جامعة بغداد

كلية التربية للعلوم الصرفة / ابن الهيثم

قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

الفصل الاول

بعض الاساسيات المهمة للمعادلات التفاضلية الاعتيادية

SOME IMPORTANT BASICS OF DIFFERENTIAL EQUATIONS

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Chapter one

SOME IMPORTANT BASICS OF DIFFERENTIAL EQUATIONS

1.1:Introduction: A differential equation is a mathematical equation that relates some function with its derivatives.

The derivatives represent their rates of change, and the equation defines a relationship between two variables.

The differential equations play an important role in many fields such as engineering, physics, economics and biology. Now, Let x be a number in the domain of the function f then we can express the first derivative of the function f for x as follows:

$$If\ \ y = f(x)\ then\ \frac{dy}{dx} = \frac{df(x)}{dx}\ or\ y' = f'(x)$$

Where the symbols $\frac{d(\dots)}{dx}$ and $(\dots)'$ represent the first derivative of the function.

1.2: Definitions

1.2.1: Differential equation

A differential equation is an equation involving derivatives or differentials.

For example:-

$$1 - \left(\frac{dy}{dx}\right)^4 + y = x$$

$$2 - x^2\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + y = 0$$

$$3-\frac{d^3y}{dx^3}-\left(\frac{d^2y}{dx^2}\right)^2+\frac{dy}{dx}=x^2+1$$

$$4-y'''+2(y'')^2+y'=cosx$$

$$5-\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial y^2}=x^2+y$$

$$6-x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u$$

1.2.2: Ordinary Differential Equation

Ordinary differential equation is a differential equation involving only ordinary derivatives (i.e.) It has derivatives of one or more dependent variables w.r.t. single independent variable. Such as equations 1,2,3 and4

1.2.3: Partial Differential Equation

A Partial differential equation is a differential equation involving partial derivatives (i.e.) It has derivatives of one or more dependent variable w.r.t. more than one independent variable.

For example the equations 5 and 6 are p.d.e

1.2.4: Order of a Differential Equation

The order of the highest order derivative in a differential equation is called the order of a diff. eq.

For example :-

(i) Equations (1) and (6) are of order one

(ii) Equations (2) and (5) are of order two

(iii) Equations (3) and (4) are of order three

1.2.5: Degree of Differential Equation

The degree of differential equation that is algebraic in its derivatives is the algebraic degree of the highest derivative shown in the equation (i.e.) when the equation is free from radicals and fractions in the dependent variable and its derivatives.

For example :-

(i) Equations (3),(4),(5) and (6) are of first degree

(ii) Equation (2) is of the third degree

(iii) Equation (1) is of the fourth degree

Other examples:- Find the order and degree of the following differential equations:

$$1 - \sqrt[3]{\left(\frac{d^2y}{dx^2}\right)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)}$$

$$2 - \sin(y') = y' + x + 3$$

Solution (1): This equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^{2/3} = \left(1+\frac{dy}{dx}\right)^{1/2}$$

$$\left[\left(\frac{d^2y}{dx^2}\right)^{2/3}\right]^6 = \left[\left(1+\frac{dy}{dx}\right)^{1/2}\right]^6$$

$$\left(\frac{d^2y}{dx^2}\right)^{4} = \left(1+\frac{dy}{dx}\right)^{3}$$

Therefore, this equation is of second order and fourth degree.

Solution (2): It hasn't degree since it is not algebraic in its derivatives.

1.2.6: Linear Differential Equation

The differential of any order shall be linear if the dependent variable and all derivatives are of the first degree and are not multiplied by each other and its general formula is

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x) \qquad \ldots (1)$$

Where $a_0, a_1, \ldots, a_n \; and \; f(x)$ are functions for x on the interval $a \le x \le b$

An equation that is not linear is said to be nonlinear

For example:-

$1 - 3y^{(3)} + 2y' = 5\sin x$ $\qquad$ *Linear*

$2 - x\dfrac{d^2y}{dx^2} - y^2 = 0$ $\quad$ *non − Linear*

$3 - y^{-1}\dfrac{d^2y}{dx^2} + 8y = e^x$ $\quad$ *non − Linear*

$4 - x^2y'' + 2xy' + y = 0$ $\quad$ *Linear*

$5 - y^{(5)} + yy' + 2x = 0$ $\quad$ *non − Linear*

$6 - y'' + 5xy' + \dfrac{1}{y} = \sqrt{x+1}$ $\quad$ *non − Linear*

1.2.7: Homogeneous Linear Differential Equation

Equation (1) is said to be homogeneous if f(x)=0

(i.e.) $a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0$ ... (2)

Therefore, the equations (2) and (4) are homogeneous and (1), (3), (5) and (6) are non-homogeneous

Note: If $a_0, a_1, \ldots, a_n$ in equation (1) are constant then the equation is said to be linear differential equation with constant coefficients.

Exercises:

Find the order, degree, linear and homogeneous of the following differential equations:

$1 - y'' + 3y' - 2y = 0$

$2 - (y''')^3 + (y'')^2 + xy = x$

$3 - (y')^4 + y^2 = 0$

$4 - \sqrt[3]{(y''')^2} = \sqrt{1 + (y')^2}$

$5 - \dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 4y = e^x$

1.3: Solution of the Differential Equation

The Solution of the differential equation is a relation between the variables of the equation and satisfies the following:

(i) Its empty of derivatives

(ii) Satisfies the differential equation

(iii) Defined on a certain interval

Example (1): Is $y(x) = A\sin 2x + B\cos 2x$ a solution of the diff. eq. $y'' + 4y = 0$ ... (3)

Sol. First, we must derive the function that given twice

$$y = A\sin 2x + B\cos 2x \qquad \dots (4)$$

$$y' = 2A\cos 2x - 2B\sin 2x \qquad \dots (5)$$

$$y'' = -4A\sin 2x - 4B\cos 2x \qquad \dots (6)$$

Substituting (4), (6) in (3), we get

$$-4A\sin 2x - 4B\cos 2x + 4(A\sin 2x + B\cos 2x)$$

$$= -4A\sin 2x - 4B\cos 2x + 4A\sin 2x + 4B\cos 2x$$

$$= 0$$

Thus, the given function satisfies the eq. (3)

$\therefore\ y(x) = A\sin 2x + B\cos 2x$ is a solution of (3)

Example (2): Prove that the function $y(x) = x\ln x - x$ ... (7)

is a solution of $xy' = x + y$ ... (8)

Sol. Deriving (7) w.r.t. x we get

$$y'(x) = x\frac{1}{x} + \ln x - 1$$

$$y'(x) = \ln x \qquad \text{... (9)}$$

Substituting (7),(9) in (8), we get

$$x\ln x = x + x\ln x - x$$

$$x\ln x = x\ln x$$

Hence, the equation (7) is a solution of the diff. eq. (8).

1.3.1: General solution of the differential equation

The general solution of the differential equation is the solution that is free of derivatives and contains a number of arbitrary constants and their number is equal to the order of the equation .

Example (3): Find the general solution of the equation $y''' = 0$

Sol. Integrating both sides three times

$$\int y''' \, dx = 0 \cdot dx \qquad \dots (10)$$

$$y'' = c_1 \qquad \dots (11)$$

$$y' = c_1 x + c_2 \qquad \dots (12)$$

$$y = \frac{c_1}{2} x^2 + c_2 x + c_3 \qquad \dots (13)$$

Where $c_1, c_2 \; and \; c_3$ are arbitrary constants.

Note that, the number of the constants are equal to the order of the equation

1.3.2: The Particular Solution

It's the solution that results after substituting the values of the arbitrary constants in the general solution.

Example (4): write the particular solution of the equation $y''' = 0 \; when \; c_1 = 2, c_2 = 2, c_3 = 0.$

Sol. : The solution of $y''' = 0$ is $y(x) = \frac{c_1}{2} x^2 + c_2 x + c_3$ (from Ex(3))

Sub. $c_1, c_2 \; and \; c_3$ in it

$$y(x) = \frac{2}{2} x^2 + 2x + 0$$

$$y(x) = x^2 + 2x$$

Remark: A general solution is a set of solutions that represent curves and are not intersected while only one of them passes through a given point of existence of these curves and at this point one real value is determined for the arbitrary constant.

Example (5): Find the general solution and the particular solution of the equation $y' = x$ ... (14)

That passes through the point $(1,2)$ and sketch the integral curves.

Sol. : Integrating (14) w.r.t. x we get

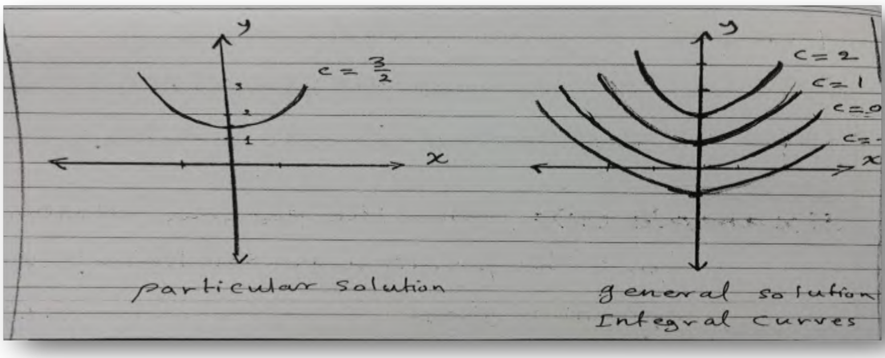
$$y = \frac{x^2}{2} + c \qquad \dots (15)$$

This is the general solution

To find the particular solution, substituting the point (1,2) in (15)

$$2 = \frac{1}{2} + c \ \rightarrow c = \frac{3}{2}, \text{ then } y = \frac{x^2}{2} + \frac{3}{2} \qquad \dots (16)$$

And this is the particular solution



1.4: Singular Solution of the Differential Equation

The singular solution is a solution that appears for some differential eq. and doesn't belong to the general solution group.

Example (6): Find the general solution and the singular solution of the equation $2y' = 3y^{1/3}$ $\quad ; x \in R$

Sol. : $2y' = 3y^{1/3} \rightarrow 2\frac{dy}{dx} = 3y^3$

$\frac{dy}{y^{1/3}} = \frac{3}{2}dx \qquad ; y \neq 0$

$y^{-1/3}dy = \frac{3}{2}dx$

Integrating both sides .

$\frac{3}{2}y^{2/3} = \frac{3}{2}x + c$

$\sqrt[3]{y^2} = x + c_1 \qquad where\ \ c_1 = \frac{2c}{3}$

y= $(x + c_1)^{3/2} \qquad ; y \neq 0$ ... (17)

this is the general solution.

Now. If $y = 0$, we note that its satisfying the diff. eq.

$y' = 0 \rightarrow 2(0) = 3(0) \rightarrow 0 = 0$

$\therefore y = 0$

is a solution to the diff. eq. but it's not belong to the general solution group, then $y = 0$ is a ***singular solution***

Exercises :

1- Find the order, degree and linearity of the following differential equations.

$a - y' + 8xy^2 = 0$

$b - (y')^2 + xy' = y^2$

$c - \sqrt{y''} = 3y' + x$

$d - y^{(4)} = \sqrt{y'}$

$e - (y'')^{1/3} = k(1 + (y')^2)^{5/2}$

2-Prove that every equation in the list K is a solution of the differential eqs. in the list H , where A,B,C are constants.

H	K
$1 - xy' = x^2 + y$	$y = x^2 + cx$
$2 - yy'' - (y')^2 = y^2 \ln y$	$\ln y = Ae^x + Be^{-x}$
$3 - xy' + y + x^4y^4e^x = 0$	$y^{-3} = x^3(3e^x + c)\ ; y \neq 0$
$4 - y'' + 3y' + 2y = 0$	$y = Ae^{-x} + Be^{-2x}$
$5 - 2(y')^2 - xy' + y = 3$	$\begin{cases} y = ct + 3 \\ x = 2t + c \end{cases}$

3-Prove that all of the equations

(i) $y = 2e^x$ $(ii) y = 3x$

$(iii) y = Ae^x + Bx \qquad , A\ and\ B\ are\ constants.$

are solutions of the diff. eq. $\quad y''(1-x) + y'x - y = 0$

4- Find the value of A (if it exist) that makes $y = Ax^3$ a solution of the diff. eqs.

$a)\ x^2y'' + 6xy' + 5y = 0 \quad , b)\ x^2y'' + 6xy' + 5y = x^2$

5- What is the values of the constant C that make $y = e^{Cx}$ a solution of the equation $y'' + 5y' + 6y = 0$

6- Find the general solution and the singular solution of the equation $(y')^2 = 4y$

1.5: Composition the Differential Equation from the General Solution

In this subject we will discuss how to find the differential equation if we know the general solution.

The method depends on the relationship of the number of arbitrary constants in the General solution group and the order of the differential equation, where the equation is derived by a number of equal times for a number of constant as shown in the following examples.

Example (7) : Find the diff. eq. that the general solution is $y = c_1x + c_2x^3, where\ c_1 and\ c_2$ are arbitrary constants.

Sol. : $y = c_1x + c_2x^3 \qquad \dots (18)$

Driving (18) twice, to get

$$y' = c_1 + 3c_2x^2 \qquad \dots(19)$$

$$y'' = 6c_2x \;\rightarrow c_2 = \frac{y''}{6x} \qquad \dots(20)$$

Sub. (20) in (19), we get

$$y' = c_1 + \frac{xy''}{2} \rightarrow c_1 = y' - \frac{1}{2}xy'' \qquad \dots(21)$$

Sub. (20) and (21) in (18)

$$y = \left(y' - \frac{1}{2}xy''\right)x + \frac{y''}{6x}x^3$$

$$= xy' - \frac{1}{2}x^2y'' + \frac{1}{6}x^2y''$$

$$= xy' - \frac{1}{3}x^2y'' \qquad \dots(22)$$

And this is the required differential equation.

Example (8) : Find the diff. eq. that the general solution is $y = Ae^x - x$

Sol. : $y = Ae^x - x \qquad \dots(23)$

Driving (23) , we get

$$y' = Ae^x - 1 \rightarrow \; Ae^x = y' + 1 \qquad \dots(24)$$

Sub. (24) in (23) we get

$$y = y' + 1 - x$$

$$y' - y - x + 1 = 0 \qquad \dots(25)$$

Note that (25) is the differential eq.

Remark: There is another way to find the differential equation from the general solution group by using some linear algebra rules and finding the parameters of the arbitrary constants and making it equal to zero.

Example (9) : Find the differential equation of Example (8) using the (determinant method)

Sol. : $y = Ae^x - x \ \rightarrow Ae^x - x - y = 0 \qquad \dots(26)$

$$y' = Ae^x - 1 \ \rightarrow Ae^x - 1 - y' = 0 \qquad \dots(27)$$

$$\begin{vmatrix} e^x & -x-y \\ e^x & -1-y' \end{vmatrix} = 0$$

$$e^x(-1-y') + (x+y)e^x = 0$$

$$-e^x - e^x y' + xe^x + ye^x = 0 \quad (e^x \neq 0)$$

Then $-1 - y' + x + y = 0 \qquad \dots(28)$

And this is the diff. eq.

Example (10) : Find the diff. eq. that the general solution is $y = c_1x + c_2x^3$using the determinant method

Sol. : $y = c_1x + c_2x^3 \ \rightarrow c_1x + c_2x^3 - y = 0 \qquad \dots(29)$

$$y' = c_1 + 3c_2x^2 \ \rightarrow \ c_1 + 3c_2x^2 - y' = 0 \qquad \dots(30)$$

$$y'' = 0 + 6c_2x \rightarrow 0 + 6c_2x - y'' = 0 \qquad \dots(31)$$

The det. is

$$\begin{vmatrix} x & x^3 & -y \\ 1 & 3x^2 & -y' \\ 0 & 6x & -y'' \end{vmatrix} = 0$$

$$x\begin{vmatrix} 3x^2 & -y' \\ 6x & -y'' \end{vmatrix} - 1\begin{vmatrix} x^3 & -y \\ 6x & -y'' \end{vmatrix} + 0\begin{vmatrix} x^3 & -y \\ 3x^2 & -y' \end{vmatrix} = 0$$

$$x(-3x^2y'' + 6xy') - (-x^3y'' + 6xy) = 0$$

$$-3x^3y'' + 6x^2y' + x^3y'' - 6xy = 0$$

$$[-2x^3y'' + 6x^2y' - 6xy = 0] \div \frac{1}{6}x$$

$$\frac{-x^2y''}{3} + xy' - y = 0$$

$$y = \frac{-x^2y''}{3} + xy' \qquad \dots(32)$$

And this is the same result in Ex.7 eq.(22)

Exercises:

1- Find the differential equation of the following curves where A,B and C are arbitrary constants.

$a)\ y = Ax^2 + A^2$

$b)\ y = Ax^2 + Bx + C$

$c)\ y = A\sin x + B\cos x$

$d)\ y = Ae^x \cos(3x + B)$

2- Find the differential equation in which the general solution is the set of equations of the circles whose centers are located on the line $y = x$ and the radius of each is equal to 1.

3- Find the differential equation of the hyperbola $xy = c$; c is an arbitrary constant.

4- Find the differential equation for the set of all straight lines in the plane.

5- Find the differential equation of the following parabolas $y^2 = 4p(x - h)$.

6- Find the differential equation for the set of all circles that contact with y-axis in the origin point.

1.6 : Existence and Uniqueness of the Solution of the differential equation.

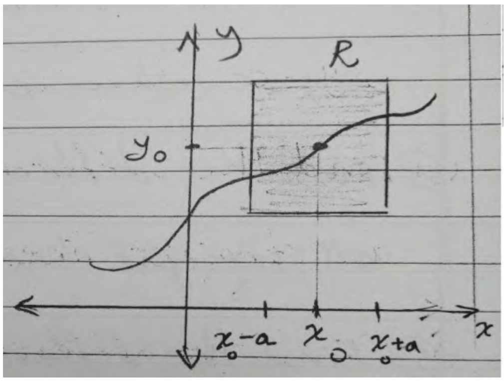
Consider the ordinary differential equation $\frac{dy}{dx} = f(x, y)$ with the initial value $y(x_0) = y_0$ where f is defined on the region

$$R = \{(x, y): |x - x_0| < a, |y - y_0| < b, \qquad a\&b > 0\}$$

If f and $\frac{\partial f}{\partial y}$ are continuous on R then the equation $\frac{\partial f}{\partial y} = f(x, y)$ has unique continuous solution $y = \Phi(\mathrm{x})$ passes from the point (x_0, y_0) for all $x, y \; in \; R$

(i.e.)

$$\begin{cases} 1) \; If \; f \; is \; continuous \; near \\ (x_0, y_0) then \; the \; solution \; is \\ exist \\ 2) \; If \; \dfrac{\partial f}{\partial y} \; is \; continuous \; near \\ (x_0, y_0) \; then \; the \; solution \; is \; unique \end{cases}$$



Example (11) : Is the solution of the equation

$$\frac{dy}{dx} = 2x \quad , y(1) = 3 \qquad \dots (33)$$

Exist and unique at (1,3)?

Sol. :

1) $\frac{dy}{dx} = 2x$ then $f(x, y) = 2x$

Its clear that f is continuous at all $x\&y$ in xy plane then the solution is exist at (1,3)

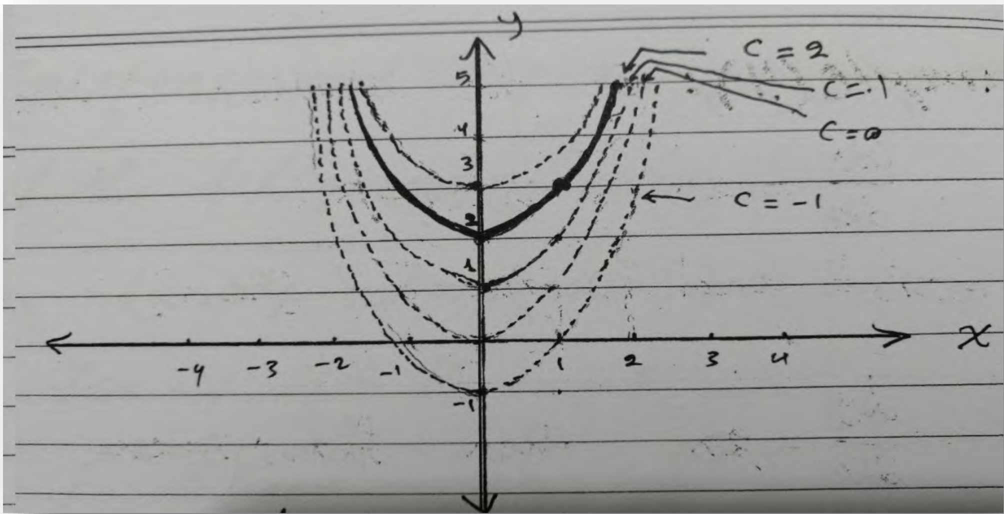
2) $\frac{\partial f}{\partial y} = 0$ and its continuous too at all $x\&y$ in xy plane then the solution is unique at (1,3) Integrating (33) to find the general Sol.

$$\frac{dy}{dx} = 2x \to dy = 2xdx \to y = x^2 + c \qquad \dots(34)$$

and this is the general solution where c is an arb. Cons.

Sub. y(1)=3 in (34), we get y(1)=$1^2 + c \to 3 = 1 + c \to c = 2$

$\therefore\ y = x^2 + 2$ is the solution passes from (1,3) and its clear that its unique.



Its clear that solution $y = x^2 + 2$ passes from the point (1,3) and it’s the only one.

Example (12) : consider $x\frac{dy}{dx} = y$ discuss the existence and uniqueness of solutions

Sol. :

$x\frac{dy}{dx} = y \rightarrow \frac{dy}{dx} = \frac{y}{x}$ ... (35)

$\therefore f(x, y) = \frac{y}{x}$ ... (36)

1) Its clear that $\frac{y}{x}$ is continuous at any point (a, b) where $a \neq 0$

So the solution is exist when $a \neq 0$

2) $\frac{\partial f}{\partial y} = \frac{1}{x}$ and its also continuous at any point (a, b) where $a \neq 0$

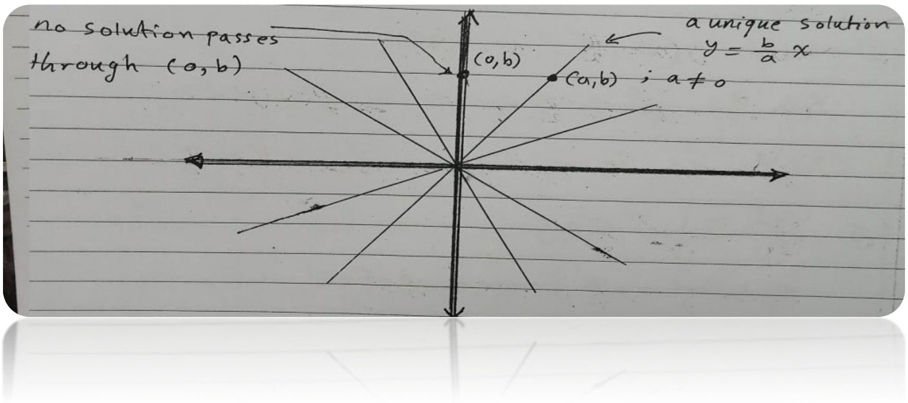
So the solution is unique when $a \neq 0$

Integrating eq. (35)

$$\frac{dy}{dx} = \frac{y}{x} \rightarrow \frac{dy}{y} = \frac{dx}{x} \rightarrow \ln y = \ln x + \ln c$$

$$\rightarrow y = cx \quad \text{... (37)}$$

Eq. (37) is the general solution where c is an arbitrary constant.



The figure shows that no solution has passes from the point (0,b)

Remark: The point that does not satisfy the condition of existence and uniqueness is called the (singular point) .

Example (13) : Does the following Initial value problem (Ivp) have a unique solution?

$$\frac{dy}{dx} = e^{x} \cos y \qquad ; y(0) = \frac{\pi}{2} \qquad \dots (38)$$

Sol: $\frac{dy}{dx} = e^{x} \cos y \rightarrow f(x,y) = e^{x} \cos y \qquad \dots . (39)$

(1) sub. $\left(0, \frac{\pi}{2}\right)$ in (39)

$$f\left(0, \frac{\pi}{2}\right) = e^{0} \cos\frac{\pi}{2} = 0 \rightarrow f(x,y) = 0$$

$f(x,y)$ is continuous near $\left(0, \frac{\pi}{2}\right)$

(2) $\frac{\partial f}{\partial y} = e^{x} \sin y$

sub. $\left(0, \frac{\pi}{2}\right)$, we get

$\frac{\partial f}{\partial y} = -1$ and its continuous near $\left(0, \frac{\pi}{2}\right)$. Then there is a unique solution at $\left(0, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = e^x \cos y \rightarrow \frac{dy}{\cos y} = e^x dx \rightarrow \sec y \, dy = e^x dx$$

$$\ln|\sec y + \tan y| = e^x + c \qquad \dots (40)$$

But $\ln|\sec y + \tan y| = \ln\left|\frac{1}{\cos y} + \tan y\right| \quad and \;\; \cos\frac{\pi}{2} = 0$

$$\rightarrow \frac{1}{\cos\frac{\pi}{2}} = \frac{1}{0}$$

$\therefore$ there is no solution passes from $\left(0, \frac{\pi}{2}\right)$ and exist in the general solution group. (40) we must look for a solution in another way.

$$\frac{dy}{dx} = 0 \qquad at \;\left(0, \frac{\pi}{2}\right)$$

Integrating both sides

$$y(x) = c \qquad \dots (41)$$

Sub. $\left(0, \frac{\pi}{2}\right) \;\; \rightarrow y(0) = c \;\rightarrow \frac{\pi}{2} = c$,sub. in (41) we get

$y(x) = \frac{\pi}{2}$ and this is the unique solution passes through the point $\left(0, \frac{\pi}{2}\right)$

Example (14) : Does the following (IVP) $\frac{dy}{dx} = x\sqrt{y-3}$

$, y(-2) = 28$ $\quad\quad\quad\quad \ldots (42)$

Have a unique solution or not?

Sol. $f(x, y) = x\sqrt{y-3}$

It's clear that f is continuous near (-2,28)

$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y-3}}$ and it's continuous near (-2,28) also.

Then the diff. eq. has a unique solution :

$\frac{dy}{dx} = x\sqrt{y-3} \;\rightarrow \frac{dy}{\sqrt{y-3}} = \text{xdx}$

$\rightarrow 2\sqrt{y-3} = \frac{x^2}{2} + \text{c}$ $\quad\quad\quad\quad \ldots (43)$

Sub. (-2,28) in (43), we get

$2\sqrt{28-3} = \frac{4}{2} + \text{c} \;\rightarrow 10 = 2 + \text{c} \;\rightarrow \text{c} = 8\,,$

$2\sqrt{y-3} = \frac{x^2}{2} + 8 \;\rightarrow \text{y} = \left(\frac{x^2}{4} + 4\right)^2 + 3$ $\quad\quad\quad\quad \ldots (44)$

Eq. (44) is the unique solution that passes through (-2,28).

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Ordinary Differential Equations

Chapter 2

أساتذة المادة

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CHAPTER TWO

The Ordinary Differential Equation of the first order and first degree

An ordinary differential equation of first order and first degree is written in one of the following forms:

المعادلة التفاضلية الاعتيادية من الرتبة الأولى والدرجة الأولى تكتب بأحدى الاشكال التالية:

$$M(x,y)dx + N(x,y)dy = 0$$

$$\textbf{or} \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\textbf{or} \quad \frac{dy}{dx} = f(x,y)$$

Such that f, M, N does not contain derivative.

Although this kind of differential equations seems simple, there is no general way of solving, but several methods depending on the type of the equation. Therefore, the equations that can be solved directly divided into several types, the most important ones are:

(رغم ان هذا النوع من المعادلات التفاضلية تبدو بسيطة الا انه لا توجد طريقة عامة للحل وانما عدة طرق حسب نوع المعادلة، وعلى ذلك تنقسم المعادلات التي يمكن إيجاد حلها بطريقة مباشرة الى عدة أنواع أهمها:)

2.1) Separable equation.

2.2) Homogenous equation.

2.3) Differential equation with linear coefficients.

2.4) Exact differential equation

2.5) integral factors.

2.6) Bernoulli's equation

2.7) Ricatt's Equation

2.8) The diff. eq. of the form $\boldsymbol{f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x)}$

2.9) Equation that is solved using a suitable substitution.

