جامعة بغداد
كلية التربية للعلوم الصرفة / ابن الهيثم
قسم الرياضيات

التفاضل والتكامل 2

المرحلة الأولى – المستوى الثاني

قسم الرياضيات

أساتذة المادة

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CALCULUS 2

CHAPTER 1: Logarithm and Exponential Fun.

The Natural Logarithm Function:

We denote it by "**ln**"

$\boldsymbol{y = \ln(x); x > 0}$

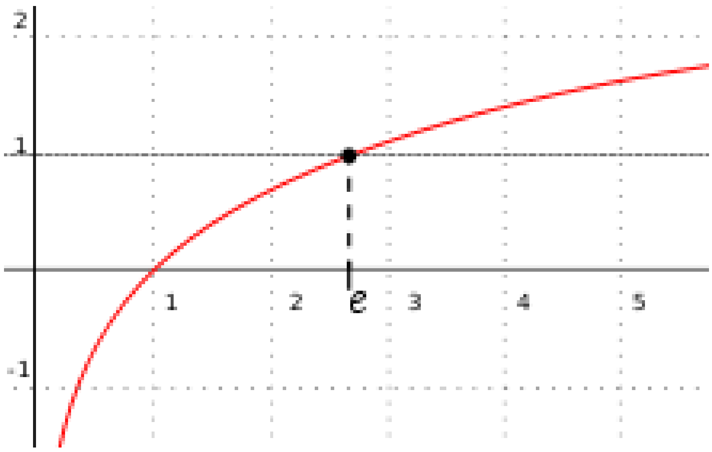
$\ln : (0, \infty) \longrightarrow \mathbb{R}$

$\ln e = 1$

$\ln 1 = 0$

$\ln 2 = 0.69$

$\ln 10 = 2.3$



Properties of ln:

Let $x > 0$ and $y > 0$, then:

1. $\ln(x.y) = \ln x + \ln y$
2. $\ln(\frac{x}{y}) = \ln x - \ln y$
3. $\ln(x^a) = a \ln x$

Examples: Evaluate the following if you know that ($\ln 2 = 0.69$):

1. $\ln 16 = \ln 2^4 = 4\ln 2 = 4.(0.69) = 2.76$

2. $\ln 8 = \ln 2^3 = 3.\ln 2 = 3.(0.69) = 2.07$

3. $\ln \frac{1}{2} = \ln 1 - \ln 2 = 0 - (0.69) = 0.69$

4. $\ln \sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2}.\ln 2 = \frac{1}{2}.(0.69) = 0.345$

The Derivative of ln

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}.\frac{du}{dx}}$$

Examples: Find y' for the following functions:

1. $y = \ln(x^2 + 2x)$

   $\Longrightarrow \quad y' = \frac{2x+2}{x^2+2x}$

2. $y = \ln(\sin x * x^2)$

   $\Longrightarrow \quad y' = \frac{1}{\sin x * x^2}.[\sin x * 2x + x^2 \cos x]$

3. $y = \ln(\tan x + \sec x)$

   $\Longrightarrow \quad y' = \frac{\sec^2 x + \sec x.\tan x}{\tan x + \sec x}$

4. $y = (\ln x)^3$

$$\implies y' = 3(\ln x)^2.\tfrac{1}{x}$$

5. $y = (\ln \tan)^5.\cos x^2$

$$\implies y' = (\ln \tan)^5.(-\sin x^2.2x) + \cos x^2.5(\ln \tan x)^4\tfrac{1}{\tan x}.\sec^2 x$$

6. $y = \ln(\cos x)$

$$\implies y' = \tfrac{-\sin}{\cos x} = -\tan x$$

7. $y = \ln(\ln x)$

$$\implies y' = \tfrac{\frac{1}{x}}{\ln x}$$

8. $y = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\tan^{-1} x}{x}$

$$\implies y' = \tfrac{1}{x} - \tfrac{1}{2}.\tfrac{2x}{1+x^2} - \tfrac{x.\frac{1}{1+x^2} - \tan^{-1} x}{x^2}$$

9. $y = x[\sin(\ln x) + \cos(\ln x)]$

$$\implies y' = x[\cos(\ln x).\tfrac{1}{x} - \sin(\ln x).\tfrac{1}{x}] + [\sin(\ln x) + \cos(\ln x)].1$$

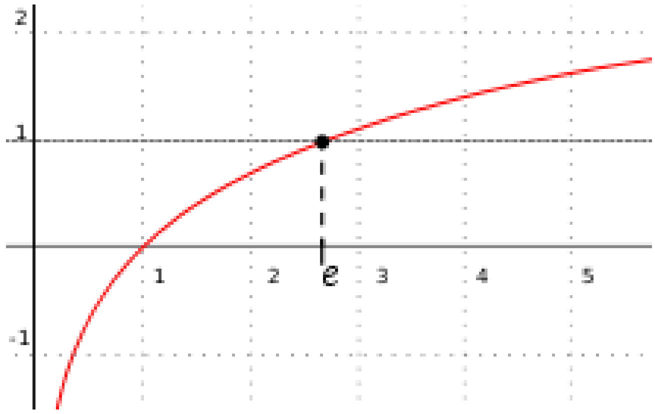
10. $y = x\sec^{-1} - \ln(x + \sqrt{x^2 - 1}); x > 1$

$$\implies y' = x.\tfrac{1}{|x|\sqrt{x^2-1}} + \sec^{-1} x.1 - \tfrac{1+\frac{1}{2}(x^2-1)^{-\frac{1}{2}}.2x}{x+\sqrt{x^2-1}}$$

The Limit of ln

$$\boxed{\lim_{x\to\infty} \ln x = +\infty}$$

$$\boxed{\lim_{x\to 0^+} \ln x = -\infty}$$



Examples: Proof the following:

1. $\lim\limits_{x\to 1}(x - \ln x) \stackrel{?}{=} 1$

   Proof: $\lim\limits_{x\to 1}(x - \ln x) = 1 - \ln 1 = 1 - 0 = \boxed{1}$

2. $\lim\limits_{x\to 1} \cos(\ln x) \stackrel{?}{=} 1$

   Proof: $\lim\limits_{x\to 1} \cos(\ln x) = \cos(\ln 1) = \cos(0) = \boxed{1}$

3. $\lim\limits_{x\to 1} \ln x^{(x+1)} \stackrel{?}{=} 0$

   Proof: $\lim\limits_{x\to 1} \ln x^{(x+1)} = \lim\limits_{x\to 1}(x+1)\ln x$

   $= \lim\limits_{x\to 1}(x+1) \lim\limits_{x\to 1} \ln x = (1+1).\ln 1 = 2.0 = \boxed{0}$

4. $\lim\limits_{x\to 0} \frac{\ln(x+1)}{x} \stackrel{?}{=} 1$

   Proof: $\lim\limits_{x\to 0} \frac{\ln(x+1)}{x} \stackrel{L'R}{=} \lim\limits_{x\to 0} \frac{\frac{1}{x+1}}{1} = \lim\limits_{x\to 1} \frac{1}{x+1} = \frac{1}{0+1} = \frac{1}{1} = \boxed{1}$

5. $\lim\limits_{x\to 1} \ln(2-x)^{2\cos x} \stackrel{?}{=} 0$

   Proof: $\lim\limits_{x\to 1} \ln(2-x)^{2\cos x} = \lim\limits_{x\to 1}[2\cos x.\ln(2-x)]$

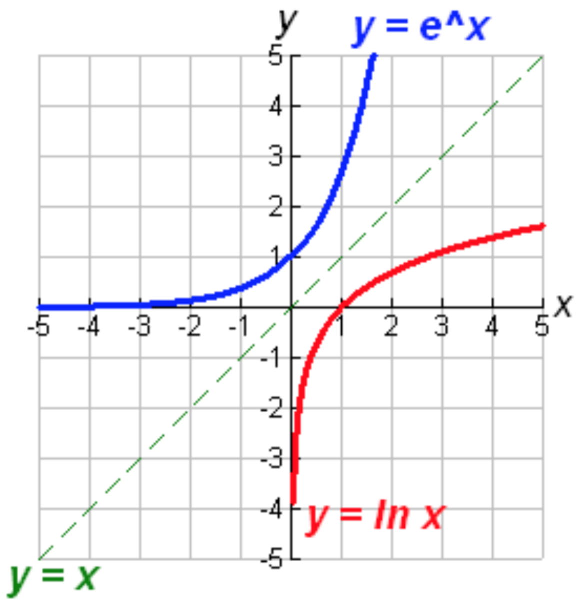
$$= 2\lim_{x\to 1}\cos x.\lim_{x\to 1}\ln(2-x)$$
$$= 2.\cos(1).\ln(2-1) = 2.\cos(1).0 = \boxed{0}$$

Problems (1.1): Find y' of the following functions:

1. $y = \ln(x\sqrt{x^2+1})$
2. $y = -5\ln(3x\sqrt{x+2})$
3. $y = t\ln t - \cos t^2$
4. $y = x^3\ln x - \frac{x}{\ln x}$
5. $y = \frac{1}{2}\ln\frac{1+w}{1-w}$
6. $y = -3\ln\frac{\sin w}{1+w^3}$
7. $y = \ln\frac{\sec x}{2+3x}$
8. $y = \ln(t^2+4) - t\tan^{-1}\frac{t}{2}$
9. $y = z(\ln z)^3$
10. $y = \csc^3 x\ln x^3 + \tan^{-1}\ln x$

The Exponential Function:

We denote it by "e^x" or "$\exp(x)$"



$\boldsymbol{y = e^x}$

$e^x : \mathbb{R} \longrightarrow (0, \infty)$

$e^x = ln^{-1}(x)$

$e^0 = 1 \ e^2 = 7.29$

Properties of e^x:

1. $e^0 = 1$
2. $e^{x_1}.e^{x_2} = e^{x_1+x_2}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^x)^r = e^{rx}, \forall r \in \mathbb{R}$
5. $e^{-x} = \frac{1}{e^x}$
6. $e^{\ln x} = x = \ln e^x$

Examples: Simplify the following:

1. $\ln(e^{-x^2}) = -x^2$

2. $\ln(e^{-\frac{1}{x}}) = \frac{1}{x}$

3. $e^{\ln\frac{1}{x}} = \frac{1}{x}$

4. $e^{2\ln x} = e^{\ln x^2} = x^2$

5. $\exp(\ln x - 2\ln y) = \exp(\ln x - \ln y^2) = \exp(\ln\frac{x}{y^2}) = \frac{x}{y^2}$

6. $e^{x+\ln x} = e^x.e^{\ln x} = e^x.x = x.e^x$

The Derivative of e^x

$$\boxed{\frac{d}{dx}e^u = e^u.\frac{du}{dx}}$$

Examples: Find y' for the following functions:

1. $y = e^{\tan^{-1}x}$

   $\Longrightarrow\ y' = e^{\tan^{-1}x}.\frac{1}{1+x^2}$

2. $y = \exp(3).\ \exp(\sin 3\theta)$

   $\Longrightarrow\ y' = \exp(3).\cos 3\theta.3 + \exp(\sin 3\theta).\ 0 = \exp(3).\cos 3\theta.3$

3. $y = e^{x^3}$

$$\implies y' = e^{x^3}.3x^2$$

4. $y = \ln \frac{e^x}{1+e^x}$

$$\implies y' = \frac{1}{\frac{e^x}{1+e^x}}.\frac{(1+e^x).e^x - e^x.e^x}{(1+e^x)^2}$$

5. $y = \frac{1}{2}(e^{\sin x} - e^{-2x})$

$$\implies y' = \frac{1}{2}(e^{\sin x}.\cos x - e^{-2x}.(-2))$$

6. $y = \sec^{-1}(e^{2x})$

$$\implies y' = \frac{1}{|e^{2x}|\sqrt{(e^{2x})^2 - 1}}.e^{2x}.2$$

7. $y = e^{\sin 2x}$

$$\implies y' = e^{\sin 2x}.\cos 2x.2$$

8. $y = e^{(\ln x)^3}$

$$\implies y' = e^{(\ln x)^3}.3(\ln x)^2.\frac{1}{x}$$

9. $y = e^{\cos(x^2)}$

$$\implies y' = e^{\cos(x^2)}. - \sin(x^2).2x$$

10. $y = e^{(3x - e^{-x^2})}$

$$\implies y' = e^{(3x - e^{-x^2})}.(3 - e^{-x^2}.(-2x))$$

Problems (1.2): Find y' of the following functions:

1. $y = x^2.e^{5x^2}$

2. $y = \exp(\pi).\, x + \exp(\cos \pi x)$

3. $y = \frac{e^z - e^{-z}}{e^z + e^z}$

4. $y = e^{\sin^{-1} x}.\sqrt{x}$

5. $y = (9t^2 - 6t + 2)\exp(3t)$

6. $y = \exp(\tan t).\ln t^5$

7. $y = w^7.\, e^{-\sqrt{w}}$

8. $y = e^{(\sqrt[3]{x} + \tan x^2)}$

9. $y = \exp(\tan^{-1}(t^3))$

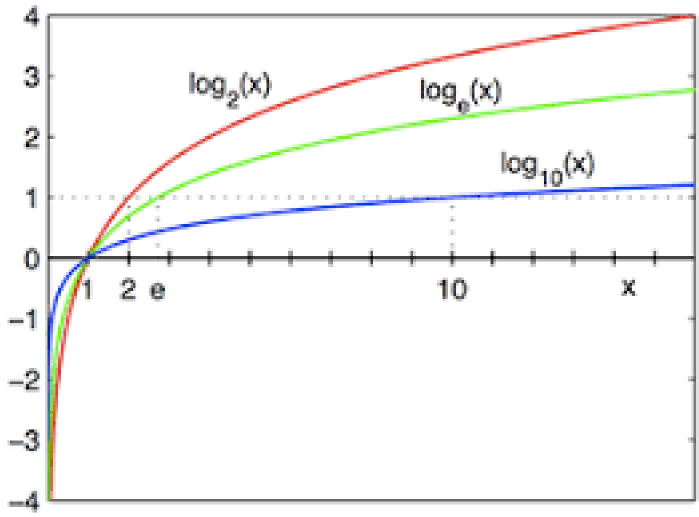
10. $y = \ln(e^{-\sin\theta} - e^{-\theta^3 + 5})$

The General Logarithm Function:

We denote it by "$\boldsymbol{\log_a x}$"

$$\boldsymbol{y = \log_a x = \frac{\ln x}{\ln a};\ x > 0, a > 0, a \neq 1}$$

$$\boldsymbol{\log_a x : (0, \infty)} \longrightarrow \mathbb{R}$$



Properties of $\log_a x$:

1. $\log_a(x.y) = \log_a x + \log_a y$

2. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$

3. $\log_a x^y = y \log_a x$

4. $\log_a a = 1 \iff \ln e = 1$

5. $\log_a 1 = 0 \iff \ln 1 = 0$

Examples: Simplify the following:

1. $\log_2 16 = \frac{\ln 16}{\ln 2} = \frac{\ln 2^4}{\ln 2} = \frac{4 \ln 2}{\ln 2} = 4$

2. $\log_{\frac{1}{7}} 49 = \frac{\ln 49}{\ln \frac{1}{7}} = \frac{\ln 7^2}{-\ln 7} = \frac{2 \ln 7}{-\ln 7} = -2$

3. $\log_{10} 10 = 1$

4. $\log_{10} 100 = \log_{10} 10^2 = 2\log_{10} 10 = 2.1 = 2$

5. $\log_{10} \frac{1}{1000} = \log_{10} 10^{-3} = -3\log_{10} 10 = -3.1 = -3$

The Derivative of $\log_a x$

$$\boxed{\frac{d}{dx}\log_a u = \frac{1}{\ln a}.\frac{1}{u}.\frac{du}{dx}}, \text{ where } u > 0,\ a > 0,\ a \neq 0.$$

Examples: Find y' for the following functions:

1. $y = \log_2(x^2 + 3x)$

   $\Longrightarrow\ y' = \frac{1}{(x^2+3x)}.\frac{1}{\ln 2}.(2x + 3)$

2. $y = \log_3 x^5 . \cos x^2$

   $\Longrightarrow\ y' = \log_3 x.(-\sin x^2.2x) + \cos x^2.\frac{1}{x^5}.\frac{1}{\ln 3}.5x^4$

3. $y = \log_7(\tan x + \sin x)$

   $\Longrightarrow\ y' = \frac{1}{(\tan x+\sin x)}.\frac{1}{\ln 7}.(\sec^2 x + \cos x)$

4. $y = \ln x . \log_{10} x$

   $\Longrightarrow\ y' = \ln x.\frac{1}{x}.\frac{1}{\ln 10}.1 + \log_{10} x.\frac{1}{x}$

5. $y = \log_a \sin^{-1} x + \frac{x}{e^x}$, where a is a constant.

