جامعة بغداد
كلية التربية للعلوم الصرفة / ابن الهيثم

التفاضل والتكامل ١

المرحلة الأولى – المستوي الأول

قسم الرياضيات

أساتذة المادة

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CHAPTER ONE: The Real Numbers $\mathbb{R}$

The subsets of $\mathbb{R}$:

1. **Natural Numbers** (denoted by $\mathbb{N}$) such that:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

2. **Intager Numbers** (denote by $\mathbb{I}$ or $\mathbb{Z}$) such that:

$$\mathbb{I} \text{ or } \mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

3. **Rational Numbers** (denoted by $\mathbb{Q}$): it is all numbers of the form $\frac{p}{q}$, such that p and q are integers and $q \neq 0$:

$$\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0\}$$

   Example: $\frac{1}{2}$, $\frac{5}{3}$, 0, $\frac{50}{10}$, $\dots$

   Note: The rational Numbers can be written as decimal from ($\frac{1}{3} = 0.333$, $\frac{1}{4} = 0.25$, $\dots$).

4. **Irrational Numbers** (denoted by $\mathbb{Q}'$): A number which is not rational is said to be irrational.

   Example: $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \pi = 3.14, \dots\}$

   Note: $\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q} \mathbb{U} \mathbb{Q}' = \mathbb{R}$

Properties of Real Numbers with Addition: $(\mathbb{R}, +)$

Let a, b, c $\in \mathbb{R}$, then:

1. $a + b \in \mathbb{R}$ (Closure)
2. $a + b = b + a$ (Commutative)
3. $a + (b + c) = (a + b) + c$ (Associative)
4. $a + 0 = 0 + a = 0$ (Identity Element)
5. $\exists(-a) \in \mathbb{R}$ such that $a+(-a) = (-a)+a = 0$ (Additive Inverse)

Properties of Real Numbers with Multiplication: $(\mathbb{R}, .)$

Let a, b, c $\in \mathbb{R}$, then:

1. $a.b \in \mathbb{R}$ (Closure)
2. $a.b = b.a$ (Commutative)
3. $a.(b.c) = (a.b).c$ (Associative)
4. $1.a = a.1$ (Multiplicative Identity)
5. $a.(b + c) = a.b + a.c$ (Distributive)

   $(b + c).a = b.a + c.a$
6. $\exists a^{-1} \in \mathbb{R}$ such that $a.a^{-1} = a.\frac{1}{a} = 1$ (Multiplication Inverse)

Intervals:-

1. **Finite intervals:-** Let $a, b \in \mathbb{R}$ such that $a < b$ then:

   (a) **Open Interval** $= \{x \in \mathbb{R} : a < x < b\} = (a, b)$

   (Note: $a \notin (a, b)$ and $b \notin (a, b)$)

   (b) **Closed Interval** $= \{x \in \mathbb{R} : a \leq x \leq b\} = [a, b]$

   (Note: $a \in [a, b]$ and $b \in [a, b]$)

   (c) **The Half Open Interval** $= \{x \in \mathbb{R} : a < x \leq b\} = (a, b]$

   (Note: $b \in (a, b]$ and $a \notin (a, b]$)

   OR:

   The Half Open Interval $= \{x \in \mathbb{R} : a \leq x < b\} = [a, b)$

   (Note: $b \notin [a, b)$ and $a \in [a, b)$)

2. **Infinite intervals:-** Let each of $a, b \in \mathbb{R}$ such that $a < b$ then:

   (a) $\{x \in \mathbb{R}$ such that $a < x < \infty$ (or $x > a$) $\} = (a, \infty)$

   (b) $\{x \in \mathbb{R}$ such that $a \leq x < \infty$ (or $x \geq a$) $\} = [a, \infty)$

   (c) $\{x \in \mathbb{R}$ such that $-\infty < x < a$ (or $x < a$) $\} = (-\infty, a)$

   (d) $\{x \in \mathbb{R}$ such that $-\infty < x \leq a$ (or $x \leq a$) $\} = (-\infty, a]$

   (e) $\{x \in \mathbb{R}$ such that $-\infty < x < \infty\} = (-\infty, \infty) = \mathbb{R}$

Inequalities:-

Let $a, b \in \mathbb{R}$, b is **greater** than a and denoted by $b > a$ **if** $b - a > 0$.

Solving Inequalities:-

Solving the inequalities means obtaining all values of x for which the inequality is true.

Properties of Inequalities:-

Let $a, b, c \in \mathbb{R}$, then:

1. if $a < b$, then $a + c < b + c$
2. if $a < b$ and $c > 0$, then $a.c < b.c$
3. if $a < b$ and $c < 0$, then $a.c > b.c$

Note :- In general, we have linear and non-linear inequalities.

Linear Inequalities Examples:-

Example 1: Solve the following inequality: $3(x + 2) < 5$?

solution:-

$3(x+2) < 5 \longrightarrow 3(x+2) < 5 \longrightarrow 3x < 5 - 6 \longrightarrow < \frac{-1}{3}$

Hence, the solution set $=\{x \in \mathbb{R} : x < \frac{-1}{3}\} = (-\infty, \frac{-1}{3})$.

Example 2: Solve the following inequality: $7 < 2x + 3 < 11$?

solution:-

$7 < 2x + 3 < 11 \longrightarrow -3 + 7 < 2x < -3 + 11 \longrightarrow 4 < 2x < 8 \longrightarrow 2 < x < 4$

Hence, the solution set $=\{x \in \mathbb{R} : 2 < x < 4\} = (2, 4)$.

<u>Non-Linear Inequalities Examples</u>:-

Example 1: Solve the following inequality: $x^2 < 25$?

<u>solution</u>:- $x^2 < 25 \rightarrow x^2 - 25 < 0 \rightarrow (x-5)(x+5) < 0$

Since the result is negative, then there are two possibilities:

<u>Either</u>:

$(x+5) > 0$ and $(x-5) < 0 \longrightarrow x > -5$ and $x < 5$

So, the solution set is $(-5, 5)$

<u>Or</u>:

$(x+5), 0$ and $(x-5) > 0 \longrightarrow x < -5$ and $x > 5$

So, the solution set is $\emptyset$

Therefore, the solution set for the inequality is

$(-5, 5) \cup \emptyset = (-5, 5)$

Example 2: Solve the following inequality: $x^2 - 5x > 6$?

<u>solution</u>:-

$x^2 - 5x > 6 \rightarrow x^2 - 5x - 6 > 0 \rightarrow (x-6)(x+1) > 0$

Since the result is Positive, then there are two possibilities:

<u>Either</u>:

$(x-6) > 0$ and $(x+1) > 0 \longrightarrow x > 6$ and $x > -1$

So, the solution set: $S_1 = \{x \in \mathbb{R} : x > 6\} = (6, \infty)$

Or:

$(x-6)<0$ and $(x+1),0 \longrightarrow x<6$ and $x<-1$

So, the solution set: $S_2=\{x\in\mathbb{R}: x<-1\}=(-\infty,-1)$

Therefore, the solution set for the inequality is:

$S=S_1\cup S_2=(6,\infty)\cup(-\infty,-1)=\mathbb{R}\setminus[-1,6]$

Absolute Value:-

The absolute value of a real number x is denoted by $|x|$ and defined as follows:

$$|x|=\sqrt{x^2}=\begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x & \text{if } x<0 \end{cases}$$

Examples: $|-8|=8$, $|\frac{-2}{3}|=\frac{2}{3}$, $|9|=9$, $|0|=0$, etc.

Properties of Absolute Value:-

1. $|-a|=|a|$

   proof: $|-a|=\sqrt{(-a)^2}=\sqrt{a^2}=|a|$

2. $||a||=|a|$

   proof: $||a||=\sqrt{|a|^2}=\sqrt{a^2}=|a|$

3. $|a.b|=|a|.|b|$

   proof: $|a.b|=\sqrt{(a.b)^2}=\sqrt{a^2.b^2}=\sqrt{a^2}.\sqrt{b^2}=|a|.|b|$

4. $|\frac{a}{b}| = \frac{|a|}{|b|}$; $b \neq 0$

   proof: $|\frac{a}{b}| = \sqrt{(\frac{a}{b})^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}$

5. $|a+b| \leq |a| + |b|$

Solving Absolute Value Inequalities:-

The absolute value of x can be written as follows:

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ \\ -x & \text{if } x < 0 \end{cases}$$

The above definition means the absolute value of any real number is a real non-negative number.

Geometrically, the absolute value of unmber x is the distance point between "x" and the origin point "0". In general, $|a-b|$ is the distance between a and b on the real number line "$\mathbb{R}$".

Remarks:

1. To solve the inequality $|x| < a$ where $a, x \in \mathbb{R}$.

   Case (1): If $x \geq 0 \Longrightarrow |x| = x$,

   but $|x| < a \Longrightarrow x < a$. $\Longrightarrow S_1 = (-\infty, a)$

   Case (2): If $x < 0 \Longrightarrow |x| = -x$, but $|x|< a \Longrightarrow -x < a \Longrightarrow x > -a. \Longrightarrow S_2 = (-a, \infty)$

Since, $S = S_1 \cap S_2$

$\Longrightarrow \{x \in \mathbb{R} : |x|< a\} = \{x \in \mathbb{R} : -a < x < a\} = (-a, a)$

Similarly,

$\Longrightarrow \{x \in \mathbb{R} : |x|\leq a\} = \{x \in \mathbb{R} : -a \leq x \leq a\} = [-a, a]$

2. To solve the inequality $|x|> a$ where $a, x \in \mathbb{R}$.

   Case (1): If $x \geq 0 \Longrightarrow |x|= x$,

   but $|x|> a \Longrightarrow x > a. \Longrightarrow S_1 = (a, \infty)$

   Case (2): If $x < 0 \Longrightarrow |x|= -x$,

   but $|x|> a \Longrightarrow -x > a \Longrightarrow x < -a. \Longrightarrow S_2 = (-\infty, -a)$

   Since, $S = S_1 \cup S_2$

   $\Longrightarrow \{x \in \mathbb{R} : |x|> a\} = (a, \infty) \cup (-\infty, -a) = \mathbb{R} \setminus [-a, a]$

   Similarly,

   $\Longrightarrow \{x \in \mathbb{R} : |x|\geq a\} = [a, \infty) \cup (-\infty, -a] = \mathbb{R} \setminus (-a, a)$

Examples:- Find the solution set for the following inequalities?

- $|x|> 3$

  solution:-

  $\{x \in \mathbb{R} : |x|> 3\} = \{x \in \mathbb{R} : x > 3$ or $x < -3\} =$

  $(3, \infty) \cup (-\infty, -3) = \mathbb{R} \setminus [-3, 3]$

- $|x| \leq 4$

  solution:-

  $\{x \in \mathbb{R} : |x| \leq 4\} = \{x \in \mathbb{R} : -4 \leq x \leq 4\} = [-4, 4]$

- $|x - 4| < 5$

  solution:-

  $\{x \in \mathbb{R} : |x - 4| < 5\} = \{x \in \mathbb{R} : -5 < x - 4 < 5\}$

  $= \{x \in \mathbb{R} : -1 < x < 9\} = (-1, 9)$

- $|7 - 4x| \geq 1$

  solution:-

  $\{x \in \mathbb{R} : |x - 4| \geq 1\} = \{x \in \mathbb{R} : 7 - 4x \geq 1 \text{ or } 7 - 4x \leq -1\}$

  $= \{x \in \mathbb{R} : -4x \geq -6 \text{ or } -4x \leq -8\}$

  $= \{x \in \mathbb{R} : x \leq \frac{3}{2} \text{ or } x \geq 2\}$

  $= (-\infty, \frac{3}{2}] \cup [2, \infty)$

  $= \mathbb{R} \setminus (\frac{3}{2}, 2)$

Problems 1.1:

1. Write the following sets equivalent interval, and test of these intervals whether they are Open, Close or Half Open Intervals:

   (a) $\{x : -20 \leq x \leq -12\}$

   (b) $\{x : -3 \leq x < 4\}$

   (c) $\{x : -1 < x < 10\}$

   (d) $\{x : -2 < x \leq 0\}$

2. Give a description of the following intervals as sets:

(a) $(3, 5)$
(b) $(-3, 0)$
(c) $[2, 7]$
(d) $[-5, -1)$
(e) $(-4, 4)$
(f) $(-0, 7]$

3. Find the solution set of the following inequalities:

(a) $x(x-3) > 4$
(b) $2 < \frac{1}{x};\ x \neq 0$
(c) $x^2 \geq 25$
(d) $x^2 - 2x - 24 < 0$
(e) $-7 \leq -3x + 5 \leq 14$
(f) $\frac{x}{x-3} < 4$
(g) $\frac{x^2+2x-35}{x+2} > 0$
(h) $6x - 4 > 7x + 2$
(i) $x^2 \leq 16$
(j) $3x^2 > 2x + 5$
(k) $x^2 > 5x + 6$
(l) $\frac{x-3}{x+2} < 5$
(m) $\frac{1}{x-2} > \frac{2}{x+3}$
(n) $\frac{x-2}{x+3} < \frac{1}{2}$

4. Find the solution set of the following inequalities:

(a) $|x| \geq 5$
(b) $|x| < 2$
(c) $|3x + 3| \geq 2$
(d) $1 \leq |\frac{x-3}{1-2x}| \leq 2$
(e) $|\frac{2-x}{x-3}| \geq 4$
(f) $|x + 1| < |3x + 4|$
(g) $\frac{|2-x|}{3x} \leq 1$
(h) $|\frac{3+2x}{3x}| \leq 1$
(i) $|x - 1| \geq 6$
(j) $|2 - 2x| \leq 7$
(k) $|\frac{4}{2x+1}| \leq 3$

CHAPTER TWO: The Functions

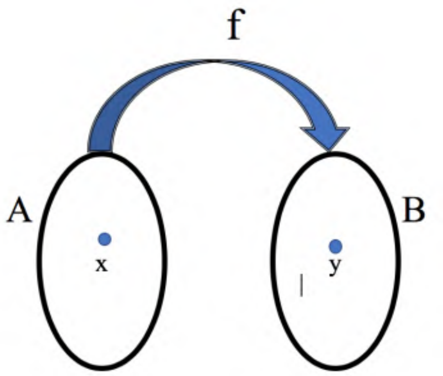
Definition: Let A and B be two non-empty sets, the relation that assigns to every element $x \in A$, with a unique value $y \in B$ is called a **function**. i.e.,

$$\boxed{f : A \longrightarrow B;\ \forall x \in A \exists!\, y \in B \text{ such that } f(y) = y}$$

Notes:

1. $A = Domain = D_f$

   $B = Co - domain = Co - D_f$

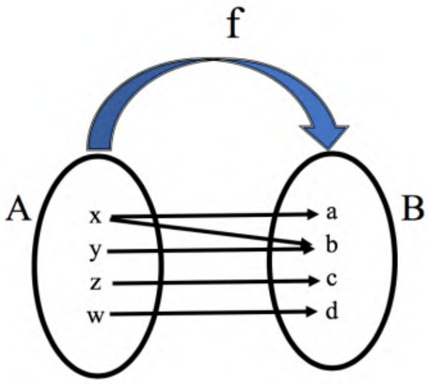


2. The set of all images $f(x) = y, \forall x \in D_f$ is called the Range of f.

   i.e., $\boxed{R_f = \{f(x) = y;\ x \in D_f\}}$

Functions and Non-functions Examples:-

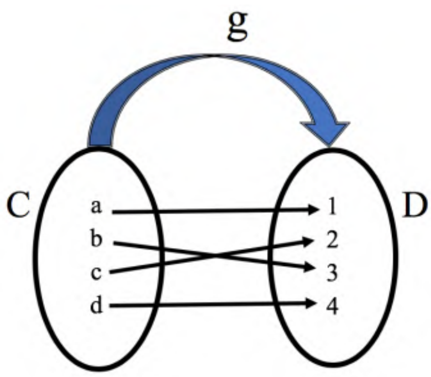
- **Example(1):**



f is not a function because $f(x) = a$ and $f(x) = b$

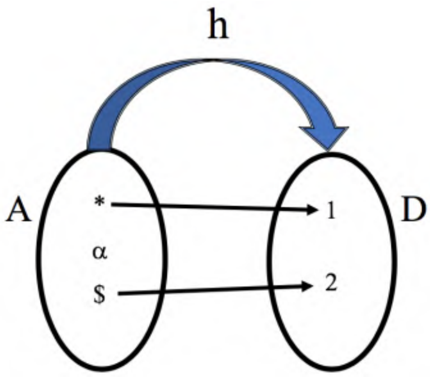
(i.e., x has two images).

- **Example(2):**



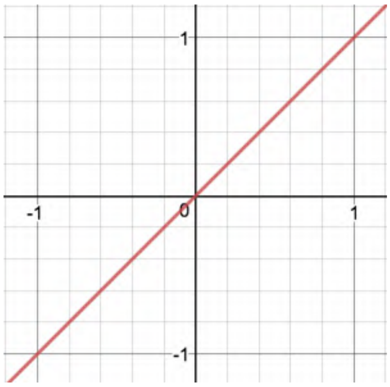
g is a function and $R_g = \{1, 3, 4\}$.

- **Example(3):**



h is not a function because $\alpha \in A$ and α has not image.

- **Example(4):** $y = x$ **is a Linear function.**

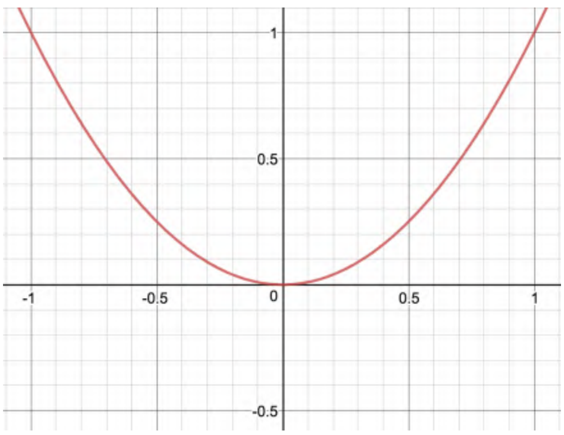


$y = f(x) = x,\ f : \mathbb{R} \longrightarrow \mathbb{R}$

$D_f = \mathbb{R} = \{x \in \mathbb{R} : -\infty < x < \infty\}$

$R_f = \mathbb{R} = \{y \in \mathbb{R} : -\infty < y < \infty\}$

- **Example(5):** $y = x^2$ **is a Non-linear function.**

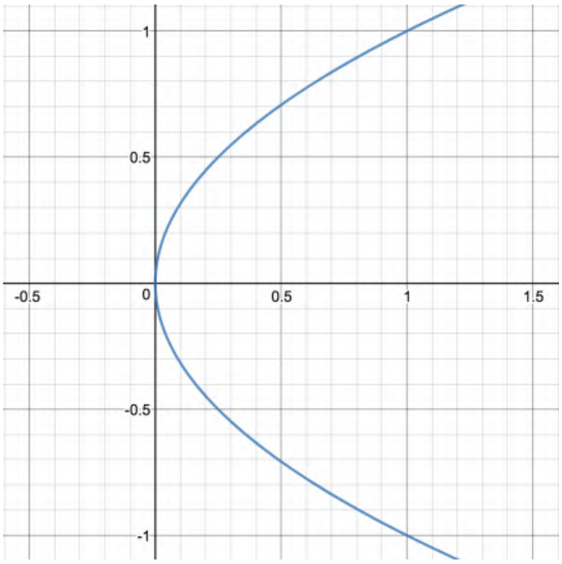


$y = f(x) = x^2,\ f : \mathbb{R} \longrightarrow [0, \infty]$

$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+ = \{y \in \mathbb{R} : y \geq 0\} = [0, \infty)$

- **Example(6): Is $y^2 = x$ a function?**

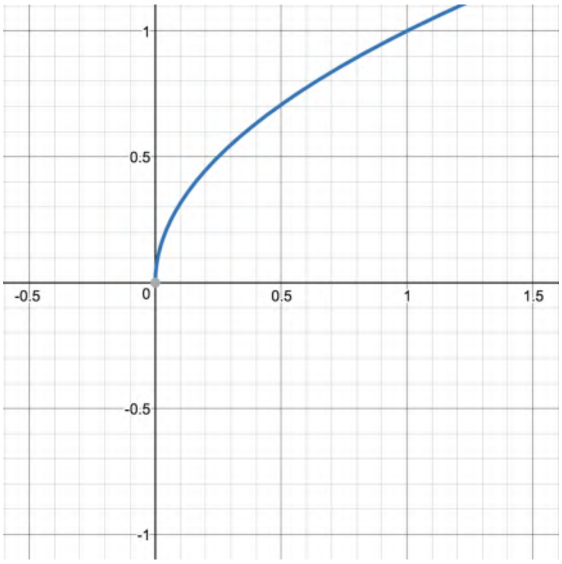


$y^2 = x \longrightarrow \sqrt{y^2} = \sqrt{x} \longrightarrow |y| = \mp\sqrt{x}$

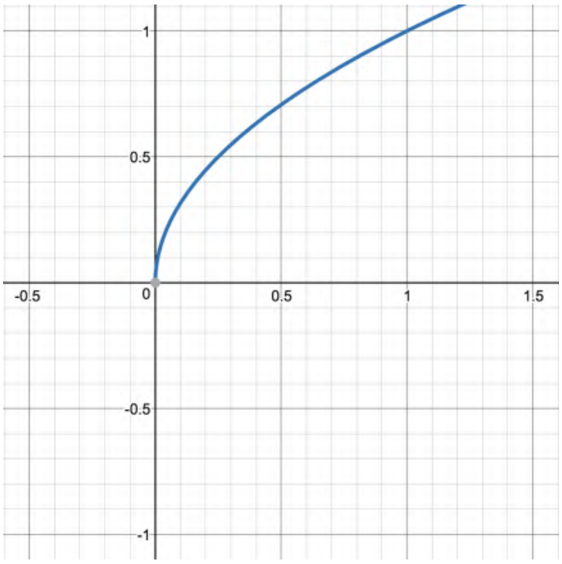
$\forall x \in D_f,\ \exists \mp \sqrt{x}$ (i.e., there are two images for each x).

Hence, "$y^2 = x$" is not a function.

However, $y_1 = +\sqrt{x}$ is a function.



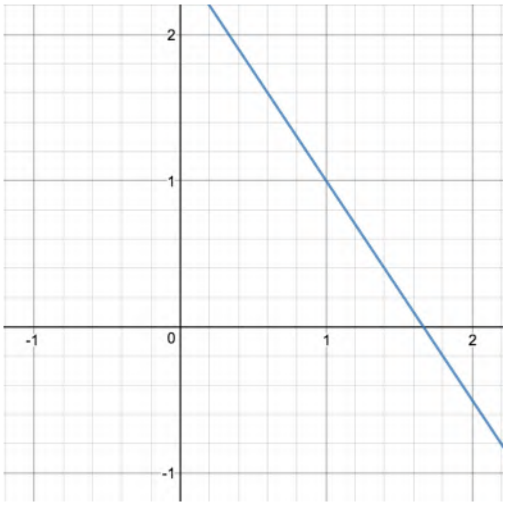
Also, $y_2 = -\sqrt{x}$ is a function.



- **Example(7):** Is 2y +3x = 5 a function?

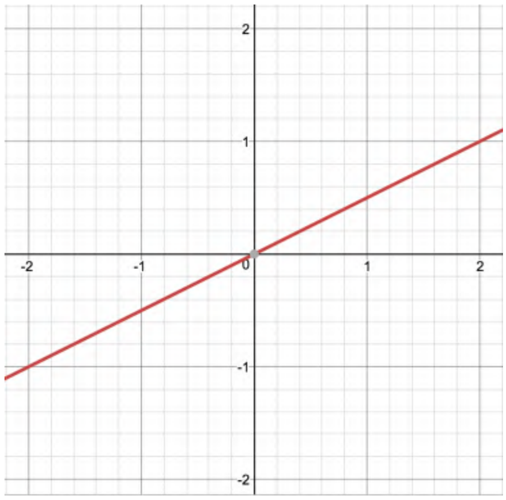
  $2y + 3x = 5 \longrightarrow 2y = 5 - 3x \longrightarrow y = \frac{5-3x}{2}$

  Since for each value of x there exits only one value of y, it is a function.



- **Example(8):** Is $\frac{x}{y} = 2$ a function?

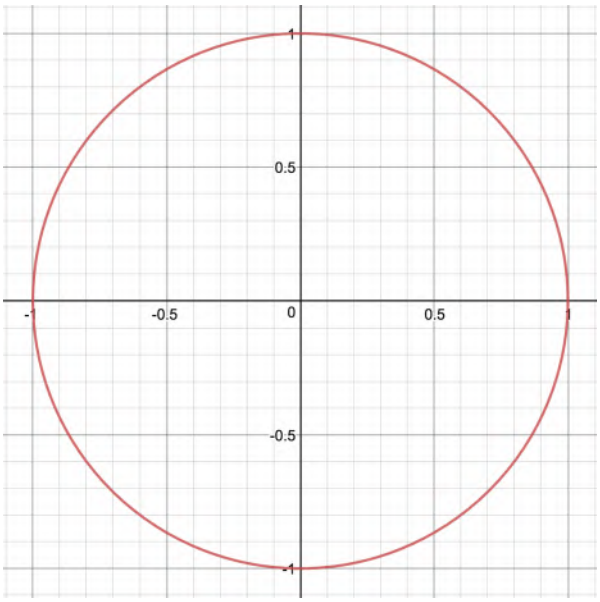
$\frac{x}{y} = 2 \implies y = \frac{1}{2}x$



Since for each value of x there exits only one value of y, it is a function.

- **Example(9):** Is $x^2+y^2=1$ a function?

$x^2+y^2=1 \longrightarrow y=\mp\sqrt{1-x^2}$

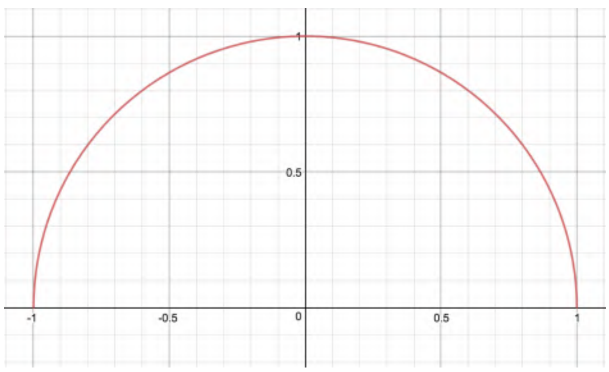


Since $\forall x \in D_f$, $\exists \mp\sqrt{1-x^2}$ (i.e., the are two images for each value of x), "$x^2+y^2=1$" is not a function.

However, $y_1=f_1(x)=+\sqrt{1-x^2}$ is a function.

$1-x^2 \geq 0 \longrightarrow x^2 \leq 1 \longrightarrow -1 \leq x \leq 1$

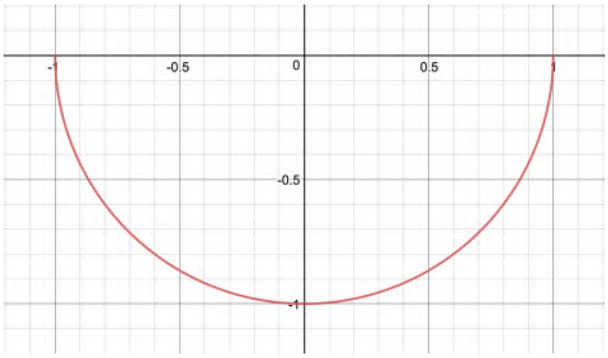
$D_{f_1}=[-1,1]$ and $R_{f_1}=[0,1]$



Also, $y_2=f_2(x)=-\sqrt{1-x^2}$ is a function.