Now let's start:

2.1 Separable of Variables:

Definition1: An equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \quad \text{or} \quad h(y)d\,y = g(x)\,dx$$

is said to be separable or to have separable variable. And if both g and h are differentiable, then:

$$\int h(y)\, dy = \int g(x)\, dx + c$$

<u>Ex 1 :</u> Solve : a)$\boldsymbol{\frac{dy}{dx} = 1 + e^{5x}}$ **b)** $\boldsymbol{\frac{dy}{dx} = sin(x)}$

<u>Solution:</u>

a) $dy = (1 + e^{5x})dx \Rightarrow y = x + \frac{1}{5}e^{5x} + c$

b) $dy = \sin x\, dx \Rightarrow y = -\cos x + c$

<u>Ex 2 :</u> Solve : $\boldsymbol{(1 + x^2)\, y\, dy + (y + 3)\, dx = 0}$

<u>Solution:</u> $[(1 + x^2)\, y\, dy + (y + 3)\, dx = 0] * \frac{1}{(1+x^2)(y+3)}$

$$\Rightarrow \left(\frac{y}{y+3}\right) dy + \frac{dx}{1+x^2} = 0$$

$$\Rightarrow \int \left(1 - \frac{3}{y+3}\right) dy + \tan^{-1} x = c$$

$$\Rightarrow \int dy - \int \frac{3}{y+3} dy + \tan^{-1} x = c$$

$$\Rightarrow \mathrm{y} - 3 \ln\ |\mathrm{y} + 3|\ + \tan^{-1} x = c,\ c \in \mathbb{R} \qquad (c\ is\ an\ arb.cons.)$$

Remark: $\frac{y}{y+3}$ can be solved by two methods

يمكن تبسيط المقدار $\frac{y}{y+3}$ بطريقتين: اما *القسمة الطويلة* او *بالتجزئة*.

1) القسمة الطويلة:

$$\begin{array}{r|l} & 1 \\ \hline y+3 & y \end{array}$$

2) التجزئة:

$$\frac{y}{y+3} = \frac{y+3-3}{y+3} = \frac{y+3}{y+3} - \frac{3}{y+3} = 1 - \frac{3}{y+3}$$

<u>Ex 3 :</u> Solve $\quad \boldsymbol{y' = e^{x-y}}$

<u>Solution:</u> $\frac{dy}{dx} = e^x . e^{-y} \Rightarrow dy = e^x . e^{-y}\, dx$

$$\Rightarrow e^y\, dy = e^x\, dx \Rightarrow \int e^y dy = \int e^x\, dx$$

$\Rightarrow\ e^y = e^x + c$, (The general solution),(c is an arb. cons.)

<u>Ex 4 :</u> Solve $\boldsymbol{x\,y\,dy + (2yx^2 + 4x^2 - y - 2)dx = 0}$

<u>Solution:</u> بواسطة التحليل نحصل على:

$$[x\,y\,dy + (y+2)(2x^2-1)dx = 0] * \frac{1}{x(y+2)}$$

$$\Rightarrow \frac{y}{y+2}\,dy + \frac{2x^2-1}{x}\,dx = 0$$

$$\Rightarrow \int (1 - \frac{2}{y+2})\,dy + \int (2x - \frac{1}{x})\,dx = \int 0$$

$\Rightarrow y - 2\ln|y+2| + x^2 - \ln|x| = c$, (The general solution of O.D.E.)

<u>Ex 5 :</u> Solve $\boldsymbol{sin^2x\,cos\,y\,dx + sin\,y\,sec\,x\,dy = 0}$

<u>Solution:</u> $\frac{\sin^2 x}{\sec x}\,dx + \frac{\sin y}{\cos y}\,dy = 0$

$$\Rightarrow \frac{\sin^2 x}{\frac{1}{\cos x}}\,dx + \frac{-\sin y}{-\cos y}\,dy = 0$$

$$\Rightarrow \int \sin^2 x \ \cos x \ dx + \int \frac{-\sin y}{-\cos y} \, dy = \int 0$$

Then the general solution is:

$-\frac{\sin^3 x}{3} - \ln \ |\cos y| = c$, (Where c is an arb. cons.)

<u>Ex 6 :</u> Solve $\boldsymbol{\frac{dy}{dx} + e^x y = e^x y^2}$

<u>Solution:</u> $\frac{dy}{dx} = e^x \, y^2 - e^x \, y \Rightarrow \frac{dy}{dx} = e^x \, (y^2 - y\,)$

$$\Rightarrow \frac{dy}{y(y-1\,)} = e^x \, dx$$

$\int \frac{1}{y(y-1\,)} \, dy$ (We use the fraction law) (قانون التجزئة)

$$\frac{A}{y} + \frac{B}{y-1} = \frac{A\,(y-1) + \ B\,y}{y(y-1)}$$

$$\Rightarrow Ay - A + B\,y \ = 1$$

$$\Rightarrow (A+B)y \ = 0 \qquad \ldots(1)$$

$$\Rightarrow -A \ = 1 \qquad \ldots(2)$$

$$\Rightarrow A + B \ = 0$$

$A = -B$

$From$ (2): $A = -1$

$From$ (1): $A + B = 0 \longrightarrow -1 + B = 0 \rightarrow -1 = -B \longrightarrow B = 1$

Now, $\int \frac{1}{(y^2 - y)} dy = \int (\frac{-1}{y} + \frac{1}{y-1}) dy$

$= \int \frac{-1}{y} dy + \int \frac{1}{y-1} dy = -ln|y| + ln|y-1|$

Then the general solution is:

$-\ln|y| + \ln|y-1| = e^x + c,$ (Where c is an arb. con.)

2.2 Homogenous differential equation

Definition2: The function $f(x, y)$ is homogeneous function of degree n if

$$f(t\,x, t\,y) = t^n f(x, y) \qquad \dots\dots.. \qquad \text{(I)}$$

where t is a constant.

Definition3: The differential equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad \ldots. \qquad \text{(II)}$$

is called homogenous if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

To solve the homogeneous differential equation:

$$M(x,y)dx + N(x,y)dy = 0$$

Can be written in the form

$$\frac{dy}{dx} = f(\frac{y}{x}) \qquad \ldots. \qquad \text{(III)}$$

Let $v = \frac{y}{x}$, then equation (III) becomes

$$\frac{dy}{dx} = f(v) \qquad \ldots. \qquad \text{(IV)}$$

Since $v = \frac{y}{x}$, then $y = vx$, and so

$$dy = vdx + xdv$$

We substitute the above expression.

The new equation is separable, we solve it and in the last we put $v = \frac{y}{x}$ to get the general solution.

<u>Ex 1 :</u> Solve $\boldsymbol{x\,y^2\,dy - (x^3 + y^3)dx = 0}$

<u>Solution:</u> $M(x,y) = (x^3 + y^3)$

$$N(x,y) = xy^2$$

$M(t\,x, t\,y) = (tx)^3 + (ty)^3 = t^3x^3 + t^3y^3$

$= t^3(x^3 + y^3) = t^3M(x,y)$, الدالة متجانسة من الدرجة الثالثة

$N(t\,x, t\,y) = t\,x\,(t\,y)^2 = tx\,(t^2\,y^2)$

$= t^3(x\,y^2) = t^3\,N(x,y)$, الدالة متجانسة من الدرجة الثالثة

So, M and N are both homogeneous, and have the same degree, so the diff. eq. is homogeneous.

Let $y = vx \longrightarrow dy = v\,dx + x\,dv$

بالتعويض في المعادلة التفاضلية نحصل على:

$xv^2x^2(vdx + xdv) - (x^3 + v^3x^3)dx = 0$

$$\Longrightarrow x^3v^3dx + x^4v^2dv - x^3dx - v^3x^3dx = 0$$

$$\Longrightarrow [x^4v^2dv - x^3dx = 0] * \frac{1}{x^4}$$

$$\Longrightarrow \int v^2dv - \int \frac{1}{x}dx = \int 0$$

$$\Longrightarrow \frac{v^3}{3} - \ln|x| = c$$

When $v = \frac{y}{x} \Longrightarrow \frac{y^3}{3x^3} - \ln|x| = c$, (The general solution)

<u>Ex 2 :</u> Solve $\boldsymbol{xy\,dx + (x^2 - 2y^2)dy = 0}$

<u>Solution:</u> The equation is homogeneous (prove it), so

Let $v = \frac{y}{x} \longrightarrow y = v\,x \longrightarrow dy = v\,dx + x\,dv$

$$x(vx)\,dx + (x^2 - 2v^2x^2)(v\,dx + x\,dv) = 0$$

$$\Longrightarrow vx^2\,dx + x^2v\,dx + x^3dv - 2v^3x^2dx - 2v^2x^3dv = 0$$

$$\Longrightarrow 2vx^2\,dx - 2v^3x^2dx + x^3(1 - 2v^2)dv = 0$$

$$\Longrightarrow 2x^2(v - v^3)dx + x^3(1 - 2v^2)dv = 0$$

$$\Rightarrow [2x^2(v-v^3)dx + x^3(1-2v^2)dv = 0\,] * \frac{1}{x^3(v-v^3)}$$

$$\Rightarrow \frac{2x^2}{x^3}dx + \frac{1-2v^2}{v-v^3}\,dv = 0$$

$$\Rightarrow \int \frac{2}{x}dx + \int \frac{1-2v^2}{v-v^3}\,dv = 0$$

$$\int \frac{1-2v^2}{v-v^3}\,dv \;=?$$

$$\frac{1-2v^2}{v-v^3} = \frac{1-2v^2}{v(1-v^2)}$$

$$= \frac{1-2v^2}{v(1-v)(1+v)}$$

$$= \frac{A}{v} + \frac{B}{1-v} + \frac{C}{1+v} \qquad \ldots (*)$$

$$= \frac{A(1-v^2) + Bv(1+v) + cv(1-v)}{v(1-v)(1+v)}$$

$$= \frac{A - Av^2 + Bv + Bv^2 + Cv - Cv^2}{v(1-v^2)}$$

$$-A + B - C = -2 \qquad \ldots \; (1)$$

$$B + C = 0 \qquad \ldots \; (2)$$

$A = 1$ ... (3)

By substituting (3) in (1), we get:

$-1 + B - C = -2 \longrightarrow B - C = -1$ .... (4)

Eq. (2) + Eq. (4):

$$B - C = -1$$

$$B + C = 0$$

$2B = -1 \longrightarrow B = \dfrac{-1}{2}$

From Eq. (2) $\longrightarrow C = \frac{1}{2}$

So, $A = 1,\ B = \frac{-1}{2}, and\ C = \frac{1}{2}$

Substituting A,B,C in eq.(*), we get:

بتعويض قيم A, B, C في المعادلة (*)، نحصل على:

$$\int \frac{1-2v^2}{v-v^3}\, dv = (\frac{1}{v} + \frac{\frac{-1}{2}}{1-v} + \frac{\frac{1}{2}}{1+v})dv$$

$$= \ln v - \tfrac{1}{2}\ln|1-v| + \tfrac{1}{2}\ln|1+v| = c$$

$$\therefore 2\ln|x| + \ln|v| - \frac{1}{2}\ln|1-v| + \frac{1}{2}\ln|1+v| = c$$

$$\Longrightarrow 2\ln|x| + \ln\left|\frac{y}{x}\right| - \frac{1}{2}\ln\left|1-\frac{y}{x}\right| + \frac{1}{2}\ln\left|1+\frac{y}{x}\right| = c, \text{ (The general sol.)}$$

<u>Ex 3 :</u> Prove the following diff. eq. is homogeneous, then find the general solution:

$$\boldsymbol{xdy - ydx = \sqrt{x^2+y^2}\,dx}$$

<u>Solution:</u> We must prove the degree of

$$M(x,y)dx = N(x,y)dy$$

$$x\,dy = \left(y+\sqrt{x^2+y^2}\right)dx \qquad (1)$$

$$\because M(tx,ty) = (ty + \sqrt{(tx)^2+(ty)^2}\,)$$

$$= ty + \sqrt{t^2(x^2+y^2)}$$

$$= ty + t\sqrt{x^2+y^2}$$

$$= t(y + \sqrt{x^2+y^2}\,)$$

$$= tM(x,y),$$

$\Longrightarrow \; M(x,y)$, متجانسة من الدرجة الاولى

$\because N(tx,ty) = tx$

$\Longrightarrow \; N(x,y)$, متجانسة من الدرجة الاولى

Then the diff. eq. is homogeneous. Now, we find the general solution:

Let $y = vx \longrightarrow dy = vdx + x\,dv$, by substituting in the diff. eq.(1), we get:

$$x\,(vdx + x\,dv) - vx\,dx = \sqrt{x^2 + v^2x^2}dx$$

$$xvdx + x^2dv - vxdx = x\,\sqrt{1+v^2}dx$$

$$[x^2dv = x\,\sqrt{1+v^2}dx] * \frac{1}{x^2\sqrt{1+v^2}}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{1}{x}\,dx$$

$$\int \frac{dv}{\sqrt{1+v^2}} = ?$$

Let $v = \tan\theta \;\longrightarrow dv = \sec^2\theta\; d\theta$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{\sec^2\theta\; d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\;\theta\; d\theta$$

$$= \ln|sec\theta + \tan\theta| = \ln\left|v + \sqrt{1+v^2}\right| = l\,n\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right|$$

Then the general solution is:

$$\therefore \quad l\,n\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \ln x + c \text{ (c is an arb. cons.)}$$

2.3 Differential Equation with Linear Coefficients (Equation that reduce to homogeneous equation)

These equations can be expressed as:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \ldots \quad (V)$$

Two lines:

$a_1x + b_1y + c_1 = 0$, and $a_2x + b_2y + c_2 = 0$ $\qquad \ldots \quad (VI)$

a) Intersect if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, (المستقيمان متقاطعان)

or: Intersect if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

b) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, (المستقيمان متوازيان)

or: Parallel if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

<u>Case (1):</u> If the two lines that in (VI) intersect , i.e. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, we seek a translation of axes using the form:

$x = x_1 + h \longrightarrow dx = dx_1$

$y = y_1 + k \longrightarrow dy = dy_1$

where (h, k) is the point of intersection

Then the substitution $x = x_1 + h, y = y_1 + k \quad (dx = dx_1, dy = dy_1)$

transform equation (V) into the homogeneous equation.

<u>Remark</u>: First we find the intersection point.

<u>Ex 1 :</u> Solve the following O.D.E. :

$$(\mathbf{2x - 3y + 4})\boldsymbol{dx} + (\mathbf{3x - 2y + 1})\boldsymbol{dy} = \mathbf{0}$$

<u>Solution:</u> $\left.\begin{matrix} 2x - 3y + 4 = 0 \\ 3x - 2y + 1 = 0 \end{matrix}\right]$ $\quad \frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-3}{-2}$

$\frac{2}{3} \neq \frac{-3}{-2}$ $\longrightarrow$ The two lines are intersection.

or: $\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} = 5 \neq 0 \longrightarrow$ The two lines intersect.

Now, we must find the intersection point:

$$2x - 3y + 4 = 0 * (-3) \qquad (1)$$
$$3x - 2y + 1 = 0 \; * (2) \qquad (2)$$

$\longrightarrow 5y - 10 = 0 \;\longrightarrow y = 2$, [by substituting in Eq.(1)]

$\longrightarrow 2x - 3(2) + 4 = 0 \;\longrightarrow x = 1$

$\therefore (h, k) = (1,2)$ [The intersection point]

Let $y = y_1 + 2 \;\longrightarrow dy = dy_1$

$x = x_1 + 1 \;\longrightarrow dx = dx_1$

By substituting in the D.E., we get:

$$(2(x_1 + 1) - 3(y_1 + 2) + 4)dx_1 + (3(x_1 + 1) - 2(y_1 + 2) + 1)dy_1 = 0$$

$$(2x_1 + 2 - 3y_1 - 6 + 4)dx_1 + (3x_1 + 3 - 2y_1 - 4 + 1)dy_1 = 0$$

$$(2x_1 - 3y_1)dx_1 + (3x_1 - 2y_1)dy_1 = 0$$

The above equation is homogeneous

Let $y_1 = vx_1 \longrightarrow dy_1 = v\,dx_1 + x_1\,dv$

$$(2x_1 - 3vx_1)dx_1 + (3x_1 - 2vx_1)(vdx_1 + x_1dv) = 0$$

$$(2x_1 - 3vx_1)dx_1 + 3x_1vdx_1 + 3x_1^2dv - 2v^2x_1dx_1 - 2vx_1^2dv = 0$$

$$2x_1dx_1 - 3vx_1dx_1 + 3x_1vdx_1 + 3x_1^2dv - 2v^2x_1dx_1 - 2vx_1^2dv = 0$$

$$[2x_1(1 - v^2)dx_1 + x_1^2(3 - 2v)dv = 0] * \frac{1}{x_1^2(1 - v^2)}$$

$$\frac{2}{x_1}dx_1 + \frac{3 - 2v}{1 - v^2}dv = 0$$

$$2\int \frac{dx_1}{x_1} + \int \frac{3 - 2v}{(1 + v)(1 - v)}dv = \int 0$$

$$\because \frac{3 - 2v}{(1 + v)(1 - v)} = \frac{A}{(1 + v)} + \frac{B}{(1 - v)} = \frac{A - Av + B + Bv}{(1 + v)(1 - v)}$$
$$= \frac{(A + B) + (B - A)v}{(1 + v)(1 - v)}$$

$\therefore \quad A + B = 3$

$B - A = -2$

$\longrightarrow B = \frac{1}{2}$, and $A = \frac{5}{2}$

$$2\int\frac{dx_1}{x_1}+\int(\frac{\frac{5}{2}}{(1+v)}+\frac{\frac{1}{2}}{(1-v)})\,dv=\int 0$$

$$4\int\frac{dx_1}{x_1}+5\int\frac{dv}{(1+v)}+\int\frac{dv}{(1-v)}=\int 0$$

$$4\ln|x_1|+5\ln|1+v|-\ln|1-v|=c_1$$

$$\ln x_1^4+\ln(1+v)^5-\ln(1-v)=c_1$$

$$\ln x_1^4+\ln\frac{(1+v)^5}{(1-v)}=c_1$$

$\ln(x_1^4\,.\frac{(1+v)^5}{(1-v)})=c_1$ (c_1 is an arb. cons.)

$x_1^4.\frac{(1+v)^5}{1-v}=e^{c_1},$ let $e^{c_1}=c$

When $v=\frac{y_1}{x_1}\longrightarrow x_1^4.\frac{\left(1+\frac{y_1}{x_1}\right)^5}{1-\frac{y_1}{x_1}}=c$

$$\longrightarrow x_1^4.\frac{\frac{(x_1+y_1)^5}{x_1^5}}{\frac{x_1-y_1}{x_1}}=c$$

$$\longrightarrow (x_1+y_1)^5=C(x_1-y_1)$$

$\longrightarrow (x-1+y-2)^5 = C(x-1-(y-2))$

$\longrightarrow (x+y-3)^5 = C(x-y+1),$ (The general solution)

<u>Case (2):</u> If the two lines that in (VI) are parallel (i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$)then the solution will be using the hypothesis $\boldsymbol{z = ax + by}$ as shown in the following example

<u>Ex 2 :</u> Solve the following O.D.E. $\boldsymbol{(x-y+2)dx = (x-y-3)dy}$

نلاحظ انها معادلة تفاضلية ذات معاملات خطية.

<u>Solution:</u>

$\because \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = -1+1 = 0$

اذن المستقيمان متوازيان

$(x-y+2)dx - (x-y-3)dy = 0$

Let $z = x - y \longrightarrow dz = dx - dy \longrightarrow dy = dx - dz$

$\longrightarrow (z+2)dx - (z-3)dy = 0$

$zdx + 2dx - (z-3)(dx - dz) = 0$

$$zdx + 2dx - zdx + zdz + 3dx - 3dz = 0$$

$$\int 5dx + \int (z-3)dz = \int 0$$

$$5x + \frac{z^2}{2} - 3z = c$$

$5x + \frac{(x-y)^2}{2} - 3(x-y) = c,$ (The general solution)

ملاحظة: يمكن حل المثال السابق(Ex 2) بأسلوب اخر.

$$(x - y + 2)dx = (x - y - 3)dy$$

$\longrightarrow \frac{dy}{dx} = \frac{x-y+2}{(x-y-3)}$ … (1)

Let $z = x - y$. To solve for $\frac{dy}{dx}$, we differentiate $z = x - y$ with respect to (w.r.t.) x to obtain $\frac{dz}{dx} = 1 - \frac{dy}{dx}$, and so $\frac{dy}{dx} = 1 - \frac{dz}{dx}$, substitute into Eq. (1) yields:

$$1 - \frac{dz}{dx} = \frac{z+2}{z-3} \Rightarrow \frac{dz}{dx} = 1 - \frac{z+2}{z-3} = \frac{-5}{z-3}$$

$$\Rightarrow (z-3)dz = -5dx \Rightarrow \frac{z^2}{2} - 3z = -5x + c$$

$\Rightarrow \frac{(x-y)^2}{2} - 3(x-y) = -5x + c,$ (The general solution)

Examples:

1) Solve the O.D.E. $(\boldsymbol{y^2+y})\boldsymbol{dx}-(\boldsymbol{x^2-x})\boldsymbol{dy}=\boldsymbol{0}$

Solution:

$$\frac{dx}{x^2-x}-\frac{dy}{y^2+y}=0$$

$$\underbrace{\int\frac{dx}{x^2-x}}_{(1)}-\underbrace{\int\frac{dy}{y^2+y}}_{(2)}=\int 0$$

Use the method of fragmentation (partial) of the fractions. نستخدم طريقة تجزئة الكسور

$$(1)\rightarrow\ \frac{1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$$

$$\rightarrow\ \frac{1}{x(x-1)}=\frac{Ax-A+Bx}{x(x-1)}=\frac{(A+B)x-A}{x(x-1)}$$

$\rightarrow\ A+B=0 \qquad \dots\ (i)$

$\quad -A=1 \qquad \dots\ (ii)$

$From$: $(ii)\rightarrow A=\ -1$, substitute in (i)

$\rightarrow\ -1 + B = 0 \rightarrow B = 1$

$(2) \rightarrow \frac{1}{y(y+1)} \rightarrow A = 1$, and $B =\ -1$ (بنفس الأسلوب اعلاه)

$$\because \int \frac{1}{x(x-1)} dx - \int \frac{dy}{y(y+1)} = \int 0$$

$$\rightarrow \int \frac{-1}{x} dx + \int \frac{1}{x-1} - \int \frac{1}{y} dy + \int \frac{1}{y+1} dy = \int 0$$

$\rightarrow -\ln|x| + \ln|x-1| - \ln|y| + \ln|y+1| = c,$ (The general solution)

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2) Find the general solution of $2x^2y' - y(2x+y) = 0$

المعادلة متجانسة (اثبت ذلك؟) **(H.W.)**

ويمكن حل المعادلة المتجانسة بأسلوب اخر وذلك بتحويل المعادلة بصورة $f\left(\frac{y}{x}\right)$ حيث ان $v = \frac{y}{x}$

$$2x^2y' = 2xy + y^2 \rightarrow y' = \frac{2xy + y^2}{2x^2}$$

$$\rightarrow y' = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2$$

Let $v = \frac{y}{x} \rightarrow y = vx \rightarrow dy = vdx + xdv \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\to v + x\frac{dv}{dx} = v + \frac{1}{2}v^2$$

$$\to \ x\frac{dv}{dx} = \frac{1}{2}v^2$$

$$\to \int \frac{dv}{\frac{1}{2}v^2} = \int \frac{dx}{x}$$

$$\to \ \frac{-2}{v} = \ln|x| + c$$

$$\to \ \frac{-2x}{y} = \ln|x| + c, \qquad \text{(The general solution)}$$

---

(3) Solve the following diff. eq. $(x^2 + y^2)dx - 2xydy = 0$

at $y(2) = 0$.

<u>**Solution:**</u>

$\because M(x, y) = \ x^2 + y^2$

$\to \ M(tx, ty) = \ (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2M(x, y)$

$\because \ N(x, y) = \ -2x\,y$

$$\rightarrow N(tx,ty) = -2(tx)(ty) = -2t^2x\,y = t^2N(x,y)$$

(اذن المعادلة التفاضلية متجانسة)

$$[(x^2+y^2)dx - 2xydy = 0] \div x^2$$

$$\left(1+\left(\frac{y}{x}\right)^2\right)dx - \left(2\left(\frac{y}{x}\right)\right)dy\ = 0$$

Let $v = \frac{y}{x} \rightarrow y = vx \rightarrow dy = v\,dx + x\,dv \rightarrow \frac{dy}{dx} = v\ + x\ \frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \rightarrow \frac{dy}{dx} = \frac{x^2}{2xy} + \frac{y^2}{2xy}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{y}\right) + \frac{1}{2}\left(\frac{y}{x}\right)$$

$$v + x\frac{dv}{dx} = \frac{1}{2}\left(\frac{1}{v}\right) + \frac{1}{2}v$$

$$x\frac{dv}{dx} = \frac{1}{2v} + \frac{1}{2}v - v$$

$$\rightarrow x\frac{dv}{dx} = \frac{1}{2v} - \frac{1}{2}v$$

$$\rightarrow x\frac{dv}{dx} = \frac{1}{2}\left(\frac{1}{v} - v\right)$$

$$\rightarrow x\frac{dv}{dx}=\frac{1}{2}(\frac{1-v^2}{v})$$

$$\rightarrow \frac{dv}{\frac{1}{2}\frac{(1-v^2)}{v}}=\frac{dx}{x}$$

$$\rightarrow \int 2\frac{v}{1-v^2}dv=\frac{dx}{x}$$

$$\rightarrow -\ln|1-v^2|=\ln|x|+c$$

$\rightarrow -\ln\left|1-\frac{y^2}{x^2}\right|=\ln|x|+c,$ (The general solution)

(الحل الخاص: واجب؟)

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4) Solve: $(x^2-y^2)dy-2xydx=0,\ when\ x=0, y=1$

<u>Solution:</u> (The equation is homo. of degree 2) (واجب؟)

Let $y=vx\rightarrow dy=vdx+xdv$

$$(x^2-v^2x^2)(vdx+xdv)-2xvxdx=0$$

$$(x^2v-v^3x^2)dx+(x^3-v^2x^3)dv-2x^2vdx=0$$

$$(-v^3x^2-x^2v)dx+(x^3-v^2x^3)dv=0$$

$$\left(-x^2(v^3+v)\right)dx + x^3(1-v^2)dv = 0 \qquad * \frac{1}{(v^3+v)x^3}$$

$$\frac{-1}{x}dx + \frac{1-v^2}{v^3+v}dv = 0$$

$$-\ln|x| + \int \frac{1}{v^3+v}dv - \int \frac{v\,.\,v}{v(v^2+1)}dv = c$$

$$-\ln|x| + \int \frac{1}{v(v^2+1)}dv - \frac{1}{2}\ln|1+v^2| = c \qquad \ldots \quad (*)$$

$$\frac{1}{v(v^2+1)} = \frac{A}{v} + \frac{Bv+C}{v^2+1} = \frac{A(v^2+1)+Bv^2+cv}{v(v^2+1)}$$

$$= \frac{A(v^2+1)+Bv^2+Cv}{v(v^2+1)} = \frac{(A+B)v^2+cv+A}{v(v^2+1)}$$

$\rightarrow A + B = 0 \quad \ldots (1)$

$C \quad = 0 \quad \ldots (2)$

$A \quad = 1 \quad \ldots (3)$

Sub. (3) in (1) $\rightarrow B = -1$

$$\rightarrow \int \frac{1}{v}dv + \int \frac{-v}{v^2+1}dv - \frac{1}{2}\ln|1+v^2| = \ln|x| + c$$

$$\ln|v| - \frac{1}{2}\ln|1+v^2| - \frac{1}{2}\ln|1+v^2| = \ln|x| + c$$

$$\ln|v| - \ln|1+v^2| = \ln|x| + c$$

$$\ln\frac{v}{v^2+1} = \ln x + c$$

$$\ln\frac{v}{1+v^2} - \ln x = c$$

$$\ln\frac{v}{x(1+v^2)} = c$$

$$\frac{v}{x(1+v^2)} = e^c$$

When $v = \frac{y}{x}$

$$\frac{\frac{y}{x}}{x(1+\left(\frac{y}{x}\right)^2} = e^c \to \frac{y}{x}.\frac{x}{x^2+y^2} = e^c \to \frac{y}{x^2+y^2} = e^C (\,The\ gen.sol.)$$

When $x = 0\ and\ y = 1 \to 1 = e^c \to c = \ln 1 = 0$

$\therefore \frac{y}{x^2+y^2} = 1, (since\ e^0 = 1)$ $\quad (\,The\ particular\ sol.)$ (الحل الخاص)

5) Find the general solution of O.D.E.

$$(2x+y-1)dx+(x+y-2)dy=0$$

<u>Solution:</u>

(نلاحظ انها معادلة تفاضلية ذات معاملات خطية)

$\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \neq 0 \rightarrow$ (اذن المستقيمان متقاطعان)

