Applied Mathematics

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Chapter One

Reviewing

Matrices

Matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. A matrix is usually shown by capital letter. There are many things we can do with it.

Operations on Matrices

Adding

To add two matrices: add two numbers in the matching positions; note that the two matrices must the same size. To add a matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with a matrix

$\boldsymbol{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Note

The negative of a matrix $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ is:

$$-\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$$

Subtracting

To subtract two matrices: subtract two numbers in the matching positions; the two matrices must the same size. Subtracting is actually is addition of a negative matrix $(\boldsymbol{A} + (-\boldsymbol{B}))$. To subtract two matrices $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Multiply by a Constant (Scalar Multiplication)

We can multiply a matrix by a constant as follows:

$$3 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

Multiply by another Matrix

To multiply two matrices, we need to do the dot product. Note that the number of columns for the first matrix must the same number of rows for the second one. For example, to multiply two matrices $\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 7 & 9 \\ 8 & 1 \end{bmatrix}$ as :

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 9 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 39 & 13 \\ 54 & 23 \\ 69 & 33 \end{bmatrix}$$

Transposing

To transpose the matrix, swap the rows and columns.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Trace of matrix

The trace of matrix $\boldsymbol{A}$ is denoted trace$\{\boldsymbol{A}\}$ and is equal to the sum of its diagonal elements, for example if $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ then trace$\{\boldsymbol{A}\} = 1 + 4 = 5$.

Determinant of a Matrix

The determinant of the matrix $\boldsymbol{A}$ denoted by$|\boldsymbol{A}|$ and defined for a matrix of size 2×2 $as\ follow,$ if $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determent is $|\boldsymbol{A}| = ad - bc$ and for a matrix of size 3×3 as follow, let $\boldsymbol{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, the determent is $|\boldsymbol{B}| = a(ei - fh) - b(di - fg) + c(dh - eg)$.

Inverse of a Matrix

The inverse of a matrix $\boldsymbol{A}$ denoted by $\boldsymbol{A^{-1}}$ and defined for a matrix of size 2×2 as follow, $let\ \boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse is $\boldsymbol{A^{-1}} = \frac{adj(A)}{|A|} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
A general $(n \times n)$ matrix can be inverted using methods such as (minors, cofactors and adjugate), the Gauss-Jorden elimination, Gaussian elimination, or LU decomposition.

- The square matrix is called invertible (or non-singular) if it have inverse and it is called singular if determinant of it equal zero.
- Let $\boldsymbol{A}$ and $\boldsymbol{D}$ be two square matrices of same size, $\boldsymbol{D}$ is called similar to $\boldsymbol{A}$ if there exists an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{D} = \boldsymbol{C^{-1}AC}$. For example, $\boldsymbol{D} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is similar to $\boldsymbol{A} = \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix}$ because there exist matrix $\boldsymbol{C} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ such that $\boldsymbol{D} = \boldsymbol{C^{-1}AC}$, $\boldsymbol{C}$ is called the modal matrix. The transformation $\boldsymbol{A}$ into $\boldsymbol{D}$ using $\boldsymbol{D} = \boldsymbol{C^{-1}AC}$ is said to be similarity transformation.

Dividing

The divide two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ define as follows:

$$\boldsymbol{A}/\boldsymbol{B} = \boldsymbol{A} \times \boldsymbol{B}^{-1}$$

Where $\boldsymbol{B}^{-1}$ is the inverse of a matrix $\boldsymbol{B}$.

Types of Matrices

- A matrix is symmetric if it is equal to its own transpose, i.e. it is symmetric across the diagonal. For example, $\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 4 & 5 \end{bmatrix}$ is symmetric (i.e. $\boldsymbol{A} = \boldsymbol{A}^{\boldsymbol{T}}$).
- A antisymmetric (skew-symmetric) matrix is a square matrix whose transpose equals its negative (i.e. $\boldsymbol{A}^{\boldsymbol{T}} = -\boldsymbol{A}$). For example, $\boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is antisymmetric.
- A matrix $\boldsymbol{A}$ is called orthogonal if $\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{A}^{-1}$ (i.e. $\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A} = \boldsymbol{I}$). For example, $\boldsymbol{A} = \begin{bmatrix} \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ is orthogonal.
- A square matrix $\boldsymbol{A}$ is said to be orthonormal if $\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{A}^{-1}$ and $|\boldsymbol{A}| = 1$. For example, $\boldsymbol{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthonormal.
- A diagonal matrix is a matrix in which the entries outside the main diagonal are all zero.
- A square matrix is called lower triangular if all the entries above the main diagonal are zero.
- A square matrix is called upper triangular if all the entries below the main diagonal are zero.
- $\boldsymbol{A}^{\boldsymbol{H}}$ denotes to the conjugate transpose or Hermitian transpose by taking the transpose of $\boldsymbol{A}$ and then complex conjugate of each entry (i.e. negating their imaginary parts but not their real parts). For example, $\boldsymbol{A} = \begin{bmatrix} 1 & -2-i \\ 1+i & i \end{bmatrix}$ then $\boldsymbol{A}^{\boldsymbol{H}} = \begin{bmatrix} 1 & 1-i \\ -2+i & -i \end{bmatrix}$.
- The square matrix $\boldsymbol{A}$ is called unitary matrix if $\boldsymbol{A}^{\boldsymbol{H}} = \boldsymbol{A}^{-1}$. The unitary matrix leave the length of a complex vector unchanged but for real matrix, unitary is

the same as orthogonal. For example, $\boldsymbol{A}=\begin{bmatrix}2^{-\frac{1}{2}} & 2^{-\frac{1}{2}} & 0\\ -i2^{-\frac{1}{2}} & i2^{-\frac{1}{2}} & 0\\ 0 & 0 & i\end{bmatrix}$ is a unitary matrix.

- Hermitian matrix (self-adjoint matrix) is a complex square matrix that equal to its own conjugate transpose (i.e. $\boldsymbol{A}=\boldsymbol{A}^{\boldsymbol{H}}$). For example, $\boldsymbol{A}=\begin{bmatrix}1 & 1-i\\ 1+i & 2\end{bmatrix}$ is Hermitian matrix.
- Skew-Hermitian matrices: $A^H = -A$;
- Normal matrices: $A^H A = A A^H$;

Special Matrices

In this section we introduce some important special matrices can be used in necessary application

Diagonally Dominant Matrix

A matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of magnitudes of all the other (non−diagonal) entries in that row (i.e. $|a_{\mathrm{ii}}| \geq \sum_{i\neq j}|a_{ij}|$). However a matrix is called strictly diagonally dominant if $|a_{\mathrm{ii}}| > \sum_{i\neq j}|a_{ij}|$ and strictly diagonally dominant is non- singular.

For example,

$\boldsymbol{A}=\begin{bmatrix}3 & -2 & 1\\ 1 & -3 & 2\\ -1 & 2 & 4\end{bmatrix}$ is diagonally dominant since $|3| \geq |-2|+|1|$, $|-3| \geq |1|+|2|, |4| \geq |-1|+|2|$. But $\boldsymbol{B}=\begin{bmatrix}-2 & 2 & 1\\ 1 & 3 & 2\\ 1 & -2 & 0\end{bmatrix}$ is not diagonally dominant since $|-2| < |2|+|1|$, $|0| < |1|+|-2|$.

$\boldsymbol{C} = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$ is strictly diagonally dominant since $|-4| > |2| + |1|$, $|6| > |1| + |2|, |5| > |1| + |-2|$.

Exercises

Classify the following matrices as diagonally dominant, strictly diagonally dominant or unknown:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -4 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} -6 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & -2 & 7 \end{bmatrix}.$$

Band Matrix

A band matrix is a spare matrix (i.e. a matrix in which most of the elements are zero) whose non-zero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on the other side.

Tridiagonal Matrix

A square matrix with non zero elements only on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal (i.e. along the subdiagonal and superdiagonal). For example, the following matrix is Tridiagonal: $A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

The tridiagonal is not necessary symmetric and it is kind of band matrix.

Monotone Matrix

A monotonic matrix of order *n* is an $(n \times n)$ matrix in which every element is either zero or contains a number from the set {1, 2, ….,*n*} (i.e. A is an $(n \times n)$ matrix is monotone if all elements of A^{-1} are nonnegative). For example, the following (2×2) matrices are monotone:

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. The monotone is non-singular matrix.

Pseudo Inverse of a Matrix

The matrix $(\boldsymbol{A^T A})^{-1}\boldsymbol{A^T}$ is called pseudo inverse of a matrix $\boldsymbol{A}$ and denoted by $pinv(\boldsymbol{A})$. The pseudo inverse can be expressed of a rectangular matrix, or not invertible square matrix.

Example 1: Find $\boldsymbol{A^{-1}}$for the following matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}$$

Solution: we see that A is rectangular matrix that we cannot be compute $\boldsymbol{A^{-1}}$ director. So, we find pseudo inverse as follow :

Firstly find $\mathbf{A^T}$ **A**,

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{adj(A^T \boldsymbol{A})}{|A^T \boldsymbol{A}|} = \frac{1}{30} \begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix}$$

$$pinv(\boldsymbol{A}) = (\boldsymbol{A^T A})^{-1} \boldsymbol{A^T} = \frac{1}{30} \begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 5 & -17 & 4 \\ 0 & 12 & 6 \end{bmatrix}$$

That is $pinv(\boldsymbol{A})$: $\begin{bmatrix} \frac{1}{6} & \frac{-17}{30} & \frac{2}{15} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$.

Chapter Two

Eigenvalues, Eigenvectors and its Applications

In this chapter we introduce definition of eigenvalues, eigenvectors, how can it's calculating and illustrate the importance of the topic by demonstrating some of its application.

Eigenvalues and Eigenvectors

Suppose that $\boldsymbol{A}$ is a square $(n \times n)$ matrix. We say that a nonzero vector v is an eigenvector (*ev*) and a *scalar* λ is its eigenvalue (*ew*) if

$$\boldsymbol{A}v = \lambda v \qquad (2.1)$$

Geometrically this means that $\boldsymbol{A}v$ is in the same or appositive direction as v, depending on the sign of λ.

Notice that Equation (2.1) can be rewritten as follows:

$$\boldsymbol{A}v - \lambda v = 0$$

since $Iv = v$, we can do the following:

$$\boldsymbol{A}v - \lambda v = \boldsymbol{A}v - \lambda Iv = (\boldsymbol{A} - \lambda I)v = 0$$

If v is nonzero, then the matrix $(\boldsymbol{A} - \lambda I)$ must be singular and

$|\boldsymbol{A} - \lambda I| = 0.$

This is called the ***characteristic equation*** (or ***characteristic polynomial*** $\boldsymbol{p(\lambda)}$).

Calculating Eigenvalues and Eigenvectors

If $\boldsymbol{A}$ is (2×2) or (3×3) matrix then we can find its eigenvalues and eigenvectors by hand.

Note

Let $\boldsymbol{A}$ is a square $(n \times n)$ matrix and λ is an eigenvalue of $\boldsymbol{A}$. The set of all eigenvectors corresponding to λ, together with zero vector is a subspace of R^n and this space is called eigenspace of λ.

Example 2: Find eigenvalues and eigenvectors for the following matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} \end{bmatrix}$.

Solution $\boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} \end{bmatrix} - \boldsymbol{\lambda} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{1} - \boldsymbol{\lambda} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} - \boldsymbol{\lambda} \end{bmatrix}$,

$|\boldsymbol{A} - \lambda I| = (1 - \lambda)(5 - \lambda) - 3(4) = \lambda^2 - 6\lambda - 7$ (*This called characteristic polynomial*),

$\lambda^2 - 6\lambda - 7 = 0 \rightarrow (\lambda - 7)(\lambda + 1) = 0 \rightarrow \lambda = 7, \lambda = -1$.

$\lambda = 7$ and $\lambda = -1$ are the eigenvalues of $\boldsymbol{A}$.

To find eigenvectors, if $\lambda = 7$, we solve the equation

$$(\boldsymbol{A} - 7\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -6x + 4y \\ 3x - 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$-6x + 4y = 0, 3x - 2y = 0$,

Hence, $(x, y) = (2, 3)$ is a solution of $3x - 2y = 0$ (or $-6x + 4y = 0$). Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 7$ are non-zero vectors of form $r_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix}, r_1 \in R\{0\}$.

The $S_1 = \{r_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix}, r_1 \in \mathrm{R}\}$ is a subspace of R^2.

To find eigenvectors, if $\lambda = -1$, we solve the equation

$$(\boldsymbol{A} - (-1)\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 2x + 4y \\ 3x + 6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$2x+4y=0, 3x+6y=0,$

Hence, (x, y) = (-2, 1) is a solution of $2x+4y=0$(or $3x+6y=0$). Thus the eigenvectors of A when $\lambda=-1$ are non-zero vectors of form

$r_2\begin{bmatrix}-2\\1\end{bmatrix}$, $r_2 \in R\{0\}$. The $S_2=\{r_2\begin{bmatrix}-2\\1\end{bmatrix}, r_2 \in$ R} is a subspace of R^2.

Example3: Find eigenvalues and eigenvectors for the following matrix $\boldsymbol{A}=\begin{bmatrix}\mathbf{1} & \mathbf{-3} & \mathbf{3}\\ \mathbf{3} & \mathbf{-5} & \mathbf{3}\\ \mathbf{6} & \mathbf{-6} & \mathbf{4}\end{bmatrix}$.

Solution: $\boldsymbol{A}-\boldsymbol{\lambda I}=\begin{bmatrix}\mathbf{1} & \mathbf{-3} & \mathbf{3}\\ \mathbf{3} & \mathbf{-5} & \mathbf{3}\\ \mathbf{6} & \mathbf{-6} & \mathbf{4}\end{bmatrix}-\boldsymbol{\lambda}\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}$

$$=\begin{bmatrix}\mathbf{1} & \mathbf{-3} & \mathbf{3}\\ \mathbf{3} & \mathbf{-5} & \mathbf{3}\\ \mathbf{6} & \mathbf{-6} & \mathbf{4}\end{bmatrix}-\begin{bmatrix}\boldsymbol{\lambda} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\lambda} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{\lambda}\end{bmatrix}$$

$$=\begin{bmatrix}\mathbf{1-\boldsymbol{\lambda}} & \mathbf{-3} & \mathbf{3}\\ \mathbf{3} & \mathbf{-5-\boldsymbol{\lambda}} & \mathbf{3}\\ \mathbf{6} & \mathbf{-6} & \mathbf{4-\boldsymbol{\lambda}}\end{bmatrix},$$

$$|\boldsymbol{A}-\lambda\boldsymbol{I}|=(1-\lambda)\{[(-5-\lambda)(4-\lambda)]-3(-6)\}+3\{3(4-\lambda)-3(6)\}$$
$$+3\{3(-6)-6(-5-\lambda)\}$$

$$=(1-\lambda)(\lambda^2+\lambda-20+18\}+3\{12-3\lambda-18\}+3\{-18+30+6\lambda\}$$
$$=-\lambda^3+3\lambda-2-9\lambda-18+18\lambda+36=-\lambda^3+12\lambda+16$$

To find the solution to $|A-\lambda I|=0$, i.e. to solve $\lambda^3-12\lambda-16=0$,

$$\lambda^3-12\lambda-16=(\lambda-4)(\ \lambda^2+4\lambda+4)=0,$$

$\lambda=4, \lambda=\frac{-4\pm\sqrt{4^2-4(1)(4)}}{2}=\frac{-4\pm0}{2}=-2$ (repeated root).

To find eigenvectors, if $\lambda=4$, we solve the equation

$$(\boldsymbol{A}-4\boldsymbol{I})v=0 \rightarrow \left(\begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix}-\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$\begin{bmatrix}-3 & -3 & 3\\ 3 & -9 & 3\\ 6 & -6 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$\begin{bmatrix} -3x-3y+3z \\ 3x-9y+3z \\ 6x-6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$-3x-3y+3z=0, 3x-9y+3z=0, 6x-6y=0,$

$$x-\frac{1}{2}z=0, y-\frac{1}{2}z=0$$

Hence, $(x, y, z) = (\frac{1}{2}, \frac{1}{2}, 1)$ is a solution of $x-\frac{1}{2}z=0, y-\frac{1}{2}z=0.$

Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 4$ is non-zero vectors of the form $r_1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}, r_1 \in R\{0\}$. The $S_1 = \{r_1 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}, r_1 \in R\}$ is a subspace of R^3.

To find eigenvectors, if $\lambda = -2$, we solve the equation

$$(\boldsymbol{A}-(-2)\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3x-3y+3z \\ 3x-3y+3z \\ 6x-6y+6z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$3x-3y+3z=0, 3x-3y+3z=0, 6x-6y+6z=0,$

$$x-y+z=0$$

Hence, $(x, y, z) = (0,1,1)$ is a solution of $x-y+z=0.$

Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = -2$ are non-zero vectors of the form $r_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, r_2 \in R\{0\}$. The $S_2 = \{r_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, r_2 \in R\}$ is a subspace of R^3.

Complex Eigenvalues

It turns out that the eigenvalues of some matrices are complex numbers, even when the matrix only contains real numbers. When this happens the complex *ew*'s must occur in conjugate pairs, i.e.,

$\lambda_{1,2} = \alpha \pm \beta i$

The corresponding *ev*'s must also come in conjugate pairs:

$w = u \pm vi$

Example 4: Find eigenvalues and eigenvectors for the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$.

Solution

$\boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} - \boldsymbol{\lambda} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{-\lambda} & \mathbf{-1} \\ \mathbf{1} & \boldsymbol{-\lambda} \end{bmatrix}$,

$|A - \lambda \boldsymbol{I}| = (-\lambda)(-\lambda) - 1(-1) = \lambda^2 + 1$,

$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \lambda^2 + 1 = 0 \rightarrow \lambda^2 = -1 \rightarrow \lambda = \pm \mathrm{i}$.

$\lambda = \mathrm{i}$ and $\lambda = -\mathrm{i}$ are the eigenvalues of $\boldsymbol{A}$.

To find eigenvectors, if $\lambda = \mathrm{i}$, we solve the equation

$(\boldsymbol{A} - i\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -ix - y \\ x - iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$-ix - y = 0, x - iy = 0 \rightarrow y = -ix$,

Hence, the eigenvectors of $\boldsymbol{A}$ when $\lambda = \mathrm{i}$ are non-zero vectors of form $r_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}$ $, r_1 \in R\{0\}$. The eigenspace = $\{\begin{bmatrix} z_1 \\ -iz_2 \end{bmatrix}, z_1, z_2 \in \mathrm{R}\}$.

To find eigenvectors, if $\lambda = -\mathrm{i}$, we solve the equation

$(\boldsymbol{A} - (-i)\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} ix - y \\ x + iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$ix - y = 0, x + iy = 0 \rightarrow y = ix,$

Hence, the eigenvectors of $\boldsymbol{A}$ when $\lambda = -\mathrm{i}$ are non-zero vectors of form $r_2 \begin{bmatrix} 1 \\ i \end{bmatrix}, r_2 \in R\{0\}$. The eigenspace = $\{\begin{bmatrix} z_1 \\ iz_2 \end{bmatrix}, z_1, z_2 \in \mathrm{R}\}$.

Notes

1. An eigenvalue of $A_{n\times n}$ is a root of the characteristic polynomial. Indeed λ is an eigenvalue of A iff det(A − λI) = 0. So there are at most n distinct eigenvalues of A.
2. Similar matrices have the same eigenvalues (**HW**).
3. If $\boldsymbol{A}$ be a diagonal matrix then its eigenvalues are the diagonal elements (**HW**).
4. If $\boldsymbol{A}$ be an upper (lower) triangular matrix then its eigenvalues are the diagonal elements (**HW**).
5. If $\boldsymbol{A}$ be a square matrix then $\boldsymbol{A}$ and $\boldsymbol{A^T}$ have the same eigenvalues (**HW**).
6. If $\boldsymbol{A}$ be a square matrix then $|\boldsymbol{A}|$ is equal to the product of all eigenvalues of A (**HW**).
7. $\boldsymbol{A}$ is a singular matrix ↔ $\lambda = 0$ be an eigenvalue of $\boldsymbol{A}$ (**HW**).
8. If A be an invertible matrix with eigenvalue λ of eigenvector v then λ^{-1} is an eigenvalue of $\boldsymbol{A^{-1}}$ with eigenvector v (**HW**).
9. The set of all the eigenvalues of A is referred to as the spectrum of A and denoted by Λ(A).