$1 - x^2 \geq 0 \longrightarrow x^2 \leq 1 \longrightarrow -1 \leq x \leq 1$ (same the above)

$D_{f_1} = [-1, 1]$ and $R_{f_1} = [-1, 0]$



How to Find the Domain and the Rang of a Function?

Remark (1): The domain of all polynomials or odd roots is all real numbers.

Example: Find the domain and the rang of the following functions?

1. $f(x) = x^3 + 2x^2 + 3x - 5$

   $D_f = \mathbb{R};\ R_f = \mathbb{R}$

2. $g(x) = \sqrt[3]{x^7 - 1}$

   $D_g = \mathbb{R};\ R_g = \mathbb{R}$

Remark (2): The domain of even root is all the real numbers such that the expression under the radical is greater than or equal to zero.

Example 1 : Let $f(x) = \sqrt{x^2 - 4}$, find D_f and R_f?

To find D_f:

$X^2 - 4 \geq 0 \implies (x-2)(x+2) \geq 0$

either: $x - 2 \geq 0 \wedge x + 2 \geq 0 \implies x \geq 2 \wedge x \geq -2 \implies [2, \infty)$

or: $x-2 \leq 0 \wedge x+2 \leq 0 \implies x \leq 2 \wedge x \leq -2 \implies (-\infty,-2]$

Hence, $D_f = (-\infty,-2] \cup [2,\infty) = \mathbb{R} \setminus (-2,2)$

<u>To find R_f:</u>

Since, $y^2 \geq 0 \implies y \in \mathbb{R}^+ \implies R_f = \mathbb{R}^+$

Example 2 : Let $g(x) = -\sqrt{2x-1}$, find D_g and R_g?

<u>To find D_g:</u>

$2x-1 \geq 0 \implies 2x \geq 1 \implies x \geq \frac{1}{2} \implies D_g = [\frac{1}{2},\infty)$

$$y = -\sqrt{2x-1} \implies y^2 = 2x-1 \implies 2x = y^2+1 \implies x = \frac{y^2+1}{2}$$

<u>To find R_g:</u>

$y \leq 0 \implies y \in \mathbb{R}^- \implies R_g = R^- = (-\infty,0]$

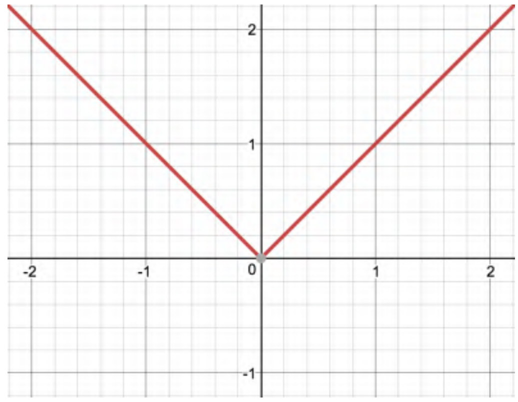
Definition: The function that is defined by more than one formula (e.g., the function are written using the brace $\{\}$, signum function absolute value function) is) called <u>**Piecewise function**</u>.

<u>**Remark (3)**</u>: The domain of the Piecewise function are the restrictions of the functions.

Example 1 : Let $f(x) = |x| = \sqrt{x^2} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0 \end{cases}$

find D_f and R_f?

$$D_f = \mathbb{R} \text{ and } R_f = \mathbb{R}^+$$



$$\textbf{Example 2 :} \text{ Let } g(x) = \begin{cases} -1 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}, \text{ find } D_g \text{ and } R_g?$$

$D_g = \mathbb{R}$

$R_g = \{-1, 3\}$

Example 3 : Let $h(x) = y = |x + 3|$, find D_h and R_h?

$$\text{since, } |x+3| = \begin{cases} x+3 & \text{if } x+3 > 0 \rightarrow x > -3 \\ 0 & \text{if } x+3 = 0 \rightarrow x = -3 \\ -(x+3) & \text{if } x+3 < 0 \rightarrow x < -3 \end{cases}$$

$D_h = \mathbb{R}$

$R_h = \mathbb{R}^+$

$$\textbf{Example 4 :} \text{ Let } f(x) = \begin{cases} x & \text{if } x < -2 \\ x+1 & \text{if } -2 \leq x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}, \text{ find } D_f \text{ and } R_f?$$

$D_f = ?$

$x < -2 \vee -2 \leq x \leq 1 \vee x > 1$

$\Longrightarrow\ (-\infty, -2) \cup [-2, 1] \cup (1, \infty) = \mathbb{R}$

$\Longrightarrow\ D_f = \mathbb{R}$

$R_f = ?$

$x < -2 \vee -1 \leq x \leq 2 \vee x > 1$

$\Longrightarrow\ (-\infty, -2) \cup [-1, 2] \cup (1, \infty) = \mathbb{R} \setminus [-2, -1)$

$\Longrightarrow\ R_f = \mathbb{R} \setminus [-2, -1)$

Example 5 : Let $w(t) = |t - 2|$, find D_w and R_w?

$$|t-2| = \begin{cases} t-2 & \text{if } t > 2 \\ 0 & \text{if } t = 2 \\ -(t-2) & \text{if } t < 2 \end{cases}$$

$D_w = \mathbb{R}$

$R_w = \mathbb{R}^+$

Remark (4): The domain of the Rational function is all the real number values except the value of x which makes the denominator equal to zero.

Example 1 : Let $f(x) = \frac{x}{x^2-1}$, find D_f and R_f?

To find D_f:

$x^2 - 1 \neq 0 \implies x^2 \neq 1 \implies \sqrt{x^2} \neq 1 \implies |x| \neq 1 \implies x \neq \mp 1$

$D_f = \mathbb{R} \setminus \{-1, 1\}$

<u>To find R_f:</u>

$y = f(x) = \frac{x}{x^2-1} \implies x = yx^2 - y \implies yx^2 - x - y = 0$

$\implies x = \frac{1 \mp \sqrt{1+4y^2}}{2y}$ (Using $x = \frac{-b \mp \sqrt{b^2-4bc}}{2a}$)

Since $2y \neq 0 \implies y \neq 0$,

and $1 + 4y^2 \geq 0 \implies y^2 \geq \frac{-1}{4} \implies y^2 \geq 0 \implies y \in \mathbb{R}$

Hence, $R_f = \mathbb{R} \setminus \{0\}$

Example 2 : Let $h(x) = \sqrt[3]{\dfrac{x+1}{x-2}}$, find D_h and R_h?

<u>To find D_h:</u>

$\sqrt[3]{x-2} \neq 0 \implies x - 2 \neq 0 \implies x \neq 2$

Hence, $D_h = \mathbb{R} \setminus \{2\}$

<u>To find R_h:</u>

$y^3 = \frac{x+1}{x-2} \implies (x-2)y^3 = x + 1$

$\implies xy^3 - 2y^3 - x - 1 = 0$

$\implies (y^3 - 1)x = 2y^3 + 1$

$\implies x = \frac{2y^3+1}{y^3-1}$

Let $y^3 - 1 \neq 0 \implies y^3 \neq 1 \implies y \neq 1$

Hence, $R_h = \mathbb{R} \setminus \{1\}$

Problems 2.1: Find the domains and ranges for the following functions?

1. $f(x) = \sqrt{\frac{1}{x} - 2}$
2. $h(x) = \frac{\sqrt{x+1}}{x-1}$
3. $l(x) = \frac{x+1}{|x-5|}$
4. $g(x) = \frac{2-x}{\sqrt{1-x}}$
5. $f(x) = \frac{1}{x^2+1} + 3$
6. $w(t) = \sqrt{x^2 + 25}$
7. $g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$

Algebraic of function:-

Let f and g be two functions, then:-

1. **Equality of functions:**

   f and g are equality $\Longleftrightarrow D_f = D_g$ and $f(x) = g(x)$

2. **The Sum of functions:**

   The sum of f and g is : $(f + g)(x) = f(x) + g(x)$

   with the domain: $D_{f+g} = D_f \cap D_g$

3. **The Difference of functions:**

   The difference between f and g is: $(f - g)(x) = f(x) - g(x)$

   with the domain: $D_{f+g} = D_f \cap D_g$

4. **The Product of functions:**

   The product of f and g is: $(f.g)(x) = f(x).g(x)$

with the domain: $D_{f.g} = D_f \cap D_g$

5. **The Division of functions:**

   The division of f and g is: $(f|g)(x) = \frac{f(x)}{g(x)}$

   with the domain: $D_{f|g} = D_f \cap D_g \setminus \{x \in \mathbb{R} : g(x) = 0\}$

   Similarly,

   $(g|f)(x) = \frac{g(x)}{f(x)}$

   $D_{g|f} = D_g \cap D_f \setminus \{x \in \mathbb{R} : g(x) = 0\}$

Example 1 : Which of the following functions are equal to the function $f(x) = \frac{x-2x^2}{x}$?

1. $g(x) = 1 - 2x$

   Solution:- $D_g = \mathbb{R}$; $D_f = \mathbb{R} \setminus \{0\}$

   Since, $D_f \neq D_g \Longrightarrow f(x) \neq g(x)$

2. $h(x) = \frac{x^2-2x^3}{x^2}$

   Solution:- $D_h = \mathbb{R} \setminus \{0\}$; $D_f = \mathbb{R} \setminus \{0\}$

   $h(x) = \frac{x^2-2x^3}{x^2} = \frac{x(x-2x^2)}{x.x} = \frac{x-2x^2}{x} = f(x)$

   Since, $D_h = D_f$ and $h(x) = f(x) \Longrightarrow h(x) = f(x)$

3. $l(x) = \sqrt{1 - 4x + 4x^2}$

   Solution:-

   $\sqrt{1 - 4x + 4x^2} = \sqrt{(1 - 2x)(1 - 2x)} = \sqrt{(1 - 2x)^2}$

$= |1-2x| \Longrightarrow \ D_l = \mathbb{R}$

Since, $D_l \neq D_f \ \Longrightarrow \ l(x) \neq f(x)$

4. $w(x) = \frac{(x^3+x)(1-2x)}{x(1+x^2)}$

   Solution:-

   $x(1+x^2) \neq 0 \ \Longrightarrow \ x \neq 0 \vee 1+x^2 \neq 0$ (i.g. , $x^2 \neq 0$ which is always true)

   $\Longrightarrow \ D_w = \mathbb{R} \setminus \{0\}$

   $w(x) = \frac{(x^3+x)(1-2x)}{x(1+x^2)} = \frac{x(x^2+1)(1-2x)}{x(1+x^2)} = \frac{x-2x^2}{x} = f(x)$

   Since, $D_w = D_f$ and $w(x) = f(x) \Longrightarrow w(x) = f(x)$

Example 2 : If $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{4-x}$, find

$f(x)+g(x)$, $f(x)-g(x)$, $f(x).g(x)$, $f(x)/g(x)$, and $g(x)/f(x)$

with domain for all.

Solution:-

Since, $f(x) = \sqrt{x+1}$,

$x+1 \geq 0 \ \Longrightarrow \ x \geq -1$

$\Longrightarrow \ D_f = [-1, \infty)$

Since, $g(x) = \sqrt{4-x^2}$,

$4-x^2 \geq 0 \ \Longrightarrow \ x^2 \leq 4 \ \Longrightarrow \ |x| \leq 2 \ \Longrightarrow \ -2 \leq x \leq 2$

$\Longrightarrow \ D_g = [-2, 2]$

Now,

$(f+g)(x) = f(x)+g(x) = \sqrt{x+1} + \sqrt{4-x^2}$

$(f-g)(x) = f(x) - g(x) = \sqrt{x+1} - \sqrt{4-x^2}$

$(f.g)(x) = f(x).g(x) = \sqrt{x+1}.\sqrt{4-x^2} = \sqrt{(x+1)(4-x^2)}$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{\sqrt{4-x^2}} = \sqrt{\frac{(x+1)}{(4-x^2)}}$

$(\frac{g}{f})(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4-x^2}}{\sqrt{x+1}} = \sqrt{\frac{(4-x^2)}{(x+1)}}$

Also,

$D_{f+g} = D_{f-g} = D_{f.g} = D_f \cap D_g = [-1\infty) \cap [-2,2] = [-1,2]$

$D_{\frac{f}{g}} = D_f \cap D_g \setminus \{x \in \mathbb{R} : g(x) = 0\}$

$= [-1,2] \setminus \{x \in \mathbb{R} : \sqrt{4-x^2} = 0\}$

$= [-1,2] \setminus \{-2,2\}$

$= [-1,2)$

$D_{\frac{g}{f}} = D_f \cap D_g \setminus \{x \in \mathbb{R} : f(x) = 0\}$

$= [-1,2] \setminus \{x \in \mathbb{R} : \sqrt{x+1} = 0\}$

$= [-1,2] \setminus \{-1\}$

$= (-1,2]$

Problems 2.2:

1. Check whether each of the following two functions equal or not?

   (a) $f(x) = \frac{2x^2+4x}{6x^2}$, $\quad g(x) = \frac{6x^3+12x^2}{6x^3}$

   (b) $v(x) = \frac{\sqrt{x+1}}{x^3}$, $\quad w(x) = \frac{\sqrt[3]{x^2-1}}{\sqrt{x^2}}$

   (c) $h(x) = \frac{2x^2+3x^{-2}}{8x}$, $\quad l(x) = \frac{2x^3+3x^{-1}}{8x^2}$

2. Find each of $f+g$, $f-g$, $f.g$, f/g, g/f, then find the domain of

each of them?

(a) $f(x) = x^2, \qquad g(x) = x + 1$

(b) $f(x) = x^3 + x, \qquad g(x) = \frac{1}{\sqrt{x+1}}$

(c) $f(x) = \frac{x}{x+1}, \qquad g(x) = \frac{x-1}{\sqrt{x}}$

Composition Functions:-

Let $f(x)$ and $g(x)$ be two functions such that $R_{g(x)} \subseteq D_{f(x)}$, then there exist a function $(f \circ g)(x)$ define in the following formula:

$$(f \circ g)(x) = f(g(x))$$

$$D_{(f \circ g)(x)} = \{x : g(x) \in D_{f(x)} \wedge x \in D_{g(x)}\}$$

Similarly, we can define $(g \circ f)(x)$ as follows:

$$(g \circ f)(x) = g(f(x))$$

$$D_{(g \circ f)(x)} = \{x : f(x) \in D_{g(x)} \wedge x \in D_{f(x)}\}$$

Note: $(f \circ g)(x) \neq (g \circ f)(x)$

Example 1: Let $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, find $f \circ g$ and $g \circ f$?

Solution:-

First, we are going to find the domain and range for $f(x)$ and $g(x)$,

$f(x) = \sqrt{x} \implies D_f = \mathbb{R}^+ = [0, \infty)$

$y = \sqrt{x} \implies y^2 = x \implies R_f = \mathbb{R}^+ = [0, \infty)$

Also,

$g(x) = x^2 + 1 \implies D_g = \mathbb{R} = [0, \infty)$

$y = x^2 + 1 \implies y = x^2 + 1 \implies x^2 = y - 1 \implies x = \mp\sqrt{y - 1}$

So, $y - 1 \geq 0 \implies y \geq 1 \implies R_g = [1, \infty)$

To find $f \circ g$: Since $R_g = [1, \infty) \subseteq [0, \infty) = D_f$, so $f \circ g$ exist.

$$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^2 + 1}$$

$D_{f \circ g} = \{x : x \in D_g \text{ and } g(x) \in D_f\}$

$= \{x : x \in \mathbb{R} \text{ and } x^2 + 1 \in \mathbb{R}^+\}$

$= \{x : x \in \mathbb{R} \wedge x \in \mathbb{R}\} = \mathbb{R}$

(Since $x^2 + 1 \geq 0 \implies x^2 \geq -1$ which is always true, and hence $x \in \mathbb{R}$)

To find $g \circ f$: Since $R_f = [0, \infty) \subseteq [0, \infty) = D_g$, so $g \circ f$ exist.

$$(g \circ f)(x) = g(f(x)) = (\sqrt{x})^2 + 1 = x + 1$$

$D_{g \circ f} = \{x : x \in D_f \wedge f(x) \in D_g\}$

$= \{x : x \in \mathbb{R}^+ \wedge \sqrt{x} \in \mathbb{R}\}$

$= \{x : x \in \mathbb{R}^+ \wedge x \in \mathbb{R}^+\} = \mathbb{R}^+$ [Since, $x \geq 0 \implies x \in \mathbb{R}^+$]

Example 2: Let $f(x) = \sqrt{x - 4}$ and $g(x) = \frac{x+1}{3-x}$, find $f \circ g$ and $g \circ f$?

Solution:-

First, we are going to find the domain and range for $f(x)$ and $g(x)$,

To find D_f:

$x - 4 \geq 0 \implies x \geq 4 \implies D_f = [4, \infty)$

To find R_f:

$y = \sqrt{x - 4} \implies y^2 = x - 4 \implies x = y^2 + 4 \implies R_f = \mathbb{R}^+$

Also,

To find D_g:

$3 - x \neq 0 \implies x \neq 3 \implies D_g = \mathbb{R} \setminus \{3\}$

To find R_g:

$y = g(x) = \frac{x+1}{3-x} \implies x + 1 = 3y - xy \implies x + xy = 3y - 1$

$\implies x = \frac{3y-1}{1+y}$

$\because y + 1 \neq 0 \implies y = -1$

Hence, $R_g = \mathbb{R} \setminus \{-1\}$

To find $f \circ g$:

$R_g = \mathbb{R} \setminus \{-1\} \nsubseteq [4, \infty) = D_f \implies f \circ g$ does not exist.

To find $g \circ f$:

$R_f = \mathbb{R}^+ \nsubseteq \mathbb{R} \setminus \{3\} = D_g \implies g \circ f$ does not exist.

Problems 2.3: Find $f \circ g$ and $g \circ f$ for the following functions:-

1. $f(x) = |x|, \qquad g(x) = -x$
2. $f(x) = \frac{x}{x+2}, \qquad g(x) = \frac{x-1}{x}$
3. $f(t) = \sqrt{x-1}, \qquad g(x) = \sqrt{1-x}$
4. $f(x) = x + 1, \qquad g(x) = 2x$
5. $f(x) = -\sqrt{x}, \qquad g(x) = x^2 + 1$
6. $f(x) = 2x + 4, \qquad g(x) = \frac{1}{2}x - 2$

7. $f(x) = x^2, \qquad g(x) = 2x + 3$

8. $f(x) = x^3, \qquad g(x) = \sqrt{1-x}$

The Greatest Integer Function:-

The function whose value at any number x is the greatest integer less than or equal to x is called the greatest integer function. It is denoted by "$\lceil \ \rceil$" such that $\lceil x \rceil \leq x$.

Examples:

$\lceil 2 \rceil = 2$

$\lceil 1.5 \rceil = 1$

$\lceil -1.5 \rceil = -2$

$\lceil 3.4 \rceil$=3

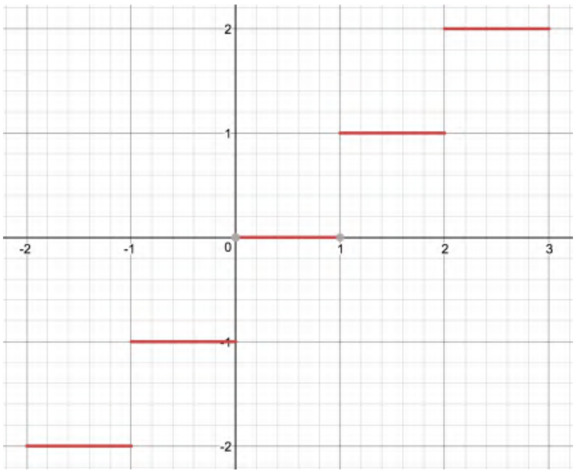
Example 1: Sketch a graph for the following function:

$f(x) = \lceil x \rceil, \forall x \in [-2, 3)$

x	$y = \lceil x \rceil$	closed point	open point
$-2 \leq x < -1$	-2	$(-2,-2)$	$(-1,-2)$
$-1 \leq x < 0$	-1	$(-1,-1)$	$(0,-1)$
$0 \leq x < 1$	0	$(0,0)$	$(1,0)$
$1 \leq x < 2$	1	$(1,1)$	$(2,1)$
$2 \leq x < 3$	2	$(2,2)$	$(3,2)$

From the above table, we can see that:

$D_f = [-2, 3)$ and $R_f = \{-2, -1, 0, 1, 2\}$



Note: In general, $f(x) = \lceil x \rceil = n$, $\forall n \in \mathbb{I}, \forall x \in [n, n+1)$ is called "**Step Function**".

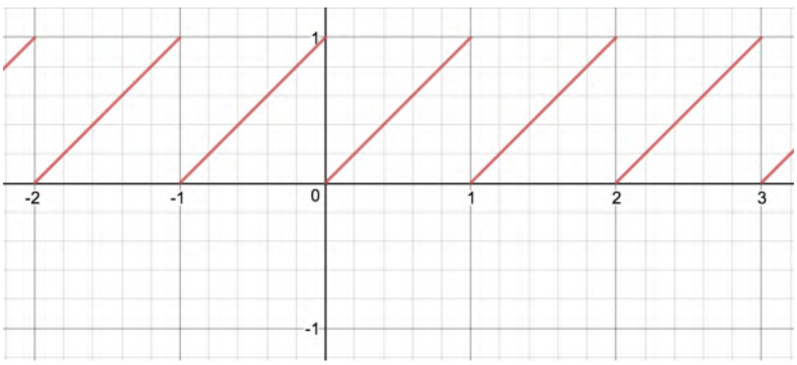
Example 2: Sketch a graph for the following function:

$f(x) = x - \lceil x \rceil, \forall x \in [-3, 3]$.

x	$\lceil x \rceil$	$y = x - \lceil x \rceil$	closed point	open point
$-3 \leq x < -2$	-3	$x+3$	$(-3,0)$	$(-2,1)$
$-2 \leq x < -1$	-2	$x+2$	$(-2,0)$	$(-1,1)$
$-1 \leq x < 0$	-1	$x+1$	$(-1,0)$	$(0,1)$
$0 \leq x < 1$	0	x	$(0,0)$	$(1,1)$
$1 \leq x < 2$	1	$x-1$	$(1,0)$	$(2,1)$
$2 \leq x < 3$	2	$x-2$	$(2,0)$	$(3,1)$
$3 = x$	3	$x-3$	$(3,0)$	

From the above table, we can see that:

$D_f = [-3, 3]$ and $R_f = \{-3, -2, -1, 0, 1, 2, 3\}$



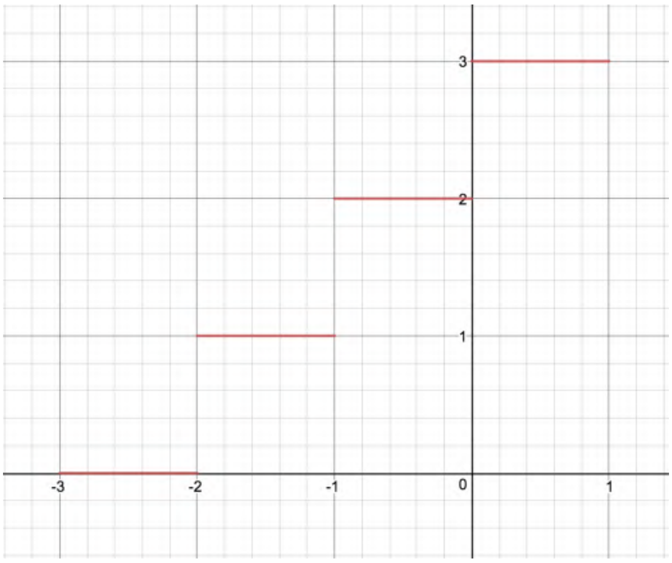
Example 3: Sketch a graph for the following function:

$f(x) = \lceil 3 + x \rceil, \forall x \in [-3, 1)$.

x	$3+x$	$y = \lceil 3+x \rceil$	closed point	open point
$-3 \leq x < -2$	$0 \leq 3+x < 1$	0	$(-3,0)$	$(-2,0)$
$-2 \leq x < -1$	$1 \leq 3+x < 2$	1	$(-2,1)$	$(-1,1)$
$-1 \leq x < 0$	$2 \leq 3+x < 3$	2	$(-1,2)$	$(0,2)$
$0 \leq x < 1$	$3 \leq 3+x < 4$	3	$(0,3)$	$(1,3)$

From the above table, we can see that:

$D_f = [-3, 1]$ and $R_f = \{0, 1, 2, 3\}$



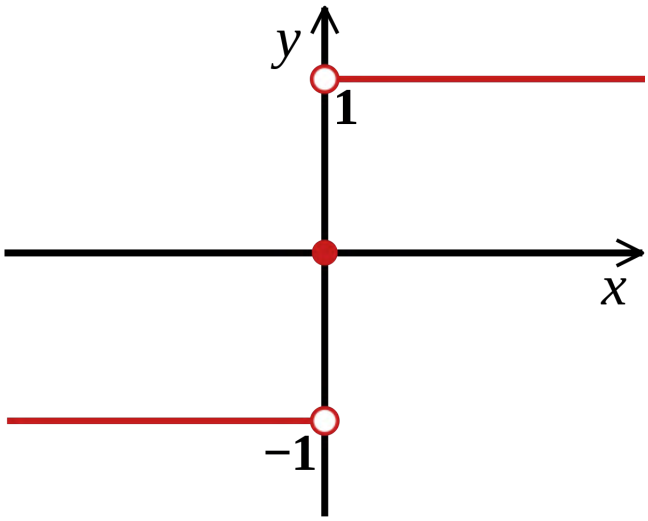
Signum Function:-

We denoted to the signum function by "$Sgn(x)$", and it is defined as follows:

$$Sgn(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

$D_f = \mathbb{R}$

$R_f\{-1, 0, 1\}$



Example 1: Find the Domain and Range and Sketch a graph for the following function:

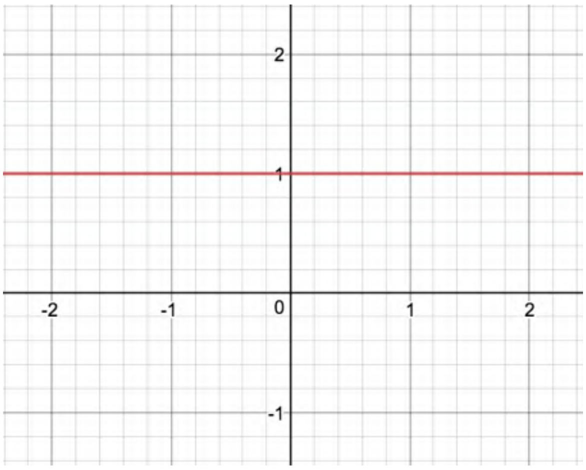
$$f(t) = Sgn(t^2 + 1)$$

Solution:-

$$f(t) = Sgn(t^2{+}1) = \begin{cases} 1 & \text{if } t^2 + 1 > 0 \implies t^2 \geq -1 \implies t^2 \geq 0 \implies t \in \mathbb{R} \\ 0 & \text{if } t^2 + 1 = 0 \implies t^2 = -1, \text{ Contradiction} \\ -1 & \text{if } t^2 + 1 < 0 \implies t^2 < -1, \text{ Contradiction} \end{cases}$$

Hence, $Sgn(t^2 + 1) = 1, \forall t \in \mathbb{R}$

$D_f = R$ and $R_f = 1$



Example 2: Find the Domain and Range and Sketch a graph for the following function:

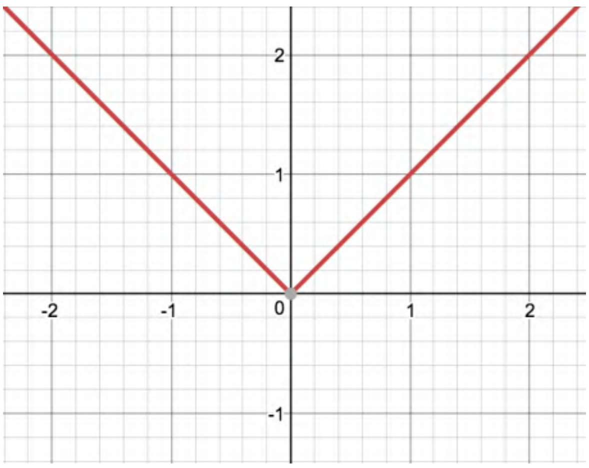
$$g(t) = tSgn(t)$$

<u>Solution</u>:-

$$g(x) = tSgn(t) = t * \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases} = \begin{cases} t & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -t & \text{if } t < 0 \end{cases} = |t|$$

Hence, $g(t) = tSgn(t) = |t|, \forall t \in \mathbb{R}$

$D_f = \mathbb{R}$ and $R_f = \mathbb{R}^+$



Odd Function:-

A function $f(x)$ is called an odd function if $f(-x) = -f(x)$

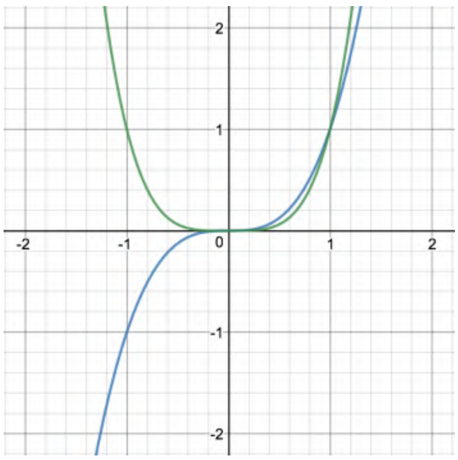
Examples:

- $f(x) = x^3$

  $\because f(-x) = (-x)^3 = -x^3 = -f(x) \implies f(x)$ is an odd function.