الان يجب استخراج نقطة التقاطع:

$2x+y+1=0 \qquad \dots (1)$

$x+y-2=0 \qquad \dots (2)$

$Eq.(1) - Eq.(2) \rightarrow x+1=0 \rightarrow x=-1$

$From: Eq.(2): -2+y-1=0 \rightarrow y=3$

$\therefore (h,k)=(-1,3)$ [The intersection point]

Let $x=x_1-1 \rightarrow dx=dx_1$

Let $y=y_1+3 \rightarrow dy=dy_1$

Sub of O.D.E.$\rightarrow$

$$(2(x_1-1)+(y_1+3)-1)dx_1+\big((x_1-1)+y_1+3-2\big)dy_1=0$$

$$(2x_1+y_1)dx_1+(x_1+y_1)dy_1=0 \qquad \ldots (*)$$

الان أصبحت المعادلة التفاضلية متجانسة

Let $v=\frac{y_1}{x_1}\rightarrow y_1=vx_1\rightarrow dy_1=vdx_1+x_1dv$

Sub. in (*) $\rightarrow$

$$(2x_1+vx_1)dx_1+(x_1+vx_1)(vdx_1+x_1dv)=0$$

$$2x_1dx_1+vx_1dx_1+x_1vdx_1+x_1^2dv+v^2x_1dx_1+vx_1^2dv=0$$

$$x_1(2+2v+v^2)dx_1+x_1^2(1+v)dv=0$$

$$\int\frac{1}{x_1}dx_1+\int\frac{2(1+v)}{2(2+2v+v^2)}dv=\int 0$$

$$\ln|x_1|+\frac{1}{2}\ln|2+2v+v^2|=c$$

$$\ln|x_1|+\frac{1}{2}\ln\left|2+2(\frac{y_1}{x_1})+\frac{y_1^2}{x_1^2}\right|=c$$

$$\ln|x+1|+\frac{1}{2}\ln\left|2+2(\frac{y-3}{x+1})+\frac{(y-3)^2}{(x+1)^2}\right|=c\,,(\text{The general sol.})$$

6) Find the general solution of

$$(\mathbf{4}\boldsymbol{x}+\mathbf{2}\boldsymbol{y}+\mathbf{3})\boldsymbol{dx}+(\mathbf{6}\boldsymbol{x}+\mathbf{3}\boldsymbol{y}-\mathbf{2})\boldsymbol{dy}=\mathbf{0}$$

(The Differential Eq. with linear coefficients)

Solution:

$4x+2y+3=0 \rightarrow 2(2x+y)+3=0$

$6x+3y-2=0 \rightarrow 3(2x+y)-2=0$

Let $z=2x+y \rightarrow dz=2dx+dy \rightarrow dy=dz-2dx$

نقوم الان بالتعويض بالمعادلة التفاضلية

قبل ذلك سنقوم بالإثبات بأن المستقيمان متوازيان باستخدام الميل، حيث ان:

ميل المستقيم الاول : $m_1 = \frac{-\text{معامل}x}{\text{معامل}y}$ & ميل المستقيم الثاني: $m_2 = \frac{-\text{معامل}x}{\text{معامل}y}$

If $m_1 = m_2 \rightarrow$ المستقيمان متوازيان

If $m_1 \neq m_2 \rightarrow$ المستقيمان متقاطعان

$$\because m_1 = \frac{-4}{2} = -2, m_2 = \frac{-6}{3} = -2$$

$\therefore m_1 = m_2, so\ the\ two\ lines\ are\ Parllel$ (متوازيان)

الان نبدأ بالتعويض:

$$(2z+3)dx + (3z-2)(dz-2dx) = 0$$

$$2zdx + 3dx + 3zdz - 6zdx - 2dz + 4dx = 0$$

$$-4zdx + 7dx + 3zdz - 2dz = 0$$

$$(-4z+7)dx + (3z-2)dz = 0$$

$$dx + \frac{(3z-2)}{(-4z+7)}dz = 0$$

$$dx + \left(\frac{-3}{4} + \frac{4}{4}.\frac{\frac{13}{4}}{-4z+7}\right)dz = 0$$

$$x - \frac{3}{4}z - \frac{13}{16}\ln|-4z+7| = c$$

$$x - \frac{3}{4}(2x+y) - \frac{13}{16}\ln|-4(2x+y)+7| = c, \qquad (The\ gen.\ sol.).$$

2.4 Exact Differential Equation:

Theorem: If M and N have continuous partial derivative in a rectangular region R then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

be an exact equation if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ (الشرط لكي تكون المعادلة التفاضلية تامة)

Ex 1: Determine whether the following equations are exact or not:

i. $(2x^2+5)dx + 3y dy = 0$

ii. $x\cos y\,dx + y\cos x\,dy = 0$

iii. $\cos y\,dx + (y^2 - x\sin y)\,dy = 0$

Sol. (i): $\because \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \rightarrow it\ is\ exact$

Sol. (ii): $\because \frac{\partial M}{\partial y} = -x\sin y, and\ \frac{\partial N}{\partial x} = -y\sin x$

$$\rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow it\ is\ not\ exact$$

Sol. (iii): $\because \frac{\partial M}{\partial y} = -\sin y, and\ \frac{\partial N}{\partial x} = -\sin y \rightarrow\ it\ is\ exact$

Remark 1: An equation in the form

$$M(x)dx + N(y)dy = 0$$

is an exact equation.

For example: $x^2 dx + \sin y\, dy\ = 0$ is exact as follows:

$\because \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \rightarrow it\ is\ exact$

Remark 2: Every separable eq.is an exact equation after separating its variables (كل معادلة قابلة للفصل هي معادلة تامة بعد فصل متغيراتها)

Remark 3: The exact diff. eq. for the function $f(x, y)$ is

$$d\ f(x, y) = \frac{df}{dx}dx + \frac{df}{dy}dy$$

if we make it equal zero, we get the exact diff. eq.

$$\frac{df}{dx}dx + \frac{df}{dy}dy = 0$$

The solution of this diff. eq. is $f(x, y) = c,$ (ثابت اختياري : c)

Solution of the Exact Diff. Eq.

Through some examples, we will show here two methods to find the general solution of the exact equation.

من خلال الامثلة سوف نوضح طريقتين لحل المعادلة التفاضلية التامة:

■ **Method 1**: By using Remark 3

Ex 2 : Solve $(\boldsymbol{x}+\mathbf{2}\boldsymbol{y})\boldsymbol{dx}+(\mathbf{2}\boldsymbol{x}+\boldsymbol{y})\boldsymbol{dy}=\mathbf{0}$

Solution: $\because \frac{\partial M}{\partial y}=2=\frac{\partial N}{\partial x} \rightarrow \textit{the diff. eq. is exact},$

and hence its solution is $f(x,y)=C$

$$\frac{\partial F}{\partial x}=x+2y \qquad \ldots \quad (1) \qquad (M)$$

$$\frac{\partial F}{\partial y}=2x+y \qquad \ldots \quad (2) \qquad (N)$$

We can use any of the above equations. to find F.

From (1): $\frac{\partial F}{\partial x}=x+2y \rightarrow F=\frac{1}{2}x^2+2xy+h(y) \quad \ldots \quad (3)$ (تكامل جزئي)

And to find $h(y)$, we use the fact that eq.(3) satisfied eq.(2), then:

$2x + h'(y) = 2x + y \rightarrow h'(y) = y \rightarrow h(y) = \frac{1}{2}y^2$

Substituting in eq.(3), we get

$F(x, y) = \frac{1}{2}\, x^2 + 2xy + \frac{1}{2}y^2$

Then the solution is: $x^2 + 4xy + y^2 = c$

Ex3 : Solve $(2xy - 3x^2)dx + (x^2 + 2y)dy = 0$

Solution: $\because \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \rightarrow the\ diff.\ eq.\ is\ exact,$

and hence its solution is $f(x, y) = c$

$\frac{\partial F}{\partial x} = 2xy - 3x^2 \qquad \dots \ (1) \qquad (M)$

$\frac{\partial F}{\partial y} = x^2 + 2y \qquad \dots \ (2) \qquad (N)$

From (1): $F(x, y) = x^2y - x^3 + h(y)$ (نكامل بالنسبة x)

$N = \frac{\partial F}{\partial y} = x^2 + h'(y) = x^2 + 2y$

$\rightarrow h'(y) = 2y \ \rightarrow \ h(y) = y^2$

Then $F(x,y) = x^2y - x^3 + y^2$,

and the general solution is: $x^2y - x^3 + y^2 = c$

Ex 4: Solve $\boldsymbol{(1 + xy^2)dx + (x^2y + y)dy = 0}$

__**Solution:**__ $\because \frac{\partial M}{\partial y} = 2xy = \frac{\partial N}{\partial x} \rightarrow the\ diff.eq.is\ exact,$

$\frac{\partial F}{\partial x} = 1 + xy^2$ ... (1) (M)

$\frac{\partial F}{\partial y} = x^2y + y$ ... (2) (N)

From (1): $F(x,y) = x + \frac{1}{2}x^2y^2 + h(y)$ (نكامل بالنسبة x)

$x^2y + h'(y) = x^2y + y \rightarrow h'(y) = y \rightarrow h(y) = \frac{1}{2}y^2$

And the general solution is: $2x + x^2y^2 + y^2 = c$

Ex 5: Solve the initial-value problem

$$\frac{3x^2y+1}{y}dx - \frac{x}{y^2}dy = 0,\ y(2) = 1$$

Solution: we can rewrite the equation as following

$(3x^2 + y^{-1})dx - xy^{-2}dy = 0$

$\frac{\partial M}{\partial y} = -y^{-2}$

$\frac{\partial N}{\partial x} = -y^{-2}$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow the\ diff.eq.is\ exact,$

$\frac{\partial F}{\partial x} = 3x^2 + y^{-1}$ ... (1) (M)

$\frac{\partial F}{\partial y} = -xy^{-2}$ ... (2) (N)

From (2): $F(x,y) = xy^{-1} + g(x)$ (نكامل بالنسبة y)

Then: $\frac{\partial F}{\partial x} = y^{-1} + g'(x) = 3x^2 + y^{-1}$

$\rightarrow g(x) = x^3, then\ F(x,y) = xy^{-1} + x^3$

And the general solution is: $xy^{-1} + x^3 = c.$

When y(2)=1, we get :

$2(1)^{-1} + 2^3 = c \Rightarrow c = 10$

Sub. in the general solution above, we get:

$xy^{-1} + x^3 = 10$

And this is the particular solution

ملخص الطريقة :

1.نضع $M = \frac{\partial F}{\partial x}$ ونكاملها جزئيا ل x فنحصل على F(x,y) حيث تتضمن دالة اختيارية ل y ولتكن h(y)

2.نشتق F(x,y) بالنسبة ل y ونساويها ب N للحصول على قيمة الدالة h(y)

3.نعوض h(y) في معادلة F(x,y) عن فنحصل على الحل العام

ملاحظة:يمكن البدء مع $N = \frac{\partial F}{\partial y}$ ونحصل على النتيجة نفسها

METHOD 2: We choose a point (a,b) that it is within the domain of the function and satisfies M(x,y) and N(x,y) and we substitute in the following:

$$F(x,y) = \int_a^x M(t,y)dt + \int_b^y N(a,t)dt = c$$

We find this integral to obtain the general solution of the diff. eq.

Ex 6: Solve $(\boldsymbol{x}+\mathbf{2}\boldsymbol{y})\boldsymbol{dx}+(\mathbf{2}\boldsymbol{x}+\boldsymbol{y})\boldsymbol{dy}=\mathbf{0}$

Solution: $M(x,y)=x+2y \Rightarrow \frac{\partial M}{\partial y}=2$

$$N(x,y)=2x+y \Rightarrow \frac{\partial N}{\partial x}=2$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow the\ equation\ is\ exact$$

Let (a,b)=(0,0),substituting in :

$$F(x,y)=\int_a^x M(t,y)dt+\int_b^y N(a,t)dt=c$$

$$=\int_0^x M(t,y)dt+\int_0^y N(0,t)dt=c$$

$$=\int_0^x (t+2y)dt+\int_0^y tdt=c$$

$$=\left(\frac{t^2}{2}+2yt\right)\Big|_0^x+\left(\frac{t^2}{2}\right)\Big|_0^y=c$$

Ex 7: Solve the following diff. eq.

$$(\boldsymbol{y}^{\mathbf{2}}+\boldsymbol{x}\boldsymbol{y}^{\mathbf{2}}+\mathbf{1})\boldsymbol{dx}+(\boldsymbol{x}^{\mathbf{2}}\boldsymbol{y}+\mathbf{2}\boldsymbol{x}\boldsymbol{y}+\boldsymbol{y})\boldsymbol{dy}=\mathbf{0}$$

Solution:

$$\frac{\partial M}{\partial y} = 2y + 2xy$$

$$\frac{\partial N}{\partial x} = 2xy + 2y$$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow the\ diff.\ eq.\ is\ exact$, let (a,b)=(0,0) ,we get:

$$F(x,y) = \int_0^x (y^2 + ty + 1)\, dt + \int_0^y t\, dt = c$$

$$= xy^2 + \frac{1}{2}x^2y^2 + x + \frac{1}{2}y^2 = c$$

the solution is: $2xy^2 + x^2y^2 + 2x + y^2 = c.$

Ex 8: Solve $\frac{3x^2y+1}{y}dx - \frac{x}{y^2}dy = 0 \quad , y(2) = 1$

Solution:

$$M(x,y) = \frac{3x^2y+1}{y} \Rightarrow \frac{\partial M}{\partial y} = -y^{-2}$$

$$N(x,y) = \frac{-x}{y^2} \Rightarrow \frac{\partial N}{\partial x} = -y^{-2}$$

Then the equation is exact.

Let (a,b)=(0,1) ,sub. in

$$F(x,y) = \int_a^x M(t,y)dt + \int_b^y N(a,t)dt = c$$

$$= \int_0^x M(t,y)dt + \int_1^y N(0,t)dt = c$$

$$= \int_0^x \frac{3t^2y+1}{y} dt + \int_1^y 0.\,dt = c$$

$$= \int_0^x (3t^2 + \frac{1}{y})dt = c$$

$$= \left(t^3 + \frac{t}{y}\right) \Big|_0^x = c$$

$= x^3 + \frac{x}{y} = c$ and this is the general solution

When y(2)=1, we get the following

$$2^3 + \frac{2}{1} = c \Rightarrow c = 10$$

Then the Particular solution is $x^3 + \frac{x}{y} = 10$

<u>H.W.:</u> Solve the following equations by using two methods for the exact Differential Equation:

1. $(x^2 + 1 - 4xy - 2y^2)dx - (2x^2 + 4xy - y^3 + 2)dy = 0$

2. $ysec^2x\ dx + tan\ x\ dy = 0$

3. $\cos y\ dx + (y^2 - x \sin y)\ dy = 0$

4. $e^x y^2 dx + 2e^x y dy = 0$

2.5: Integral factors:

If the equation ($M(x,y)dx + N(x,y)dy = 0$) is not exact, then we multiply both sides by a factor $I(x,y)$ that turns it into an exact, and this factor is called the integral factor.

(اذا كانت المعادلة غير تامة، عندها نقوم بضرب طرفي المعادلة ب $I(x,y)$ لتحويلها الى تامة)

ملاحظة: 1.يسمى $I(x,y)$ عامل التكامل.

2. وظيفة عامل التكامل هي تحويل المعادلة غير التامة الى معادلة تامة

3. يختلف عامل التكامل من معادلة الى اخرى حسب شكل المعادلة كما مبين بالجدول الاتي:

N.	$M(x,y)dx + N(x,y)dy$	I(x,y) عامل التكامل	التعويض المناسب Z

1 i ii	اذا كان $\frac{M_y - N_x}{N} = f(x) \Rightarrow$ $\frac{M_y - N_x}{-M} = f(y) \Rightarrow$	$I(x) = e^{\int f(x)dx}$ $I(y) = e^{\int f(y)dy}$	تصبح المعادلة تامة وتحل حسب التامة
2	في حال ان $\frac{M_y - N_x}{N} \neq f(x)$ $\frac{M_y - N_x}{-M} \neq f(y)$	نضرب المعادلة ب $x^m y^n$ ونعوض في $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ لايجاد قيم لمعرفة عامل التكامل المطلوب	تصبح المعادلة تامة وتحل حسب التامة
3	$ydx - xdy$ Or: $xdy - ydx$	$(xy)^{-1}, x^{-2}, y^{-2}$ Or: $(ax^2 + bxy + cy^2)^{-1}$	$ln\frac{x}{y}$ or $\frac{x}{y}$ or $\frac{y}{x}$ or: $\int \left\{a + b\left(\frac{y}{x}\right) + c(\frac{y}{x})^2\right\} d(\frac{y}{x})$
4	$pydx + qxdy$	$x^{p-1}y^{q-1}$	$z = x^p y^q$
5	$ydx + xdy$	1	$z = xy$
6	$pxdx + qydy$	1	$z = \frac{1}{2}(px^2 + qy^2)$
7	$dy + P(x)ydx$ or : $dx + \propto(y)xdy$	$I = e^{\int p(x)dx}$	$z = y.e^{\int p(x)dx}$

	وتسمى المعادلة من هذا النوع **بالمعادلة الخطية**	$\boldsymbol{I = e^{\int \propto(y)dy}}$	$\boldsymbol{z = xe^{\int \propto(y)dy}}$

Now, we will explain each of these cases using examples

Case 1:

Multiplying the equation $(\boldsymbol{M(x,y)dx + N(x,y)dy = 0})$ by I(x,y) we get:

$(\boldsymbol{I\,.\,M(x,y)dx + I\,.\,N(x,y)dy = 0})$, where I, M, N are functions of x and y.

Deriving IM w.r.t. y and IN w.r.t. x

$$\rightarrow \frac{\partial(I\,M)}{\partial y} = I\,M_y + I_y M$$

$$\frac{\partial(I\,N)}{\partial x} = I\,N_x + I_x N$$

$$\rightarrow \frac{\partial(I\,M)}{\partial y} = \frac{\partial(I\,N)}{\partial x},$$ (بأعتبار ان I هو عامل التكامل)

then $I\,M_y + I_y M = I\,N_x + I_x N$

$$\rightarrow I\left[M_y - N_x\right] = I_x\, N - I_y M \quad \ldots\ldots\ldots\ldots.. \qquad (*)$$

Now we have two cases

1. *If $I(x, y) = I(x)$, that is I is a function just for (x).*

Then

$$I_x = \frac{dI}{dx} \quad , \quad I_y = 0$$ (عامل التكامل دالة ل x فقط)

Sub. in (*), we get

$$I[M_y - N_x] = N\,\frac{dI}{dx}$$

$$\rightarrow \frac{dI}{I} = \frac{M_y - N_x}{N}\,dx$$

Let $P(x) = \frac{M_y - N_x}{N}$ (function a lone for x)

وبتكامل الطرفين نحصل على:

$$\ln I = \int P(x)dx$$

$$\rightarrow I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int\left(\frac{M_y - N_x}{N}\right)dx}$$ (عامل التكامل)

2. If $I(x, y) = I(y)$, that is I is a function just for (y).Then

$I_y = \frac{dI}{dy}$, $I_x = 0$ (عامل التكامل دالة ل y فقط)

Sub. in (*) we get:

$I[M_y - N_x] = -I_y M$

$\frac{I_y}{I} = \frac{M_y - N_x}{-M}$

$I(y) = e^{\int P(y)dy}$ where $P(y) = \frac{M_y - N_x}{-M}$ (function a lone for y)

$\rightarrow I(y) = e^{\int (\frac{M_y - N_x}{-M})dx}$ (*عامل التكامل*)

Ex1: Solve $(3x^3 + 2y)dx + \left(2x \ln 3x + \frac{3x}{y}\right) dy = 0$

Sol.: $M = 3x^3 + 2y$, $N = 2x \ln 3x + \frac{3x}{y}$

$M_y = 2$, $N_x = 2 + 2 \ln 3x + \frac{3}{y}$

$M_y - N_x = -\left(2 \ln 3x + \frac{3}{y}\right) \neq 0$

$\therefore$ *The equation is not exact.*

$$\frac{M_y - N_x}{N} = \frac{2\ln 3x + \frac{3}{y}}{x(2\ln 3x + \frac{3}{y})} = -\frac{1}{x} = P(x)$$

$$\rightarrow I(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln\frac{1}{x}} = \frac{1}{x}$$ (عامل التكامل)

(بضرب طرفي المعادلة في $\frac{1}{x}$ تصبح تامة)

$$\left(3x^2 + \frac{2y}{x}\right)dx + \left(2\ln 3x + \frac{3}{y}\right)dy = 0$$

$$\because \quad \frac{\partial M}{\partial y} = \frac{2}{x} = \frac{\partial N}{\partial x} \quad \rightarrow \quad \textit{The eq. is exact}$$

الان نقوم بحل المعادلة التفاضلية التامة (H.W.)

Ex 2: Solve $\boldsymbol{y(2x + y)dx + (3x^2 + 4xy - y)dy = 0}$

Sol.: $M = y\,(2x + y), \quad N = 3x^2 + 4xy - y$

$M_y = 2x + 2y, \quad N_x = 6x + 4y$

$M_y - N_x = -4x - 2y = -2(2x + y) \neq 0$

The equation is not exact.

We must find the integration factor.

$$\frac{M_y - N_x}{-M} = \frac{-2(2x+y)}{-y(2x+y)} = \frac{2}{y}$$

$$\rightarrow I = e^{\int \frac{2}{y} dy} = e^{2\ln y} = e^{\ln y^2} = y^2$$ (عامل التكامل)

(نضرب طرفي المعادلة بعامل التكامل حتى تصبح تامة)

$$y^3(2x+y)dx + y^2(3x^2+4xy-y)dy = 0$$

$$M = 2xy^3 + y^4, \quad N = 3x^2y^2 + 4xy^3 - y^3$$

$$M_y = 6xy^2 + 4y^3, \quad N_x = 6xy^2 + 4y^3$$

<u>Solution:</u>

1. Integrating M(x,y) for x: نكامل بالنسبة ل x

$$M = \frac{\partial F}{\partial x} = 2xy^3 + y^4$$

$$F(x,y) = 2\,\frac{x^2}{2}y^3 + xy^4 + h(y) \qquad \ldots \quad (*)$$

$$= x^2y^3 + xy^4 + h(y)$$

2. Deriving for y: نشتق بالنسبة ل y

$$\frac{\partial F}{\partial y} = 3x^2y^2 + 4xy^3 + h'(y)$$

3. Set $\frac{\partial F}{\partial y} = N$ نساوي. $\frac{\partial F}{\partial y}$ الى N

$$3x^2y^2 + 4xy^3 + h'(y) = 3x^2y^2 + 4xy^3 - y^3$$

$$\rightarrow h'(y) = -y^3$$

$$\rightarrow h(y) = \frac{-y^4}{4} = -\frac{1}{4}\,y^4$$

$$\rightarrow F = x^2y^3 + xy^4 - \frac{1}{4}\,y^4$$

The solution is : $x^2y^3 + xy^4 - \frac{1}{4}\,y^4 = c$

<u>Case (2):</u> If the above two integration factors state are not exist.

بمعنى انه عامل التكامل (I) ليس دالة الى x ولا دالة الى y بشكل منفصل فعلينا هنا استخراج عامل التكامل.

Suppose that the integrating factor $\boldsymbol{I(x,y) = x^m\, y^n}$, and we find the value of m and that maxes the differential equation exact.

اذن الهدف هو الوصول الى قيمة m,n حتى نحصل على عامل التكامل.

وسوف نوضح ذلك بأستخدام المثال الاتي:

Ex3: solve the diff. eq.

$$(x^2 + xy^2)\frac{dy}{dx} - 3xy + 2y^3 = 0$$

Sol: (يتم بالبداية ترتيب المعادلة).

$$(2y^3 - 3xy)dx + (x^2 + xy^2)dy = 0 \qquad \dots \quad (*)$$

$$\frac{\partial M}{\partial y} = 6y^2 - 3x \neq \frac{\partial N}{\partial x} = 2x + y^2 \quad \rightarrow The\ diff.\ eq.\ is\ not\ exact.$$

الان نبحث عن عامل التكامل

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{6y^2 - 3x - 2x - y^2}{-(2y^3 - 3xy)} \neq I(y)$$

And so

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6y^2 - 3x - 2x - y^2}{x^2 + xy^2} \neq I(x)$$

Let $I(x,y) = x^m y^n\ is\ integrating\ factor$

$$(2x^m y^{n+3} - 3x^{m+1} y^{n+1})dx + (x^{m+2}\, y^n + x^{m+1}\, y^{n+2})dy = 0$$

$$\frac{\partial M}{\partial y} = 2(n+3)x^m y^{n+2} - 3(n+1)x^{m+1} y^n$$

$$= x^m y^n[(2n+6)y^2 - (3n+3)x]$$

$$\frac{\partial N}{\partial x} = (m+2)x^{m+1} y^n + (m+1)x^m y^{n+2}$$

$$= x^m y^n[(m+2)x + (m+1)y^2]$$

$$Since\ it\ is\ exact \rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\rightarrow\ x^m y^n[(2n+6)y^2 - (3n+3)x] =\ x^m y^n[(m+2)x + (m+1)y^2]$$

$$\rightarrow\ (2n+6)y^2 - (3n+3)x =\ (m+2)x + (m+1)y^2$$

(نقوم الان بتساوي المعاملات)

$$m + 2 =\ -3n - 3 \qquad \rightarrow m + 1 =\ -3n - 4 \qquad (1)$$

$$m + 1 =\ 2n + 6 \qquad \rightarrow m + 1 =\ 2n + 6 \qquad (2)$$

بمساواة المعادلتين (1) و(2) نحصل على:

$$\rightarrow\ 2n + 6 =\ -3n - 4$$

$\rightarrow\ 5n = -10$

$\rightarrow\ n = -2$

From (1) $\rightarrow m + 1 = -3(-2) - 4 \rightarrow m = 1$

$\therefore I(x, y) = xy^{-2}$

$(2xy - 3x^2y^{-1})dx + (x^3y^{-2} + x^2)dy = 0$

$$\frac{\partial M}{\partial y} = 2x + 3x^2y^{-2}$$

$$\frac{\partial N}{\partial x} = 2x + 3x^2y^{-2}$$

$\because\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow The\ eq.\ become\ exact.$ (حل المعادلة التامة يترك للطالب)

<u>Remark:</u> Some integral factors can be deduced for certain amounts of differential rule like cases 3-6 in the table.

(يمكن استنتاج بعض عوامل التكامل لبعض المقادير من قواعد التفاضل مثل مشتقة حاصل قسمة دالتين ومشتقة حاصل ضرب دالتين وغيرهما والتي ستظهر في الحالات من 3-6 في الجدول

For example:

$\boldsymbol{d\left(\frac{y}{x}\right) = \frac{xdy-ydx}{x^2}}$, this means $(\frac{1}{x^2})$ is an integral factor of $xdy - ydx.$

Case 3: When the left side of the equation is in the form $\boldsymbol{xdy - ydx}$ or $\boldsymbol{ydx - xdy}$:

Ex 4: Prove that the integral factor of $\boldsymbol{xdy - ydx + f(x)dx = 0}$ is $\boldsymbol{x^{-2}}$

Proof:

$$\therefore [x\,dy - y\,dx + f(x)dx = 0] * \frac{1}{x^2}$$

$$\frac{xdy - ydx}{x^2} + \frac{f(x)}{x^2}\,dx = 0$$

$$\frac{1}{x}\,dy - \frac{y}{x^2}\,dx + \frac{1}{x^2}\,f(x)dx = 0$$

$$(\frac{1}{x^2}\,f(x) - \frac{y}{x^2})\,dx + \frac{1}{x}\,dy\ = 0$$

$$\because\ \frac{\partial M}{\partial y} = -\frac{1}{x^2} = \frac{\partial N}{\partial x}\ \ \rightarrow\ \ Exact$$

Then x^{-2} is an integral factor of the equation above

And the general solution is:

$$\int d(\frac{y}{x}) + \int \frac{1}{x^2} f(x)\ dx = 0$$

$$\rightarrow \frac{y}{x} + \int \frac{1}{x^2} f(x)\ dx = c$$

And by the same way one can prove that $(\frac{1}{y^2})$ is an integral factor of

$$xdy - ydx + f(y)dy = 0.$$

In general, we can convert $xdy - ydx$ or $ydx - xdy$ to exact diff. eq. by dividing on one of the following:

x^2 , y^2 , xy , $x^2 + y^2$, $x^2 - y^2$, ...