10. The maximum modulus of the eigenvalues is called spectral radius and denoted by ρ(A), that is:

$$\rho(\mathrm{A}) = \max_{\lambda \in \Lambda(A)} |\lambda|.$$

Cayley-Hamilton Theorem

Arthur Cayley (16 August 1821 – 26 January 1895) was a British mathematician

Let $\boldsymbol{A}$ be a square $(n \times n)$ matrix with characteristic polynomial

$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots . + c_{n-1}\lambda + c_n$ and $\lambda^n + c_1\lambda^{n-1} + \cdots . + c_{n-1}\lambda + c_n = 0$ then $\boldsymbol{A}^n + c_1\boldsymbol{A}^{n-1} + \cdots . + c_{n-1}\boldsymbol{A} + c_n\boldsymbol{I_n} = 0$.

Example 5: Apply Cayley-Hamilton Theorem on the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$.

Solution: $p(\lambda) = \lambda^2 - 3\lambda + 2$, by Cayley-Hamilton Theorem

$$\boldsymbol{A}^2 - 3\boldsymbol{A} + 2\boldsymbol{I_2} = 0,$$

$$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 3 & -9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Exercises

1. Find eigenvalues and eigenvectors for the following matrices
$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$
$\boldsymbol{E} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{G} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}, \boldsymbol{H} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \boldsymbol{K} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix},$
$\boldsymbol{L} = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$
2. Find eigenvalues and eigenvectors for the following matrices:
$\boldsymbol{A} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{bmatrix}$ and
$\boldsymbol{D} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & -2 & 3 \end{bmatrix}.$
3. Apply Cayley-Hamilton Theorem on the following matrices:
$\boldsymbol{A} = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

Eigenvalues and Eigenvectors of symmetric matrix

A matrix is symmetric if it is equal to its own transpose, in symmetric matrix the upper right left and the lower left half of the matrix are mirror images of each other about the diagonal. A$(n \times n)$ symmetric matrix not only has a nice structure, but it also satisfied the following:

- It has exactly n eigenvalues (not necessary distinct).
- There exists a set of n eigenvectors, one for each eigenvalue, that are mutually orthogonal.
- A symmetric matrix has n eigenvalues and there exist n linearly independent eigenvectors (because of orthogonal) even if the eigenvalues are not distinct.

Example 6: Find eigenvalues and eigenvectors for the matrix

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{5} & \mathbf{3} \\ \mathbf{3} & \mathbf{5} \end{bmatrix}.$$

Solution: $\boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix} \mathbf{5} & \mathbf{3} \\ \mathbf{3} & \mathbf{5} \end{bmatrix} - \boldsymbol{\lambda} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$

$$= \begin{bmatrix} \mathbf{5} & \mathbf{3} \\ \mathbf{3} & \mathbf{5} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{5} - \boldsymbol{\lambda} & \mathbf{3} \\ \mathbf{3} & \mathbf{5} - \boldsymbol{\lambda} \end{bmatrix},$$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = (5-\lambda)(5-\lambda) - 3(3) = \lambda^2 - 10\,\lambda + 25 - 9 = \lambda^2 - 10\,\lambda + 16$$

$$= (\lambda - 8)(\lambda - 2) = 0 \rightarrow \lambda = 8, 2$$

To find eigenvectors, if $\lambda = 8$, we solve the equation

$$(\boldsymbol{A} - 8\boldsymbol{I})v = 0 \quad \rightarrow \left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -3x + 3y \\ 3x - 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$-3x + 3y = 0, 3x - 3y = 0,$$

$x = 1, y = 1$. Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 8$ is nonzero vectors of form $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find eigenvectors, if $\lambda = 2$, we solve the equation:

$$(\boldsymbol{A} - 2I)v = 0 \rightarrow \left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3x + 3y \\ 3x + 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$3x + 3y = 0, 3x + 3y = 0,$

$x = 1, y = -1$. Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 2$ are non-zero vectors of form $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus we have two orthogonal eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (linearly independent).

Example 7: Find eigenvalues and eigenvectors for the matrix

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Solution: $\boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{4} \\ \mathbf{2} & \mathbf{0} & \mathbf{2} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \end{bmatrix} - \boldsymbol{\lambda} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

$$= \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{4} \\ \mathbf{2} & \mathbf{0} & \mathbf{2} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\lambda} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{3} - \boldsymbol{\lambda} & \mathbf{2} & \mathbf{4} \\ \mathbf{2} & -\boldsymbol{\lambda} & \mathbf{2} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} - \boldsymbol{\lambda} \end{bmatrix},$$

$$|\boldsymbol{A} - \boldsymbol{\lambda I}| = (3 - \lambda)\{[(-\lambda)(3 - \lambda)] - 2(2)\} - 2\{2(3 - \lambda) - 2(4)\} \\ + 4\{2(2) - 4(-\lambda)\}$$

$$= (3-\lambda)(\lambda^2-3\lambda-4\} - 2\{6-2\lambda-8\} + 4\{4+4\lambda\}$$
$$= -\lambda^3+6\lambda^2-5\lambda-12+4\lambda+4+16+16\lambda = -\lambda^3+6\lambda^2+15\lambda+8$$

To find the solution to $|A-\lambda I| = 0$, i.e. to solve $(-\lambda^3+6\lambda^2+15\lambda+8=0 \rightarrow (\lambda^3-6\lambda^2-15\lambda-8=0)$,

$\lambda^3-6\lambda^2-15\lambda-8=(\lambda-8)(\lambda+1)^2=0 \rightarrow \lambda = 8,-1,-1$

To find eigenvectors, if $\lambda = 8$, we solve the equation

$$(\boldsymbol{A}-8\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -5x+2y+4z \\ 2x-8y+2z \\ 4x+2y-5z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$-5x+2y+4z=0, 2x-8y+2z=0, 4x+2y-5z=0,$

$x=2, y=1, z=2$. Thus the eigenvectors of $\boldsymbol{A}$ when $\lambda = 8$ are non-zero vectors of form $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

To find eigenvectors, if $\lambda = -1$, we solve the equation :

$$(\boldsymbol{A}+\boldsymbol{I})v = 0 \rightarrow \left(\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 4x+2y+4z \\ 2x+y+2z \\ 4x+2y+4z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$4x+2y+4z=0, 2x+y+2z=0, 4x+2y+4z=0,$

This system reduces to single equation $(2x + y + 2z = 0)$ since the other two equations are twice this one. There are two parameters here (x and z), thus eigenvectors for $\lambda = -1$ must have the form $(y = -2x - 2z)$ which corresponds to the vectors of form $\begin{bmatrix} s \\ -2s - 2t \\ t \end{bmatrix}$. We must choose values of s and t that yield two orthogonal vectors (the third one is $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$). First, choose anything, let $s = 1$ and $t = 0$, the eigenvector is $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. Now find a vector $\begin{bmatrix} x \\ -2x - 2z \\ z \end{bmatrix}$ such that: $0 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} . \begin{bmatrix} x \\ -2x - 2z \\ z \end{bmatrix} = x + 4x + 4z + 0 = 5x + 4z,$ we can choose $x = 4$ and $z = -5$. Thus we have two orthogonal vectors $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$ that corresponds to the two eigenvalue $\lambda = -1$.

Note that since this matrix is symmetric we do indeed have three eigenvalues and a set of three orthogonal (and thus linearly independent) eigenvectors (one for each eigenvalue).

Exercises

Find eigenvalues and eigenvectors for the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ 1 & 30 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} 4 & -8 & 0 \\ -8 & 0 & 8 \\ 0 & 8 & -4 \end{bmatrix}$$

Applications of Eigenvalues and Eigenvectors

In this section some applications of eigenvalues and eigenvectors are introduce to illustrate the importance of topic.

Positive Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be positive definite if the scalar $(v^T \ \boldsymbol{A} \ v)$ is positive for every nonzero column vector v of n real numbers. However, a Hermitian matrix $\boldsymbol{A}$ is said to be positive definite if the scalar

$(v^H \ \boldsymbol{A} \ v)$ is real and positive for all nonzero column vector v of n complex numbers.

Example 8: Find if the following matrices are positive definite:

$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, and { $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ **HW**}.

Solution: The identity matrix $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is positive definite because a real matrix, it is symmetric, and for any non-zero column vector v with real entries a and b, $v^T\ \boldsymbol{I}\ v = [a \quad b] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^2 + b^2$ is positive but as a complex matrix, for any non-zero column vector v with complex entries a and b,

$v^H\ \boldsymbol{I}\ v = [a^* \ \ b^*] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^*a + b^*b = |a|^2 + |b|^2$ is positive and one of a and b is not zero.

The symmetric matrix $\boldsymbol{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite because a real matrix, and for any nonzero column vector v with <u>real</u> entries a , b and c,

$$v^T\ \boldsymbol{B}\ v = [a \quad b \quad c] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} =$$

$[(2a-b) \quad (-a+2b-c) \quad (-b+2c)] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2a^2 - ab - ab + 2b^2 - bc - bc + 2c^2 = a^2 + (a^2 - 2ab + b^2) + (b^2 - 2ac + c^2) + c^2 = a^2 + (a-b)^2 + (b-c)^2 + c^2$ is positive.

Other way to knowing the matrix is positive definite or not illustrated in the following definition

Definition

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be positive definite if all the eigenvalues of the matrix $\boldsymbol{A}$ is positive.

<u>Example 9</u>: The following matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not positive definite since its eigenvalues is 1 and -3

Positive Semi Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be positive semi definite if the scalar $(v^T\ \boldsymbol{A}\ v \geq 0)$ (i.e. non- negative) for every nonzero column vector v of n real numbers but If $\boldsymbol{A}$ is complex matrix then $\boldsymbol{A}$ is said to be positive semi definite if the scalar $(v^H\ \boldsymbol{A}\ v \geq 0)$. For examples, the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is positive semi definite because a real matrix, it is symmetric, and for any nonzero column vector v with real entries a and b,

$$v^T \boldsymbol{A}\ v = [a \quad b] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \geq 0$$

Other way to knowing the matrix is positive semi definite or not illustrated in the following definition

Definition

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be positive semi definite if all the eigenvalues $\boldsymbol{\lambda}$ of the matrix $\boldsymbol{A}$ is as $\boldsymbol{\lambda} \geq \boldsymbol{0}$.

Negative Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be negative definite if the scalar $(v^T\ \boldsymbol{A}\ v)$ is negative for every nonzero column vector v of n real numbers. However, a Hermitian matrix $\boldsymbol{A}$ is said to be negative definite if the scalar

$(v^H\ \boldsymbol{A}\ v)$ is real and negative for all nonzero column vector v of n complex numbers. For examples, the matrix $\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is negative definite because a real matrix, it is symmetric, and for any nonzero column vector v with real entries a , b and c,, $v^T\ \boldsymbol{A}\ v = [a \quad b \quad c] \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} < 0$.

Other way the eigenvalue of $\boldsymbol{A}$ is $-3, -2\ and\ -1$, i.e., all $\boldsymbol{\lambda} < \boldsymbol{0}$.

Negative Semi Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be negative semi definite if the scalar $(v^T\ \boldsymbol{A}\ v \leq 0)$ (i.e. non- positive) for every nonzero column vector v of n real numbers but If $\boldsymbol{A}$ is complex matrix then $\boldsymbol{A}$ is said to be negative semi definite if the scalar $(v^H\ \boldsymbol{A}\ v \leq 0)$. For examples, the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ is negative semi definite because a real matrix, it is symmetric, and for any non-zero column vector v with real entries a and b, $v^T\ \boldsymbol{A}\ v = [a \quad b] \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq 0$.

Notes

- The example1 above $\boldsymbol{B}$ shows that a matrix in which some elements are negative may still be positive definite. Conversely, a matrix whose entries are positive is not necessary positive definite and the following example as, $\boldsymbol{C}=\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$ for which $\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=-2<0$.
- For any real invertible matrix $\boldsymbol{A}$, the $\boldsymbol{A^T A}$ is a positive definite.
- A symmetric matrix is a positive definite ↔ all eigenvalues are positive.
- A symmetric matrix is a negative definite ↔ all eigenvalues are negative.

Exercises:

Classify the following matrices as positive definite, negative definite, positive semi definite or negative semi definite:

$$\boldsymbol{A}=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix},\boldsymbol{B}=\begin{bmatrix}-4 & -6\\-3 & -5\end{bmatrix},\boldsymbol{C}=\begin{bmatrix}5 & 4\\1 & 2\end{bmatrix},\boldsymbol{D}=\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix},$$

$$\boldsymbol{E}=\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix},\boldsymbol{F}=\begin{bmatrix}-3 & 0\\0 & 0\end{bmatrix},\boldsymbol{G}=\begin{bmatrix}2 & -4\\-1 & 2\end{bmatrix},\boldsymbol{H}=\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix},\boldsymbol{K}=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix},$$

$$\boldsymbol{L}=\begin{bmatrix}5 & -3\\3 & -1\end{bmatrix},\boldsymbol{M}=\begin{bmatrix}5 & 4 & 1\\4 & 5 & 1\\1 & 1 & 2\end{bmatrix},\boldsymbol{N}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 2\\-2 & 0 & 3\end{bmatrix},\boldsymbol{Q}=\begin{bmatrix}-5 & -2\\-4 & -1\\-3 & 0\end{bmatrix}.$$

Diagonalization of a Matrix with Distinct Eigenvalues

A square matrix $\boldsymbol{A}$ is said to be diagonalizable if there exists an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{D}=\boldsymbol{C^{-1}AC}$ is a diagonal matrix.

Example10: Prove that the matrix $\boldsymbol{A}=\begin{bmatrix}\mathbf{7} & \mathbf{-10}\\\mathbf{3} & \mathbf{-4}\end{bmatrix}$ is diagonalizable.

Solution: $\boldsymbol{\lambda_1=2}$ and eigenvectors $\boldsymbol{v_1=r_1}\begin{bmatrix}\mathbf{2}\\\mathbf{1}\end{bmatrix}$

$\lambda_2=1$ and eigenvectors $v_2=r_2\begin{bmatrix}5\\3\end{bmatrix}$

There exists $\boldsymbol{C}=\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$ such that $\boldsymbol{C^{-1}AC}=\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}^{-1}\begin{bmatrix}7 & -10\\3 & -4\end{bmatrix}\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}=\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}\begin{bmatrix}7 & -10\\3 & -4\end{bmatrix}\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}=\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$ is a diagonal matrix.

Notes

- The product $\boldsymbol{D} = \boldsymbol{C^{-1}AC}$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}$.
- $\boldsymbol{A}$ is a diagonalizable $\leftrightarrow$ it has linearly independent the eigenvectors.
- Matrix Powers: $\boldsymbol{A}$ is similar to a diagonal matrix $\boldsymbol{D} = \boldsymbol{C^{-1}AC}$ then $\boldsymbol{A^k} = \boldsymbol{CD^kC^{-1}}$.
- If a matrix $\boldsymbol{A}$ with distinct eigenvalues then $\boldsymbol{A}$ is diagonalizable.
- The eigenvalues of $\boldsymbol{A}$ lies on the main diagonal of similar matrix
  $\boldsymbol{D} = \boldsymbol{C^{-1}AC}$.
- If $\boldsymbol{A}$ is a symmetric matrix then eigenvectors that associated to distinct eigenvalues of $\boldsymbol{A}$ are orthogonal.

Example11: Let A$=\begin{bmatrix} \boldsymbol{-4} & \boldsymbol{-6} \\ \boldsymbol{3} & \boldsymbol{5} \end{bmatrix}$, $\{\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-2} \\ \boldsymbol{0} & \boldsymbol{-2} & \boldsymbol{0} \\ \boldsymbol{-2} & \boldsymbol{0} & \boldsymbol{3} \end{bmatrix} \boldsymbol{HW}\}$,

1. Prove that $\boldsymbol{A}$ is diagonalizable,
2. Find the diagonal matrix $\boldsymbol{D}$ similar to $\boldsymbol{A}$, and
3. Find $\boldsymbol{A^5}$.

Solution: $\boldsymbol{\lambda_1 = 2}$ and eigenvectors $\boldsymbol{v_1 = r_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$\lambda_2 = -1$ and eigenvectors $v_2 = r_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Since $\boldsymbol{A}$ has two distinct eigenvalues then $\boldsymbol{A}$ is diagonalizable.

- Select two linearly independent eigenvectors
$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$$
- $\mathbf{C} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$, $\boldsymbol{C^{-1}AC} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} =$
  $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \boldsymbol{D},$
  The main diagonal of $\boldsymbol{D}$ has the distinct eigenvalues of $\boldsymbol{A}$.
- $\boldsymbol{D^5} = \begin{bmatrix} 2^5 & 0 \\ 0 & (-1)^5 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix},$

$$\boldsymbol{A^5} = \boldsymbol{CD^5C^{-1}} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} -32 & 2 \\ 32 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -30 & -66 \\ 33 & 65 \end{bmatrix}.$$

<u>Example 12</u>: Prove that the matrix $\boldsymbol{A} = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ is not diagonalizable.

<u>Solution</u>: $\boldsymbol{\lambda = 2}$ (A repeated root) and eigenvector $\boldsymbol{v = r}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$,
a matrix $\boldsymbol{A}$ it does not have two distinct eigenvalues then $\boldsymbol{A}$ is not diagonalizable.

<u>Some Applications in Genetics</u>

Using the diagonalization of a matrix to compute its powers, the propagation of an inherited trait in successive generations is investigated. In application, we shall examine the inheritance of traits in animals or plants. It is assumed to be governed by a set of two genes **A** and **a.** Under autosomal inheritance each individual in the population (**AA**, **Aa**, and **aa**).

<u>Autosomal Inheritance</u>

In the Table1 below, we list the probabilities of the possible genotypes of the offspring for all possible combinations of the genotypes of the parents.

Table1

	AA-AA	AA-Aa	AA-aa	Aa-Aa	Aa-aa	aa-aa
AA	1	$1/2$	0	$1/4$	0	0
Aa	0	$1/2$	1	$1/2$	$1/2$	0
aa	0	0	0	$1/4$	$1/2$	1

<u>Example13</u>: Suppose a farmer has a large population of plants consisting of some distribution of all three possible genotypes (**AA**, **Aa**, and **aa**). He desires to undertake a breeding program in which each plant in the population is always fertilized with a plant of genotype **AA**. We want to derive an expression for the distribution of the three possible genotype in the population after any number of generations. For n= 0,1,2,…., let us set

a_n= fraction of plants of **AA** in n-th generation,

b_n= fraction of plants of **Aa** in n-th generation,

c_n= fraction of plants of **aa** in n-th generation.

Thus a_0, b_0 and c_0 be a specify initial distribution of genotype. We have

$$a_n+b_n+c_n=1 \quad \text{for all } n=0,1,2,\ldots..$$

From Table 1 above, we have equation 1, as follows:

for all n= 1,2,….

$$a_n = a_{n-1} + \frac{1}{2} b_{n-1}$$

$$b_n = c_{n-1} + \frac{1}{2} b_{n-1}$$

$$c_n = 0$$

We can rewrite equation 1 in matrix notation as follows:

$$X^{(n)} = MX^{(n-1)} \text{ , n=1,2,3,….. (2)}$$

Where

$$X^{(n)} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}, X^{(n-1)} = \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From equation 2, we get

$X^{(n)} = MX^{(n-1)} = M^2X^{(n-2)}$=….=$M^nX^{(0)}$…….(3)

By diagonalization

$$\boldsymbol{M = C^{-1}AC,}$$

$$\boldsymbol{M^n = CD^nC^{-1}} \text{ , for all n=1, 2,…. (4)}$$

Where

$$D^n = \begin{bmatrix} d_{11} & 0 & . & . & 0 \\ 0 & d_{22} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & d_{kk} \end{bmatrix}^n = \begin{bmatrix} {d_{11}}^n & 0 & . & . & 0 \\ 0 & {d_{22}}^n & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & {d_{kk}}^n \end{bmatrix}$$

The eigenvalues of $\boldsymbol{M}$ **is** : $\boldsymbol{\lambda_1 = 1}$, $\boldsymbol{\lambda_2 = \frac{1}{2}}$ and $\boldsymbol{\lambda_3 = 0}$, the corresponding eigenvectors

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$,

From equation 4

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we have, n=1,2,…..

$$\boldsymbol{X^{(n)}} = \boldsymbol{CD^nC^{-1}X^{(0)}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & 1/2^n & 0 \\ 0 & -1/2^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} -1 & -1 + 1/2^n & -1 + 1/2^{n-1} \\ 0 & -1/2^n & -1/2^{n-1} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} -a_0 + \left(-1 + 1/2^n\right)b_0 + (-1 + 1/2^{n-1})c_0 \\ (-1/2^n)b_0 + (-1/2^{n-1})c_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} -(a_0 + b_0 + c_0) + \left(1/2^n\right)b_0 + (1/2^{n-1})c_0 \\ (-1/2^n)b_0 + (-1/2^{n-1})c_0 \\ 0 \end{bmatrix}$$

Using the fact that a_0+b_0+c_0=1, we thus have

$$a_n = -1 + \left(1/2^n\right)b_0 + (1/2^{n-1})c_0$$

$$b_n = \left(-1/2^n\right)b_0 + (-1/2^{n-1})c_0$$

$$c_n = 0$$

Since $(1/2^n) \to 0$ as $n \to \infty$, it follows from these equations that

$a_n \to -1, b_n \to 0, c_n = 0$

n approaches infinity, that is, in the limit all plants in the population will be **AA**.