- $g(x) = x^4$

  $\because g(-x) = (-x)^4 = x^4 \neq -g(x) \implies g(x)$ is NOT an odd function.



Note: For odd function, $D_f = R_f = \mathbb{R}$

Even Function:-

A function $f(x)$ is called an Even function if $f(-x) = f(x)$

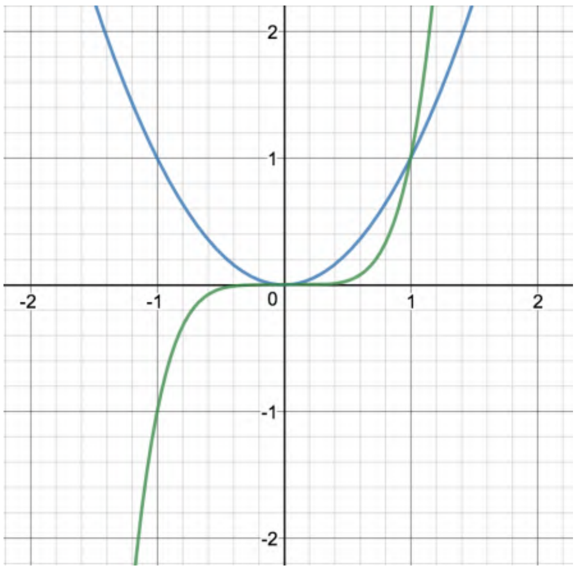
Examples:

- $h(x) = x^2$

  $\because h(-x) = (-x)^2 = x^2 = h(x) \implies h(x)$ is an even function.

- $t(x) = x^5$

  $\because t(-x) = (-x)^5 = -x^5 \neq t(x) \implies t(x)$ is NOT an even function.



Note: For even function, $D_f = \mathbb{R}$, but $R_f = \mathbb{R}^+$

Shifting Function:-

Let $y = f(x)$ s.t. $x \in \mathbb{R}$, and let $c \in \mathbb{R}$, then:

1. $g(x) = f(x) + c$ [Shifting to the **top** c unit]

2. $g(x) = f(x) - c$ [Shifting to the **bottom** c unit]

3. $g(x) = f(x + c)$ [Shifting to the **left** c unit]

4. $g(x) = f(x - c)$ [Shifting to the **right** c unit]

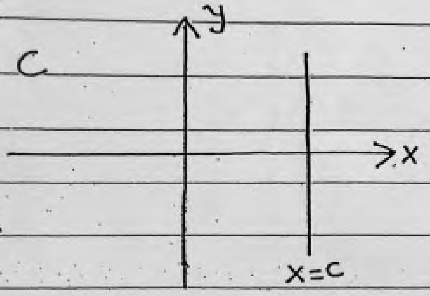
5. $g(x) = -f(x)$ [reflect around x-axes]

6. $g(x) = f(-x)$ [reflect around y-axes]

اشكال دوال بصورة عامة :

{1} $x = c$

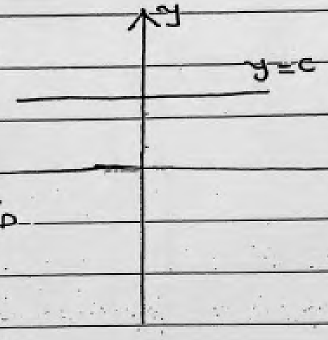
خط شاقولي



{2} $y = c$

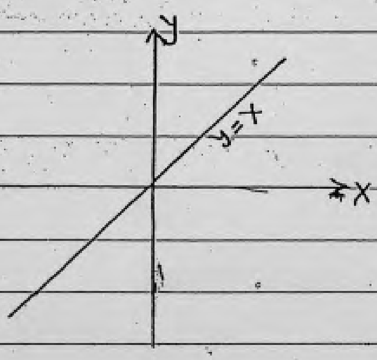
تسمى دالة ثابتة

خط افقي



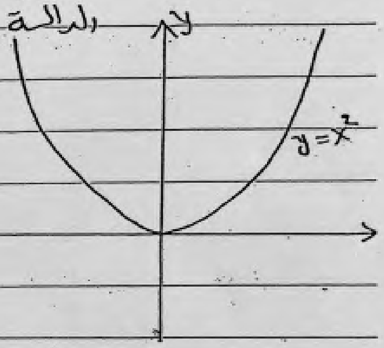
{3} $f(x) = y = x$

الدالة خطية



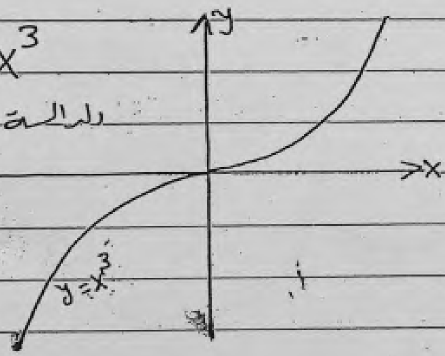
{4} $y = x^2$

الدالة التربيعية



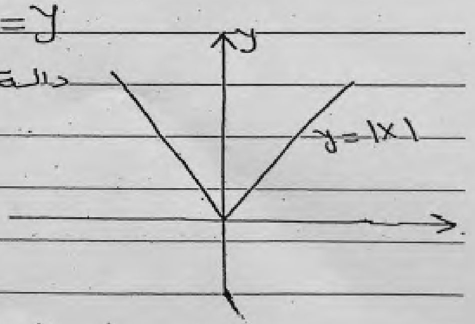
{5} $y = x^3$

الدالة التكعيبية



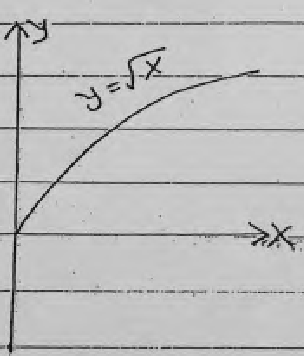
{6} $|x| = y$

دالة المطلق



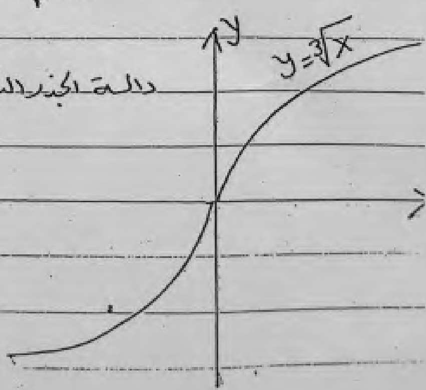
{7} $y = \sqrt{x}$

دالة الجذر التربيعي



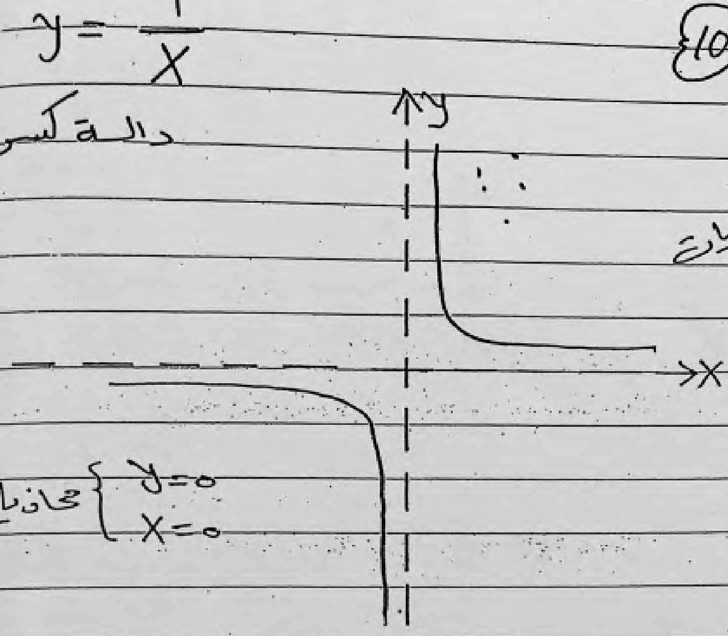
{8} $y = \sqrt[3]{x}$

دالة الجذر التكعيبي

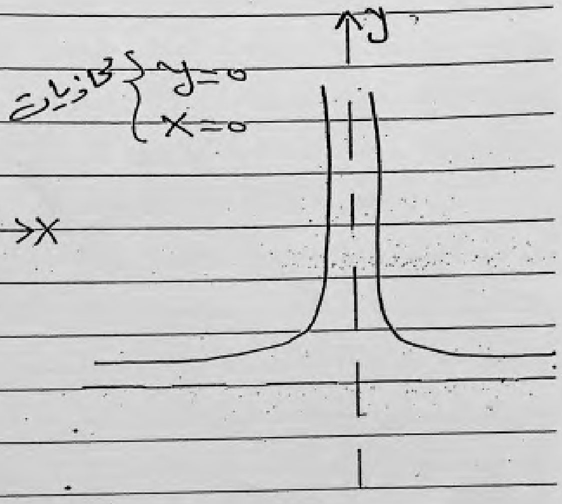


(9) $y = \frac{1}{x}$

دالة كسرية



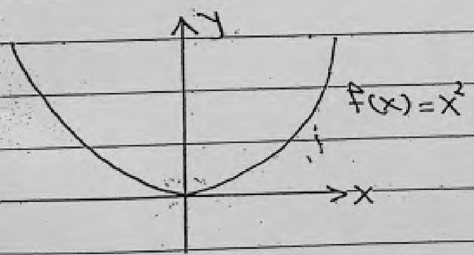
(10) $y = \frac{1}{x^2}$



مثال (1): ارسم مخطط الدالة $f(x)=y=x^2$ ثم جد:

$g(x) = f(x)+1$, $k(x) = f(x)-1$, $h(x) = f(x+1)$, $t(x) = f(x-1)$, $L(x) = -f(x)$, $m(x) = f(-x)$

sol

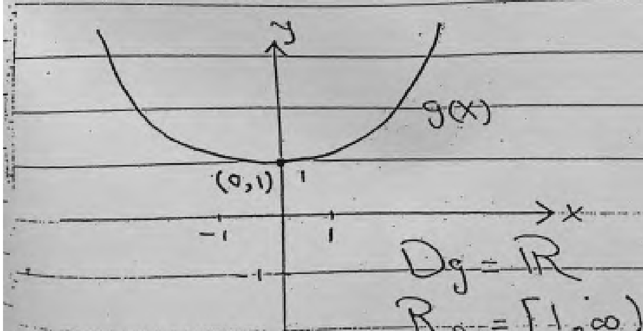


$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+$

$= [0,\infty)$

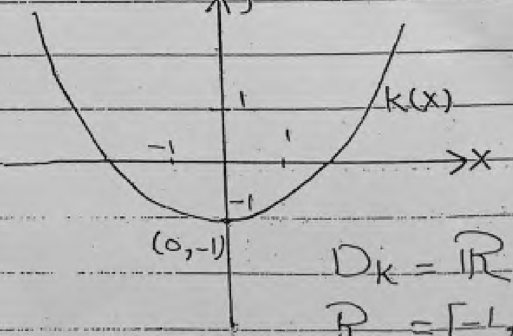
(1) $g(x) = f(x)+1 = x^2+1$



$D_g = \mathbb{R}$

$R_g = [1,\infty)$

(2) $k(x) = f(x)-1 = x^2-1$

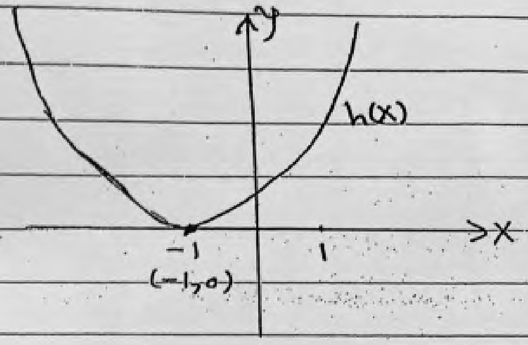


$D_k = \mathbb{R}$

$R_k = [-1,\infty)$

$x+1=0 \Rightarrow x=-1$

{3} $h(x) = f(x+1) = (x+1)^2$

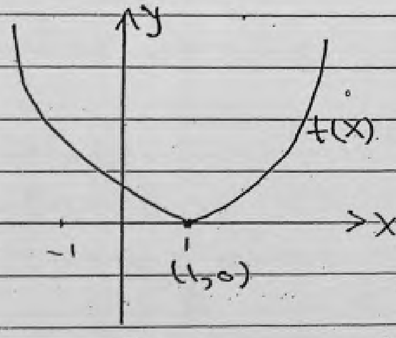


$D_h = \mathbb{R}$

$R_h = \mathbb{R}^+ = [0,\infty)$

$x-1=0 \Rightarrow x=1$

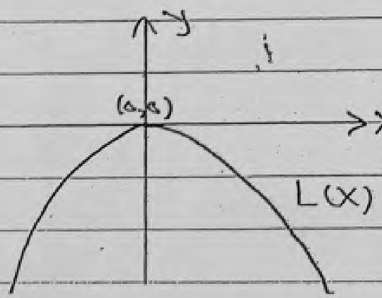
{4} $t(x) = f(x-1) = (x-1)^2$



$D_t = \mathbb{R}$

$R_t = \mathbb{R}^+ = [0,\infty)$

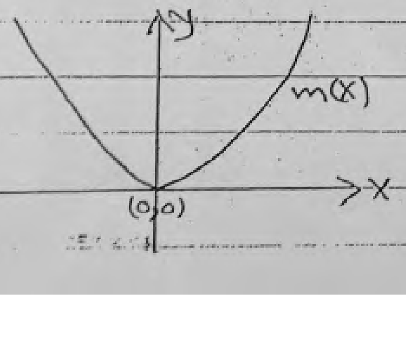
{5} $L(x) = -f(x) = -x^2$



$D_L = \mathbb{R}$

$R_L = \mathbb{R}^- = (-\infty,0]$

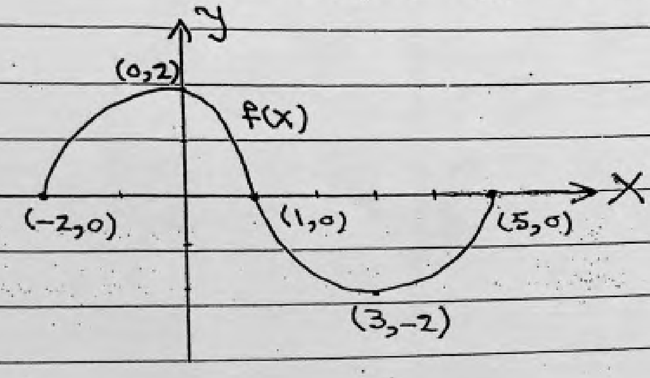
{6} $m(x) = f(-x) = (-x)^2$



$D_m = \mathbb{R}$

$R_m = \mathbb{R}^+ = [0,\infty)$

مثال (٧): أرسم مخطط الدالة $f(3-x)$ اذا كانت $f(x)$ معطاه في الشكل المجاور:



$D_f = [-2,5]$

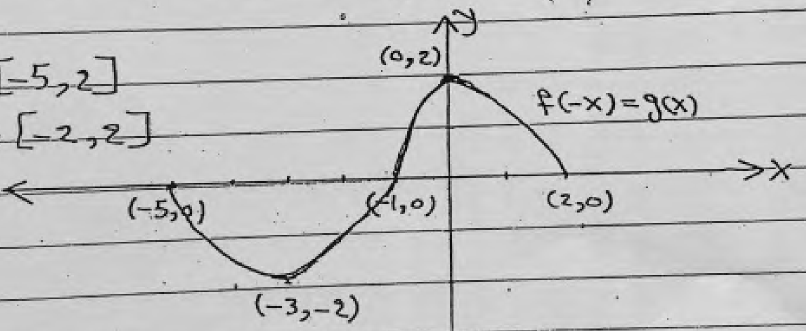
$R_f = [-2,2]$

sol

(١) نعكس المخطط حول محور y للحصول على مخطط $f(-x)$

$D_g = [-5,2]$

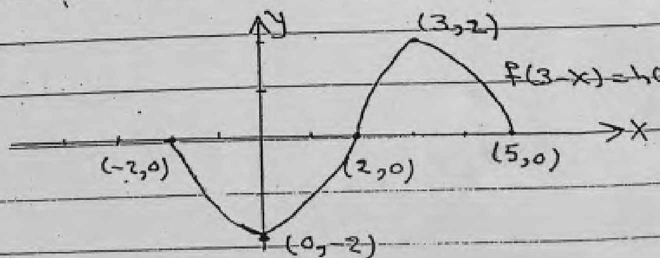
$R_g = [-2,2]$



(٢) ازاحة المخطط الى اليمين ثلاث وحدات للحصول على مخطط الدالة $f(3-x)$

$D_h = [-2,5]$

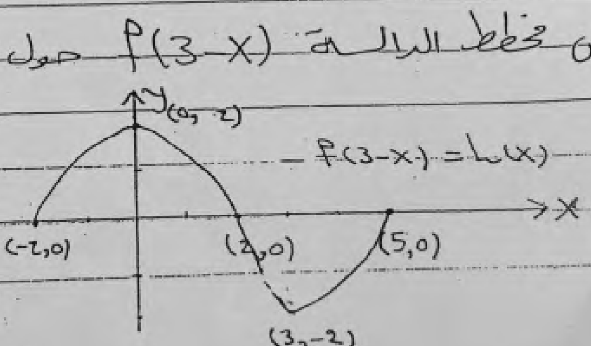
$R_h = [-2,2]$



(٣) نعكس مخطط الدالة $f(3-x)$ حول محور السينات.

$D_L = [-2,5]$

$R_L = [-2,2]$

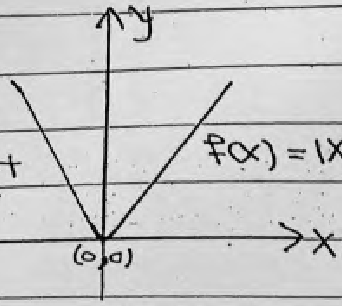


مثال (٣): أرسم مخطط الدالة $y = -|2-x| + 4$

{1} $f(x) = |x|$

$D_f = \mathbb{R}$

$R_f = \mathbb{R}^+$

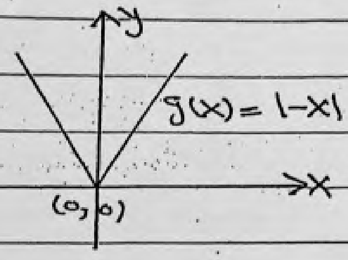


{2} $g(x) = |-x|$

نعكس مخطط الدالة حول محور y

$D_g = \mathbb{R}$

$R_g = \mathbb{R}^+$



{3} $h(x) = |2-x|$

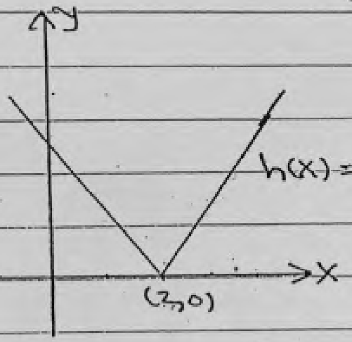
$2 - x = 0$

$\boxed{x = 2}$

ازاحة نحو اليمين

$D_h = \mathbb{R}$

$R_h = \mathbb{R}^+$

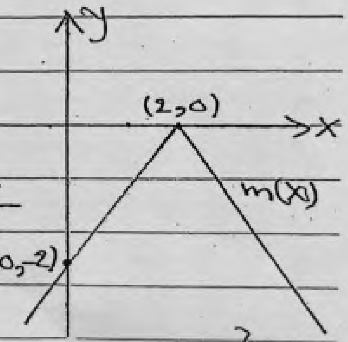


{4} $m(x) = -|2-x|$

نعكس مخطط الدالة حول محور x

$D_m = \mathbb{R}$

$R_m = \mathbb{R}^-$



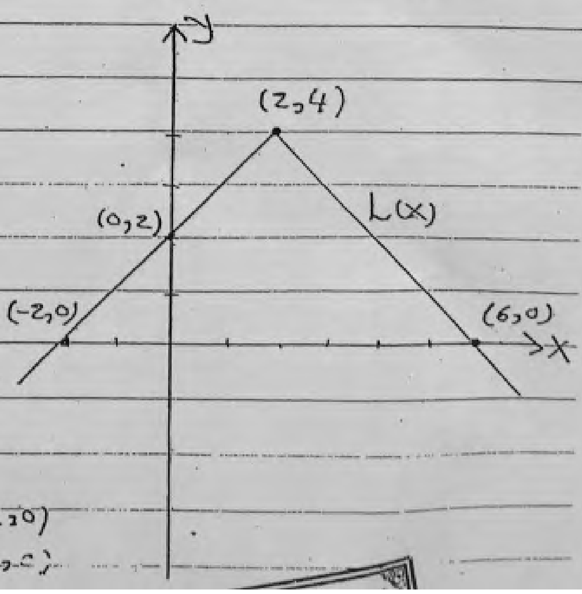
$x = 0$

$\Rightarrow y = -|2-0| = -|2| = -2 \Big\} \Rightarrow (0,$

{5} $L(x) = -|2-x| + 4$

$D_L = \mathbb{R}$

$R_L = (-\infty, 4]$

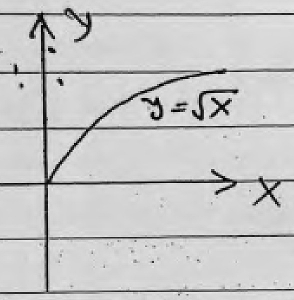


$\left.\begin{array}{l} x = 0 \\ y = -|2-0| + 4 = 2 \end{array}\right\} \Rightarrow (0,2)$

$y = 0$

$|2-x| = 4$: $\left.\begin{array}{l} 2 - x = 4 \Rightarrow x = -2 \\ 2 - x = -4 \Rightarrow x = 6 \end{array}\right\} \Rightarrow \begin{array}{l}(-2,0) \\ (6,0)\end{array}$

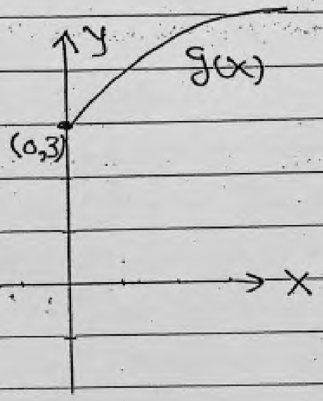
مثال (٤): الشكل الآتي يمثل مخططاً للدالة $y=\sqrt{x}$ ، أرسم مخططاً لكل من الدوال الآتية :



$D_f = \mathbb{R}^+$

$R_f = \mathbb{R}^+$

{1} $g(x) = \sqrt{x} + 3$



$D_g = \mathbb{R}^+$

$R_g = [3, \infty)$

{2} $h(x) = \sqrt{x} - 2$

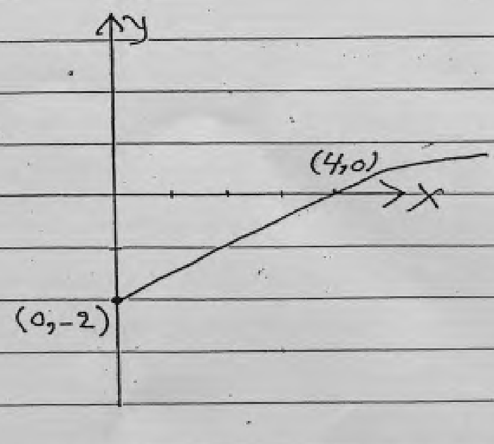
$y = 0 \Rightarrow 0 = \sqrt{x} - 2$

$\Rightarrow \sqrt{x} = 2$

$\Rightarrow x = 4 \Rightarrow (4,0)$

$D_h = \mathbb{R}^+$

$R_h = [-2, \infty)$

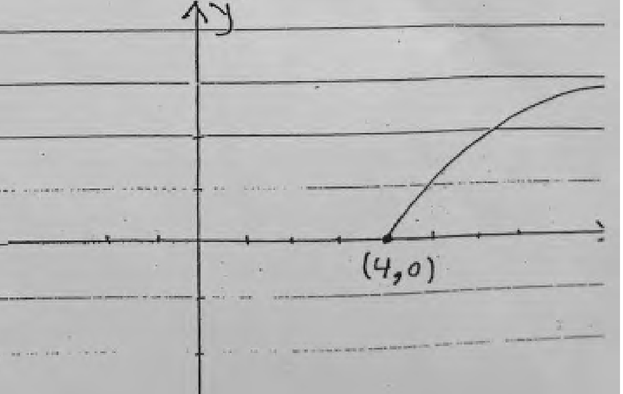


{3} $m(x) = \sqrt{x-4}$

$x - 4 = 0 \Rightarrow x = 4$

$D_m = [4, \infty)$

$R_m = \mathbb{R}^+$



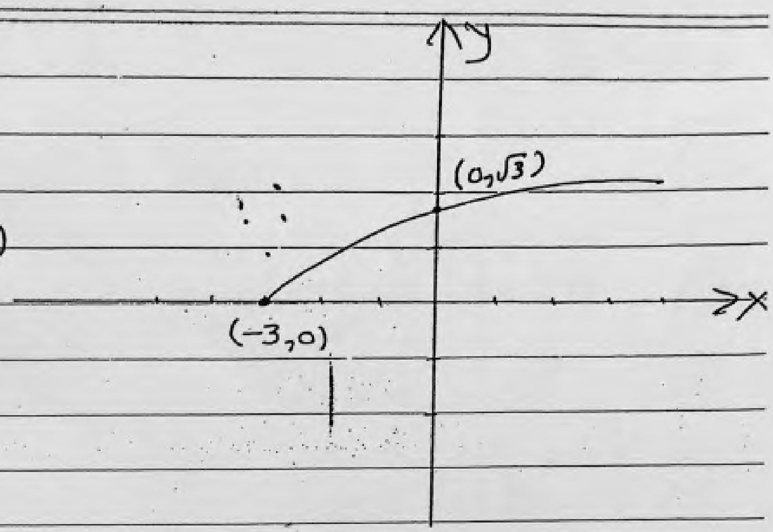
{4} $\sqrt{x+3} = t(x)$

$x+3=0 \Rightarrow x=-3$

$x=0 \Rightarrow y=\sqrt{3} \Rightarrow (0,\sqrt{3})$

$D_t = [-3,\infty)$

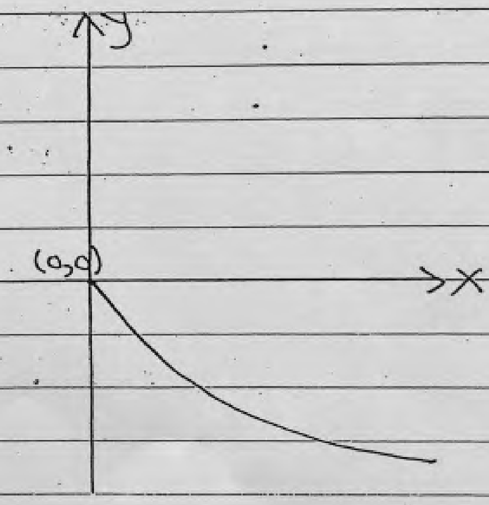
$R_t = \mathbb{R}^+$



{5} $h(x) = -\sqrt{x}$

$D_h = \mathbb{R}^+$

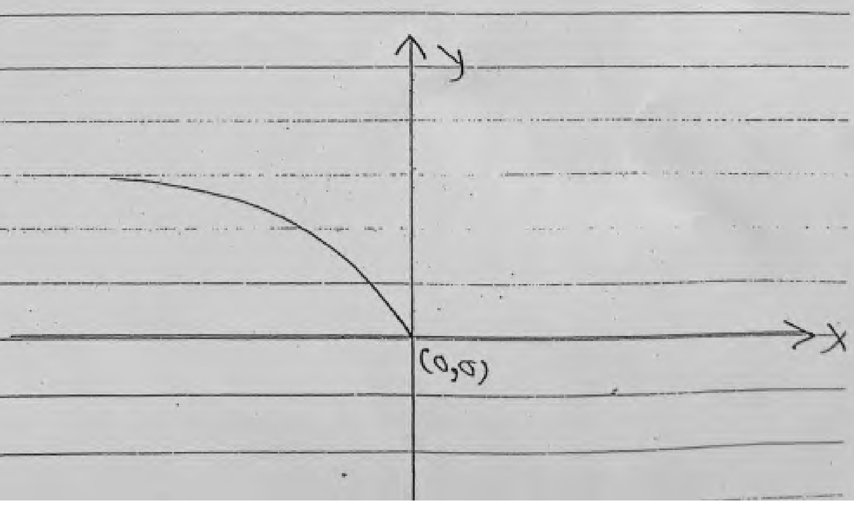
$R_h = \mathbb{R}^-$
$= (-\infty, 0]$



{6} $k(x) = \sqrt{-x}$

$D_k = \mathbb{R}^-$
$= (-\infty, 0]$

$R_k = \mathbb{R}^+$



CHAPTER THREE: Limits and Continuity

Definition: If the values of $f(x)$ approaches the value L as x approaches c, we say that f has **limit** equal to L as x approaches c, and we write it as:

$$\lim_{x \to c} f(x) = L$$

Example: Let $f(x) = x^2 + 3$, find the limit of $f(x)$ as x approaches 2.