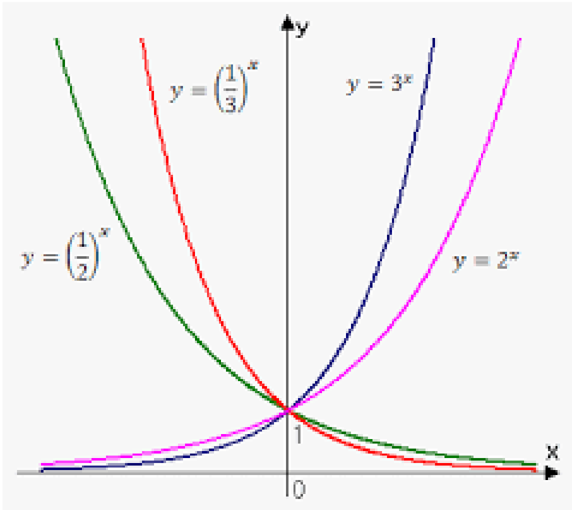
$\implies y' = \frac{1}{\sin^{-1} x} \cdot \frac{1}{\ln a} \cdot \frac{1}{\sqrt{1-x^2}} + \frac{e^x.1 - x.e^x.1}{e^{2x}}$

Problems (1.3): Find y' of the following functions:

1. $y = x \log_5 x - \cos x^2$
2. $y = e^{-4t} \log_2(1-t) + \log_a(3t^2)$, where a is a constant.
3. $y = \log_4(\cos 3w) \sec \sqrt{w}$
4. $y = \log_{11}(\tan(3\pi t))$
5. $y = \sqrt[3]{1 + \log_3(z^5)}$
6. $y = \log_n(\cos^{-1} \theta) + \sin \theta \ . \ \log_3 t^2$, where n is a constant.
7. $y = (\log_6(5z))^3$
8. $y = \sin(\log_2 \theta) + \sec \theta^2$
9. $y = \log_b(we^{w^2}) + \sin^{-1} w$, where b is a constant.
10. $y = \cot(\log_7 t^2).\frac{1}{t^3}$

The General Exponential Function:

We denote it by "$\boldsymbol{a^x}$"



$$\boldsymbol{y = a^x = e^{x \ln a};\ a > 0}$$

$$\boldsymbol{a^x : \mathbb{R} \longrightarrow (0, \infty)}$$

Properties of $\boldsymbol{a^x}$:

1. $a^1 = a$, where $a > 0$
2. $a^0 = 1$
3. $a^u . a^v = a^{u+v}$
4. $(a^{\frac{m}{n}})^n = a^m$
5. $(a.b)^u = a^u . b^u$, where $a > 0$ and $b > 0$

The Derivative of $\boldsymbol{a^x}$

$$\because a^u = e^{u \ln a}$$

$$\xrightarrow{\frac{d}{dx}} \frac{d}{dx} a^u = \frac{d}{dx}(e^{u \ln a})$$

$$\longrightarrow \frac{d}{dx}a^u = a^u . \ln a . \frac{du}{dx}$$

$$\therefore \boxed{\boldsymbol{\frac{d}{dx}a^u = a^u . \ln a . \frac{du}{dx}}}$$

Examples: Find y' for the following functions:

1. $y = 2^{x^2+\sec x}$

   $\Longrightarrow \ y' = 2^{x^2+\sec x} . \ln 2 . (2x + \sec x \tan x)$

2. $y = 4^{sin^{-1}x}$

   $\Longrightarrow \ y' = 4^{sin^{-1}x} . \ln 4 . \frac{1}{\sqrt{1-x^2}}$

3. $y = x^\pi . \pi^x$

   $\Longrightarrow \ y' = x^\pi . \pi^x . \ln \pi . 1 + \pi^x . \pi . x^{\pi-1} . 1$

Example: Find the value of x, where $3^x = 2^{x+1}$

Solution: $3^x = 2^{x+1}$

$\xRightarrow{\ln} \ \ln 3^x = \ln 2^{x+1}$

$\Longrightarrow \ x \ln 3 = (x+1) \ln 2$

$\Longrightarrow \ x \ln 3 = x \ln 2 + \ln 2$

$\Longrightarrow \ x \ln 3 - x \ln 2 = \ln 2$

$$\Longrightarrow \; x(\ln 3 - \ln 2) = \ln 2 \;\Longrightarrow\; x(\ln \tfrac{3}{2}) = \ln 2 \;\Longrightarrow\; \boxed{x = \frac{\ln 2}{\ln \frac{3}{2}}}$$

Problems (1.4): Find y' of the following functions:

1. $y = 5^{x^2+x-1}$

2. $y = 6^{\sin w + \ln w + 3}$

3. $y = 2^{\sec \sqrt{t}}$

4. $y = e^{5x}\, 3^{\tan x}$

5. $y = \ln \frac{x^4}{1+x^3} + 7^{\frac{x^2}{2}}$

6. $y = 2^{-t^2}\, \cos t^3$

7. $y = \pi^{\cos x}\, e^{\sqrt{x}} - 5^{-4x^3}$

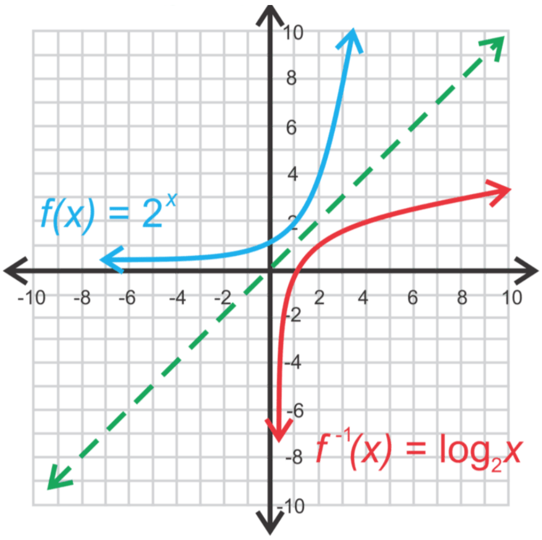
8. $y = \ln(1 + e^{2x})\, 5^{x^2}$

9. $y = -4 \ln w + \frac{4}{w} - 2^{sin(\sqrt{w})}$

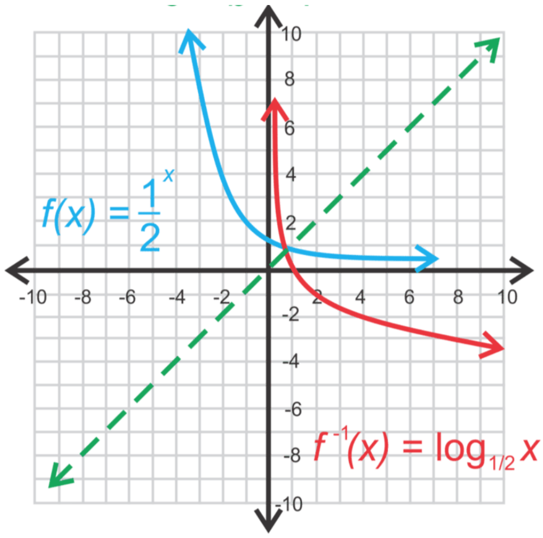
10. $y = 8^{\cos \pi\theta}\, \ln \sqrt{\theta} + 3^{\sec(5\theta)}$

The Relation between a^x and $\log_a x$

$$\boxed{\log_a y = x \iff y = a^x; a > 0 \;\&\; a \neq 0}$$



$a > 1$



$0 < a < 1$

Logarithmic Differentiation

If in a function where the base and the exponent are variables, we should change the form before we take the derivative.

Examples: Find y' for the following:

1. $y = x^{x^2}$

$$\xrightarrow{ln} \ln y = \ln x^{x^2}$$

$$\longrightarrow \ln y = x^2 \ln x$$

$$\xrightarrow{\frac{d}{dx}} \frac{y^{'}}{y} = x^2\frac{1}{x} + \ln x.2x$$

$$\longrightarrow y^{'} = y * (x^2\frac{1}{x} + \ln x.2x)$$

$$\xrightarrow{y=x^{x^2}} y^{'} = x^{x^2} * (x^2\frac{1}{x} + \ln x.2x)$$

2. $y = (x^2+1)^{\ln x}$

$$\xrightarrow{ln} \ln y = \ln(x^2+1)^{\ln x}$$

$$\longrightarrow \ln y = \ln x \ln(x^2+1)$$

$$\xrightarrow{\frac{d}{dx}} \frac{y^{'}}{y} = \ln x.\tfrac{2x}{x^2+1} + \ln(x^2+1).\tfrac{1}{x}$$

$$\longrightarrow y^{'} = y * (\ln x.\tfrac{2x}{x^2+1} + \ln(x^2+1).\tfrac{1}{x})$$

$$\longrightarrow y^{'} = (x^2+1)^{\ln x} * (\ln x.\tfrac{2x}{x^2+1} + \ln(x^2+1).\tfrac{1}{x})$$

3. $y = (\sin x)^{\tan x}$

$$\xrightarrow{ln} \ln y = \ln(\sin x)^{\tan x}$$

$$\longrightarrow \ln y = \tan x.\ln(\sin x)$$

$$\xrightarrow{\frac{d}{dx}} \frac{y^{'}}{y} = \tan x.\tfrac{\cos x}{\sin x} + \ln(\sin x).\sec^2 x$$

$$\longrightarrow y^{'} = y * (\tan x\tfrac{\cos x}{\sin x} + \ln(\sin x).\sec^2 x)$$

$$\longrightarrow y^{'} = (\sin x)^{\tan x} * (\tan x.\tfrac{\cos x}{\sin x} + \ln(\sin x).\sec^2 x)$$

$$\longrightarrow y^{'} = (\sin x)^{\tan x} * (\tfrac{\sin x}{\cos x}.\tfrac{\cos x}{\sin x} + \ln(\sin x).\sec^2 x)$$

$$\longrightarrow y^{'} = (\sin x)^{\tan x} * (1 + \sec^2 x.\ln(\sin x))$$

4. $y = \sqrt[5]{\dfrac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}}$

$$\longrightarrow y = \left(\tfrac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}\right)^{\frac{1}{5}}$$

$$\xrightarrow{ln} \ln y = \tfrac{1}{5} * \ln\left(\tfrac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}\right)$$

$$\longrightarrow \ln y = \tfrac{1}{5} * \ln\left((x+2)^3(x+1)^2\right) - \ln\left((x-4)^2(x+3)\right)$$

$$\longrightarrow \ln y = \tfrac{1}{5} * \Big(\ln(x+2)^3 + \ln(x+1)^2 \Big) - \Big(\ln(x-4)^2 + \ln(x+3) \Big)$$

$$\longrightarrow \ln y = \tfrac{1}{5} * \Big(3\ln(x+2) + \ln(x+1)^2 - 2\ln(x-4) - \ln(x+3) \Big)$$

$$\xrightarrow{\frac{d}{dx}} \frac{y'}{y} = \frac{1}{5}\left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3} \right)$$

$$\longrightarrow y' = y * \frac{1}{5}\left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3} \right)$$

$$\longrightarrow y' = \sqrt[5]{\frac{(x+2)^3(x+1)^2}{(x-4)^2(x+3)}} * \frac{1}{5}\left(\frac{3}{x+2} + \frac{2}{x+1} - \frac{2}{x-4} - \frac{1}{x+3} \right)$$

Problems (1.5):

1. Find y' for the following:

   (a) $y = x^x$

   (b) $y = e^{\sqrt{z^2+e^z}}$

   (c) $y = x^{\sqrt{5}+\ln x}$

   (d) $y = n^{\ln n}$

   (e) $y = (\tan t)^{3sin^{-1}t}$

2. Using L'R proof the following:

   (a) $\lim\limits_{x\to\infty} \frac{3x^2+x+4}{5x^2+8x} = \frac{3}{5}$

   (b) $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{\cos x} = 0$

   (c) $\lim\limits_{x\to\infty} \frac{\ln x}{x} = 0$

(d) $\lim\limits_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{x^{-1}} = 1$

(e) $\lim\limits_{x\to 1} \frac{x\ln x - x + 1}{(x-1)\ln x} = \frac{1}{2}$

(f) $\lim\limits_{x\to -\infty} \frac{x^2}{e^x} = 0$

(g) $\lim\limits_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3} = \frac{1}{6}$

(h) $\lim\limits_{x\to -\infty} \frac{3x-2}{e^{x^2}} = 0$

CHAPTER 2: Indefinite Integration

2.1 General Indefinite Integration:

In calculus, an antiderivative of a function $f(x)$ is a differentiable function $F(x)$ whose derivative is equal to the original function $f(x)$.

$$F'(x) = f(x) \implies \frac{dF(x)}{dx} = f(x)$$

$$\implies dF(x) = f(x)dx$$

$$\xRightarrow{\int} \int dF(x) = \int f(x)dx$$

$$\implies \boxed{F(x) = \int f(x)dx + C}, \text{ where } C \in \mathbb{R}.$$

Example: $y' = 2x \implies \frac{dy}{dx} = 2x$

$$\implies dy = 2x \ dx \xRightarrow{\int} \int dy = \int 2x \ dx \implies \boxed{y = \frac{x^2}{2} + C}$$

Indefinite Integrals Properties:

Let $f(x)$ be a function and $x \in \mathbb{R}$, then:

i. $\int af(x) = a\int f(x)dx$, where a is a constant.

ii. $\int \Big(f(x) \mp g(x)\Big)dx = \int f(x)dx \mp \int g(x)dx$

Notes:

- $\int \Big(f(x) * g(x)\Big) dx \neq \int f(x)dx * \int g(x)dx$
- $\int \frac{f(x)}{g(x)} dx \neq \frac{\int f(x)dx}{\int g(x)dx}$

Rules: In general,

i. $\boxed{\int dx = x + c}$

ii. $\boxed{\int u^n du = \frac{u^{n+1}}{n+1} + C}$, where $C \in \mathbb{R}$.