And the general solution of this amounts is $ax^2 + bxy + cy^2$

<u>Ex 5:</u> Prove that $\frac{1}{ax^2+bxy+cy^2}$ is an integral factor of $(xdy - ydx)$, set $a \neq 0, b \neq 0, c \neq 0$ at the same time.

$$xdy - y\,dx \,.\frac{1}{ax^2 + bxy + cy^2} = \frac{xdy - y\,dx}{ax^2 + bxy + cy^2}$$

$$= \frac{\frac{xdy - y\,dx}{x^2}}{a + b\frac{y}{x} + c\left(\frac{y}{x}\right)^2} = \frac{d(\frac{y}{x})}{a + b\frac{y}{x} + c\left(\frac{y}{x}\right)^2}$$

Let $\frac{y}{x} = z$

$\frac{dz}{a+bz+cz^2} = d\,f(z)$ (تفاضل تام)

Ex 6: Find the general solution of

$$\boldsymbol{xdy - ydx = x^4y^2dx}$$

Sol: $\left(\underbrace{-y - x^4y^2}_{M}\right)dx + \underbrace{x}_{N}\,dy = 0$

$$\frac{\partial M}{\partial y} = -1 - 2x^4y, \quad \frac{\partial N}{\partial x} = 1$$

$$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \rightarrow \quad The\ Diff.\ Eq.\ is\ not\ exact.$$

لذلك نحتاج الى عامل التكامل: لإيجاد الحل العام نضرب الطرفين ب $(\frac{1}{x^2})$

$$\frac{xdy - ydx}{x^2} = x^4\frac{y^2}{x^2}dx$$

$$d(\frac{y}{x}) = x^4\left(\frac{y}{x}\right)^2 dx$$

$Let\ z = \frac{y}{x} \rightarrow\ dz = x^4 z^2 dx$

$\rightarrow \int z^{-2}\, dz = \int x^4\, dx$

$\rightarrow \frac{-1}{z} = \frac{x^5}{5} + c$

$\rightarrow \frac{-x}{y} = \frac{x^5}{5} + c$ (الحل العام)

طريقة حل أخرى: يمكن أيضا" ان نقسم على y^2

$\frac{xdy - ydx}{y^2} = x^4 dx$

$\rightarrow \int d(\frac{-x}{y}) = \int x^4 dx$

$\rightarrow \frac{-x}{y} = \frac{x^5}{5} + c$ (الحل العام)

Ex 7: Find the general solution of

$$\boldsymbol{xdy - ydx = y^3(x^2 + y^2)dy}$$

Sol:

$$\underbrace{-y}_{M}\,dx + \left[\underbrace{x - y^3(x^2 + y^2)}_{N}\right] dy = 0$$

$$\frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 1 - 2y^3x$$

$$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \rightarrow \quad The\ diff.\ eq.\ is\ not\ exact.$$

لذلك نحتاج الى عامل التكامل: لإيجاد الحل العام نضرب الطرفين ب $(\frac{1}{x^2+y^2})$

$$\frac{xdy - ydx}{x^2 + y^2} = y^3dy$$

$$\frac{\frac{xdy - ydx}{x^2}}{1 + \frac{y^2}{x^2}} = y^3dy$$

$$\frac{d(\frac{y}{x})}{1 + \left(\frac{y}{x}\right)^2} = y^3dy$$

$$Let\ z = \frac{y}{x} \quad \rightarrow \quad \int \frac{dz}{1+z^2} = \int y^3dy$$

$$\rightarrow \quad \tan^{-1} z = \frac{1}{4}y^4 + c$$

$$\rightarrow \quad \tan^{-1}(\frac{y}{x}) = \frac{1}{4}y^4 + c$$ (الحل العام)

Case 4: When the equation contains the term($\boldsymbol{pydx + qxdy}$)

first we know that

$$d(x^p y^q) = qx^p y^{q-1}dy + p\,y^q x^{p-1}dx \qquad \text{(حاصل ضرب دالتين)}$$

$$= x^{p-1}y^{q-1}(qxdy + pydx)$$

$$= x^{p-1}y^{q-1}(pydx + qxdy)$$

لذلك يكون عامل تكامل المقدار $(pydx + qxdy)$ هو $(x^{p-1}y^{q-1})$

والتعويض المناسب هو $(z = x^p y^q)$

Ex8: Find the general solution of

$$\boldsymbol{xdy - 3ydx = x^4y^{-1}dx}$$

Sol:

$$\left.\begin{array}{c} \boldsymbol{xdy - 3ydx = x^4y^{-1}dx} \\ pydx + qxdy \end{array}\right] \quad \text{نقارن}$$

$$\therefore \quad p = -3, \quad q = 1$$

$$\therefore \quad Integrate\ factor\ is \quad x^{-3-1}y^{1-1} = x^{-4}$$

$$z = x^{-3}y \;\rightarrow\; z = \frac{y}{x^3} \rightarrow \; y = z\,.\,x^3$$

$$\rightarrow \quad dz = x^{-3}dy - 3yx^{-4}dx$$

By multiplying the end of equation by $\frac{1}{x^4}$

(نضرب طرفي المعادلة بعامل التكامل)

$$\underbrace{x^{-3}dy - 3x^{-4}ydx}_{dz} = y^{-1}dx$$

$$\rightarrow \quad dz = \frac{dx}{y} \quad \rightarrow \quad dz = \frac{dx}{z.x^3}$$

$$zdz = x^{-3}dx \quad \rightarrow \quad \frac{z^2}{2} = \frac{x^{-2}}{-2} + c$$

$$\rightarrow \quad \frac{(x^{-3}y)^2}{2} = \frac{x^{-2}}{-2} + c, \quad (The\;\; general\;\; solution)$$

Ex9: Find the general solution of

$$\boldsymbol{xdy - ydx = x^2y^3dx}$$

Sol:

$$\left.\begin{matrix} xdy - ydx = x^2y^3dx \\ pydx + qxdy \end{matrix}\right] \quad \text{نقارن}$$

$$\therefore \quad p = -1, \quad q = 1$$

$$\therefore \quad Integrate\ factor\ is \quad x^{-1-1}y^{1-1} = \frac{1}{x^2}$$

$$z = x^{-1}y \ \rightarrow \ z = \frac{y}{x}$$

By multiplying the end of equation by $\frac{\mathbf{1}}{\boldsymbol{x^2}}$

$$\frac{xdy - ydx}{x^2} = y^3dx$$

$$d\left(\frac{y}{x}\right) = \frac{x^3}{x^3}y^3dx \qquad (x^3 \text{نضرب ونقسم على})$$

Let $z = \frac{y}{x} \quad \rightarrow \quad dz = x^3z^3dx$

$$\rightarrow \quad \int z^{-3}\, dz = \int x^3dx$$

$$\rightarrow \quad \frac{z^{-2}}{-2} = \frac{x^4}{4} + c$$

$$\rightarrow \quad \frac{-1}{2z^2} = \frac{x^4}{4} + c$$

$$\rightarrow \quad \frac{-1}{2(\frac{y}{x})^2} = \frac{x^4}{4} + c, \quad (The\ \ general\ \ solution)$$

Ex 10: Find the general solution of

$$xdy + 2ydx = e^{x}dx$$

Sol: p=2 , q=1

I=$x^{p-1}y^{q-1}$ حسب الحالة 4 من الجدول

$= x^{2-1}y^{1-1} = x$

التعويض المناسب هو $z = x^{p}y^{q} = x^{2}y$

Multiplying both sides by I=x ,we get:

$$x^{2}dy + 2xydx = xe^{x}dx$$

$$d(x^{2}y) = xe^{x}dx \Rightarrow dz = xe^{x}dx$$

Integrating both sides, we get:

$$z = \int xe^{x}dx + c$$

$$= xe^{x} - \int e^{x}dx + c$$

$$= xe^{x} - e^{x} + c$$

Replacing z to get the following general solution:

$$x^{2}y = xe^{x} - e^{x} + c$$

Case 5: When the equation contains the term $(xdy + ydx)$

The integral factor in this case equal 1 this means that we just need some math operations here.

Ex11: Solve $\boldsymbol{xdy + ydx = (5x - 2x^2y)dx}$

Sol: Let $z = xy \quad \rightarrow \quad dz = xdy + ydx$

$\rightarrow \quad y = \frac{z}{x}$

substituting in the original equation بالتعويض بالمعادلة التفاضلية:

$$\rightarrow \quad dz = \left(5x - 2x^2\left(\frac{z}{x}\right)\right)dx$$

$$\rightarrow \quad dz = (5 - 2z)xdx$$

$$\rightarrow \quad \int \frac{dz}{(5-2z)} = \int xdx$$

$$\rightarrow \quad -\frac{1}{2}\ln|5 - 2z| = \frac{1}{2}x^2 + c$$

$\rightarrow \quad -\frac{1}{2}\ln|5 - 2xy| = \frac{1}{2}x^2 + c$, (The general solution)

Ex12: Solve $\boldsymbol{x^2\frac{dy}{dx} + xy + \sqrt{1 - x^2y^2} = 0}$

Sol: $x^2dy + xydx + \sqrt{1 - x^2y^2}dx = 0$

$$x(xdy + ydx) + \sqrt{1 - x^2y^2}dx = 0$$

$$\frac{(xdy+ydx)}{\sqrt{1-x^2y^2}} + \frac{dx}{x} = 0$$

$\sin^{-1}(xy) + \ln|x| = c,$ (The general solution)

Case 6: When the equation contains the term ***(xdx+ydy)***

The integral factor in this case equal 1 this means that we just need some math operations here.

Ex13: **Solve** $\boldsymbol{xdx + ydy = 3\sqrt{x^2 + y^2}\, y^2dy}$

Sol: $\frac{xdx+ydy}{\sqrt{x^2+y^2}} = 3y^2dy$

$$\frac{1}{2}\ \frac{2xdx+2ydy}{\sqrt{x^2+y^2}} = 3y^2dy$$

$$\frac{1}{2}\int(x^2 + y^2)^{-1/2}(2xdx + 2ydy) = \int 3y^2dy$$

$\sqrt{x^2 + y^2} = y^3 + c$ and this is the general solution

Homework: Solve the following diff. equations

14) Solve $\boldsymbol{xdx + ydy = y^2(x^2 + y^2)dy}$

15) $\boldsymbol{xdx + ydy = (x^2 + y^2)^3(xdy - ydx)}$

Case 7: Linear Differential Equation:

We have defined the general form of linear differential equation of order n to be:

$$\boldsymbol{a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)}$$

We remind you, that linearity means that all coefficients are functions of x only, and that y and its derivatives are raised to the first power.

Def: (Linear Differential Equation of First Order and First Degree)

A differential eq. of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad a_1(x) \neq 0 \qquad .... \quad (1)$$

is said to be a linear first order differential equation.

Dividing by $a_1(x)$, we get a standard form of a linear first- order equation.

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad .... \quad (2)$$

is said to be a linear first-order differential equation of x.

And a differential eq. of the form

$$\frac{dx}{dy} + \alpha(y)x = \beta(y), \qquad .... \quad (3)$$

is said to be a linear first-order differential equation of y.

To find solution of eq. (2), we can write it as:

$$[P(x)y - Q(x)]dx + dy = 0$$

We try to make it exact.

$M = P(x)y - Q(x) \quad , \quad N = 1$

$M_y = P(x) \quad , \quad N_x = 0$

$M_y - N_x = P(x) \neq 0$

i.e. the equation is not exact, then

$I = I(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int P(x)dx}$

is integral factor.

Multiply the ends of diff. eq. by the integral factor to be exact:

$$e^{\int P(x)dx} P(x)ydx + e^{\int P(x)dx} dy = \; e^{\int P(x)dx} Q(x)dx$$

i.e., $d\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} Q(x)dx$

And by integrating the two ends of the equation, we get:

$$e^{\int P(x)dx}\, y = \; e^{\int P(x)dx}\, Q(x)dx + c$$

We can simply the form of equation

$$I(x)y = \int I(x)Q(x)dx + c$$

Similarly, we can conclude the solution of the linear diff. eq.

$$\frac{dx}{dy} + \alpha(y)x = \beta(y)$$

is

$$I(y)x = \int I(y)\beta(y)dy + K$$

where $I(y) = e^{\int \alpha(y)dy}$ is the integrate factor.

Ex16: Find the general solution of the following diff. eq.

$$\boldsymbol{x\frac{dy}{dx} + 2y = x^2}$$

Sol.: We put the equation as $\quad \frac{dy}{dx} + P(x)y = Q(x)$

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

$$\rightarrow \qquad P(x) = \frac{2}{x} \quad , \quad Q(x) = x$$

1) $\int P(x)dx = \int \frac{2}{x}dx = \ln x^2$

$\therefore I(x) = e^{\int P(x)dx} = e^{\ln x^2} = x^2 \qquad (Integral\ \ Factor)$

2) $\int I(x)\, Q(x)dx = \int x^2 x dx = \int x^3 dx = \frac{1}{4}x^4$

Then the solution is :

$$Iy = \int I(x)\, Q(x)dx + c$$

$$\rightarrow \quad x^2y = \frac{1}{4}x^4 + c, \quad (The\ general\ solution)$$

Ex17: Find the general solution of the following diff. eq.

$$(\boldsymbol{y} + \boldsymbol{y^2})\boldsymbol{dx} - (\boldsymbol{y^2} + \boldsymbol{2xy} + \boldsymbol{x})\boldsymbol{dy} = \boldsymbol{0}$$

Sol.: We put the equation a $\quad \frac{dx}{dy} + \alpha(y)\, x = \beta(y)$

Dividing the diff. eq. by $\quad y + y^2 \quad$, we get:

$$\frac{dx}{dy} - \frac{(y^2 + 2xy + x)}{(y + y^2)} = 0$$

$$\rightarrow \quad \frac{dx}{dy} - \frac{2y+1}{y+y^2}\, x = \frac{y^2}{y+y^2}$$

$$\rightarrow \quad \alpha(y) = -\frac{2y+1}{y+y^2} \quad , \quad \beta(y) = \frac{y^2}{y+y^2} = \frac{y}{y+1}$$

$$I(y) = e^{\int \alpha(y)dy} = e^{-\int \frac{2y+1}{y+y^2}dy} = e^{-\ln(y^2+y)} = e^{\ln(\frac{1}{y^2+y})}$$

$$= \frac{1}{y^2+y}(\,Integrate\ \ Factor)$$

$$\rightarrow \; Ix = \int I(y)\, \beta(y) dy + c \Rightarrow \frac{1}{y^2+y}\, .x = \int \frac{1}{y^2+y} \cdot \frac{y}{y+1} dy$$

$$\Rightarrow \quad \frac{1}{y^2+y} \,.\, x = \int \frac{1}{(y+1)^2} dy$$

Then the general solution is :

$$\frac{1}{y^2+y} \,.\, x = -\frac{1}{y+1} + c$$

Ex18: Solve: $\boldsymbol{\frac{dy}{dx} - \frac{3}{x} y = x^2}$

Sol.: $I(x) = e^{\int -\frac{3}{x} dx} = e^{-3 \ln x} = \frac{1}{x^3}$

$$\rightarrow \qquad \frac{1}{x^3}\, y = \int \frac{1}{x^3}\, x^2 dx + c$$

$$\rightarrow \qquad \frac{y}{x^3} = \int \frac{1}{x}\, dx + c$$

$$\rightarrow \qquad \frac{y}{x^3} = \ln|x| + c, \qquad (The\ \ general\ \ solution).$$

Ex19: Solve: $\boldsymbol{dx - 2xdy = e^{4y}\, dy}$

Sol.: $\rightarrow \frac{dx}{dy} - 2x = e^{4y}$

$$I(y) = e^{\int -2dy} = e^{-2y}$$

$$\rightarrow \qquad e^{-2y} \,.\, x = \int e^{-2y} \,.\, e^{4y}\, dy + c$$

$$\rightarrow \qquad e^{-2y} \,.\, x = \int e^{2y}\, dy + c$$

$$\rightarrow \qquad e^{-2y} \,.\, x = \frac{1}{2}\, e^{2y} + c \qquad (The\ \ gen.\ sol.).$$

Ex20: Solve: $\boldsymbol{\cos y = y'(\cos^2 y - x)}$

Sol.: $\rightarrow \frac{dx}{dy} \cos y = \cos^2 y - x$

$$\rightarrow \quad \frac{dx}{dy} + \frac{1}{\cos y} \,.\, x = \cos y$$

$$\alpha(y) = \sec y \quad , \quad \beta(y) = \cos(y)$$

$$I(y) = e^{\int \sec y dy} = e^{\ln|\sec y + \tan y|} = \sec y + \tan y$$

$$\rightarrow \qquad I(y) \,.\, x = \int (\sec y + \tan y) \cos y\, dy + c$$

$$\rightarrow \qquad I\,(y).\ x = \int (1 + \sin y) dy + c$$

$$\rightarrow \qquad I\,(y).\ x = y - \cos y + c$$

$$\rightarrow \qquad (\sec y + \tan y)\ .\ x = y - \cos y + c$$

$$\rightarrow \qquad x = \frac{y - \cos y + C}{(\sec y + \tan y)} \qquad (The\ \ general\ \ sol.).$$

2.6. A nonlinear equation can be reduced to a linear

Equation of Bernoulli:

The diff. eq.

(A) $\frac{dy}{dx} + P(x)\,y = Q(x)y^n,\ s.t.\ n \neq 0, 1$

is called Bernoulli's equation, where n is any real number, Bernoulli's equation is nonlinear, and we can change it to linear as following:

1) By dividing on y^n, we find that:

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x) \qquad \text{... (1)}$$

2) Suppose $z = y^{1-n}$, the derivative parties for x, we get:

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \quad \rightarrow \quad y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx} \qquad \text{... (2)}$$

نعوض الان في معادلة رقم (1)

$$\frac{1}{1-n}\ \frac{dz}{dx} + P(x)\, z = Q(x)$$

Multiplying by $(1-n)$, we get:

$$\frac{dz}{dx} - (1-n)\, P(x)\, z = (1-n)\, Q(x) \qquad \dots (3)$$

Which is linear eq. in z, solving (3) for z.

Ex1: Find the solution of the equation

$$\boldsymbol{dy + 2xydx = xe^{-x^2}y^3dx}$$

Sol.:

$$\frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \qquad \text{(معادلة برنولي)}$$

$$y^{-3}\ \frac{dy}{dx} + 2xy^{-2} = xe^{-x^2} \qquad \dots.\ (*)$$

Let $z = y^{-2} \rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \rightarrow -\frac{1}{2}\frac{dz}{dx} = y^{-3}\frac{dy}{dx}$ (نعوض في *)

$\rightarrow \qquad -\frac{1}{2}\frac{dz}{dx} + 2x\, z = x\, e^{-x^2}$

$$\rightarrow \qquad \frac{dz}{dx} - 4x\, z = -2\, x\, e^{-x2}$$

$I = e^{\int -4xdx} \quad \rightarrow \quad I = e^{-2x^2}$ (عامل التكامل)

$$z.I = \int I.Q(x)dx + c$$

$$z\,.\,e^{-2x^2} = -2\int xe^{-x^2}\, e^{-2x^2} dx + c$$

$$= -2\int x\, e^{-3x^2} dx + c$$

$$= \frac{1}{3}\, e^{-3x^2} + c$$

The solution is:

$$z\,.\,e^{-2x^2} = \frac{1}{3}\, e^{-3x^2} + c$$

$$\rightarrow \qquad y^{-2}\,.\,e^{-2x^2} = \frac{1}{3}\, e^{-3x^2} + c$$

$$\rightarrow \qquad \frac{e^{-2x^2}}{y^2} = \frac{1}{3}\, e^{-3x^2} + c \qquad (\textit{The general solution})$$

(B) The second formula of Bernoulli's eq.

$$\frac{dx}{dy} + g(y)\,x = h(y)x^n$$

وبنفس الخطوات السابقة للصيغة الأولى:

$$\frac{dx}{dy} + g(y)\,x = h(y)x^n \qquad * \,(x^{-n})$$

Let $z = x^{1-n} \to \frac{dz}{dy} = (1-n)x^{-n}\frac{dx}{dy}$

(وبعدها تحول المعادلة الى معادلة خطية)

Ex2: Solve $\boldsymbol{\frac{dx}{dy} - 2yx = 6y^2 . e^{-2y^2} . x^3}$

Sol.: $\frac{dx}{dy} - 2yx = 6y^2 . e^{-2y^2} . x^3 \qquad * \,(x^{-3})$

$$x^{-3} . \frac{dx}{dy} - 2yx^{-2} = 6y^2 . e^{-2y^2} \qquad .... \quad (*)$$

Let $z = x^{-2} \to \frac{dz}{dy} = -2x^{-3}\frac{dx}{dy} \to -\frac{1}{2}\frac{dz}{dy} = x^{-3}\frac{dx}{dy}$ (نعوض في *)

$$-\frac{1}{2}\frac{dz}{dy} - 2y\,z = 6y^2 . e^{-2y^2} \qquad * \,(-2)$$

$$\to \qquad \frac{dz}{dy} + 4y\,z = -12y^2 . e^{-2y^2} \qquad \text{(معادلة خطية)}$$

$$I = e^{\int 4y dy} = e^{2y^2} \qquad \text{(عامل التكامل)}$$

$$I.z = \int I.h(y)\, dy + C$$

$$e^{2y^2} z = \int e^{2y^2}. -12y^2 \ e^{-2y^2} dy + C$$

$$\rightarrow \ e^{2y^2} x^{-2} = -12 \frac{y^3}{3} + C \qquad (\textit{The general solution})$$

Ex3: Find the general sol. of

$$\boldsymbol{\frac{dy}{dx} - \left(1 + \frac{1}{x}\right) y = -2e^x y^2}$$

Sol.: $$y^{-2}\frac{dy}{dx} - \left(1 + \frac{1}{x}\right) y^{-1} = -2e^x \qquad .... \quad (*)$$

Let $z = y^{-1} \rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx} \rightarrow -\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$ (نعوض في *)

$$-\frac{dz}{dx} - \left(1 + \frac{1}{x}\right) z = -2e^x \qquad * (-1)$$

$$\frac{dz}{dx} + \left(1 + \frac{1}{x}\right) z = 2e^x \quad \text{(معادلة خطية)}$$

$$I = e^{\int \left(1+\frac{1}{x}\right) dx} \rightarrow \ I = e^{x+\ln x} \rightarrow I = xe^x \quad \text{(عامل التكامل)}$$

$$z.I = \int 2e^{x} \cdot I\, dx$$

$$\rightarrow \qquad z.xe^{x} = \int 2e^{x} \cdot xe^{x}\, dx$$

$$\rightarrow \qquad z.xe^{x} = \int \underbrace{2xe^{2x}\, dx}_{\text{تكامل التجزئة}}$$

Let $u = 2x \rightarrow du = 2dx$

$dv = e^{2x}dx \rightarrow v = \frac{1}{2}e^{2x}$

$$\because \int u\, dv = u.v$$

$$- \int v\, du \quad \rightarrow \int \underbrace{2xe^{2x}\, dx}_{u\,.dv}$$

$$= xe^{2x} - \int e^{2x}dx = xe^{2x} - \frac{1}{2}e^{2x}$$

Hence, the general solution is:

$$\frac{x}{4}\, e^{x} = e^{2x}\left(x - \frac{1}{2}\right) + C$$

Ex4: Solve $\quad \boldsymbol{ydx - 4xdy = y^6dy} \qquad \div\, ydy$

Sol.: $\rightarrow \quad \frac{dx}{dy} - \frac{4}{y}\,x = y^5$

$$I = \int e^{-\frac{4}{y}dy} = e^{-4\ln y} = y^{-4} \quad \text{(عامل التكامل)}$$

$$\rightarrow \quad x.\,y^{-4} = \int y^{-4}\,y^5 dy + C$$

$$\rightarrow \quad x.\,y^{-4} = \int y\,dy + C$$

$$\rightarrow \quad x.\,y^{-4} = \frac{y^2}{2} + C, \quad (The\ \ general\ \ sol.)$$

Ex5: Solve $\boldsymbol{dx + \frac{2}{y}\,x\,dy = 2x^2y^2\,dy}$

Sol.: $\frac{dx}{dy} + \frac{2}{y}\,x = 2x^2y^2$ (برنولي)

$$x^{-2}\frac{dx}{dy} + \frac{2}{y}\,x^{-1} = 2y^2 \qquad \ldots\ (*)$$

Let $z = x^{-1} \rightarrow \frac{dz}{dy} = -x^{-2}\frac{dx}{dy} \rightarrow -\frac{dz}{dy} = x^{-2}\frac{dx}{dy}$ (نعوض في *)

$$-\frac{dz}{dy} + \frac{2}{y}\,z = 2y^2$$

$$\frac{dz}{dy} - \frac{2}{y}\, z = -2y^2 \qquad (\text{ linear})$$

Let $I = e^{\int -\frac{2}{y}dy} = \frac{1}{y^2}$ (عامل التكامل)

$$\frac{1}{y^2}\, .z = \int -2y^2 \,.\frac{1}{y^2} dy + C$$

$$\frac{1}{y^2}\, x^{-1} = \int -2 \; dy + C$$

$$\frac{1}{y^2 x} = -2y + C, \quad (The \;\; general \;\; solution)$$

2.7. Ricatt's Equation

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x) \qquad \dots \quad (1)$$

Such that P, Q, R are functions of x, then the Eq. (1) becomes linear when $P(x) = 0$. So Eq. (1) becomes Bernolli when $R(x) = 0$.

نلاحظ ان معادلة ريكاتي اعم من معادلة برنولي والمعادلة الخطية. ولإيجاد حل معادلة ريكاتي لابد من ان نعلم حلاً خاصاً وليكن y_1 حيث ان $y_1 = y_1(x)$.

ويكون الحل العام لمعادلة ريكاتي باستخدام التعويض:

$$y = y_1 + \frac{1}{z}$$

$$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{z^2}\frac{dz}{dx}$$

Substituting in Eq.(1),we get:

$$\frac{dy_1}{dx} - \frac{1}{z^2}\frac{dz}{dx} = P(x)\left(y_1 + \frac{1}{z}\right)^2 + Q(x)\left(y_1 + \frac{1}{z}\right) + R(x)$$

$$\rightarrow \frac{dy_1}{dx} - \frac{1}{z^2}\frac{dz}{dx} =$$

$$P(x)y_1^2 + 2P(x)y_1\frac{1}{z} + P(x)\frac{1}{z^2} + Q(x)y_1 + Q(x)\frac{1}{z} + R(x)$$

Remark: y_1 is a particular solution of the equation, and by multiplying by z^2we get:

$$\frac{1}{z^2}\frac{dz}{dx} = 2P(x)y_1\frac{1}{z} + P(x)\frac{1}{z^2} + Q(x)\frac{1}{z}$$

$$\rightarrow \quad \frac{dz}{dx} + \left(2P(x)y_1 + Q(x)\right)z = -P(x)$$

(وهي معادلة خطية في z وتحل كما في السابق)

Ex1: Find the general sol. of

$$\boldsymbol{2x^2\frac{dy}{dx} = (x-1)(y^2-x^2)+2xy} \quad \dots (*)$$

Where $y = x$ is particular solution.