Example 14:

Let each plant is fertilized with a plant of its own genotype.

So, $X^{(n)} = M^n X^{(0)}$

$$M = \begin{bmatrix} 1 & 1/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/4 & 1 \end{bmatrix}$$

The eigenvalues of $\boldsymbol{M}$ **is :** $\boldsymbol{\lambda_1 = 1}$, $\boldsymbol{\lambda_2 = 1}$ and $\boldsymbol{\lambda_3 = 1/2}$, the corresponding eigenvectors

$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}/$

$\boldsymbol{X^{(n)} = CD^n C^{-1} X^{(0)}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 1/2^n \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2^n \\ 0 & 0 & 1/2^{n-1} \\ -1 & 1^n & 1/2^n \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -2 & 2 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 2 & 1 - 1/2^n & 0 \\ 0 & 1/2^{n-1} & 0 \\ -2 - 2 & -1 + 2 - 1/2^n & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 2a_0 + \left(-1 - 1/2^n\right) b_0 \\ (-1/2^n) b_0 \\ (-2 - 2) a_0 + \left(-1 + 2 - 1/2^n\right) b_0 + 2c_0 \end{bmatrix}$$

$$a_n = 2a_0 + \left(-1 - 1/2^n\right) b_0$$

$$b_n = (-1/2^n) b_0$$

$$c_n = (-4) a_0 + \left(1 - 1/2^n\right) b_0 + 2c_0$$

Since $\left(1/2^n\right) \to 0$ as $n \to \infty$, it follows from these equations that

$a_n \to 2a_0 - b_0, b_n \to 0, c_n \to -4a_0 + b_0 + 2c_0$

The population in the limit contains only **AA** and **aa**.

Autosomal Recessive Diseases

In human population, there are many genetic diseases governed by autosomal inheritance in which a normal gene **A** dominates an abnormal gene **a**. Genotype **AA** is a normal individual, genotype **Aa** is a carrier of the diseases, though is not afflicted with the disease, and genotype **aa** is afflicted with the disease.

	AA	**Aa**	**aa**
A	1	1/2	0
a	0	1/2	1

Let us now determine the fraction of carriers in future generations, we set

$$X^{(n)} = \begin{bmatrix} a_n \\ b_n \end{bmatrix}, n = 1,2, \ldots.$$

Where

a_n= fraction of population of **AA** in n-th generation,

b_n= fraction of population of **Aa** (carriers in n-th generation).

$$X^{(n)} = MX^{(n-1)}, n = 1,2, \ldots.$$

Where

$$M = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$$

The eigenvalues of $\boldsymbol{M}$ **is** : $\boldsymbol{\lambda_1 = 1}$ and $\boldsymbol{\lambda_2 = \frac{1}{2}}$ the corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

$D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

So

$$X^{(n)} = MX^{(0)}, n = 1,2, \ldots.$$

$\boldsymbol{X^{(n)} = CD^nC^{-1}X^{(0)}}$= $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 - 1/2^n \\ 0 & 1/2^n \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_0 + b_0 - 1/2^n b_0 \\ 1/2^n\, b_0 \end{bmatrix}$$

Since $a_0 + b_0 = 1$, we have

$a_n = 1 - 1/2^n b_0$, $b_n = 1/2^n b_0$, n=1,2,….

as $n \to \infty$, we have $a_n \to 1, b_n \to 0$

The population in the limit contains only **AA**.

Exercises

1- Prove that the following matrices are diagonalizable, Find the diagonal matrix D similar to $\boldsymbol{A}$ and $\boldsymbol{A^{23}}$:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$$

2- In Example 11, suppose the plants are always fertilized with a plant of **Aa.** Derive formulas for the fractions of the plants of **AA, Aa** and **a.** Also find the limiting genotype distribution as **n** tends to ∞.

2.2.7 Orthogonally and Diagonalizable of a Matrix

A square matrix $\boldsymbol{A}$ is said to be orthogonally diagonalizable if there exists an orthogonal matrix $\boldsymbol{C}$ such that $\boldsymbol{D} = \boldsymbol{C^T A C}$ is a diagonal matrix.

Notes:

- A square matrix $\boldsymbol{A}$ is said to be orthogonally diagonalizable ↔ $\boldsymbol{A}$ is a symmetric matrix.
- A square matrix $\boldsymbol{C}$ is said to be orthogonal ↔ the columns (rows) of $\boldsymbol{C}$ is an orthonormal set.
- The eigenvalues of a square matrix $\boldsymbol{A}$ lies on the main diagonal of $\boldsymbol{D} = \boldsymbol{C^{-1} A C} = \boldsymbol{C^T A C}$, where $\boldsymbol{C}$ is an orthogonal matrix.
- Norm of vector $v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is denoted and define as follows:

$$\left|\left|\boldsymbol{v}\right|\right| = \sqrt{{x_1}^2 + {x_2}^2 + .. + {x_n}^2}$$

Example15: Is $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ orthogonally diagonalizable.

Solution: $\lambda_1 = \mathbf{-2}$ and eigenvector $v_1 = r_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, take $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

$\lambda_2 = 4$ and eigenvector $v_2 = r_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$, take $v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$,

$\lambda_3 = -1$ and eigenvector $v_3 = r_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, take $v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$,

The set $\{v_1, v_2, v_3\}$ is linearly independent (**HW**) and orthogonal but not orthonormal. We can normalize these vectors as follows:

$v_{11} = \frac{v_1}{||v_1||}$, $v_{22} = \frac{v_2}{||v_2||}$, $v_{33} = \frac{v_3}{||v_3||}$

The set $\{v_{11}, v_{22}, v_{33}\}$= $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}\right\}$ is an orthonormal set in R^3.

Let $\boldsymbol{C} = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ is orthogonal matrix because $\boldsymbol{C^{-1}} = \boldsymbol{C^T}$ (**HW**) (or by previous notes). Hence $\boldsymbol{C^{-1}AC} = \boldsymbol{C^TAC} \rightarrow \boldsymbol{C^TAC} =$

$\begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ which is diagonal matrix then $\boldsymbol{A}$ is orthogonally diagonalizable.

Exercises

Find orthogonally diagonalizable for the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{E} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

Singular Value Decomposition(SVD)

The singular value decomposition is a factorization of a real or complex matrix.

If a matrix $\boldsymbol{A}$ has a matrix of eigenvectors $\boldsymbol{C}$ that is not invertible, for example, the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the noninvertible matrix of eigenvectors $\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\boldsymbol{A}$ does not have an eign decomposition (matrix diagonalization).

However, if $\boldsymbol{A}$ is an $(m \times n)$ real matrix with $m > n$, then $\boldsymbol{A}$ can be written using (SVD) of the form:

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\boldsymbol{T}}$$

Where $\boldsymbol{U}$ is $(m \times r)$ matrix, $\boldsymbol{S}$ is $(r \times r)$ matrix and $\boldsymbol{V}$ is $(n \times r)$ matrix.

properties of SVD

It is always possible to decompose a real matrix $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\boldsymbol{T}}$, where

- $\boldsymbol{U}, \boldsymbol{S}$ and $\boldsymbol{V}$ unique
- $\boldsymbol{U}$ and $\boldsymbol{V}$ have the column orthonormal (i.e. the add square element of each must equal 1),
  - $\boldsymbol{U}^{\boldsymbol{T}}\boldsymbol{U} = \boldsymbol{I}$ and $\boldsymbol{V}^{\boldsymbol{T}}\boldsymbol{V} = \boldsymbol{I}$,
  - Columns are orthogonal unit vectors (i.e dot product each two columns must equal 0).
- $\boldsymbol{S}$ diagonal (i.e. it has entries only along the diagonal),
  - Entries (singular values are positive) and sorted in decreasing order.

Where the two identity matrices may have different dimensions and $\boldsymbol{S}$ is a diagonal matrix and it has entries only along the diagonal and are arranged in descending order.

Calculating the (SVD) consists of finding the eigenvalues and eigenvectors of

$\boldsymbol{A}\ \boldsymbol{A}^{\boldsymbol{T}}$ and $\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}$. The eigenvectors of $\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}$ make up the columns of $\boldsymbol{V}$, the eigenvectors of $\boldsymbol{A}\ \boldsymbol{A}^{\boldsymbol{T}}$ make up the columns of $\boldsymbol{U}$. Also, the singular values in **S** are square roots of eigenvalues from $\boldsymbol{A}\ \boldsymbol{A}^{\boldsymbol{T}}$ or $\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}$. The singular values are the diagonal entries of the $\boldsymbol{S}$ matrix and are arranged in descending order. The singular values are always real numbers. If the matrix $\boldsymbol{A}$ is a real matrix, then $\boldsymbol{U}$ and $\boldsymbol{V}$ are also real.

$\boldsymbol{A = USV^T, AV = US}$

Note

For the complex matrix $\boldsymbol{A}$, the (SVD) is in the form:

$$\boldsymbol{A = USV^H}$$

Where $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary matrices, $\boldsymbol{V^H}$is the conjugate transpose of $\boldsymbol{V}$, and $\boldsymbol{S}$ is a diagonal matrix whose elements are the singular values of the original matrix (i.e. the square roots of the eigenvalues of $\boldsymbol{A^H A}$ are called singular values).

If $\boldsymbol{A}$ is a complex matrix, then there always exists such a decomposition with positive singular values.

Example16: Find the (SVD) for the following matrix: $\boldsymbol{A} = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$.

Solution:

$$\boldsymbol{A^T A} = \begin{bmatrix} 5 & -1 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix}$$

$$\left|\boldsymbol{A^T A} - \lambda \mathbf{I}\right| = 0 \to \begin{vmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{vmatrix} = 0 \to (26-\lambda)(74-\lambda) - 1384 = 0$$
$$\to 1924 - 100\lambda + \lambda^2 - 324 = 0 \to \lambda^2 - 100\lambda + 1600 = 0$$
$$\to (\lambda - 20)(\lambda - 80) = 0 \to \lambda = 80{,}20$$

$\boldsymbol{A^T A} - 80\boldsymbol{I} = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix} - \begin{bmatrix} 80 & 0 \\ 0 & 80 \end{bmatrix} = \begin{bmatrix} -54 & 18 \\ 18 & -6 \end{bmatrix}$, $r_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not orthonormal, $v_1 = \frac{\mathrm{r}_1}{||\mathrm{r}_1||} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$ is orthonormal vector.

$\boldsymbol{A^T A} - 20\boldsymbol{I} = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix} - \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} 6 & 18 \\ 18 & 54 \end{bmatrix}$, $r_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is not orthonormal, $v_2 = \frac{\mathrm{r}_2}{||\mathrm{r}_2||} = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$ is orthonormal vector.

$$\boldsymbol{V} = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}, \boldsymbol{S} = \begin{bmatrix} \sqrt{80} & 0 \\ 0 & \sqrt{20} \end{bmatrix} = \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{bmatrix}$$

$$\boldsymbol{AV} = \boldsymbol{US} \rightarrow \boldsymbol{AV} = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \rightarrow \boldsymbol{AV} = \begin{bmatrix} 2\sqrt{10} & -\sqrt{10} \\ 2\sqrt{10} & \sqrt{10} \end{bmatrix} \rightarrow$$

$$\boldsymbol{AV} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{bmatrix} = \boldsymbol{US} \rightarrow \boldsymbol{U} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

<u>Example 17</u>: Find the (SVD) for the following matrix: $\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$.

<u>Solution</u>: $\boldsymbol{A^T A} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$

$$\left|\boldsymbol{A^T A} - \lambda\mathbf{I}\right| = 0 \rightarrow \begin{bmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{bmatrix} = 0$$

$$\rightarrow (10-\lambda)[(10-\lambda)(2-\lambda) - 16] + 2[0 - 2(10-\lambda)] = (10-\lambda)[\lambda^2 - 12\lambda + 4] - 20 + 2\lambda = 0 \rightarrow \lambda(\lambda-10)(\lambda-12) = 0 \rightarrow \lambda = 0,10,12$$

When $\lambda = 12$,

$\boldsymbol{A^T A} - 12\boldsymbol{I} = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix}$, $r_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is not orthonormal, $v_1 = \frac{r_1}{||r_1||} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ is orthonormal vector.

When $\lambda = 10$,

$\boldsymbol{A^T A} - 10\boldsymbol{I} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix}$, $r_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ is not orthonormal, $v_2 = \frac{r_2}{||r_2||} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$ is orthonormal vector.

When $\lambda = 0$,

$\boldsymbol{A^T A} - 0\boldsymbol{I} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$, $r_2 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ is not orthonormal, $v_2 = \frac{r_2}{||r_2||} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}$ is orthonormal vector.

$$\boldsymbol{V} = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}, \boldsymbol{S} = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

$$\boldsymbol{AV} = \boldsymbol{US} \rightarrow \boldsymbol{AV} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix} \rightarrow \boldsymbol{AV} = \begin{bmatrix} \sqrt{6} & \sqrt{5} & 0 \\ \sqrt{6} & -\sqrt{5} & 0 \end{bmatrix} \rightarrow$$

$$\boldsymbol{AV} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} = \boldsymbol{US} \rightarrow \boldsymbol{U} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Exercises

Find the (SVD) for the following matrices: $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$.

Chapter Three

<u>Bimatrices</u>

<u>Definition of Bimatrices</u>

A bimatrix $\boldsymbol{A_B}$ is defined as the union of two rectangular away of numbers $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ arranged into rows and columns. It is written as follows $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ where $\boldsymbol{A_1} \neq \boldsymbol{A_2}$

with $\boldsymbol{A_1} = \begin{bmatrix} a^1{}_{11} & a^1{}_{12} & \dots & a^1{}_{1n} \\ a^1{}_{21} & a^1{}_{22} & \dots & a^1{}_{2n} \\ \vdots & \vdots & ::: & \vdots \\ . & . & \dots & . \\ a^1{}_{m1} & a^1{}_{m2} & \dots & a^1{}_{mn} \end{bmatrix}$ and $\boldsymbol{A_2} = \begin{bmatrix} a^2{}_{11} & a^2{}_{12} & \dots & a^2{}_{1n} \\ a^2{}_{21} & a^2{}_{22} & \dots & a^2{}_{2n} \\ \vdots & \vdots & ::: & \vdots \\ . & . & \dots & . \\ a^2{}_{m1} & a^2{}_{m2} & \dots & a^2{}_{mn} \end{bmatrix}$

$\cup$ is just the notational convenience (symbol) only. $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are called as the component matrices of the bimatrix $\boldsymbol{A_B}$.

The above array is called a *m* by *n* bimatrix (written as $\boldsymbol{B}(m \times n)$) since each of $\boldsymbol{A_i}$ (i = 1, 2) has *m* rows and *n* columns. It is to be noted a bimatrix has no numerical value associated with it. It is only a convenient way of representing a pair of arrays of numbers.

<u>Notes</u>

- If $\boldsymbol{A_1} = \boldsymbol{A_2}$ then $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ is not a bimatrix. A bimatrix $\boldsymbol{A_B}$ is denoted by $(\boldsymbol{a^1}_{ij}) \cup (\boldsymbol{a^2}_{ij})$.
- If both $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are $(m \times n)$ matrices then the bimatrix $\boldsymbol{A_B}$ is called the $(m \times n)$ rectangular bimatrix.
- If both $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are square matrices then $\boldsymbol{A_B}$ is called the square bimatrix.
  - If one of the matrices in the bimatrix $\boldsymbol{A_B}$ is square and other is rectangular or both $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are rectangular matrices say $(m_1 \times n_1)$, $(m_2 \times n_2)$ with $m_1 \neq m_2$ or $n_1 \neq n_2$ then $\boldsymbol{A_B}$ is called the mixed bimatrix.
  - A bimatrix whose all elements are zero is called zero (null) bimatrix and

it is denoted by $\boldsymbol{O_B} = \boldsymbol{O_1} \cup \boldsymbol{O_2} = \boldsymbol{0_1} \cup \boldsymbol{0_2}$.

- We make an assumption the zero bimatrix is a union of two zero matrices even if $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are one and the same (i.e. $\boldsymbol{A_1} = \boldsymbol{A_2} = (\boldsymbol{0})$).
- The unit (identity) square bimatrix denoted by $\boldsymbol{I_B} = \boldsymbol{I^{m\times m}}_{\boldsymbol{1}} \cup \boldsymbol{I^{m\times m}}_{\boldsymbol{2}}$.
- Identity mixed square bimatrix denoted by $\boldsymbol{I_B} = \boldsymbol{I^{m\times m}}_{\boldsymbol{1}} \cup \boldsymbol{I^{n\times n}}_{\boldsymbol{2}}$.

Example 18: Classify each the following bimatrices

a) $\boldsymbol{A_B} = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$, b) $\boldsymbol{A'_B} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cup \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$,

c) $\boldsymbol{A''_B} = [3 \quad -2 \quad 0 \quad 1 \quad 1] \cup [1 \quad 1 \quad -1 \quad 2 \quad 1]$,

d) $\boldsymbol{A^1}_{\boldsymbol{B}} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, e) $\boldsymbol{A^2}_{\boldsymbol{B}} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$,

f) $\boldsymbol{A^3}_{\boldsymbol{B}} = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$, and g) $\boldsymbol{A^4}_{\boldsymbol{B}} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \cup \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

Solution: a) (2×3) rectangular bimatrix, b) is a column bimatrix , c) is a row bimatrix, d) (3×3) square bimatrix, e) mixed square bimatrix ,f) mixed bimatrix and g) mixed rectangular bimatrix .

Operations on Bimatrices

Here the operations on Bimatrices are introduce

Equal

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two bimatrice, $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ are equal (i.e. $\boldsymbol{A_B} = \boldsymbol{C_B}$) $\leftrightarrow$ $\boldsymbol{A_1} = \boldsymbol{C_1}$ and $\boldsymbol{A_2} = \boldsymbol{C_2}$.
$\boldsymbol{A_B}$ is not equal to $\boldsymbol{C_B}$ (i.e. $\boldsymbol{A_B} \neq \boldsymbol{C_B}$) $\leftrightarrow$ $\boldsymbol{A_1} \neq \boldsymbol{C_1}$ or $\boldsymbol{A_2} \neq \boldsymbol{C_2}$.

Example 19: Let

1- $\boldsymbol{A_B} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$,

2- $\boldsymbol{A_B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 0 & 4 & -2 \\ -3 & 0 & 0 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Solution: 1- and 2- $\boldsymbol{A_B} \neq \boldsymbol{C_B}$.

Multiply by a Constant (Scalar Multiplication)

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\lambda \in R$ be a scalar then

$$\lambda \boldsymbol{A_B} = \begin{bmatrix} \lambda a^1{}_{11} & \lambda a^1{}_{12} & \dots & \lambda a^1{}_{1n} \\ \lambda a^1{}_{21} & \lambda a^1{}_{22} & \dots & \lambda a^1{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ \lambda a^1{}_{m1} & \lambda a^1{}_{m2} & \dots & \lambda a^1{}_{mn} \end{bmatrix} \cup \begin{bmatrix} \lambda a^2{}_{11} & \lambda a^2{}_{12} & \dots & \lambda a^2{}_{1n} \\ \lambda a^2{}_{21} & \lambda a^2{}_{22} & \dots & \lambda a^2{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ \lambda a^2{}_{m1} & \lambda a^2{}_{m2} & \dots & \lambda a^2{}_{mn} \end{bmatrix},$$

The product $\lambda \boldsymbol{A_B}$ is another ($m \times n$) bimatrix. If $\boldsymbol{A_B}$ is ($m \times n$) bimatrix then $\lambda \boldsymbol{A_B} = [\lambda \boldsymbol{a^1}_{ij}] \cup [\lambda \boldsymbol{a^2}_{ij}] = [\boldsymbol{a^1}_{ij}\lambda] \cup [\boldsymbol{a^2}_{ij}\lambda] = \boldsymbol{A_B}\lambda.$

<u>Example 20</u>: 1- Let $\boldsymbol{A_B} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & -1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, find $\lambda \boldsymbol{A_B}$ when $\lambda = 3$,

2- Let $\boldsymbol{A_B} = [3 \quad 2 \quad 1 \quad -4] \cup [0 \quad 1 \quad -1 \quad 0]$, find $\lambda \boldsymbol{A_B}$ when $\lambda = -2$,

<u>Solution</u> :

1- $3\boldsymbol{A_B} = \begin{bmatrix} 6 & 0 & 3 \\ 9 & 9 & -3 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & -3 \\ 6 & 3 & 0 \end{bmatrix}$,

2- $(-2)\boldsymbol{A_B} = [-6 \quad -4 \quad -2 \quad 8] \cup [0 \quad -2 \quad 2 \quad 0]$.