Examples: Evaluate the following integrals:

1) $\int 3dx$

$= 3 \int dx$

$= \boxed{3x + C}$

2) $\int 5x^2 dx$

$= 5 \int x^2 dx$

$= \boxed{5\, \frac{x^3}{3} + C}$

3) $\int 6x^{-3} dx$

$= 6 \int x^{-3} dx$

$= 6\,\frac{x^{-2}}{-2} + C$

$= \boxed{\dfrac{-3}{x^2} + C}$

4) $\int \frac{3\pi}{x^5} dx$

$= \int 3\pi x^{-5} dx$

$= 3\pi \ \int x^{-5} dx$

$= 3\pi\,\frac{x^{-4}}{-4} + C$

$= \boxed{\dfrac{-3\pi}{4x^4} + C}$

5) $\int (2x+3)dx$

$= \int 2x dx + \int 3 dx$

$= x^2 + C_1 + 3x + C_2$

$= \boxed{x^2 + 3x + C}$, where $C = C_1 + C_2$.

6) $\int \sqrt{2x+1} dx$

$= \frac{1}{2} \int (2x+1)^{\frac{1}{2}}.2dx$

$= \frac{1}{2} \frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \boxed{\dfrac{1}{3}\sqrt{(2x+1)^3} + C}$

7) $\int (x^2 - \sqrt{x}) dx$

$= \int x^2 dx - \int x^{\frac{1}{2}} dx$

$= \boxed{\dfrac{x^3}{3} - \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C}$

8) $\int t^2(1+2t^3)^{\frac{-2}{3}}dt$

$$= \frac{1}{6}\int(1+2t^3)^{-\frac{2}{3}}.6t^2dt$$

$$= \frac{1}{6}\frac{(1+2t^3)^{\frac{1}{3}}}{\frac{1}{3}} + C$$

$$= \boxed{\frac{1}{2}\sqrt[3]{1+2t^3} + C}$$

9) $\int \frac{x^3-2x^7}{5x^5}dx$

$$= \int \frac{x^3}{5x^5} - \frac{2x^7}{5x^5}dx$$

$$= \frac{1}{5}\int x^{-2}dx - \frac{2}{5}\int x^2dx$$

$$= \frac{1}{5}\frac{x^{-1}}{-1} - \frac{2}{5}\frac{x^3}{3} + C$$

$$= \boxed{\frac{-1}{5x} - \frac{2}{15}x^3 + C}$$

10) $\int \frac{(z+1)dz}{\sqrt[3]{z^2+2z+2}}$

$$= \int(z^2+2z+2)^{-\frac{1}{3}}(z+1)dz$$

$$= \frac{1}{2}\int(z^2+2z+2)^{-\frac{1}{3}}2(z+1)dz$$

$$= \frac{1}{2}\frac{(z^2+2z+2)^{\frac{2}{3}}}{\frac{2}{3}} + C$$

$$= \boxed{\frac{3}{4}\sqrt[3]{(z^2+2z+2)^2} + C}$$

Problems (2.1): Evaluate the following integrals:

1. $\int \frac{dx}{(3x+2)^2}$

2. $\int \frac{3r}{\sqrt{1-r^2}} dr$

3. $\int \frac{-1+5x}{x^{\frac{1}{4}}} dx$

4. $\int \frac{y}{\sqrt{2y^2+1}}\, dy$

5. $\int (\sqrt{x} + \frac{1}{\sqrt{x}}) dx$

6. $\int \sqrt{2+5y}\, dy$

7. $\int (2+7t)^{\frac{2}{3}} dt$

8. $\int (3x-1)^{22} dx$

9. $\int \frac{-7x}{(x^2-16)^8}\, dx$

10. $\int z\, \sqrt{2z^2+1}\, dz$

11. $\int (x^2+5x) dx$

12. $\int (3x^3 + 2x^{-5} + 4x - 9)\, dx$

13. $\int \frac{y}{(19-2y^2)^{\frac{1}{3}}} dy$

14. $\int 7w^2 \sqrt{2w^3-5}\, dw$

15. $\int \frac{\theta+1}{\theta^3}\, d\theta$

16. $\int \sqrt{y^2+2y}\, (y+1)\, dy$

17. $\int \frac{z^5+z^{-2}-z-\pi}{z^3}\, dz$

18. $\int \frac{3t}{\sqrt{2-t^2}}\, dt$

19. $\int \sqrt[3]{r^3+\pi}\;\, r^2\, dr$

20. $\int \frac{3y^2}{\sqrt{y^3+7}} dy$

2.2 Integrals of Trigonometric Functions:

We can derive all the Trigonometric integration forms from the derivative Trigonometric forms as follows:

(1) $\frac{d}{du}\sin(u) = \cos(u) \xrightarrow{\int} \boxed{\int \cos(u)du = \sin(u) + C}$

(2) $\frac{d}{du}\cos(u) = -\sin(u) \xrightarrow{\int} \boxed{\int \sin(u)du = -\cos(u) + C}$

(3) $\frac{d}{du}\tan(u) = \sec^2(u) \xrightarrow{\int} \boxed{\int \sec^2(u)du = \tan(u) + C}$

(4) $\frac{d}{du}\cot(u) = -\csc^2(u) \xrightarrow{\int} \boxed{\int \csc^2(u)du = -\cot(u) + C}$

(5) $\frac{d}{du}\sec(u) = \sec(u)\tan(u) \xrightarrow{\int} \boxed{\int \sec(u)\tan(u)du = \sec(u) + C}$

(6) $\frac{d}{du}\csc(u) = -\csc(u)\cot(u) \xrightarrow{\int} \boxed{\int \csc(u)\cot(u)du = -\csc(u) + C}$

Examples: Evaluate the following integrals:

1) $\int \sin(3x)dx$

$= \frac{1}{3}\int \sin(3x).3dx$

$= \boxed{-\frac{1}{3}\cos(3x) + C}$

2) $\int \cos(2t)dt$

$= \frac{1}{2}\int \cos(2t).2\,dt$

$$= \boxed{\frac{1}{2}\sin(2t) + C}$$

3) $\int x\sec^2(x^2)dx$

$$= \frac{1}{2}\int \sec^2(x^2).2xdx$$
$$= \boxed{\frac{1}{2}\tan(x^2) + C}$$

4) $\int \cot(5x)\csc(5x)dx$

$$= \frac{1}{5}\int \cot(5x)\csc(5x).5dx$$
$$= \boxed{-\frac{1}{5}\csc(5x) + C}$$

5) $\int \frac{1}{\sqrt{x}}\csc^2(\sqrt{x})dx$

$$=$$

6) $\int t^2\tan^4(t^3)dt$

$$=$$

Remark(1): Sometimes we should do some algebra to evaluate the integral.

Examples: Evaluate the following integrals:

1) $\int \sec^3(x)\tan(x)dx$

$$= \int \underbrace{\sec^2(x)}_{u}\,\underbrace{\sec(x)\tan(x)dx}_{du}$$

$$= \boxed{\frac{\sec^3(x)}{3} + C}$$

2) $\int \csc^7(x)\cot(x)dx$

$=$

3) $\int \frac{\cos(2x)}{\sin^3(2x)}dx$

$$= \frac{1}{2}\int \underbrace{(\sin(2x))}_{u}{}^{-3}\,\underbrace{\cos(2x).2dx}_{du}$$

$$= \frac{1}{2}\frac{(\sin(2x))^{-2}}{-2} + C$$

$$= \boxed{-\frac{1}{4\sin^2(2x)} + C}$$

4) $\int \frac{6-\cos(3x)}{\sin^2(3x)}dx$

$$= \int \frac{6}{\sin^2(3x)}dx - \int \frac{\cos(3x)}{\sin^2(3x)}dx$$

$$= 6\int \frac{1}{\sin^2(3x)}dx - \int \frac{1}{\sin(3x)}\frac{\cos(3x)}{\sin(3x)}dx$$

$$= 6\int \csc^2(3x)dx - \int \csc(3x)\cot(3x)dx$$

$$= \frac{6}{3}\int \csc^2(3x).3dx - \frac{1}{3}\int \csc(3x)\cot(3x).3dx$$

$$= 2(-\cot(3x)) - \frac{1}{3}(-\csc(3x)) + C$$

$$= \boxed{-2\cot(3x) + \frac{1}{3}\csc(3x) + C}$$

5) $\int \frac{\sec^2(2w)}{\tan^3(2w)}dw$

$=$

6) $\int \frac{\pi}{\sin^2(5z)}dz$

$=$

Remark(2):

A. When the power of $\sin(x)$ or $\cos(x)$ is **odd**, we use:

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

B. When the power of $\sin(x)$ or $\cos(x)$ is **even**, we use:

$$\boxed{\sin^2(x) = \frac{1}{2}(1 - \cos 2x)} \text{ or } \boxed{\cos^2(x) = \frac{1}{2}(1 + \cos 2x)}$$

Examples: Evaluate the following integrals:

1) $\int \sin^3(x)dx$

$= \int \sin(x)\sin^2(x)dx$

$= \int \sin(x)(1 - \cos^2(x))dx$

$= \int \sin(x)dx - \int \underbrace{\cos^2(x)}_{u}\underbrace{\sin(x)dx}_{du}$

$= \boxed{\cos(x) - \frac{\cos^3(x)}{3} + C}$

2) $\int \cos^2(y)dy$

$= \int \frac{1}{2}(1 + \cos(2y))dy$

$= \frac{1}{2}\int dy + \frac{1}{2}.\frac{1}{2}\int \cos(2y).2dy$

$= \boxed{\frac{1}{2}y + \frac{1}{4}\sin(2y) + C}$

3) $\int \sin^2(y)dy$

$=$

4) $\int \cos^4(x)dx$

$$= \int \Big(\cos^2(x)\Big)^2 dx$$

$$= \int \Big(\tfrac{1}{2}(1+\cos(2x))\Big)^2 dx$$

$$= \tfrac{1}{4}\int \Big((1+2\cos(2x)+\cos^2(2x))\Big)dx$$

$$= \tfrac{1}{4}\int 1dx + \tfrac{1}{4}\int 2\cos(2x)dx + \tfrac{1}{4}\int \cos^2(2x)dx$$

$$= \tfrac{1}{4}\int dx + \tfrac{1}{2}\int \cos(2x)dx + \tfrac{1}{4}\int \tfrac{1}{2}(1+\cos 4x)dx$$

$$= \tfrac{1}{4}\int dx + \tfrac{1}{2}.\tfrac{1}{2}\int \cos(2x).2dx + \tfrac{1}{8}\int dx + \tfrac{1}{8}.\tfrac{1}{4}\int \cos 4x.4dx$$

$$= \boxed{\frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C}$$

Remark(3):

A. If the powers of $\tan(x)$ & $\cot(x)$ is even, we use:

use $\boxed{\tan^2(x) = \sec^2(x) - 1}$

B. If the powers of $\sec(x)$ & $\csc(x)$ is even, we use:

use $\boxed{\sec^2(x) = 1 + \tan^2(x)}$

Examples: Evaluate the following integrals:

1) $\int \tan^2(3x)dx$

$$= \int(\sec^2(3x) - 1)dx$$

$$= \frac{1}{3}\sec^2(3x).3dx - \int dx$$

$$= \boxed{\frac{1}{3}\tan(3x) - x + C}$$

2) $\int \cot^2(5x)dx$

$$=$$

3) $\int \sec^4(x)dx$

$$= \int \sec^2(x)\sec^2(x)dx$$

$$= \int \left(1 + \tan^2(x)\right)\sec^2(x)dx$$

$$= \int \sec^2(x)dx + \int \tan^2(x)\sec^2(x)dx$$

$$= \boxed{\tan(x) + \frac{\tan^3(x)}{3} + C}$$

4) $\int \csc^4(x)dx$

$$= \int \csc^2(x)\csc^2(x)dx$$

$$= \int \left(1 + \cot^2(x)\right)\csc^2(x)dx$$

$$= \int \csc^2(x)dx + \int \cot^2(x)\csc^2(x)dx$$

$$= \boxed{-\cot(x) - \frac{\cot^3(x)}{3} + C}$$

Problems (2.2): Evaluate the following integrals:

1. $\int \sin(2t)dt$
2. $\int x\sin(2x^2)dx$
3. $\int \cos(3\theta-1)d\theta$
4. $\int 4\cos(3y)dy$
5. $\int (x-2\sin^2 x)dx$
6. $\int \pi\sin^2(3t)dt$
7. $\int \cos^3 x dx$
8. $\int \frac{1}{\cos^2(3w)}dw$
9. $\int \frac{2}{3}\cot^2(5y)dy$
10. $\int 3\tan^2(2x)dx$
11. $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}dx$
12. $\int \cos(2x+4)dx$
13. $\int (3\cos(x)-4x^3+2)dx$
14. $\int \sin^2(x)\cos(x)dx$
15. $\int \sin(x)\cos^3(x)dx$
16. $\int \cos^2(2y)\sin(2y)dy$
17. $\int \sin^4\theta d\theta$
18. $\int \Big(1-\sin^2(3t)\Big)\cos(3t)dt$
19. $\int \frac{3-\sin x}{\cos^2 x}dx$
20. $\int 2\sin(z)\cos(z)dz$
21. $\int \sqrt{2+\sin(3t)}\cos(3t)dt$
22. $\int \frac{\sin(\frac{z-1}{3})}{\cos^2(\frac{z-1}{3})}dz$
23. $\int (3\sin(2x)+4\cos(3x))dx$
24. $\int \sin(t)\cos(t)\Big(\sin(t)+\cos(t)\Big)dt$

2.3 Integrals of Inverse Trigonometric Functions :

We can derive all the integration forms from our derivatives forms as follows:

(1) $\frac{d}{du}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \xrightarrow{\int} \boxed{\int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u) + C}$

(2) $\frac{d}{du}\cos^{-1}(u) = -\frac{1}{\sqrt{1-u^2}} \xrightarrow{\int} \boxed{\int \frac{1}{\sqrt{1-u^2}}du = -\cos^{-1}(u) + C}$

(3) $\frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2} \xrightarrow{\int} \boxed{\int \frac{1}{1+u^2}du = \tan^{-1}(u) + C}$

(4) $\frac{d}{du}\cot^{-1}(u) = -\frac{1}{1+u^2} \xrightarrow{\int} \boxed{\int \frac{1}{1+u^2}du = -\cot^{-1}(u) + C}$

(5) $\frac{d}{du}\sec^{-1}(u) = \frac{1}{|u|\sqrt{u^2-1}} \xrightarrow{\int} \boxed{\int \frac{1}{|u|\sqrt{u^2-1}}du = \sec^{-1}(u) + C}$

(6) $\frac{d}{du}\csc^{-1}(u) = -\frac{1}{|u|\sqrt{u^2-1}} \xrightarrow{\int} \boxed{\int \frac{1}{|u|\sqrt{u^2-1}}du = -\csc^{-1}(u) + C}$

Examples: Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{1-4x^2}}$

$= \frac{1}{2}\int \frac{2dx}{\sqrt{1-(2x)^2}}$

$= \boxed{\frac{1}{2}\sin^{-1}(2x) + C}$ or $\boxed{\frac{1}{2}\cos^{-1}(2x) + C}$

