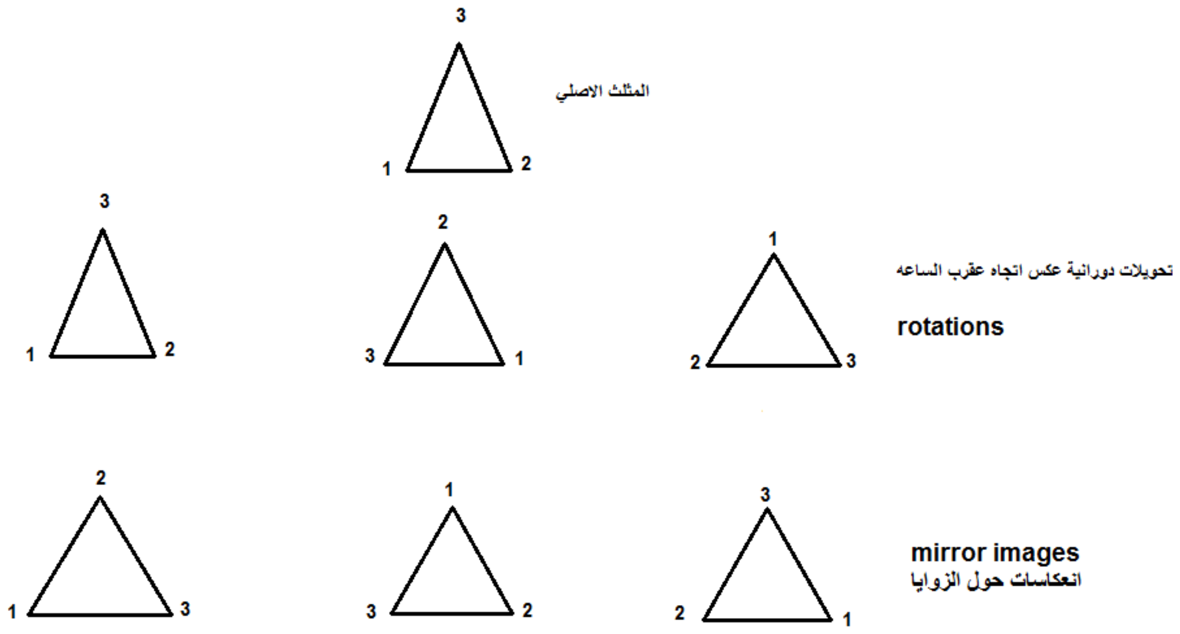
This group is called symmetric group.

o	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_3	f_1	f_6	f_4	f_5
f_3	f_3	f_1	f_2	f_5	f_6	f_4
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_6	f_4	f_3	f_1	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1

(S_3 , o) is not Comm. Group.

Also (S_3,o) is called the group of symmetries of on equilateral triangle .

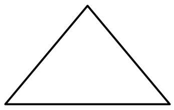
(زمرة تناظر المثلث متساوي الساقين)



Definition : (**The dihedral group D_n of order 2*n***)

The n^{th} dihedral group is the group of symmetries of the regular *n*-gon. $o(D_n) = 2n$

D_3 : is the third dihedral group.

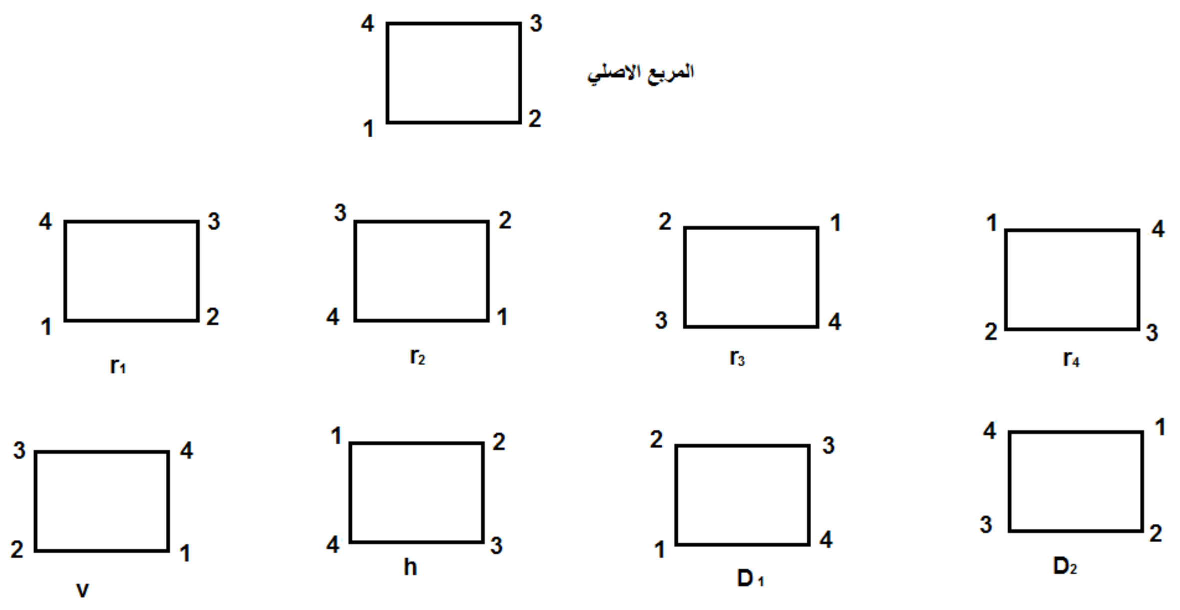


, $O(D_3) = (2)(3) = 6$ elements.

Example . The group of symmetries of square D_4 or Gs, $o(D_4) = 8$

$G_s = D_4 = \{r_1, r_2, r_3, r_4, v, D_1, D_2\}$, where r_i are a clockwise rotation

V, h, D_1, D_2 are mirror images



(1) Write all elements of Gs as a permutation.

(2) Is (Gs , o) comm. group? Use table (H.W.)