Addition

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two ($m \times n$) bimatrices. The sum $\boldsymbol{D_B}$ of the bimatrices $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ is defined as follows:

$$\boldsymbol{D_B} = \boldsymbol{A_B} + \boldsymbol{C_B} = (\boldsymbol{A_1} \cup \boldsymbol{A_2}) + (\boldsymbol{C_1} \cup \boldsymbol{C_2}) = [\boldsymbol{A_1} + \boldsymbol{C_1}] \cup [\boldsymbol{A_2} + \boldsymbol{C_2}]$$

Where $\boldsymbol{A_1} + \boldsymbol{C_1}$ and $\boldsymbol{A_2} + \boldsymbol{C_2}$ are the usual addition on matrices (i.e. if

$\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2} = [\boldsymbol{a^1}_{ij}] \cup [\boldsymbol{a^2}_{ij}]$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2} = [\boldsymbol{c^1}_{ij}] \cup [\boldsymbol{c^2}_{ij}]$,

$$\boldsymbol{D_B} = \boldsymbol{A_B} + \boldsymbol{C_B} = [\boldsymbol{a^1}_{ij} + \boldsymbol{c^1}_{ij}] \cup [\boldsymbol{a^2}_{ij} + \boldsymbol{c^2}_{ij}] =$$

$$\begin{bmatrix} \boldsymbol{a^1}_{11} + \boldsymbol{c^1}_{11} & \boldsymbol{a^1}_{12} + \boldsymbol{c^1}_{12} & \dots & \boldsymbol{a^1}_{1n} + \boldsymbol{c^1}_{1n} \\ \boldsymbol{a^1}_{21} + \boldsymbol{c^1}_{21} & \boldsymbol{a^1}_{22} + \boldsymbol{c^1}_{22} & \dots & \boldsymbol{a^1}_{2n} + \boldsymbol{c^1}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ \boldsymbol{a^1}_{m1} + \boldsymbol{c^1}_{m1} & \boldsymbol{a^1}_{m2} + \boldsymbol{c^1}_{m2} & \dots & \boldsymbol{a^1}_{mn} + \boldsymbol{c^1}_{mn} \end{bmatrix} \cup$$

$$\begin{bmatrix} a^2{}_{11}+c^2{}_{11} & a^2{}_{12}+c^{22}{}_{12} & \dots & a^2{}_{1n}+c^2{}_{1n} \\ a^2{}_{21}+c^2{}_{21} & a^2{}_{22}+c^2{}_{22} & \dots & a^2{}_{2n}+c^2{}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & \dots & . \\ a^2{}_{m1}+c^2{}_{m1} & a^2{}_{m2}+c^2{}_{m2} & \dots & a^2{}_{mn}+c^2{}_{mn} \end{bmatrix}.$$

Notes

- The sum of two bimatrices is not in general bimatrix. For example, let $\boldsymbol{A_B} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$, and $\boldsymbol{C_B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix}$, $\boldsymbol{A_B} + \boldsymbol{C_B} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ is not bimatrix because $[\boldsymbol{A_1} + \boldsymbol{C_1}] = [\boldsymbol{A_2} + \boldsymbol{C_2}]$.

- If $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ be two mixed bimatrix then $(\boldsymbol{A_B} + \boldsymbol{C_B})$ is always mixed bimatrix.
- If $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ be two $(m \times n)$ bimatrices then $\boldsymbol{A_B} + \boldsymbol{C_B} = \boldsymbol{C_B} + \boldsymbol{A_B}$. Also if $\boldsymbol{A_B}$, $\boldsymbol{C_B}$ and $\boldsymbol{D_B}$ be three $(m \times n)$ bimatrices then $\boldsymbol{A_B} + (\boldsymbol{C_B} + \boldsymbol{D_B}) = (\boldsymbol{A_B} + \boldsymbol{C_B}) + \boldsymbol{D_B}$.

<u>Example 21</u>: Let 1- $\boldsymbol{A_B} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & -1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, and

$$\boldsymbol{C_B} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix},$$

2- $\boldsymbol{A_B} = [3 \quad 2 \quad 1 \quad -4 \quad 0] \cup [0 \quad 1 \quad -1 \quad 0 \quad 1]$,

$\boldsymbol{C_B} = [1 \quad 1 \quad 1 \quad 1 \quad 1] \cup [5 \quad -1 \quad 2 \quad 0 \quad 3]$, find $\boldsymbol{A_B} + \boldsymbol{C_B}$,

<u>Solution</u>: 1- $\boldsymbol{A_B} + \boldsymbol{C_B} = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 5 & -2 \end{bmatrix} \cup \begin{bmatrix} 3 & 4 & 0 \\ 2 & 3 & -1 \end{bmatrix}$,

2- $\boldsymbol{A_B} + \boldsymbol{C_B} = [4 \quad 3 \quad 2 \quad -3 \quad 1] \cup [5 \quad 0 \quad 1 \quad 0 \quad 4]$.

<u>Example 22</u>: Let $\boldsymbol{A_B} = \begin{bmatrix} 6 & -1 \\ 2 & 2 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}$, find $\boldsymbol{A_B} + \boldsymbol{A_B}$.

<u>Solution</u>: $\boldsymbol{A_B} + \boldsymbol{A_B} = \begin{bmatrix} 12 & -2 \\ 4 & 4 \\ 2 & -2 \end{bmatrix} \cup \begin{bmatrix} 6 & 2 \\ 0 & 4 \\ -2 & 6 \end{bmatrix} = \boldsymbol{2}\ \boldsymbol{A_B}$.

Subtraction

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two ($m \times n$) bimatrices. The subtraction of the bimatrices $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ is defined as follows:

$$\begin{aligned}\boldsymbol{A_B} - \boldsymbol{C_B} &= \boldsymbol{A_B} + (-\boldsymbol{C_B}) = (\boldsymbol{A_1} \cup \boldsymbol{A_2}) + (-\boldsymbol{C_1} \cup -\boldsymbol{C_2}) \\ &= [\boldsymbol{A_1} - \boldsymbol{C_1}] \cup [\boldsymbol{A_2} - \boldsymbol{C_2}] = [\boldsymbol{A_1} + (-\boldsymbol{C_1})] \cup [\boldsymbol{A_2} + (-\boldsymbol{C_2})].\end{aligned}$$

Where $\boldsymbol{A_1} + (-\boldsymbol{C_1})$ and $\boldsymbol{A_2} + (-\boldsymbol{C_2})$ are the usual addition on matrices.

Example 23: Let $\boldsymbol{A_B} = [1 \quad 2 \quad 3 \quad -1 \quad 2 \quad 1] \cup [3 \quad -1 \quad 2 \quad 0 \quad 3 \quad 1]$,

$\boldsymbol{C_B} = [-1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0] \cup [2 \quad 0 \quad -2 \quad 0 \quad 3 \quad 0]$, find $\boldsymbol{A_B} - \boldsymbol{C_B}$.

Solution: $\boldsymbol{A_B} - \boldsymbol{C_B} = [2 \quad 1 \quad 2 \quad -2 \quad 1 \quad 1] \cup [1 \quad -1 \quad 4 \quad 0 \quad 0 \quad 1]$.

Notes

- The subtract of two bimatrices is not in general bimatrix. For example, let $\boldsymbol{A_B} = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 2 \\ -1 & 1 & 2 \end{bmatrix}$, and $\boldsymbol{C_B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, $\boldsymbol{A_B} - \boldsymbol{C_B} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is not bimatrix because $[\boldsymbol{A_1} - \boldsymbol{C_1}] = [\boldsymbol{A_2} - \boldsymbol{C_2}]$.

- If $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ be two mixed bimatrix then $(\boldsymbol{A_B} - \boldsymbol{C_B})$ is always mixed bimatrix.

Multiplication of Two Bimatrices

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two square bimatrices. The multiplication of the bimatrices $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ is defined as follows:

$$\boldsymbol{A_B}.\boldsymbol{C_B} = (\boldsymbol{A_1} \cup \boldsymbol{A_2}).(\boldsymbol{C_1} \cup \boldsymbol{C_2}) = [\boldsymbol{A_1}.\boldsymbol{C_1}] \cup [\boldsymbol{A_2}.\boldsymbol{C_2}].$$

Example 24: a) Let $\boldsymbol{A_B} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$,

b) $\boldsymbol{A_B} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$,

c) $\boldsymbol{A_B} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & -4 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ -1 & 2 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{bmatrix}$,

find $\boldsymbol{A_B}.\ \boldsymbol{C_B}$.

Solution:

a) $A_B.\ C_B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} =$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix},$$

b) $A_B.\ C_B = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 3 & 1 \\ 8 & -1 & -5 \\ 6 & 0 & -3 \end{bmatrix} \cup \begin{bmatrix} 5 & -1 \\ 1 & -1 \end{bmatrix},$$

c) $A_B.\ C_B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 0 \\ -1 & 2 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 5 & 5 \\ -1 & 2 \\ -1 & 3 \\ 13 & -6 \end{bmatrix} \cup \begin{bmatrix} 7 & 4 & 3 \\ 1 & -3 & -1 \end{bmatrix}.$$

Notes:

- The multiply of two bimatrices is not in general bimatrix.
- If $A_B = (A_1)^{m\times n} \cup (A_2)^{p\times q}$ be mixed rectangular bimatrix and $C_B = (C_1)^{n\times m} \cup (C_2)^{q\times p}$ be another mixed rectangular bimatrix then $(A_B.C_B)$ is square bimatrix.

Transpose

Let $A_B = A_1 \cup A_2$,to transpose the bimatrix, swap the rows and columns of each matrix A_1 and A_2 (i.e. $A^T{}_B = A^T{}_1 \cup A^T{}_2$).

- Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices and $D_B = A_B + C_B$ then $D^T{}_B = A^T{}_B + C^T{}_B$.
- If A_B and C_B be two bimatrices then $(A_B C_B)^T = C^T{}_B A^T{}_B$.
- If $A_B, C_B, \dots, N_B$ be bimatrices such that their product $(A_B\ C_B \dots\ N_B)$ is well defined then $(A_B C_B \dots N_B)^T = N^T \dots C^T{}_B A^T{}_B$.

Example 25: Let $\boldsymbol{A_B} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$,

$\boldsymbol{C_B} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \cup \begin{bmatrix} 3 & 3 \\ 0 & 0 \\ 1 & -1 \\ -2 & 0 \end{bmatrix}$, Find $\boldsymbol{A^T}_{\mathbf{B}}, \boldsymbol{C^T}_{\mathbf{B}}$ and $(\boldsymbol{A_B C_B})^T$.

Solution: $\boldsymbol{A^T}_{\mathbf{B}} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 1 & 0 \\ 4 & 2 \end{bmatrix}$, $\boldsymbol{C^T}_{\mathbf{B}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & -2 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 1 & -2 \\ 3 & 0 & -1 & 0 \end{bmatrix}$

$\boldsymbol{A_B C_B} = \begin{bmatrix} 9 & 2 & 5 & 2 \\ 0 & 1 & 1 & -2 \\ 3 & 2 & 3 & -2 \end{bmatrix} \cup \begin{bmatrix} -1 & 5 \\ -4 & 0 \end{bmatrix}$, $(\boldsymbol{A_B C_B})^T = \begin{bmatrix} 9 & 0 & 3 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \\ 2 & -2 & -2 \end{bmatrix} \cup \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix}$

$$\boldsymbol{C^T}_{\mathbf{B}} \boldsymbol{A^T}_{\mathbf{B}} = \begin{bmatrix} 9 & 0 & 3 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \\ 2 & -2 & -2 \end{bmatrix} \cup \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix}$$

Then $(\boldsymbol{A_B C_B})^{\boldsymbol{T}} = \boldsymbol{C^T}_{\mathbf{B}} \boldsymbol{A^T}_{\mathbf{B}}$.

Some Basic Properties of Bimatrices

- Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two ($m \times n$) bimatrices. The sum $\boldsymbol{A_B} + \boldsymbol{C_B} = (\boldsymbol{A_1} \cup \boldsymbol{A_2}) + (\boldsymbol{C_1} \cup \boldsymbol{C_2}) = [\boldsymbol{A_1} + \boldsymbol{C_1}] \cup [\boldsymbol{A_2} + \boldsymbol{C_2}]$ is bimatrix $\leftrightarrow$ $[\boldsymbol{A_1} + \boldsymbol{C_1}] \neq [\boldsymbol{A_2} + \boldsymbol{C_2}]$.
- Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two ($m \times n$) bimatrices. The $\boldsymbol{A_B} - \boldsymbol{C_B} = (\boldsymbol{A_1} \cup \boldsymbol{A_2}) - (\boldsymbol{C_1} \cup \boldsymbol{C_2}) = [\boldsymbol{A_1} - \boldsymbol{C_1}] \cup [\boldsymbol{A_2} - \boldsymbol{C_2}]$ is bimatrix $\leftrightarrow$ $[\boldsymbol{A_1} - \boldsymbol{C_1}] \neq [\boldsymbol{A_2} - \boldsymbol{C_2}]$.
- Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ and $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ be two square bimatrices. The $\boldsymbol{A_B}.\boldsymbol{C_B} = [\boldsymbol{A_1}.\boldsymbol{C_1}] \cup [\boldsymbol{A_2}.\boldsymbol{C_2}]$ is bimatrix $\leftrightarrow$ $[\boldsymbol{A_1}.\boldsymbol{C_1}] \neq [\boldsymbol{A_2}.\boldsymbol{C_2}]$.
- If $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ be two ($m \times m$) square bimatrices. In general $\boldsymbol{A_B}.\boldsymbol{C_B} \neq \boldsymbol{C_B}.\boldsymbol{A_B}$.

Example 26: Let $\boldsymbol{A_B} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$, Find $\boldsymbol{A_B}.\boldsymbol{C_B}$ and $\boldsymbol{C_B}.\boldsymbol{A_B}$.

Solution: $\boldsymbol{A_B}.\ \boldsymbol{C_B} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{C_B}.\boldsymbol{A_B} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} . \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boldsymbol{A_B}.\boldsymbol{C_B} \neq \boldsymbol{C_B}.\boldsymbol{A_B}$.

- ❖ In some cases for the bimatrices $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ only one type of product $\boldsymbol{A_B}.\boldsymbol{C_B}$ may defined and $\boldsymbol{C_B}.\boldsymbol{A_B}$ may not be even defined.

<u>**Example 27**</u>: Let $\boldsymbol{A_B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}, \boldsymbol{C_B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 4 \end{bmatrix}$.

<u>**Solution:**</u> $\boldsymbol{A_B}.\ \boldsymbol{C_B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 0 \\ 3 & 0 \\ 1 & -1 \end{bmatrix} \cup \begin{bmatrix} 8 & -1 & 6 \\ 9 & 0 & 3 \\ 5 & -2 & 9 \end{bmatrix},$$

But $\boldsymbol{C_B}.\ \boldsymbol{A_B}$ is not define.

- ❖ Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$, $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ and $\boldsymbol{D_B} = \boldsymbol{D_1} \cup \boldsymbol{D_2}$ be three square bimatrices:

  $\boldsymbol{A_B}(\boldsymbol{C_B}\boldsymbol{D_B}) = (\boldsymbol{A_B}\boldsymbol{C_B})\ \boldsymbol{D_B} = \boldsymbol{A_B}\boldsymbol{C_B}\boldsymbol{D_B}$ (Associative law)

  Where $\boldsymbol{A_B}(\boldsymbol{C_B}\boldsymbol{D_B}) = \boldsymbol{A_B}(\boldsymbol{C_1}\boldsymbol{D_1} \cup \boldsymbol{C_2}\boldsymbol{D_2}) = \boldsymbol{A_1}(\boldsymbol{C_1}\boldsymbol{D_1}) \cup \boldsymbol{A_2}(\boldsymbol{C_2}\boldsymbol{D_2}) = (\boldsymbol{A_1}\boldsymbol{C_1})\boldsymbol{D_1} \cup (\boldsymbol{A_2}\boldsymbol{C_2})\boldsymbol{D_2} = (\boldsymbol{A_1}\boldsymbol{C_1} \cup \boldsymbol{A_2}\boldsymbol{C_2})\boldsymbol{D_B} = (\boldsymbol{A_B}\boldsymbol{C_B})\ \boldsymbol{D_B}$.

- ❖ Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$, $\boldsymbol{C_B} = \boldsymbol{C_1} \cup \boldsymbol{C_2}$ and $\boldsymbol{D_B} = \boldsymbol{D_1} \cup \boldsymbol{D_2}$ be three square bimatrices:

  $\boldsymbol{A_B}(\boldsymbol{C_B} + \boldsymbol{D_B}) = \boldsymbol{A_B}\boldsymbol{C_B} + \boldsymbol{A_B}\boldsymbol{D_B}$ (Distributive law)

  Where $\boldsymbol{A_B}(\boldsymbol{C_B} + \boldsymbol{D_B}) = \boldsymbol{A_B}\{(\boldsymbol{C_1} + \boldsymbol{D_1}) \cup (\boldsymbol{C_2} + \boldsymbol{D_2})\} = (\boldsymbol{A_1}(\boldsymbol{C_1} + \boldsymbol{D_1})) \cup (\boldsymbol{A_2}(\boldsymbol{C_2}+\boldsymbol{D_2})) = (\boldsymbol{A_1}\boldsymbol{C_1} + \boldsymbol{A_1}\boldsymbol{D_1}) \cup (\boldsymbol{A_2}\boldsymbol{C_2}+\boldsymbol{A_2}\boldsymbol{D_2}) = (\boldsymbol{A_1}\boldsymbol{C_1} \cup \boldsymbol{A_2}\boldsymbol{C_2})+(\boldsymbol{A_1}\boldsymbol{D_1} \cup \boldsymbol{A_2}\boldsymbol{D_2}) = \boldsymbol{A_B}\boldsymbol{C_B} + \boldsymbol{A_B}\boldsymbol{D_B}$.

- ❖ Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ be a $(m \times m)$ square bimatrix. $\boldsymbol{A_B}$ is called diagonal if each of $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are $(m \times m)$ diagonal matrices. The identity bimatrix is diagonal bimatrix. But if $\boldsymbol{A_B}$ mixed square bimatrix, $\boldsymbol{A_B}$ is called mixed diagonal bimatrix if both $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are diagonal matrices.

- Diagonal bimatrix cannot be defined in case of rectangular bimatrix or mixed bimatrix which is not mixed square bimatrix.
- For every bimatrix $\boldsymbol{A_B}$ there is exist a zero bimatrix $\boldsymbol{O_B}$ such that

$$\boldsymbol{A_B} + \boldsymbol{O_B} = \boldsymbol{O_B} + \boldsymbol{A_B} = \boldsymbol{A_B}$$

- $\boldsymbol{O_B A_B} = \boldsymbol{O_B}$, this is true only in case of square bimatrix or mixed square bimatrix only.
- If $\boldsymbol{A_B}$ is square bimatrix or mixed square bimatrix then

$$\boldsymbol{A_B A_B} = \boldsymbol{A^2}_{\boldsymbol{B}} \;, \quad \boldsymbol{A_B A_B A_B} = \boldsymbol{A^3}_{\boldsymbol{B}} \text{ and so on.}$$

- This type of product does not exist in case of rectangular bimatrix or mixed rectangular bimatrix.
- For any scalar λ , the square bimatrix $\boldsymbol{A^{m\times m}}_{\boldsymbol{B}}$ is called a scalar bimatrix if

$$\boldsymbol{A^{m\times m}}_{\boldsymbol{B}} = \lambda \boldsymbol{I^{m\times m}}_{\boldsymbol{B}}.$$

- If $\boldsymbol{A_B} = \boldsymbol{A^{m\times m}}_{\boldsymbol{1}} \cup \boldsymbol{A^{n\times n}}_{\boldsymbol{2}}$ then scalar bimatrix of $\boldsymbol{A_B}$ is defined as:

$$\boldsymbol{A_B} = \lambda \boldsymbol{I_B} = \lambda \boldsymbol{I^{m\times m}}_{\boldsymbol{1}} \cup \lambda \boldsymbol{I^{n\times n}}_{\boldsymbol{2}}$$

- Null the bimatrix can be got for any form of bimatrices $\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ provided the product $(\boldsymbol{A_B C_B})$ is defined and $\boldsymbol{A_B C_B} = [0]$.