2) $\int \frac{dt}{1+t^2}$

$= \boxed{\tan^{-1}(t) + C}$ or $\boxed{-\cot^{-1}(t) + C}$

3) $\frac{dx}{x\sqrt{4x^2-1}}$

$$= \frac{2dx}{2x\sqrt{(2x)^2-1}}$$

$$= \boxed{\sec^{-1}|2x|+C} \text{ or } \boxed{-\csc^{-1}|2x|+C}$$

4) $\frac{-dx}{\sqrt{4-25x^2}}$

$$= \frac{-dx}{\sqrt{4(1-\frac{25}{4}x^2)}}$$

$$= \frac{-dx}{2\sqrt{1-(\frac{5}{2}x)^2}}$$

$$= \frac{-1}{2}.\frac{2}{5}.\frac{\frac{5}{2}dx}{\sqrt{1-(\frac{5}{2}x)^2}}$$

$$= \boxed{-\frac{1}{5}\sin^{-1}(\frac{5}{2}x)+C} \text{ or } \boxed{\frac{1}{5}\cos^{-1}(\frac{5}{2}x)+C}$$

5) $\int \frac{\cos(x)dx}{\sqrt{1-\sin^2(x)}}$

$$= \sin^{-1}(\sin(x)) + C$$

$$= \boxed{x+C}$$

6) $\int \frac{\tan^{-1}(x)}{1+x^2}dx$

$$= \int \tan^{-1}(x).\frac{dx}{1+x^2}$$

$$= \boxed{\frac{(\tan^{-1}(x))^2}{2}+C}$$

7) $\frac{\sqrt{\sec^{-1}(x)}}{x\sqrt{x^2-1}}dx$

$$= (\sec^{-1}(x))^{\frac{1}{2}}.\frac{1}{x\sqrt{x^2-1}}dx$$

$$= \boxed{\frac{(\sec^{-1}(x))^{\frac{3}{2}}}{\frac{3}{2}}+C}$$

Problems (2.3): Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{1-(2+9z)^2}}dz$
2. $\int d(\csc^{-1}(t))$
3. $\int \frac{x}{1+16x^4}dx$
4. $\int (\cos^{\frac{1}{2}}\sin(x) - \frac{13}{1+x^2})dx$
5. $\int \frac{x}{9x+x^3}dx$
6. $\int \frac{3}{16z^2+4}dz$
7. $\int \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}}dx$
8. $\int \frac{(\sin^{-1}(x))^2}{\sqrt{1-x^2}}dx$
9. $\int \frac{\sin(\tan^{-1}(x))}{1+x^2}dx$
10. $\int \frac{1}{1+25x^2}dx$
11. $\int \frac{-1}{\sqrt{1-16w^2}}dw$
12. $\int d(\sec^{-1}(t))$
13. $\int \frac{-3t}{9t^4+9}dt$
14. $\int (\cos^3(2x)\sin(2x) - \frac{x^3}{\pi})dx$
15. $\int \frac{1}{25+t^2}dt$
16. $\int \frac{1}{81w^2+9}dw$
17. $\int \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}}dx$
18. $\int \frac{\left(\cos^{-1}(2z)\right)^2}{\sqrt{1-4z^2}}dz$
19. $\int \frac{\sin(\sin^{-1}(3x))}{\sqrt{1-9x^2}}dx$
20. $\int \frac{-\pi}{3+27t^2}dt$

2.4 Integrals of Logarithmic Functions :

$$\because \frac{d}{du}\ln(u) = \frac{1}{u}\,du \xrightarrow{\int} \boxed{\int \frac{1}{u}du = \ln|u| + C},\ u \neq 0$$

Examples: Evaluate the following integrals:

1) $\int \frac{2}{x}dx$

$= 2\int \frac{1}{x}dx$

$= \boxed{2\ln|x| + C}$

2) $\int (\frac{3}{x^2} + \frac{5}{x})dx$

$= 3\int x^{-2}dx + 5\int \frac{1}{x}dx$

$= 3\frac{x^{-1}}{-1} + 5\ln|x| + C$

$= \boxed{\dfrac{-3}{x} + 5\ln|x| + C}$

3) $\int \frac{x}{(2x^2+3)}dx$

$= \frac{1}{4}\int \frac{4x}{(2x^2+3)}dx$

$= \boxed{\dfrac{1}{4}\ln|2x^2+3| + C}$

4) $\int \frac{\ln(x)}{x}dx$

$= \int \ln(x).\frac{1}{x}dx$

$= \boxed{\dfrac{(\ln(x))^2}{2} + C}$

5) $\int \frac{1}{x \ln x} dx$

$$= \int \frac{\frac{1}{x}}{\ln(x)} dx$$

$$= \boxed{\ln|\ln(x)| + C}$$

6) $\int \frac{e^x}{1+2e^x} dx$

$$= \frac{1}{2} \int \frac{2e^x}{1+2e^x} dx$$

$$= \boxed{\frac{1}{2} \ln|1 + 2e^x| + C}$$

7) $\int \frac{\sec^2(x)}{\tan(x)} dx$

$$= \boxed{\ln|\tan(x)| + C}$$

8) $\int \frac{\sec(2x)\tan(2x)}{\sec(2x)} dx$

$$= \frac{1}{2} \int \frac{2\sec(2x)\tan(2x)}{\sec(2x)} dx$$

$$= \boxed{\ln|\sec(2x)| + C}$$

9) $\int \tan(u) du$

$$= -\int \frac{-\sin(u)}{\cos(u)} du$$

$$= \boxed{-\ln|\cos(u)| + C}$$

10) $\int \cot(u) du$

$$= \int \frac{\cos(u)}{\sin(u)} du$$

$$= \boxed{\ln|\sin(u)| + C}$$

11) $\int \sec(u) du$

$$= \int \sec(u).\frac{\big(\sec(u)+\tan(u)\big)}{\big(\sec(u)+\tan(u)\big)}du$$
$$= \int \frac{\sec^2(u)+\sec(u)\tan(u)}{\tan(u)+\sec(u)}du$$
$$= \boxed{\ln\Big|\tan(u)+\sec(u)\Big|+C}$$

12) $\int \csc(u)du$

$$= \int \csc(u).\frac{\big(\csc(u)+\cot(u)\big)}{\big(\csc(u)+\cot(u)\big)}du$$
$$= -\int \csc(u).\frac{\big(-\csc(u)-\cot(u)\big)}{\big(\csc(u)+\cot(u)\big)}du$$
$$= -\int \frac{\csc^2(u)+\csc(u)\cot(u)}{\cot(u)+\csc(u)}du$$
$$= \boxed{-\ln\Big|\cot(u)+\csc(u)\Big|+C}$$

Problems (2.4): Evaluate the following integrals:

1. $\int \frac{1}{x-3}dx$

5. $\int \frac{\cos(x)}{\sin(x)}dx$

2. $\int \frac{dx}{x.\ln^5(x)}$

6. $\int \frac{\sin(x)}{2-\cos(x)}dx$

3. $\int \frac{x.dx}{4x^2+1}$

7. $\int \frac{x}{1-x^2}dx$

4. $\int \frac{2x-5}{x}dx$

8. $\int \frac{\ln^2(x)}{x}dx$

9. $\int \frac{5dx}{\sqrt{1-9x^2}}$

10. $\int \frac{x+10}{x^2}dx$

11. $\int \frac{y^2+2y+1}{(y+1)^3}dy$

12. $\int \frac{dx}{2-3x}$

13. $\int \frac{x^2}{4-x^3}dx$

14. $\int \frac{x}{x+1}dx$

15. $\int \frac{x^{\frac{3}{2}}}{4x^{\frac{5}{2}}} + 6dx$

16. $\int \frac{\sin(\theta)}{1+7\cos(\theta)}d\theta$

17. $\int \frac{ds}{\tan^{-1}(s)+s^2\tan^{-1}(s)}$

18. $\int \frac{\ln(x)}{4x\ln(2)}dx$

19. $\int \frac{w^2+2w-1}{w+4}dw$

20. $\int \frac{\ln(3)\cos(x)}{-5-\sin(x)}dx$

2.5 Integrals of General Logarithmic Function :

$$\because \tfrac{d}{du}\log_a(u) = \tfrac{1}{u.\ln(a)}\, du \xrightarrow{\int} \boxed{\boldsymbol{\int \frac{1}{u.\ln(a)}du = \log_a(u) + C}}$$

where $a > 0$ and $a \neq 1$ (i.e., $\ln(a) \neq 0$)

Examples: Evaluate the following integrals:

1) $\int \frac{x}{x^2 \ln 5}dx$

$$= \tfrac{1}{2}\int \tfrac{2x}{x^2 \ln 5}dx$$

$$= \boxed{\frac{1}{2}\log_5(x^2) + C}$$

2) $\int \frac{\cos(3t)}{\sin(3t)\ln 4}dt$

$$= \tfrac{1}{3}\int \tfrac{3\cos(3t)}{\sin(3t)\ln 4}dt$$

$$= \boxed{= \frac{1}{3}\log_4(\sin(3t)) + C}$$

3) $\int \frac{1}{(\sqrt{1-x^2})\sin^{-1}(x)\ln 3}dx$

$$= \boxed{\log_3(\sin^{-1}(x)) + C}$$

4) $\int (\sqrt{w} - \frac{1}{\sqrt{1-4w^2}\cos^{-1}(2w).\ln 3})dw$

$=$

Problems (2.5): Evaluate the following integrals:

1. $\frac{\cos(x)}{\sin(x)\ln 3}dx$

2. $\int \frac{1}{e^{2x}\ln 3}.e^{2x}dx$

3. $\int \frac{1}{e^{\sin(x)}\ln 7}.e^{\sin(x)}\cos(x)dx$

4. $\int \frac{\frac{1}{1+x^2}}{\tan^{-1}(x)\ \ln 11}dx$

5. $\int \frac{x^3}{x^4\ln 5}dx$

6. $\int \frac{1}{\cos^{-1}(w)\ln 2}\frac{1}{\sqrt{1-w^2}}dw$

7. $\int \frac{1}{\cot^{-1}(x)\ln 4}\cdot\frac{1}{1+x^2}dx$

8. $\int \frac{1}{2t\ln 3}dt$

9. $\int (\cos(4z)-\frac{1}{\sqrt{1-z^2}\sin^{-1}(z).\ln 2})dz$

10. $\int \frac{\sin(5w)}{\cos(5w)\ln 7}dw$

2.6 Integrals of Exponential Function :

$$\because \frac{d}{du}e^{(u)} = e^{(u)}du \xrightarrow{\int} \boxed{\int \boldsymbol{e^{(u)}du = e^{(u)} + C}}$$

Examples: Evaluate the following integrals:

1) $\int e^{2x}dx$

$= \frac{1}{2}\int 2e^{2x}dx$

$= \boxed{\frac{1}{2}e^{2x} + C}$

2) $\int e^{\sin(3x)}.\cos(3x)dx$

$= \frac{1}{3}\int 3.e^{\sin(3x)}.\cos(3x)dx$

$= \boxed{\frac{1}{3}e^{\sin(3x)} + C}$

3) $\int \frac{4}{e^{3x}}dx$

$= \int \frac{4}{-3}.(-3).e^{-3x}dx$

$= \boxed{\frac{4}{-3}e^{-3x} + C}$

4) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

$= \boxed{\ln|e^x + e^{-x}| + C}$

5) $\int e^{2x}\sin(e^{2x})dx$

$= \frac{1}{2}\int 2.e^{2x}\sin(e^{2x})dx$

$$= \boxed{-\cos(e^{2x}) + C}$$

6) $\int \frac{e^{\sin^{-1}(2x)}}{\sqrt{1-4x^2}}$

$$= \frac{1}{2}\int \frac{2.e^{\sin^{-1}(2x)}}{\sqrt{1-(2x)^2}}$$

$$= \boxed{e^{\sin^{-1}(2x)} + C}$$

7) $\int \frac{e^{3w}}{5-2e^{3w}} dw$

$$= \frac{1}{-6}\int \frac{-6e^{3w}}{5-2e^{3w}} dw$$

$$= \boxed{\frac{-1}{6}\ln|5-2e^{3w}| + C}$$

8) $\int \frac{e^{4t-\pi}}{e^{2t}} dt$

$=$

9) $\int \tan(e^{2x})e^{2x} dt$

$=$

10) $\int x^2 e^{2x^3} dt$

$=$

Problems (2.6): Evaluate the following integrals:

1. $\int xe^{x^2}dx$

2. $\int \frac{e^x}{1+2e^x}dx$

3. $\int (x+1)e^{x^2+2x}dx$

4. $\int (xe^{x^2} + e^{-3x}\sin(e^{-3x}))dx$

5. $(e^{5x}\cos(e^{5x}) - e^{\frac{x}{2}})dx$

6. $\frac{e^{\cot^{-1}(3x)}}{1+9x^2}dx$

7. $\frac{e^{7x}}{3-e^{7x}}dx$

8. $\int e^{\frac{x}{3}}\cos(e^{\frac{x}{3}})dx$

9. $\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx$

10. $\int \frac{e^{\tan^{-1}(2t)}}{1+4t^2}\,dt$

2.7 Integrals of general Exponential Function :

$$\because \frac{d}{du}a^{(u)} = a^{(u)}\ln(a)du \xrightarrow{\int} \boxed{\int a^{(u)}du = \frac{a^{(u)}}{\ln(a)} + C}$$

where $a > 0$ and $a \neq 1$ (i.e., $\ln(a) \neq 0$)

Examples: Evaluate the following integrals:

1) $\int 3^x dx$

$$= \boxed{\frac{3^x}{\ln 3} + C}$$

2) $\int 5^{2t-2}dx$

$$= \frac{1}{2}\int 5^{2t-2}2dx$$

$$= \boxed{\frac{1}{2}\frac{5^{2t-2}}{\ln 5} + C}$$

3) $\int \cos(\theta).4^{-\sin(\theta)}d\theta$

$$= -\int -\cos(\theta).4^{-\sin(\theta)}d\theta$$

$$= \boxed{\frac{4^{-\sin(\theta)}}{\ln 4} + C}$$

4) $\int 6^{\ln|\cos(x)|}.\tan(x)dx$

$$= -\int 6^{\ln|\cos(x)|}.\left(\frac{-\sin(x)}{\cos(x)}\right)dx$$

$$= \boxed{-\frac{6}{\ln 6} + C}$$

5) $\int 5^t \frac{2}{\sqrt{t}} dt$

$=$

Problems (2.7): Evaluate the following integrals:

1. $\int x\, 2^{-x^2} dx$

2. $\int 3^{\ln|\sin(\theta)|} \cot(\theta) d\theta$

3. $\int \frac{7^{\ln(z)}}{z} dz$

4. $\int 6^{\tan^{-1}(t)} \frac{1}{1+t^2} dt$

5. $\int 3^{w^2}\, 2w dw$

6. $\int 3^{\sqrt{t}} \frac{1}{\sqrt{t}} dt$

7. $\int 4^{\sin(2x)} \cos(2x) dx$

8. $\int (r+1) 2^{r^2+2r} dr$

9. $\int 4^{e^{3x}} e^{3x} dx$

10. $\int 9^{\ln|\cos z^2|} \tan(z^2) dz$

CHAPTER 3 : Techniques of Integration

3.1 Power of Trigonometric Functions

A. To find the integral for: $\boxed{\displaystyle\int \sin^n(x)\cos^m(x)}$

case(1) If one of the powers is odd (or both are odd), then we use the following form:

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

---

case(2) If both of the powers are even, then we use one of the following forms:

$$\boxed{\sin^2(x) = \frac{1}{2}(1 - \cos 2x)} \text{ or } \boxed{\cos^2(x) = \frac{1}{2}(1 + \cos 2x)}$$

Examples: Evaluate the following integrals:

1) $\int \sin^2(x)\cos^3(x)dx \qquad \Big[n : \text{odd} \;\&\; m : \text{even} \;\longrightarrow\; case(1)\Big]$

$$= \int \sin^2(x)\cos(x)\cos^2(x)dx$$

$$= \int \sin^2(x)\cos(x)\Big(1 - \sin^2(x)\Big)dx$$

$$= \int \underbrace{\sin^2(x)}_{u} \underbrace{\cos(x)dx}_{du} - \int \underbrace{\sin^4(x)}_{u} \underbrace{\cos(x)dx}_{du}$$

$$= \boxed{\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$$

2) $\boldsymbol{\int \sin^3(x)\cos^3(x)dx}$ $\quad \Big[n : \text{odd} \;\&\; m : \text{odd} \;\longrightarrow\; case(1)\Big]$

$$= \int \sin^3(x)\cos(x)\cos^2(x)dx$$

$$= \int \sin^3(x)\cos(x)\Big(1 - \sin^2(x)\Big)dx$$

$$= \int \underbrace{\sin^3(x)}_{u} \underbrace{\cos(x)dx}_{du} - \int \underbrace{\sin^5(x)}_{u} \underbrace{\cos(x)dx}_{du}$$

$$= \boxed{\frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6} + C}$$

3) $\boldsymbol{\int \sin^2(x)\cos^2(x)dx}$ $\quad \Big[n : \text{even} \;\&\; m : \text{even} \;\longrightarrow\; case(2)\Big]$

$$= \int \tfrac{1}{2}(1 - \cos(2x))\tfrac{1}{2}(1 + \cos(2x))dx$$

$$= \int \tfrac{1}{4}(1 - \cos^2(2x))dx$$

$$= \int \tfrac{1}{4}dx - \int \tfrac{1}{4}\cos^2(2x)dx$$

$$= \int \tfrac{1}{4}dx - \int \tfrac{1}{4}.\tfrac{1}{2}(1 + \cos(4x)dx$$

$$= \int \tfrac{1}{4}dx - \int \tfrac{1}{8}dx - \int \tfrac{1}{8}\cos(4x)dx$$

$$= \tfrac{1}{4}\int dx - \tfrac{1}{8}\int dx - \tfrac{1}{32}\int \cos(4x).4dx$$

$$= \boxed{\frac{x}{4} - \frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

B. To find the integral for: $\boxed{\int \tan^n(x)\sec^m(x)}$

case(1): If the powers $\underline{\sec(x)}$ is even, we divide the $\underline{\sec(x)}$ as:

$\boxed{\sec^m(x) = \cos^{m-2}(x)\sec^2(x)}$ & use $\boxed{\sec^2(x) = 1 + \tan^2(x)}$

---

case(2): If the powers $\underline{\tan(x)}$ is odd, we divide the $\underline{\tan(x)}$ as:

$\boxed{\tan^n(x) = \tan^{n-1}(x)\tan(x)}$ & use $\boxed{\tan^2(x) = \sec^2(x) - 1}$

Examples: Evaluate the following integrals:

1) $\int \sec^4(x)\tan^2(x)dx$

$$= \int \sec^2(x)\sec^2(x)\tan^2(x)dx$$

$$= \int \left(1 + \tan^2(x)\right)\sec^2(x)\tan^2(x)dx$$

$$= \int \underbrace{\tan^2(x)}_{u}\underbrace{\sec^2(x)dx}_{du} + \int \underbrace{\tan^4(x)}_{u}\underbrace{\sec^2(x)dx}_{du}\,dx$$

$$= \boxed{\frac{\tan^3(x)}{3} + \frac{\tan^5(x)}{5} + C}$$

2) $\int \tan^3(x)\sqrt{\sec(x)}dx$

$$= \int \tan^2(x)\tan(x)\sqrt{\sec(x)}dx$$

$$= \int \left(\sec^2(x) - 1\right)\tan(x)\sec^{\frac{1}{2}}(x)dx$$

$$= \int \sec^2(x)\tan(x)\sec^{\frac{1}{2}}(x)dx - \int \tan(x)\sec^{\frac{1}{2}}(x)dx$$

$$= \int \underbrace{\sec^{\frac{3}{2}}(x)}_{u}\underbrace{\sec(x)\tan(x)dx}_{du} - \int \underbrace{\sec^{\frac{-1}{2}}(x)}_{u}\underbrace{\sec(x)\tan(x)(x)dx}_{du}$$

$$= \boxed{\frac{\sec^{\frac{5}{2}}(x)}{\frac{5}{2}} - \frac{\sec^{\frac{1}{2}}(x)}{\frac{1}{2}} + C}$$

C. To find the integral for: $\boxed{\displaystyle\int \cot^n(x)\csc^m(x)}$

case(1) If the power of $\underline{\csc(x)}$ is even, we divide the $\underline{\csc(x)}$ as:

$\boxed{\csc^m(x) = \csc^{m-2}(x)\csc^2(x)}$ & we use $\boxed{\csc^2(x) = 1 + \cot^2(x)}$

---

case(2) If the power of $\underline{\cot(x)}$ is odd, we divide the $\underline{\cot(x)}$ as:

$\boxed{\cot^n(x) = \cot^{n-1}(x)\cot(x)}$ & we use $\boxed{\cot^2(x) = \csc^2(x) - 1}$

Examples: Evaluate the following integrals:

1) $\int \csc^4(x)\cot^2(x)dx$

$$= \int \csc^2(x)\csc^2(x)\cot^2(x)dx$$

$$= \int \left(1 + \cot^2(x)\right)\csc^2(x)\cot^2(x)dx$$

$$= \int \underbrace{\cot^2(x)}_{u}\underbrace{\csc^2(x)dx}_{du} + \int \underbrace{\cot^4(x)}_{u}\underbrace{\csc^2(x)dx}_{du}\,dx$$

$$= \boxed{-\frac{\cot^3(x)}{3} - \frac{\cot^5(x)}{5} + C}$$

2) $\displaystyle\int \frac{\cot^3(x)}{\sqrt{\csc(x)}}dx$

$$= \int \cot^2(x)\cot(x)\csc^{\frac{-1}{2}}(x)dx$$

$$= \int \Big(\csc^2(x) - 1\Big)\cot(x)\csc^{\frac{-1}{2}}(x)dx$$

$$= \int \csc^{\frac{3}{2}}(x)\cot(x)dx - \int \cot(x)\csc^{\frac{-1}{2}}(x)dx$$

$$= \int \underbrace{\csc^{\frac{1}{2}}(x)}_{u}\underbrace{\csc(x)\cot(x)dx}_{du} - \int \underbrace{\csc^{\frac{-3}{2}}(x)}_{u}\underbrace{\csc(x)\cot(x)(x)dx}_{du}$$

$$= \boxed{\frac{\csc^{\frac{3}{2}}(x)}{\frac{3}{2}} - \frac{\csc^{\frac{-1}{2}}(x)}{\frac{-1}{2}} + C}$$

Problems (3.1): Evaluate the following integrals:

1. $\int \cos^2(3\theta)\, d\theta$

2. $\int \sin^4(2x)\, dx$

3. $\int \cos^4(w)\, dw$

4. $\int \frac{3\tan^3(x)}{\sqrt{\sec(x)}}dx$

5. $\int \cos^3(2x)\, dx$

6. $\int \frac{\cos^3(t)}{\sin^2(t)}\, dt$

7. $\int \frac{\sin^3(x)}{\cos^2(x)}\, dx$

8. $\int \frac{\cos(x)}{(1+sin(x))^2}\, dx$

9. $\int \sin^3(\theta)\cos^2(\theta)\, d\theta$

10. $\int \sin^2(3x)\cos(3x)\, dz$

11. $\int \cos^{\frac{2}{3}}(3x)\sin^5(x)\,dx$

12. $\int \sin^2(w)\cos^4(w)\,dw$

13. $\int \frac{\cos(2t)}{\sin^4(t)}\,dt$

14. $\int \cot^3(\theta)\,dx$

15. $\int tan^2(4t)\,dt$

16. $\int \csc^4(x)\,dx$

17. $\int \csc^2(2z)\cot(2z)\,dz$

18. $\int \sec^4(3x)\tan(3x)\,dx$

19. $\int \frac{1}{\cos^2(w)}\,dw$

20. $\int \tan^3(x)\sec(x)\,dx$

3.2 Integration by Parts

$$\because d(u.v) = ud(v) + vd(u)$$

$$\implies ud(v) = d(u.v) - vd(u)$$

$$\xRightarrow{\int} \int ud(v) = \int d(u.v) - \int vd(u) \implies \boxed{\int ud(v) = u.v - \int vd(u)}$$

Examples: Evaluate the following integrals:

1) $\int \ln(x)dx = ?$

let $u = \ln(x) \implies du = \frac{dx}{x}$

$dv = dx \implies v = \int dx = x$

$\because \int ud(v) = u.v - \int vd(u)$

$\implies \int \ln(x)dx = \ln(x).x - \int x.\frac{dx}{x}$

$= \ln(x).x - \int dx$

$= \boxed{\ln(x).x - x + C}$

2) $\int xe^x dx = ?$

let $u = x \implies du = dx$

$dv = e^x dx \implies v = \int e^x dx = e^x$

$\because \int ud(v) = u.v - \int vd(u)$

$\implies \int xe^x dx = x.e^x - \int e^x dx$

$= \boxed{xe^x - e^x + C}$

3) $\int \boldsymbol{x^2 e^x dx} = ?$

let $u = x^2 \implies du = 2xdx$

$dv = e^x dx \implies v = \int e^x dx = e^x$

$\because \int ud(v) = u.v - \int vd(u)$

$\implies \int x^2 e^x dx = x^2.e^x - \int e^x.2xdx$

$= x^2.e^x - 2\int e^x.xdx$

$= x^2.e^x - 2(xe^x - e^x + C)$

$= \boxed{x^2.e^x - 2xe^x + 2e^x + C}$

4) $\int \boldsymbol{e^x \sin(2x) dx} = ?$

let $u = e^x \implies du = e^x dx$

$dv = \sin(2x)dx \implies v = \frac{1}{2}\int \sin(2x).2dx = \frac{-1}{2}\cos(2x)$

$\because \int ud(v) = u.v - \int vd(u)$

$\implies \int e^x \sin(2x)dx = e^x.\frac{-1}{2}\cos(2x) - \int \frac{-1}{2}\cos(2x).e^x dx$

$\implies \int e^x \sin(2x)dx = e^x.\frac{-1}{2}\cos(2x) + \frac{1}{2}\underbrace{\int \cos(2x).e^x\, dx}_{u.dv}$

let $u = e^x \implies du = e^x dx$

$dv = \cos(2x)dx \implies v = \frac{1}{2}\sin(2x)$

$\therefore \int \cos(2x).e^x = e^x.\frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)e^x dx$

$\implies \int e^x \sin(2x)dx = e^x.\frac{-1}{2}\cos(2x) + \frac{1}{2}[e^x.\frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)e^x dx]$

$$\Longrightarrow \underbrace{\int e^x \sin(2x)dx} = e^x.\tfrac{-1}{2}\cos(2x) + \tfrac{1}{4}e^x.\sin(2x) - \tfrac{1}{4}\underbrace{\int e^x \sin(2x)dx}$$

$$\Longrightarrow \tfrac{5}{4}\int e^x \sin(2x)dx = \tfrac{-1}{2}e^x.\cos(2x) + \tfrac{1}{4}e^x.\sin(2x)$$

$$\overset{*\frac{4}{5}}{\Longrightarrow} \int e^x \sin(2x)dx = \boxed{\frac{-2}{5}e^x.\cos(2x) + \frac{1}{5}e^x.\sin(2x) + C}$$

5) $\boldsymbol{\int \frac{x^3}{(x^2+1)^{\frac{3}{2}}}dx = ?}$

$$\int \frac{x^3}{(x^2+1)^{\frac{3}{2}}}dx = \int \underbrace{x^2}_{u} \underbrace{\frac{x}{(x^2+1)^{\frac{3}{2}}}dx}_{dv}$$

let $u = x^2 \Longrightarrow du = 2xdx$

$dv = \frac{x}{(x^2+1)^{\frac{3}{2}}}dx$

$$\Longrightarrow v = \int \frac{x}{(x^2+1)^{\frac{3}{2}}}dx = \int x.(x^2+1)^{\frac{-3}{2}}dx$$

$$= \tfrac{1}{2}\int 2x.(x^2+1)^{\frac{-3}{2}}dx = \tfrac{1}{2}\frac{(x^2+1)^{\frac{-1}{2}}}{\frac{-1}{2}} = \frac{-1}{(x^2+1)^{\frac{1}{2}}}$$

$$\because \int ud(v) = u.v - \int vd(u)$$

$$\Longrightarrow \frac{x^3}{(x^2+1)^{\frac{3}{2}}}dx = x^2.\frac{-1}{(x^2+1)^{\frac{1}{2}}} - \int \frac{-1}{(x^2+1)^{\frac{1}{2}}}.2xdx$$

$$= \frac{-x^2}{(x^2+1)^{\frac{1}{2}}} - \int (x^2+1)^{\frac{-1}{2}}.2xdx$$

$$= \boxed{\frac{-x^2}{(x^2+1)^{\frac{1}{2}}} + \frac{(x^2+1)^{\frac{1}{2}}}{\frac{1}{2}} + C}$$

6) $\int \sec^3(x)dx = ?$

$$\int \sec^3(x)dx = \int \underbrace{\sec(x)}_{u}\underbrace{\sec^2(x)dx}_{dv}$$

$$\text{let } u = \sec(x) \implies du = \sec(x)\tan(x)dx$$

$$dv = \sec^2(x)dx \implies v = \int \sec^2(x)dx = \tan(x)$$

$$\because \int ud(v) = u.v - \int vd(u)$$

$$\implies \int \sec^3(x)dx = \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)dx$$

$$= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx$$

$$= \sec(x)\tan(x) - \int [\sec^2(x) - 1]\sec(x)dx$$

$$\implies \int \sec^3(x)dx = \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx$$

$$\implies 2\int \sec^3(x)dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$$

$$\implies \int \sec^3(x)dx = \boxed{\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + K}$$

Problems (3.2): Evaluate the following integrals:

1. $\int e^{\ln\sqrt{x}}\,dx$
2. $\int \ln\sqrt{x-1}\,dx$
3. $\int e^{x}\cos(2x)\,dx$
4. $\int x\sec^2(x)\,dx$
5. $\int \ln(x+\sqrt{1+x^2})\,dx$
6. $\int x\ \tan^2(x)\,dx$
7. $\int \frac{\tan^{-1}(x)}{x^2}\,dx$
8. $\int x\ \cos^2(x)\,dx$
9. $\int x^2\sin(x)\,dx$
10. $\int z\ \sin^2(z)\,dz$
11. $\int x\ \ln\sqrt{x+2}\,dx$
12. $\int (x+1)^2\,e^{x}\,dx$
13. $\int e^{2t}\cos(e^{t})\,dt$
14. $\int x\ \ln(x^3+x)\,dx$
15. $\int x^3\,e^{x^2}\,dx$
16. $\int x^2\sin(1-x)\,dx$
17. $\int z\ \sin^2(z)\,dz$
18. $\int x\ \ln(\sqrt[3]{3x+1})\,dx$
19. $\int \cos(\ln x)\,dx$
20. $\int e^{2x}\ \sin(3x)\,dx$
21. $\int\ x\sin(x)\,dx$
22. $\int\ \theta\cos(3\theta)\,d\theta$
23. $\int\ x\sqrt{4x+2}\,dx$
24. $\int\ x\tan^{-1}x\,dx$

3.3 Integration by Partial Fractions:

If the integrand (a function that is to be integrated/the expression after the integral sign) is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, the fraction needs to be expressed in partial fractions before integration takes place.