Sol.: Let $y = x + \frac{1}{z} \quad \rightarrow \quad \frac{dy}{dx} = 1 - \frac{1}{z^2}\frac{dz}{dx}$

Note: Eq. (*) is Ricatt's eq. (Prove that (H.W.))

$$2x^2\left(1 - z^{-2}\frac{dz}{dx}\right) = (x-1)[\left(x+\frac{1}{z}\right)^2 - x^2] + 2x(x+\frac{1}{z})$$

$$2x^2 - 2\frac{x^2}{z^2}\frac{dz}{dx} = (x-1)\left(\frac{2x}{z}+\frac{1}{z^2}\right) + 2x^2 + \frac{2x}{z}$$

$$\left[-2\frac{x^2}{z^2}\frac{dz}{dx} = \frac{2x^2}{z} + \frac{x}{z^2} - \frac{2x}{z} - \frac{1}{z^2} + \frac{2x}{z}\quad\right] * \frac{z^2}{-2x^2}$$

$$\frac{dz}{dx} = -z - \frac{1}{2x} + \frac{1}{2x^2}$$

$$\frac{dz}{dx} + z = \frac{1}{2x^2} - \frac{1}{2x} \qquad \text{(معادلة خطية)}$$

$$I(x) = e^{\int dx} \rightarrow I(x) = e^x \qquad \text{(عامل التكامل)}$$

$$z.e^x = \frac{1}{2}\int (\frac{e^x}{x^2} - \frac{e^x}{x})dx + C$$

نستخدم تكامل التجزئة ل $(\int \frac{e^x}{x}\, dx)$:

Let $u = \frac{1}{x},\ \ dv = e^x dx$

$$\rightarrow \quad z\,.\,e^x = \frac{1}{2}[\int \frac{e^x}{x^2}dx - \frac{e^x}{x} - \int \frac{e^x}{x^2}dx] + C$$

$$\rightarrow \quad z\,.\,e^x = -\frac{1}{2}\frac{e^x}{x} + C$$

$$\because y = x + \frac{1}{z} \quad \rightarrow \quad \frac{1}{z} = y - x \rightarrow \quad z = \frac{1}{y-x}$$

So, the general sol. is:

$$\frac{e^x}{y-x} = -\frac{1}{2}\frac{e^x}{x} + c$$

Ex2: Find the general sol. of

$$\boldsymbol{x^2\frac{dy}{dx} = x^2y^2 + xy - 3} \qquad \boldsymbol{s.t.\ y = \frac{1}{x}} \quad \text{(حل خاص)}$$

Sol.:

$$\frac{dy}{dx} = y^2 + \frac{1}{x}y - \frac{3}{x^2}$$ (معادلة ديكاتي)

Let $y = y_1 + \frac{1}{z} \quad \rightarrow \quad y = \frac{1}{x} + \frac{1}{z}$

$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{1}{z^2}\frac{dz}{dx}$$ (نعوض في المعادلة)

$$x^2\left(-\frac{1}{x^2} - \frac{1}{z^2}\frac{dz}{dx}\right) = x^2\left(\frac{1}{x} + \frac{1}{z}\right)^2 + x\left(\frac{1}{x} + \frac{1}{z}\right) - 3$$

$$-1 - \frac{x^2}{z^2}\frac{dz}{dx} = x^2\left(\frac{1}{x^2} + \frac{2}{xz} + \frac{1}{z^2}\right) + 1 + \frac{x}{z} - 3] \qquad * (z^2)$$

$$\rightarrow \qquad -x^2\frac{dz}{dx} = z^2 + 2xz + x^2 + z^2 + xz - 2z^2$$

$$\rightarrow \qquad -x^2\frac{dz}{dx} = z^2 + 3xz + x^2 \qquad * \left(-\frac{1}{x^2}\right)$$

$$\rightarrow \qquad \frac{dz}{dx} = -3\frac{z}{x} - 1$$

$$\rightarrow \qquad \frac{dz}{dx} + \frac{3}{x}z = 1$$ (معادلة خطية) (التكملة واجب)

2. 8: The diff. eq. of the form $\boldsymbol{f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x)}$

Let:

$$\underbrace{\boldsymbol{f'(y)}}_{\text{مشتقتها}}\boldsymbol{\frac{dy}{dx} + P(x)}\,\underbrace{\boldsymbol{f(y)}}_{\text{الدالة}} = \boldsymbol{Q(x)} \qquad \dots \quad (\mathbf{1})$$

Such that $P(x)$ & $Q(x)$ are function of x.

To solve this equation we use :لحل هذا النوع من المعادلات فأننا نستخدم التعويض
the following substitution:

$$\text{Let } z = f(y) \qquad \dots \quad (2)$$

Deriving for x ,we get   بالتفاضل ل x نحصل على :

$$\frac{dz}{dx} = f'(y)\frac{dy}{dx} \qquad [\text{بالتعويض في } (1)]$$

$$\frac{dz}{dx} + P(x)z = Q(x) \qquad (\text{وهي معادلة خطية في } z)$$

Ex1: Find the general sol. of

$$\boldsymbol{e^y\frac{dy}{dx} + e^y = x}$$

Sol.: Let $z = e^y \quad \rightarrow \quad \frac{dz}{dx} = e^y\frac{dy}{dx}$ (نعوض في المعادلة التفاضلية)

$\rightarrow \qquad \frac{dz}{dx} + z = x$ (معادلة خطية)

$I(x) = e^{\int dx} \quad \rightarrow \quad I(x) = e^x$

$\rightarrow \qquad ze^x = \int x \,. e^x dx + c$

$\rightarrow \qquad e^y e^x = xe^x - e^x + c$ (c is an arb. cons.)

And this is the general sol.

Ex2: Find the general sol. of

$$\boldsymbol{3x(1-x^2)y^2\frac{dy}{dx} + (2x^2-1)y^3 = a\,x^3, \quad a \text{ is a constant}}$$

Sol.: When we divide the eq. by $x(1-x^2)$, we get:

$$3y^2\frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}y^3 = \frac{a\,x^2}{(1-x^2)} \qquad \ldots \quad (*)$$

Let $z = y^3 \;\rightarrow\; \frac{dz}{dx} = 3y^2\frac{dy}{dx}$

بالتعويض في المعادلة (*)، نحصل على:

$$\frac{dz}{dx} + \frac{2x^2-1}{x(1-x^2)}z = \frac{a\,x^2}{(1-x^2)}$$ (معادلة خطية)

$$I(x) = e^{\int \frac{2x^2-1}{x(1-x^2)}dx} = e^{\ln \frac{1}{x\sqrt{1-x^2}}} = \frac{1}{x\sqrt{1-x^2}} \qquad \text{(عامل التكامل)}$$

Then the general solution is:

$$\frac{z}{x\sqrt{1-x^2}} = \int \frac{1}{x\sqrt{1-x^2}} \cdot \frac{a\,x^2}{(1-x^2)} dx + c$$

$$= a \int \frac{x}{(1-x^2)^{\frac{3}{2}}} dx + c$$

$$= a\,\frac{1}{\sqrt{1-x^2}} + c$$

$$\because z = y^3 \quad \rightarrow \quad \frac{y^3}{x\sqrt{1-x^2}} = a\,\frac{1}{\sqrt{1-x^2}} + c$$

2.9) Equation that is solved using a suitable substitution.

هنالك صور أخرى غير قياسية يمكن بتعويضات مناسبة تحويلها الى صورة قابلة للفصل وليس هنالك قاعدة عامة لمثل هذه التعويضات فكل معادلة لها ظروفها الخاصة التي توحي بالتعويضات المناسبة وكشأن كل تعويض فليس أي تعويض يؤدي الى حل المسألة بل يحتاج الامر الى مهارة وفطنة للوصول الى التعويض المناسب , كما مبين في الامثلة الاتية:

Ex (1) : Solve $\boldsymbol{y' = \cos(x+y)}$

Solution: سوف نستخدم الفرضية.

Let $z = x + y \rightarrow dz = dx + dy \rightarrow dy = dz - dx$

$$\rightarrow \frac{dy}{dx} = \cos z \rightarrow \frac{dz - dx}{dx} = \cos z$$

$$\rightarrow dz - dx = \cos z \, dx$$

$$\rightarrow dz = \cos z \, dx + dx$$

$$\rightarrow [dz = (\cos z + 1)dx] \cdot \frac{1}{\cos z + 1}$$

$$\rightarrow \frac{dz}{\cos z + 1} = dx$$

$$\rightarrow \frac{dz}{\cos z + 1} . \frac{\cos z - 1}{\cos z - 1} = dx \qquad (conjucate\text{: استخدام المرافق})$$

$$\rightarrow \frac{\cos z - 1}{\cos^2 z - 1} dz = dx$$

$$\rightarrow \frac{\cos z - 1}{-\sin^2 z} dz = dx$$

$$\rightarrow -\sin^{-2} z \, (\cos z - 1) \, dz = dx$$

$$\rightarrow (-\sin^{-2} z) \, (\cos z) \, dz + \int \sin^{-2} z \, dz - \int dx = \int 0$$

$$\rightarrow \frac{-\sin^{-1} z}{-1} + \int \csc^2 z \, dz - x = c$$

$$\rightarrow \frac{1}{\sin z} - \cot z - x = c$$

Returning z in terms of x & y we get

$$\frac{1}{\sin x + y} - \cot(x+y) - x = c, \quad \text{(and this is the general solution)}$$

Ex (2): Solve $\boldsymbol{\frac{dy}{dx} = y^2 + 2y + 2}$

<u>Solution:</u>

$$\frac{dy}{y^2+2y+2} = dx \rightarrow \frac{dy}{y^2+2y+1+1} = dx \rightarrow \frac{dy}{(y+1)^2+1} = dx$$

$$\rightarrow \tan^{-1}(y+1) = x + c, \qquad \text{(The general solution)}$$

Ex (3): Solve $\boldsymbol{\frac{dy}{dx} - 3(3x+y)^2 = 0}$

Sol: Let $z = 3x + y \rightarrow y = z - 3x \rightarrow dy = dz - 3dx \rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 3$

بالتعويض بالمعادلة التفاضلية:

$$\rightarrow \quad \frac{dz}{dx} - 3 + 3z^2 = 0$$

$\rightarrow \quad dz - 3dx + 3z^2 dx = 0$

$\rightarrow \quad dz - 3(1 - z^2)dx = 0$

$\rightarrow \quad \dfrac{dz}{1 - z^2} - 3dx = 0$

$\rightarrow \quad \tan^{-1} z - 3x = c$

$\rightarrow \quad \tan^{-1}(3x + y) - 3x = c,$ (The general solution)

Where c is an arb.cons.

Ex4: Solve $\boldsymbol{tan^2(x + y)dx - dy = 0}$

Sol: Let $z = x + y \rightarrow dz = dx + dy \rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

بالتعويض بالمعادلة التفاضلية:

$$\boldsymbol{tan^2(x + y) - \frac{dy}{dx} = 0}$$

$\rightarrow \quad tan^2(z) - \dfrac{dz}{dx} + 1 = 0$

$\rightarrow \quad tan^2(z) + 1 = \dfrac{dz}{dx}$

$$\rightarrow \qquad dx = \frac{dz}{1 + tan^2(z)}$$

$$\rightarrow \qquad dx = \frac{dz}{1 + \frac{\sin^2 z}{\cos^2 z}}$$

$$\rightarrow \qquad dx = \frac{dz}{\frac{\cos^2 z + \sin^2 z}{\cos^2 z}}$$

$$\rightarrow \qquad dx = \frac{dz}{\frac{1}{\cos^2 z}}$$

$$\rightarrow \qquad dx = \cos^2 z \; dz$$

$$\rightarrow \qquad dx = \frac{1 + \cos\,(2z)}{2}\; dz$$

$$\rightarrow \qquad dx = \frac{1}{2}\,[1 + \cos(2z)]dz$$

$$\rightarrow \qquad \int dx = \int \frac{1}{2}\,[dz + \cos(2z)]dz$$

$$\rightarrow \qquad x = \frac{1}{2}\left[z + \frac{\sin(2z)}{2}\right] + c$$

$$\rightarrow \qquad x = \frac{1}{2}\left[(x+y) + \frac{\sin(2(x+y))}{2}\right] + c,$$ (The general solution)

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قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

Ordinary Differential Equations

CHAPTER FOUR

Solution of The Differential Equations of The First Order and Higher Degree

اساتذة المادة

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Solution of The Differential Equations of The First Order and Higher Degree

The general form of the differential equation of the first order and the degree n is:

$$\boldsymbol{p^n + a_1(x,y)p^{n-1} + \cdots + a_{n-1}(x,y)p + a_n(x,y) = 0} \quad \dots\dots\dots \quad \textbf{(I)}$$

Where n=2,3,4,... and $p = \frac{dy}{dx}$

The differential equations of this type are divided into three cases:

4.1: Equation solvable for $\boldsymbol{p}$

4.2: Equation solvable for $\boldsymbol{y}$

4.3: Equation solvable for $\boldsymbol{x}$

Here we will discuss each of these three types with examples:

4.1: Equation solvable for $\boldsymbol{p}$:

In this type we can analyze the left side of equation (I), which is considered to be polynomial for p in the form of n linear factors, so we can write equation (I) as the form:

$$\boldsymbol{(p - F_1)(p - F_2)\dots(p - F_n) = 0} \quad \dots\dots\dots\dots \quad \dots\dots\dots\dots\dots. \quad \text{(II)}$$

Where $F_1, F_2, \dots, F_n$ are functions of x and y,

Then equivalent each factor of equation (II) by zero to obtain n of differential equations of order 1 and degree1

في هذه الحالة نقوم بتحليل المعادلة الى حاصل ضرب عدد من العوامل جميعها من الرتبة الاولى والدرجة الاولى باستخدام طرق التحليل المعروفة ثم ناخذ كل عامل على حدة ونرجع p الى المشتقة $\frac{dy}{dx}$ ونكامل

نكرر العملية نفسها لبقية الحدود ثم نضرب الحدود الناتجة فنحصل على الحل العام.

Ex1: Solve $\boldsymbol{x^2p^2 + xyp - 6y^2 = 0}$

Sol: $(xp + 3y)(xp - 2y) = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$

$$xp + 3y = 0 \Rightarrow xp = -3y \Rightarrow p = \frac{-3y}{x} \Rightarrow \frac{dy}{dx} = \frac{-3y}{x} \Rightarrow \frac{dy}{y} = \frac{-3dx}{x}$$

Integrating both sides, we get:

$ln|y| = -3ln|x| + lnc \Rightarrow$

$ln|y| = ln|x|^{-3} + lnc \Rightarrow y = cx^{-3} \Rightarrow y - cx^{-3} = 0$ ... ... (2)

Now take $(xp - 2y) = 0$ then

$$xp = 2y \Rightarrow x\frac{dy}{dx} = 2y \Rightarrow \frac{dy}{y} = \frac{2dx}{x}$$

Integrating both sides, we get:

$y = cx^2 \Rightarrow y - cx^2 = 0$ ... ... (3)

From (2) and (3), we get:

$(y - cx^{-3})(y - cx^2) = 0$ ... ... (4)

And this is the general solution.

Ex2: Solve the following equation $p^2 + py = x^2 + xy$

Sol: rearrange the equation

$p^2 + py - x^2 - xy = 0$ ... ... (1)

$\Rightarrow p^2 - x^2 + py - xy = 0$

$\Rightarrow (p - x)(p + x) + y(p - x) = 0$

$\Rightarrow (p - x)(p + x + y) = 0$ ... ... (2)

Take the first factor,

$$(p - x) = 0 \Rightarrow p = x \Rightarrow \frac{dy}{dx} = x \Rightarrow dy = xdx$$

Integrating both sides to get:

$y = \frac{x^2}{2} + c \Rightarrow y - \frac{x^2}{2} - c = 0$ ... ... (3)

Take the second factor, $dv = e^x dx \Rightarrow v = e^x$

$p + x + y = 0 \Rightarrow \frac{dy}{dx} + y = -x$ *and this is a linear eq.*

$I = e^{\int dx} = e^x$

Substitute in :

$e^x y = \int -xe^x dx + c$ $\qquad u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = e^x$

$e^x y = -xe^x + e^x + c$

$e^x y + xe^x - e^x - c = 0 \qquad \dots\dots\dots\dots\dots\dots..(4)$

From (3)&(4), we get:

$$\left(y - \frac{x^2}{2} - c\right)(e^x y + xe^x - e^x - c) = 0 \qquad \dots\dots\dots\dots\dots\dots.(5)$$

So eq. (5) is the general sol.

4.2: Equation solvable for *y*

This type of equations can be written as:

$y = F(x, p) \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (III)$

Differentiating for x we get :

$$\frac{dy}{dx} = \frac{dF}{dx} + \frac{dF}{dp} \cdot \frac{dp}{dx} = F(x, p, \frac{dp}{dx})$$

$$\Rightarrow p = F\left(x, p, \frac{dp}{dx}\right) \qquad \dots\dots\dots\dots\dots\dots.. \qquad (IV)$$

And this equation is of the first order and the first degree to solve it, we analyze the equation into a several factors, one of which contains $\frac{dp}{dx}$ and we get from it the general solution $\emptyset(x, p, c) = 0$

the rest contains p and we get from it the singular solution.

Ex3: Solve $\boldsymbol{2yp - 3x = xp^2}$

Sol: rearrange the equation

$$2y = 3\frac{x}{p} + xp \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

Differentiating both sides:

$$2\frac{dy}{dx} = 3\frac{p - x\frac{dp}{dx}}{p^2} + x\frac{dp}{dx} + p \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

$$2p = \frac{3}{p} - \frac{3x}{p^2} \cdot \frac{dp}{dx} + x\frac{dp}{dx} + p \Rightarrow$$

$$\left[2p - p = \frac{3}{p} - \frac{3x}{p^2} \cdot \frac{dp}{dx} + x\frac{dp}{dx}\ \right] \quad \times p^2$$

$$p^3 - 3p - xp^2\frac{dp}{dx} + 3x\frac{dp}{dx} = 0$$

$$p(p^2 - 3) - x\frac{dp}{dx}(p^2 - 3) = 0$$

$$(p^2 - 3)(p - x\frac{dp}{dx}) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3)$$

Either $(p - x\frac{dp}{dx}) = 0 \Rightarrow pdx = xdp \Rightarrow \frac{dx}{x} = \frac{dp}{p} \Rightarrow lnp = lnx + lnc$

$$\Rightarrow p = cx \quad \ldots\ldots\ldots\ (4)$$

Substituting (4)in the original equation, we get:

$$2cxy - 3x = c^2x^3 \Rightarrow$$

$$\boldsymbol{y = \frac{cx^2}{2} + \frac{3}{2c}} \qquad \ldots\ \ldots\ ..(5)$$

Equation (5) is the general solution

Or $p^2 - 3 = 0 \Rightarrow p^2 = 3 \Rightarrow p = \pm\sqrt{3}$ $\quad \ldots\ldots\ldots \quad (6)$

Sub. In the original equation, we get:

$$\pm 2\sqrt{3}y - 3x = 3x \Rightarrow$$

$$\boldsymbol{y = \pm\sqrt{3}x} \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots \qquad (7)$$

We note that equation (7) does not contain arbitrary constants, so it does not represent a general solution , but rather a singular solution.

نلاحظ ان المعادلة (7) لا تحوي ثوابت اختيارية وعليه هي لا تمثل حلا عاما وانما تمثل حلا منفردا

Ex4: Solve $\boldsymbol{16x^2 + 2p^2y - p^3x = 0}$ …….. **(1)**

Sol: Dividing on p^2we get:

$$2y = px - 16\frac{x^2}{p^2}$$

Deriving for x we get:

$$2\frac{dy}{dx} = p + x\frac{dp}{dx} - 32\frac{x}{p^2} + 32\frac{x^2}{p^3}\frac{dp}{dx}$$

$$[\,2p = p + x\frac{dp}{dx} - 32\frac{x}{p^2} + 32\frac{x^2}{p^3}\frac{dp}{dx} \qquad] \times p^3$$

$$p^4 + 32xp - xp^3\frac{dp}{dx} - 32x^2\frac{dp}{dx} = 0$$

$$p(p^3 + 32x) - x(p^3 + 32x)\frac{dp}{dx} = 0$$

$$(p^3 + 32x)\left(p - x\frac{dp}{dx}\right) = 0$$

Either $\left(p - x\frac{dp}{dx}\right) = 0 \Rightarrow p = x\frac{dp}{dx} \Rightarrow \frac{dp}{p} = \frac{dx}{x} \Rightarrow$

$p = cx$ … … …. (2)

Substituting eq. (2) in (1) we get:

$16x^2 + 2c^2x^2y - c^3x^4 = 0$ …………… (3)

Equation (3) is the general solution,

or $p^3 + 32x = 0 \Rightarrow p^3 = -32x \Rightarrow p = \sqrt[3]{-32x}$ ……………(4)

Substituting eq. (4) in (1) we get:

$$16x^2 + 2(32x)^{\frac{2}{3}}y + 32x^2 = 0$$

$$y = \frac{-48x^2}{2(32x)^{\frac{2}{3}}}$$

$$\boldsymbol{y = \frac{-3\,x^{\frac{4}{3}}}{\sqrt[3]{2}}} \quad \text{……………..} (5)$$

And this is the singular solution.

In this type (equation solvable for y) we have two special cases,

4.2.1. Clairaut's equation

4.2.2. Lagrange's equation

And we will discuss them in the following:

4.2.1. Clairaut's equation: معادلة كليروت

It is a differential equation of the form:

$$\boldsymbol{y = px + f(p)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (V)$$

we can solve it by differentiating it for x.

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \Rightarrow$$

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \Rightarrow$$

$$x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0 \Rightarrow$$

$$(x + f'(p))\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0 \Rightarrow p = c \ \ substituting\ in\ (V), we\ get:$

$$\boldsymbol{y = cx + f(c)} \qquad \ldots\ldots\ldots\ldots.. (VI)$$

And this is the general solution.

Or $\left(x + f'(p)\right) = 0 \Rightarrow f'(p) = -x$

Taking $(f')^{-1}$for both sides to get:

$$p = (f')^{-1}(-x)$$

Sub. In (V), we get:

$$\boldsymbol{y = x(f')^{-1}(-x) \ + f(f')^{-1}(-x)} \qquad \ldots\ldots\ldots\ldots\ldots\text{(VII)}$$

And this is the singular solution.

Ex1: Solve $\boldsymbol{y = px + \cos p}$

Sol: $y = px + \cos p$ ……………………(1)

Deriving eq. (1):

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - sinp \cdot \frac{dp}{dx} \quad \text{………………….} \ (2)$$

$$p = p + x\frac{dp}{dx} - sinp \cdot \frac{dp}{dx} \ \Rightarrow$$

$$x\frac{dp}{dx} - sinp \cdot \frac{dp}{dx} = 0 \Rightarrow (x - sinp) \cdot \frac{dp}{dx} = 0$$

Either $(x - sinp) = 0 \Rightarrow \sin p = x \Rightarrow p = sin^{-1}x$

Sub. in (1), we get:

$$\boxed{y = xsin^{-1}x + \cos(sin^{-1}x) \quad \dots\dots\dots\dots\dots (2)}$$

And this is the singular sol.

Or $\frac{dp}{dx} = 0 \Rightarrow p = c$

Sub. in (1), we get:

$$\boxed{y = cx + \cos c \quad \dots\dots\dots\dots\dots (3)}$$

And this is the general sol.

Ex 2: Find the general solution of $y = px + \sqrt{4 + p^2}$

Sol: $y = px + \sqrt{4 + p^2}$ ……….. (1)

Deriving for x, we get:

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{2\sqrt{4+p^2}}(2p\frac{dp}{dx}) \Rightarrow$$

$$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{4+p^2}}(p\frac{dp}{dx}) \Rightarrow$$

$$x\frac{dp}{dx} + \frac{1}{\sqrt{4+p^2}}(p\frac{dp}{dx}) = 0 \Rightarrow$$

$$(x + \frac{1}{\sqrt{4+p^2}}p)\frac{dp}{dx} = 0$$

To get the general sol., we take $\frac{dp}{dx} = 0 \Rightarrow p = c$

Sub. in (1), we get:

$$\boxed{y = cx + \sqrt{4 + c^2} \qquad \ldots\ldots\ldots\ldots\ldots \quad (2)}$$

Eq. (2) represent the general sol.

4.2.2. Lagrange's equation معادلة لاكرانج

It is a differential equation of the form:

$$\boldsymbol{y = xf(p) + g(p) \qquad ; f(p) \neq p \ \& \ p = \frac{dy}{dx} \qquad \ldots.. \quad (VIII)}$$

To find the general solution , we differentiate equation (VIII):

$$\frac{dy}{dx} = f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx} \Rightarrow$$

$$p = f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx} \Rightarrow$$

$$p = f(p) + (xf'(p) + g'(p))\frac{dp}{dx} \Rightarrow$$

$$\left(xf'(p) + g'(p)\right)\frac{dp}{dx} = p - f(p) \Rightarrow$$

$$\frac{dx}{dp} - \frac{f'(p)}{p - f(p)}x = \frac{g'(p)}{p - f(p)}$$

Where $f'(p) = \frac{d(f(p))}{dp}$, $g'(p) = \frac{d(g(p))}{dp}$,

This equation is a linear differential equation of order 1 with two variables x , p. and I=$e^{\int -\frac{f'(p)}{p-f(p)}dp}$, Q=$\frac{g'(p)}{p-f(p)}$

Ex1: find the general solution of the differential equation

$$\boldsymbol{y = 2px + p^3 \qquad \ldots\ldots\ldots \quad (1)}$$

Sol: Equation (1) in Lagrange form where $f(p) = 2p$ & $g(p) = p^3$, deriving both sides:

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 3p^2\frac{dp}{dx} \Rightarrow$$

$$p = 2p + (2x + 3p^2)\frac{dp}{dx} \Rightarrow$$

$$-p = (2x + 3p^2)\frac{dp}{dx} \Rightarrow$$

$$\frac{dx}{dp} = \frac{(2x+3p^2)}{-p} \Rightarrow \frac{dx}{dp} + \frac{2x}{p} = -3p \qquad (and\ this\ is\ a\ linear\ eq.)$$

$I = e^{\int \frac{2}{p}dp} = p^2$, $Q(p) = -3p$

Hence, we get:

$$p^2 x = \int (-3p)\, p^2 dp + c$$

$$p^2 x = \frac{-3}{4}p^4 + c \qquad \text{...........................} \qquad (2)$$

نرتب المعادلة (2) ثم نستخدم اكمال المربع لايجاد قيمة p

$$p^4 + \frac{4}{3}p^2 x = \frac{4}{3}c$$

$$p^4 + \frac{4}{3}p^2 x + \frac{4}{9}x^2 = \frac{4}{3}c + \frac{4}{9}x^2$$

$$(p^2 + \frac{2}{3}x)^2 = \frac{4}{3}c + \frac{4}{9}x^2$$

نجذر الطرفين

$$p^2 = \pm\sqrt{\frac{4}{3}c + \frac{4}{9}x^2} - \frac{2}{3}x \Rightarrow$$

$$p = \pm(\pm\sqrt{\frac{4}{3}c + \frac{4}{9}x^2} - \frac{2}{3}x)^{\frac{1}{2}} \qquad \text{............} \qquad (3)$$

Sub. (3) in (1), we get:

$$y = \pm 2x(\pm\sqrt{\frac{4}{3}c + \frac{4}{9}x^2} - \frac{2}{3}x)^{\frac{1}{2}} \pm (\pm\sqrt{\frac{4}{3}c + \frac{4}{9}x^2} - \frac{2}{3}x)^{\frac{3}{2}} \text{ ....}(4)$$

And this is the general solution.

4.3: Equation solvable for *x*

This equation can be written as

$$\boldsymbol{x = f(y,p)} \qquad \ldots\ldots\ldots\ldots\ldots \qquad \boldsymbol{(IX)}$$

Deriving for y, we get:

$$\frac{dx}{dy} = \frac{df}{dy} + \frac{df}{dp}\frac{dp}{dy} = F\left(y,p,\frac{dp}{dy}\right) \Rightarrow$$

$$\frac{1}{p} = F\left(y,p,\frac{dp}{dy}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (X)$$

Eq. (X) is a diff. eq. of order 1 and degree 1 whose solution is :

$$\emptyset(y,p,c) = 0 \qquad \ldots\ldots\ldots\ldots\ldots \qquad (XI)$$

From eqs. (IX) &(XI) we get the general solution.

من المعادلتين (IX) &(XI) نحصل على الحل العام عن طريق تعويض احداهما بالاخرى وحذف المتغير p

Ex1: Solve $\boldsymbol{p^3 - 2xyp + 4y^2 = 0} \qquad \ldots\ldots\ldots\ldots\ldots\ \boldsymbol{(1)}$

Sol: Rearrange equation(1) as:

$$2x = \frac{p^2}{y} + 4\frac{y}{p} \qquad \ldots\ldots\ldots\ldots(2)$$

Deriving for y, we get:

$$2\frac{dx}{dy} = \frac{2yp\frac{dp}{dy} - p^2}{y^2} + 4\frac{p - y\frac{dp}{dy}}{p^2} \Rightarrow$$

$$\left[\frac{2}{p} = \frac{2p}{y}\cdot\frac{dp}{dy} - \frac{p^2}{y^2} + \frac{4}{p} - \frac{4y}{p^2}\cdot\frac{dp}{dy}\ \right] \qquad \times p^2y^2 \quad \Rightarrow$$

$$2py^2 = 2p^3y\frac{dp}{dy} - p^4 + 4py^2 - 4y^3\frac{dp}{dy}$$

$$2py^2 - p^4 + 2y(p^3 - 2y^2)\frac{dp}{dy} = 0 \quad \Rightarrow$$

$$p(2y^2 - p^3) - 2y(2y^2 - p^3)\frac{dp}{dy} = 0 \Rightarrow$$

$$(p - 2y\frac{dp}{dy})(2y^2 - p^3) = 0$$

Taking the first term $\left(p - 2y\frac{dp}{dy}\right) = 0 \Rightarrow \frac{dy}{y} = 2\frac{dp}{p}$

Integrating both sides, we get:

$$ln|y| + lnc = 2ln|p| \Rightarrow lnp^2 = ln|cy| \Rightarrow$$

$$p^2 = cy \quad \Rightarrow p = \pm\sqrt{cy} \qquad \dots\dots\dots\dots\dots\dots\dots \quad (3)$$

Sub (3) in (2):

$$2x = \frac{cy}{y} + 4\frac{y}{\sqrt{cy}} \Rightarrow$$

$$2x - c = 4\frac{y}{\sqrt{cy}} \Rightarrow (2x-c)^2 = \frac{16y}{c} \Rightarrow$$

$$y = \frac{c(2x-c)^2}{16} \qquad \dots\dots\dots\dots\dots\dots(4)$$

Eq. (4)is the general sol.