Definition: A permutation f of a set A is called a cycle of length n if there exist $a_1, a_2, \ldots\ldots, a_n \in A$ such that

$f(a_1) = a_2$, $f(a_2) = a_3, \ldots, f(a_{n-1}) = a_n$, $f(a_n) = a_1$ and $f(x) = x$,

for $x \in A$ but $x \notin \{a_1, a_2, \ldots\ldots, a_n\}$. We write $f = (a_1, a_2, \ldots, a_n)$.

Example: If $A = \{1, 2, 3, 4, 5\}$, then

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} = (1354)(2) = (1354)$$

Observe that

(1354) = (3541) = (5413) = (4135).

Example: (2) Let A = {1, 2, 3, 4, 5, 6} be a set of a group S_6 . Then

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix} = (142)o(3)o(56) = (142)o(56)$$

And

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 5 & 2 & 1 \end{pmatrix} = (16)o(245)o(3) = (16)o(245)$$

These permutations above are not cycles.

Theorem: Every permutation f of a finite set A is a product of disjoint cycles.

Definition: A cycle of length 2 is a transposition.

Example: The permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$ = (24) is a transposition.

Property: any permutation can be expressed as the product of transpositions.

(i.e.) $(a_1 a_2 \ldots a_n) = (a_1 a_2)(a_1 a_3) \quad \ldots..(a_1 a_n)$

Therefore any cycle is a product of transpositions.

Example: We see that (16) (2 5 3) = (16) (2 5) (2 3).

Definition: A permutation is even or odd according as it can be written as the product of an even or odd number of transpositions .

Example (1) Let $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in P_3$

Is f even or odd permutation .

Ans . $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ = (1 3 2) = (13) (12)

f has 2 transpositions $\Rightarrow f$ is an even perm.

Example(2): Determine an even and odd permutations of P_4.

(H.W)

Definition: "Alternating group " زمرة التباديل

The Alternating group on n letters, denoted by A_n is the group consisting of all even permutations in the symmetric group S_n.

$$o\,(A_n) = \frac{n\,!}{2} \quad , \quad A_n \subset S_n$$

Example(1): Let $S_3 = \{ f_1, f_2, f_3, f_4, f_5, f_6 \}$, then

$A_3 = \{i \, , f_2, f_3\}$ is a sub group of S_3

$o\,(A_3) = \frac{6}{2} = 3$

Example(2): Find A_4 from S_4

(H. W.)

Chapter Two

Subgroups and Cyclic Groups الزمر الجزئية والزمر الدائرية

Definition (1):

Let(G,∗) be agroup and H⊆G, H is anon-empty subset of G. Then(H,∗) is a subgroup of (G,∗) if (H,∗) is itself a group.

Definition (2)

Let (G,∗) be agroup and H ⊆G, Then (H,∗) is subgroup of G if :

(1) ∀a,b∈H⇒a∗b∈H

(2) The identity element of G is an element of H.e∈G⇒ e∈H

(3)∀a∈H⇒a^{-1}∈H

Remark (1):

Each group (G,∗) has at least two subgroup ({e},∗) and (G,∗), these subgroups are known trivial subgroup and improper, any subgroup different from these subgroups known a proper subgroup.

Examples (1):

1. (Z,+) is a proper subgroup of (R,+)

2. H={1, – 1} ⊆ {1,–1,i , –i}, then (H,.) is a subgroup of ({1,–1, i ,–i}, .)

3. H=$\{\bar{0},\bar{2}\}$⊆Z_4

   (H,$+_4$) is a proper subgroup of (Z_4,$+_4$). But $\{\bar{0},\bar{3}\}$ is not subgroup of (Z_4,$+_4$).

4. (Q\{0},×)is a subgroup of (R\{0},×).

Theorem (1): Let (G,∗) be a group and H≠ϕ, H⊆G. Then (H,∗) is a subgroup of (G,∗) iff $a*b^{-1} \in H, \forall\, a, b \in H$

Proof:

(⟹) let (H,∗) be a subgroup and a,b∈H, then

$a, b^{-1} \in H \Longrightarrow a*b^{-1} \in H$ (since ∗closure)

(⟸) Let $a*b^{-1} \in H$ T.P. (H,∗) is subgroup

(1) Since $H \neq \phi \Longrightarrow \exists\, b \in H$ s.t. $b*b^{-1} \in H \Longrightarrow e \in H$.

(2) Since b∈H and $e \in H \Longrightarrow e*b^{-1} \in H \Longrightarrow b^{-1} \in H$

(3) Let a∈ H and $b^{-1} \in H$ (by2) $\Longrightarrow a*(b^{-1})^{-1} \in H \Longrightarrow a*b \in H$

∴By definition (2) (H,∗) is a subgroup of (G,∗)

Example (2): Let (Z,+) be agroup and H={5a: a∈Z}. Show that (H,+) is a subgroup of (Z,+)

Solution: By The above, let x + y ∈ H, T.P. $x+y^{-1} \in H$

$x \in H \Longrightarrow x=5a, a \in Z$, $y \in H \Longrightarrow y=5b, b \in Z$

$x+y^{-1} = 5a + (5b)^{-1} = 5a+5(-b)$

$= 5\underbrace{(a-b)}_{\in Z} \in H$

⟹(H,+) is a subgroup of (Z,+)

Theorem (2): If $(H_i,*)$ is the collection of subgroups of (G,∗), then $(\cap H_i,*))$ is also subgroup of (G,∗)

Proof:

(1) Since $\exists\, e \in H_i, \forall i \Longrightarrow e \in \cap H_i \Longrightarrow \cap H_i \neq \phi$

(2) Let $x, y \in \cap H_i$ T.P. $x*y^{-1} \in \cap H_i$

Since $x, y \in \cap H_i \Longrightarrow x, y \in H_i \forall i$

$\Longrightarrow x*y^{-1} \in H_i, \forall i$ (since H_i subgroups)

$\Longrightarrow x*y^{-1} \in \cap H_i$

∴ $(\cap H_i,*)$ is subgroup of (G,∗)

Theorem (3): Let $(H_i,*)$ is the collection of subgroups of $(G,*)$ and let H_k and $H_j \in \{H_i\}$ such that $\exists H_\ell \in \{H_i\}$, $H_k \subseteq H_\ell$ and $H_j \subseteq H_\ell$ then $(\cup H_i,*)$ is also subgroup.