<u>Example 28</u>: Let $\boldsymbol{A_B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \boldsymbol{C_B} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 3 & 1 \end{bmatrix} \cup \begin{bmatrix} 5 \\ 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}.$

Find $\boldsymbol{A_B C_B}$.

<u>Solution</u>: $\boldsymbol{A_B}.\ \boldsymbol{C_B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 4 & 3 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cup [0]$$

<u>Symmetric and Skew Symmetric Bimatrices</u>

A bimatrix is *symmetric* if it is equal to its own transpose, (i.e. for the bimatrix $\boldsymbol{A_B}$, the component matrices of $\boldsymbol{A_B}$ are also symmetric, $\boldsymbol{A_B} = \boldsymbol{A^T}_{\mathbf{B}}$). A symmetric bimatrix must be either a square bimatrix or a mixed square bimatrix. Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ be a $(m \times m)$ square bimatrix. This is an m^{th} order square bimatrix. This will not have $2m^2$ arbitrary elements since $a^1{}_{ij} = a^1{}_{ji}$ and $a^2{}_{ij} = a^2{}_{ji}$ where $\boldsymbol{A_1} = (a^1{}_{ij})$ and $\boldsymbol{A_2} = (a^2{}_{ij})$ both below and above the main diagonal. The number above the main diagonal of $\boldsymbol{A_B}$ is (m^2-m) and the diagonal elements are also arbitrary. Thus the total number of arbitrary elements in an m^{th} order square symmetric bimatrix is $(m^2-m+2m)=m(m+1)$. But if $\boldsymbol{A_B}=\boldsymbol{A}^{m\times m}{}_{\boldsymbol{1}}\cup\boldsymbol{A}^{n\times n}{}_{\boldsymbol{2}}$ be a mixed square bimatrix then it has a total number $(\frac{m(m+1)}{2}+\frac{n(n+1)}{2})$ arbitrary elements.

<u>Example 29</u>: Let $\boldsymbol{A_B}=\begin{bmatrix}3&0&2\\0&1&-1\\2&-1&-5\end{bmatrix}\cup\begin{bmatrix}0&1&2\\1&-5&3\\2&3&0\end{bmatrix}$,

$C_B=\begin{bmatrix}2&0\\0&1\end{bmatrix}\cup\begin{bmatrix}3&1&2&4\\1&0&-1&2\\2&-1&1&-4\\4&2&-4&8\end{bmatrix}$, find if A_B and C_B are symmetric bimatrices.

<u>Solution</u>: $\boldsymbol{A^T}_{\mathbf{B}}=\begin{bmatrix}3&0&2\\0&1&-1\\2&-1&-5\end{bmatrix}\cup\begin{bmatrix}0&1&2\\1&-5&3\\2&3&0\end{bmatrix}=\boldsymbol{A_B},\ \boldsymbol{A_B}$ has $(3(3+1)=3(4)=12)$ arbitrary elements.

$\boldsymbol{C^T}_{\mathbf{B}}=\begin{bmatrix}2&0\\0&1\end{bmatrix}\cup\begin{bmatrix}3&1&2&4\\1&0&-1&2\\2&-1&1&-4\\4&2&-4&8\end{bmatrix}=\boldsymbol{C_B},\ \boldsymbol{C_B}$ has $(\frac{2(2+1)}{2}+\frac{4(4+1)}{2}=\frac{2(3)}{2}+\frac{4(5)}{2}=\frac{6}{2}+\frac{20}{2}=3+10=13)$ arbitrary elements.

$\boldsymbol{A_B}$ and $\boldsymbol{C_B}$ are symmetric bimatrices.

- A ***skew-symmetric*** is a bimatrix $\boldsymbol{A_B}$ for which $(\boldsymbol{A_B}=-\boldsymbol{A^T}_{\mathbf{B}})$ where $(-\boldsymbol{A^T}_{\mathbf{B}}=-\boldsymbol{A^T}_{\boldsymbol{1}}\cup-\boldsymbol{A^T}_{\boldsymbol{2}})$ (i.e. the component matrices $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ of $\boldsymbol{A_B}$ are also skew-symmetric).

- If the m^{th} order skew-symmetric bimatrix have the diagonal elements of $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are zero (i.e. $a^1{}_{ii}=a^2{}_{ii}=0$) then the number of arbitrary elements is $2m(m-1)$.
- If $\boldsymbol{A_B}=\boldsymbol{A}^{m\times m}{}_{\boldsymbol{1}}\cup\boldsymbol{A}^{n\times n}{}_{\boldsymbol{2}}$ is a mixed square bimatrix then $\boldsymbol{A_B}$ is called skew-symmetric $\boldsymbol{A_B}=-\boldsymbol{A^T}_{\mathbf{B}}$, i.e.,
  $(\boldsymbol{A_1}^{m\times m}=-(\boldsymbol{A}^{m\times m}{}_{\boldsymbol{1}})^T)$ and $(\boldsymbol{A_2}^{n\times n}=-(\boldsymbol{A}^{n\times n}{}_{\boldsymbol{2}})^T)$. This bimatrix has $(m(m-1)+n(n-1))$ arbitrary elements.

Example 30: Let $A_B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & 13 \\ -3 & 0 & -2 \\ -13 & 2 & 0 \end{bmatrix}$,

$C_B = \begin{bmatrix} 0 & -1 & -2 & -4 \\ 1 & 0 & 1 & -2 \\ 2 & -1 & 0 & 4 \\ 4 & 2 & -4 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find if A_B and C_B are skew-symmetric bimatrices.

Solution: $-A^T{}_B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & 13 \\ -3 & 0 & -2 \\ -13 & 2 & 0 \end{bmatrix} = A_B,$ A_B has $2(3)(3-1) = 6(2) = 12$ arbitrary elements.

$-C^T{}_B = \begin{bmatrix} 0 & -1 & -2 & -4 \\ 1 & 0 & 1 & -2 \\ 2 & -1 & 0 & 4 \\ 4 & 2 & -4 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = C_B$, C_B has
$4(4-1) + 2(2-1) = 4(3) + 2(1) = 12 + 2 = 14)$ arbitrary elements.

A_B and C_B are skew-symmetric bimatrices.

Subbimatrix

Let $A_B = A_1^{m\times n} \cup A_2^{p\times q}$ be a bimatrix. If we cross out all but k_1 rows and s_1 columns of ($m \times n$) matrix A_1 and cross out all but k_2 rows and s_2 columns of ($p\times q$) matrix A_2 the resulting ($k_1 \times s_1$) and ($k_2 \times s_2$) bimatrix is called a Subbimatrix of A_B.

Example 31: Let $A_B = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 6 & 0 & 1 & 2 \\ -1 & 6 & -1 & 0 \end{bmatrix} \cup \begin{bmatrix} 3 & 8 & 3 & 6 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & -1 & 3 \\ -1 & 4 & 0 & 0 & 2 \end{bmatrix}$.

Solution: $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 6 & -1 \end{bmatrix} \cup \begin{bmatrix} 3 & 8 \\ 1 & 1 \\ 2 & 1 \\ -1 & 4 \end{bmatrix}$ is a subbimatrix of A_B.

Bideterminant

Let $A_B = A_1 \cup A_2$ be a square bimatrix. The bideterminant of a square bimatrix is an ordered pair (d_1 , d_2) where $d_1 = |A_1|$ and $d_2 = |A_2|$.
$|A_B| = (d_1, d_2)$ where d_1 and d_2 are real maybe positive or negative or zero.

Example 32: Let $\boldsymbol{A_B} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 4 & 5 \\ -2 & 0 \end{bmatrix}$, find $|\,A_B|$.

Solution: $|\,A_B| = (0,10)$.

- If A_B and C_B be square bimatrices of order *n* then their product $D_B = A_B\, C_B$.

$$D_B = \boldsymbol{A_B C_B} = (\boldsymbol{A_1 C_1}) \cup (\boldsymbol{A_2 C_2})$$
$$|\,\boldsymbol{D_B}| = |\,\boldsymbol{A_B C_B}| = |\,\boldsymbol{A_1}||\,\boldsymbol{C_1}| \cup |\,\boldsymbol{A_2}||\,\boldsymbol{C_2}|$$

i.e., the determinant of the product is the product of the determinants.

Example 33: Let $\boldsymbol{A_B} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cup \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}$, and $\boldsymbol{C_B} = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix} \cup \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$, find $|\,\boldsymbol{A_B\, C_B}|$.

Solution: $|\,\boldsymbol{A_B\, C_B}| = (-80,-39)$.

$$|\,\boldsymbol{A_1}| = \boldsymbol{5}, |\,\boldsymbol{A_2}| = \boldsymbol{-3}, |\,\boldsymbol{C_1}| = \boldsymbol{-16}, |\,\boldsymbol{C_2}| = \boldsymbol{13},$$
$$|\,\boldsymbol{A_B C_B}| = |\,\boldsymbol{A_1}||\,\boldsymbol{C_1}| \cup |\,\boldsymbol{A_2}||\,\boldsymbol{C_2}| = (-80,-39)$$

- If A_B and C_B be rectangular bimatrices then product $\boldsymbol{D_B} = \boldsymbol{A_B\, C_B}, \boldsymbol{A_B C_B} = (\boldsymbol{A_1 C_1}) \cup (\boldsymbol{A_2 C_2})$
  $|\,\boldsymbol{A_B C_B}| = |\,\boldsymbol{A_1 C_1}| \cup |\,\boldsymbol{A_2 C_2}| = (\boldsymbol{d_1}, \boldsymbol{d_2})$ where $\boldsymbol{d_1} = |\,\boldsymbol{A_1 C_1}|$ and $\boldsymbol{d_2} = |\,\boldsymbol{A_2 C_2}|$.

Example 34: Let $\boldsymbol{A_B} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 0 & 3 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, and

$\boldsymbol{C_B} = \begin{bmatrix} 3 & 0 \\ 9 & 2 \\ 1 & 7 \end{bmatrix} \cup \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 5 & -1 \end{bmatrix}$, find $|\,\boldsymbol{A_B\, C_B}|$.

Solution: $\boldsymbol{A_B\, C_B} = \begin{bmatrix} \boldsymbol{44} & \boldsymbol{43} \\ \boldsymbol{9} & \boldsymbol{21} \end{bmatrix} \cup \begin{bmatrix} 5 & 1 \\ 7 & 1 \end{bmatrix}$

$|\,A_B\, \boldsymbol{C_B}| = (537,-2)$.

Biinverse of Bimatrix

Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$ be a square bimatrix, if there exists a square bimatrix $\boldsymbol{A_B^{-1}} = \boldsymbol{A_1^{-1}} \cup \boldsymbol{A_2^{-1}}$ which satisfied the following:

$\boldsymbol{A_B\, A_B^{-1}} = \boldsymbol{A_1 A_1^{-1}} \cup \boldsymbol{A_2 A_2^{-1}} = \boldsymbol{I} \cup \boldsymbol{I}$, then $\boldsymbol{A_B^{-1}}$ is called the biinverse or bireciprocal of $\boldsymbol{A_B}$.

- It is most important to note that even $\boldsymbol{A_B}$ be a mixed square bimatrix then also $\boldsymbol{A_B^{-1}}$ exists by $\boldsymbol{I_1} \cup \boldsymbol{I_2} = \boldsymbol{I_B}$ will be such $\boldsymbol{I_1} \neq \boldsymbol{I_2}$.
- It is most important to note that: $A_B^{-1} \neq \frac{1}{A_B}$

Example 35: Let $\boldsymbol{A_B} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$, and $\boldsymbol{C_B} = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find $\boldsymbol{A_B^{-1}}$ and $\boldsymbol{C_B^{-1}}(\boldsymbol{HW})$.

Solution: $\boldsymbol{A_B^{-1}} = \begin{bmatrix} 1 & 0 \\ -2/3 & 1/3 \end{bmatrix} \cup \begin{bmatrix} 1/2 & -1 \\ 1/2 & 0 \end{bmatrix} =$

$\boldsymbol{A_B A^{-1}{}_B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Properties of biinverse

1- $(\boldsymbol{A_B C_B})^{-1} = \boldsymbol{C_B^{-1} A_B^{-1}}$

2- $\boldsymbol{C_B^{-1} A_B^{-1} A_B C_B} = \boldsymbol{C_B^{-1} C_B} = \boldsymbol{I_B}$

Example 36: Let $\boldsymbol{A_B} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, and $\boldsymbol{C_B} = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 5 \\ 6 & 4 \end{bmatrix}$, find $(\boldsymbol{A_B C_B})^{-1}$.

Solution: $\boldsymbol{A_B^{-1}} = \begin{bmatrix} 1 & 0 \\ -2/3 & 1/3 \end{bmatrix} \cup \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$$\boldsymbol{C_B^{-1}} = \begin{bmatrix} -1/8 & 5/8 \\ 1/4 & -1/4 \end{bmatrix} \cup \begin{bmatrix} -37/30 & 7/15 \\ 3/5 & -1/5 \end{bmatrix}$$

$$(\boldsymbol{A_B C_B})^{-1} = \left\{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\right\}^{-1} \cup \left\{\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 6 & 4 \end{bmatrix}\right\}^{-1} = \left\{\begin{bmatrix} 2 & 5 \\ 10 & 13 \end{bmatrix}\right\}^{-1} \cup \left\{\begin{bmatrix} 6 & 14 \\ 18 & 37 \end{bmatrix}\right\}^{-1} = \begin{bmatrix} -13/24 & 5/24 \\ 5/12 & -1/12 \end{bmatrix} \cup \begin{bmatrix} -37/30 & 7/15 \\ 3/5 & -1/5 \end{bmatrix} = \boldsymbol{C_B^{-1} A_B^{-1}}.$$

3- $(\boldsymbol{A_B^{-1}})^{-1} = \boldsymbol{A_B}$

Example 37: Let $\boldsymbol{A_B} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, find $(\boldsymbol{A_B^{-1}})^{-1}$.

Solution: $\boldsymbol{A_B^{-1}} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ -2/3 & 1/3 \end{bmatrix}$

$(\boldsymbol{A_B^{-1}})^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \boldsymbol{A_B}$.

The square bimatrix $\boldsymbol{A_B}$ is non- bisingular if $|A_B| \neq (0,0)$. If $|A_B| = (0,0)$ then the bimatrix $\boldsymbol{A_B}$ is bisingular. Let $\boldsymbol{A_B} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$, if one of $\boldsymbol{A_1}$ or $\boldsymbol{A_2}$ is non- singular matrix then the bimatrix $\boldsymbol{A_B}$ is called semi bisingular.

Example21: Let $\boldsymbol{A_B} = \begin{bmatrix} 0 & 7 \\ 0 & 5 \end{bmatrix} \cup \begin{bmatrix} 3 & 8 \\ 6 & 16 \end{bmatrix}$, $\boldsymbol{C_B} = \begin{bmatrix} 1 & 5 \\ 5 & 25 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\boldsymbol{D_B} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cup \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}$

Solution: $\boldsymbol{A_B}$ is bisingular since $|A_B| = (0,0)$,

$\boldsymbol{C_B}$ is semi bisingular since $\begin{vmatrix} 1 & 5 \\ 5 & 25 \end{vmatrix} = 0$ but $\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3$and $\boldsymbol{D_B}$ is non bisingular since $|D_B| = (5,-3)$.

Partition of Bimatrices

For several reasons we introduce the notion of partition bimatrices in subbimatrices. Some of the main reasons are;

(1) The partitioning may simplify the writing or printing of A_B.

(2) It exhibits some particular structure of A_B which is of interest.

(3) It simplifies computation.

Example 38: Let $A_B = A_1 \cup A_2$ be a bimatrix, i.e., A_B

$$= \left[\begin{array}{ccc|c} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & & a_{26} \\ \hline a_{31} & a_{32} & \cdots & a_{36} \\ \vdots & & & \\ a_{91} & a_{92} & & a_{96} \end{array}\right] \cup \left[\begin{array}{cc|ccc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \end{array}\right].$$

we have the following partitions if we imagine A_B to be divided up by lines as shown. Now the bimatrix is partitioned into

$$A^1{}_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{15} \\ a_{21} & a_{22} & \cdots & a_{25} \end{bmatrix} \cup \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A_B^{11} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{15} \\ a_{21} & a_{22} & \cdots & a_{25} \end{bmatrix} \cup \begin{bmatrix} b_{13} & b_{14} & b_{15} \\ b_{21} & b_{24} & b_{25} \end{bmatrix}$$

$$A_B^{3} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{15} \\ a_{21} & a_{22} & \cdots & a_{25} \end{bmatrix} \cup \begin{bmatrix} b_{31} & b_{32} \end{bmatrix}$$

$$A_B^{4} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{15} \\ a_{21} & a_{22} & \cdots & a_{25} \end{bmatrix} \cup [b_{33}\ b_{34}\ b_{35}]$$

$$A_B^{5} = \begin{bmatrix} a_{16} \\ a_{26} \end{bmatrix} \cup \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A_B^{6} = \begin{bmatrix} a_{16} \\ a_{26} \end{bmatrix} \cup \begin{bmatrix} b_{13} & b_{14} & b_{15} \\ b_{23} & b_{24} & b_{25} \end{bmatrix}$$

$$A_B^{7} = \begin{bmatrix} a_{16} \\ a_{26} \end{bmatrix} \cup [b_{31}\ \ b_{32}]$$

$$A_B^{8} = \begin{bmatrix} a_{16} \\ a_{26} \end{bmatrix} \cup [b_{33}\ \ b_{34}\ \ b_{35}]$$

$$A_B^9 = \begin{bmatrix} a_{31} & a_{32} & \cdots & a_{35} \\ a_{41} & a_{42} & \cdots & a_{45} \\ & & & \\ a_{91} & a_{92} & & a_{95} \end{bmatrix} \cup \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A_B^{10} = \begin{bmatrix} a_{31} & a_{32} & \cdots & a_{35} \\ a_{41} & a_{42} & \cdots & a_{45} \\ \vdots & \vdots & & \vdots \\ a_{91} & a_{92} & \cdots & a_{95} \end{bmatrix} \cup \begin{bmatrix} b_{13} & b_{14} & b_{15} \\ b_{21} & b_{24} & b_{25} \end{bmatrix}$$

$$A_B^{11} = \begin{bmatrix} a_{31} & a_{32} & \cdots & a_{35} \\ a_{41} & a_{42} & \cdots & a_{45} \\ \vdots & \vdots & & \vdots \\ a_{91} & a_{92} & \cdots & a_{95} \end{bmatrix} \cup \begin{bmatrix} b_{31} & b_{32} \end{bmatrix}$$

$$A_B^{12} = \begin{bmatrix} a_{31} & a_{32} & \cdots & a_{35} \\ a_{41} & a_{42} & \cdots & a_{45} \\ \vdots & \vdots & & \vdots \\ a_{91} & a_{92} & \cdots & a_{95} \end{bmatrix} \cup \begin{bmatrix} b_{33} & b_{34} & b_{35} \end{bmatrix}$$

$$A_B^{13} = \begin{bmatrix} a_{36} \\ \vdots \\ a_{96} \end{bmatrix} \cup \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A_B^{14} = \begin{bmatrix} a_{36} \\ \vdots \\ a_{96} \end{bmatrix} \cup \begin{bmatrix} b_{13} & b_{14} & b_{15} \\ b_{23} & b_{24} & b_{25} \end{bmatrix}$$

$$A_B^{15} = \begin{bmatrix} a_{36} \\ \vdots \\ a_{96} \end{bmatrix} \cup \begin{bmatrix} b_{31} & b_{32} \end{bmatrix}$$

and

$$A_B^{14} = \begin{bmatrix} a_{36} \\ \vdots \\ a_{96} \end{bmatrix} \cup \begin{bmatrix} b_{33} & b_{34} & b_{35} \end{bmatrix}.$$

With the lines shown for A_B there are 16 partition some mixed bimatrices some mixed square matrices and so on.