Remark(1):

IF the "degree of denominator" $>$ "degree of enumerator", then we use the partial fraction as follows:

i) If we have $\boldsymbol{\frac{1}{ax^2+bx+c}}$, and we are unable to analyze it, then its numerator will be always one degree less.

Examples:

- $\frac{1}{x^2+3x+2} =$
- $\frac{1}{(x-3)(x^2+3x+2)} =$
- $\frac{1}{(x-3)(x^2+x+1)} =$
- $\frac{1}{(x-3)(x^3+x^2+2)} =$
- $\frac{1}{(x-3)(x^2+5)(x^4+3)} =$

ii) If we have $\frac{\mathbf{1}}{\boldsymbol{(x-a)^n}}$ where n is a positive constant, then we analyze it as follows:

$$\boldsymbol{\frac{1}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_n}{(x-a)^n}}$$

Examples:

- $\frac{1}{(x-3)^2} =$
- $\frac{1}{(x-3)^3} =$
- $\frac{1}{(x-3)^4} =$

Examples:

1) $\boldsymbol{\int \frac{1}{(x^2-2x-3)} dx = ?}$

$$\frac{1}{(x^2-2x-3)} = \frac{1}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$$

$$= \frac{Ax+A+Bx-3B}{(x-3)(x+1)} = \frac{(A+B)x+(A-3B)}{(x-3)(x+1)}$$

$$\Longrightarrow \ A+B=0$$

$$\mp A \pm 3B = \mp 1$$

(by subtracting)

$$\Longrightarrow \ 4B=-1 \ \Longrightarrow \ \boxed{B=\frac{-1}{4}}$$

$$\because A+B=0 \ \Longrightarrow \ A=-B \ \Longrightarrow \ \boxed{A=\frac{1}{4}}$$

$$\frac{1}{(x^2-2x-3)} = \frac{A}{(x-3)} + \frac{B}{(x+1)} = \frac{\frac{1}{4}}{(x-3)} + \frac{\frac{-1}{4}}{(x+1)}$$

$$\therefore \int \frac{1}{(x^2-2x-3)}dx$$

$$= \int \frac{\frac{1}{4}}{(x-3)} + \frac{\frac{-1}{4}}{(x+1)}dx$$

$$= \frac{1}{4}\int \frac{1}{(x-3)}dx + \frac{-1}{4}\int \frac{1}{(x+1)}dx$$

$$= \boxed{\frac{1}{4}\ln|x-3| + \frac{-1}{4}\ln|x+1| + C}$$

2) $\int \frac{1}{x^2(x-2)}dx = ?$

$$\frac{1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$\xrightarrow{*x^2(x-2)} 1 = Ax(x-2) + B(x-2) + Cx^2$$

Let $x = 0 \implies \boxed{B = \frac{-1}{2}}$

Let $x = 2 \implies \boxed{C = \frac{1}{4}}$

Comparing coefficients of x^2 gives:

$A + C = 0 \implies \boxed{A = \frac{-1}{4}}$

$$\therefore \int \frac{1}{x^2(x-2)}dx = \int \left(\frac{\frac{-1}{4}}{x} + \frac{\frac{-1}{2}}{x^2} + \frac{\frac{1}{4}}{x-2}\right)dx$$

$$= \frac{-1}{4}\int \frac{1}{x}dx + \frac{-1}{2}\int \frac{1}{x^2}dx + \frac{1}{4}\int \frac{1}{x-2}dx$$

$$= \boxed{\frac{-1}{4}\ln|x| + \frac{-1}{2}\frac{x^{-1}}{-1} + \frac{1}{4}\ln|x-2| + C}$$

3) $\int \frac{1}{x^2+3x-4} dx = ?$

$=$

4) $\int \frac{x^2}{(2x+1)(x+2)^2} dx = ?$

$$\frac{x^2}{(2x+1)(x+2)^2} = \frac{A}{(2x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$\xrightarrow{*(2x+1)(x+2)^2} x^2 = A(x+2)^2 + B(2x+1)(x+2) + C(2x+1)$$

$$\text{Let } x = -2 \implies \boxed{C = \frac{-4}{3}}$$

Let $x = \frac{-1}{2} \implies \boxed{A = \dfrac{1}{9}}$

Comparing coefficients of x^2 gives:

$A + 2B = 1 \implies \dfrac{1}{9} + 2B = 1 \implies \boxed{B = \dfrac{4}{9}}$

$\therefore \int \frac{x^2}{(2x+1)(x+2)^2} dx$

$= \int \left(\frac{\frac{1}{9}}{(2x+1)} + \frac{\frac{4}{9}}{(x+2)} + \frac{\frac{-4}{3}}{(x+2)^2} \right) dx$

$= \frac{1}{9} \int \frac{1}{(2x+1)} dx + \frac{4}{9} \int \frac{1}{(x+2)} dx + \frac{-4}{3} \int \frac{1}{(x+2)^2} dx$

$= \frac{1}{18} \int \frac{2}{(2x+1)} dx + \frac{4}{9} \int \frac{1}{(x+2)} dx + \frac{-4}{3} \int (x+2)^{-2} dx$

$= \boxed{\dfrac{1}{18} \ln(2x+1) + \dfrac{4}{9} \ln(x+2) + \dfrac{-4}{3} . \dfrac{(x+2)^{-1}}{-1} + C}$

Remark(2):

IF the "degree of enumerator" $\geq$ "degree of denominator", then we use the long division as follows:

- $\frac{x+4}{x+8} = 1 + \frac{-4}{x+8}$
- $\frac{x^2+4}{x^2+8} = 1 + \frac{-4}{x^2+8}$
- $\frac{x^2+4}{x+8} = (x-8) + \frac{68}{x+8}$
- $\frac{x^3+4}{x+8} = (x^2 - 4x + 32) + \frac{252}{x+8}$

Examples:

1) $\int \frac{x-3}{x+5}dx = ?$

$=$

2) $\int \frac{5-x^2}{16+x^2}dx = ?$

$=$

3) $\int \frac{x^2-1}{x^2+9}dx = ?$

$=$

4) $\int \frac{x^2+4}{x+8}dx = ?$

$\because \frac{x^2+4}{x+8} = (x-8) + \frac{68}{x+8}$

$$\int \frac{x^2+4}{x+8}dx = \int \left((x-8) + \frac{68}{x+8}\right)dx$$

$$= \int \left((x-8) + \frac{68}{x+8}\right)dx$$

$$= \int (x-8)dx + \int \frac{68}{x+8}dx$$

$$= \boxed{\frac{x^2}{2} - 8x + 68\ln\left|x+8\right| + C}$$

5) $\int \frac{x^3+4}{x+8} dx = ?$

Problems (3.3): Evaluate the following integrals:

1. $\int \frac{x}{x^2+4x-5}\, dx$

2. $\int \frac{x}{x^2-2x-3}\, dx$

3. $\int \frac{(x+1)}{x^2+4x-5}\, dx$

4. $\int \frac{x^2}{x^2+2x-1}\, dx$

5. $\int \frac{1}{x(x+1)^2}\, dx$

6. $\int \frac{1}{(x+1)(x^2+1)}\, dx$

7. $\int \frac{1}{x(x^2+2x+1)}\, dx$

8. $\int \frac{\sin\theta}{\cos^2\theta+\cos\theta-2}\, d\theta$

9. $\int \frac{e^t}{e^{2t}+3e^t+2}\, dt$

10. $\int \frac{(3x-7)}{(x-1)(x-2)(x-3)}\, dx$

11. $\int \frac{\sin(x)}{\cos^2(x)-5\cos(x)+4}\, dx$

12. $\int \frac{1}{x^2-6x+5}\, dx$

13. $\int \frac{x}{(x+2)(x+1)^2}\,dx$

14. $\int \frac{2x-1}{(x^2-1)(x+2)}\,dx$

15. $\int \frac{x-1}{(x+1)(x^2+1)}\,dx$

16. $\int \frac{5x-3}{x^2-2x-3}\,dx$

17. $\int \frac{\pi^2}{x^2-9}\,dx$

18. $\int \frac{1}{(x-1)^2}\,dx$

19. $\int \frac{x}{x^2-4x-5}\,dx$

20. $\int \frac{(2x+41)}{x^2+5x-14}\,dx$

21. $\int \frac{x}{(x-1)(x+1)^2}\,dx$

22. $\int \frac{2x^2+3}{x^2-1}\,dx$

23. $\int \frac{8}{x^4+2x^3}\,dx$

24. $\int \frac{x}{x^4-16}\,dx$

25. $\int \frac{3x^3+3x^2+3x+2}{x^3(x+1)}\,dx$

26. $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}\,dx$

27. $\int \frac{\sec^2(t)}{\sec^2(t)-3\tan(t)+1}\,dt$

28. $\int \frac{3}{x^2-4}\,dx$

29. $\int \frac{x^2+20}{x^2-16}\,dx$

30. $\int \frac{\pi}{x^2-4x-12}\,dx$

31. $\int \frac{2x+3}{x+4}\,dx$

32. $\int \frac{x^2+3}{x^2+4}\,dx$

33. $\int \frac{x^3+x^2+3}{x^2+4}\,dx$

34. $\int \frac{x^2+20}{x^2-16}\,dx$

35. $\int \frac{6x+2}{x-1}\,dx$

36. $\int \frac{3+x^2}{4+x^2}\,dx$

37. $\int \frac{3+5x}{3+x}\,dx$

38. $\int \frac{x^2+x+1}{x^2+1}\,dx$

39. $\int \frac{3x^2-5}{x-2}\,dx$

40. $\int \frac{3x^2-10}{x^2-4x+4}\,dx$

3.4 Integration by Trigonometric Substitution:

In this section, we see how to integrate expressions like

$$\boxed{\sqrt{a^2 - u^2}}, \boxed{\sqrt{a^2 + u^2}}, \text{ and } \boxed{\sqrt{u^2 - a^2}}$$

depending on the function we need to integrate, we substitute one of the following trigonometric expressions to simplify the integration:

- For $\boxed{\sqrt{\boldsymbol{a^2 - u^2}}}$, use $\boxed{\boldsymbol{u = a \sin \theta}}$
- For $\boxed{\sqrt{\boldsymbol{a^2 + u^2}}}$, use $\boxed{\boldsymbol{u = a \tan \theta}}$
- For $\boxed{\sqrt{\boldsymbol{u^2 - a^2}}}$, use $\boxed{\boldsymbol{u = a \sec \theta}}$

Examples: Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{9-x^2}} = ?$

$\because \sqrt{9 - x^2} \equiv \sqrt{a^2 - u^2} \implies$ we use $\boxed{u = a \sin \theta}$

$\because a = 3 \text{ and } u = x \implies \boxed{x = 3 \sin \theta}$

$\implies \boxed{\theta = \sin^{-1}(\frac{x}{3})}$ and $\boxed{dx = 3 \cos \theta d\theta}$

$\therefore \sqrt{9 - x^2} = \sqrt{9 - (3 \sin \theta)^2} = \sqrt{9 - 9 \sin^2 \theta}$

$= \sqrt{9(1 - \sin^2 \theta)} = 3\sqrt{1 - \sin^2 \theta} = 3\sqrt{\cos^2 \theta} = \boxed{3 \cos \theta}$

$\therefore \int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3 \cos \theta d\theta}{3 \cos \theta} = \int d\theta = \theta + C = \boxed{\sin^{-1}(\frac{x}{3}) + C}$

2) $\int \frac{dx}{\sqrt{25+x^2}} = ?$

$$\because \sqrt{25+x^2} \equiv \sqrt{a^2+u^2} \implies \text{we use } \boxed{u = a\tan\theta}$$

$$\because a = 5 \text{ and } u = x \implies \boxed{x = 5\tan\theta}$$

$$\implies \boxed{\theta = \tan^{-1}(\frac{x}{5})} \text{ and } \boxed{dx = 5\sec\theta d\theta}$$

$$\therefore \sqrt{25+x^2} = \sqrt{25+(5\tan\theta)^2} = \sqrt{25+25\tan^2\theta}$$

$$= \sqrt{25(1+\tan^2\theta)} = 5\sqrt{1+\tan^2\theta} = 5\sqrt{\sec^2\theta} = \boxed{5\sec\theta}$$

$$\therefore \int \frac{dx}{\sqrt{25+x^2}} = \int \frac{5\sec^2\theta d\theta}{5\sec\theta} = \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C = \boxed{\ln|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}| + C}$$

3) $\int \frac{dx}{x\sqrt{x^2-16}} = ?$

$$\because \sqrt{x^2-16} \equiv \sqrt{u^2-a^2} \implies \text{we use } \boxed{u = a\sec\theta}$$

$$\because a = 4 \text{ and } u = x \implies \boxed{x = 4\sec\theta}$$

$$\implies \boxed{\theta = \sec^{-1}(\frac{x}{4})} \text{ and } \boxed{dx = 4\sec\theta\tan\theta d\theta}$$

$$\therefore \sqrt{x^2-16} = \sqrt{(4\sec\theta)^2-16} = \sqrt{16\sec^2\theta-16}$$

$$= \sqrt{16(\sec^2\theta-1)} = 4\sqrt{\sec^2\theta-1} = 4\sqrt{\tan^2\theta} = \boxed{4\tan\theta}$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-16}} = \int \frac{4\sec\theta\tan\theta d\theta}{4\sec\theta * 4\tan\theta} = \frac{1}{4}\int d\theta = \frac{1}{4}\theta + C = \boxed{\frac{1}{4}\sec^{-1}(\frac{x}{4}) + C}$$

Problems (3.4): Evaluate the following integrals:

1. $\int \frac{dx}{\sqrt{1-4x^2}}$
2. $\int \frac{dx}{\sqrt{4+x^2}}$
3. $\int \frac{dx}{\sqrt{4-(x-1)^2}}$
4. $\int \frac{xdx}{4+x^2}$
5. $\int \frac{dx}{x\sqrt{25+x^2}}$
6. $\int \frac{(x+1)dx}{\sqrt{4-x^2}}$
7. $\int \frac{dx}{\sqrt{2-5x^2}}$
8. $\int \frac{\sin\theta dx}{\sqrt{2-\cos^2\theta}}$
9. $\int \frac{dx}{x\sqrt{36-x^2}}$
10. $\int \frac{dx}{\sqrt{(9-x^2)^{\frac{3}{2}}}}$
11. $\int \frac{dx}{\sqrt{81+x^2}}$
12. $\int \frac{dx}{(x^2-25)^{\frac{3}{2}}}$
13. $\int \sqrt{x^2+4}$
14. $\int \frac{dx}{\sqrt{9+x^2}}$
15. $\int \frac{x^2dx}{\sqrt{4-x^2}}$
16. $\int \frac{xdx}{\sqrt{16-x^2}}$
17. $\int \frac{xdx}{\sqrt{(x^2-1)^{\frac{3}{2}}}}$
18. $\int \sqrt{49-x^2}$
19. $\int \frac{dx}{\sqrt{(x^2+1)^2}}$
20. $\int \frac{(2x^2+3)dx}{\sqrt{(x^2+1)^2}}$

3.5 Integration by Completing the Square:

By completing the square, we may rewrite any quadratic polynomial

$$ax^2 + bx + c$$

in the form

$$(x + k_1)^2 + k_2$$

where k_1 and k_2 may be positive or negative.