Now taking the second term $(2y^2 - p^3) = 0$, we get:

$$p^3 = 2y^2 \Rightarrow p = \sqrt[3]{2y^2}$$

Sub in eq. (2)

$$2x = \frac{(2y^2)^{2/3}}{y} + 4\frac{y}{\sqrt[3]{2y^2}} \qquad \dots\dots\dots\dots\dots (5)$$

And this is the singular solution.

Ex 2: Solve the following diff. eq.

$$\boldsymbol{p^2 x = 2yp - 3} \qquad \dots\dots\dots\dots\dots (1)$$

Sol: Dividing on p^2

$$x = \frac{2y}{p} - \frac{3}{p^2} \qquad \dots\dots\dots\dots\dots (2)$$

Deriving for y

$$\frac{dx}{dy} = \frac{2}{p} - 2yp^{-2}\frac{dp}{dy} + 6p^{-3}\frac{dp}{dy} \Rightarrow$$

$$\left[\frac{1}{p} = \frac{2}{p} + (-2yp^{-2} + 6p^{-3})\frac{dp}{dy} \qquad \right] \quad \times p^3$$

$$p^2 - 2p^2 = (6 - 2yp)\frac{dp}{dy} \quad \Rightarrow$$

$-p^2 = (6-2yp)\frac{dp}{dy} \qquad \Rightarrow$

$-p^2 dy = 6dp - 2ypdp \;\Rightarrow$

$[\; 2ypdp - p^2 dy = 6dp \;] \qquad \div -p^4$

$\frac{p^2 dy - 2ypdp}{p^4} = \frac{-6}{p^4} dp \qquad \Rightarrow$

$d\left(\frac{y}{p^2}\right) = \frac{-6}{p^4} dp$

Integrating both sides:

$\frac{y}{p^2} = 2p^{-3} + c$

$y = 2p^{-1} + cp^2$ ....(3)

Sub. In eq. (1)

$p^2 x = 2(2p^{-1} + cp^2)p - 3 \Rightarrow$

$p^2 x = 2cp^3 + 1$ ....(4)

هنا ايجاد p صعب جدا بهذا الشكل لذا سنستفيد من معادلة (3)

From eq. (3):

$cp^3 = py - 2$

Sub. In (4):

$p^2 x = 2py - 3 \;\Rightarrow\; p^2 x - 2py + 3 = 0$ بالدستور

$p = \frac{y \pm \sqrt{y^2 - 3x}}{x}$

Sub. In (3), we get:

$y = \frac{2x}{y \pm \sqrt{y^2-3x}} + c\left(\frac{y \pm \sqrt{y^2-3x}}{x}\right)^2$ ....(5)

Eq. (5) represent the general sol.

■**Question**: Can you solve example (2) using the second case? Is the result you will get equal to the output you have?

Exercises:

Solve the following equations:

1. $y = 3px + 6y^2p^2$

2. $xp^2 + (y - 1 - x^2)p - x(y - 1) = 0$

3. $y^2(1 + p^2) = 1$

4. $y = px + \sqrt{4 + p^2}$

5. $y^2p^2 - 3xyp + 2x^2 = 0$

6. $p^2 - p - 6 = 0$

7. $p^2 - 4xp - 12x^2 = 0$

8. $p^2 + xp + yp + xy = 0$

9. $py - 2p^4 + 2 = 0$

10. $y = p\sin p + \cos p$

11. $x - 2p - lnp = 0$

12. Find the general solution of:

a. $(y - px)^2 = \sin(y - px) + p^2$

b. $e^{y-px} = (y - px)^2 - p^3$

Hint: derive for x

13. $y = px + p^2$

14. $px = y + p^3$

15. $4y - 4px\, ln|x| = p^2x^2$

16. $yp^2 - 2xp + y = 0$

17. $y = 2xp + p\ ln|p|$

جامعة بغداد

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قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

Ordinary Differential Equations

CHAPTER FIVE

The Ordinary Differential Equations of Higher Order

اساتذة المادة

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Section 5.1: Linear Differential equations of order n

Definition (5.1.1): The general form of linear differential equation (LDE) of order n is:

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=f(x) \quad \ldots\ldots\ldots. \ (1)$$

Where $a_0(x)\neq 0$

■If all the coefficients $a_0, a_1, \ldots, a_n$ are constants then eq. (1) is called ***linear equation with constant coefficients.***

■but if at least one of the coefficients is a function of *x* then eq. (1) is called ***linear equation with variable coefficients.***

■In eq.(1); If *f(x)*=0 then eq.(1) becomes ***homogeneous equation*** and has the form

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=0 \quad \ldots\ldots\ldots \ (2)$$

المعادلة (2) معادلة تفاضلية خطية متجانسة لان *f(x)*=0

وعليه المعادلة (1) تسمى معادلة تفاضلية غير متجانسة عندما *f(x)*≠0

Example: The following equations are LDE with constant coefficients

المعادلات الاتية هي معادلات تفاضلية خطية ذات معاملات ثابتة

1) $y''+y'+y=x^3$
2) $y''+y=\cos x$
3) $y''-3y'+y=4x^3+4$
4) $\frac{d^4y}{dx^4}+\frac{d^3y}{dx^3}+\frac{dy}{dx}=0$ نلاحظ ان المعادلة 4 متجانسة

Theorem5.1.1: If the functions $y_1, y_2, \ldots, y_n$ are solutions of the homo. eq. (2) and $c_1, c_2, \ldots, c_n$ are constants, then

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n \quad \ldots\ldots.(3)$$

Is a solution of eq. (2) also.

Definition(5.1.2): The functions $y_1, y_2, \ldots, y_n$ are called (***linear dependent***) on the set I if we found numbers ($c_1, c_2, \ldots c_n$) are not all equal to zero where

$$c_1y_1+c_2y_2+\cdots+c_ny_n=0 \quad \ldots\ldots\ldots.(4)$$

And it is called (***linear independent***) on the set I if we found numbers $(c_1, c_2, \dots c_n\ )$ that all equal to zero (i.e.)

$c_1 = c_2 = \cdots = c_n = 0$ where:

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

linear dependent معتمدة خطيا

linear independent مستقلة خطيا

Section 5.2: The Wronskian Determinant محدد رونسكي

The Wronskian determinant of differentiable functions $y_1, y_2, \dots, y_n$ on the interval I is:

$$W = W(y_1, y_2, \dots y_n) = \begin{vmatrix} y_1 & , & y_2 & , \dots , & y_n \\ y'_1 & , & y'_2 & , \dots , & y'_n \\ y''_1 & , & y''_2 & , \dots , & y''_n \\ & & . & & \\ & & . & & \\ & & . & & \\ y^{(n-1)}{}_1 & , & y^{(n-1)}{}_2 & , \dots , & y^{(n-1)}{}_n \end{vmatrix}$$

Theorem 5.2.1: If the functions $y_1, y_2, \dots, y_n$ are solutions of the homogeneous equation (2) then if:

■ $W = W(y_1, y_2, \dots y_n) = 0$ then $y_1, y_2, \dots, y_n$ are ***linearly dependent solutions***

And if

■ $W = W(y_1, y_2, \dots y_n) \neq 0$ then $y_1, y_2, \dots, y_n$ are ***linearly independent solutions***

Ex1: Prove that the functions $e^x, 2e^x, e^{-x}$ are linearly dependent on the interval $(-\infty, \infty)$.

Sol: $W(e^x, 2e^x, e^{-x}) = \begin{vmatrix} e^x & 2e^x & e^{-x} \\ e^x & 2e^x & -e^{-x} \\ e^x & 2e^x & e^{-x} \end{vmatrix}$

$$= e^x \begin{vmatrix} 2e^x & -e^{-x} \\ 2e^x & e^{-x} \end{vmatrix} - 2e^x \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} + e^{-x} \begin{vmatrix} e^x & 2e^x \\ e^x & 2e^x \end{vmatrix}$$

$$= e^x(2+2) - 2e^x(1+1) + e^{-x}(0)$$

$$= 4e^x - 4e^x = 0$$

Then $e^{x}, 2e^{x}, e^{-x}$ are linearly dependent.

Ex2: Prove that $\mathbf{\sin x}$, $\mathbf{\cos x}$ are two linearly independent solutions of the diff. eq. $y'' + y = 0 \ \ in \ \ -\infty < x < \infty$.

Sol:

Let $y_1 = \sin x$, $y_2 = \cos x$ then

$y'_1 = \cos x$, $y'_2 = -sinx$ and

$y''_1 = -\sin x$, $y''_2 = -\cos x$

So

$$y'' + y = 0$$

1- $y''_1 + y_1 = -cosx + cosx = 0$
2- $y''_2 + y_2 = -sinx + sinx = 0$

كلاهما حل للمعادلة التفاضلية الان نتاكد هل ان الحلول مرتبطة ام مستقلة خطيا؟

$$w(cosx, \sin x) = \begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = \cos^2 x + sin^2 x = 1$$

So the sol. Are linearly independent

Ex3: prove that the function $xe^{x}, x^{2}e^{x}$ are linearly dependent or linearly independent.

Sol:

$$\begin{vmatrix} xe^{x} & x^{2}e^{x} \\ xe^{x} + e^{x} & x^{2}e^{x} + 2xe^{x} \end{vmatrix} = xe^{x}(x^{2}e^{x} + 2xe^{x}) - x^{2}e^{x}(xe^{x} + e^{x})$$

$$= x^{3}e^{x} + 2x^{2}e^{2x} - x^{3}e^{2x} - x^{2}e^{2x}$$

$$= x^{2}e^{2x} \neq 0$$

Then the given functions are linearly independent.

الدوال مستقلة خطيا

H.W1: prove that $\boldsymbol{e^{x}cosx}, \boldsymbol{e^{x}sinx}$ are linearly dependent or linearly independent.

H.W2: prove that $\boldsymbol{y_1 = e^{ax}cosbx, y_2 = e^{ax}sinbx}$ are sol. of diff. eq.

$$\boldsymbol{\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0}$$

then show that the sol. is linearly independent.

<u>Section(5-3) Differential operator:المؤثر التفاضلي</u>

Let D denote differentiation with respect to x that is

$Dy = \frac{dy}{dx}, D^2y = \frac{d^2y}{dx}$ and so on , that is, for positive integral k

$, D^k y = \frac{d^k y}{dx^k}$, k is a positive number.

ان هذه الدوال التفاضلية لاتعني شيئا الا اذا اثرت على دالة ما للمتغير المستقل

For example, $D(cos4x) = -4sin4x$

$$D(5x^3 - 6x^2) = 15x^2 - 12x$$

The expression: $L = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$

Is called a differential operator of order n.

It may be defined as that operator which, when applied to any function y, yields the result:

$$(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y$$

Where $a_0, a_1, \dots, a_{n-1}, a_n$ are coefficients

Properties of operator D: خواص المؤثر التفاضلي

1.Commutative property الخاصية الابدالية

Ex1: let $A = D + 2\ and\ B = 3D - 1\ , prove\ that\ AB = BA$

التحقق من ان المؤثر التفاضلي ابدالي ام لا

Sol: $By = (3D - 1)y = 3\frac{dy}{dx} - y$

And $A(By) = (D + 2)\left(3\frac{dy}{dx} - y\right) = 3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6\frac{dy}{dx} - 2y$

$$= 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y$$

$$= (3D^2 + 5D - 2)y$$

Hence $A.B = (D + 2)(3D - 1) = (3D^2 + 5D - 2)$

Now consider $Ba. \Rightarrow Ay = (D + 2)y = \frac{dy}{dx} + 2y$

$$B(Ay) = (3D - 1)\left(\frac{dy}{dx} + 2y\right) = \frac{3d^2y}{dx^2} + 6\frac{dy}{dx} - \frac{dy}{dx} - 2y$$

$$= 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y$$

$$= (3D^2 + 5D - 2)y$$

Hence $BA = AB$

اذن المؤثر التفاضلي D ابدالي

Remark: The effect factors of a linear differential equation with fixed coefficients are commutative. But if some of these factors are functions then the effect factors may be non- commutative.

عوامل التاثير للمعادلات التفاضلية الخطية ذات المعاملات الثابتة ابدالية اما اذا كانت بعض المعاملات دوالا فيمكن ان تكون عوامل التاثير ليست ابدالية

Ex2.: let $G = xD + 2\ and\ H = D - 1$

$$G(Hy) = (xD + 2)\left(\frac{dy}{dx} - y\right) = x\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \frac{dy}{dx} - 2y$$

$$= x\frac{d^2y}{dx^2} + (-x)\frac{dy}{dx} - 2y$$

So $GH = xD^2 + (2-x)D - 2$

On other hand

$$H(Gy) = (D-1)(x\frac{dy}{dx} + 2y)$$

$$= x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2\frac{dy}{dx} - x\frac{dy}{dx} - 2y$$

$$= x\frac{d^2y}{dx^2} + (3-x)\frac{dy}{dx} - 2y$$

Then $HG = xD^2 + (3-x)D - 2$

$\therefore HG \neq GH$ ليست ابدالية

More properties of the operator D:

2) $D^m f(x) + D^n g(x) = D^n g(x) D^m f(x)$

3) $D^m . D^n = D^n . D^m = D^{m+n}$

(i.e.) $D^m[D^n f(x)] = D^n[D^m f(x)] = D^{m+n} f(x)$

4) $D[c_1 f(x) + c_2 g(x)] = c_1 Df(x) + c_2 Dg(x)$

Where $f(x), g(x)$differential function and c_1, c_2 are constant.

■Theorems about the operator D:

Theorm1: if b was a number then

$$\boldsymbol{f(D)e^{bx} = f(b)e^{bx}}$$

Proof:

$$De^{bx} = be^{bx}, D^2e^{bx} = b^2e^{bx} \ldots\ldots D^n e^{bx} = b^n e^{bx}$$

$$\because f(D)e^{bx} = (D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n)e^{bx}$$

$$= (b^n + a_1 b^{n-1} + \cdots + a_{n-1}b + a_n)e^{bx} = f(b)e^{bx}$$

Example: find $(D^2 + 3D + 2)e^{3x}$

Sol. : $b = 3$, $f(D) = D^2 + 3D + 2$

$$f(3) = 3^3 + 3.3 + 2 = 20$$

$$(D^2+3D+2)e^{3x}=f(3)e^{3x}=20e^{3x}$$

<u>Theorm2</u>: if it is $\boldsymbol{y}$ differentiable function and $\boldsymbol{b}$ is a number then

$$\boldsymbol{f(D).\{e^{bx}y\}=e^{bx}f(D+b)y}$$

Proof:

$$D\{e^{bx}y\}=e^{bx}Dy+be^{bx}y=e^{bx}(D+b)y$$

Also $D^2\{e^{bx}y\}=D\{e^{bx}(D+b)y\}=e^{bx}\{(D+b)(D+b)y\}$

$$=e^{bx}(D+b)^2y$$

By the continue as the same way we get

$$D^n\{e^{bx}y\}=e^{bx}(D+b)^ny$$

$$\therefore f(D)\{e^{bx}y\}=e^{bx}f(D+b)y$$

Example: find $(D^2-4D+1)\{e^{2x}y\}$

Sol.:

$$\because b=2 \quad , \qquad f(D)=D^2-4D+1$$

$$\Longrightarrow f(D+b)=f(D+2)=(D+2)^2-4(D+2)+1$$

$$D^2+4D+4-4D-8+1=D^2-3$$

$$\therefore (D^2-4D+1)\{e^{2x}y\}=e^{2x}(D^2-3)y$$

$$=e^{2x}f(D+2)y$$

<u>Theorm3</u>: if it is $\boldsymbol{b}$ number then

a- $\boldsymbol{f(D^2)\sin bx=f(-b^2)\sin bx}$
b- $\boldsymbol{f(D^2)\,cos\,bx=f(-b^2)\cos bx}$

Proof:

$$D(\sin bx)=b\cos bx$$

$$D^2(sinbx)=-b^2\sin bx$$

$$D^3(sin\,bx)=D^2.D\,sin\,bx=(-b^2).D\,sin\,bx$$

$$D^4(\sin bx) = (-b^2)^2 \sin bx = b^4 \sin bx$$

$$\because f(D^2)\sin bx = (D^{2n} + a_1 D^{2(n-1)} + \cdots + a_{n-1}D^2 + a_n)\sin bx$$

By substitute of b , we get:

$$f(D^2)\sin bx = -[(-b^2)^n + a_1(-b^2)^{n-1} + \cdots + a_{n-1}(-b^2) + a_n]\sin bx$$
$$= f(-b^2)\sin bx$$

And by the same away we prove $f(D^2)\cos bx = f(-b^2)\cos bx$

Example: find

a- $(D^4 + 3D^2 - 1)\sin 2x$
b- $(D^4 + 2D^2)\cos 2x$

Sol.a:

$$b = 2, \qquad f(D^2) = D^4 + 3D^2 - 1$$

$$\Longrightarrow f(-2^2) = \{(-2^2)^2 + (-2^2) - 1\} = 16 - 12 - 1 = 3$$

$\therefore (D^4 + 3\mathrm{D}^2 - 1)\sin 2x = 3\sin 2x$

Sol.b:

$(D^4 - 2D^2)\cos 2x = \{(-2^2) - 2(-2^2)\}\cos 2x$

$= \{16 + 8\}\cos 2x = 24\cos 2x$

Example: **prove that** $(\boldsymbol{D}+\mathbf{1})(\boldsymbol{D}^2+\mathbf{2})\,\mathbf{sin}\,\mathbf{2}\boldsymbol{x} = (\boldsymbol{D}^2+\mathbf{2})(\boldsymbol{D}+\mathbf{1})\,\mathbf{sin}\,\mathbf{2}\boldsymbol{x}$

Sol.:

$(D+1)[D^2 \sin 2x + 2\sin 2x] = (D+1)[-4\sin 2x + 2\sin 2x]$

$= -4D\sin 2x + 2D\sin 2x - 4\sin 2x + 2\sin 2x$

$= -8\sin 2x + 4cos2x - 4sin2x + 2sin2x$

$= -4cos2x - 2sin2x$

الاتجاه الثاني واجب بيتي

H.W

1- Prove $(\mathrm{D}+1)(\mathrm{D}+2\mathrm{x})\mathrm{y} = (\mathrm{D}+2\mathrm{x})(\mathrm{D}+1)\mathrm{y}$

2- Is $(D+x)(D+2x)e^{x}=(D+2x)(D+x)e^{x}$
3- Find $(D^2+1)^2e^{2x}$
4- Find $(D^4+2D^2+1)\cos 3x$
5- Find $(D^3+2D^2)\{\sin 2x+e^{6x})y\}$

(5-4) solution the linear differential equation by reduce order to the first order

حل المعادلة التفاضلية الخطية بتخفيض رتبتها الى الرتبة الاول

قبل اعطاء الصيغة العامة لهذه الطريقة سوف نعطي بعض الامثلة التوضيحية

Example1: **solve the eq.** $\boldsymbol{y''-2y'-3y=5e^{-2x}}$

سوف نتبع الخطوات الاتية

Sol: $(D^2-2D-3)y=5e^{-2x}$ 1- نكتب المعادلة بدلالة المؤثر D

$(D+1)(D-3)y=5e^{-2x}$ 2- نفرض فرضية $(D-3)y=u$

3- نعوض هذه الفرضية في المعادلة السابقة فنحصل على معادلة خطية

$(D+1)u=5e^{-2x} \qquad \longrightarrow \frac{du}{dx}+u=5e^{-2x}$

$I=e^{\int dx}=e^{x}$ integral factor

$u.e^{x}=\int e^{x}.5e^{-2x}dx+c$

$[u.e^{x}=-5e^{-x}+c].e^{-x}$

$\longrightarrow u=-5e^{-2x}+ce^{-x}$

$\longrightarrow (D-3).y=u \longrightarrow (D-3)y=-5e^{-2x}+ce^{-x}$

$$I=e^{-3x} \quad ; \quad e^{-3x}.y=\int e^{-3x}(-5e^{-2x}+ce^{-x})dx+c_1$$

$$\longrightarrow e^{-3x}.y=\left(e^{-5x}-\frac{c}{4}e^{-4x}+c_1\right)$$

$\longrightarrow y=e^{-2x}-\frac{c}{4}e^{-x}+c_1e^{3x}$ the general sol.

Example2: solve $\boldsymbol{y'''+2y''+5y'-6y=0}$

Sol:

$(D^3 + 2D - 5D - 6)y = 0$

$D = -1$ is the root of equation

اي اذا نعوض (-1) سوف تصفر المعادلة بحيث يحقق المعادلة

$D = -1 \Longrightarrow (D + 1) = 0$

$\frac{D^3+2D^2-5D-6}{D+1} = D^2 + D - 6$

$\Longrightarrow (D + 1)(D^2 + D - 6).y = 0$

$(D + 1)(D + 3)(D - 2).y = 0$

Let $u = (D + 3)(D - 2)y$ نعوض في المعادلة

$\Longrightarrow (\mathrm{D} + 1)\mathrm{u} = 0 \to \frac{du}{dx} + u = 0$

$\to \int \frac{du}{u} + \int dx = 0 \quad \to \ln u + x = c$

$\to u = e^{c-x}$

$\to u = e^c.e^{-x} \quad , \quad e^c = A$

$\to u = Ae^{-x}$

يتم التعويض عن u في الفرضية

$(D + 3)(D - 2)y = u$

$(D + 3)(D - 2)y = Ae^{-x}$

Let $u_1 = (D - 2)y$

$(D + 3)u_1 = Ae^{-x} \quad \longrightarrow \frac{du_1}{dx} + 3u_1 = Ae^{-x}$ linear

$e^{3x}.u_1 = \int e^{3x}.Ae^{-x}dx + c_1$

عامل التكامل e^{3x}

$e^{3x}.u_1 = A\int e^{2x}dx + c_1 \to (e^{3x}u_1 = \frac{A}{2}e^{2x} + c_1)e^{-3x}$

$\to u_1 = \frac{A}{2}e^{-x} + c_1e^{-3x} \quad ; \quad let\ \frac{A}{2} = c_2$

$\to u_1 = c_2e^{-x} + c_1e^{-3x}$

نعوض عن قيمة u_1 في الفرضية

$\rightarrow (D-2)y = c_1 e^{-3x} + c_2 e^{-x} \rightarrow \dfrac{dy}{dx} - 2y = c_1 e^{-3x} + c_2 e^{-x}$

خطة الحل واجب بيتي

$\rightarrow y = \dfrac{-c_1}{5} e^{-3x} + \dfrac{-c_2}{3} e^{-x} + c_3 e^{2x}$

$\rightarrow y = Ke^{-3x} + Be^{-x} + c_3 e^{2x}$ الحل العام

The general rule of (5-4)

We can find a general solution for linear diff. eq. with fixed coefficients

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x)$$

Where we first write the differential equation with the effect of D

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = f(x)$$

ثم نكتب الجزء الايسر بدلالة عوامل التاثير

$$(D-m_1)(D-m_2) \ldots (D-m_n) y = f(x)$$

Let $u_1 = (D-m_2)(D-m_3) \ldots (D-m_n) y$

نعوض قيمة u_1 في المعادلة السابقة

$\Longrightarrow (D-m_1) u_1 = f(x)$ linear equation

The general solution of the above equation is

$$u_1 . 2^{-m_1 x} = \int e^{-m_1 x} . f(x) dx + c$$

$$u_1 = e^{m_1 x} \int e^{-m_1 x} f(x) dx + c e^{-m_1 x}$$

نعوض في الفرضية لنحصل على معادلة خطية من الرتبة (n-1)

$$(D-m_2) - (D-m_3) \ldots (D-m_n) . y = e^{m_1 x} \int e^{-m_1 x} f(x) dx + c$$

$(D-m_2) u_2 = e^{m_1 x} \int e^{-m_1 x} f(x) dx$ $\qquad e^{-m_1 x}$ عامل التكامل

$$u_2 . e^{-m_2 x} = \int e^{-m_2 x} . e^{m_1 x} \int e^{-m_1 x} . f(x)(dx)^2$$

ونستمر بهذا الاسلوب حيث سيكون الحل العام للمعادلة التفاضلية الخطية ذات المعاملات الثابتة

$$y = e^{mnx} \int e^{-m_n x} . e^{m_{n-1}} \int \dots \int e^{-m_1 x} f(x)(dx)^2$$

تحسب التكاملات بصورة متتالية

Example: $(D-2)^3 y = 2x$ نقوم بالحل باستخدام الاسلوب السابق

Solution: $(D-2)(D-2)(D-2)y = 2x$

Let $u_1 = (D-2)(D-2)y \rightarrow (D-2)u_1 = 2x$

$\rightarrow \frac{du_1}{dx} - 2u_1 = 2x \quad ; \; I = e^{-2x}$ Integration factor

$$\int (e^{-2x} du_1 - 2u_1 e^{-2x} dx) = \int 2x e^{-2x} dx$$

$$\Longrightarrow e^{-2x} u_1 = \left[-x e^{-2x} - \frac{1}{2} e^{-2x} + c_1 \right] * e^{2x}$$

$$\longrightarrow u_1 = -x - \frac{1}{2} + c_1 e^{2x}$$

$$(D-2)(D-2)y = -x - \frac{1}{2} + c_1 e^{ex}$$

Let $u_2 = (D-2)y \Longrightarrow (D-2)u_2 = -x - \frac{1}{2} + c_1 e^{2x}$

$$I = e^{-2x}$$

$$\Longrightarrow e^{-2x} u_2 = \int -x e^{-2x} dx + \int -\frac{1}{2} e^{-2x} + \int c_1 dx$$

$$\Longrightarrow e^{-2x} u_1 = \frac{x}{2} e^{-2x} + \frac{1}{4} e^{-2x} + \frac{1}{4} e^{-2x} c_1 x + c_2$$

$$\Longrightarrow u_2 = \frac{x}{2} + \frac{1}{4} + \frac{1}{4} + c_1 x e^{2x} + c_2 e^{2x}$$

$$\Longrightarrow u_2 = \frac{x}{2} + \frac{1}{2} + c_1 x e^{2x} + c_2 e^{2x}$$

$$\Longrightarrow (D-2)y = \frac{x}{2} + \frac{1}{2} + c_1 x e^{2x} + c_2 e^{2x}$$

وبنفس الاسلوب نحصل على

$y = -\frac{x}{4} - \frac{3}{8} + (c_1 \frac{x^2}{2} + c_2 x + c_3) e^{2x}$ الحل العام

Exercises :

1- $(D^2 - 1)y = x$
2- $y'' - 6y' + 8y = e^x$
3- $(D^2 - 4D + 3)y = 2x + 1$
4- $(D^3 - 8)y = x$

(5-5)The solution of the homogenous linear diff. eq. of the higher order and constant coefficient

$$\boldsymbol{\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0} \qquad \text{.....………(1)}$$

Where $a_1, a_2, \ldots, a_n$ are constants

Or $(\boldsymbol{D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = 0}$

We can write this eq. by diff. operator as a form

معادلة بدلالة المؤثر

$$f(D)y = (D - m_1)(D - m_2) \ldots (D - m_2)y - 0 \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

And the following equation is called the (Auxiliary equation)

$$f(m) = (m - m_1)(m - m_2) \ldots (m - m_n) = 0$$

As for the equation

$$\boldsymbol{m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n = 0} \ \ldots\ldots\ldots\ldots\ldots\mathbf{(3)}$$

Is called (the characteristic equation) المعادلة المميزة

لاعطاء طريقة عامة لحل هذا النوع من المعادلات نبدا بدراسة حل المعادلة التفاضلية المتجانسة ذات معاملات ثابتة من الرتبة الثانية اولا ثم نقوم بتعميمها بعد ذلك للرتبة n

$$\boldsymbol{\frac{d^2 y}{dx^n} + a_1 \frac{dy}{dx} + a_2 y = 0} \ldots \ldots \ldots \ldots \ldots \mathbf{(4)}$$ معادلة خطية متجانسة من الرتبة الثانية

ولحل هذه المعادلة نستخدم طريقة التخفيض للرتبة الاولى ونحل كما في السابق

نكتب هذه المعادلة بدلالة عوامل التاثير بالصورة

$$(D^2 + a_1 D + a_2)y = (D - m_1)(D - m_2)y = 0$$

Let $u = (D - m_2)y$ .... (*)

$$\Rightarrow (D - m_1)u = 0 \Rightarrow \frac{du}{dx} - m_1 u = 0$$

$I = e^{\int -m_1 dx} \qquad = e^{-m_1 x}$ integrate factor

نضرب الطرفين بعامل التكامل

$$e^{-m_1x}du - m_1ue^{-m_1x}\ dx = 0$$

$$\int d(e^{-m_1x}.u) = \int 0.dx$$

$\rightarrow e^{-m_1x}.u = a \Rightarrow u = ae^{m_1x}$ (*) نعوض في المعادلة

$$\Rightarrow (D - m_2)y = ae^{m_1x} \quad \Rightarrow \frac{dy}{dx} - m_2y = ae^{m_1x}$$

$$I = e^{-m_2x}$$

$$\int (e^{-m_2x}dy - m_2ye^{-m_2x}\ dx) = \int ae^{(m_1-m_2)x}\ dx$$

$$\rightarrow e^{-m_2x}y = \int ae^{(m_1-m_2)x}\ dx \qquad \ldots\ldots\ldots\ldots(5)$$

$$\rightarrow \left\{e^{-m_2x}y = \frac{a}{m_1-m_2}e^{(m_1-m_2)x} + c\right\} \qquad * \ e^{m_2x}$$

$$y = \frac{a}{m_1-m_2}e^{m_1x} + ce^{m_2x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Let $c_1 = \frac{a}{m_1-m_2}$ and $c_2 = c$

Then we get:

$$y = c_1e^{m_1x} + c_2e^{m_2x} \qquad \ldots\ldots\ldots\ldots\ldots(7)$$

نلاحظ ان هذا الحل يكون عندما $m_1 \neq m_2$

اما في حال ان $m_1 = m_2$ فنعوض في المعادلة (5) فنحصل على المعادلة الاتية :

$$\rightarrow e^{-m_2x}y = \int a\ dx$$

$$\rightarrow e^{-m_2x}y = ax + c$$

$$\rightarrow y = axe^{m_2x} + ce^{m_2x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

The solutions that obtained in eqs. (7) and (8) are called the complementary function.