Proof:

(1) Since $\exists\ e \in H_i$ for some $i \Rightarrow e \in \cup H_i \Rightarrow \cup H_i \neq \phi$

(2) Let $x,y \in \cup H_i$, then $x,y \in H_k$ or $x,y \in H_j$, so $x,y \in H_\ell$

$\Rightarrow x*y^{-1} \in H_\ell$, (since H_ℓ subgroup)

$\Rightarrow x*y^{-1} \in \cup H_i$

$\therefore$ $(\cup H_i,*)$ is subgroup of $(G,*)$

Theorem (4): Let $(H_1,*)$ and $(H_2,*)$ are two subgroupsof $(G,*)$ then $(H_1 \cup H_2,*)$, is asubgroup of $(G,*)$ iff $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Proof:

$(\Rightarrow)$ Let $(H_1 \cup H_2,*)$ is a subgroup, T.P. $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Suppose that $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$

$\therefore \exists\ a \in H_1$, $a \notin H_2$ and $\exists\ b \in H_2$, $b \notin H_1$

$\therefore a*b \in H_1 \cup H_2 \Rightarrow a*b^{-1} \in H_1 \cup H_2$

$\Rightarrow a*b^{-1} \in H_1$ or $a*b^{-1} \in H_2$

$\Rightarrow a, b \in H_1$ or $a, b \in H_2$ C! (تناقض)

$\therefore H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

$(\Leftarrow)$ Let $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$ T.P. $(H_1 \cup H_2,*)$ is asubgroup

If $H_1 \subseteq H_2 \Rightarrow H_1 \cup H_2 = H_2$ is asubgroup.

If $H_2 \subseteq H_1 \Rightarrow H_1 \cup H_2 = H_1$ is asubgroup

$\therefore H_1 \cup H_2$ is asubgroup in two cases.

Remark (2): $(H_1 \cup H_2,*)$ need not be a subgroup of $(G,*)$.

For example: $H_1=\{r_1, r_3\}$ is a subgroup of G_s, and $H_2=\{r_1, v\}$ is a subgroup of G_s.

But $H_1 \cup H_2 = \{r_1, r_3, v\}$ is not a subgroup of G_s, since $r_3 \circ v = h \notin H_1 \cup H_2$

Definition (3): Let $(G,*)$ be a group and $(H,*)$, $(K,*)$ be two subgroups of G, then the product of H and K is the set:

$$H*K=\{h*k : h \in H, k\in K\}$$

Notes(1):

(1) $H*H$ is write H^2

(2) If $H=\{a\}$, then $H*K=a*K$. If $K =\{b\}$, then $H*K=H*b$.

(3)$H\cup K \subseteq H*K$.

Theorem (5): Let $(G,*)$ be a group and $(H,*)$, $(K,*)$ are two subgroups of $(G,*)$, then

(1) $H*K \neq \phi \wedge H*K \subseteq G$

(2)$H\subseteq H*K$ and $K\subseteq H*K$

(3)$(H*K,*)$ is a subgroup of $(G,*)$ iff $H*K=K*H$

(4) If $(G,*)$ is commutative group,then $(H*K,*)$ is asubgroup of $(G,*)$.

Proof:

(1)$\because$ $e \in H \quad \wedge e \in K \Rightarrow e * e= e\in H*K$

$$\therefore H*K \neq \phi$$

And let $x\in H*K \Longrightarrow x=a*b \ni a\in H \subseteq G$ and $b\in K\subseteq G$

$\Longrightarrow a\in G \quad \wedge \quad b \in G$

$\Longrightarrow a*b =x\in G$

$$\therefore H*K \subseteq G$$

(2) Let $x \in H \Longrightarrow x=x*e \in H*K$

$\Longrightarrow x\in H*K$

$$\therefore H\subseteq H*k$$

Similarly $K \subseteq H*K$

(3) (⇒) suppose (H∗K,∗) is asubgroup of (G,∗) T.P. H∗K=K∗H

(i.e.) H∗K ⊆ K∗H ∧ K∗H ⊆H∗K

Let x ∈ H∗K ⟹x =a∗b ∋ a∈H ∧ b ∈ K

Since H∗K is subgroupof G⟹x^{-1}∈ H∗K

Let x^{-1}= c ∗ d ∋ c ∈H∧ d∈K

x =$(x^{-1})^{-1}$ = $(c*d)^{-1}$ = $d^{-1}*c^{-1}$∋d^{-1} ∈K ∧c^{-1}∈ H

∴x = $d^{-1}*c^{-1}$∈ K∗H

∴H∗K⊆K∗H

K∗H ⊆ H∗K **(H.W.)**

(⇐) Let H∗K=K∗H T.P.(H∗K,∗) is subgroup of(G,∗)

H∗K≠ ϕand H∗K⊆G (by 1)

Let x,y ∈H∗K T.P.x∗y^{-1}∈H∗K

x∈H∗K ⟹x=a∗b ∋a∈ H ∧b ∈ K

y∈H∗K ⟹y=c∗d ∋ c ∈ H ∧d ∈ K

$x*y^{-1}$= $(a*b)*(c*d)^{-1}$

$=(a*b)*(d^{-1}*c^{-1})$

$=a*(\underbrace{b*d^{-1}}_{\in K})*\underbrace{c^{-1}}_{\in H}$

∴$(b*d^{-1})*c^{-1}$∈ K∗H=H∗K

∴$(b*d^{-1})*c^{-1}$∈ H∗K

⇒∃ p∈ H , ℓ ∈K ∋ $(b*d^{-1})*c^{-1}$ = p∗ℓ

∴$a*(b*d^{-1})*c^{-1}$ $=\underbrace{a*p}_{\in H}*\underbrace{\ell}_{\in K}$ ∈H∗K

∴x∗y^{-1}∈H∗k

∴(H∗K,∗) is subgroup of (G,∗)

(4) If (G,∗) is commutative group, then (H∗K,∗) is subgroup of (G,∗)

Proof:H∗K≠ϕand H∗K ⊆G   (by 1)