$x \to 2^+$	x	3	2.5	2.3	2.1	2.01	2.001	2.0001	
(from the right)	$f(x)$	12	9.25	8.25	7.44	7.040	7.004	7.0007	$\simeq 7$
$x \to 2^-$	x	1	1.2	1.4	1.5	1.9	1.99	1.999	
(from the left)	$f(x)$	4	4.44	4.96	5.95	5.98	6.98	6.999	$\simeq 7$

From the table, we notice that:

- When x approaches 2 from the right, $f(x)$ approaches 7 (i.e., $\lim_{x \to 2^+} f(x) = 7$).
- When x approaches 2 from the left, $f(x)$ approaches 7 (i.e., $\lim_{x \to 2^-} f(x) = 7$).

Properties of Limits: Let $\lim_{x \to c} f_1(x) = L_1$ and $\lim_{x \to c} f_2(x) = L_2$ where c, K, L_1, $L_2 \in \mathbb{R}$, then:

1. $\lim_{x \to c}[f_1(x) \mp f_2(x)] = L_1 \mp L_2$

2. $\lim\limits_{x\to c}[f_1(x) * f_2(x)] = L_1 * L_2$

3. $\lim\limits_{x\to c} K * f_1(x)] = K * L1$

4. $\lim\limits_{x\to c} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$, where $L_2 \neq 0$

Examples: Evaluate the following limits:

1. $\lim\limits_{n\to 5} \frac{\sqrt{4+n}-2}{n} = \frac{\sqrt{4+5}-2}{5} = \boxed{\frac{1}{5}}$

2. $\lim\limits_{x\to 2} \frac{x^2+2x+4}{x+2} = \frac{2^2+2.2+4}{2+2} = \frac{12}{4} = \boxed{3}$

3. $\lim\limits_{x\to 5} \frac{x^2-25}{3(x-5)} = \lim\limits_{x\to 5} \frac{(x+5)(x-5)}{3(x-5)} = \lim\limits_{x\to 5} \frac{x+5}{3} = \frac{5+5}{3} = \boxed{\frac{10}{3}}$

4. $\lim\limits_{h\to 0} \frac{(2+h)^2-4}{h} = \lim\limits_{h\to 0} \frac{4+4h+h^2-4}{h} = \lim\limits_{h\to 0} \frac{h(4+h)}{h}$
$= \lim\limits_{h\to 0} 4 + h = 4 + 0 = \boxed{0}$

5. $\lim\limits_{n\to 0} \frac{\sqrt{4+n}-2}{n} = \lim\limits_{n\to 0} \frac{\sqrt{4+n}-2}{n} . \frac{\sqrt{4+n}-2}{\sqrt{4+n}-2}$
$= \lim\limits_{n\to 0} \frac{4+n-4}{n(\sqrt{4+n}+2)} = \lim\limits_{n\to 0} \frac{n}{n(\sqrt{4+n}+2)}$
$= \lim\limits_{n\to 0} \frac{1}{(\sqrt{4+n}+2)} = \frac{1}{\sqrt{4+0}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$

Right and Left Hand-Side Limits:

Sometimes the value of a function $f(x)$ lend to different limits as x approaches c from different sides.

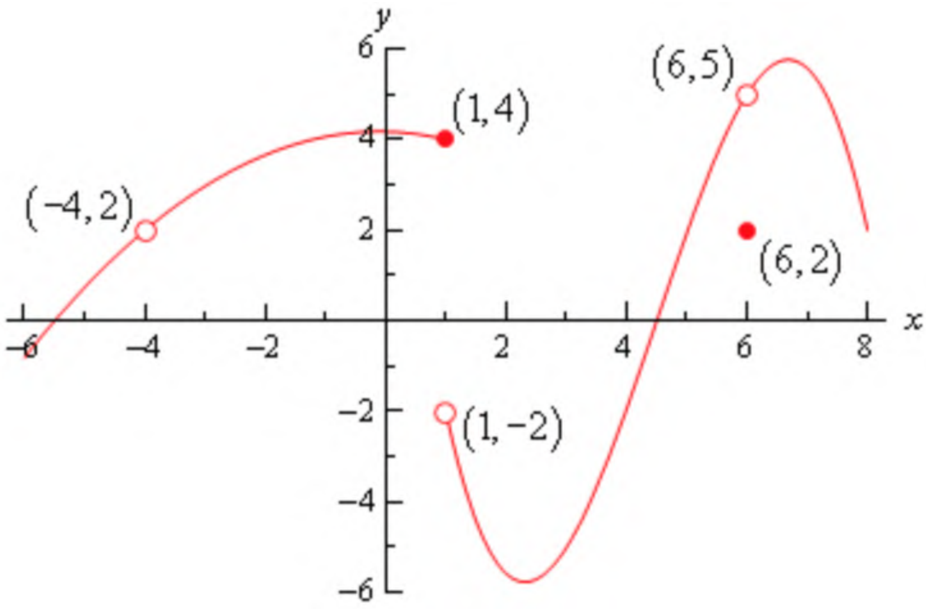
Theorem: Suppose $f(x)$ is defined on an open interval that containing c. Then $\lim_{x \to c} f(x)$ is defined if and only if $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c-} f(x)$ are both defined and equal.

i.e.,

$$\boxed{\lim_{x \to c} f(x) = L \Longleftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c-} f(x) = L}$$

Note If $\lim_{x \to c^+} f(x) \neq \lim_{x \to c-} f(x) \Longrightarrow \lim_{x \to c} f(x)$ "**DOES NOT EXIST**"

Example 1: Evaluate the following, where $f(x)$ is defined as shown below.



- $f(6) = 2$
- $\lim_{x \to 6^-} f(x) = 5$
- $\lim_{x \to 6^+} f(x) = 5$

$\Longrightarrow \lim_{x \to 6} f(x) = 5$

- $f(1) = 4$
- $\lim_{x \to 1^-} f(x) = 4$
- $\lim_{x \to 1^+} f(x) = -2$

$\Longrightarrow \lim_{x \to 1} f(x)$ "Does Note Exists"

Example 2: Let $f(x) = \begin{cases} x^2 - 4 & if \;\; x \leq 3 \\ 5 & if \;\; x > 3 \end{cases}$

Find $\lim_{x \to 3^+} f(x)$, $\lim_{x \to 3^-} f(x)$, and $\lim_{x \to 3} f(x)$

Solution:-

- $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} \begin{cases} x^2-4 & if\ x \le 3 \\ 5 & if\ x > 3 \end{cases}$

  $= \lim\limits_{x\to 3^+} 5 = 5$

- $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} \begin{cases} x^2-4 & if\ x \le 3 \\ 5 & if\ x > 3 \end{cases}$

  $= \lim\limits_{x\to 3^-} x^2 - 4 = 3^2 - 4 = 9 - 4 = 5$

- $\lim\limits_{x\to 3} f(x) = ?$

  $\because \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^-} f(x) = 5 \Longrightarrow \lim\limits_{x\to 3} f(x) = 5$

Example 3: Lef $g(x) = \begin{cases} \sqrt{x+4}-1 & if\ x < 0 \\ -2 & if\ x = 0 \\ \frac{x}{x+3} & if\ x > 0 \end{cases}$

Find $\lim\limits_{x\to 0^+} g(x)$, $\lim\limits_{x\to 0^-} g(x)$, and $\lim\limits_{x\to 0} g(x)$

Solution:-

- $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} \begin{cases} \sqrt{x+4}-1 & if\ x < 0 \\ -2 & if\ x = 0 \\ \frac{x}{x+3} & if\ x > 0 \end{cases}$

  $= \lim\limits_{x\to 0^+} \frac{x}{x+3} = \frac{0}{0+3} = 0$

- $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} \begin{cases} \sqrt{x+4} - 1 & if\ x < 0 \\ -2 & if\ x = 0 \\ \frac{x}{x+3} & if\ x > 0 \end{cases}$

$= \lim\limits_{x \to 0^-} \sqrt{x+4} - 1 = \sqrt{0+4} - 1 = 2 - 1 = 1$

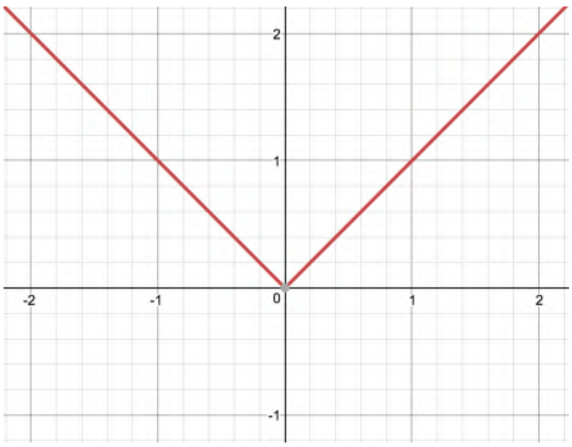
- $\lim\limits_{x \to 0} g(x) = ?$

$\because \lim\limits_{x \to 0^+} g(x) = 0 \neq 1 = \lim\limits_{x \to 0^-} g(x) \Longrightarrow \lim\limits_{x \to 0} g(x)$ **"DOES NOT EXISTS"**

Example 4: Evaluate $\lim\limits_{x \to 0} |x|$?

Solution:-

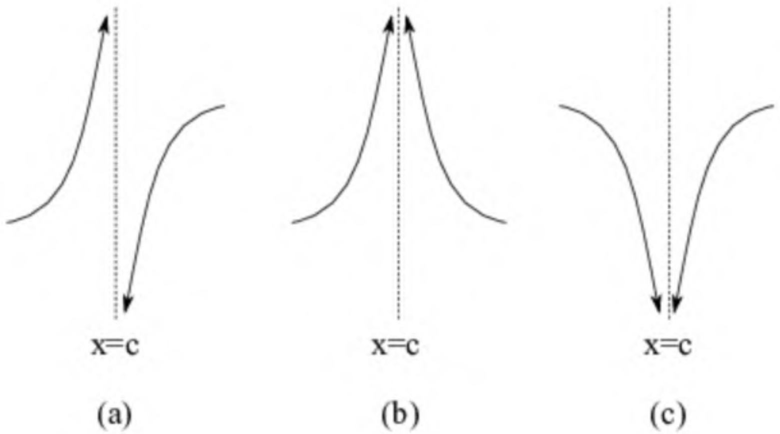
$$\lim_{x \to 0} |x| = \lim_{x \to 0} \begin{cases} x & if x \geq 0 \\ -x & if x < 0 \end{cases}$$



- $\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^+} (x) = 0$
- $\lim\limits_{x \to 0^-} |x| = \lim\limits_{x \to 0^-} (-x) = 0$

$\because \lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^-} |x| = 0 \Longrightarrow \lim\limits_{x \to 0} |x| = 0$

Infinite ($\mp\infty$) Limits : Let $f(x)$ be defined as follows, then:



In (a): $\lim_{x\to c^+} f(x) = +\infty$ and $\lim_{x\to c^-} f(x) = -\infty$

$\Longrightarrow \lim_{x\to c} f(x)$ "**DOES NOT EXIST**"

In (b): $\lim_{x\to c^+} f(x) = +\infty$ and $\lim_{x\to c^-} f(x) = +\infty$

$\Longrightarrow \lim_{x\to c} f(x) = +\infty$

In (c): $\lim_{x\to c^+} f(x) = -\infty$ and $\lim_{x\to c^-} f(x) = -\infty$

$\Longrightarrow \lim_{x\to c} f(x) = -\infty$

Remark:

$\boxed{\dfrac{0}{(\mp)\text{ value}} = 0}$ $\boxed{\dfrac{(+)\text{value}}{0} = +\infty}$ $\boxed{\dfrac{(-)\text{value}}{0} = -\infty}$

Example 1: Evaluate $\lim\limits_{x\to 0} \frac{1}{x^2}$?

Solution:-

- $\lim\limits_{x \to 0^+} \frac{1}{x^2} = \frac{1}{\text{“a } \textbf{positive} \text{ value that is very close to zero”}} = +\infty$

- $\lim\limits_{x \to 0^-} \frac{1}{x^2} = \frac{1}{\text{“a } \textbf{positive} \text{ value that is very close to zero”}} = +\infty$

  $\because \lim\limits_{x \to 0^+} \frac{1}{x^2} = \lim\limits_{x \to 0^-} \frac{1}{x^2} = +\infty \implies \lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$

Example 2: Evaluate $\lim\limits_{x \to 2} \frac{1}{x-2}$?

Solution:-

- $\lim\limits_{x \to 2^+} \frac{1}{x-2} = \frac{1}{\text{“a } \textbf{positive} \text{ value that is very close to zero”}} = +\infty$

- $\lim\limits_{x \to 2^-} \frac{1}{x-2} = \frac{1}{\text{“a } \textbf{negative} \text{ value that is very close to zero”}} = -\infty$

  $\because \lim\limits_{x \to 2^+} \frac{1}{x-2} \neq \lim\limits_{x \to 2^-} \frac{1}{x-2} \implies \lim\limits_{x \to 2} \frac{1}{x-2}$ “**DOES NOT EXIST**”

Example 3: Evaluate $\lim\limits_{x \to -1} \frac{x}{1+x}$?

Solution:-

- $\lim\limits_{x \to -1^+} \frac{x}{1+x} = \frac{\textbf{negative} \text{ value}}{\text{“a } \textbf{positive} \text{ value that is very close to zero”}} = -\infty$

- $\lim\limits_{x \to -1^-} \frac{x}{1+x} = \frac{\textbf{negative} value}{\text{“a } \textbf{negative} \text{ value that is very close to zero”}} = +\infty$

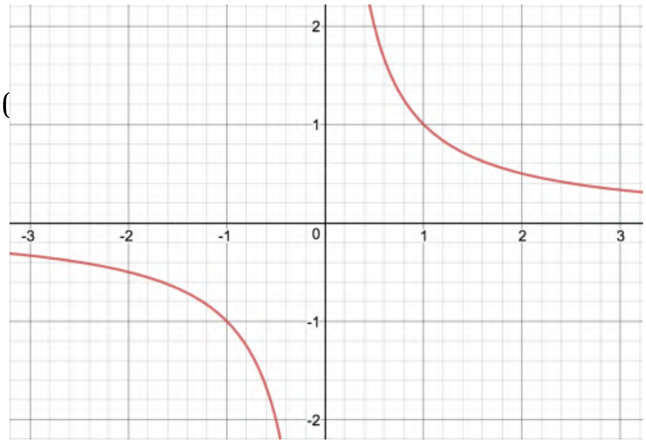
  $\because \lim\limits_{x \to -1^+} \frac{x}{1+x} \neq \lim\limits_{x \to -1^-} \frac{x}{1+x} \implies \lim\limits_{x \to -1} \frac{x}{1+x}$ “**DOES NOT EXIST**”

Evaluating Limits at Infinite ($\mp\infty$):

Remark (1): $\boxed{\dfrac{1}{\mp\infty} = 0}$

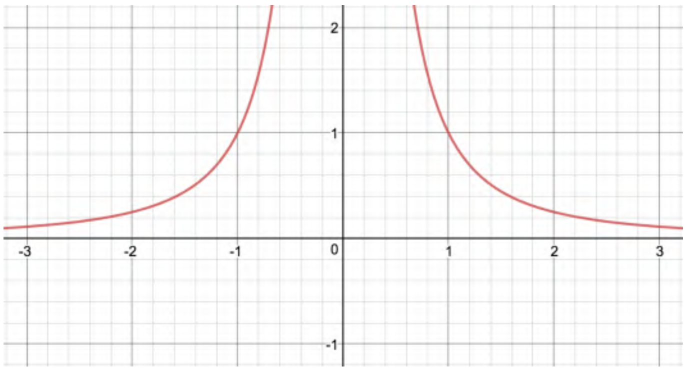
Examples:

- $\lim\limits_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$ $\lim\limits_{x\to-\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$



- $\lim\limits_{x\to\infty} \frac{1}{x^2} = \frac{1}{(\infty)^2} = \frac{1}{\infty} = 0$

  $\lim\limits_{x\to-\infty} \frac{1}{x^2} = \frac{1}{(-\infty)^2} = \frac{1}{\infty} = 0$



Remark (2): To find the limit of a rational function as $x \to \infty$ (when the limit exists), we divided the numerator and denominator by the highest power of x in the denominator.

Examples:

- $$\lim_{x\to\infty} \frac{x^2+2x+1}{5x^2+2} = \lim_{x\to\infty} \frac{\frac{x^2+2x+1}{x^2}}{\frac{5x^2+2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{2x}{x^2}+\frac{1}{x^2}}{\frac{5x^2}{x^2}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{2}{x^2}+\frac{1}{x^2}}{\frac{5}{1}+\frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1+0+0}{5+0} = \boxed{\frac{1}{5}}$$

- $$\lim_{x\to\infty} \frac{x-2}{2x^2-7x+5} = \lim_{x\to\infty} \frac{\frac{x}{x^2}-\frac{2}{x^2}}{\frac{2x^2}{x^2}-\frac{7x}{x^2}+\frac{5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^2}}{\frac{2}{1}-\frac{7}{x}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{0}{2-0+0} = \boxed{0}$$

- $$\lim_{x\to\infty} \frac{x^5+x^2+2}{x^3+1} = \lim_{x\to\infty} \frac{\frac{x^5+x^2+2}{x^3}}{\frac{x^3+1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{x^2+\frac{1}{x}+\frac{2}{x^3}}{1+\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\infty+0+0}{1+0} = \boxed{+\infty}$$

- $$\lim_{x\to\infty} \frac{-4x^3+7x}{2x^2-3x-10} = \lim_{x\to\infty} \frac{\frac{-4x^3+7x}{x^2}}{\frac{2x^2-3x-10}{x^2}}$$

$$= \lim_{x\to\infty} \frac{-4x+\frac{7}{x}}{2-\frac{3}{x}-\frac{10}{x^2}} = \frac{-\infty+0}{2-0-0} = \boxed{-\infty}$$

Problems (3.1):

1. Evaluate the following limits:

(a) $\lim_{x\to 2} \frac{x^2}{x^3-9}$

(b) $\lim_{x\to -3} \frac{\sqrt{2x+22}-4}{x+3}$

(c) $\lim_{x\to 4} \frac{1}{x^2-16}$

(d) $\lim_{x\to 0^+} \frac{1}{3x}$

(e) $\lim_{x\to 0^+} \frac{1}{x}$

(f) $\lim_{x\to \infty} \frac{2x+3}{5x+7}$

(g) $\lim_{x\to 3^+} \frac{1}{x-3}$

(h) $\lim_{x\to -\infty} \frac{10x^5+x^4+31}{x^6}$

(i) $\lim_{x\to \infty} (\frac{-x}{7x+4} + \frac{5x+2}{2x^3-1})$

(j) $\lim_{x\to -2^-} \frac{x^2-2}{x-2}$

(k) $\lim_{x\to -\infty} \frac{9-x\ -x^3}{3+2x+x^2}$

2. Let $f(x) = \begin{cases} \frac{x-2}{x-1} & if\ x \leq 0 \\ \frac{1}{x^2} & if\ x > 0 \end{cases}$

Find:

(a) $f(0)$

(b) $\lim_{x\to +\infty} f(x)$

(c) $\lim_{x\to -\infty} f(x)$

(d) $\lim_{x\to 0^+} f(x)$

(e) $\lim_{x\to 0^-} f(x)$

(f) Does the Limit exists at $x = 0$?

3. Let $g(x) = \begin{cases} 3-x & if\ x < 2 \\ 6 & if\ x = 2 \\ \frac{x}{2} & if\ x > 2 \end{cases}$

Find:

(a) $g(2)$

(b) $g(3)$

(c) $g(-1)$

(d) $\lim_{x\to 2^+} g(x)$

(e) $\lim_{x \to 2^-} g(x)$

(f) Does the Limit exists at $x = 2$?

Continuous Function

We say f is a continuous function at the point x_0 if there is no interrupt at x_0, and f is a continuous function at the interval x_0 if there is no any interrupt in this interval.

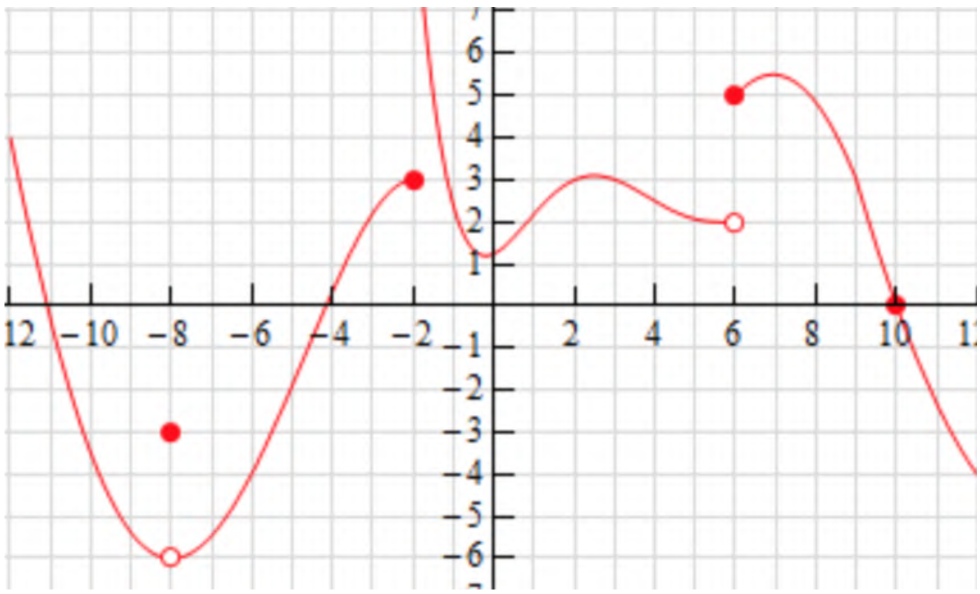
Definition: f is a continuous function at $x_0 \iff$

1. $f(x_0)$ exists
2. $\lim\limits_{x \to x_0} f(x)$ exists $\Big($**i.e.,** $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x) = L\Big)$
3. $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Definitions:

- If $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$, then $f(x)$ is continuous from the right at x_0
- If $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$, then $f(x)$ is continuous from the left at x_0
- $f(x)$ is continuous at $x_0 \iff f(x)$ is continuous from the right and the left.

Example 1: Let $f(x)$ be defined as shown below. Check if $f(x)$ is continuous at x_0 where $x_0 = 10, 6, -2, -8$.



1. - $f(10) = 0$
   - $\lim\limits_{x \to 10^+} f(x) = 0$

     $\lim\limits_{x \to 10^-} f(x) = 0$

     $\because \lim\limits_{x \to 10^+} f(x) = \lim\limits_{x \to 10^-} f(x) = 0 \implies \lim\limits_{x \to 10} f(x) = 0$ "Exists"
   - $\because \lim\limits_{x \to 10} f(x) = f(10) \implies f(x)$ is **continuous** at 10

   $\boxed{\textbf{Note}}$ $f(x)$ is continuous from the right and from the left at $x_0 = 10$

2. - $f(6) = 5$
   - $\lim\limits_{x \to 6^+} f(x) = 5$

     $\lim\limits_{x \to 6^-} f(x) = 2$

     $\because \lim\limits_{x \to 6^+} f(x) \neq \lim\limits_{x \to 6^-} f(x) \implies \lim\limits_{x \to 6} f(x)$ "Does Not Exists"

- $\because \lim_{x \to 6} f(x)$ "Does Not Exists" $\implies$ $f(x)$ is **discontinuous** at 6

$\boxed{\textbf{Note}}$ $f(x)$ is continuous from the right at $x_0 = 6$

3. - $f(-2) = 3$
   - $\lim_{x \to -2^+} f(x) = +\infty$

     $\lim_{x \to -2^-} f(x) = 3$

     $\because \lim_{x \to -2^+} f(x) \neq \lim_{x \to -2^-} f(x) \implies \lim_{x \to -2} f(x)$ "Does Not Exists"
   - $\because \lim_{x \to -2} f(x)$ "Does Not Exists" $\implies$ $f(x)$ is **discontinuous** at -2

$\boxed{\textbf{Note}}$ $f(x)$ is continuous from the left at $x_0 = -2$

4. - $f(-8) = -3$
   - $\lim_{x \to -8^+} f(x) = -6$

     $\lim_{x \to -8^-} f(x) = -6$

     $\because \lim_{x \to -8^+} f(x) = \lim_{x \to -8^-} f(x) = -6 \implies \lim_{x \to -8} f(x) = -6$ "Exists"
   - $\because \lim_{x \to 10} f(x) \neq f(10) \implies$ $f(x)$ is **discontinuous** at 10

$\boxed{\textbf{Note}}$ $f(x)$ is Not continuous Neither form the right nor from the left at $x_0 = -8$

Example 2: Let $f(x) = \begin{cases} x+1 & if\ x \geq 0 \\ 1 & if\ x < 0 \end{cases}$

Is $f(x)$ continuous at $x_0 = 0$?