There are three cases of the roots of characteristic eq.

The first case 1: Different Real Roots

If the roots of the characteristic eq.(3) are different and real , let $m_1 and\ m_2$ are two roots of the equation and $m_1 \neq m_2$; $m_1, m_2 \in R$

يكون جذرا المعادلة المميزة مختلفين وحقيقيين

From eq.(7) we note that the complementary function y_c is:

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

In general, for n roots $(m_1, m_2, \dots, m_n)$ the complementary function y_c is:

$$\boldsymbol{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}} \qquad \dots\dots\dots\dots (9)$$

Where $c_1, c_2, \dots, c_n$ are an arbitrary constants.

<u>Example</u>1: find the general sol of the eq.

$$\boldsymbol{y'' + 3y' - 4y = 0}$$

Sol.:

$$(D^2 + 3D - 4)y = 0$$

the Ch. Eq. is نكتب بدلالة المعادلة المساعدة

$$\Rightarrow m^2 + 3m - 4 = 0$$

$$(m + 4)(m - 1) = 0 \Rightarrow \quad m = -4 \quad , \quad m = 1$$

اعداد حقيقية ومختلفة

$\therefore$ the general sol. is

$$y = y_c = c_1 e^{-4x} + c_2 e^{x}$$ الحل العام

Where c_1, c_2 are two arbitrary constants

Example 2 : Solve $\boldsymbol{y''' + 2y'' + 5y' - 6y = 0}$

Sol.:

$(D^3 + 2D^2 - 5D - 6)y = 0$

The Ch. Eq. is

$m^3 + 2m^2 - 5m - 6 = 0$

$(m+1)(m+3)(m-2) = 0$

$\quad\downarrow \qquad\quad \downarrow \qquad\quad \downarrow$

$m_1 = -1, m_2 = -3\,, m_3 = 2$ (three different real roots)

The general sol. is :

$y = c_1e^{-x} + c_2e^{-3x} + c_3e^{2x}$ where $c_1, c_2\ and c_3$ are an arbitrary constants.

The second case 2: Equal Real Roots

If the roots of the characteristic eq.(3) are equal and real suppose that

$m_1 = m_2 \quad ; \quad m_1, m_2 \in R$

Where c_1, c_2 are an arbitrary constant

From eq.(8) we note that the complementary function y_c is:

$y = y_c = c_1e^{m_1x} + c_2xe^{m_1x}$

Or $y = y_c = (c_1 + c_2x)\ e^{m_1x}$

In general, for n roots $(m_1, m_2, \dots, m_n)$ the complementary function y_c is:

$$\boldsymbol{y = y_c = (c_1 + c_2x + \cdots + c_nx^{n-1})\ e^{m_1x}} \qquad \dots\dots(\mathbf{10})$$

Where $c_1, c_2, \dots, c_n$ are an arbitrary constants.

Example3: Find the general solution of the eq.

$$\boldsymbol{y'' - 4y' + 4y = 0}$$

Sol.: $(D^2 - 4D + 4)y = 0$

The Ch. Eq. is:

$$m^2 - 4m + 4 = 0 \qquad \rightarrow \qquad (m-2)^2 = 0$$

$\Longrightarrow m = 2,2$ جذران حقيقيان ومكرران

The general sol. is:

$$y = (c_1 + c_2 x)e^{2x}$$

Where c_1, c_2 are an arbitrary constant

Example4: Find the general solution of the eq

$$(\boldsymbol{D^3 - 5D^2 + 7D - 3})\boldsymbol{y = 0}$$

Sol.: The Ch. Eq. is:

$$m^3 - 5m^2 + 7m - 3 = 0$$

$(m-1)(m-3)(m-3) = 0$

$(m-1)^2(m-3) = 0$

$\downarrow \qquad\qquad \downarrow$

$m_1 = m_2 = 1, m_3 = 3$

جذران حقيقيان مكرران وجذر مختلف

The general solution is:

$y = (c_1 + c_2 x)e^{x} + c_3 e^{3x}$

Where c_1, c_2, c_3 are an arbitrary constants.

<u>The third case : Complex roots</u>

If the roots of the characteristic eq. is complex numbers, suppose m_1 *and* m_2 are complex numbers then

If was $m_1 = a + ib \quad , \quad m_2 = a - ib$

The sol. Of the diff. eq. of the first case is

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x} = c_1 e^{ax}.e^{ibx} + c_2 e^{ax}.e^{-ibx}$

$$= e^{ax}(c_1 e^{ibx} + c_2 e^{-ibx})$$

(Euler's low) قانون اويلر

$$e^{ibx} = cos\, bx + i \sin bx$$

$e^{-ibx} = cos\, bx + i \sin bx$

$\Rightarrow y = e^{ax}[c_1(\cos bx + i \sin bx) + c_2(\cos bx + i \sin bx)]$

$= e^{ax}[c_1 \cos bx + c_1 i \sin bx + c_2 \cos bx - c_2 i \sin bx]$

$= e^{ax}[(c_1 + c_2)\cos bx + i(c_1 - c_2)\sin bx]$

$= e^{ax}[A \cos bx + B \sin bx]$

Where $A = c_1 + c_2$, $B = i(c_1 - c_2)$

∴قانون حالة الجذور العقدي لمعادلة من الرتبة الثانية هو.

$$\boldsymbol{y = y_c = e^{ax}[A \cos bx + B \sin bx]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(11)}$$

Example5 : Find the general sol of the eq.

$$\boldsymbol{y'' + 2y' + 5y = 0}$$

Sol.:

$(D^2 + 2D + 5)y = 0$ نكتب بدلالة المعادلة المساعدة

$m^2 + 2m + 5 = 0$

$\therefore m = \frac{-2 \pm \sqrt{4-20}}{2} = -1 + 2i$

∴ the general sol. $y = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$

Example 6: find the general sol of the eq.

$$\boldsymbol{y'' + 9y = 0}$$

Sol.: $(D^2 + 9)y = 0$

$m^2 + 9 = 0 \qquad \Rightarrow \qquad m^2 = -9 \qquad \Rightarrow \qquad m = \pm 3i$

The general sol.

$y = c_1 \cos 3x + c_2 \sin 3x$

■**Generalization of the homogenous diff. eq. of the order *n***

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

The char. Eq. is:

$$(D - m_1)(D - m_2) \dots (D - m_n) = 0$$

1- If the roots of char. Eq. real and different

اذا كانت جذور المعادلة المميزة حقيقية ومختلفة

Where $m_1, m_2, \dots, m_n$ are the n-th roots of the equation. The solution is

$\boldsymbol{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}}$ **قانون**

2- If the roots of char. Eq. are real and equal

اذا كانت جذور المعادلة المميزة حقيقية ومكررة

$\boldsymbol{y = (c_1 + c_2 x + c_3 x^2 \dots + c_n x^{n-1}) e^{mx}}$ **قانون**

3- If the roots of the char. Eq. are complex numbers and equal, then the sol. is:

$\boldsymbol{y = e^{ax}\left[(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right) \cos bx + (c_{k+1} + c_{k+2} x + c_{k+3} x^2 + \cdots + c_n x^{k-1}) \sin bx}$ **; n=2k** **قانون**

For example:

If n=6→$m_1 = m_2 = m_3 = a + ib$ *and* $m_4 = m_5 = m_6 = a - ib$

Then the complementary solution is:

$y = e^{ax}[(c_1 + c_2 x + c_3 x^2) \cos bx + (c_4 + c_5 x + c_6 x^2) \sin bx$

Example7: find the general sol of the eq.

$$\boldsymbol{(D - 2)^3 (D + 3)^2 (D - 4) y = 0}$$

Sol.: $(m-2)^3(m+3)^2(m-4)=0$

$$\Rightarrow \quad m=2,2,2,-3,-3,4$$

The general sol. is

$$y=(c_1+c_2x+c_3x^2)e^{2x}+(c_4+c_5x)e^{-3x}+c_6e^{4x}$$

Example 8 : Solve the equation:

$$(\boldsymbol{D^4+8D^2+16})\boldsymbol{y}=\boldsymbol{0}$$

Sol.: $m^4+8m^2+16=0$ نكتب بدلالة المعادلة المميزة

$(m^2+4)^2=0 \qquad \Rightarrow \qquad m=\pm 2i \ , \ \pm \ 2i$

$(i.e.)\quad m_1=m_2=2i\,, m_3=m_4=-2i$

جذور عقدية مكررة

$$y=(c_1+c_2x)cos2x+(c_3+c_4x)sin2x$$

Example 9: solve the equation:

$$(\boldsymbol{D^3-3D^2+9D+13})\boldsymbol{y}=\boldsymbol{0}$$

Sol.: $m^3-3m^2+9m+13=0$

$m=-1 \ , \ (-1)^3-3(-1)^2+9(-1)+13=0$ بالتخمين

$$-1-3-9+13=0$$

تحل بالدستور

$$(m+1)(m^2-4m+13)=0$$

$$\begin{array}{r|l} & m^2-4m+13 \\ \hline m+1 & m^3-3m^2+9m+13 \\ & \underline{\pm m^3 \pm m^2} \\ & -4m^2+9m \\ & \underline{\pm 4m^2 \pm 4m} \\ & 13m+13 \\ & \underline{\pm 13m \pm 13} \\ & 0+0 \end{array}$$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{+4\pm\sqrt{16-4.1.13}}{2.1}$$

$$=\frac{4\pm\sqrt{-36}}{2}=2\pm 3i$$

$m_1=-1 \quad , \quad m_2=2+3i \quad , \quad m_3=2-3i$

Then the general sol. Is

$$y = c_1 e^{-x} + c_2 e^{2x} cos3x + c_3 e^{2x} sin3x$$

Example: $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

$$m^3 - 4m^2 + m + 6 = 0$$

$$m = -1 \quad \Rightarrow \quad (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$

$$\Rightarrow (m+1)(m^2 - 5m + 6) = 0$$

$$(m+1)(m-3)(m-2) = 0$$

$\Rightarrow m_1 = -1, m_2 = 3, m_3 = 2$ جذور حقيقية مختلفة

الحل العام هو

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{2x}$$

$$\begin{array}{r|l}
 & m^2 - 5m + 6 \\
\hline
m+1 & m^3 - 4m^2 + m + 6 \\
 & \pm m^3 \pm m^2 \\
\hline
 & -5m^2 + m \\
 & \pm 5m^2 \pm 5m \\
\hline
 & 6m + 6 \\
 & \pm 6m \pm 6 \\
\hline
 & 0 + 0
\end{array}$$

(5-6) the non-homogenous linear equation with constant coefficients

المعادلة الخطية اللامتجانسة ذات المعاملات الثابتة

The general solution of the non-homogenous linear equation is:

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x) \qquad \dots\dots\dots(1)$$

Can be write it as the form:

$$\boldsymbol{f(D)y = f(x)} \qquad \dots\dots\dots(\mathbf{2})$$

$$\boldsymbol{f(D)y = 0} \ \dots\dots\dots\dots(3)$$

الحل العام للمعادلة المتجانسة والذي نسميه بالدالة المتممة $y_c(x)$ مضافا اليه الحل الخاص للمعادلة اللامتجانسة والذي يسمى بالتكامل الخاص $y_p(x)$ وعلى هذا فان الحل العام للمعادلة (2) هو:

$$\boldsymbol{y = y_c(x) + y_p(x)}$$

لقد درسنا بالبند السابق طريقة ايجاد الدالة المتممة للمعادلة اللامتجانسة وذلك بوضع $f(D)y = 0$ ويبقى علينا ايجاد $I(x)$ الذي هو الحل الخاص للمعادلة اللامتجانسة ان واحدة من طرق ايجاد الحل الخاص هو فرض معادلات محدودة تعتمد على طبيعة الدالة $f(x)$ كأن تكون دالة اسية او مثلثية او متعددة حدود

(5-6-1)

1- **<u>Un determined coefficient method:</u>** **<u>طريقة المعاملات غير المحدودة</u>**

<u>The first case</u> If the $f(x)$ was as polynomial متعددة الحدود

The method for finding the particular solution for non- homogenous diff. eq. is the particular solution is polynomial that degree equal to the degree of $f(x)$ then after that we find the coefficient of the polynomial that was gave

---

Example1 : **Solve the eq.** $\boldsymbol{3y'' - 5y' - 2y = 6x^2 - 7}$

Sol.: we first find the complementary function ($y_c(x)$)

$$3y'' - 5y' - 2y = 0$$

ثم نكتب المعادلة المميزة وهي

$$3m^2 - 5m - 2 = 0$$

$$(3m+1)(m-2) = 0 \quad \Rightarrow \quad m = -\frac{1}{3} \quad , \quad m = 2$$

Then the complete function is

$$y_c(x) = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x}$$

The particular sol. $y_p(x)$ we note that $f(x) = 6x^2 - 7$ its polynomial of the 2-degree so we suppose the solution is a polynomial of 2-degree is

$y = ax^2 + bx + c$

we must find the coefficient a, b, c , but the first we find y, y', y'' then we make it up in the original equation

$y' = 2ax + b \quad ; \quad y'' = 2a$

بالتعويض في المعادلة الاصلية

$$3(2a) - 5(2ax + b) - 2(ax^2 + bx + c) = 6x^2 - 7$$

نساوي معاملات قوى x بالطريقة التالية

$$-2a = 6 \quad \Rightarrow \quad a = -3$$

$$-10a - 2b = 0 \quad \Rightarrow \quad -10(-3) - 2b = 0 \quad \Rightarrow \quad b = 15$$

$$6a - 5b - 2c = -7 \quad \Rightarrow \quad 6(-3) - 5(15) - 2c = -7$$

$$\Rightarrow \quad -18 - 75 - 2c = -7$$

$$\Rightarrow \quad c = -43$$

The particular sol. Is $y_p(x) = -3x^2 + 15x - 43$

and the general sol. Is $y = y_c(x) + y_p(x)$

$\Rightarrow \quad y = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x} - 3x^2 + 15x - 43$ الحل العام

<u>The second case</u> : If the function is $f(x) = be^{ax}$

لكي نجد الحل الخاص يجب ان نلاحظ الاتي

- اذا لم يكن a جذرا للمعادلة المميزة فعند ذلك نفرض الحل الخاص للمعادلة اللامتجانسة الفرضية التالية
- $y = Ae^{ax}$ a ليس جذر للمعادلة

حيث ان A هو المجهول ونجده بالتعويض عن y واشتقاقه في المعادله المعطاة

- اما اذا كان a احد جذور المعادلة المميزة وغير مكرر فان الفرضية للحل الخاص للمعادلة اللامتجانسة ستكون
- $y = Axe^{ax}$ a جذر غير مكرر

- واذا كان a احد جذور المعادلة المميزة ومكرر n من المرات فان الفرضية للحل الخاص للمعادلة اللامتجانسة ستكون
- $y = Ax^n e^{ax}$ a جذر مكرر

Example2 : Solve the eq. $3y'' - 5y' - 2y = 3e^{3x}$

Sol.: we find the char. Eq.

$$3y'' - 5y' - 2y = 0 \quad \Rightarrow \quad 3m^2 - 5m - 2 = 0$$

$$\Rightarrow \quad (3m+1)(m-2) = 0 \quad \Rightarrow m = -\frac{1}{3} \ , \quad m = 2$$

$y_c = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x}$

$\because a = 3$ not root of the char. Eq. so we suppose the function as :

$$y = Ae^{3x} \quad \rightarrow \quad y' = 3Ae^{3x} \quad ; \quad y'' = 9Ae^{3x}$$

We sub. $y \ , \ y' \ , \ y''$ by the diff. eq.

$(27 - 15 - 2)Ae^{3x} = 5e^{3x} \quad \rightarrow \ A = \frac{5}{10} = \frac{1}{2}$

So the particular solution is

$y_p(x) = \frac{1}{2} e^{3x}$ and the general sol is

$y = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x} + \frac{1}{2} e^{3x}$

Example2: Find the sol. of $\boldsymbol{3y'' - 5y' - 2y = 5e^{2x}}$

Sol.: suppose the particular sol. is

$y = Axe^{2x}$

Since $a = 2$ is root of the char. Eq. from the last example where $m = -\frac{1}{3}, 2$

نلاحظ ان الجذر غير مكرر اي ان جذر المعادلة المميزة هو احد جذور المعادلة الاصلية ولكنه غير مكرر

$y' = Ae^{2x} + 2Axe^{2x}$

$y'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$

نقوم بالتعويض عن y'', y', y بالمعادلة الاصلية

$$3(2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}) - 5(Ae^{2x} + 2Axe^{2x}) - 2Axe^{2x} = 5e^{2x}$$

$$(12 - 10 - 2)Axe^{2x} + (12 - 5)Ae^{2x} = 5e^{2x}$$

$$\Rightarrow \; 0 + 7Ae^{2x} = 5e^{2x} \;\Rightarrow\; A = \frac{5}{7}$$

$\therefore$ the particular sol. is $y_p(x) = \frac{5}{7}xe^{2x}$

Then the general sol. is $\quad y = c_1e^{2x} + c_2e^{-\frac{1}{3}x} + \frac{5}{7}xe^{2x}$ الحل العام

Example 3: Solve the eq. $\quad \boldsymbol{y'' - 4y' + 4y = x^2e^{2x}}$

Sol.: the char. Eq. of homogenous eq. is

$y'' - 4y' + 4y = 0 \quad \Rightarrow \quad m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m - 2)(m - 2) = 0$

وبما ان $a = 2$ وان جذور المعادلة المميزة هو 2 ومكرر مرتين اذن نفرض الحل الخاص بدالة

$y = Ax^2e^{2x}$

$y' = 2Ax^2e^{2x} + 2Axe^{2x}$

$y'' = 4Ax^2e^{2x} + 4Axe^{2x} + 2Ae^{2x} + 4Axe^{2x}$

وبالتعويض عن y, y', y'' بالمعادلة التفاضلية نجد ان

$$4Ax^2e^{2x} + 4Axe^{2x} + 2Ae^{2x} + 4Axe^{2x} - 4(2Ax^2e^{2x} + 2Axe^{2x}) + 4(Ax^2e^{2x}) = x^2e^{2x}$$

$4Ax^2 + 4Ax + 2A + 4Ax - 8Ax^2 - 8Ax + 4Ax^2 = x^2$

$$\Rightarrow \quad 2A = x^2 \qquad \Rightarrow \quad A = \frac{x^2}{2}$$

Then the particular solution is:

$y_p = \frac{1}{2}x^4e^{2x}$

The general sol. is :

$y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2} x^4 e^{2x}$

<u>The third case</u> If the function was $f(x) = b\sin ax$ *or* $f(x) = b\cos ax$

To find the particular sol. of non-homogenous equation as this case suppose that

$\rightarrow$ $y = A\cos ax + B\sin ax$

اذا لم يكن $m = ai$ جذرا للمعادلة المميزة

Where A , B unknown

— If it is not $m = ai$ root of char. Eq.

ثم نعوض عن y واشتقاقاته بالمعادلة التفاضلية حيث نجد B , A

But if it is $m = ai$ one of root of char. Eq. and not repeater suppose that as this case

$\rightarrow$ $y = x(A\cos ax + B\sin ax)$

اذا كان $m = ai$ جذرا للمعادلة المميزة وغير مكرر

If it is $m = ai$ one of root char. Eq. and repeater suppose that as this case

$\rightarrow$ $y = x^n(A\cos ax + B\sin ax)$

Example 4: Solve the eq. $3y'' - 5y' - 2y = 4\sin 2x$

Solution:

نلاحظ انه ليس للمعادلة المميزة جذورا عقدية حيث $m = -\frac{1}{3}, 2$

The particular sol. is

$$y = A\cos 2x + B\sin 2x$$

$$y' = -2A\sin 2x + 2B\cos 2x$$

$$y'' = -4A\cos 2x - 4B\sin 2x$$

ثم نعوض عن y'' , y' , y بالمعادلة الاصلية ونجد ان

$$3(-4A\cos 2x - 4B\sin 2x) - 5(-2A\sin 2x + 2B\cos 2x)$$
$$- 2(A\cos 2x + B\sin 2x) = 0\cos 2x + 4\sin 2x$$

نساوي معاملات $\cos 2x$, $\sin 2x$ نحصل على

$$-14A - 10B = 0$$

$$10A - 14B = 4$$

$$\Rightarrow A = \frac{5}{37} \qquad , \qquad B = \frac{-7}{37}$$

$\therefore$the particular sol. is $\quad y = \frac{5}{37}\cos 2x - \frac{7}{37}\sin 2x$

$\therefore$ the general sol. is

$$y = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x} + \frac{5}{37}\cos 2x - \frac{7}{37}\sin 2x$$

(5-6-1)

Inverse operator method: طريقة المؤثر العكسي

The Particular sol. of the linear differential eq. is

$\boldsymbol{F(D)y = f(x)}$ ……………………………………………………… (1)

With the constant coefficient is given as the form

$\boldsymbol{y = \frac{1}{F(D)} f(x)}$ ……………………………………………… ……………(2)

Where $F(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$.

And $\frac{1}{F(D)}$ is called the inverse operator

وباستخدام خواص المؤثر التي تم شرحها سابقا

1- $F(D)e^{bx} = F(b)e^{bx}$
2- $F(D)\{e^{bx}y\} = e^{bx}F(D+b)y$
3- $F(D^2)\sin bx = F(-b^2)\sin bx$
4- $F(D^2)\cos bx = F(-b^2)\cos bx$

So to find the particular sol. for non-homogenous eq. we take the following cases
لايجاد الحل الخاص للمعادلة التفاضلية الخطية اللامتجانسة سوف نتعامل مع الحالات الاتية

The first case الحالة الاولى

If $\boldsymbol{f(x) = e^{bx}}$ then there are two cases

1- If $F(b) \neq 0$, then the particular sol. of the eq. is:

$$\boxed{\boldsymbol{F(D)y = e^{bx} \quad \Rightarrow \quad y = \frac{1}{F(D)}.e^{bx} = \frac{1}{F(b)}e^{bx}} \quad \ldots\ldots\ldots\ldots (3)}$$

لان الدالة $F(b)$ في المقام لكي يتحقق الشرط لابد ان تكون لاتساوي صفر

2- If $F(b) = 0$, then the particular sol. will be as a form

$$\boxed{\boldsymbol{y = \frac{1}{G(b)} \cdot \frac{x^r}{r!} e^{bx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(4)}}$$

where $F(D) = (D - b)^r G(D)$, r is a positive integer.

اي في الحالة الثانية عندما يكون المؤثر =صفر فلايجوز تطبيق الحالة الاولى عندئذ نحلل المؤثر $F(D)$فنحصل على عامل من نوع $(D-b)^r$ وهو المتسبب في جعل المؤثر= صفر ويبقى من المعادلة $G(b) \neq 0$ وتكون النتيجة حسب القانون اعلاه

Example1: **Solve the eq.** $\boldsymbol{y''' - 2y'' - 5y' + 6y = e^{4x}}$

Sol.: the Eq. is

$(D^3 - 2D^2 - 5D + 6)y = e^{4x}$

1-to find y_c

$(D - 1)(D - 3)(D + 2)y = 0$

the char. Eq. is

$(m - 1)(m - 3)(m + 2) = 0$

$\rightarrow \; m = 1 \; ,3 \; ,-2$

$\therefore \; y_c = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$

Where $c_1, c_2 \& c_3$ are arbitrary constants

2- the particular sol. y_p is :

$$y = \frac{1}{(D-1)(D-3)(D+2)} . e^{4x}$$

$F(D) = (D - 1)(D - 3)(D + 2)$, b=4

$\rightarrow F(4) = (4 - 1)(4 - 3)(4 + 2) = 18 \neq 0$

Note that $F(b) \neq 0$

$\therefore y_p = \frac{1}{F(b)} e^{bx} \rightarrow y_p = \frac{1}{18} e^{4x}$

The general sol. is

$y = c_1e^x + c_2e^{3x} + c_3e^{-2x} + \frac{1}{18}e^{4x}$

Example 2: Solve the eq. $\boldsymbol{y''' - 2y'' - 5y' + 6y = e^{3x}}$

Sol.: the comp. fun.y_c is: من المثال السابق

$y_c = c_1e^x + c_2e^{3x} + c_3e^{-2x}$

To find the particular sol.y_p since b=3

$F(b) = (b-1)(b-3)(b+2) \to F(3) = (3-1)(3-3)(3+2) = 0$
الحالة الثانية

$$y_p = \frac{1}{(D-1)(D-3)(D+2)}.e^{3x}$$

G(3)=(3-1)(3+2)=10≠ 0 هو الذي يصفر المقام وان $(D-3)$نلاحظ ان

r=1

The part. Sol.y_p is

$y_p = \frac{1}{G(b)} \cdot \frac{x^r}{r!}e^{bx} = \frac{1}{10}xe^{3x}$

<u>Or:</u>

$$y_p = \frac{1}{(D-3)^1}\left\{\frac{1}{(D-1)(D+2)}e^{3x}\right.$$

$$= \frac{1}{(D-3)^1}\left\{\frac{1}{10}e^{3x}\right\} = \frac{1}{10}\frac{x^1}{1i}e^{3x} = \frac{1}{10}xe^{3x}$$

<u>**Remark:**</u> $\frac{1}{D}f(x) = \int f(x)dx$

Prove: let $\frac{1}{D}f(x) = z$ $\Rightarrow$ $f(x) = Dz = \frac{dz}{dx}$

$\therefore \quad z = \int f(x)dx = \frac{1}{D}f(x)$

$\frac{1}{D}$ التاثير العكسي للمؤثر D اي ان $\frac{1}{D}$ يمثل التكامل بالنسبة الى x بينما $\frac{1}{D^k}$ يمثل التكامل بالنسبة الى x عدد k من المرات

$$\therefore \ \frac{1}{D^1} = \int dx = x$$

$$\therefore \ \frac{1}{(D-3)^2} = \iint (dx)^2$$

$$= \frac{1}{2}x^2$$

$$\therefore \ \frac{1}{(D-3)^3} = \iiint (dx)^3 = \frac{1}{3}x^3$$

Note: $\frac{1}{D-3}e^{3x} = \frac{x^1}{1i}e^{3x}$

Proof:

Let $\quad z = \frac{1}{D-3}e^{3x}$

$$\Rightarrow \quad \frac{dz}{dx} - 3z = e^{3x} \qquad \longrightarrow \qquad z.e^{-3x} = \int e^{-3x}dx$$

$$\Rightarrow \quad z.e^{-3x} = x \qquad \longrightarrow \qquad z = xe^{3x}$$

Example3: Solve the eq. $\boldsymbol{y''' + y'' - y' - y = e^x}$

Sol.: $(D^3 + D^2 - D - 1)y = 0$

$$D(D^2 - 1) + (D^2 - 1)y = 0$$

$$(D^2 - 1)(D + 1)y = 0$$

$$(D - 1)(D + 1)(D + 1)y = 0$$

$$\rightarrow \ (D - 1)(D + 1)^2 y = 0$$

Then the char. Eq. is

$(m-1)(m+1)^2 = 0$

$m_1 = m_2 = -1, m_3 = 1$

$y_c = (c_1 + c_2 x)e^{-x} + c_3 e^x$

To find the particular sol.