Let x,y ∈H∗K T.P. $x*y^{-1}$∈H∗K

x∈H∗K ⟹x=a∗b ∋a∈ H ∧ b ∈ K

y∈H∗K ⟹y=c∗d ∋ c ∈ H ∧ d ∈ K

$x*y^{-1}= (a*b)*(c*d)^{-1}$

$=(a*b)*(d^{-1}*c^{-1})$

$=(a*b)*(c^{-1}*d^{-1})$ (since G is commutative)

$=a*(b*)*d^{-1}$(∗ is associative)

$=(a*c^{-1})*(b*d^{-1})$ (∗ is commutativeand associative)

∴$x*y^{-1}$∈ H∗K

∴(H∗K,∗) is a subgroup of (G,∗)

Example (3): In (Z_8,$+_8$), Let H={$\bar{0},\bar{4}$}and K={$\bar{0},\bar{2},\bar{4},\bar{6}$}. Find H $+_8$K

Solution: H$+_8$K={$\bar{0},\bar{2},\bar{4},\bar{6}$}.

Notes (2): Let (H,∗) and (K,∗) are two subgroupof (G,∗), then :

(1)H∗K≠K∗H

(2)(H∗K,∗) need not be subgroup of (G,∗). Give example **(H.W.)**

Exercises: Is (H,∗) a subgroup of (G,∗) each of the following:

(1) (Z_8, $+_8$), H={$\bar{0},\bar{6}$}. Find H^2.

(2) (Z_4, $+_4$), H={$\bar{0},\bar{1},\bar{2}$}. Find H^2.

Definition (4): The center of a group (G,∗) denoted by cent(G) or C(G) is the set C(G)={c∈G: c∗x =x∗c, ∀x∈G}   العناصر التي تتبادل مع كل عناصر الزمرة

Note (3):C(G)≠ϕ, since ∃e∈G s.t.

e∗x=x∗e ∀x∈G ⇒e∈ C(G)

Examples (4):

(1) The group $(R\backslash\{0\}, .)$

$C(R)=R$ since R with multiplication is commutative

(2) The group $(S_3,\circ)$, $C(S_3)=\{f_1\}$

Since $C(S_3)=\{f\in S_3: f\circ g = g\circ f\ \forall g\in S_3\}=\{f_1\}$

Theorem (6): Let $(G,*)$ be agroup. Then $(cent(G),*)$ is a subgroupof $(G,*)$.

Proof:

$cent(G) \neq \phi$ (by note (3))

$C(G)=\{a\in G: x*a=a*x,\ \forall x\in G\}\subseteq G$

Let $a,b \in cent(G)$ T.P. $a*b^{-1}\in cent(G)$

$a\in cent(G)\Rightarrow a*x = x*a,\ \forall x\in G$

$b\in cent(G)\Rightarrow b*x = x*b,\ \forall x\in G$

T.P. $(a*b^{-1}) *x=x*(a*b^{-1})\ \forall x\in G$

$(a*b^{-1}) *x= a*(b^{-1}*x)$

$= a*(x^{-1}*b)^{-1}$

$=a*(b*x^{-1})^{-1}$(since $b \in cent (G)$)

$= a*(x*b^{-1})$

$=(a*x)*b^{-1}$

$=(x*a)*b^{-1}$(since $b \in cent (G)$)

$= x*(a*b^{-1})$

$\therefore (a*b^{-1})\in cent (G)$

$\therefore$ (cent (G),$*$) is asubgroupof $(G,*)$

Theorem(7): Let $(G,*)$ be agroup. Then

$cent(G)=G\Leftrightarrow G$ is a commutative group.

Proof:

$(\Rightarrow)$ $\forall a\in G \Rightarrow a\in cent(G)$

$\therefore a*x = x*a, \forall x \in G$

$\therefore a*x = x*a, \forall x, a \in G$

$\therefore$G is commutativegroup

($\Leftarrow$) suppose that G is commutative group T.P. cent(G) =G

(i.e) T.P. cent(G) $\subseteq G \wedge G \subseteq$ cent(G)

By definitionof cent(G) we havecent(G)$\subseteq$G.

T.P. G$\subseteq$cent(G)

Let $x \in G$, G is commutative group$\Rightarrow x*a = a*x, \forall a \in G$

$\therefore x \in$ cent(G)$\Rightarrow G \subseteq$cent(G)

$\therefore$ cent G=G

Cyclic Groups (الزمر الدوارة أو (الزمر الدائرية

Definition (5): Let (G,$*$) be agroup and $a \in G$, the cyclic subgroup of G generated by the a is denoted by $<a>$ and defined as

$$<a>=\{a^k : k \in Z\}=\{\ldots, a^{-1}, a^0, a^1, \ldots\}$$

G=$<a>$ is called cyclic group.

- تسمى الزمرة دائرية او دوارة اذا امكن توليدها من عنصر واحد او اذا وجد عنصر يولدها

Definition (6):A group (G,$*$) is called cyclic group generated by $a$iff$\exists a \in G$ such that

$$G=<a>=\{a^k : k \in Z\}$$

Examples (5): In (Z_9,$+_9$)find the cyclic subgroup generated by$\bar{2}$,$\bar{3}$,$\bar{1}$

$<\bar{2}>=\{a^k : k \in Z\}=\{\ldots,(\bar{2})^{-3}, (\bar{2})^{-2}, (\bar{2})^{-1}, (\bar{2})^{0}, (\bar{2})^{1}, (\bar{2})^{2}, (\bar{2})^{3}, \ldots\}$

$$=\{\ldots,\bar{3},\bar{5},\bar{7},\bar{0},\bar{2},\bar{4},\bar{6},\ldots\} = \{\bar{0},\bar{1},\bar{2},,\ldots,\bar{8}\} = Z_9$$

$\therefore Z_9$ is cyclic groupgenerated by $\bar{2}$

$<\bar{3}>=\{\ldots,(\bar{3})^{-3},(\bar{3})^{-2},(\bar{3})^{-1},(\bar{3})^{0},(\bar{3})^{1},(\bar{3})^{2},(\bar{3})^{3},\ldots\}$