Remark(1): After completing the square, we either integrate directly or use the trigonometric substitution.

Remark(2): We use the completing the square method, if

1. We are unable to analyze expression $ax^2 + bx + c$

2. We have the expression $ax^2 + bx$

3. We have the expression ax^2+bx+c or ax^2+bx under roots ($\sqrt{\ldots}$, $\sqrt[3]{\ldots}$, *etc*).

Examples: Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{2x-x^2}}$ =?

$$\sqrt{2x - x^2} = \sqrt{-(x^2 - 2x)} = \sqrt{-(x^2 - 2x + 1 - 1)}$$

$$= \sqrt{-((x-1)^2 - 1)} = \sqrt{1 - (x-1)^2}$$

$$\therefore \int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1 - (\underbrace{x-1}_{u})^2}}$$

$$= \boxed{-\cos(x-1) + C}$$

2) $\int \frac{dx}{4x^2 + 4x + 2} = ?$

$$4x^2 + 4x + 2 = 4(x^2 + x + \tfrac{1}{2})$$

$$= 4(x^2 + x + \underbrace{\frac{1}{4} - \frac{1}{4}}_{=0} + \tfrac{1}{2}) = 4((x + \tfrac{1}{2})^2 + \tfrac{1}{4}) = 4(x + \tfrac{1}{2})^2 + 1$$

$$\therefore \int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{4(x + \frac{1}{2})^2 + 1}$$

$$= \frac{1}{2} \int \frac{2dx}{1 + (2x + \frac{1}{2})^2}$$

$$= \boxed{\frac{1}{2} \tan^{-1}(2x + \frac{1}{2}) + C} \text{ or } \boxed{\frac{-1}{2} \cot^{-1}(2x + \frac{1}{2}) + C}$$

3) $\int \frac{dx}{\sqrt{21 - 4x - x^2}} = ?$

$$\sqrt{21 - 4x - x^2} = \sqrt{21 - (x^2 + 4x + 4 - 4)} = \sqrt{25 - (x+2)^2}$$

$$\therefore \int \frac{dx}{\sqrt{21 - 4x - x^2}} = \int \frac{dx}{\sqrt{25 - (x+2)^2}} = \int \frac{dx}{5\sqrt{1 - (\frac{x+2}{5})^2}}$$

$$= \boxed{\sin^{-1}(\frac{x+2}{5}) + C}$$

4) $\int \frac{(x+1)dx}{\sqrt{x^2-6x+4}} = ?$

$$\sqrt{x^2-6x+4} = \sqrt{x^2-6x+9-9+4}$$

$$= \sqrt{(x-3)^2-5} \equiv \sqrt{u^2-a^2}$$

$u = x-3 \longrightarrow du = dx$ and $a = \sqrt{5}$

we will use $u = a\sec\theta$

$\longrightarrow x-3 = \sqrt{5}\sec\theta$

$\longrightarrow x = \sqrt{5}\sec\theta + 3 \longrightarrow dx = \sqrt{5}\sec\theta\tan\theta d\theta$

$$\therefore \int \frac{(x+1)dx}{\sqrt{x^2-6x+4}} = \frac{(\sqrt{5}\sec\theta+3+1)(\sqrt{5}\sec\theta\tan\theta d\theta)}{\sqrt{\sqrt{5}\sec\theta-5}}$$

$$= \int \frac{5\sec^2\theta\tan\theta+4\sqrt{5}\sec\theta\tan\theta d\theta}{\sqrt{5\underbrace{(\sec^2\theta-1)}_{\tan^2\theta}}}$$

$$= \int \frac{5\sec^2\theta\tan\theta+4\sqrt{5}\sec\theta\tan\theta d\theta}{\sqrt{5}\tan\theta}$$

$$= \int \sqrt{5}\sec^2\theta d\theta + \int 4\sec\theta\tan d\theta$$

$$= \sqrt{5}\tan\theta + 4\ln\left|\sec\theta+\tan\theta\right| + C$$

$$= \boxed{\sqrt{5}\frac{\sqrt{(x-3)^2-5}}{\sqrt{5}} + 4\ln\left|\frac{(x-3)}{\sqrt{5}} + \frac{\sqrt{(x-3)^2-5}}{\sqrt{5}}\right| + C}$$

Problems (3.5): Evaluate the following integrals:

1. $\int \frac{dx}{x^2-2x+5}$

2. $\int \frac{x\,dx}{\sqrt{x^2-2x+5}}$

3. $\int \frac{(x+1)dx}{\sqrt{2x-x^2}}$

4. $\int \frac{(x-1)dx}{\sqrt{x^2-4x+3}}$

5. $\int \frac{x\,dx}{\sqrt{5+4x-x^2}}$

6. $\int \frac{dx}{\sqrt{x^2-2x-8}}$

7. $\int \frac{(1-x)dx}{\sqrt{8+2x-x^2}}$

8. $\int \frac{x\,dx}{\sqrt{x^2+4x+5}}$

9. $\int \frac{x\,dx}{x^2+4x+5}$

10. $\int \frac{(2x+3)\,dx}{4x^2+4x+5}$

11. $\int \frac{dx}{2x^2-12x+26}$

12. $\int \frac{dx}{\sqrt{21-4x-x^2}}$

13. $\int \frac{dx}{\sqrt{6+5x-x^2}}$

14. $\int \frac{dx}{x^2+2x+5}$

CHAPTER 4 : The Definite Integrals and Its Application

4.1 Definite Integrals:

If $a, b \in \mathbb{R}$ and $F(x)$ is an anti-derivative for $f(x)$, then:

$$\boxed{\int_a^b f(x) = F(x)\Big]_a^b = F(b) - F(a)}$$

a is called lower limit, and b is called upper limit for the integral.

Properties for Definite Integrals:

1. $\int_a^b (k_1 f \mp k_2 g) = k_1 \int_a^b f \mp k_2 \int_a^b g$

2. $\int_a^b f = \int_a^c f + \int_c^b f$, where $c \in [a, b]$

3. $\int_a^b f = -\int_b^a f$

4. $\int_a^a f = 0$

5. If f is an EVEN function $\Longrightarrow \int_{-a}^{a} f = 2\int_0^a f$

6. If f is an ODD function $\Longrightarrow \int_{-a}^{a} f = 0$

Examples: Evaluate the following integrals:

1) $\int_1^6 (3x^2 + 2x)dx$

$$= (3\frac{x^3}{3} + 2\frac{x^2}{2})\Big]_1^6$$

$$= (x^3 + x^2)\Big]_1^6$$

$$= (6^3 + 6^2) - (1^3 + 1^2) = 252 - 2 = \boxed{250}$$

2) $\int_0^2 \sqrt{4x+1}dx$

$$= \frac{1}{4}\frac{(4x+1)^{\frac{3}{2}}}{\frac{3}{2}}\Big]_0^2$$

$$= \frac{1}{6}\sqrt{(4x+1)^3}\Big]_0^2$$

$$= \frac{1}{6}\left(\sqrt{(4*2+1)^3} - \sqrt{(4*0+1)^3}\right) = \frac{1}{6}(27-1) = \frac{26}{6} = \boxed{\frac{13}{3}}$$

3) $\int_0^{\pi} \sin(x)dx$

$$= -\cos(x)\Big]_0^{\pi}$$

$$= -\cos(\pi) - \cos(0) = -(-1) + 1 = \boxed{2}$$

4) $\int_0^{\frac{\pi}{4}} \csc^2(x)dx$

$$= \cot(x)\Big]_0^{\frac{\pi}{4}} = (\cot(\frac{\pi}{4}) - \cot(0))$$

$$= 1 - \infty = \boxed{-\infty}$$

5) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos(x)}{\sin^3 x)}dx$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(x)\sin^{-3}(x)dx$$

$$= \frac{\sin^{-2}(x)}{-2}\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{-1}{2}\left[\frac{1}{\sin^2(x)}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{-1}{2}\left[\frac{1}{\sin^2(\frac{\pi}{2})} - \frac{1}{\sin^2(\frac{\pi}{4})}\right]$$

$$= \frac{-1}{2}\left[\frac{1}{1} - \frac{1}{(\frac{1}{\sqrt{2}})^2}\right] = \frac{-1}{2}[1-2] = \boxed{\frac{1}{2}}$$

Problems (4.1): Evaluate the following integrals:

1. $\int_{-1}^{1}(x+1)^2 dx$

2. $\int_{0}^{1}(x^2-2x+3)dx$

3. $\int_{1}^{2}(2w+5)dw$

4. $\int_{1}^{2}(3-\frac{6}{x^2})dx$

5. $\int_{0}^{1}\sqrt{1+x}dx$

6. $\int_{1}^{2}\frac{1}{t}dt$

7. $\int_{1}^{2}\frac{2t}{t^2+1}dt$

8. $\int_{-1}^{3}(x^2-1)dx$

9. $\int_{0}^{\frac{\pi}{4}}\sec^2(x)dx$

10. $\int_{\frac{\pi}{3}}^{\frac{\pi}{4}}(1+\tan^2(\theta))d\theta$

11. $\int_{1}^{3}5e^{2z}dz$

12. $\int_{0}^{\pi}\cos(\frac{w}{2})dw$

13. $\int_0^{\pi} \sin^2(\theta)d\theta$

14. $\int_0^{\frac{2\pi}{w}} \cos^2(wt)dt$

15. $\int_0^1 \frac{1}{(2x+1)^3}dx$

16. $\int_0^{\frac{\pi}{6}} \frac{\sin(2z)}{\cos^2(2z)}dz$

17. $\int_0^5 2^x dx$

18. $\int_{-\pi}^0 \cos^2(2\theta)d\theta$

19. $\int_0^4 \pi e^{3x}dx$

20. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cot\theta d\theta$

21. $\int_0^{\frac{\pi}{2}} \csc(\theta)\cot(\theta)d\theta$

22. $\int\limits_0^{\frac{\pi}{3}} x\sin(x)\, dx$

23. $\int\limits_0^{\frac{\pi}{2}} \theta\cos(3\theta)\, d\theta$

24. $\int\limits_0^{3} x\sqrt{4x+2}\, dx$

25. $\int\limits_0^{\frac{1}{2}} x\tan^{-1} x\, dx$

26. $\int\limits_0^{3} x\sqrt{4x+2}\, dx$

4.2 Multiple Integrals:

The multiple integral is a definite integral of a function of more than one real variable, for example, $f(x, y)$ or $f(x, y, z)$. Integrals of a function of two variables over a region in $\mathbb{R}^2$ are called **double** integrals, and integrals of a function of three variables over a region of $\mathbb{R}^3$ are called **triple** integrals. i.e.,

$$\int_y \int_x f(x, y)\, dx\, dy$$

$$\int_z \int_y \int_x f(x, y, z)\, dx\, dy\, dz$$

Examples: Evaluate the following integrals:

1) $\int_0^\pi \int_0^x \sin(y)\, dy\, dx$

$= \int_0^\pi -\cos(y)\Big]_0^x dx$

$= \int_0^\pi (-\cos(x) + 1) dx$

$= (-\sin(x) + x)\Big]_0^\pi$

$= (-\sin(\pi) + \pi) - (-\sin(0) + 0)$

$= (-0 + \pi) - (0 + 0) = \boxed{\pi}$

2) $\int_0^1 \int_{\frac{-1}{2}}^0 \int_{x-y}^{x+y+1} dz\, dy\, dx$

$$= \int_0^1 \int_{\frac{-1}{2}}^{0} z\Big]_{x-y}^{x+y+1} \, dy \, dx$$

$$= \int_0^1 \int_{\frac{-1}{2}}^{0} \Big((x+y+1)-(x-y)\Big) \, dy \, dx$$

$$= \int_0^1 \int_{\frac{-1}{2}}^{0} \Big(1+2y\Big) \, dy \, dx$$

$$= \int_0^1 (y+y^2)\Big]_{\frac{-1}{2}}^{0} \, dx$$

$$= \int_0^1 \Big[0-\Big(\tfrac{-1}{2}+(\tfrac{-1}{2})^2\Big)\Big] \, dx$$

$$= \int_0^1 \tfrac{1}{4} \, dx$$

$$= \tfrac{1}{4}x\Big]_0^1$$

$$= \tfrac{1}{4}(1-0) = \boxed{\frac{1}{4}}$$

Problems (4.2): Evaluate the following integrals:

1. $\int_0^{\pi} \int_{-1}^{0} \frac{2}{\pi} \, dy \, dx$

2. $\int_{-1}^{1} \int_0^2 (1-6x^2y) \, dx \, dy$

3. $\int_0^2 \int_{-1}^{1} \int_y^{x+2} dz \, dy \, dx$

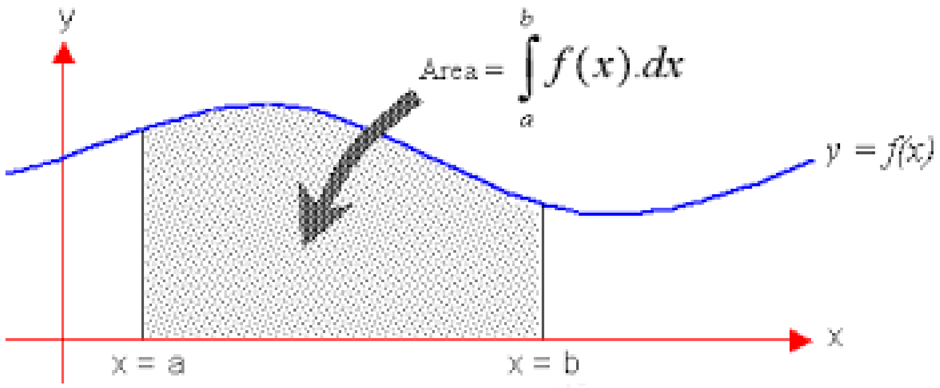
4. $\int_0^2 \int_0^{4-y^2} \int_0^{y} \left(\frac{3}{2}\right) \, dz \, dx \, dy$

5. $\int_{-1}^{1} \int_0^{1-x} \int_{4x^2}^{5-x^2} \pi \, dz \, dy \, dx$

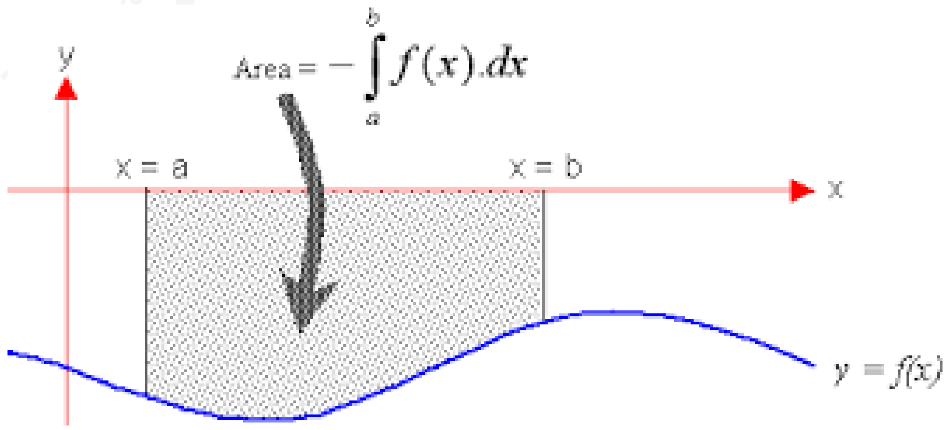
6. $\int_0^2 \int_{-1}^{1} \int_y^{x+2} dz \, dy \, dx$

4.3 Area Under a Curve

The area under a curve between two points can be found by doing a definite integral between the two points. To find the area under the curve $y = f(x)$ between $x = a$ and $x = b$, integrate $y = f(x)$ between the limits of a and b.