Note that the rule for addition of partitioned bimatrices is the same as the rule for addition of ordinary bimatrices if the subbimatrices are conformable for addition. In other words the matrices can be added by blocks. Addition of bimatrices A_B and B_B when partitioned is possible only when A_B and B_B are of the same type, i.e., both are m × n rectangular bimatrix or both A_B and B_B are both square bimatrix or both A_B and B_B are either mixed square matrices or mixed rectangular matrices with computable addition.

Example 39: Let

$$A_B = \left[\begin{array}{cc|cc} 3 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ \hline 0 & 1 & 0 & 2 \end{array}\right] \cup \left[\begin{array}{cc|c} 3 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \\ \hline 0 & 0 & 5 \end{array}\right]$$

$$B_B = \left[\begin{array}{cc|cc} 5 & -1 & 3 & 2 \\ -1 & 0 & 0 & 5 \\ \hline 1 & 1 & 5 & -2 \end{array}\right] \cup \left[\begin{array}{cc|c} 6 & 2 & -3 \\ 0 & 1 & 2 \\ 1 & 1 & 6 \\ \hline 0 & -1 & 6 \end{array}\right]$$

$A_B + B_B$ as block sum of bimatrices is given by:

$$A_B + B_B = \left[\begin{array}{cc|cc} 8 & -1 & 4 & 4 \\ 0 & 1 & 0 & 8 \\ \hline 1 & 2 & 5 & 0 \end{array}\right] \cup \left[\begin{array}{cc|c} 9 & 2 & -2 \\ 0 & 2 & 3 \\ 3 & 2 & 6 \\ \hline 0 & -1 & 11 \end{array}\right].$$

Example 40: Let

$$A_B = \left[\begin{array}{ccc|cc} 3 & 1 & 1 & 2 & 5 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ \hline 0 & -1 & 2 & 0 & -2 \end{array}\right] \cup \left[\begin{array}{c|c} 3 & 1 \\ 2 & 1 \\ \hline 2 & 2 \end{array}\right]$$

$$B_B = \left[\begin{array}{ccc|cc} -1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 & 1 \\ 2 & 1 & 2 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array}\right] \cup \left[\begin{array}{c|c} 0 & -1 \\ 2 & 1 \\ \hline 3 & 4 \end{array}\right].$$

Now if the bimatrices are divided into blocks by these lines then the addition of these can be carried out for both mixed bimatrices are compatible with respect to addition and the block division also a compatible or a similar division.

$$A_B + B_B = \left[\begin{array}{ccc|cc} 2 & 1 & 1 & 2 & 6 \\ 2 & 2 & 1 & 0 & -1 \\ -3 & 2 & 4 & 0 & 1 \\ 3 & 2 & 3 & 0 & 2 \\ \hline 1 & 0 & 3 & 1 & -1 \end{array}\right] \cup \left[\begin{array}{c|c} 3 & 0 \\ 4 & 2 \\ \hline 5 & 6 \end{array}\right].$$

So the sum of the bimatrices is also divided into blocks.

__**Example 41**__**:** Let

$$A_B = \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 9 & 2 & -1 \\ \hline 3 & 1 & 2 \end{array}\right] \cup \left[\begin{array}{cc|c} 3 & 5 & 1 \\ \hline 5 & 4 & 0 \end{array}\right]$$

and

$$B_B = \left[\begin{array}{cc|c} -1 & 0 & 1 \\ 1 & 4 & 2 \\ \hline 2 & 2 & 2 \\ 3 & -1 & 0 \end{array}\right] \cup \left[\begin{array}{c|cc} 0 & 1 & 7 \\ \hline 0 & 3 & 8 \end{array}\right].$$

Clearly $A_B + B_B$ cannot be added as blocks for the sum of these two bimatrices exist but as block the addition is not compatible.

Now we define bimatrix multiplication as block bimatrices.

__**Example 42**__**:** Let

$$A_B = \left[\begin{array}{cc|cc} 3 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 \end{array}\right] \cup \left[\begin{array}{cc|cc} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$B_B = \left[\begin{array}{cc|cc} 3 & 0 & 0 & 6 \\ -1 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & -1 \\ 2 & 1 & 0 & 1 \end{array}\right] \cup \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{array}\right].$$

two 4 × 4 square bimatrices and divided into blocks as shown by the lines, i.e.,

$$A_B = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \cup \begin{bmatrix} A_5 & A_6 \\ A_7 & A_8 \end{bmatrix}$$

And

$$B_B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \cup \begin{bmatrix} B_5 & B_6 \\ B_7 & B_8 \end{bmatrix}$$

$$A_B B_B = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_2B_4 \end{bmatrix} \cup \begin{bmatrix} A_5B_5 + A_6B_7 & A_5B_6 + A_6B_8 \\ A_7B_5 + A_8B_7 & A_7B_6 + A_6B_8 \end{bmatrix}$$

The bimatrix which is partitioned. It is to be noted all bimatrices in general need be compatible with block multiplication. This sort of block multiplication can be defined even in the case of mixed square bimatrices.

Special Types of Bimatrices

Now we define some special types of bimatrices called overlap row (row overlap) and column overlap bimatrices as follow:

Definition

Let $A_B = A_1 \cup A_2$ be a rectangular bimatrix. This bimatrix A_B is said to be a row overlap rectangular bimatrix if the matrices A_1 and A_2 have at least one row with same entries.

Example 43: Let A_B define as follow:

$$A_B = \begin{bmatrix} 3 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 4 & 4 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 4 & -1 & 2 & 2 & 3 \\ 4 & 4 & 2 & 2 & 3 \end{bmatrix}$$

a bimatrix which is a 4 × 5 bimatrix. This is a row over lap bimatrix for we have the rows (0 1 1 1 1) and (4 4 2 2 3) to be rows with same entries of both the matrices.

Likewise we define column overlap bimatrix as follow:

Definition

Bimatrix $A_B = A_1 \cup A_2$ is said to be column overlap bimatrix if the matrices A_1 and A_2 has at least a column with same entries.

Example 44: Let A_B define as follow:

$$A_B = \begin{bmatrix} 3 & 1 & 0 & 1 & 3 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 1 & 1 & -1 & 4 \end{bmatrix} \cup \begin{bmatrix} -4 & 1 & 2 & 0 & 1 \\ 4 & 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 0 & 5 \end{bmatrix}$$

be the 3×5 bimatrix. This is a column overlap bimatrix for $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

is the column which is both in A_1 and A_2.

Now we define row overlap and column overlap rectangular bimatrix (row column overlap) as follow:

Definition

Let $A_B = A_1 \cup A_2$ be a $m \times n$ (rectangular) bimatrix where A_1 and A_2 have columns which have same entries and A_1 and A_2 have rows which have same entries, then the bimatrix is called as the row-column overlap bimatrix.

We illustrate this by the following example:

Example 45: Let

$$A_B = \begin{bmatrix} 3 & 1 & 0 & 2 & 4 & 5 \\ 6 & 3 & 6 & 3 & 6 & 3 \\ 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 6 & -2 & 0 & 1 & 3 & 5 \\ 6 & 3 & 6 & 3 & 6 & 3 \\ 7 & 5 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The bimatrix A_B has (6 3 6 3 6 3) and (0 1 0 1 0 1) as common rows and

$$\begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

are common columns. Thus A_B is a row-column overlap bimatrix.

Thus the concept of row overlap bimatrix, column overlap bimatrix, and row column overlap bimatrix would be helpful in future applications.

Fuzzy Bimatrix

Definition

Fuzzy matrix is a matrix where all entries in [0, 1]

For example

Let A = $\begin{bmatrix} 1 & 0.22 \\ 0.001 & 0.9 \end{bmatrix}$ is a square fuzzy matrix

B= $\begin{bmatrix} 0.03 & 0.5 & 1 \\ 0 & 0.2 & 0.812 \end{bmatrix}$ is a 2×3 matrix

Definition

Let $A = A_1 \cup A_2$ where A_1 and A_2 are two distinct fuzzy matrices with entries from the interval [0, 1]. Then $A = A_1 \cup A_2$ is called the fuzzy bimatrix. It is important to note the following:

1. If both A_1 and A_2 are fuzzy matrices we call A a fuzzy bimatrix.
2. If only one of A_1 or A_2 is a fuzzy matrix and other is not a fuzzy matrix then we call $A = A_1 \cup A_2$ as the semi fuzzy bimatrix. (It is clear all fuzzy matrices are trivially semi fuzzy matrices).

In other words if both A_1 and A_2 are $m \times n$ fuzzy matrices then we call $A = A_1 \cup A_2$ an $m \times n$ fuzzy bimatrix or a rectangular fuzzy bimatrix.

If A_1 and A_2 are both $n \times n$ fuzzy matrices then we call $A = A_1 \cup A_2$ a square or an $n \times n$ fuzzy bimatrix. In the fuzzy bimatrix $A = A_1 \cup A_2$, if both A_1 and A_2 are square matrices but of different order say A_1 is a $n \times n$ matrix and A_2 is $s \times s$ matrix then we call $A = A_1 \cup A_2$ a mixed fuzzy square bimatrix. (Similarly one can define mixed square semi fuzzy bimatrix).

Likewise if both A_1 and A_2 are rectangular matrices say A_1 is an $m \times n$ matrix and A_2 is a $p\times q$ matrix then we call $A = A_1 \cup A_2$ a mixed fuzzy rectangular bimatrix. (If $A = A_1 \cup A_2$ is a semi fuzzy bimatrix then we call A the mixed rectangular semi fuzzy bimatrix).

Notation: We denote a fuzzy bimatrix by $A_F = A_1 \cup A_2$.

Example 46: Let A_F is the 3 × 3 square fuzzy bimatrix;

$$A_F = \begin{bmatrix} 0 & .1 & 0 \\ .1 & .2 & .1 \\ .3 & .2 & .1 \end{bmatrix} \cup \begin{bmatrix} .2 & .1 & .1 \\ .1 & 0 & .1 \\ .2 & .1 & .2 \end{bmatrix}$$

Example 47: Let A_F is a mixed square fuzzy bimatrix:

$$A_F = \begin{bmatrix} .2 & 0 & 1 \\ .4 & .2 & 1 \\ .3 & 1 & .2 \end{bmatrix} \cup \begin{bmatrix} .3 & 1 & 0 & .4 & .5 \\ 0 & 0 & 1 & .8 & .2 \\ 1 & 0 & 0 & .1 & .2 \\ .1 & .3 & .3 & .5 & .4 \\ .2 & .1 & .3 & 0 & 1 \end{bmatrix}$$

Example 48: Let A_F is a mixed rectangular fuzzy bimatrix. We denote A_F by $A_F = A_1 \cup A_2 = A_1^{2\times 5} \cup A_2^{5\times 4}$ such as:

$$A_F = \begin{bmatrix} .3 & 1 & .5 & 1 & .9 \\ .6 & 0 & .2 & .3 & .4 \end{bmatrix} \cup \begin{bmatrix} 1 & .2 & 0 & 0 \\ .3 & 1 & .2 & 1 \\ .4 & 1 & 0 & 0 \\ .3 & .3 & .2 & 1 \\ 1 & .5 & .7 & .6 \end{bmatrix}$$

Example 49: Let A_F is 4 × 2 rectangular fuzzy bimatrix

$$A_F = \begin{bmatrix} .3 & 1 \\ 1 & .2 \\ .5 & 0 \\ .3 & .6 \end{bmatrix} \cup \begin{bmatrix} .3 & .7 \\ 1 & 1 \\ .4 & 1 \\ .2 & 0 \end{bmatrix}$$

Example 50: Let A_F is a square semi fuzzy bimatrix

$$A_F = \begin{bmatrix} .3 & 1 & 1 & 1 \\ 0 & 0 & .1 & .2 \\ 0 & 0 & 0 & .3 \\ .3 & 1 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 5 & 0 & -1 & 2 \end{bmatrix}$$

<u>Example 51</u>: Let A_F is a rectangular mixed semi fuzzy bimatrix.

$$A_F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Thus as in case of bimatrices we may have square, mixed square, rectangular or mixed rectangular fuzzy (semi fuzzy) bimatrices.

Now we proceed on to define operators on these types of bimatrix.

<u>Sum of Two Fuzzy Bimatrices</u>

One defines the algebraic sum of two fuzzy (semi fuzzy) bimatrices *A* and *B*, denoted $\mathbf{D_F}$ by the expression:

Let $\boldsymbol{A}_F=\boldsymbol{A_1}\cup \boldsymbol{A_2}$ and $\boldsymbol{B}_F=B_1\cup B_2$ be two $(m \times n)$ fuzzy bimatrices. The sum $\boldsymbol{D_F}$ of the fuzzy bimatrices $\boldsymbol{A_F}$ and B_F is defined as follows:

$\boldsymbol{D}_F = \boldsymbol{A_F} + \boldsymbol{B_F} = (\boldsymbol{A_1}\cup\boldsymbol{A_2}) + (\mathbf{B_1}\cup\mathbf{B_2}) = [\boldsymbol{A_1}+B\ ] \cup [\boldsymbol{A_2}+B_2]$

Where $\boldsymbol{A_1} + B_1$ and $\boldsymbol{A_2} + B_2$ are the usual addition on matrices. So the results of summation may be not fuzzy bimatrix (or semi fuzzy bimatrix) always

<u>Example 52</u>: Let A_F and B_F are two fuzzy bimatrices such that:

$A_F = \begin{bmatrix} 0 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \cup \begin{bmatrix} 1 & 0.2 \\ 0.66 & 0.9 \end{bmatrix}$ **and** $B_F = \begin{bmatrix} 1 & 0.21 \\ 0.271 & 0.1 \end{bmatrix} \cup \begin{bmatrix} 0.1 & 0.6 \\ 0.08 & 0.02 \end{bmatrix}$

Then

$\mathbf{D_F} = \begin{bmatrix} \mathbf{1} & \mathbf{0.91} \\ \mathbf{0.471} & \mathbf{0.4} \end{bmatrix} \cup \begin{bmatrix} \mathbf{1.1} & \mathbf{0.8} \\ \mathbf{0.74} & \mathbf{0.92} \end{bmatrix}$, We see that $\mathbf{D_F}$ is semi fuzzy bimatrix.

<u>Example 53</u>: Let A_F and B_F are two fuzzy bimatrices such that:

$A_F = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.23 & 0.9 & 1 \\ 0.8 & 0.3 & 0.06 \end{bmatrix} \cup \begin{bmatrix} 0.9 & 0.03 \\ 0.51 & 0.11 \\ 0.77 & 0.61 \end{bmatrix}$ and

$B_F = \begin{bmatrix} 0.7 & 0.5 & 0.8 \\ 0.163 & 0.08 & 0 \\ 0.12 & 0.7 & 0.1 \end{bmatrix} \cup \begin{bmatrix} 0.012 & 0.88 \\ 0.12 & 0.7 \\ 0.14 & 0 \end{bmatrix}$ Then

$D_F = \begin{bmatrix} 0.9 & 1 & 0.8 \\ 0.393 & 0.98 & 1 \\ 0.92 & 1 & 0.16 \end{bmatrix} \cup \begin{bmatrix} 0.912 & 0.91 \\ 0.63 & 0.81 \\ 0.91 & 0.61 \end{bmatrix}$ is a fuzzy bimatrix

<u>Example 54</u>: Let A_F and B_F are two fuzzy bimatrices such that:

$A_F = \begin{bmatrix} 0.09 & 0.34 & 0.136 \\ 1 & 0 & 0.2 \end{bmatrix} \cup \begin{bmatrix} 0.9 & 0.63 \\ 0.55 & 0.41 \\ 0.7 & 0 \end{bmatrix}$ and

$B_F = \begin{bmatrix} 0.13 & 0.04 & 0.23 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \cup \begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0 \\ 0.1 & 0.3 \end{bmatrix}$ Then

$\mathbf{D_F} = \begin{bmatrix} 0.22 & 0.38 & 0.366 \\ 1.2 & 0.1 & 0.9 \end{bmatrix} \cup \begin{bmatrix} 1.2 & 1.13 \\ 0.95 & 0.41 \\ 0.8 & 0.3 \end{bmatrix}$ it is clear that $\mathbf{D_F}$ is bimatrix**.**

Note that: in the same manner we can define the Subtraction, Scalar Multiplication and Multiplication of two fuzzy (semi fuzzy) bimatrices.

The Multiplication of two fuzzy bimatrices may be fuzzy bimatrices.

Chapter Four

Graph Theory

Introduction

The basic idea of graphs was introduced in 18^{th} century by the great Swiss mathematician Leonhard Euler who was asked to find a nice path across the famous Konigsberg bridges. German city of Konigsberg (now it is Russian Kaliningrad) was situated on the river, it had a park situated on the banks of the river and two islands mainland and islands were joined by seven bridges. A problem was whether it was possible to take a walk through the town in such a way as to cross over each bridge exactly once.

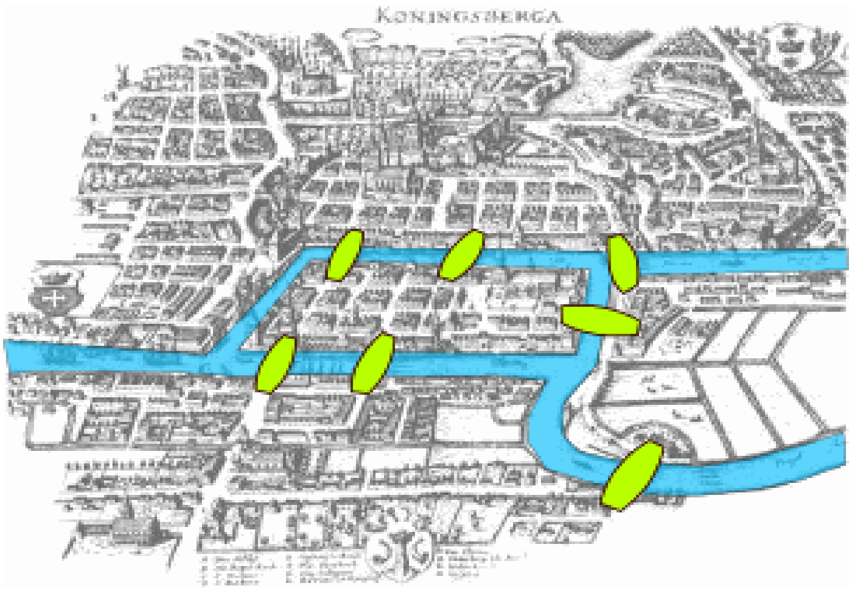


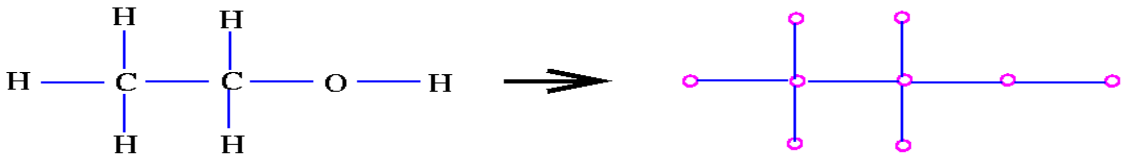
Figure1: The Konigsberg bridges

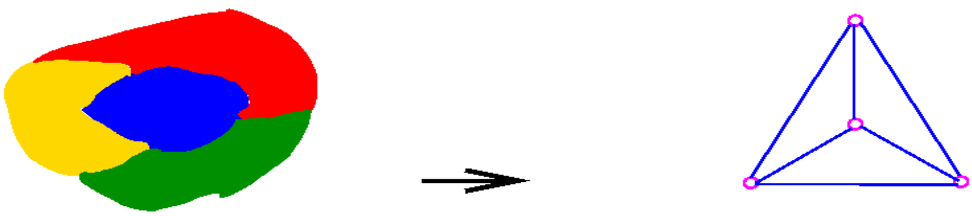
Graph theory is one of the most branches of modern mathematics with wide applications to combinatorial problems and to classical algebraic problems. Graph theory has applications in diverse areas such as social sciences, linguistics, physical sciences, communication, engineering etc. Besides this, graph theory plays an important role in several areas computer science such as switching theory, logical design, artificial intelligence, formal languages, computer graphics, compiler writing, information organization and retrieval etc.

The graph theory is a branch of discrete mathematics and it was used in Flight Networks, Road Networks, Rail Networks ,Design of Computer Chips, Map Colouring, , Internet (Google, Yahoo,…),and Computer Network Security.