Solution:

- $f(0) = 0 + 1 = 1$
- $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (x+1) = 0 + 1 = 1$

  $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} (1) = 1$

  $\because \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = 1 \implies \lim\limits_{x \to 0} f(x) = 1$ "Exists"
- $\because \lim\limits_{x \to 0} f(x) = f(0) \implies f(x)$ is **continuous** at 0

Example 3: Let $f(x) = \begin{cases} \frac{x^2-4}{x-2} & if\ x \neq 2 \\ 3 & if\ x = 2 \end{cases}$

Is $f(x)$ continuous at $x_0 = 2$?

Solution:

- $f(2) = 3$
- $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \frac{x^2-4}{x-2} = \lim\limits_{x \to 2^+} \frac{(x-2)(x+2)}{x-2} = \lim\limits_{x \to 2^+} (x+2) = 4$

  $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \frac{x^2-4}{x-2} = \lim\limits_{x \to 2^-} \frac{(x-2)(x+2)}{x-2} = \lim\limits_{x \to 2^-} (x+2) = 4$

$\because \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 4 \implies \lim_{x\to 2} f(x) = 4$ "Exists"

- $\because \lim_{x\to 2} f(x) \neq f(2) \implies f(x)$ is **discontinuous** at 2

Example 4: Let $f(x) = \begin{cases} \frac{x^3-1}{x-1} & if\ x \neq 1 \\ K & if\ x = 1 \end{cases}$

be a continuous function at $x_0 = 1$, Find the value of K?

Solution:

$\because f(x)$ is continuous function at $x_0 = 1 \implies f(1) = \lim_{x\to 1} f(x)$

$\because f(1) = K$, and

$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3$

Hence, $\lim_{x\to 1} f(x) = f(1) \implies \boxed{K = 3}$

Example 5: Let $f(x) = \begin{cases} x^2 - 2 & if\ x \leq 2 \\ \mathrm{C}x + 3 & if\ x > 2 \end{cases}$

be a continuous function at $x_0 = 2$, Find the value of C?

Solution:

$\because f(x)$ is continuous function at $x_0 = 2 \implies \lim_{x\to 2} f(x)$ exists

$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x)$

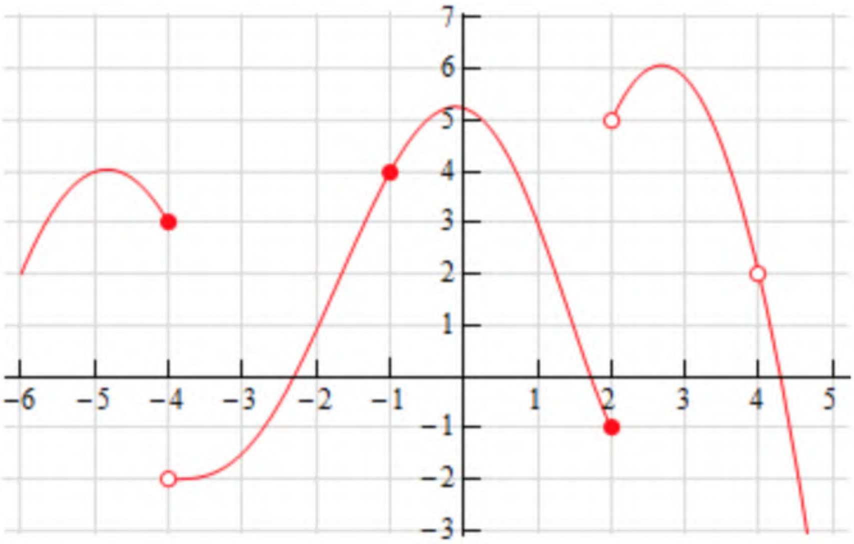
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 - 2 = 2^2 - 2 = 2$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} Cx + 3 = 2C + 3$$

$\because \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} f(x) \implies 2 = 2C + 3 \implies 2C = -1 \implies \boxed{C = \frac{-1}{2}}$

Problems (3.2):

1. Let $f(x)$ be defined as shown below. Check if $f(x)$ is continuous at x_0 where $x_0 = -4, -1, 2, 4$. If it is not continuous, is it right (or left) continuous at x_0? why?



2. Let $f(x) = \begin{cases} \frac{x^3+3x}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$
is $f(x)$ is continuous function at $x_0 = 0$? If $f(x)$ is discontinuous, redefine $f(x)$ to be continuous at $x_0 = 0$?

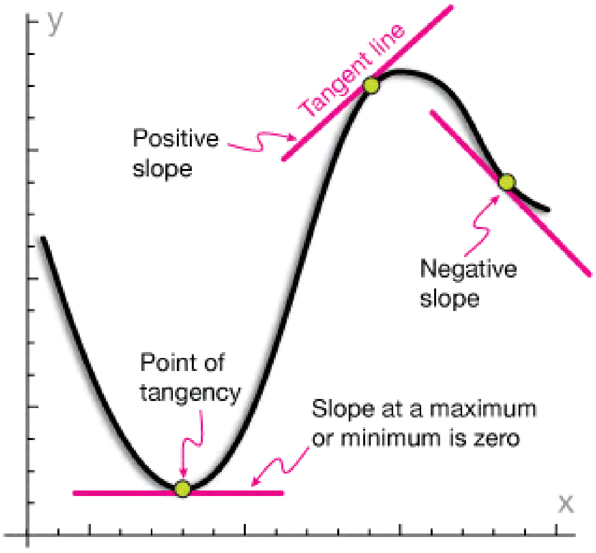
3. Let $f(x) = \begin{cases} ax + 3 & if\ x \geq 1 \\ 3x^2 + 1 & if\ x < 1 \end{cases}$
be a continuous function at $x_0 = 1$, find the value of a?

4. Let $f(x) = \begin{cases} 2x + M & if\ x \leq -1 \\ x^2 + N & if\ x > -1 \end{cases}$
be a continuous function at $x_0 = -1$ and $f(2) = 7$, find the values

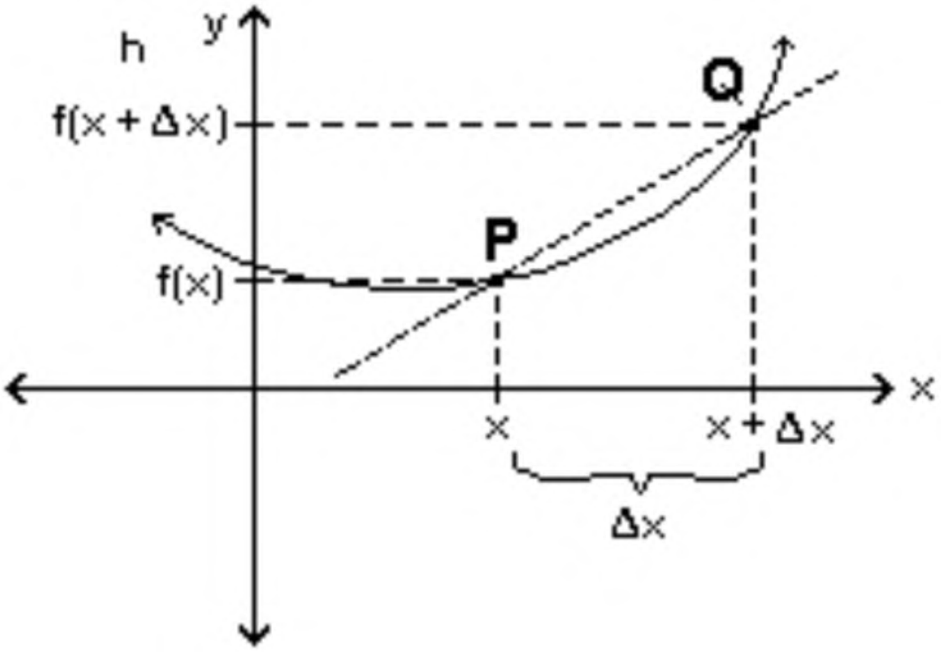
of M and N?

CHAPTER FOUR: Differentiation

For each point on the curve $y = f(x)$, there is a single straight tangent line at the point; The slop of straight tangent of the curve $y = f(x)$ at the point $(x, f(x))$ represents the derivative at that point.



Let $P(x, f(x))$ be a fixed point on the curve; and $Q(x+\Delta x, f(x+\Delta x))$ be another point, so $\Delta y = f(x + \Delta x) - f(x)$.



Note that: At Δx, decreasing length (close to zero) the straight secant PQ more and more applicability begins on the straight tangent at the point $(x, f(x))$. When $(\Delta x \to 0)$, knowing that the slop straight tangent at the point $(x, f(x))$ represents a derived function at that point.

$$m_{tan} = \lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\boxed{f^{'}(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

Remark: When the value of the limit exist, the function is called differentiable function, and $f^{'}$ is called the derivative of f at x.

Remark: The equation of the tangent line at a point (x_1, y_1) is given by the following form:

$$\boxed{(y - y_1) = m_{tan}(x - x_1)}$$

Definition: The normal line of a curve is the line that is perpendicular to the tangent of the curve at a particular.

$$\boxed{m_{\perp} = \frac{-1}{m_{tan}}}$$

Remark: The equation of the normal line at a point (x_1, y_1) is given by the following form:

$$\boxed{(y - y_1) = m_{\perp}(x - x_1)}$$

$\boxed{\textbf{Note}}$ $f^{'}(x) = y^{'} = \frac{dy}{dx} = \frac{df(x)}{dx}$

Example 1: Let $f(x) = 4x - 2$, find $f'(x)$ by using the definition?

Solution:-

$f^{'}(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x))}{\Delta x}$

$\because f(x) = 4x - 2,\ f(x + \Delta x) = 4(x + \Delta x) - 2$

$$\begin{aligned} \Longrightarrow f^{'}(x) &= \lim_{\Delta x \to 0} \frac{[4(x + \Delta x) - 2] - [4x - 2])}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{4x + 4\Delta x - 2 - 4x + 2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{4\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} 4 = 4 \end{aligned}$$

Example 2: Let $f(x) = \sqrt{x}$, find the equation of the tangent line and normal line at the point $(4, 2)$ by using the definition?

__**Solution**__:-

We need to find: $m_{tan}]_{(4,2)} = f^{'}(x)]_{(4,2)}$

$$\begin{aligned}
\Longrightarrow f^{'}(x) &= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} . \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

$\Longrightarrow m_{tan} = \frac{1}{2\sqrt{x}} \Longrightarrow m_{tan}]_{(4,2)} = f^{'}(x)]_{(4,2)} = \frac{1}{2\sqrt{4}}$

Now, we need to find the equation of the tangent line at the point $(x_1, y_1) = (4, 2)$

$(y - y_1) = m_{tan}(x - x_1)$

$\Longrightarrow y - 2 = \frac{1}{4}(x - 4)$

$\Longrightarrow y = \frac{1}{4}x + 1$

Next, we need to find the equation of the normal line at the point $(x_1, y_1) = (4, 2)$

$\because m_{\perp} = \frac{-1}{m_{tan}} \longrightarrow m_{\perp} = \frac{-1}{\frac{1}{4}} = -4$

$(y - y_1) = m_{\perp}(x - x_1)$

$\Longrightarrow \; y - 2 = -4(x - 4)$

$\Longrightarrow \; y = -4x + 18$

Problems 4.1:

1. Find $f'(x)$ by using the definition of the following function:-

   (a) $f(x) = x^2$ (b) $f(x) = 4 - \sqrt{x+3}$

2. Let $f(x) = x^2$, find the equation of the tangent line and normal line at the point $(3, 9)$ by using the definition.

3. Let $f(x) = \sqrt{x+3}$, find the equation of the tangent line at $x = 2$.

Differentiable VS. Continuous:

Theorem: If f(x) is a differentiable function at x_0, then it is a continuous function at x_0.

Proof: To prove $f(x)$ is continuous function at x_0,

we need to show: $\lim_{x\to 0} f(x) = f(x_0)$ $\Big($i.e., $\lim_{x\to 0}[f(x) - f(x_0)] = 0\Big)$

Suppose that:

$\Delta x = x - x_0 \implies x = x_0 + \Delta x \implies f(x) = f(x_0 + \Delta x)$

Hence, when $x \to 0$, $\Delta x \to 0$

$$\begin{aligned}
\lim_{x\to 0}[f(x) - f(x_0)] &= \lim_{x\to 0}[f(x_0 + \Delta x) - f(x_0)] \\
&= \lim_{x\to 0}\left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.\Delta x\right] \\
&= \lim_{x\to 0}\left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.\lim_{x\to 0}\Delta x\right] \\
&= f^{'}(x_0).0 = 0
\end{aligned}$$

Note The inverse of the above theorem is not true.

(i.e., If f(x) is a continuous at x_0, then it is not necessary to be differentiable at x_0)

Example: Let $f(x) = |x|$, and $x_0 = 0$.

From the above plot $f(x) = |x|$ is continuous at $x_0 = 0$.

However, $f(x) = |x|$ is **not differentiable** at $x_0 = 0$.

Proof:

$$\because |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

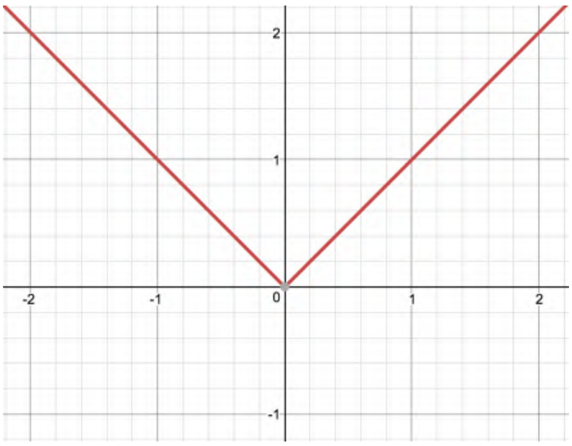
$$\therefore |\Delta x| = \begin{cases} \Delta x & \Delta x \geq 0 \\ -\Delta x & \Delta x < 0 \end{cases}$$

$$\begin{aligned} f^{'}(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{|x+\Delta x| - |x|}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{|0+\Delta x| - |0|}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} \end{aligned}$$

Hence, $L^{+} = \lim_{\Delta x \to 0^{+}} = 1$ & $L^{-} = \lim_{\Delta x \to 0^{-}} = -1$

Since, $L^{+} \neq L^{-} \implies$ The limit does not exists.

$\therefore f(x)$ is not a differentiable function at $x_0 = 0$



General Theorems of Differentiation:

$\boxed{\textbf{NOTE}}$: The following theorems are going to be given without proofs. The proofs can be find in any calculus book.

Theorem(1): If $f(x) = c$, c be a constant, then $f^{'}(x) = 0$.

Theorem(2): If $f(x)$ is a differentiable function at x, and let c be a constant, then $(c.f)$ is differentiable at x and $(c.f)^{'}(x) = c.f^{'}(x)$.

Theorem(3): If $f(x)$ and $g(x)$ are a differentiable functions at x, then $(f + g)$ is differentiable at x and $(f + g)^{'}(x) = f^{'}(x) + g^{'}(x)$.

Remark: In general, If $f_1, f_2, \ldots, f_n$ are differentiable function at x, then $(f_1, f_2, \ldots, f_n)^{'}(x) = f_1^{'}(x) \mp f_2^{'}(x) \mp \ldots \mp f_n^{'}(x)$.

Theorem(4): If $f(x) = x^n$ where $n > 0$, then $f^{'}(x) = nx^{n-1}$.

Theorem(5): If $f(x)$ and$g(x)$ are two differentiable functions at x, then $f.g$ is differentiable function at x and

$(f.g)^{'}(x) = f(x).g^{'}(x) + f^{'}(x).g(x)$.

Remark: In general, if f, g and h are differentiable functions at x, then:

$$
\begin{aligned}
(f.g.h)'(x) &= f(x).(g.h)'(x) + f'(x).(g.h)(x) \\
&= f(x).(g(x).h'(x) + g'(x).h(x)) + f'(x).(g.h)(x) \\
&= f(x).g(x).h'(x) + f(x).g'(x).h(x)) + f'(x).g(x).h(x)
\end{aligned}
$$

Theorem(6): If $f(x)$ and$g(x)$ are two differentiable functions at x, then $\frac{f(x)}{g(x)}$ and $g(x) \neq 0$, then $\frac{f(x)}{g(x)}$ is differentiable function at x and
$(\frac{f}{g})'(x) = \frac{g(x).f'(x) - f(x).g'(x)}{(g(x))^2}$

Theorem(7): If $g(x)$is a differentiable functions at x, $f(x)$ is differentiable functions at $f(x)$, and $h = f \circ g$, then
$h'(x) = (f \circ g)'(x) = f'\Big(g(x)\Big)g'(x)$

Theorem(8): If $f(x)$ is a differentiable functions at x, and $y = (f(x))^n$ where $n \in \mathbb{Z}$, then
$y' = \frac{dy}{dx} = \Big((f(x))^n\Big)' = n\Big(f(x)\Big)^{n-1}.f'(x)$

Problems (4.2):

1. Find derivative of the following functions:

   (a) $y = (\frac{x^2+3}{x+1})^4$

   (b) $y = (2\sqrt{x} - 1)^3$

(c) $y = \sqrt{3 - x^2}$

(d) $f(w) = \sqrt{w} + \sqrt[3]{w} + \sqrt[4]{w}$

(e) $f(x) = (x^3 + 2)^2(1 - x^2)^3$

(f) $f(x) = \frac{(1+2x^3)(1+x^4)}{x^2}$

(g) $f(x) = \sqrt{x + \sqrt{1 + \sqrt{x}}}$

(h) $f(t) = t^3 - \frac{1}{t^2+1}$

(i) $f(t) = \frac{\sqrt{t^2+1}}{(t+2)^4}$

(j) $f(z) = z^2(z^2 + 1)^{-\frac{1}{3}}$

2. Let $f(x) = x$ and $g(x) = x^2$, what is the value of x that makes the tangent line of two curves are parallel.

3. Let $f(x) = \frac{1}{\sqrt{x}}$, what is the value of x that make the tangent of the curve when it is parallel to the line $x + 8y = 10$

Chain Rule:

1. If $y = f(x)$ and $x = g(t)$, then $\boxed{\frac{dy}{dt} = \frac{dy}{dx} . \frac{dx}{dt}}$

2. If $y = f(x)$ and $t = g(x)$, then $\boxed{\frac{dy}{dt} = \frac{\frac{dy}{dx}}{\frac{dt}{dx}}}$

Example 1: Let $y = 3x - 1$ and $x = 2t$, find $\frac{dy}{dt}$?

Solution:-

$$\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$$
$$= (3).(2) = \boxed{6}$$

OR: y = 3x-1 = 3(2t)-1 = 6t-1 = $\boxed{6}$

Example 2: Let $y = t^2 - 1$ and $x = 2t + 3$, find $\frac{dy}{dx}$?

Solution:-

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = \boxed{t}$$

Problems (4.3): Find $\frac{dy}{dx}$ for the following functions:

1. $y = u^3 + 1, \qquad u = x^2 + 3$
2. $y = 3t^2 - 1, \qquad x = 6t - 1$
3. $y = \frac{t^2}{1+t}, \qquad x = \frac{t}{2+t}$
4. $y = t^2, \qquad x = \frac{t}{1-t}$
5. $y = z^{\frac{2}{3}}, \qquad z = x^2 + 1$
6. $y = w^2 - w^{-1}, \qquad w = 3x$
7. $y = 2v^3 + \frac{2}{v^3}, \qquad v = (2x+2)^{\frac{2}{3}}$
8. $y = \frac{u^2}{u^2+1}, \qquad u = \sqrt{2x+1}$

Implicit Differentiation

Example 1: Let $x^2 + xy + y^5 = 0$, find $\frac{dy}{dx}$ and $\frac{dx}{dy}$?

Solution:-

To find $\frac{dy}{dx}$, we derive implicitly for x by considering y is an implicit function of x.

$\because x^2 + xy + y^5 = 0$

$\xRightarrow{\frac{d}{dx}} 2x\frac{dx}{dx} + (x\frac{dy}{dx} + y\frac{dx}{dx}) + 5y^4\frac{dy}{dx} = 0$

$\Longrightarrow\ 2x + xy' + y + 5y^4y' = 0$

$\Longrightarrow\ xy' + 5y^4y' = -2x - y$

$\Longrightarrow\ (x + 5y^4)y' = -2x - y\ \Longrightarrow \boxed{y' = \dfrac{dy}{dx} = \dfrac{-2x - y}{x + 5y^4}}$

To find $\frac{dx}{dy}$, we derive implicitly for y by considering x is an implicit function of y.

$\because x^2 + xy + y^5 = 0$

$\xRightarrow{\frac{d}{dy}} 2x\frac{dx}{dy} + (x\frac{dy}{dy} + y\frac{dx}{dy}) + 5y^4\frac{dy}{dy} = 0$

$\Longrightarrow\ 2x\frac{dx}{dy} + x + y\frac{dx}{dy} + 5y^4 = 0$

$\Longrightarrow\ x + 5y^4 = -2x\frac{dx}{dy} - y\frac{dx}{dy}$

$\Longrightarrow\ x + 5y^4 = (-2x - y)\frac{dx}{dy}\ \Longrightarrow \boxed{\dfrac{dx}{dy} = \dfrac{x + 5y^4}{-2x - y}}$

Note that: $\boxed{\dfrac{dx}{dy} = x' = \dfrac{1}{y'} = \dfrac{1}{\frac{dy}{dx}}}$

Example 2: Find the equation of the tangent line and normal line of the curve $x^2 + y^2 = 2$ at $(1, 1)$.

Solution:- $\because x^2 + y^2 = 2$

$\xRightarrow{\frac{d}{dx}} 2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0 \implies 2x + 2yy' = 0$

$\implies y' = \frac{-2x}{2y} \implies y' = \frac{-x}{y}$

Hence, $y']_{(1,1)} = m_{tan}]_{(1,1)} = \frac{-1}{1} = -1$

The equation of the tangent line: $(y - y_1) = m_{tan}(x - x_1)$

$\implies (y - 1) = -1(x - 1)$

$\implies (y - 1) = -x + 1$

$\implies y = -x + 2$

Since, $m_{\perp}]_{(1,1)} = \frac{-1}{m_{tan}]_{(1,1)}} \implies m_{\perp}]_{(1,1)} = \frac{-1}{-1} = 1$

The equation of the normal line: $(y - y_1) = m_{\perp}(x - x_1)$

$\implies (y - 1) = 1(x - 1)$

$\implies (y - 1) = x - 1$

$\implies y = x$

Problems (4.4):

1. Find the slop of the tangent line of the curve $x^2 + xy + y^2 = 7$ at the point $(1, 2)$.

2. Find the slop of the tangent line of the circle equation $8x^2 + 8y^2 = 232$ at the point $(-5, 2)$.

3. Find the equation of the tangent line and the normal line of the curve $xy^2 + yx^2 + y^2 = 0$ at the point $(1, 1)$.

4. Find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ for the following functions:

   (a) $x^3y^2 + 2xy - x + 3y = 6$

   (b) $x^2 + x^3 = y + y^4$

   (c) $\frac{1}{x} + \frac{1}{y} = x + y$

   (d) $x^2 - \sqrt{xy} + y^2 = 6$

   (e) $x^3 + y^3 - 9xy = 0$

   (f) $x^2y + yx^2 = 3y^3$

   (g) $2 - y^3 + x^2y = 5$

   (h) $(1 + x^2y)^3 + x\sqrt{y} = 9$

High-Order Derivative

Let $y = f(x)$, then:

$f^{'}(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{dy}{dx} = y^{'} = y^{(1)}$ [**First Derivative**]

$f^{''}(x) = \lim\limits_{\Delta x \to 0} \frac{f^{'}(x+\Delta x)-f^{'}(x)}{\Delta x} = \frac{d^2y}{dx^2} = y^{''} = y^{(2)}$ [**Second Derivative**]

$f^{'''}(x) = \lim\limits_{\Delta x \to 0} \frac{f^{''}(x+\Delta x)-f^{''}(x)}{\Delta x} = \frac{d^3y}{dx^3} = y^{'''} = y^{(3)}$ [**Third Derivative**]

$\vdots$

$f^{(n)}(x) = \lim\limits_{\Delta x \to 0} \frac{f^{(n-1)}(x+\Delta x)-f^{(n-1)}(x)}{\Delta x} = \frac{d^{(n)}y}{dx^{(n)}} = y^{(n)}$ [n^{th} **Derivative**]

Notes: $\boxed{\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dx}\right)}, \boxed{\frac{d^3y}{dx^3} = \frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)}, \ldots, \boxed{\frac{d^ny}{dx^n} = \frac{dy}{dx}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right)}$

Example: Let $y = 2x^3 + x^2 - 1$, Find $y^{(1)}$, $y^{(2)}$,$y^{(3)}$, $y^{(4)}$ and $y^{(5)}$?

Solution:-

$\because y = 2x^3 + x^2 - 1$

$\Longrightarrow y^{(1)} = 6x^2 + 2x$

$\Longrightarrow y^{(2)} = 12x + 2$

$\Longrightarrow y^{(3)} = 12$

$\Longrightarrow y^{(4)} = 0 \Longrightarrow y^{(5)} = 0$

Problems (4.5): Find $y^{'}$, $y^{''}$ and $y^{'''}$ for the following:

1. $y = x^7 - x^2 + 4x + 33$

2. $y = -4 + 2x^2 - 7x^3 + x^4$

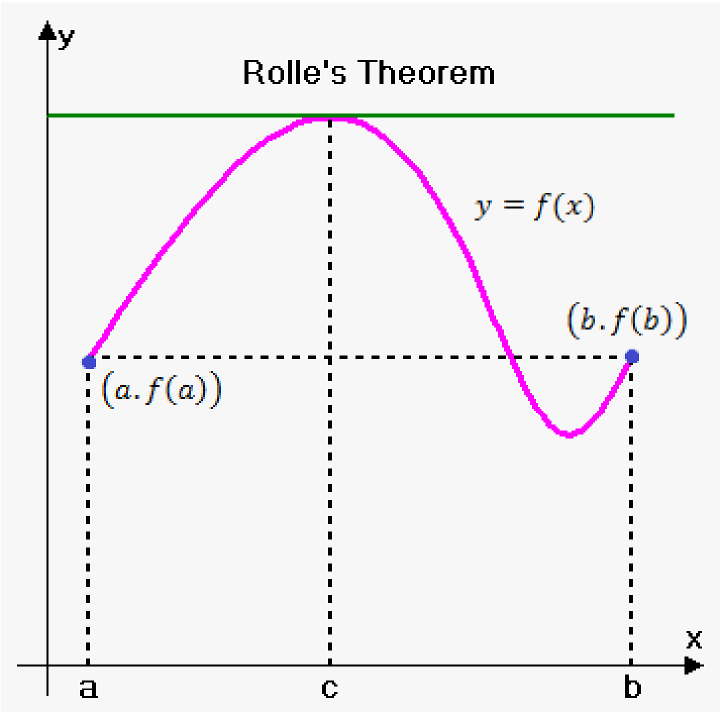
3. $y = \frac{1}{2}x^2 - 100$

4. $y = x^3 - 9x - 5$

5. $y = -x^3 - 9x^2 - 23$

6. $y = -3x^2 - 4x^3 + x^4$

Rolle's Theorem: Let $f(x)$ be a continuous function on $[a, b]$, and f is differentiable on (a, b). If $f(a) = f(b)$, then $f^{'}(c) = 0$ such that $c \in (a, b)$.