$=\{\ldots,\bar{3},\bar{6},\bar{0},\bar{3},\bar{6},\bar{0},\ldots\}=\{\bar{0},\bar{3},\bar{6}\}$ is cyclic subgroup of Z_9

$<\bar{1}>=\{\ldots,(\bar{1})^{-3},(\bar{1})^{-2},(\bar{1})^{-1},(\bar{1})^{0},(\bar{1})^{1},(\bar{1})^{2},(\bar{1})^{3},\ldots\}$

$=\{\ldots,\bar{6},\bar{7},\bar{8},\bar{0},\bar{1},\bar{2},\bar{3},\ldots\}=\{\bar{0},\bar{1},\bar{2},,\ldots,\bar{8}\}=Z_9$

$\therefore Z_9$ is cyclic groupgenerated by $\bar{1}$

Examples (6): In (Z,+) fined cyclic groupgenerated by1, 2, – 1

$<1>=\{1^k : k\in Z\}=\{\ldots,1^{-3},1^{-2},1^{-1},1^{0},1^{1},1^{2},1^{3},\ldots\}$

$=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}=Z$

$<2>=\{2^k : k\in Z\}=\{\ldots,2^{-3},2^{-2},2^{-1},2^{0},2^{1},2^{2},2^{3},\ldots\}$

$=\{\ldots,-6,-4,-2,0,2,4,6,\ldots \neq Z$

$<-1>=\{(-1)^k : k\in Z\}$

$=\{\ldots,(-1)^{-3},(-1)^{-2},(-1)^{-1},(-1)^{0},(-1)^{1},(-1)^{2},(-1)^{3},\ldots\}$

$=\{\ldots,2,1,0,-1,-2,\ldots\}=Z$

$\therefore$(Z,+) is cyclic groupgenerated by 1 and – 1

Examples (7): Is $(S_3,\circ)$ cyclic group ?

$<f_1>=\{f_1\}\neq S_3$

$<f_2>=\{f_2^k : k\in Z\}=\{\ldots,f_2^{-2},f_2^{-1},f_2^{0},f_2^{1},f_2^{2},\ldots\}$

$=\{\ldots,f_2,f_3,f_1,f_2,f_3,\ldots\}=\{f_1,f_2,f_3\}\neq S_3$

$<f_3>=\{f_1,f_2,f_3\}\neq S_3$

$<f_4>=\{f_1,f_4\}\neq S_3$

$<f_5>=\{f_1,f_5\}\neq S_3$

$<f_6>=\{f_1,f_6\}\neq S_3$

$\therefore (S_3,\circ)$ is not cyclic group.

Examples (8): In(Z_6,$+_6$) find cyclic group generated by $\bar{1},\bar{2},\bar{5}$ **(H.W.)**

Theorem (8): Every cyclic group is commutative.

Proof: Let (G,$*$) be acyclic group

$\therefore \exists a \in$G s.t. G=$<a>=\{a^k: k \in Z\}$T.P. G is commutative group

Let $x,y \in$G T.P. $x*y = y*x$, $\forall x,y \in$G

$\because x \in G = <a> \Rightarrow x = a^m \ni m \in Z$ and $y \in G = <a> \Rightarrow y = a^n \ni n \in Z$

$x*y = a^m * a^n = a^{m+n} = a^{n+m} = a^n * a^m = y * x$

$\therefore$G is commutativegroup

The convers of this theorem is not true, for example:

(G={e,a,b,c},$*$) s.t. $a^2 = b^2 = c^2 = e$

$a^2 = e \Rightarrow a*a = e \Rightarrow a^{-1} = a$

$b^2 = e \Rightarrow b*b = e \Rightarrow b^{-1} = b$

$c^2 = e \Rightarrow c*$c$= e \Rightarrow c^{-1} =$c

$e^{-1} = e \Rightarrow x^{-1} = x \forall x \in$G

$\therefore$ (G,$*$) is commutativegroup

But (G,$*$) is not cyclic group since:

$< e > = \{e\} \neq$G

$< a > = \{a^k : k \in Z\} = \{e,a\} \neq$G

$< b > = \{b^k : k \in Z\} = \{e,b\} \neq$G

$< c > = \{c^k : k \in Z\} = \{e,c\} \neq$G

$\therefore$(G,$*$) is not cyclic

Theorem (9): $<a> = <a^{-1}> \forall a \in$G

Proof:

$< a > = \{a^k : k \in Z\} = \{ (a^{-1})^{-k} \because -k\in Z\}$

$= \{ (a^{-1})^m \because$m$= -$k$\in Z\}$

$= <a^{-1}>$

Theorem (10): If (G,∗) is afinite group of order$n$generated by$a$, then G= $< a >$={a^k:k∈Z} ={$a^1, a^2, \ldots\ldots, a^n = e$}such that $n$is least positive integer $\ni a^n = e\,, (i.e.)$

o(a)=n= o(G) (رتبة العنصر الذي يولد الزمرة = رتبة الزمرة)

Examples (9): Show that (Z_n,$+_n$)is cyclic group.

$$Z_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$$

بما ان الزمرة منتهية فتكتب بالشكل :

o(Z_n) = n ,T.P. Z_n. =<$\bar{1}$>

<$\bar{1}$>={$(\bar{1})^k$:k∈Z}={$(\bar{1})^1, (\bar{1})^2, (\bar{1})^3, (\bar{1})^n = \bar{0}$}

={$\bar{1}, \bar{2}, \bar{3}, \ldots, \bar{n} = \bar{0}$}=$Z_n$

Z_n. =<$\bar{1}$>ando(Z_n)= o($\bar{1}$)=n.

Definition (7):(Division Algorithm for Z)خوارزمية القسمة

If a and b are integers with b>0, then there is a unique pair of integersqandr such that:

a =bq + r where 0 ≤ r <b

The number q is called the quotient and r is called the remainder when a is divided by b.

Examples (10): Find the quotientq and remainder r when 38 is divided by 7 according to the division algorithm.

Answer: 38=7(5)+3 0≤3≤7

∴q =5 and r=3

Examples (11):a=23 ,b=7

23=7(3)+2 0≤2≤7

q =3 ,r=2

Examples (12): a=15 , b=2

15=(2)(7)+1 0≤1≤2

q =7 ,r=1

Theorem (11): A subgroup of acyclic group is cyclic.