Definition of Graph

A ***graph*** G = (V, E) consists of a (finite) set denoted by V, or by V(G) and a collection E, or E(G), of unordered pairs {u, v} of distinct elements from V. Each element of V is called a vertex or a point or a node, and each element of E is called an edge or a line or a link.



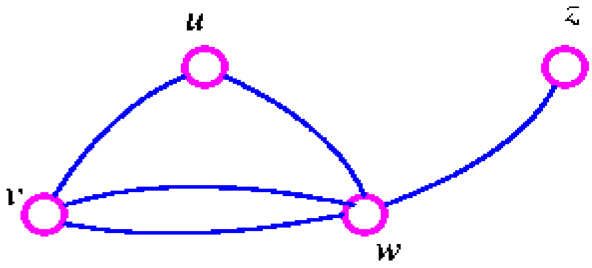


Directed and Undirected Graph

A graph G = (V, E) is ***directed*** if the edge set is composed of ordered vertex (node) pairs. A graph is ***undirected*** if the edge set is composed of unordered vertex pair.

Vertex Cardinality

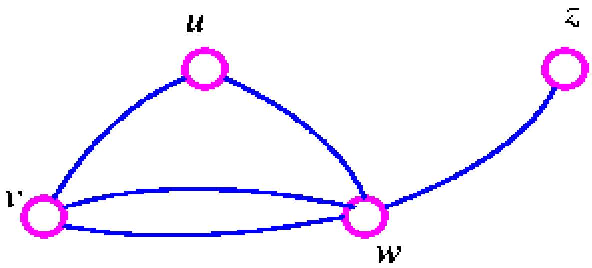
The number of vertices, the ***cardinality*** of V, is called the ***order*** of graph and denoted by |V|. We usually use *n* to denote the order of G. The number of edges, the cardinality of E, is called the ***size*** of graph and denoted by |E|. We usually use *m* to denote the size of G.



The above graph has |V|=4, |E|=5.

Vertex Degree

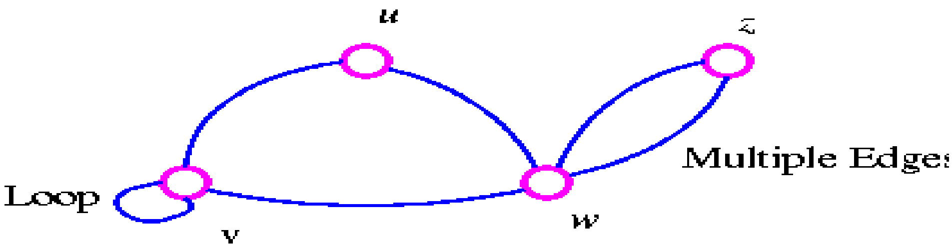
The ***Degree*** of vertex v is the number of edges incident on v.



The above graph has *deg*(*u*) = 2, *deg*(*v*) = 3, *deg*(w) = 4 and *deg*(z) = 1.

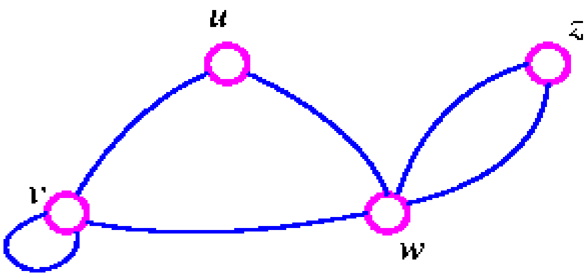
Loop and Multiple Edges

A ***loop*** is an edge whose endpoints are equal i.e., an edge joining a vertex to itself is called a loop. We say that the graph has ***multiple edges*** if in the graph two or more edges joining the same pair of vertices.

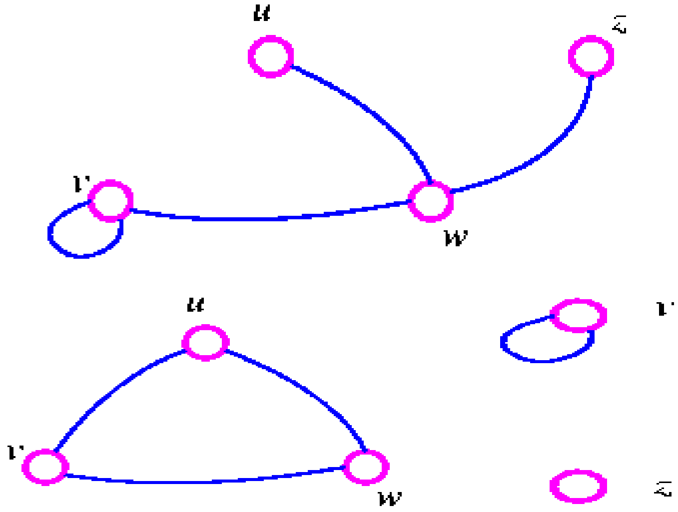


Sub Graph

Let *G* be a graph with vertex set *V*(*G*) and edge set E(G). A ***sub graph*** of *G* is a graph all of whose vertices belong to *V*(*G*) and all of whose edges belong to *E*(*G*). For example, in the graph below:

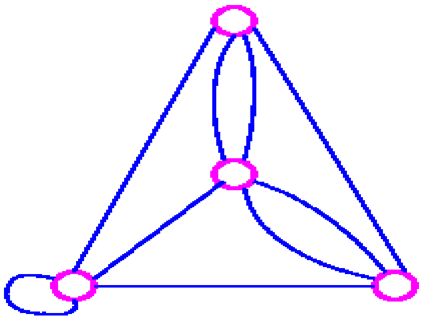


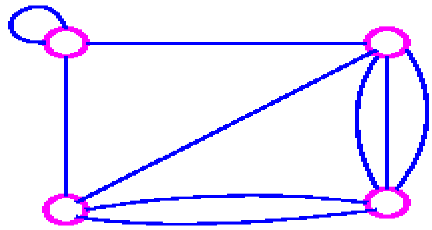
Where *V*(*G*) = {*u*, *v*, *w*, *z*} and *E*(*G*) = (*uv*, *uw*, *vv*, *vw*, *wz*, *zw*} then the following four graphs are sub graphs of G.



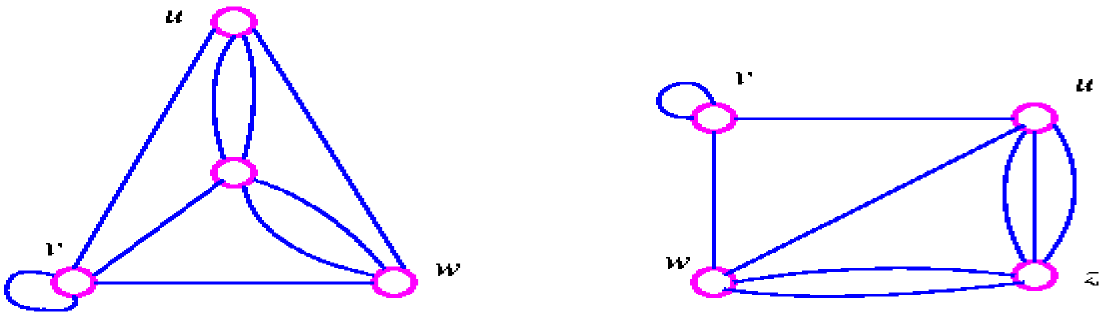
Isomorphic Graphs

Two graphs G and H are isomorphic if H can be obtained from G by relabelling the vertices - that is, if there is a one-to-one correspondence between the vertices of G and those of H, such that the number of edges joining any pair of vertices in G is equal to the number of edges joining the corresponding pair of vertices in H. For example, two unlabelled graphs, such as:





The two graphs are isomorphic if labels can be attached to their vertices so that they become the same graph.

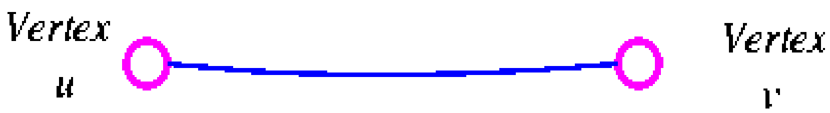


The word isomorphic derives from the Greek for same and form.

Kinds of Graph

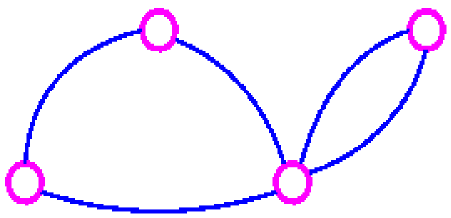
Simple Graph

A graph with no loops or multiple edges is called a ***simple*** graph. We specify a simple graph by its set of vertices and set of edges, treating the edge set as a set of unordered pairs of vertices and write $e = uv$ (or $e = vu$) for an edge e with endpoints $\boldsymbol{u}$ and $\boldsymbol{v}$.

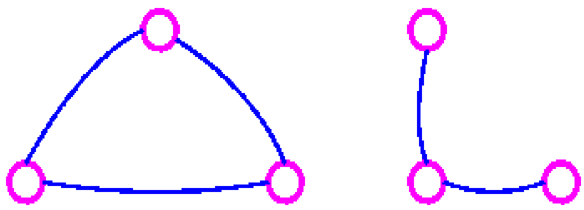


Connected Graph

A graph that is in one piece is said to be ***connected***, whereas one which splits into several pieces is disconnected.



Connected Non–Simple Graph

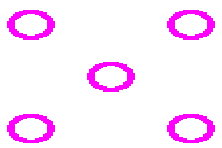


Disconnected Simple Graph

A graph G is connected if there is a path in G between any given pair of vertices, otherwise it is disconnected. Every disconnected graph can be split up into a number of connected sub graphs, called components.

Regular Graph

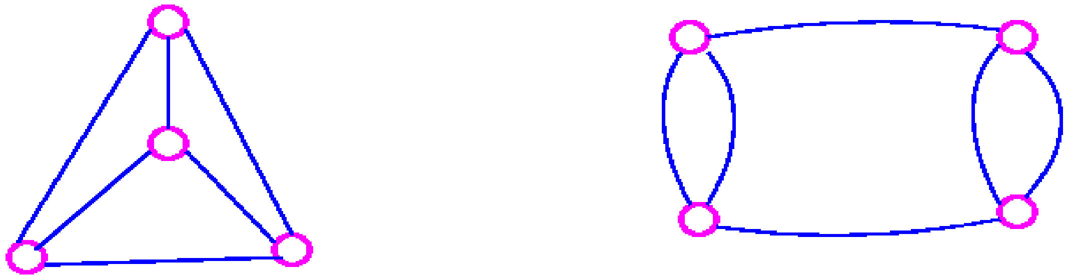
A graph is ***regular*** if all the vertices of *G* have the same degree. In particular, if the degree of each vertex is *r*, the *G* is regular of degree *r*.



Graph with degree $r = 0$

Graphs with degrees $r = 2$

Graphs with degree r = 2



Graphs with degrees r = 3

Graphs with degrees r = 4

Lemma(1)

In any graph, the sum of all the vertex-degree is equal to twice the number of edges.

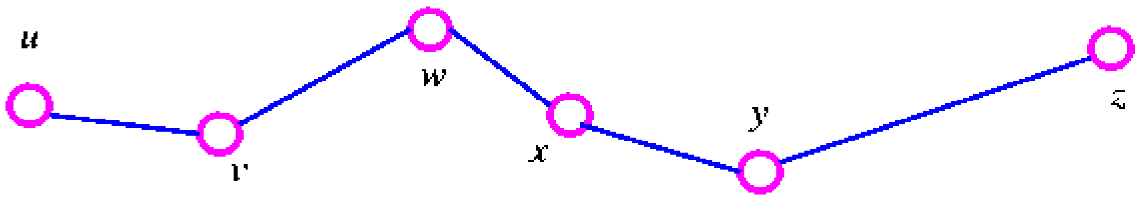
Proof: Since each edge has two ends, it must contribute exactly 2 to the sum of the degrees. The result follows immediately.

The Following are the consequences (results) of the above lemma (1).

- In any graph, the sum of all the vertex - degree is an even number.
- In any graph, the number of vertices of odd degree is even.
- If G is a graph which has n vertices and is regular of degree r, then G has exactly 1/2 nr edges.

Walk

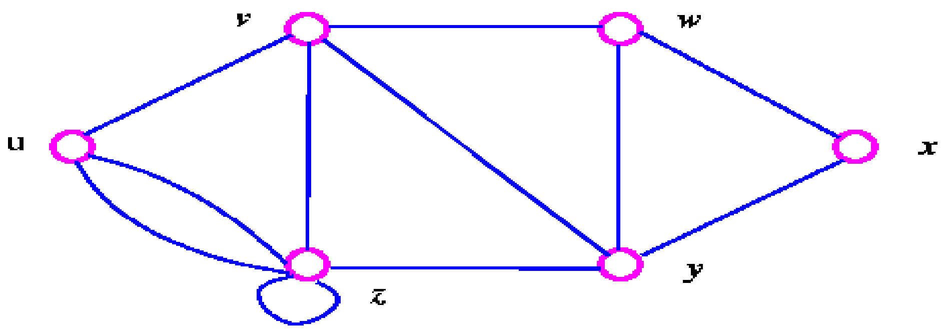
A ***walk*** of length k in a graph *G* is a succession of *k* edges of G of the form: *uv*, *vw*, *wx*, . . . , *yz* .



We denote this walk by uvwx yz and refer to it as a walk between *u* and *z.*

Trail and Path

If all the edges (but not necessarily all the vertices) of a **walk** are different, then the walk is called a ***trail***. If, in addition, all the vertices are different, then the trail is called ***path***.



- The walk *vzzywxy* is a trail since the vertices y and z both occur twice.
- The walk *vwxyz* is a path since the walk has no repeated vertices.

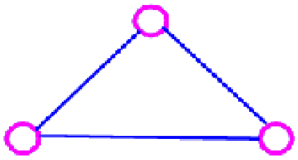
Complete Graph

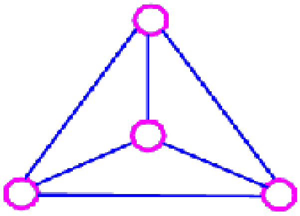
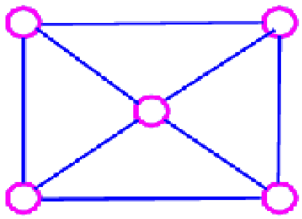
A ***complete graph*** is a graph in which every two distinct vertices are joined by exactly one edge. The complete graph with *n* vertices is denoted by K_n .

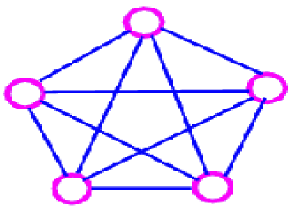
The following are the examples of complete graphs:

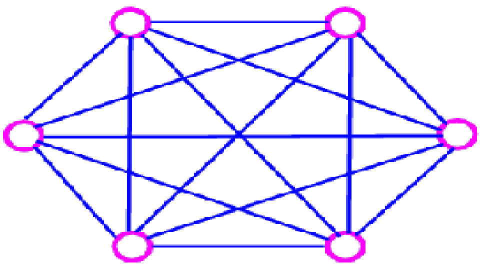
K_1 

K_2 

K_3 

K_4  OR 

K_5 

K_6 

The graph K_n is regular of degree n-1, and therefore has $n(n-1)/2$ edges, by consequence 3 of the lemma(1) .

Non Directed Graph

The graph in which the edges are not directed is known as a non directed graph.

Directed Graph

The graph in which the edges are directed is known as a directed

Trivial Graph

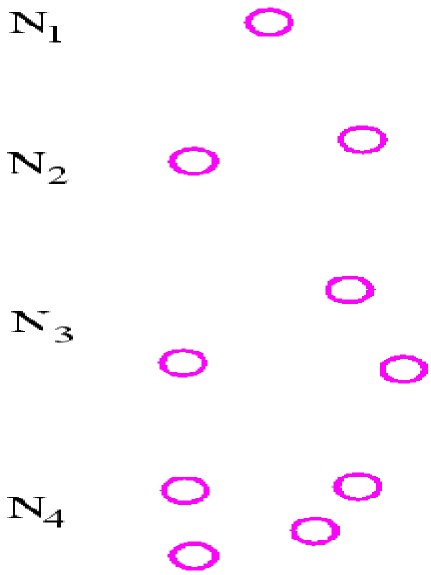
The graph which has only one vertex.

● **a**

Null Graph

A ***null graph*** is a graph containing no edges. The null graph with n vertices is denoted by N_n .

The following are the examples of null graphs:



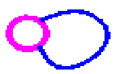
Note that: N_n is regular of degree 0.

Cycle Graph

A ***cycle graph*** is a graph consisting of a single cycle. The cycle graph with n vertices is denoted by C_n .

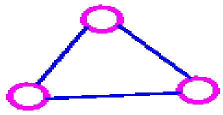
The following are the examples of cycle graphs:

C_1

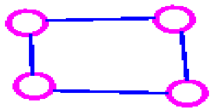


C_2

C_3



C_4



Note that: C_n is regular of degree 2, and has *n* edges.

Path Graph

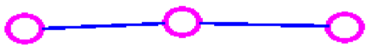
A ***path graph*** is a graph consisting of a single path. The path graph with n vertices is denoted by P_n .

The following are the examples of path graphs:

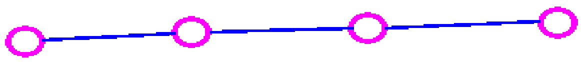
P_1

P_2

P_3



P_4

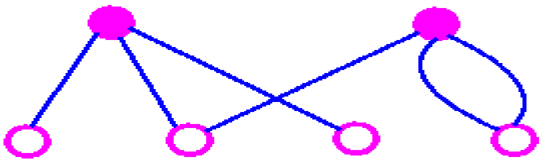


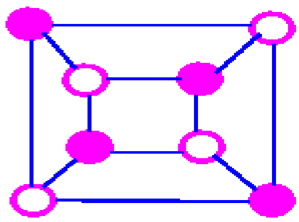
Note that: path graph, P_n , has *n*-1 edges, and can be obtained from cycle graph, C_n by removing any edge.

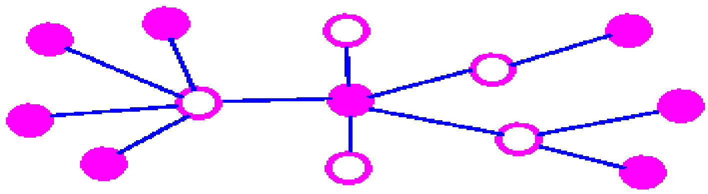
Bipartite Graph

A ***bipartite graph*** is a graph that's vertex-set can be split into two sets in such a way that each edge of the graph joins a vertex in first set to a vertex in second set.

The examples of bipartite graphs are:



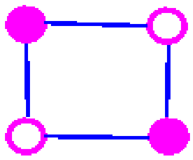


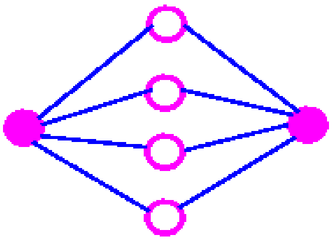
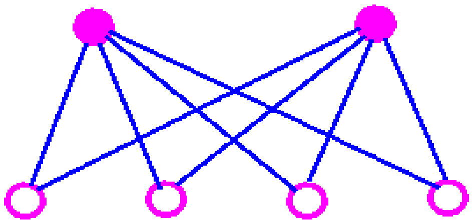


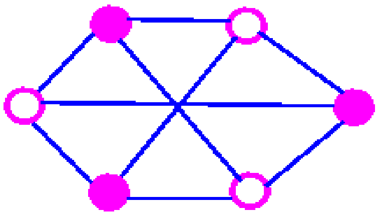
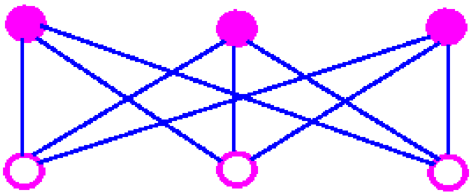
Complete Bipartite Graph

A ***complete bipartite graph*** is a bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one edge. The complete bipartite graph with *r* vertices of first set and s vertices of second set is denoted by $K_{r,s}$.

The following are some examples:

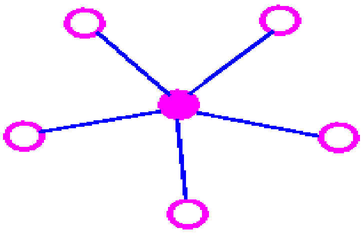
$K_{2,2}$ 

$K_{2,4}$  OR 

$K_{3,3}$  OR 

Note that: $K_{r,s}$ has r + s vertices (r vertices of degree s, and s vertices of degree r) and rs edges. Note also that $K_{r,s} = K_{s,r}$.