<u>**Remark:**</u> If the area is above x-axis,then the area is positive, and if the area under the x-axis, the area is negative, so we should change the sign to positive value by adding a negative sign or by taking the absolute value.



Remark: To avoid the negative value, we will take the absolute value:

$$\boxed{\textbf{Area} = \left| \int_{x=a}^{x=b} f(x)dx \right|}$$

Example (1): Find the area bounded by $y = x^2$ and $x = 1$ and $x = 3$?

Solution:

$$\begin{aligned}
\text{Area} &= \left| \int_{x=1}^{x=3} x^2 dx \right| \\
&= \left| \left[\frac{x^3}{3} \right]_{x=1}^{x=3} \right| \\
&= \left| \frac{3^3}{3} - \frac{1^3}{3} \right| \\
&= \left| 9 - \frac{1}{3} \right| = \boxed{8\frac{2}{3}}\ unit^2
\end{aligned}$$

Example (2): Find the total area between the curve $y = x^3$ and $x = -2$ and $x = 2$?

Solution:

If we simply integrated x^3 between $x = -2$ and $x = 2$, we would get:

$$\text{Area} = \left| \int_{x=-2}^{x=2} x^3 dx \right| = \left| \left[\frac{x^4}{4} \right]_{x=-2}^{x=2} \right| = \left| \frac{16}{4} - \frac{16}{4} \right| = 0$$

So, instead we have to split the graph up and do two separate integrals:

$$A1 = \left| \int_{x=0}^{x=2} x^3 dx \right| = \left| \left[\frac{x^4}{4} \right]_0^2 \right| = \left| \frac{16}{4} - 0 \right| = 4$$

$$A2 = \left| \int_{x=-2}^{x=0} x^3 dx \right| = \left| \left[\frac{x^4}{4} \right]_{-2}^0 \right| = \left| 0 - \frac{16}{4} \right| = \left| -4 \right| = 4$$

Hence, Area $= A1 + A2 = 4 + 4 = \boxed{8}\ unit^2$

Example (3): Find the area bounded by the line $x + y = 1$ and the coordinate axes?

Solution:

$$\because x + y = 2 \implies y = 2 - x$$

$$y = 0 \longrightarrow x = 2 \implies (2, 0)$$

$$\begin{aligned} \text{Area} &= \left| \int_{x=0}^{x=2} (2 - x) dx \right| \\ &= \left| \left(x - \frac{x^2}{2}\right) \right]_{x=0}^{x=2} \Bigg| \\ &= \left| (0 - 0) - \left(2 - \frac{2^2}{2}\right) \right| \\ &= \left| -2 + 2 \right| = \boxed{4}\ unit^2 \end{aligned}$$

Another way:

$$\because x + y = 2 \implies x = 2 - y$$

$$x = 0 \longrightarrow y = 2 \implies (2, 0)$$

$$\text{Area} = \left| \int_{y=0}^{y=2} (2 - y) dy \right|$$

$$= \left| \left[y - \frac{y^2}{2} \right]_{y=0}^{y=2} \right|$$

$$= \left| (0 - 0) - (2 - \frac{2^2}{2}) \right|$$

$$= \left| -2 + 2 \right| = \boxed{4}\ unit^2$$

Problems (4.3):

1. Find the total area bounded by the curve $y = x^3 - 4x$ and $x - axis$.

2. Find the area bounded by $y = y4 - x^2$ and x-$axis$.

3. Find the area bounded by $x = y^2 - y^3$ and y-$axis$.

4. Find the area bounded by $\sqrt{x} + \sqrt{y} = 1$ and the two axes.

5. Prove that the area under one carve of $y = sin(x)$ equals to 2 $units^2$.

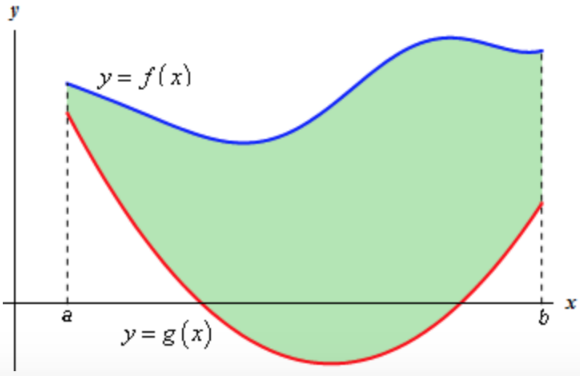
6. Find the area bounded by $y = x^2 - 4x$ and x-$axis$.

7. Find the area bounded by $x = 8 - 2y - y^2$ and y-$axis$.

4.4 Area Between Two Curves

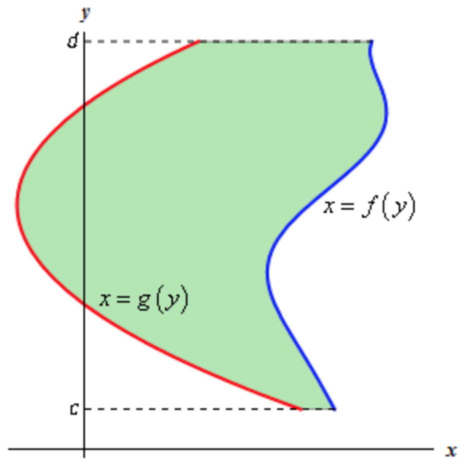
In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



$$\textbf{Area} = \left| \int_{x=a}^{x=b} \Big[f(x) - g(x) \Big] \, dx \right|$$

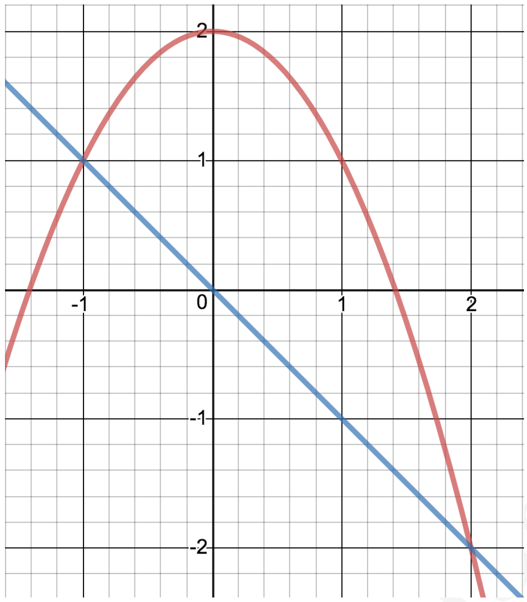
The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



$$\textbf{Area} = \left| \int_{y=a}^{y=b} \Big[f(y) - g(y) \Big] \, dy \right|$$

Example (1): Find the area between the curve $y = 2 - x^2$ and the line $y = -x$?

Solution:



$$y_1 = y_2 \implies 2 - x^2 = -x$$

$$\implies x^2 - x - 2 = 0$$

$$\implies (x-2)(x+1) = 0 \implies x = 2 \text{ and } x = 1$$

$$\text{Area} = \left| \int_{x=-1}^{x=2} \Big[f(x) - g(x) \Big] dx \right|$$

$$= \left| \int_{x=-1}^{x=2} \Big[(2 - x^2) - x \Big] dx \right|$$

$$= \left| \int_{x=-1}^{x=2} \Big[2 - x^2 - x \Big] dx \right|$$

$$= \left| \left[2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{x=-1}^{x=2} \right| = \boxed{\frac{27}{6}} \; units^2$$

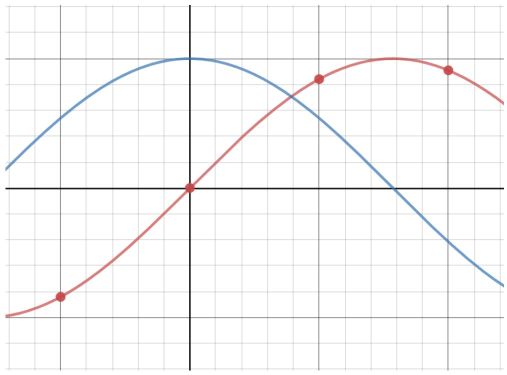
Example (2): Find the area of the triangular shaped region in the first quarter bounded by the y-axis and the curves $y = \sin(x)$ and $y = \cos(x)$?

Solution:

$y_1 = y_2$

$\longrightarrow \cos(x) = \sin(x)$

$\longrightarrow x = \frac{\pi}{4}$



$$\text{Area} = \left| \int_{x=0}^{x=\frac{\pi}{4}} \Big[f(x) - g(x) \Big] dx \right|$$

$$= \left| \int_{x=0}^{x=\frac{\pi}{4}} \Big[\cos(x) - \sin(x) \Big] dx \right|$$

$$= \left| \Big[\sin(x) + \cos(x) \Big]_{x=0}^{x=\frac{\pi}{4}} \right|$$

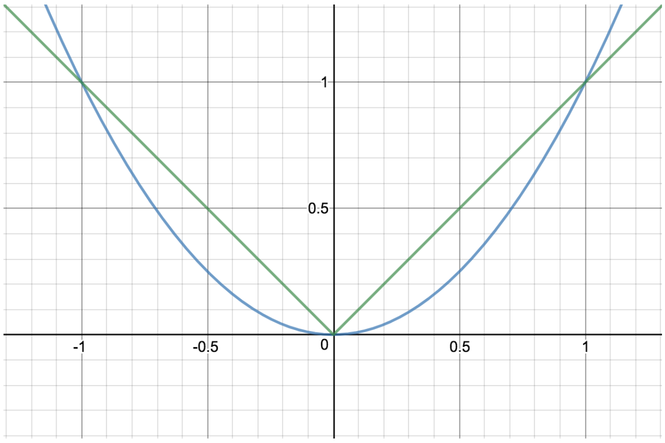
$$= \left| \Big[\Big(\sin(\tfrac{\pi}{4}) + \cos(\tfrac{\pi}{4}) \Big) - \Big(\sin(0) + \cos(0) \Big) \Big] \right|$$

$$= \left| \Big[\Big(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} \Big) - \Big(0 + 1 \Big) \Big] \right|$$

$$= \boxed{\frac{2}{\sqrt{2}} + 1} \; units^2$$

Example (3): Find the area bounded between the two carves $y = x^2$ and $y = |x|$?

Solution:



$y_1 = y_2$

$\longrightarrow |x| = x^2 \longrightarrow \sqrt{x} = x^2$

$\longrightarrow x^2 = x^4 \longrightarrow x^2 - x^4 = 0$

$\longrightarrow x^2(1 - x^2) = 0$

$\longrightarrow x^2(1 - x)(1 + x) = 0$

$x^2 = 0 \implies x = 0 \longrightarrow (0, 0)$

$(x - 1) = 0 \implies x = 1 \longrightarrow (1, 1)$

$(x + 1) = 0 \implies x = -1 \longrightarrow (-1, 1)$

$$\text{A1} = \left| \int_{x=-1}^{x=0} \Big[f(x) - g(x) \Big] dx \right|$$

$$= \left| \int_{x=-1}^{x=0} \Big[-x - x^2 \Big] dx \right|$$

$$= \left| \Big[-\frac{x^2}{2} - \frac{x^3}{3} \Big]_{x=-1}^{x=0} \right| = \left| \Big[0 - (\tfrac{-1}{2} - \tfrac{-1}{3}) \Big] \right| = \left| \tfrac{1}{2} - \tfrac{1}{3} \right| = \boxed{\frac{1}{6}}$$

$$\text{A2} = \left| \int_{x=0}^{x=1} \Big[f(x) - g(x) \Big] dx \right|$$

$$= \left| \int_{x=0}^{x=1} \Big[x - x^2 \Big] dx \right|$$

$$= \left| \Big[\frac{x^2}{2} - \frac{x^3}{3} \Big]_{x=0}^{x=1} \right| = \left| \Big[(\tfrac{1}{2} - \tfrac{1}{3}) - 0 \Big] \right| = \left| \tfrac{1}{2} - \tfrac{1}{3} \right| = \boxed{\frac{1}{6}}$$

$$\text{Area} = \text{A1} + \text{A2} = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{2}{6} = \boxed{\frac{1}{3}} \; units^2$$

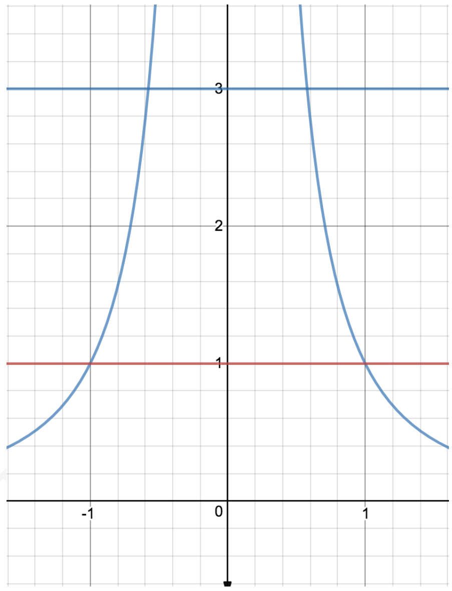
Example (4): Find the area bounded between carve $y = \frac{1}{x^2}$ and the two lines $y = 1$ and $y = 3$?

Solution:

$y = \frac{1}{x^2}$

$\longrightarrow x^2 = \frac{1}{y}$

$\longrightarrow x = \mp \frac{1}{\sqrt{y}}$



$$\text{Area} = \left| \int_{y=1}^{y=3} \Big[f(x) - g(x) \Big] dx \right|$$

$$=$$

Problems (4.4):

1. Find the area bounded by the curve $y = \sqrt{x}$ and the line $y = x$.

2. Find the area bounded by the curve $y = x^3$ and the lines $x = -1$ and $x = -3$.

3. Find the area bounded by the curves $y = x - x^2$ and $y = x^2 - x$.

4. Find the area bounded by the curve $x = 4y - y^2 - 3$ and the line $x = -3$.

5. Find the area bounded by the curve $y = x^3$ and the line $y = x$ in the first quarter.

6. Find the area bounded between carve $y = \frac{1}{x}$ and the two lines $y = 2$ and $y = 3$?

7. Find the area bounded by the curves $y = e^x$ and $y = e^{-x}$ and the lines $y = 2$ and $y = 4$.

8. Find the area bounded by the curves $y = x^2+2$ and the $y = x+5$.