Example 1: Let $f(x) = x^2 - 3x + 2$. Show that $f(x)$ satisfies Rolle's theorem on $[1, 2]$.

Solution:-

$f(x)$ is continuous on $[1, 2]$. (because $f(x)$ is a polynomial function)

$f(x)$ is differentiable on $(1, 2)$. (because $f(x)$ is a polynomial function)

$a = 1$ and $b = 2$

$f(a) = f(1) = 1^2 - 3(1) + 2 = 0$

$f(b) = f(2) = 2^2 - 3(2) + 2 = 0$

$\implies f(a) = f(b)$

From above Rolle's theorem is satisfied, and hence $\exists c \in (1, 2)$ s.t. $f^{'}(c) = 0$

$\because f^{'}(x) = 2x - 3$

$\implies f^{'}(c) = 2c - 3 = 0$

$\implies 2c - 3 = 0 \implies c = \frac{3}{2} \in (1, 2)$

Example 2: Let $f(x) = 1 - |x|$. Show that $f(x)$ does not satisfy Rolle's theorem on $[-1, 1]$.

Solution:-

$f(x)$ is continuous on $[-1, 1]$. (because $f(x)$ is a polynomial function)

But, $f(x)$ is not differentiable at $x = 0$?

proof:

$f^{'}(x) = \lim\limits_{\Delta x \to 0} \frac{1 - |x + \Delta x| - 1 + |x|}{\Delta x}$

$f^{'}(0) = \lim\limits_{\Delta x \to 0} \frac{-|x + \Delta x| + |x|}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{-|\Delta x|}{\Delta x}$

$L^{+} = \lim\limits_{\Delta x \to 0^{+}} \frac{-\Delta x}{\Delta x} = -1$

$L^{-} = \lim\limits_{\Delta x \to 0^{-}} \frac{-(-\Delta x)}{\Delta x} = 1$

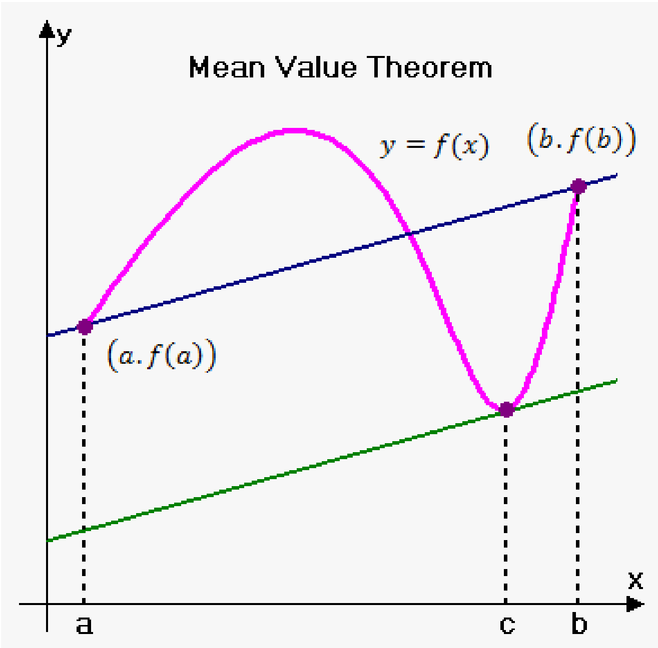
$\because L^{+} \neq L^{-} \implies$ the limit does not exist at 0.

Hence, $f^{'}(0)$ does not exist.

Therefore, $f(x)$ does not satisfy Rolle's theorem on $[-1, 1]$.

The Mean Value Theorem: Let $f(x)$ be a continuous function on $[a, b]$, and f is differentiable on (a, b), then there exist at least one point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Note: Rolle's theorem is a special case from the Mean Value Theorem.

Example 1: Find the value of c that satisfies the Mean Value Theorem, where:

$f(x) = x^2$, $x \in [0, 2]$.

Solution:-

$f(x)$ is continuous on $[0, 2]$.(because $f(x)$ is a polynomial function)

$f(x)$ is differentiable on $(0, 2)$

Since, $f(x)$ is continuous on $[0,2]$ and differentiable on $(0,2)$, then by the Mean Value Theorem there exist at least one point $c \in (a,b)$ such that:

$$f^{'}(c) = \frac{f(b) - f(a)}{b - a}$$

$\because a = 0 \implies f(a) = f(0) = 0^2 = 0$

$\because b = 2 \implies f(b) = f(2) = 2^2 = 4$

$f^{'}(x) = 2x$

$f^{'}(c) = 2c$

$f^{'}(c) = \frac{f(b)-f(a)}{b-a} = \frac{4-0}{2-0} = \frac{4}{2} = 2$

$2c = 2 \implies c = 1 \in (0,2)$

Example 2: Let $f(x) = x^3 - 3x$, and $f : [a,0] \longrightarrow \mathbb{R}$ where f satisfies the Mean Value Theorem at $c = -1$, find the value of a.

Solution:-

$$f^{'}(c) = \frac{f(b) - f(a)}{b - a}$$

$f^{'}(x) = 3x^2 - 3 \implies f^{'}(c) = f^{'}(-1) = 3(-1)^2 - 3 = 0$

$\because a =?$ and $b = 0$

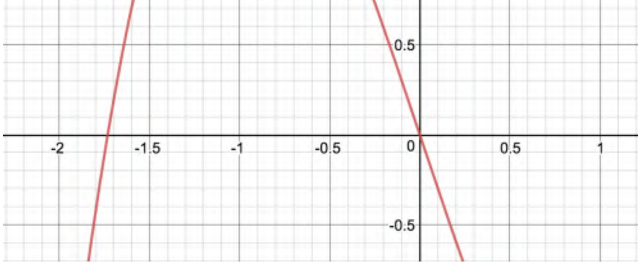
Hence, $0 = \frac{f(0)-f(a)}{0-a}$

$\implies \frac{(0^3-3(0))-(a^3-3a)}{0-a} = 0$

$\implies \frac{0-(a^3-3a)}{0-a} = 0$

$\implies \frac{(a^3-3a)}{a} = 0$

$\implies a^2 - 3 = 0$



$\Longrightarrow\ a^2 = 3$

$\Longrightarrow\ a^2 = \mp\sqrt{3}\ \Longrightarrow\ a = -\sqrt{3} = -1.7$

Problems (4.6):

1. Check whether the following functions satisfy the Rolle's theorem or not?

   (a) $f(x) = (2-x)^2$ on $[0,4]$

   (b) $f(x) = 9x + 3x^2 - x^3$ on $[-1,1]$

2. Find the value of c that satisfies the Mean Value Theorem, where: $f(x) = x^2 - 6x + 4$, $x \in [-1,7]$.

3. Let $f(x) = x^2 - 4x$, and $f : [0,b] \longrightarrow \mathbb{R}$ where f satisfies the Mean Value Theorem at $c = 2$, find the value of b.

L'Hopitals Rule:

Let f and g be differentiable functions at x_0, and

$\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$, where $\lim\limits_{x\to x_0} g'(x) \neq 0$. (OR $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\mp\infty}{\mp\infty}$)

Then:

$$\boxed{\lim_{x\to x_0} \frac{f(x)}{g(x)} \overset{L'R}{=} \lim_{x\to x_0} \frac{f'(x)}{g'(x)}}$$

Example 1: Find $\lim\limits_{x\to 1}\frac{x^2-3x+2}{x^2-1}$.

<u>**Solution**</u>:-

$\because \lim\limits_{x\to 1} x^2-3x+2=0$ and $\lim\limits_{x\to 1} x^2-1=0$

$\therefore \lim\limits_{x\to 1}\frac{x^2-3x+2}{x^2-1}=\lim\limits_{x\to 1}\frac{2x-3}{2x}=\boxed{\dfrac{1}{2}}$

Example 2: Find $\lim\limits_{x\to 0}\frac{2-\sqrt{x+4}}{x}$.

<u>**Solution**</u>:-

$\because \lim\limits_{x\to 0} 2-\sqrt{x+4}=0$ and $\lim\limits_{x\to 0} x=\boxed{0}$

$\lim\limits_{x\to 0}\frac{2-\sqrt{x+4}}{x}$

$=\lim\limits_{x\to 0}\frac{0-\frac{-1}{2}(x+4)^{\frac{-1}{2}}}{1}$

$=-\frac{1}{2}.\lim\limits_{x\to 0}\frac{1}{\sqrt{x+4}}=-\frac{1}{2}.\frac{1}{2}=\boxed{-\dfrac{1}{4}}$

Another Method: by multiplying by the conjugate:

$\lim\limits_{x\to 0}\frac{2-\sqrt{x+4}}{x}$

$=\lim\limits_{x\to 0}\frac{2-\sqrt{x+4}}{x}.\frac{2+\sqrt{x+4}}{2+\sqrt{x+4}}$

$=\lim\limits_{x\to 0}\frac{4-(x+4)}{x(2-\sqrt{x+4})}$

$=\lim\limits_{x\to 0}\frac{-x}{x(2+\sqrt{x+4})}=\frac{-1}{2+\sqrt{0+4}}=\frac{-1}{2+2}=\boxed{\dfrac{-1}{4}}$

<u>**Problems (4.7)**</u>: Find the following limits if it exists:

1. $\lim\limits_{x\to 2}\frac{x^2+2x-8}{x^2-9x+14}$ 2. $\lim\limits_{x\to 0}\frac{\sqrt{x+9}-3}{x}$

3. $\lim\limits_{x\to1} \frac{x^2+5x+4}{x^2-4x-5}$

5. $\lim\limits_{x\to1} \frac{x^2-3x+2}{x^2-1}$

4. $\lim\limits_{x\to0} \frac{4x^3+3x^2-8x+1}{x^3-2x^2+3x-6}$

6. $\lim\limits_{x\to0} \frac{x^3+4x^2-5x}{x^3-2x}$

Increasing and Decreasing Functions:

Definition: A function f is defined on an interval $[a, b]$ is said to be **increasing** on $[a, b]$ if $\forall x_1, x_2 \ni a \leq x_1 < x_2 \leq b \implies f(x_1) < f(x_2)$.

Example: Let $f(x) = x^3$, on $[-2, 2]$.

$-2 < -1 \implies f(-2) = -8 < -1 = f(-1)$

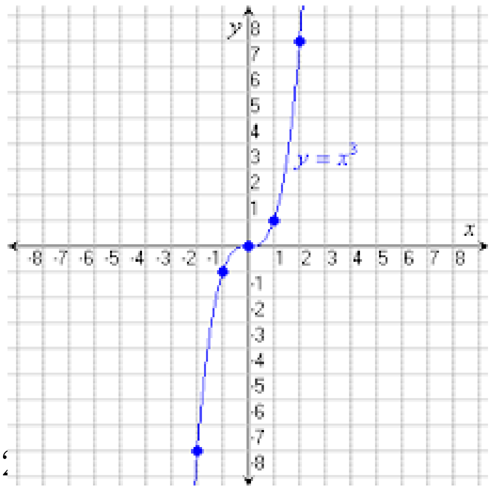
$-1 < 1 \implies f(-1) = -1 < 1 = f(1)$

$1 < 2 \implies f(1) = 1 < 8 = f(2)$

$\vdots$

$\because \forall a, b \in [-2, 2] \implies f(a) < f(b)$

$\therefore f(x)$ is an increasing function on $[-2, 2]$



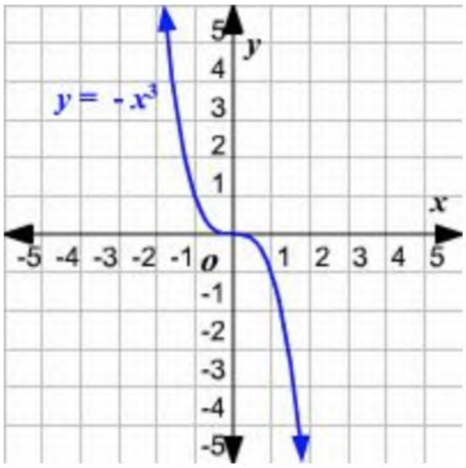
Definition: A function f is defined on an interval $[a, b]$ is said to be **decreasing** on $[a, b]$ if $\forall x_1, x_2 \ni a \leq x_1 < x_2 \leq b \implies f(x_1) > f(x_2)$.

Example: Let $f(x) = -x^3$, on $[-2, 2]$.

$-2 < -1 \implies f(-2) = 8 > 1 = f(-1)$

$-1 < 1 \implies f(-1) = 1 > -1 = f(1)$

$1 < 2 \implies f(1) = -1 > -8 = f(2)$



$\because \forall a, b \in [-2, 2] \implies f(a) > f(b)$

$\therefore f(x)$ is a decreasing function on $[-2, 2]$.

Definition: Let f be defined and continuous function on $[a, b]$, and let $x_0 \in [a, b]$, then $(x_0, f(x_0))$is said to be a **Critical Point** of $f \iff f'(x_0) = 0$ or $f'(x)$ is not defined.

Example 1: Let $f(x) = x^2$ be defined and continuous on [-1,1]. Find the critical points (if exists)?

Solution:

$f'(x) = 2x$

When $f^{'}(x) = 0 \implies 2x = 0 \implies x = 0$

Hence, $(x_0, f(x_0)) = (0, 0)$ is a critical point.

Example 2: Let $f(x) = \frac{x^3}{3} - \frac{x^2}{2}$ be defined and continuous on all the real numbers. Find the critical points (if exists)?

Solution:

$f'(x) = x^2 - x$

When $f^{'}(x) = 0 \implies x^2 - x = 0 \implies x(x-1) = 0 \implies x = 0$ or $x = 1$

Hence, $(0, f(0)) = (0, 0)$ and $(1, f(1)) = (1, -\frac{1}{6})$ are the critical points.

Example 3: Let $f(x) = |x|$ be defined and continuous on $[-1, 1]$. Find the critical points?

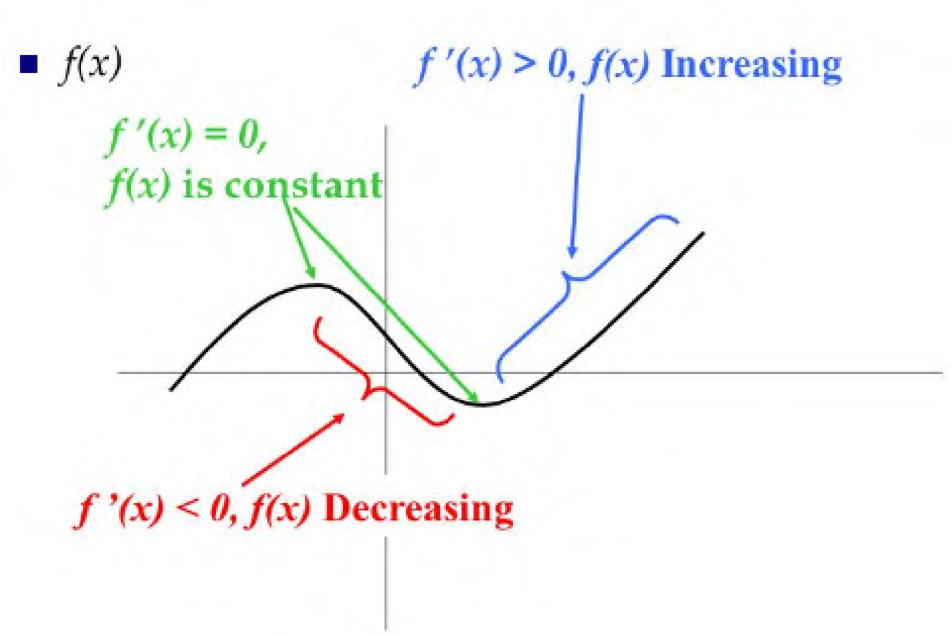
Solution:

$0 \in [-1, 1]$, but $f'(0)$ Does Not Exists

Hence, $(0, f(0)) = (0, 0)$ is a critical point.

Theorem: Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b), then:

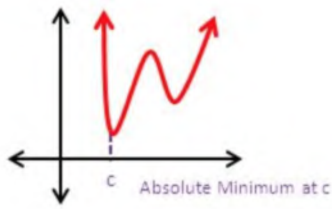
1. If $f'(x) > 0$ $\forall x \in (a, b)$, then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ $\forall x \in (a, b)$, then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ $\forall x \in (a, b)$, then f is constant on $[a, b]$.



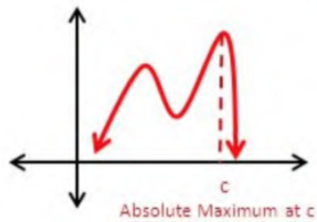
Maximum and Minimum Points:-

Definitions:

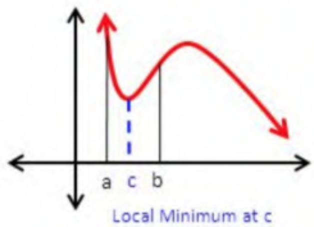
Absolute Minimum – occurs at a point c if $f(c) \leq f(x)$ for x all values in the domain.



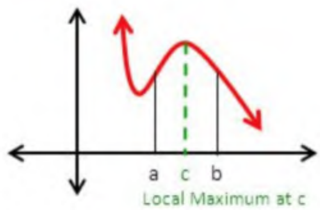
Absolute Maximum – occurs at a point c if $f(c) \geq f(x)$ for all x values in the domain.



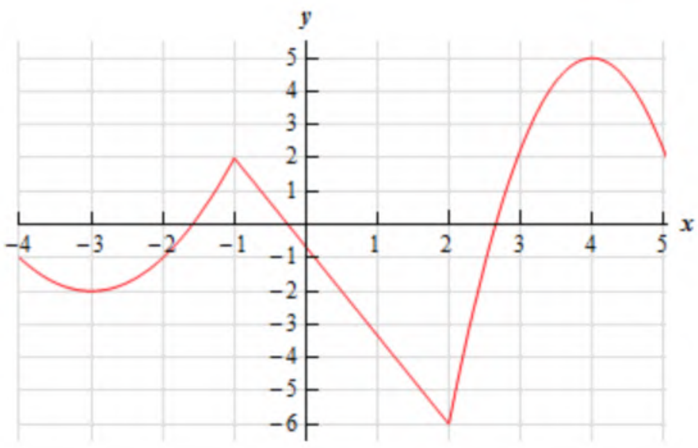
Local Minimum – occurs at a point c in an open interval, (a, b), in the domain if $f(c) \leq f(x)$ for all x values in the open interval.



Local Maximum – occurs at a point c in an open interval, (a, b), in the domain if $f(c) \geq f(x)$ for all x values in the open interval.



Example: Let $f(x)$ be define on $[-4, 5]$ as given in the following plot. Find the absolute maximum, absolute minimum, local maximum and local minimum points.



Solution:

Absolute maximum: $(4, 6)$

Absolute minimum: $(2, -6)$

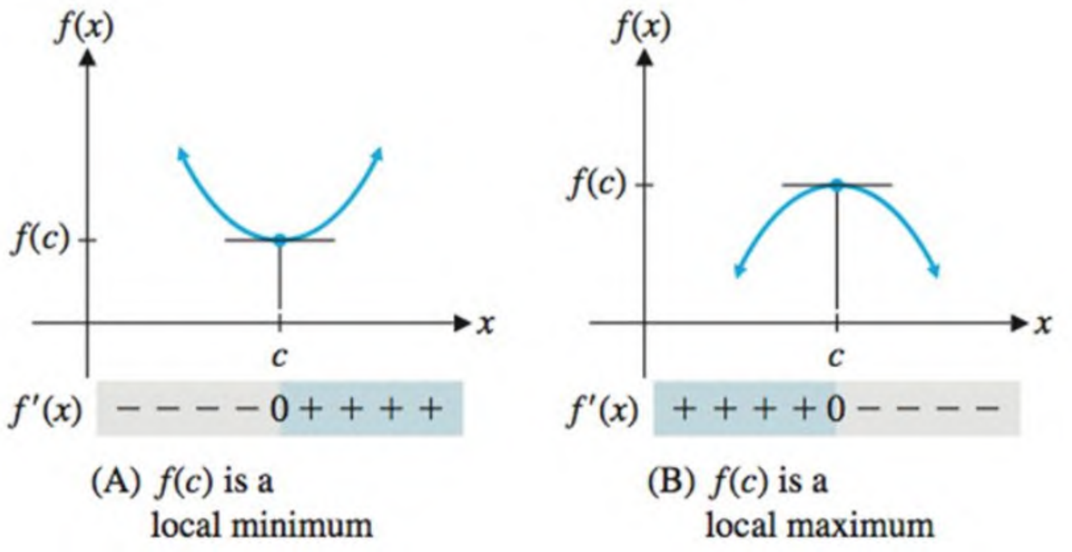
Local maximum: $(-1, 2)$ and $(4, 5)$

Local maximum: $(-3, -2)$ and $(2, -6)$

First Derivative Test

- If the sign changes from " $+$ " to " $-$ " at c, then c is called a **local maximum point**.

- If the sign changes from " $-$ " to " $+$ " at c, then c is called a **local minimum point**.



Example 1: Let $f(x) = x^3 - 6x^2 + 1$. Using the **First Derivative Test**, find the local maximum and minimum points.

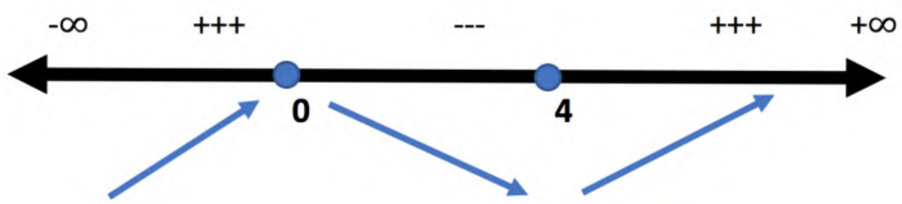
<u>Solution</u>:

First, we need to find the critical points ("$f'(x) = 0$"):

$\because f(x) = x^3 - 6x^2 + 1 \implies f'(x) = 3x^2 - 12x$

$f'(x) = 0 \implies 3x^2 - 12x = 0 \implies 3x(x-4) = 0$

Hence, $f(x)$ has critical points at $x = 0, 4$.



Increasing Intervals: $(-\infty, 0)$ and $(4, \infty)$

Decreasing Interval: $(0, 4)$

$f(x)$ has local maximum at $x = 0$, and $(0, 1)$ is a local maximum point.

$f(x)$ has local minimum at $x = 4$, and $(4, -31)$ is a local minimum point.

Example 2: Let $f(x) = x^3 - 6x^2 + 9x - 8$ on $(0, 5)$. Using the **First Derivative Test**, find the local maximum and minimum points.

Solution:

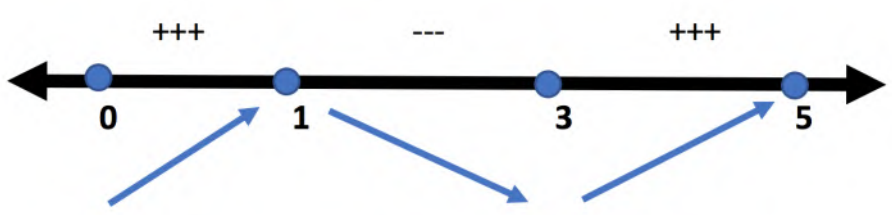
First, we need to find the critical points ("$f'(x) = 0$"):

$\because f(x) = x^3 - 6x^2 + 9x - 8 \implies f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0 \implies 3x^2 - 12x + 9 = 0$

$\implies 3(x^2 - 4x + 3) = 0 \implies (x - 1)(x - 3) = 0$

Hence, $f(x)$ has critical points at $x = 1, 3$.



Increasing Intervals: $(0, 1)$ and $(3, 5)$

Decreasing Interval: $(1, 3)$

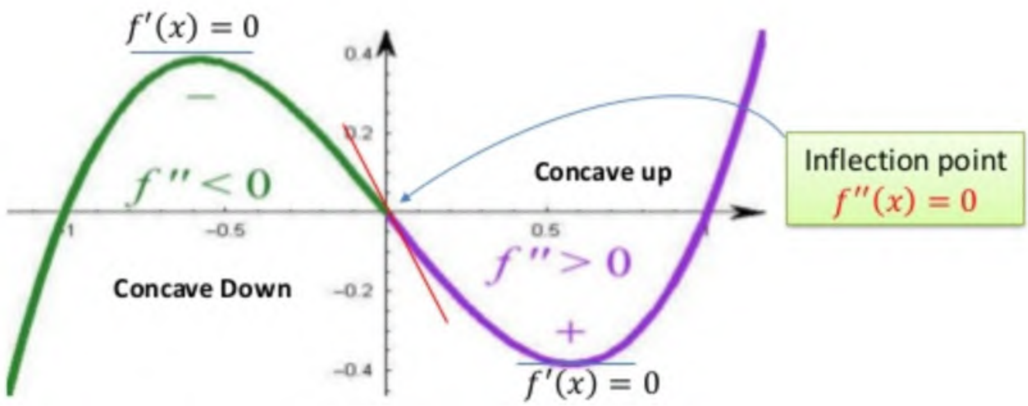
$f(x)$ has local maximum at $x = 1$, and $(1, -4)$ is a local maximum point.

$f(x)$ has local minimum at $x = 3$, and $(3, -8)$ is a local minimum point.

Definition: An **inflection point** is a point on a curve where the curve changes from being Concave Down (going up, then down) to Concave Up (going down, then up), or the other way around.

Second Derivative Test

Assume that $f'(a) = 0$; so, the point $(a, f(a))$ is suspicious to be a maximum or minimum. If $f''(a) > 0$, the point is a minimum point and if $f''(a) < 0$, the point is a maximum point.



Example 1: Let $f(x) = x^3 - 6x^2 + 1$. Using the **Second Derivative Test**, find the local maximum and minimum points.

Solution:

First, we need to find the critical points ("$f^{'}(x) = 0$"):

$\because f(x) = x^3 - 6x^2 + 1 \implies f^{'}(x) = 3x^2 - 12x$

$f^{'}(x) = 0 \implies 3x^2 - 12x = 0 \implies 3x(x-4) = 0$

Hence, $f(x)$ has critical points at $x = 0, 4$.