Proof: Let G be acyclic group generated by $a$and let H be asubgroup of G.

If H={e}, then H =<e>is cyclic

If H≠{e} and H≠G (H is proper subgroup)

Then

x∈H ⇒ x=a^m , m∈Z

x^{-1}∈H ⇒x^{-1} =a^{-m} , – m∈Z

Let m be aleast positive integer, such that $a^m \in H$

T.P. H=$<a^m> = \{(a^m)^g : g \in Z\}$

T.P. H⊆$<a^m>$∧$<a^m>$⊆H

Let y∈ H⇒y=a^s , s∈Z

By division algorithmof sand m

s =mg+r ⇒r=s – mg

∴$a^r = a^{s-mg} = a^s * (a^{-m})^g$ 0≤r≤m

∴a^r∈ H but 0≤r<m

r=0⇒s=mg

$$a^s = (a^m)^g \in < a^m >$$

∴y=$a^s \in <a^m>$⇒H⊆$<a^m>$

T.P.$<a^m>$⊆H

Let x∈$< a^m >$⇒x=$(a^m)^g$, g∈Z

$a^m \in H \Rightarrow (a^m)^g$ ∈H

∴x∈H⇒$<a^m>$⊆H

∴(H,∗) is cyclic subgroup.

Corollary (1): If (G,$*$) is a finite cyclic group of order n generated by a , then every subgroup of G is cyclic generated by$a^m \ni m|n$

Proof:suppose (G,$*$) is afinite, o(G)=n

G= $< a >$=$\{a^1, a^2, \ldots\ldots, a^n = e\}$

Let (H,$*$)be asubgroup of (G,$*$). Then (H,$*$) is cyclic (by Theorem 11) such that H=$<a^m>$

T.P.$m|n$(n=mg , g ∈ Z)

e ∈H ⇒a^n∈H,a^m∈H, by division algorithmof $n$and$m$

⇒n=mg+r   0≤r<m

r =n – mg⇒$a^r = a^n * (a^m)^{-g}$

⇒$a^r = (a^m)^{-g}$∈H

But 0≤r<m

⇒ If r=0 ⇒n=mg

∴$m|n$

Examples (13): Find all subgroup of (Z_{15},$+_{15}$)

Answer:o(Z_{15})=15 , H=<$(\bar{1})^m$>∋$m|n$

H=<$(\bar{1})^m$>∋m|15

m=1,3,5,15

If m=1 ⇒H_1=<$\bar{1}$>= Z_{15}

If m=3⇒ H_2=<$(\bar{1})^3$>=$\{\bar{3}, \bar{6}, \bar{9}, \overline{12}, \bar{0}\}$

If m= 5⇒ H_3=<$(\bar{1})^5$>=$\{\bar{5}, \overline{10}, \bar{0}\}$

If m= 15⇒ H_4=<$(\bar{1})^{15}$>=$\{\bar{0}\}$ = <$\bar{0}$>

(H.W.) Find all subgroup of (Z_8,$+_8$).

Corollary (2): If (G,$*$) is finite cyclic group of prime order, then G has no proper subgroup.

Proof: Let (G,$*$) be finite groupsuch that o(G)=p (p prime number)

G= <a>={$a^1, a^2, \ldots\ldots, a^p$=e}

Let (H,∗)be cyclic subgroup

∴H=< a^m >∋m|p⇒$m = 1$ or $m = $ p

If m=1 ⇒ H=<a>= G (not proper subgroup)

If m=p ⇒H=<a^p=e>= {e} (not proper subgroup)

∴G has no proper subgroup.

Examples (14): Find all subgroup of (Z_7,$+_7$)

Answer:o(Z_7)=7, let H=<$(\bar{1})^m$>∋m|7

∴ m=1 , m =7

If m=1 ⇒H_1=<$\bar{1}$>= Z_7

If m=7⇒ H_2=<$(\bar{1})^7$>={$\bar{0}$}

Definition (8): [g.c.d(x,y)] القاسم المشترك الاكبر

A positive integer$c$is said to be a greatest common divisor of two non-zero number x and y

iff**(1)** c|x ∧c|y

(2)if$a$|x ∧a|y ⇒a|c

(g.c.d(x,y) =c)

Examples (15): Find (g.c.d.(12,18))

Answer:g.c.d(12,18)=6 since

(1) 6|12 ∧ 6|18

(2) 3|12 ∧ 3|18 ⇒ 3|6

or 1|12∧1|18 ⇒ 1|6

or 2|12 ∧2|18 ⇒2|16

Remark (3): If (G,$*$) is finite cyclic group of order n generated by a ,then the generators of G is a^k such that g.c.d (k,n) =1.

Examples (16): Find all generators of (Z_6,$+_6$)

Answer: o (Z_6) =6 , $Z_6 =<\bar{1}>$

$Z_6 =<(\bar{1})^k>$ s.t.g.c.d(k,6)=1,k=1,2,3,4,5

k=1 $\Rightarrow$g.c.d(1,6)= 1 $\Rightarrow Z_6 =<\bar{1}>$

k=2 $\Rightarrow$g.c.d(2,6)$\neq$ 1$\Rightarrow Z_6 \neq <(\bar{1})^2> =<\bar{2}>$

k=3$\Rightarrow$g.c.d(3,6)$\neq$ 1 $\Rightarrow Z_6 \neq <(\bar{1})^3> =<\bar{3}>$

k=4$\Rightarrow$g.c.d(4,6)$\neq$1 $\Rightarrow Z_6 \neq <<(\bar{1})^4> =<\bar{4}>$

k=5 $\Rightarrow$g.c.d(5,6)=1 $\Rightarrow Z_6 =<(\bar{1})^5>=<\bar{5}>$

The generators of Z_6 are $\{\bar{1},\bar{5}\}$

Theorem (12): If(G,$*$)is an infinite cyclic group generated by a, then:

(1) a and a^{-1} are only generators of G

(2) Every subgroup of G except {e} is an infinite subgroup.