<u>**An Important Note**</u>**:** A complete bipartite graph of the form $K_{1,s}$ is called a ***star graph***.

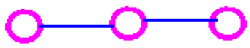


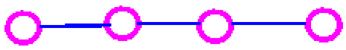
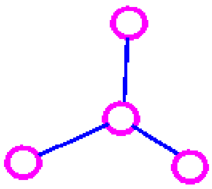
<u>Tree Graph</u>

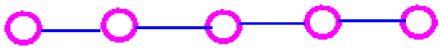
A ***tree*** is a connected graph which has no cycles.

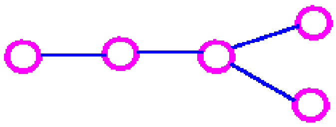
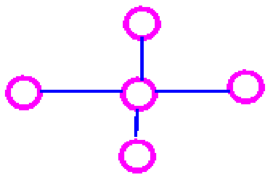
n = 1 

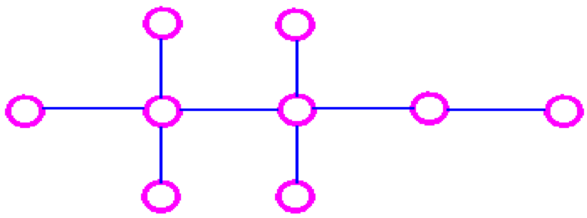
n = 2 

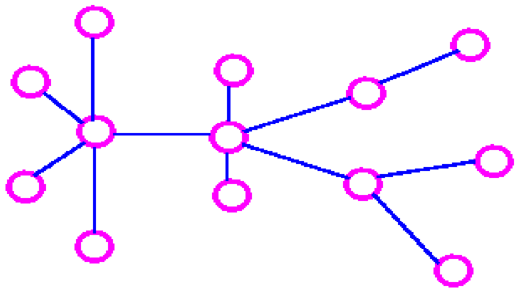
n = 3 

n = 4  OR 

n = 5 

OR  OR 

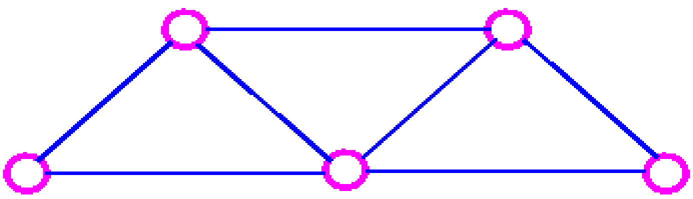




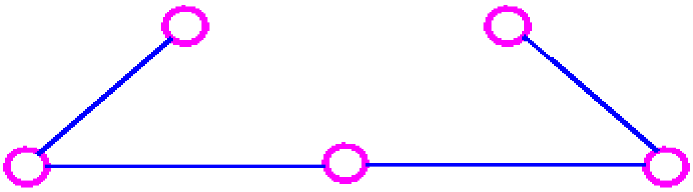
<u>Spanning Tree</u>

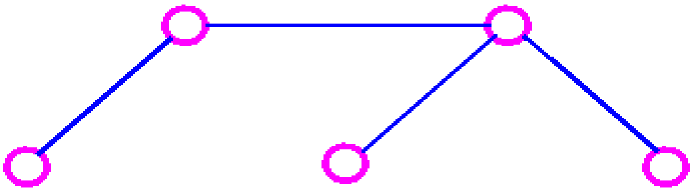
If *G* is a connected graph, the ***spanning*** tree in *G* is a sub graph of *G* which includes every vertex of *G* and is also a tree.

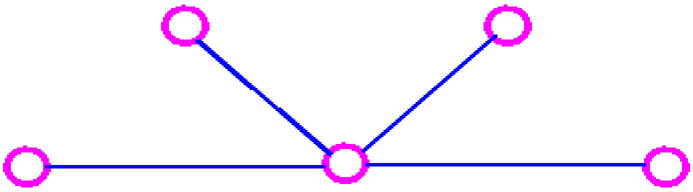
Consider the following graph:



The following are the three of its spanning trees:

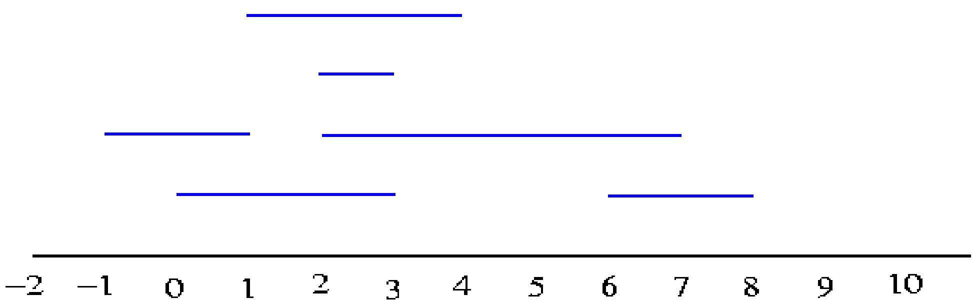




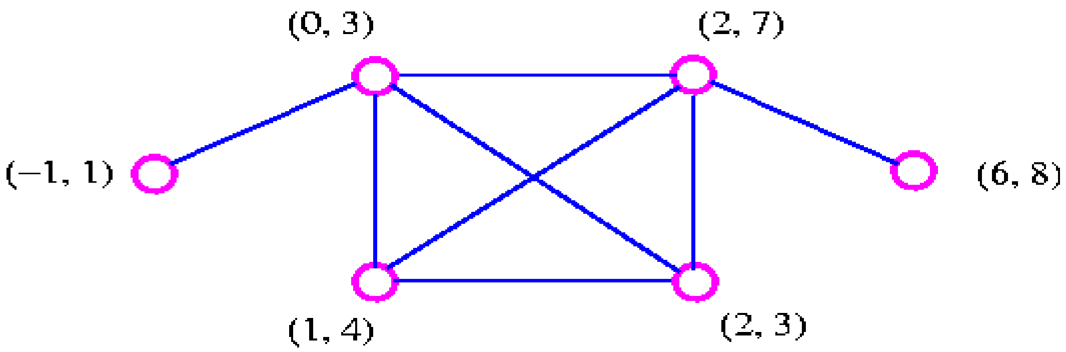


Interval Graph

Consider the intervals (0, 3), (2, 7), (-1, 1), (2, 3), (1, 4), (6, 8) which may be illustrated as:



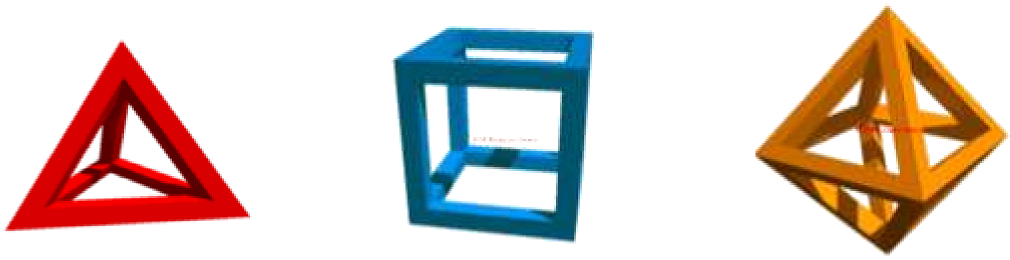
We can construct the resulting interval graphs by taking the interval as vertices, join two of these vertices by an edge whenever the corresponding intervals have at least one point in common.



Note that: since the intervals (-1, 1) and (1, 4) are open intervals, they do not have a point in common.

The Platonic Graph

The following regular solids are called the Platonic solids:

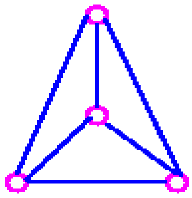


Tetrahedron Hexahedron (cube) Octahedron

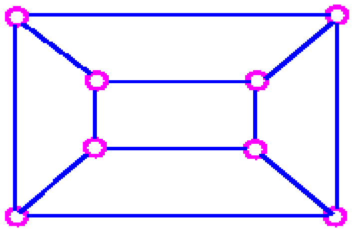
Dodecahedron Icosahedron

The name Platonic arises from the fact that these five solids were mentioned in Plato's Timaeus. A Platonic graph is obtained by projecting the corresponding solid on to a plane as the following:

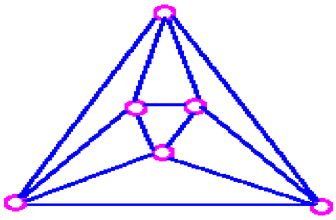
Tetrahedron



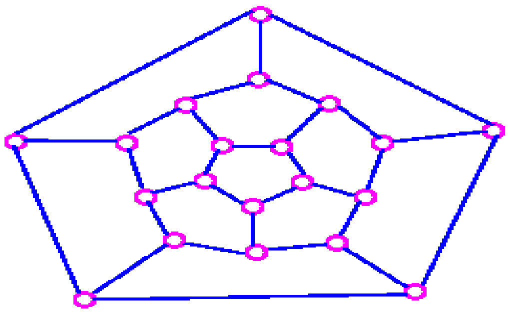
Cube



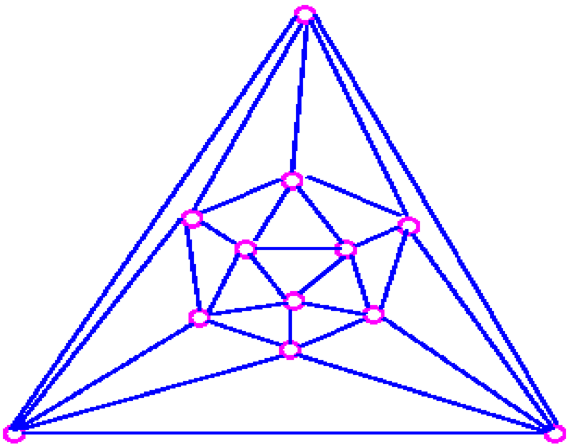
Octahedron



Dodecahedron

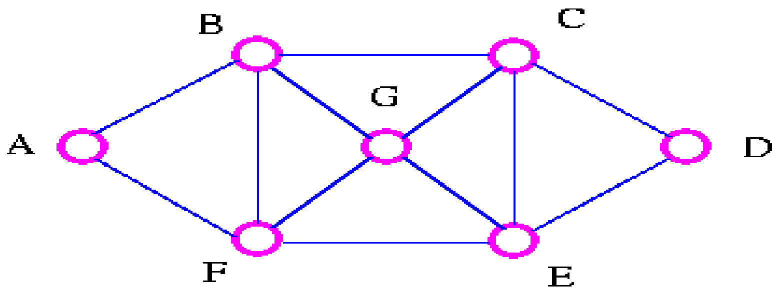


Isosahedron



Euler Graph

Consider the following road map:



The explorer's problem: An explorer wants to explore all the routes between number of cities. Can a tour be found which traverses each route only once? Particularly, find a tour which starts at A, goes along each road exactly once, and ends back at A.

Examples of such tour are:

A B C D E F B G C E G F A
A F G C D E G B C E F B A

The explorer travels along each road (edges) just once but may visit a particular city (vertex) several times.

The Traveller's Problem

A traveller wants to visit a number of cities. Can a tour be found which visits each city only once? Particularly, find a tour which starts at A, goes to each city exactly once, and ends back at A.

Examples of such tour are

A B C D E G F A
A F E D C G B A

The travellers visit each city (vertex) just once but may omit several of the roads (edges) on the way.

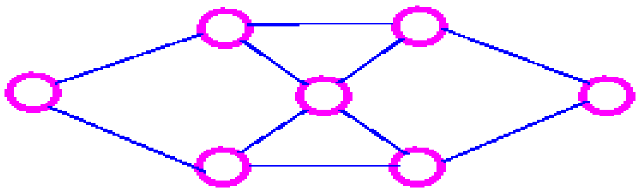
Eulerian Trail

A connected graph G is ***Eulerian*** if there is a closed trail which includes every edge of G, such a trail is called an Eulerian trail.

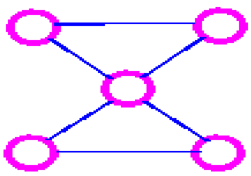
Hamiltonian Cycle

A connected graph G is ***Hamiltonian*** if there is a cycle which includes every vertex of G; such a cycle is called a Hamiltonian cycle.

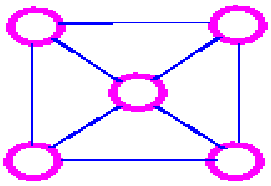
Consider the following examples:



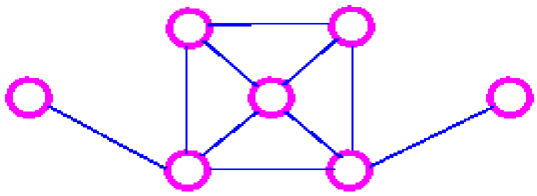
This graph is **Both** Eulerian and Hamiltonian.



This graph is Eulerian, but **Not** Hamiltonian.



This graph is a Hamiltonian, but **Not** Eulerian.



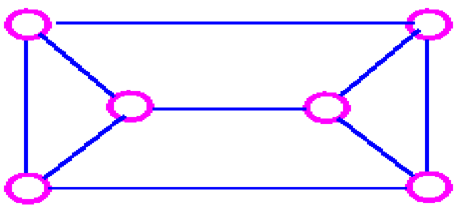
This graph is **Neither** Eulerian **Nor** Hamiltonian

Theorem

Let G be a connected graph. Then G is Eulerian if and only if every vertex of G has even degree.

Dirac's Theorem

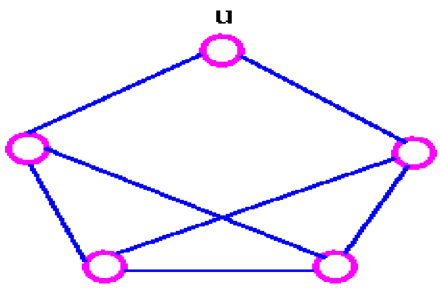
Let G be a simple graph with n vertices where $n \geq 3$. If $\deg(v) \geq 1/2\ n$ for each vertex v, then G is Hamiltonian. For example



$n = 6$ and $\deg(v) = 3$ for each vertex, so this graph is Hamiltonian by Dirac's theorem.

Ore's Theorem

Let G be a simple graph with n vertices where $n \geq 2$ if $\deg(v) + \deg(w) \geq n$ for each pair of non-adjacent vertices v and w, then G is Hamiltonian. For example

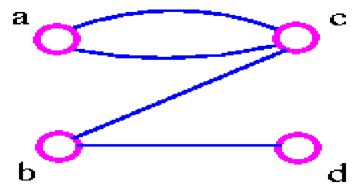


$n = 5$ but $\deg(u) = 2$, so Dirac's theorem does not apply. However, $\deg(v) + \deg(w) \geq 5$ for all pairs of vertices v and w (infact, for all pairs of vertices v and w), so this graph is Hamiltonian by Ore's theorem.

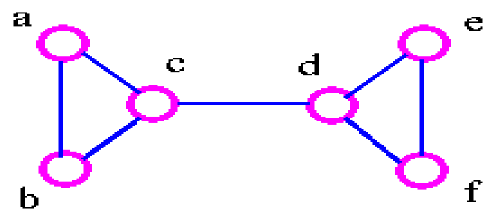
Note that: if $\deg(v) \geq 1/2\ n$ for each vertex, then $\deg(v) + \deg(w) \geq n$ for each pair of vertices v and w. It follows that Dirac's theorem can be deduced from Ore's theorem.

Bridge

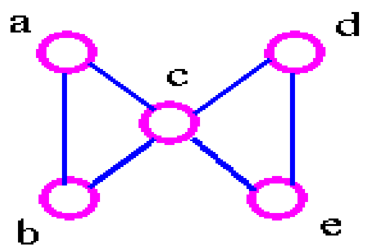
A ***bridge*** is a single edge whose removal disconnects a graph.



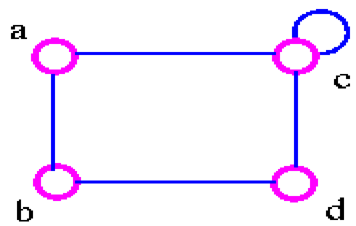
The above graph G1 can be split up into two components by removing one of the edges bc or bd. Therefore, edge bc or bd is a bridge.



The above graph G2 can be disconnected by removing a single edge, cd. Therefore, edge cd is a bridge.



The above graph G3 cannot be disconnected by removing a single edge, but the removal of two edges (such as ac and bc) disconnects it.



The above graph G4 can be disconnected by removing two edges such as ac and dc.

Edge Connectivity

The edge-connectivity $\lambda(G)$ of a connected graph G is the smallest number of edges whose removal disconnects G. When $\lambda(G) \geq k$, the graph G is said to be k-edge-connected.

For example, the edge connectivity of the above four graphs G1, G2, G3, and G4 are as follows:

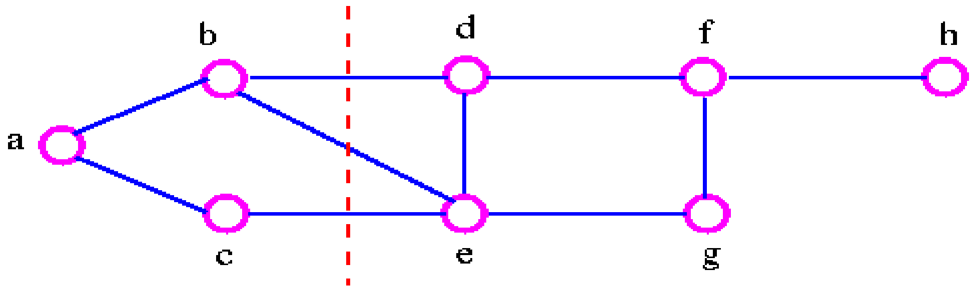
- *G1* has edge-connectivity 1.
- *G2* has edge connectivity 1.
- *G3* has edge connectivity 2.
- *G4* has edge connectivity 2.

Cut Set

A ***cut set*** of a connected graph G is a set S of edges with the following properties :

- The removal of all edges in S disconnects G.
- The removal of some (but not all) of edges in S does not disconnects G.

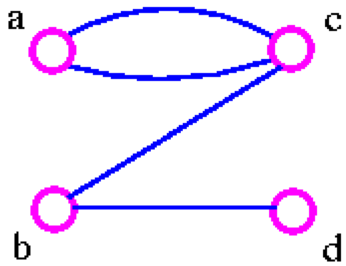
As an example consider the following graph:



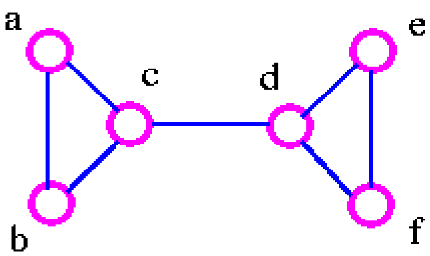
We can disconnect G by removing the three edges bd, bc, and ce, but we cannot disconnect it by removing just two of these edges. Note that a cut set is a set of edges in which no edge is redundant.

Vertex Connectivity

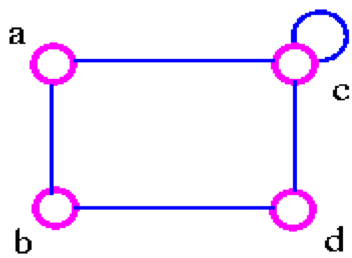
The connectivity (or vertex connectivity) **K**(*G*) of a connected graph *G* (other than a complete graph) is the minimum number of vertices whose removal disconnects *G*. When **K**(*G*) ≥ *k*, the graph is said to be *k*-connected (or *k*-vertex connected). When we remove a vertex, we must also remove the edges incident to it. As an example consider following graphs:



The above graph G can be disconnected by removal of single vertex (either b or c). The G has connectivity 1.



The above graph G can be disconnected by removal of single vertex (either c or d). The vertex c or d is a cut-vertex. The G has connectivity 1.



The above G cannot be disconnected by removing a single vertex, but the removal of two non-adjacent vertices (such as b and c) disconnects it. The G has connectivity 2.

Cut-Vertex

A ***cut-vertex*** is a single vertex whose removal disconnects a graph.

It is important to note that the above definition breaks down if G is a complete graph, since we cannot then disconnect G by removing vertices.