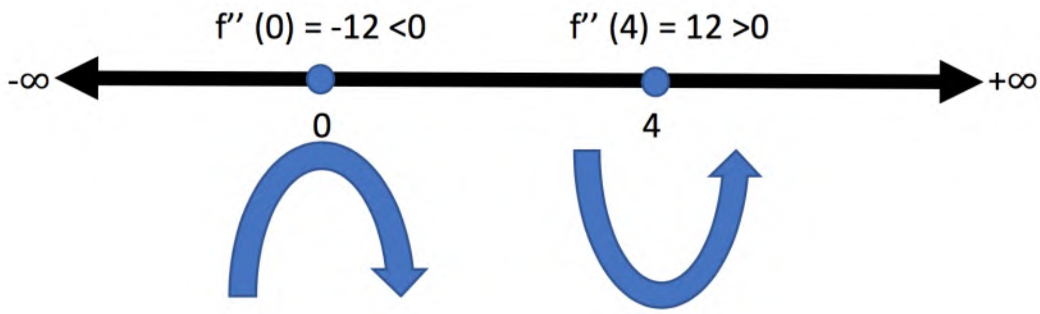
$\because f'(x) = 3x^2 - 12x \implies f^{''}(x) = 6x - 12$

$f^{''}(x) = 0 \implies 6x - 120 \implies x = 2.$

$\because f^{''}(0) = -12 \implies f(x)$ "Concave Down" on $(-\infty, 2)$, and has local Maximum at "$x = 0$".

$\because f^{''}(4) = 12 \implies f(x)$ "Concave Up" on $(2, \infty)$, and has local Minimum at "$x = 4$".

$f(x)$ has an inflection point at $x = 2$ because the function concave down then concave up.



Example 2: Let $f(x) = x^3 - 6x^2 + 9x - 8$ on $(0, 5)$. Using the **Second Derivative Test**, find the local maximum and minimum points.

Solution:

First, we need to find the critical points ("$f'(x) = 0$"):

$\because f(x) = x^3 - 6x^2 + 9x - 8 \implies f'(x) = 3x^2 - 12x + 9$

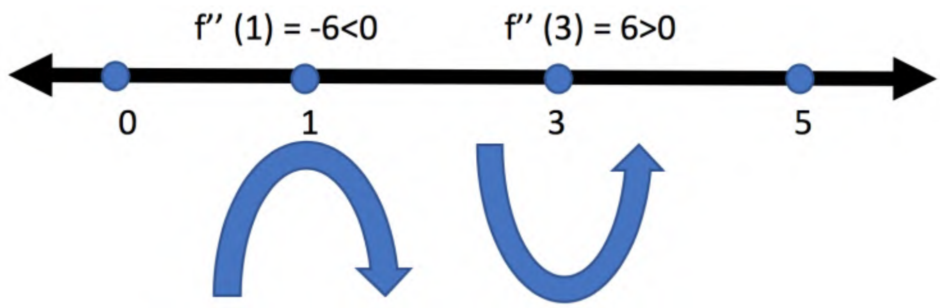
$f'(x) = 0 \implies 3x^2 - 12x + 9 = 0$

$\implies 3(x^2 - 4x + 3) = 0 \implies (x-1)(x-3) = 0$

Hence, $f(x)$ has critical points at $x = 1, 3$.

$\because f'(x) = 3x^2 - 12x + 9 \implies f''(x) = 6x - 12$

$f''(x) = 0 \implies 6x - 12 = 0 \implies x = 2$



$\because f''(1) = -6 \implies f(x)$ "Concave Down" on $(0, 2)$,

and has local Maximum at "$x = 1$".

$\because f''(3) = 6 \implies f(x)$ "Concave Up" on $(2, 5)$,

and has local Minimum at "$x = 3$".

$f(x)$ has an inflection point at $x = 2$ because the function concave down then concave up.

Problems (4.8):

1. By using the **First Derivative Test**, check whether the critical points are local maximum or minimum points, and specify the increasing and decreasing intervals.

   (a) $f(x) = \frac{1}{2}x^2 - x,\ x \in [0, 2]$

   (b) $f(x) = \frac{x^3}{3} + \frac{5}{2}x^2 + 6x,\ x \in \mathbb{R}$

   (c) $f(x) = x^3 - x,\ x \in [-2, 2]$

2. By using the **Second Derivative Test**, check whether the critical points are local maximum or minimum points, and specify the concave up and concave down intervals.

   (a) $f(x) = \frac{x^3}{6} - \frac{x^2}{2} - 4,\ x \in [-2, 5]$

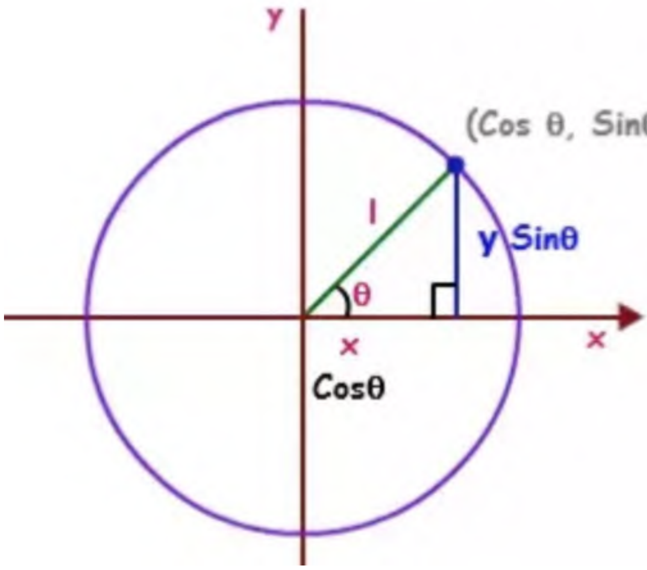
   (b) $f(x) = \frac{x^3}{3} + \frac{5}{2}x^2 + 6x,\ x \in \mathbb{R}$

   (c) $f(x) = \frac{x^4}{12} - \frac{x^3}{6} - x^2,\ x \in [-3, 3]$

We will define six trigonometric functions in terms of the central angle θ drawn in the center circle $(0,0)$ and radius r.

In the central angle θ with one of its sides is applied to the x-axis and the other side is drawn from the origin point and cut the circumference of the circle at the point $p(x,y)$, then:

- Sine: $\sin\theta = \frac{y}{r}$
- Cosine: $\cos\theta = \frac{x}{r}$
- Tangent: $\tan\theta = \frac{y}{x}$
- Cotangent: $\cot\theta = \frac{x}{y}$
- Secant: $\sec\theta = \frac{r}{x}$
- Cosecant: $\cos\theta = \frac{r}{y}$



From the previous definition definitions, a relation can be found between trigonometric functions as follows:

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{\csc\theta}{\sec\theta}$

- $\sec\theta = \frac{1}{\cos\theta}$

- $\csc\theta = \frac{1}{\sin\theta}$

And since the equation of the circle center $(0,0)$ and radius r is:

$x^2 + y^2 = r^2$

$\because$ $\boxed{x = r\cos\theta}$ and $\boxed{y = r\sin\theta}$

$\Longrightarrow\ r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\Longrightarrow\ r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\Longrightarrow\ \boxed{\cos^2\theta + \sin^2\theta = 1} \qquad \ldots \qquad (1)$

<u>**Note:**</u> From the above equation, we can derive the following forms:

- If we divide eq.(1) by $\cos\theta$ $\Longrightarrow$ $\boxed{1 + \tan^2\theta = \sec^2\theta}$

- If we divide eq.(1) by $\sin\theta$ $\Longrightarrow$ $\boxed{\cot^2\theta + 1 = \csc^2\theta}$

Laws of Sum and Subtract two Angles:

Let A and B be any two angels, then:

- $\sin(A+B) = \sin A\cos B + \sin B\cos A$

- $\sin(A-B) = \sin A\cos B - \sin B\cos A$

- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\cos(A-B) = \cos A \cos B + \sin A \sin B$
- $\tan(A \mp B) = \frac{\tan A \mp \tan B}{1 + \tan A \tan B}$

<u>Note:</u> Now, we can use the laws of sum and subtract two angles to derive the following forms:

- $\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$

  $\Longrightarrow \boxed{\sin(2\theta) = 2\sin\theta\cos\theta} \qquad \ldots \qquad (2)$

- $\cos(2\theta) = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

  $\Longrightarrow \boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta} \qquad \ldots \qquad (3)$

<u>Note:</u> From eq.(1) and eq (3), we can derive the following:

- eq.(1) + eq. (3) $\Longrightarrow 2\cos^2\theta = 1 + cos2\theta$
- eq.(1) - eq. (3) $\Longrightarrow 2\sin^2\theta = 1 - cos2\theta$

<u>Remark:</u> Trigonometric function are divided into two types (Odd Functions and Even Functions) as follows:

- **Odd Functions:**

$$\sin(-\theta) = -\sin\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

$$\csc(-\theta) = -\csc\theta$$

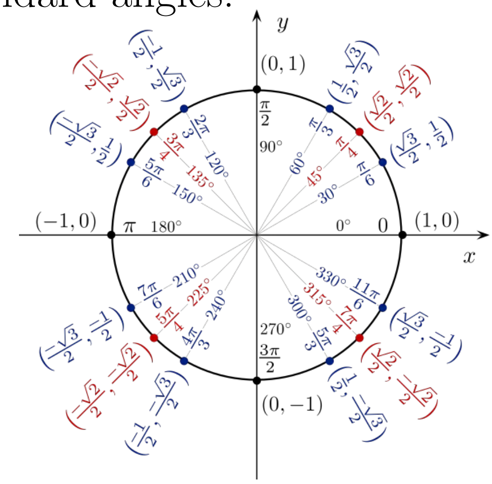
- **Even Functions:**

$$\cos(-\theta) = \cos\theta$$

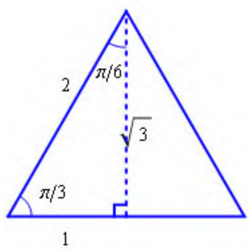
$$\sec(-\theta) = \sec\theta$$

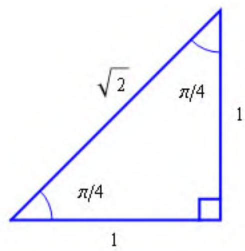
Rules: $\sin\theta$, $\cos\theta$ and $\tan\theta$ for some standard angles:

θ	$0 = 2\pi$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	∞	0	$-\infty$



θ	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$





Rules: The $\sin\theta$, $\cos\theta$, and $\tan\theta$ take positive and negative signs depends on the position in which quarter.

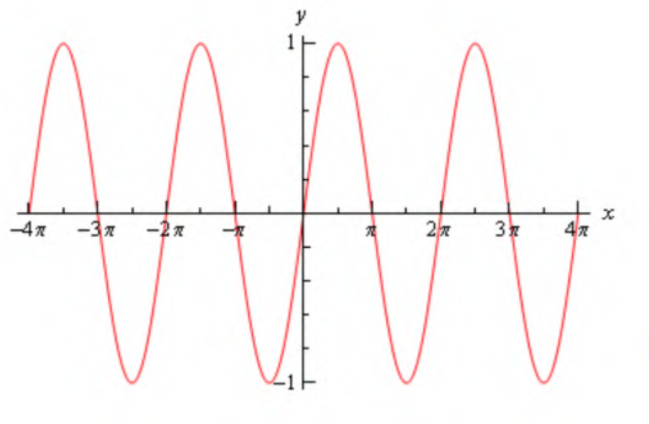
Graphs of Trigonometric Functions:

1. $y = \sin\theta$

   Domain $:= \mathbb{R}$

   Range $:= [-1, 1]$

   Period $:= 2\pi$

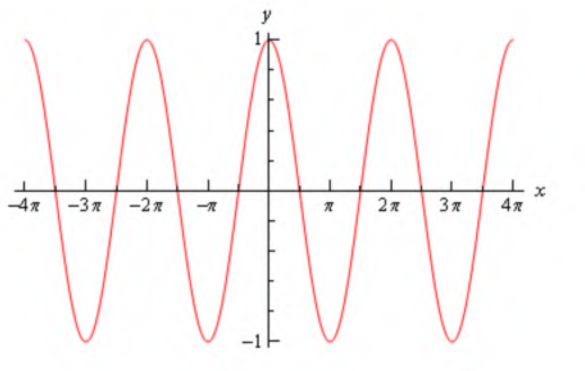


2. $y = \cos\theta$

   Domain $:= \mathbb{R}$

   Range $:= [-1, 1]$

   Period $:= 2\pi$

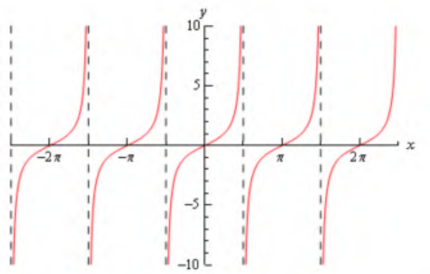


3. $y = \tan\theta$

   Domain $:= \mathbb{R} \setminus \{\frac{\pi}{2} + n\pi : n = 0, \mp 1, \dots\}$

   Range $:= \mathbb{R}$

   Period $:= \pi$

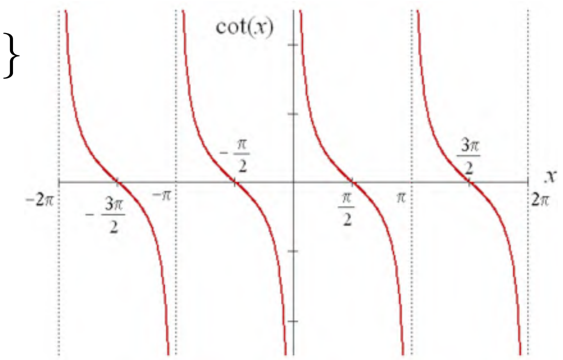


4. $y = \cot\theta$

   Domain $:= \mathbb{R} \setminus \{n\pi : n = 0, \mp 1, \dots\}$

   Range $:= \mathbb{R}$

   Period $:= \pi$

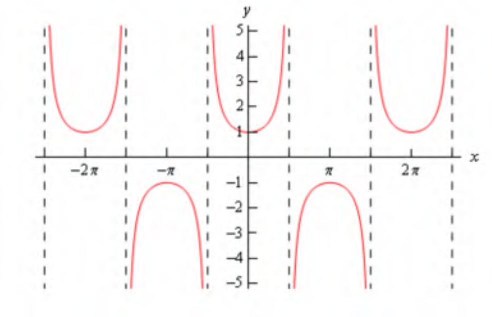


5. $y = \sec\theta$

Domain $:= \mathbb{R} \setminus \{\frac{\pi}{2} + n\pi : n = 0, \mp 1, \ldots$

Range $:= \mathbb{R} \setminus (-1, 1)$

Period $:= 2\pi$

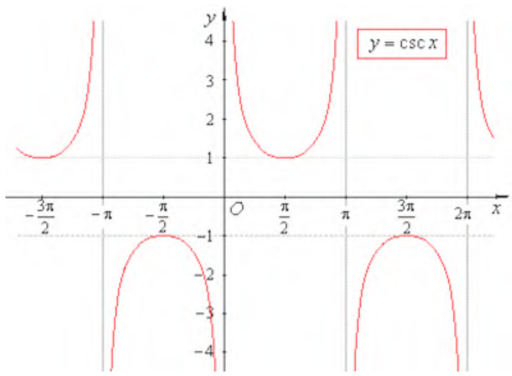


6. $y = \csc\theta$

Domain $:= \mathbb{R} \setminus \{n\pi : n = 0, \mp 1, \ldots\}$

Range $:= \mathbb{R} \setminus (-1, 1)$

Period $:= 2\pi$



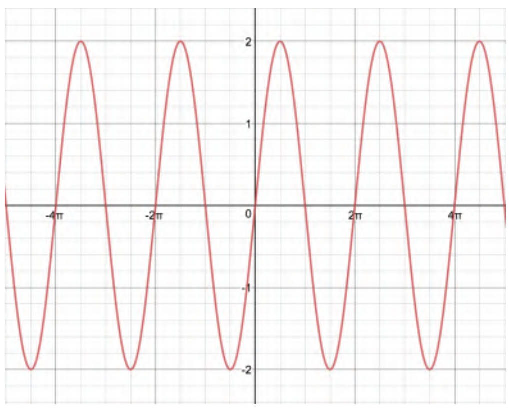
Shifting Trigonometric Functions

Examples: Plot the following functions

(1) $y = 2\sin\theta$

$\because D_{\sin\theta} := \mathbb{R} \implies D_{2\sin\theta} := \mathbb{R}$

$\because R_{\sin\theta} := [-1, 1] \implies R_{2\sin\theta} := [-2, 2]$



(2) $y = \cos\frac{\theta}{2}$

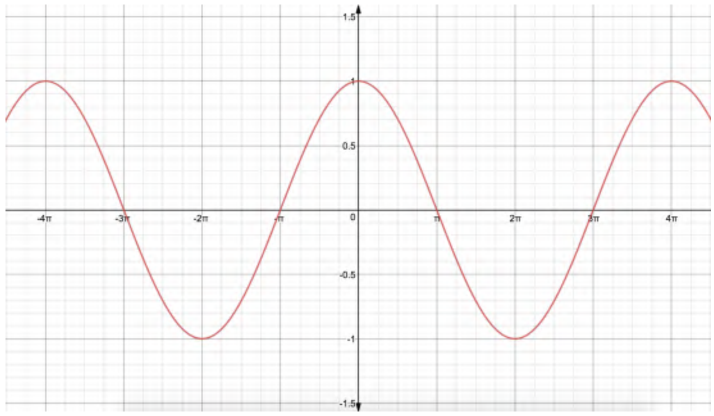
$\because D_{\cos\theta} := \mathbb{R} \implies D_{\cos\frac{\theta}{2}} := \mathbb{R}$

but, $\because -2\pi < \frac{\theta}{2} < 2\pi$

$\implies -4\pi < \theta < 4\pi$

$\because R_{\cos\theta} := [-1, 1]$

$\implies R_{\cos\frac{\theta}{2}} := [-1, -1]$



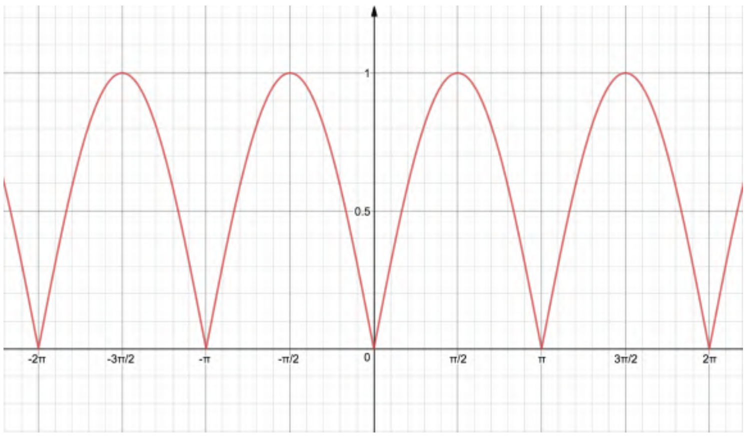
(3) $y = |\sin\theta|$

$\because D_{\sin\theta} := \mathbb{R}$

$\implies D_{|\sin\theta|} := \mathbb{R}$

$\because R_{\sin\theta} := [-1, 1]$

$\implies R_{|\sin\theta|} := [0, 1]$



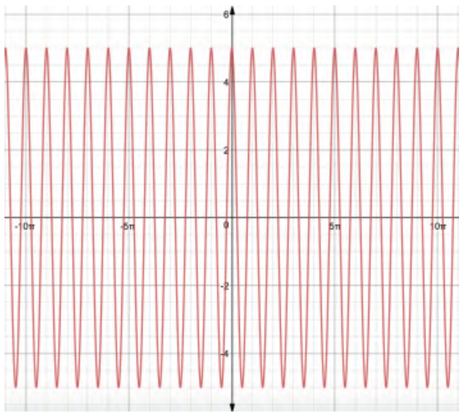
(4) $y = 5\cos(2\theta)$

$\because D_{\cos\theta} := \mathbb{R}$

$\implies D_{5\cos(2\theta)} := \mathbb{R}$

but, $\because -2\pi < 2\theta < 2\pi \implies -\pi < \theta < \pi$

$\because R_{cos} := [-1, 1] \implies R_{5\cos(2\theta)} := [-5, 5]$



Problems (5.1): Graph the following functions:

1. $y = \sin(\frac{\theta}{2})$

2. $y = \cos(3\theta)$

3. $y = 1 + \sin(\theta)$

4. $y = \frac{1+\cos(2\theta)}{2}$

5. $y = |sin(4\theta)|$

6. $y = 2sin(\theta + \pi)$

Limits of Trigonometric Functions:

Theorems:

1. $\boxed{\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1}$

2. $\boxed{\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0}$

proof: "Homework"

Result: $\boxed{\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta}=1}$

proof:

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$$
$$=\lim_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}}{\theta}$$
$$=\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}.\frac{1}{\cos\theta}\right)$$
$$=\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right).\left(\lim_{\theta\to 0}\frac{1}{\cos\theta}\right)$$
$$=\Big(1\Big).\left(\frac{1}{\cos(0)}\right)=\boxed{1}$$

Examples: Find the following limits?

- $\lim\limits_{x\to 0}\frac{\sin x}{2x}=\,?$

  $$\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{1}{2}.\frac{\sin x}{x}=\frac{1}{2}.\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{2}.1=\boxed{\frac{1}{2}}$$

- $\lim\limits_{t\to 0}\frac{\sin 3t}{t}=\,?$

  $$\lim_{t\to 0}\frac{\sin 3t}{t}=\lim_{t\to 0}\frac{3.\sin 3t}{3.t}=3.\lim_{t\to 0}\frac{\sin 3t}{3t}=3.1=\boxed{3}$$

- $\lim\limits_{z\to 0}\frac{\sin 5z}{4z}=\,?$

  $$\lim_{z\to 0}\frac{\sin 5z}{4z}=\lim_{z\to 0}\frac{5}{4}.\frac{\sin 5z}{5.z}==\frac{5}{4}.\lim_{z\to 0}\frac{\sin 5z}{5.z}=\frac{5}{4}.1=\boxed{\frac{5}{4}}$$

- $\lim\limits_{x\to\infty}x.\sin\frac{1}{x}=\,?$

  Let $y=\frac{1}{x}\implies x=\frac{1}{y}$

$\because x \to \infty \implies y \to 0$

Hence, $\lim\limits_{x\to\infty} x.\sin\frac{1}{x} = \lim\limits_{y\to 0}\frac{1}{y}.\sin y = \lim\limits_{y\to 0}.\frac{\sin y}{y} = \boxed{1}$

- $\lim\limits_{x\to 0}\frac{1-\cos x}{2x} = ?$

  $\lim\limits_{x\to 0}\frac{1-\cos x}{2x} = \lim\limits_{x\to 0}\frac{-1}{2}.\frac{\cos x-1}{x} = \frac{-1}{2}.\lim\limits_{x\to 0}\frac{\cos x-1}{x} = \frac{-1}{2}.0 = \boxed{0}$

- $\lim\limits_{\theta\to 0}\frac{3\tan\theta}{2\theta} = ?$

  $\lim\limits_{\theta\to 0}\frac{3\tan\theta}{2\theta} = \lim\limits_{\theta\to 0}\frac{3}{2}.\frac{\tan\theta}{\theta} = \frac{3}{2}.\lim\limits_{\theta\to 0}\frac{\tan\theta}{\theta} = \frac{3}{2}.1 = \boxed{\frac{3}{2}}$

Problems (5.2): Evaluate the following limits, if it exist?

1. $\lim\limits_{w\to 0}\frac{3\sin(5w)}{7w}$
2. $\lim\limits_{x\to 0}\frac{\sin(5x)}{x\cos(5x)}$
3. $\lim\limits_{t\to 0}\frac{5t}{\tan(6t)}$
4. $\lim\limits_{\theta\to 0}\frac{\sec\theta-\cos\theta}{\theta^2}$
5. $\lim\limits_{h\to 2}\frac{\cos\frac{\pi}{h}}{h-2}$
6. $\lim\limits_{z\to 0}\Big(\tan(2z).\csc(4z)\Big)$
7. $\lim\limits_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$
8. $\lim\limits_{t\to 0}\frac{-2\tan t}{5t}$

Differentiation of Trigonometric Functions:

Let u be a function of x, then:

1. $\frac{d}{dx}\Big(\sin(u)\Big) = \cos(u).\frac{du}{dx}$
2. $\frac{d}{dx}\Big(\cos(u)\Big) = -\sin(u).\frac{du}{dx}$
3. $\frac{d}{dx}\Big(\tan(u)\Big) = \sec^2(u).\frac{du}{dx}$
4. $\frac{d}{dx}\Big(\cot(u)\Big) = -\csc^2(u).\frac{du}{dx}$
5. $\frac{d}{dx}\Big(\sec(u)\Big) = \sec(u).\tan(u).\frac{du}{dx}$
6. $\frac{d}{dx}\Big(\csc(u)\Big) = -\csc(u).\cot(u).\frac{du}{dx}$

Examples: Find the derivatives of the following functions?

- $y = \frac{sinx}{x}$

  $\frac{dy}{dx} = \frac{x.\cos x - \sin x.1}{x^2} = \frac{x\cos x - \sin x}{x^2}$

- $y = \frac{2}{\cos(3t)}$

  $\frac{dy}{dt} = \frac{\cos(3t).0 - 2.(-\sin(3t).3)}{\cos^2(3t)} = \frac{6\sin(3t)}{\cos^2(3t)}$

- $y = \cot(z^2)$

  $\frac{dy}{dz} = -\csc^2(z^2).2z$

- $y = \sec^2(5x)$

  $\frac{dy}{dx} = 2.\sec(5x).\sec(5x).\tan(5x).5 = 10\sec^2(5x).\tan(5x)$

- $y = \sin(\cos w)$

  $\frac{dy}{dw} = \cos(\cos w) - \sin w.1 = \cos(\cos w) - \sin w$

Problems (5.3): Find the derivative of the following functions?

1. $y = (\frac{\sin\sqrt{x}}{\sqrt{x}})^3$
2. $y = 4\cos^2(-3w)$
3. $y = \sin^2(\frac{3}{z}) + \cos^2(z^2)$
4. $y = \sqrt[3]{9x + \cos(2x)}$
5. $y = \frac{\sqrt{2t}}{\cos(3t)}$
6. $y = x^3.\sin(2x^2 + 3)$
7. $y = (\cos^2(1+t) + \sqrt{t+5})^5$
8. $y = \frac{7\sqrt{\sec(3\theta)}}{\theta^2}$
9. $y = 2\sin(\frac{z}{2}) - (x\cos(\frac{2}{z}))^3$
10. $y = \sin(3t).\cos(5t^2)$

Implicit Differentiation of Trigonometric Functions:

Examples: Find y' of the following functions?

- $x\sin(2y) = y.\cos(2x)$

$$\Longrightarrow x.\cos(2y).2y' + \sin(2y).1 = y.(-\sin(2x)).2 + \cos(2x).y'$$

$$\Longrightarrow x.\cos(2y).2y' - \cos(2x).y' = y.(-\sin(2x)).2 - \sin(2y)$$

$$\Longrightarrow \Big(2x.\cos(2y) - \cos(2x)\Big).y' = -2y.\sin(2x) - \sin(2y)$$

$$\Longrightarrow \boxed{y' = \frac{-2y.\sin(2x) - \sin(2y)}{(2x.\cos(2y) - \cos(2x))}}$$

- $\cot(xy) + xy = 0$

$$\Longrightarrow -\csc^2(xy)(xy' + y.1) + xy' + y = 0$$

$$\Longrightarrow -x\csc^2(xy)y' - y\csc^2(xy) + xy' + y = 0$$

$$\Longrightarrow \Big(-x\csc^2(xy) + x\Big)y' = y\csc^2(xy) - y$$

$$\Longrightarrow y' = \tfrac{y\csc^2(xy)-y}{-x\csc^2(xy)+x}$$

$$\Longrightarrow y' = \tfrac{y(\csc^2(xy)-1)}{-x(\csc^2(xy)-1)}$$

$$\Longrightarrow y' = \boxed{-\frac{y}{x}}$$

Problems (5.4): Find y' of the following functions?