Proof(1):

Suppose G=<a> T.P.G=<a^{-1}>

Let $a \in$G $\ni$ G=<a>=$\{\ldots,a^{-2},a^{-1},a^0,a^1,a^2,\ldots\}$

Let b$\in$ G $\ni$ G=<b>=$\{\ldots.,b^{-2},b^{-1},b^0,b^1,b^2,\ldots\}$

$a\in$ G=<b>$\Rightarrow a=b^r$,r$\in$ Z …(1)

b$\in$G=<a>$\Rightarrow$ b=a^s,s $\in$ z …(2)

Put (1) in (2) $\Rightarrow$ $b=(b^r)^s \Rightarrow b^1=b^{rs} \Rightarrow b^1=b^{rs}$

1=rs $\Rightarrow$ r=s=1 or r=s= – 1

If r=s=1 $\Rightarrow a$=b $\Rightarrow$ G=<a>

If r=s= – 1$\Rightarrow$ b=$a^{-1}$$\Rightarrow$ G=<a^{-1}>

Proof (2): Let (H,∗)be a subgroupof (G,∗) ∋ H≠ {e}T.P.(H,∗)is infinite

Suppose that (H,∗)is finite ∋o(H)=k

(H,∗)is cyclic subgroup

$H=<a^m>=\{(a^m)^1,(a^m)^2,\dots,(a^m)^k=e\}$

$a^{mk}=e \Rightarrow o(a)=mk$

∴o(a)=o(G) c! متناقضه (G=<a>,G is finite)

∴(H,∗)is infinite.

Definition (9):H المجموعات المشاركة للزمرة الجزيئة

Let (H,∗) be a subgroup of a group (G,∗).The set

a∗H= {a ∗ h : h ∈ H}of G is the left coset of H containing a , while the subset

a∗H= {a ∗ h : h ∈ H} is the right cosetof H containing a.

Examples (17): If$(Z_6,+_6)$,a=$\bar{1}$,H={$\bar{0},\bar{2},\bar{4}$}, then

$\bar{1}+_6H=\{\bar{1},\bar{3},\bar{5}\}$, $H+_6\bar{1}=\{\bar{1},\bar{3},\bar{5}\}$

$\bar{3}+_6H=\{\bar{3},\bar{5},\bar{1}\}$, $H+_6\bar{3}=\{\bar{3},\bar{5},\bar{1}\}$

Notes(4):

(1) a∗H is not subgroup in general. Give an example **(H.W.)**

(2) a∗H = H ∗ain general, for example

$(S_3,\circ)$,H={f_1,f_4} , a=f_2

$f_2\circ H=\{f_2,f_5\}$, $H\circ f_2=\{f_2,f_6\}$

$f_2\circ H\neq H\circ f_2$

Theorem (13): Let (H,∗) be a subgroup of (G,∗)and a∈G,then

(1) H is itself left coset of H in G.

Proof: e ∈ G, e∗H={e∗h:h ∈ H} =H

(2) If $(G,*)$ is abelian group,then $a*H = H*a$

Proof: $a*H=\{a*h : h\in H\}=\{h*a : h\in H\}=H*a$

The converse is not true, for example: $(S_3,\circ)$, $H=\{f_1,f_2,f_3\}$ $a = f_4$

$f_4 \circ H=\{f_4,f_5,f_6\}$ and $H\circ f_4=\{f_4,f_6,f_5\}$

$\therefore f_4 \circ H = H\circ f_4$ but $(S_3,\circ)$ is not abelian group.

(3) $a\in a*H$

Proof: $a=a*e \in a*H$

(4) $a*H = H \Leftrightarrow a \in H$

Proof: $(\Rightarrow)$ Suppose $a*H = H$, then by (3) we get $a \in H$

$(\Leftarrow)$ Suppose $a\in H$ T.P. $a*H=H$

We must prove that $a*H\subseteq H \wedge H\subseteq a*H$

T.P. $a*H\subseteq H$

Let $x \in a*H \Rightarrow x=a*h \in H$ (since $a \in H \wedge h\in H$)

$\therefore a*H\subseteq H$

T.P. $H\subseteq a*H$

Let $b \in H \Rightarrow b=e*b$

$=(a*a^{-1})*b$

$=a*\underbrace{(a^{-1}* b)}_{\in H}\Rightarrow b\in a*H$

$\therefore H\subseteq a*H$

Thus $a*H=H$

(5) $a*H = b*H \Leftrightarrow a^{-1}* b\in H$

Proof: $(\Rightarrow) a*H= b*H$

$a^{-1}*(a*H)=a^{-1}*(b*H)$

$(a^{-1}*a)*H=(a^{-1}*b)*H$

$H=(a^{-1}*b)*H$

By (4) $\Rightarrow a^{-1}*b \in H$

$(\Leftarrow)$

Suppose that $a^{-1}*b \in H$

By (4) $\Rightarrow$ $(a^{-1}*b)*H = H$

$\Rightarrow b*H=a*H$

Remark (4): Every coset (left or right)of a subgroupH of a group $(G,*)$ has the same number of elements as H.

(6) $a*H = b*H \quad \vee \quad (a*H) \cap (b*H)=\phi$

Proof: Suppose $(a*H) \cap (b*H)=\phi$

T.P.$a*H = b*H$

$\exists x \ni x \in a*H \quad \wedge x \in b*H$

$x = a* h_1 \wedge x=b*h_2 \ni h_1,h_2 \in H$

$a* h_1=b*h_2 \Rightarrow h_1=a^{-1}*b *h_2$

$\Rightarrow h_1*h_2^{-1}=a^{-1}*b \in H$

by(5) $\Rightarrow a*H = b*H$

or supposea$* H \neq b*H$ T.P.$(a*H) \cap (b*H)=\phi$

suppose$(a*H) \cap (b*H)\neq\phi$

$\therefore \exists x \in a* H \quad \wedge x \in b*H$

$x = a*h_1 \wedge x = b*h_2$

$a^{-1}*b = h_1*h_2^{-1} \Rightarrow a^{-1}*b \in H$

$\Rightarrow a*H = b*H$ c! تناقض

$\therefore (a*H) \cap (b*H)=\phi$

(7) The set of all distinct left coset of H in G form a partition on G.