1. $y \sin x + x \sin y = y^2$
2. $\sec^2 y + \csc^2 y = 4$
3. $y = \tan(x + y)$
4. $y^2 = \sin^4(2x) + \cos^4(2x)$
5. $\cos(x^2y^2) = x$
6. $x^2y = \frac{\cot y}{1+\csc y}$
7. $\sqrt{xy} + \csc(-xy) = y$
8. $y(3 + \tan y)^{\frac{1}{3}} = x + 5$
9. $y = \tan y + \sec^2(xy) + \cot(x^2 + y^2)$
10. $x^2 = \sin y + \sin(2y)$

Evaluating Limits of Trigonometric Functions using $\mathbf{L^{'}}$Hopitals Rule:

Examples: Evaluate the following limits?

- $\lim\limits_{x\to 0} \frac{x^2+2x}{\sin(2x)} = ?$

  $\lim\limits_{x\to 0} \frac{x^2+2x}{\sin(2x)} \overset{L^{'}R}{=} \lim\limits_{x\to 0} \frac{2x+2}{\cos(2x).2}$

  $= \lim\limits_{x\to 0} \frac{x+1}{\cos(2x)} = \frac{\lim\limits_{x\to 0}(x+1)}{\lim\limits_{x\to 0}\cos(2x)} = \frac{0+1}{\cos(0)} = \frac{1}{1} = \boxed{1}$

- $\lim\limits_{h\to 2} \frac{\cos(\frac{\pi}{h})}{h-2} = ?$

  $\lim\limits_{h\to 2} \frac{\cos(\frac{\pi}{h})}{h-2} \overset{L^{'}R}{=} \lim\limits_{h\to 2} \frac{-\sin(\frac{\pi}{h}).\frac{h.0-\pi.1}{h^2}}{1-0}$

  $= \lim\limits_{h\to 2} \Big(-\sin(\frac{\pi}{h}).\frac{-\pi}{h^2}\Big) = -\sin(\frac{\pi}{2}).\frac{-\pi}{4} = -1.\frac{-\pi}{4} = \boxed{\frac{\pi}{4}}$

- $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = ?$

  $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} \overset{L^{'}R}{=} \lim\limits_{x\to 0} \frac{0-(-\sin x)}{2x}$

  $= \lim\limits_{x\to 0} \frac{1}{2}.\frac{\sin x}{x} = \frac{1}{2}\lim\limits_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}.1 = \boxed{\frac{1}{2}}$

- $\lim\limits_{\theta\to \frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos(2\theta)} = ?$

  $\lim\limits_{\theta\to \frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos(2\theta)} \overset{L^{'}R}{=} \lim\limits_{\theta\to \frac{\pi}{2}} \frac{-\cos\theta}{-2\sin(2\theta)}$

$$= \frac{1}{2}. \lim_{\theta \to \frac{\pi}{2}} \frac{\cos\theta}{\sin(2\theta)} \overset{L'R}{=} \frac{-1}{2}. \lim_{\theta \to \frac{\pi}{2}} \frac{-\sin\theta}{2\cos(2\theta)}$$

$$= \frac{-1}{4} \lim_{\theta \to \frac{\pi}{2}} \frac{\sin\theta}{\cos(2\theta)} = \frac{-1}{4}.\frac{\sin(\frac{\pi}{2})}{\cos(2\frac{\pi}{2})} = \frac{-1}{4}.\frac{1}{-1} = \boxed{\frac{1}{4}}$$

Problems (5.5):

1. Evaluate the following limits?

   (a) $\lim_{x \to 0} \frac{1-\cos^2 x}{2x^2}$

   (b) $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta.\sin\theta}$

   (c) $\lim_{t \to 0} \frac{\sec t-\cos t}{t^2}$

   (d) $\lim_{y \to 0} \frac{1-\cos y}{\sin y}$

2. Proof the following:

   (a) $\lim_{\theta \to \frac{\pi}{2}} \frac{\sin\theta-1}{\cos\theta} = 0$

   (b) $\lim_{z \to 0} \frac{\sin(2z)+7z^2-2z}{z^2(z+1)^2} = 7$

   (c) $\lim_{w \to -4} \frac{sin(\pi w)}{w^2-16} = -\frac{\pi}{8}$

   (d) $\lim_{x \to 1^+} \frac{x-1}{cot(\frac{\pi}{2}x)} = -\frac{2}{\pi}$

Suppose f be a one-to-one (i.e. 1-1) and onto function.

$$f : X \longrightarrow Y$$

- f is 1-1 $\iff x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

  $\overset{OR}{\iff} f(x_1) = f(x_2) \implies x_1 = x_2$

- f is onto $\iff \forall x \in X\ \exists y \in Y \ni y = f(x)$

- $f : X \longrightarrow Y \ni y = f(x)$

  $\iff f^{-1} : Y \longrightarrow X \ni x = f(y)$

---

(1) The Inverse of Sine Function:

Let $y = \sin(x)$

$\sin(x) : \mathbb{R} \to [-1, 1]$

We are going to define a new function which is inverse sine , and we denote it by $\sin^{-1}$ or arcsin.

$\therefore \sin^{-1} y = \sin^{-1}(\sin x) \implies \sin^{-1} y = x$

$\therefore y = \sin x \iff x = \sin^{-1} y$

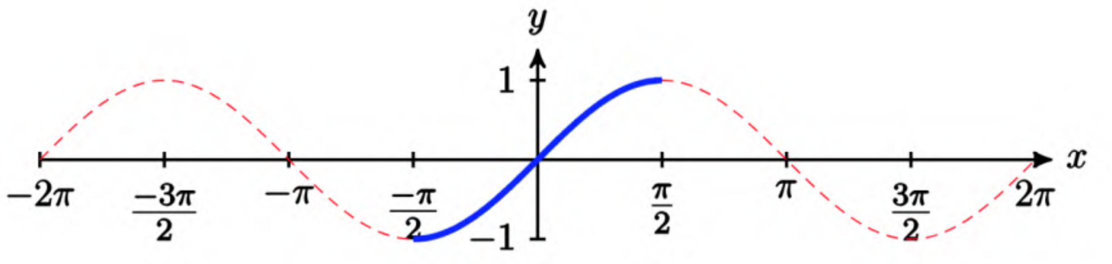
$\sin(x) : [-\frac{\pi}{2}, -\frac{\pi}{2}] \Longrightarrow [-1, 1]$

$\because$ sin is 1-1 and onto, $\Longrightarrow \exists \sin^{-1} \ni$

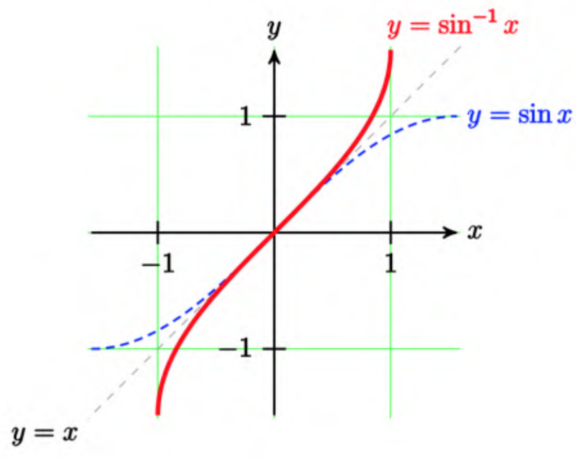
$\sin^{-1} : [-1, 1] \Longrightarrow [-\frac{\pi}{2}, -\frac{\pi}{2}]$

$D_{\sin^{-1}} = [-1, 1] = R_{\sin x}$

$R_{\sin^{-1}} = [-\frac{\pi}{2}, -\frac{\pi}{2}] = D_{\sin x}$



<u>Note</u>: $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$



<u>Remark</u>: $\sin^{-1}$ is an odd function.

$$\left(\text{i.e., } \sin^{-1}(-x) = -\sin^{-1}(x)\right)$$

---

(2) The Inverse of Cosine Function:

Let $y = \cos(x)$

$\cos(x) : \mathbb{R} \to [-1, 1]$

We are going to define a new function which is inverse cosine , and we denote it by $\cos^{-1}$ or arccos.

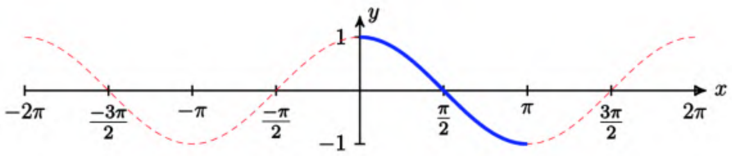
$\therefore \cos^{-1} y = \cos^{-1}(\cos x) \implies \cos^{-1} y = x$

$\therefore y = \cos x \iff x = \cos^{-1} y$

$\cos : [0, \pi] \implies [-1, 1]$

$\because$ cos is 1-1 and onto, $\implies \exists \cos^{-1} \ni$

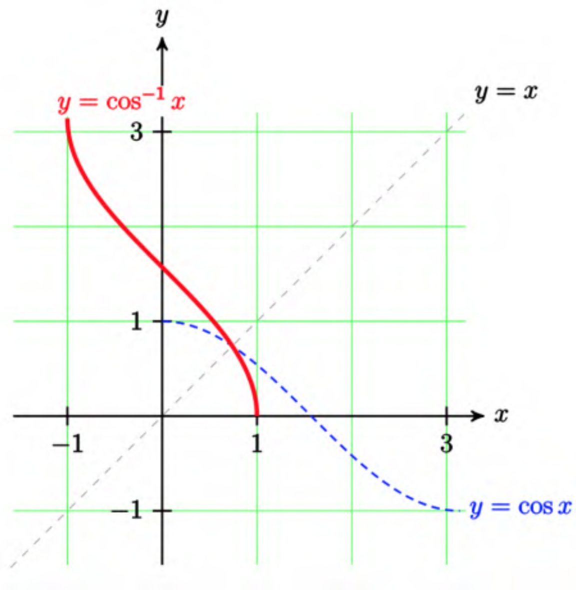
$\cos^{-1} : [-1, 1] \implies [0, \pi]$



$D_{\cos^{-1}} = [0, \pi] = R_{\cos x}$

$R_{\cos^{-1}} = [-1, 1] = D_{\cos x}$

Note: $\cos^{-1}(x) \neq \frac{1}{\cos(x)}$



Remark: $\cos^{-1}$ is neither even nor odd function.

Note: $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

(3) The Inverse of Tangent Function:

Let $y = \tan(x)$

$\tan(x) : \mathbb{R} \setminus \{x : x = \frac{\pi}{2} + n\pi; n \in \mathbb{I}\} \longrightarrow \mathbb{R}$

We are going to define a new function which is inverse cosine , and we denote it by $\tan^{-1}$ or arctan.

$\therefore \tan^{-1} y = \tan^{-1}(\tan x) \implies \tan^{-1} y = x$

$\therefore y = \tan x \iff x = \tan^{-1} y$

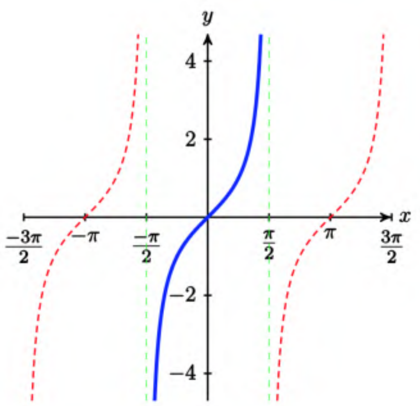
$\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$

$\because$ tan is 1-1 and onto, $\implies \exists \tan^{-1} \ni$

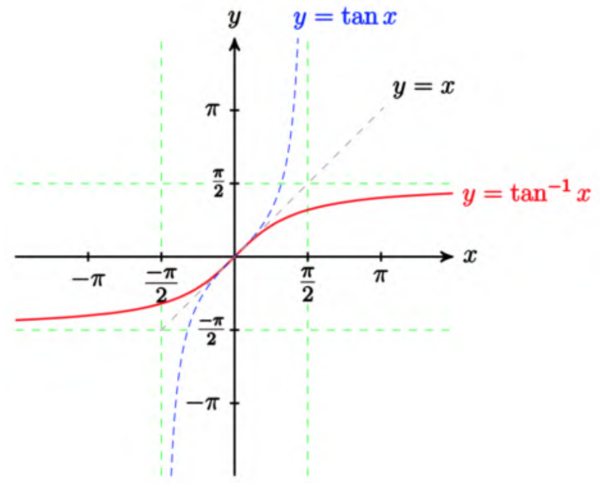
$\tan^{-1} : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$

$D_{\tan^{-1}} = \mathbb{R} = R_{\tan x}$

$R_{\tan^{-1}} = (-\frac{\pi}{2}, \frac{\pi}{2}) = D_{\tan x}$



Note: $\tan^{-1}(x) \neq \frac{1}{\tan(x)}$



Remark: $\tan^{-1}$ is an odd function.

$$\left(\text{i.e., } \tan^{-1}(-x) = -\tan^{-1}(x)\right)$$

The Derivative of Inverse trigonometric Functions:

Let u be a function of x, then:

1. $\frac{d}{dx}\left(\sin^{-1}(u)\right) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
2. $\frac{d}{dx}\left(\cos^{-1}(u)\right) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$
3. $\frac{d}{dx}\left(\tan^{-1}(u)\right) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$
4. $\frac{d}{dx}\left(\cot^{-1}(u)\right) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$
5. $\frac{d}{dx}\left(\sec^{-1}(u)\right) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$
6. $\frac{d}{dx}\left(\csc^{-1}(u)\right) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

Examples: Find the derivatives of the following functions?

- $f(x) = \sin^{-1}(x^2)$

  $\Longrightarrow\ f^{'}(x) = \frac{1}{\sqrt{1-(x^2)^2}}.2x$

- $g(t) = \cos^{-1}(\sqrt{t})$

  $\Longrightarrow\ g^{'}(t) = \frac{-1}{\sqrt{1-(\sqrt{t})^2}}.\frac{1}{2}t^{\frac{-1}{2}}$

- $y = \sin^{-1}\sqrt{1-\sqrt{\theta}}$

  $\Longrightarrow\ y^{'} = \frac{1}{\sqrt{1-(\sqrt{1-\sqrt{\theta}})^2}}.\frac{1}{2}(1-\sqrt{\theta})^{\frac{-1}{2}}.\frac{-1}{2}\theta^{\frac{-1}{2}}$

- $f(x) = \cot^{-1}\left(\frac{1-x}{1+x}\right)$

  $\Longrightarrow\ f^{'}(x) = \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}.\frac{(1+x).(-1)-(1-x).1}{(1+x)^2}$

- $g(x) = \sec^{-1}(\frac{\sqrt{1+x^2}}{x})$

  $\Longrightarrow\ g^{'}(x) = \frac{1}{|\frac{\sqrt{1+x^2}}{x}|\sqrt{(\frac{1+x^2}{x})^2-1}}.\frac{x.\frac{1}{2}(1+x^2)^{\frac{-1}{2}}.2x-\sqrt{1+x^2}.1}{x^2}$

- $y = x.\csc^{-1}(\frac{1}{x}) + \sqrt{1-x^2}$

  $\Longrightarrow\ y^{'} = x.\frac{-1}{|\frac{1}{x}|\sqrt{\frac{1}{x^2}-1}}.\frac{x.0-1.1}{x^2} + \csc^{-1}(\frac{1}{x}).1 + \frac{1}{2}(1-x^2)^{\frac{-1}{2}}.-2x$

Problems (6.1):

1. Find y' of the following functions?

   (a) $y = \sin^{-1} \frac{x-1}{x+1}$

   (b) $y = \theta.(\sin^{-1}(\theta))^2 - 2x + 2\sqrt{1-\theta}.\sin^{-1}(\theta)$

   (c) $y = t.\cos^{-1}(2t) - \frac{1}{2}\sqrt{1-4t^2}$

   (d) $y = \frac{\cos^{-1}(2x)}{\sqrt{1+4x^2}}$

   (e) $y = \cos^{-1}(\frac{3}{t}) + \frac{t}{1-t^2}$

   (f) $y = \sec^{-1}(\sqrt{w^2+4})$

   (g) $y = \sin(\tan^{-1} x)$

   (h) $y = \tan^{-1}(3\tan 2z)$

   (i) $y = \sec^{-1}(5x^2)$

   (j) $y = \frac{\cot^{-1}(3\theta)}{1+\theta^2}$

2. Find y' of the following functions?

   (a) $x \sin y + x^3 = \tan^{-1} y$

   (b) $\sin^{-1}(xy) = \cos^{-1}(x+y)$

   (c) $\cos 3y - \cos^{-1}(y) - \frac{\sin x}{2x} = 0$

Algebra of Inverse Trigonometric Functions:

Some properties for the inverse trigonometric functions:

- $\cot^{-1}(x) = \tan^{-1}(\frac{1}{x})$
- $\sec^{-1}(x) = \cos^{-1}(\frac{1}{x})$
- $\csc^{-1}(x) = \sin^{-1}(\frac{1}{x})$

Examples: Evaluate the following:

1. $\cos(\cos^{-1}(\frac{1}{2})) = ?$

   $\therefore \cos(\cos^{-1}(\frac{1}{2})) = \cos\cos^{-1}(\frac{1}{2}) = \mathrm{I}(\frac{1}{2}) = \boxed{\dfrac{1}{2}}$

2. $\sin(\cos^{-1}\frac{\sqrt{2}}{2}) = ?$

   Let $\alpha = \cos^{-1}\frac{\sqrt{2}}{2}$

   $\Longrightarrow \alpha = \cos^{-1}\frac{1}{\sqrt{2}} \Longrightarrow \cos\alpha = \frac{1}{\sqrt{2}} \Longrightarrow \alpha = \frac{\pi}{4} = 45°$

   $\therefore \sin(\cos^{-1}(\frac{\sqrt{2}}{2})) = \sin(\frac{\pi}{4}) = \boxed{\dfrac{1}{\sqrt{2}}}$

3. $\csc(\sec^{-1}(2)) = ?$

   Let $\alpha = \sec^{-1}(2)$

$\Longrightarrow \alpha = \sec^{-1}(2) = \cos^{-1}(\frac{1}{2}) \Longrightarrow \cos\alpha = \frac{1}{2} \Longrightarrow \alpha = \frac{\pi}{3} = 60°$

$\therefore \csc(\sec^{-1}(2)) = \csc(\frac{\pi}{3}) = \frac{1}{\sin(\frac{\pi}{3})} = \frac{1}{\frac{\sqrt{3}}{2}} = \boxed{\frac{2}{\sqrt{3}}}$

4. $\cot(\sin^{-1}(\frac{1}{2})) = ?$

   Let $\alpha = \sin^{-1}\frac{1}{2}$

   $\Longrightarrow \sin\alpha = \frac{1}{2} \Longrightarrow \alpha = 2\pi - \frac{\pi}{6}$

   $\therefore \cot(\sin^{-1}(\frac{1}{2})) = \cot(2\pi - \frac{\pi}{6}) = -\cot\frac{\pi}{6}$

   $= -\frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \boxed{-\sqrt{3}}$

5. $\cos(\sin^{-1}(\frac{8}{10})) = ?$

   Let $\alpha = \sin^{-1}(\frac{8}{10})$

   $\Longrightarrow \cos(\sin^{-1}(\frac{8}{10})) = \cos(\alpha) = \frac{6}{10}$

6. $\cos(\sin^{-1}(\frac{1}{3}) - \tan^{-1}(\frac{1}{2}) = ?$

   Let $\alpha = \sin^{-1}(\frac{1}{3}) \Longrightarrow \sin\alpha = \frac{1}{3}$

   Let $\beta = \tan^{-1}(\frac{1}{2}) \Longrightarrow \tan\beta = \frac{1}{2}$

   $\because \cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

   $\therefore \cos(\sin^{-1}(\frac{1}{3}) - \tan^{-1}(\frac{1}{2}))$

   $= \cos(\alpha - \beta)$

$$= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$= \tfrac{2\sqrt{2}}{3}.\tfrac{2}{\sqrt{5}} + \tfrac{1}{3}.\tfrac{1}{\sqrt{5}}$$

$$= \boxed{\frac{4\sqrt{2}+1}{3\sqrt{5}}}$$

Problems (6.2): Evaluate the following?

1. $\sec(\cos^{-1}(\frac{1}{2}))$

2. $\cos(\cot^{-1}(1))$

3. $\tan(\sin^{-1}(-\frac{1}{2}))$

4. $\csc(\sin^{-1}(\frac{1}{\sqrt{2}}))$

5. $\cos^{-1}(-\sin(\frac{\pi}{6}))$

6. $\cos(\cos^{-1}(\frac{3}{4}) - \cot^{-1}(\frac{1}{4}))$

Hyperbolic Functions:

1. $\sinh(x) = \frac{e^x - e^{-x}}{2}$

2. $\cosh(x) = \frac{e^x + e^{-x}}{2}$

3. $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4. $\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

5. $sech(x) = \frac{2}{e^x + e^{-x}}$

6. $csch(x) = \frac{2}{e^x - e^{-x}}$

Remarks:

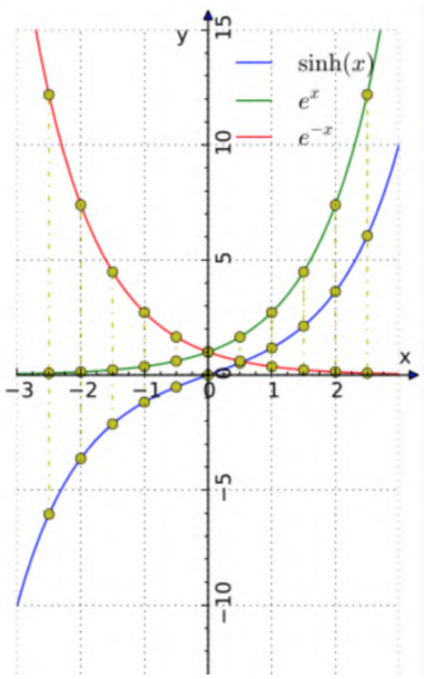
- $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$
- $\coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)}$
- $sec(x) = \frac{1}{\cosh(x)}$
- $csc(x) = \frac{1}{\sinh(x)}$

The Graph of Hyperbolic Functions:

1. $y = \sinh(x)$

   Domain $:= \mathbb{R}$

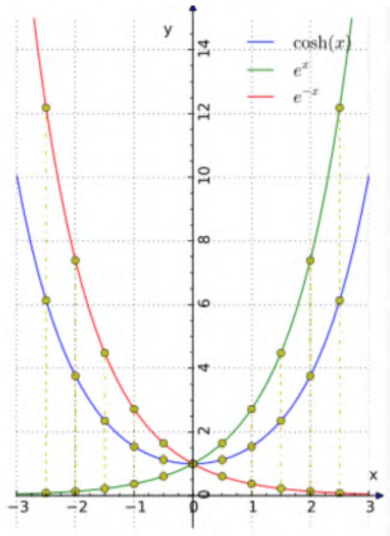
   Range $:= \mathbb{R}$



2. $y = \cosh(x)$

   Domain $:= \mathbb{R}$

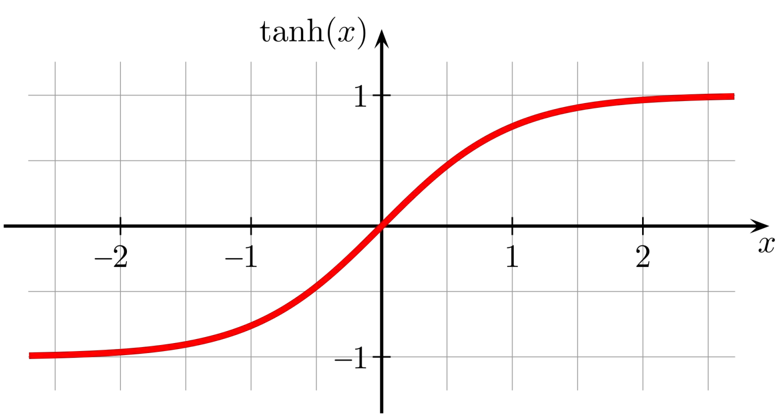
   Range $:= [1, \infty)$



3. $y = \tanh(x)$

   Domain $:= \mathbb{R}$

   Range $:= (-1, 1)$



Some Facts about Hyperbolic Functions:

1. $\cosh^2(x) - \sinh(x) = 1$

2. $1 - \tanh^2(x) = sech^2(x)$

3. $\coth^2(x) - 1 = csch^2(x)$

4. $\cosh(-x) = \cosh(x)$ " Even Function",

   $\sinh(-x) = -\sinh(x)$ " Odd Function",

   $\tan(-x) = -\tan(x)$" Odd Function"

5. $\cosh(x) + \sinh(x) = e^x$,

   $\cosh(x) - \sinh(x) = e^{-x}$

6. $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$,

   $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$,

   $\tanh(x + y) = \frac{\tanh(x)+\tanh(y)}{1-\tanh(x)\tanh(y)}$

7. $\cosh^2(x) = \frac{1}{2}(\cosh(2x) + 1)$,

   $\sinh^2(x) = \frac{1}{2}(\cosh(2x) - 1)$

8. $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$,

   $\sinh(2x) = 2\sinh(x)\cosh(x)$

The Derivative of Hyperbolic Functions:

Let u be a function of x, then:

1. $\frac{d}{dx}\Big(sinh(u)\Big) = cosh(u).\frac{du}{dx}$

2. $\frac{d}{dx}\Big(cosh(u)\Big) = sinh(u).\frac{du}{dx}$

3. $\frac{d}{dx}\Big(tanh(u)\Big) = sech^2(u).\frac{du}{dx}$

4. $\frac{d}{dx}\Big(coth(u)\Big) = -csch^2(u).\frac{du}{dx}$

5. $\frac{d}{dx}\Big(sech(u)\Big) = -sech(u).tanh(u).\frac{du}{dx}$

6. $\frac{d}{dx}\Big(csch(u)\Big) = -csch(u).coth(u).\frac{du}{dx}$

Examples: Find the derivatives of the following functions:

- $\sinh(3x)$

  $\Longrightarrow \; y^{'} = 3\cosh(3x)$

- $y = \cosh^2(5x)$

  $\Longrightarrow \; y^{'} = 2\cosh(5x).\sinh(5x).5$

- $\tanh(2x)$

  $\Longrightarrow \; y^{'} = sech^2(2x).2$

- $y = \coth(\tan(x))$

  $\Longrightarrow \; y' = -csch^2(\tan x).\sec^2 x$

- $y = sech^3 x$

  $\Longrightarrow \; y' = 3sech^2(x).(-sech(x)\tanh(x).1)$

- $y = 4csch(\frac{x}{4})$

  $\Longrightarrow \; y' = 4.(-csch(\frac{x}{4})).\coth(\frac{x}{4}).\frac{1}{4}$

Problems (6.3): Find y' of the following:

1. $y = \frac{\cosh(x)}{x}$
2. $y = e^w.\cosh(w)$
3. $y = \tanh(\frac{4t+1}{5})$
4. $y = \tanh^{-1}(\frac{1}{x})$
5. $y = \coth(\frac{1}{\theta})$
6. $y = \cosh^2(5x) - \sinh^2(5x)$
7. $\sinh(y) = \tanh(x)$
8. $y = \sinh^2(3w)$
9. $\sin^{-1}(x) = sech(y)$
10. $\tan(x) = \tanh^2(y)$
11. $\sinh(y) = \sec(x)$
12. $y^2 + x\cosh y + \sinh^2 x = 50$
13. $y = csch^3(\sqrt{2x})$
14. $x = \cosh(\cos(y))$