Proof: T.P. $G=\cup_{a\in G} a*H$ and $a_i*H \cap a_j*H =\phi$

$\because a_i*H , a_j*H$ are distinct

$\therefore a_i*H \cap a_j*H =\phi$ T.P. $G=\cup_{a\in G} a*H$

$a*H \subseteq G \quad \forall a \in G$ (by definition of coset)

$\Rightarrow \cup_{a\in G} a*H \subseteq G \quad \ldots(1)$

$\forall a\in G \Rightarrow a \in a*H \Rightarrow a \in \cup_{a\in G} a*H$

$\therefore G\subseteq \cup_{a\in G} a*H \quad \ldots(2)$

From (1)and (2) $\Rightarrow G=\cup_{a\in G} a*H$

Example (17): The group $(Z_6,+_6)$ is abelian. Find the partition of Z_6 into coset of the subgroup $H=\{\bar{0},\bar{3}\}$

Answer: $Z_6=\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}$

$\bar{0} +_6 H=\{\bar{0},\bar{3}\} =H$

$\bar{1} +_6 H=\{\bar{1},\bar{4}\}$

$\bar{2} +_6 H=\{\bar{2},\bar{5}\}$

$\bar{3} +_6 H=\{\bar{3},\bar{0}\}$

$\bar{4} +_6 H=\{\bar{4},\bar{1}\}$

$\bar{5} +_6 H=\{\bar{5},\bar{2}\}$

$\therefore$All the cosets of H are :$\{\bar{0},\bar{3}\}$, $\{\bar{1},\bar{4}\}$, $\{\bar{2},\bar{5}\}$ and since $(Z_6,+_6)$ is abelian group, then the left coset is equal the right coset.

Example (18):(H.W.)

In $(S_3,\circ)$,let $H= \{f_1,f_4\}$.Find the partitions of S_3 into left cosets of H and the partitions into right cosets of H.

Definition (10): Let (H,∗) be a subgroup of a group (G,∗).The number of left cosets or right cosetsof H in G is called the index of H in G and denotedby [G:H].

Remark (5): If (G,∗) is a finite group. Then [G:H]=$\frac{o(G)}{o(H)}$.

Example (19):$(S_3,\circ)$,H= $\{f_1,f_2, f_3\}$

∴ $[S_3:H] =\frac{o(S_3)}{o(H)} = \frac{6}{3} =2$.

Example (20):$(Z_6,+_6)$, H=$\{\bar{0},\bar{3}\}$

∴$[Z_6:H]=\frac{6}{2} =3$

Theorem (14):(LagrangeTheorem)

Let H be a subgroup of a finite group (G,∗).Then the order of H is adivisor of the order of G .

Proof:

Let G be afinite group ∋o(G)=n and H be a subgroup of G∋o(H)=m.

T.P.o(H) | o(G)   (T.P. m|n , n=mk)

Since G is finite⇒[G:H] = k

Let $a_1* H,a_2*H, \dots,a_k*H$ are left cosets of H

$a_1* H \cup a_2*H\cup \dots \cup a_k*H =G$ and

$a_i*H \cap a_j*H=\phi$

$o(a_1* H) + o(a_2*H) + \dots + o(a_k*H) = o(G)$

$$\underbrace{m + m + \cdots + m}_{k-\text{times}} = n$$

mk=n ⇒ m|n ⇒o(H)| o(G)

Corollary (1): If $(G,*)$ is finite group, then the order of any element of G divides the orderof G.

Proof:

Suppose that $(G,*)$ is finite $\ni o(G) = n$.

Let $a \in G \Rightarrow$ a is finite order such that $o(a) = m$ T.P. $o(a) | o(G)$.

Since $a \in G \Rightarrow H = \langle a \rangle$ cyclic group.

$H = \{a, a^2, \ldots, a^m = e\}$

$o(H) = o(a) = m \Rightarrow o(H) | o(G)$ (by Lagrange theorem)

$\therefore\ o(a) | o(G)$

Corollary (2): If $(G,*)$ is afinite group,then $a^{0(G)} = e \quad \forall\, a \in G$.

Proof:

Suppose that $o(G) = n$, let $a \in G \ni o(a) = m$

By Corollary (1) of Lagrange theorem $\Rightarrow o(a) | o(G)$

$\Rightarrow m | n$

$\Rightarrow n = mk$

$a^{o(G)} = a^n = (a^m)^k = e^k = e$

$\therefore a^{0(G)} = e \, \forall\, a \in G$.

Corollary (3): Every group of prime order is cyclic.

Proof: Let $(G,*)$ be finite $\ni o(G) = p$

By corollary (1)of Lagrange theorem $\Rightarrow o(a) | p \quad \forall\, a \in G$.

$o(a) = 1$ or p

If $o(a) = 1 \Rightarrow a = e$

If $o(a) = p \Rightarrow o(a) = o(G) \Rightarrow G = \langle a \rangle$

$\therefore (G,*)$ is cyclic group

Corollary (4): Every group of order less than 6 is commutative.

Proof:

Let(G,*) be a finite group ∋o(G)<6

o(G)=1or 2 or 3 or4 or 5 or6

If o(G)=1 ⇒G={e} ⇒ G is commutative

If o(G)=2or 3 or 5

By corollary (3) of Lagrange theoremG is cyclic⇒G is commutative

Ifo(G)=4

∴ o(a) = 1 or 2 or 4

If o(a)=1 ⇒ a=e

If o(a) =2 ∀ a ∈ G⇒ a^2 =e ⇒a=a^{-1} ∀ a ∈ G

∴G is commutative group

If o(a)=4 ⇒o(a)= o(G) ⇒ G=<a>

∴G is cyclic ⇒ G is commutative group.

Exercises:

(1) Find all subgroupsof (Z_5,$+_5$).

(2) Let (Z_8,$+_8$) be agroup and H=<$\overline{2}$>. Is H a subgroup of Z_8?

(3) If H={$\overline{0}$,$\overline{6}$,$\overline{12}$,$\overline{18}$}, show that (H,$+_{24}$) is a cyclic subgroupof (Z_{24},$+_{24}$). Also list the elements of each coset of H in Z_{24}.