$F(b) = (b-1)(b+1)^2$, b=1

$F(1) = (1-1)(1+1)^2=0$

$=0, \therefore r=1 \quad =4=G(1)$

$y_p = \frac{1}{(D-1)(D+1)^2} e^x$

$= \frac{1}{4}\frac{x}{1!} e^x = \frac{x}{4} e^x$

Then the general sol. is:

$y = y_C + y_P$

$= (c_1 + c_2 x)e^{-x} + c_3 e^x + \frac{x}{4} e^x$

Example4: Solve the eq. $\boldsymbol{y''' - 8y = 10e^{3x}}$

Solution: $(D^3 - 8)y = 0 \quad \rightarrow \quad (D-2)(D^2 + 2D + 4)y = 0$

The char. Eq. $(m-2)(m^2 + 2m + 4) = 0$

$\rightarrow \; m = 2, -1 \pm i\sqrt{3}$

$y_c = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x)$

The particular sol. is

$F(D) = D^3 - 8$

$F(b) = b^3 - 8 \rightarrow F(3) = 27 - 8 = 19 \neq 0$

$$y_p = 10\frac{1}{f(D)}e^{3x} = 10\frac{1}{f(3)}e^{3x} = \frac{10}{19}e^{3x}$$

The general sol. is:

$$y = c_1 e^{2x} + e^{-x}\left(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x\right) + \frac{10}{19} e^{3x}$$

The second case الحالة الثانية

If $\boldsymbol{f(x) = \sin bx}$ *or* $\boldsymbol{f(x) = \cos bx}$ to find the particular solution in this case we can using one of the three cases

A) from $e^{ibx} = \cos bx + i \sin bx$ we get:

$$\boldsymbol{cosbx = \frac{e^{ibx}+e^{-ibx}}{2}} \quad \text{and} \quad \boldsymbol{sinbx = \frac{e^{ibx}-e^{-ibx}}{2i}}$$

After that we find the particular solution of e^{ibx} as in the first case when $F(D) = 0$.

اي ان في الحالة الاولى سوف نستخدم قانون اويلر لتحويل الدوال المثلثية(sin bx, cos bx) الى دوال اسية والعمل على الحالة الاولى ويتم اللجوء الى هذه الطريقة في حالة ان تكون $(F(D) = 0)$ اي ان المقام يصبح صفرا بعد التعويض وهذا لايجوز لذلك نلجا الى الطريقة الاولى

B) If $f(x) = \sin bx$ *or* $\cos bx$ then the part. Sol. is:

$$\left.\begin{aligned} &\boldsymbol{y_p = \frac{1}{F(D^2)}\sin bx \Rightarrow y_p = \frac{1}{F(-b^2)}\sin bx \quad , \quad F(-b^2) \neq 0} \\ &\boldsymbol{or\ \ y_p = \frac{1}{F(D^2)}\cos bx \Rightarrow y_p = \frac{1}{F(-b^2)}\cos bx \quad , \quad F(-b^2) \neq 0} \end{aligned}\right] \quad (5)$$

C) If we have the following formula:

$\frac{1}{D^3+a^3}.\cos bx$ or when F(D) of odd order then the particular Sol. is:

عندما يكون المؤثر ذو رتبة فردية او ظهر فيه رتب فردية فالحل الخاص سيكون باستخدام طريقة العامل المرافق كالاتي:

$$\frac{1}{D^3+a^3}.\cos bx = \frac{1}{a^3-b^2D}.\cos bx$$

$$= \frac{a^3+b^2D}{a^6-b^4D^2}.\cos bx \qquad (\text{نضرب بالعامل المرافق})$$

$$= \frac{a^3+b^2D}{a^6+b^6}.\cos bx$$

$$= \frac{1}{a^6+b^2}(a^3\cos bx - b^3\sin bx)$$

Example1: **Solve the diff. eq.** $(\boldsymbol{D^2+4})\boldsymbol{y} = \mathbf{sin}\,\boldsymbol{4x}$

Solution: the complementary function is:

$$y_c = c_1\cos 2x + c_2\sin 2x$$

The particular sol is

$F(D^2) = D^2 + 4$

$F(-b^2) = -b^2 + 4$

$y_p = \frac{1}{D^2+4}\sin 4x = \frac{1}{-(4)^2+4}\sin 4x = -\frac{1}{12}\sin 4x$

The general solution is $y = y_c + y_p$

$y = c_1\cos 2x + c_2\sin 2x - \frac{1}{12}\sin 4x$

Example2: Solve the diff. eq. $\boldsymbol{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \cos 3x}$

Solution: $(D^2 - 2D + 1)y = 0$

$(D-1)^2y = 0$

the char. Eq. is $(m-1)^2 = 0 \quad \rightarrow \quad m = 1,1$

$\therefore \; y_c(x) = (c_1 + c_2x)e^x$

الحل الخاص هو

$$y_p = \frac{1}{D^2 - 2D + 1}.\cos 3x$$

$$= \frac{1}{-3^2 - 2D + 1}.\cos 3x$$

$$= -\frac{1}{2}.\frac{1}{(D+4)}.\cos 3x = -\frac{1}{2}\frac{D-4}{(D^2-4^2)}\cos 3x$$

$$= -\frac{1}{2}(D-4)\frac{1}{-3^2-4^2}\cos 3x = \frac{1}{50}(D\cos 3x - 4\cos 3x)$$

$$= \frac{1}{50}(-3\sin 3x - 4\cos 3x)$$

$$\therefore y = y_c(x) + y_p(x)$$

$$= (c_1 + c_2 x)e^x + \frac{1}{50}(-3\sin 3x - 4\cos 3x)$$

Example3: Solve the eq. $\boldsymbol{y'' + 4y = 8\cos 2x}$

Solution: the char. Eq. is $m^2 + 4 = 0 \quad \rightarrow \quad m = \pm 2i$

$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$

We know that $F(D^2) = D^2 + 4 \rightarrow F(-2^2) = -2^2 + 4 = 0$; b=2

بما ان المقام صفر لايجوز الحل بالطرق المباشرة فنستخدم قانون اويلردالة cos2x

$$\Longrightarrow \quad (D^2+4)y = 8\frac{e^{2ix} + e^{-2ix}}{2}$$

حيث نجد الجزء الحقيقي يمثل الحل الخاص للدالة(e^{2ix}) لان الدالة الاصلية $\cos x$

$$y_p = \underbrace{\frac{4}{(D-2i)(D+2i)}}_{u} e^{2ix} + \underbrace{\frac{4}{(D-2i)(D+2i)}}_{w} e^{-2ix}$$

To find u : b=2i then:

$$u = \frac{4}{4i}\cdot\frac{x}{1!}e^{2ix} = \frac{1}{i}xe^{2ix}$$

To find w; b=-2i then:

$$w = \frac{4}{-4i}\cdot\frac{x}{1!}e^{-2ix} = -\frac{1}{i}xe^{-2ix}$$

$$y_p = u + w = \frac{1}{i} x e^{2ix} - \frac{1}{i} x e^{-2ix}$$

$$= 2\frac{xe^{2ix} - xe^{-2ix}}{2i}$$

$$= 2x \sin 2x$$

Example4: solve the eq. $(D^2 + 3D - 4)y = \sin 3x$

Solution: the char. Eq. $(m-1)(m+4) = 0 \quad \rightarrow y_c = c_1 e^x + c_2 e^{-4x}$

الان نجد الحل الخاص

$$y_p = \frac{1}{D^2 + 3D - 4} . \sin 3x$$

$$= \frac{1}{-9+3D-4} \sin 3x = \frac{1}{3D-13} \sin 3x = \frac{3D+13}{9D^2-169} \sin 3x$$

$$= \frac{-1}{250}(3D + 13) \sin 3x$$

$$= \frac{-1}{250}(9 \cos 3x + 13 \sin 3x)$$

The third case الحالة الثالثة

If the $f(x)$ is a polynomial of x to find the particular solution of the eq. $F(D)y = x^m$ where m is a positive integer number we write $\frac{1}{F(D)}$ as a power ascending to D

For example if $F(D) = 1 - D$

a- $\frac{1}{f(D)} = \frac{1}{1-D} = 1 + D + D^2 + D^3 + \cdots + D^m + \cdots$

b- $\frac{1}{f(D)} = \frac{1}{1+D} = 1 - D + D^2 - D^3 + \cdots + D^m + \cdots$

c- $\frac{1}{f(D)} = \frac{1}{(1+D)^2} = 1 - 2D + 3D^2 - 4D^3 + \cdots + (m+1)D^m + \cdots$

d- $\frac{1}{f(D)} = \frac{1}{(1-D)^2} = 1 + 2D + 3D^2 + 4D^3 + \cdots + (m+1)D^m + \cdots$

Since the $D^{m+r}x^m = 0$ the particular solution is

$$y = \frac{1}{f(D)}.x^m = (1 + D + D^2 + D^3 + \cdots + D^m)x^m$$
$$= x^m + mx^{n-1} + \cdots + m!$$

Example to explain: $D^2x^1 = 0$, $D^3x^2 = 0$

Example1: solve the diff. eq. $y'' + 4y = 8x^3$

Solution: $D^4 + 4 = 0 \quad \rightarrow \; D = \pm 2i$

the char. Eq. is وان الدالة المتممة هي

$m^4 + 4 = 0 \quad \rightarrow \; m = \pm 2i$

$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$

The part. Sol. is

$$y_p = \frac{8}{4+D^2}.x^3 = \frac{8}{4\left(1+\frac{D^2}{4}\right)}.x^3$$

$$= 2\left(1 - \left(\frac{D^2}{4}\right) + \left(\frac{D^2}{4}\right)^2 + \cdots\right)x^3 = 2\left(x^3 - \frac{6x}{4}\right) = 2x^3 - 3x$$

The general sol. is

$y = c_1 \cos 2x + c_2 \sin 2x + 2x^3 - 3x$

Example 2: Solve the diff. eq. $(2D^2 - 5D - 3)y = x^2 + 2x - 1$

Solution: $y_c = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$ الدالة المتممة

$$y_p = \frac{1}{2D^2 - 5D - 3}(x^2 + 2x - 1) = \frac{1}{-3\left(1 + \left(\frac{5}{3}D - \frac{2}{3}D\right)^2\right)}.(x^2 + 2x - 1)$$

$$= -\frac{1}{3}\left(1 - \left(\frac{5}{3}D - \frac{2D^2}{3}\right) + \left(\frac{5}{3}D - \frac{2D^2}{3}\right)^2 + \cdots\right)(x^2 + 2x - 1)$$

$$= -\frac{1}{3}(x^2 + 2x - 1) - \frac{5}{3}(2x + 2) - \frac{2}{3}(2) + \frac{25}{9}(2)$$

$$= -\frac{1}{3}x^2 - 4x + \frac{11}{9}$$

The fourth case الحالة الرابعة

If the $f(x) = e^{ax}v(x)$ where

$$v(x) = \begin{cases} x^m \\ \sin bx \\ \cos bx \end{cases}$$

فان الحل الخاص هو

$$\boldsymbol{y_p(x) = \frac{1}{F(D)}.e^{ax}.v(x) = e^{ax}\frac{1}{F(D+a)}.v(x)} \quad \text{..............................(6)}$$

Example1: Solve the diff. eq. $(\boldsymbol{D^2 + 2D + 5})\boldsymbol{y = xe^x}$

Solution: $(D^2 + 2D + 5)y = 0$

The char. Eq. is:

$\rightarrow \quad m^2 + 2m + 1 = -4$

$(m+1)^2 = -4 \quad \rightarrow \quad m + 1 = \pm 2i \rightarrow \; m = -1 \pm 2i$

$\therefore \; y_c(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$

$$y_p(x) = \frac{1}{D^2+2D+5}.x.e^x = e^x\frac{1}{(D+1)^2+2(D+1)+5}x$$

$$= e^x\frac{1}{8\left(1+\frac{D}{2}+\frac{D^2}{8}\right)}.x$$

$$= \frac{1}{8}e^x\left(1 - \frac{1}{2}D + \cdots\right)x$$

$$= \frac{1}{8}e^x\left(x - \frac{1}{2}\right)$$

The general sol. is:

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{8}e^{x}\left(x - \frac{1}{2}\right)$$

Example2: Solve the eq. $\boldsymbol{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^{3x}}$

Solution:

$(D^2 - 2D + 1)y = 0$

لايجاد الدالة المتممة $y_c(x)$ نكتب المعادلة المميزة

$m^2 - 2m + 1 = 0 \quad \rightarrow \quad (m-1)^2 = 0 \quad \rightarrow \quad m_1 = m_2 = 1$

$\therefore \quad y_c(x) = (c_1x + c_2)e^{x}$

لايجاد الحل الخاص نكتب بدلالة المؤثر D

$$(D^2 - 2D + 1)y = x^2e^{3x}$$

$$\rightarrow \quad y_p = \frac{1}{D^2 - 2D + 1}.x^2.e^{3x}$$

$$= e^{3x}\frac{1}{(D+3)^2 - 2(D+3) + 1}.x^2$$

$$= e^{3x}\frac{1}{D^2 + 6D + 9 - 2D - 6 + 1}.x^2$$

$$= e^{3x}\frac{1}{D^2 + 4D + 4}.x^2$$

$$= e^{3x}\frac{1}{4\left(1 + D + \frac{D^2}{4}\right)}.x^2$$

$$= \frac{1}{4}e^{3x}\left(1 - \left(\frac{D^2}{4} + D\right) + \left(\frac{D^2}{4} + D\right)^2 + \cdots\right).x^2$$

$$= \frac{1}{4}e^{3x}\left(x^2 - \frac{1}{2} - 2x + 2\right)$$

$$= \frac{1}{4}e^{3x}\left(x^2 - 2x + \frac{3}{2}\right)$$

$$\rightarrow \quad y = (c_1x + c_2)e^{x} + \frac{1}{4}e^{3x}(x^2 - 2x + \frac{3}{2})$$

Example3: solve the diff. eq. $(\boldsymbol{D^2+4D+4})\boldsymbol{y}=\boldsymbol{e^x}\,\mathbf{sin}\,\boldsymbol{2x}$

Solution: $y_c=c_1e^{-2x}+c_2xe^{-2x}$

The particular sol. is

$$y_p=e^x\frac{1}{(D+1)^2+4(D+1)+4}.\sin 2x$$

$$=e^x\frac{1}{D^2+2D+1+4D+4+4}.\sin 2x$$

$$=e^x\frac{1}{D^2+6D+9}\sin 2x$$

$$=e^x\frac{1}{-4+6D+9}.\sin 2x$$

$$=e^x\frac{1}{6D+5}\sin 2x$$

$$=e^x\frac{6D-5}{36D^2-25}\sin 2x$$

$$=e^x\frac{-1}{169}(6D-5)\sin 2x$$

$$y_p=-\frac{e^x}{169}(12\cos 2x-5\sin 2x)$$

The general sol. is:

$$y=y_c+y_p$$

$$y=c_1e^{-2x}+c_2xe^{-2x}-\frac{e^x}{169}(12\cos 2x-5\sin 2x)$$

The fifth case الحالة الخامسة

If $\boldsymbol{f(x)=x^m v(x)}$ where $\boldsymbol{v(x)}=\mathbf{cos}\,\boldsymbol{ax}$ **or** $\boldsymbol{v(x)}=\mathbf{sin}\,\boldsymbol{ax}$

to find the particular solution for this case we take the following examples.

Example1: Solve the eq. $(\boldsymbol{D^2+1})\boldsymbol{y}=\boldsymbol{x^2\sin 2x}$

Solution: To find y_c, we have

$$m^2+1=0 \quad \rightarrow \quad m=\pm i$$

$$\therefore \quad y_c(x)=c_1\cos x+c_2\sin x$$

$$y_p(x)=\frac{1}{D^2+1}.x^2.\sin 2x=e^{2ix}.\frac{1}{(D+2i)^2+1}x^2$$

$$=e^{2ix}\frac{1}{D^2+4iD-3}.x^2=e^{2ix}\frac{1}{-3\left(1-\frac{4i}{3}D-\frac{D^2}{3}\right)}.x^2$$

$$=-\frac{1}{3}e^{2ix}\left[1+\frac{4iD}{3}+\frac{D^2}{3}+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)+\cdots\right].x^2$$

$$=-\frac{1}{3}e^{2ix}\left(1+\frac{4iD}{3}+\frac{D^2}{3}-\frac{16D^2}{9}+\cdots\right)x^2$$

$$=-\frac{1}{3}e^{2ix}\left(x^2+\frac{8ix}{3}-\frac{26}{9}\right)$$

$$=-\frac{1}{3}(\cos 2x+i\sin 2x)\left[\left(x^2-\frac{26}{9}\right)+\frac{8ix}{3}\right]$$

$$=-\frac{1}{3}\left[\left(x^2-\frac{26}{9}\right)\sin 2x+\frac{8x}{3}\cos 2x\right]$$

$$y=y_c(x)+y_p(x)$$

Example2: Find the general sol. of $(\boldsymbol{D+1})\boldsymbol{y}=\boldsymbol{x\sin x}$

Solution: The char. Eq. is: $m+1=0 \quad \rightarrow \quad m=-1$

$\rightarrow \; y_c=c_1e^{-x}$ the comp. sol.

The particular sol.

$$\rightarrow \quad y_p=\frac{1}{D+1}x\sin x$$

$$\rightarrow \quad y_p=\frac{1}{D+1}\{xe^{ix}\}$$

$$\rightarrow \quad y_p=e^{ix}\frac{1}{D+i+1}\cdot x$$

$$\to \quad y_p = e^{ix}\frac{1}{(i+1)(1+\frac{D}{i+1})}x$$

$$\to \quad y_p = \frac{e^{ix}}{i+1}.\frac{1-i}{1-i}\left(1-\frac{D}{i+1}+\cdots\right)x$$

$$\to \quad y_p = \frac{e^{ix}(1-i)}{2}\left(x-\frac{1}{i+1}\right)$$

$$= \frac{e^{ix}}{2}(x(1-i)-\frac{1-i}{1-i}.\frac{1-i}{1-i}$$

$$= \frac{\cos x + i\sin x}{2}(x(1-i)+i))$$

$$= \frac{x}{2}(\sin x - \cos x) + \frac{1}{2}\cos x$$

$$y = y_c + y_p$$

في حالة عدم امكانية تطبيق الطريقة السابقة نستخدم القانون التالي

$$\boldsymbol{y = \frac{1}{F(D)}.xu(x) = x\frac{1}{F(D)}.u(x) - \frac{F'(D)}{[F(D)]^2}.u(x)}$$

Example: Find the p. sol. of the diff. eq. $(\boldsymbol{D^2+3D+2})\boldsymbol{y} = \boldsymbol{x\cos 2x}$

Sol: $y_p = x.\frac{1}{D^2+3D+2}.\cos 2x - \frac{2D+3}{(D^2+3D+2)^2}.\cos 2x$

$$= x.\frac{1}{3D-2}.\cos 2x - \frac{2D+3}{D^4+6D^3+13D^2+12D+4}.\cos 2x$$

$$= x.\frac{1}{3D-2}.\cos 2x - \frac{2D+3}{(-4)^2+6(-4)D+13(-4)+12D+4}.\cos 2x$$

$$=x.\frac{1}{3D-2}.\cos 2x - \frac{2D+3}{16-24D-52+12D+4}.\cos 2x$$

$$=x.\frac{1}{3D-2}.\cos 2x - \frac{2D+3}{-12D-32}.\cos 2x$$

$$= x.\frac{1}{3D-2}.\cos 2x + \frac{1}{4}\frac{2D+3}{3D+8}.\cos 2x$$

$$=x.\frac{1}{3D-2}.\cos 2x + \frac{1}{4}\frac{(2D+3)(3D-8)}{9D^2-64}.\cos 2x$$

$$= x.\frac{3D+2}{9D^2-4}.\cos 2x + \frac{1}{4}.\frac{-1}{100}(6D^2-7D-24)\cos 2x$$

$$= x\frac{-1}{38}(3D+2)\cos 2x + \frac{1}{4}.\frac{-1}{100}(6D^2-7D-24)\cos 2x$$

$$= \frac{-x}{40}(-6\sin 2x + 2\cos 2x) + \frac{-1}{400}(-24\cos 2x + 14\sin 2x - 24\cos 2x)$$

$$= \frac{x}{20}(3\sin 2x - \cos 2x) + \frac{1}{200}(24\cos 2x - 7\sin 2x)$$

$$= \frac{3x}{20}\sin 2x - \frac{x}{20}\cos 2x + \frac{12}{100}\cos 2x - \frac{7}{200}\sin 2x$$

$$= \frac{30x-7}{200}\sin 2x - \frac{5x-12}{100}\cos 2x$$

Exercises

Solve the following diff. equations.

1- $(D^2+5D-6)y = e^{3x}$

2- $(D^2+5D-6)y = e^{-6x}$

3- $(D^4+8D^2)y = \sin 2x$

4- $(D^2+3D+2)y^2 = 0$

5- $(D^2+2D+2)y = \cos 5x$

6- $y''' - 27y = e^{2x}$

7- $y''' - 27y = x^4$

8- $(D^2+5D)y = e^x x^2$

9- $(5D^2+15)y = e^{2x}\cos x$

10- $(2D+1) = x\sin 2x$

Section (2)

Reducing the order of a differential equation

The second order differential equation is in the form

$$\boldsymbol{F(x, y, y', y'') = 0} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

And it can be reduced to the first order according to its type, we have two types here:

The first type: if the dependent variable y does not appear in the equation

اذا لم يظهر المتغير المعتمد y في المعادلة (بمعنى يظهر خالي من المشتقات)

Then we suppose that

$$y' = p \quad \Rightarrow \quad y'' = \frac{dp}{dx} = p'$$

Then equation (1) will be

$$\boldsymbol{G(x, p, p') = 0} \qquad \ldots\ldots\ldots\ldots\ldots \textbf{(2)}$$

And this is an equation of first order can be solved as in ch.2 to get p, then we return the variable p and solve it to get an equation in term of x and y represents the general sol.

مما سبق نفهم انه لحل معادلة تفاضلية من الرتبة الثانية ذات معاملات متغيرة باستخدام طريقة تخفيض الرتبة توجد حالتين

أ- عندما لا يظهر المتغير المعتمد y في المعادلة

$$y' = p \quad \Rightarrow \quad y'' = \frac{dp}{dx} = p'$$

نعوض في المعادلة الاصلية فنحصل على معادلة من الرتبة الاولى تحل كما في الفصل الثاني من حيث كونها قابلة للفصل او خطية ...ثم نقوم بارجاع p بدلالة المشتقة $\frac{dy}{dx}$ ونحل المعادلة الناتجة ايضا كما في الفصل الثاني فنحصل على الحل العام

Example (1): solve the following eq.

$$\boldsymbol{x^2y'' - (y')^2 - 2xy' = 0} \qquad \textbf{..........(3)}$$

Solution:

Note that eq. (3) does not contain the variable y

Let $y' = p \quad \Rightarrow \quad y'' = \frac{dp}{dx} = p'$

Sub. in (3)

$$x^2\frac{dp}{dx} - p^2 - 2xp = 0$$

$$x^2\frac{dp}{dx} - 2xp = p^2 \qquad \div\ x^2$$

$$\frac{dp}{dx} - \frac{2}{x}p = \frac{1}{x^2}p^2 \qquad \text{............ (4) برنولي}$$

$$p^{-2}\frac{dp}{dx} - \frac{2}{x}p^{-1} = \frac{1}{x^2} \qquad \text{............... (5)}$$

Let $z = p^{-1} \quad \Rightarrow \quad \frac{dz}{dx} = -p^{-2}\frac{dp}{dx}$

Sub. In (5)

$$\left[-\frac{dz}{dx} - \frac{2}{x}z = \frac{1}{x^2}\right] * -1$$

$$\frac{dz}{dx} + \frac{2}{x}z = \frac{-1}{x^2} \qquad \text{linear}$$

$$I.F = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$x^2z = \int -\frac{1}{x^2}.x^2dx + c_1$$

$$x^2z = -\int dx + c_1$$

$$x^2z = -x + c_1 \qquad \text{............(6)}$$

$$z = \frac{-x+c_1}{x^2}$$

Replacing $z = p^{-1}$

$$\frac{1}{p} = \frac{-x + c_1}{x^2} \quad \Rightarrow \quad p = \frac{x^2}{c_1 - x}$$

$$dy = \left(-x - c_1 + \frac{c_1^2}{c_1 - x}\right) dx$$

Integrating both sides

$y = \frac{-x^2}{2} - c_1 x - c_1^2 \ln|c_1 - x| + c_2$ $\qquad$ $c_1 \& c_2$ are constant

The second type: If the independent variable x does not appear in the equation
اذا لم يظهر المتغير المستقل x في المعادلة

Then we suppose that

$$y' = p \quad \Rightarrow \quad y'' = \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dx} \cdot \frac{dy}{dy} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \frac{dp}{dy}$$

Then equation (1) will be

$\boldsymbol{E\left(y, p, p\frac{dp}{dy}\right) = 0}$ **................ (7)**

And this equation is of the first order with dependent variable p and independent variable y solve it to get an equation with p & y then return the variable $p = \frac{dy}{dx}$ and solve it to get the general solution.

Example (2): Solve the following eq.

$\boldsymbol{yy'' + 2y' - (y')^2 = 0}$ **...........(8)**

Solution: نلاحظ ان x غير ظاهرة بالمعادلة

$$y' = p \quad \Rightarrow \quad y'' = p \frac{dp}{dy}$$

Sub. in eq. (8), we get:

$$yp\frac{dp}{dy} + 2p - p^2 = 0$$

$$p\left(y\frac{dp}{dy}+2-p\right)=0$$

$p=0 \quad \Rightarrow \frac{dy}{dx}=0 \qquad \Rightarrow \quad y=b$ Singular sol.

Or

$y\dfrac{dp}{dy}+2-p=0 \qquad \div y$

$\frac{dp}{dy}-\frac{1}{y}p=\frac{-2}{y}$ linear

$I=e^{\int -\frac{1}{y}dy}=e^{-\ln y}=y^{-1}$

$y^{-1}p=\displaystyle\int y^{-1}.\frac{-2}{y}dy+c_1$

$[y^{-1}p=2y^{-1}+c_1]*y$

$p=2+c_1y \quad p$ بارجاع

$\dfrac{dy}{dx}=2+c_1y$

$\frac{dy}{2+c_1y}=dx \qquad \frac{1}{c_1}\ln|2+c_1y|=x+c_2$ where c_1 & c_2 are constants

And this the general sol.

<u>Exercises:</u>

Solve the following equations:

1) $2y''-(y')^2+1=0$
2) $y''+k^2y=0$
3) $xy''=y'$
4) $y''-k^2y=0$
5) $yy''+(y')^2=0$
6) $yy''+(y')^3=0$

كلية التربية للعلوم الصرفة ابن الهيثم

قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

CHAPTER SIX

LAPLACE TRANSFORM

اساتذة المادة

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طباعة

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Chapter six

About the Laplace transform

The Laplace transform method solves differential equations and corresponding initial and boundary value problems, Laplace transform reduce the problem with differential to an algebraic problem.

Definition:

Let $f(x)$ be an arbitrary function defined for $x \geq 0$, then:

$$\boldsymbol{F(p) = L[f(x)] = \int_0^\infty f(x)e^{-px}\,dx} \qquad \text{---------------------- } \mathbf{(6.1)}$$

Where p is a real number is the (Laplace transform) of $f(x)$.

Remark:

The original function f depends on x and the new function F depends on p.

- $f(x)$ in (6.1) is called the inverse transform of $F(p)$ and will be denoted by $L^{-1}[F(p)]$

(i.e.)

$$\mathrm{F}(p) = L[f(x)] \Leftrightarrow f(x) = L^{-1}[F(p)]$$

Theorem: linearity of Laplace transforms

$$\boldsymbol{L[af(x) + bg(x)] = aL[f(x)] + bL[g(x)]} \qquad \text{--------------- } \mathbf{(6.2)}$$

Where **a** & **b** are constants.

Laplace transform of some functions:

1- Let $\boldsymbol{f(x) = 1}$ when $\boldsymbol{x \geq 0}$ **,** then $\boldsymbol{L[1] = \frac{1}{p}\,;\, p > 0}$

Proof: *from the definition (6.1), we get:*

$$L[f(x)] = L[1] = \int_0^\infty 1.\,e^{-px}\,dx$$

$$= \frac{1}{-p}e^{-px}\Big]_0^\infty$$

$$= \frac{1}{-p}[\lim_{x-\infty} e^{-px} - e^{0}]$$

$$= \frac{1}{-p}(-1) = \frac{1}{p} \qquad , p > 0$$

2- If $\boldsymbol{f(x) = x}$ $\quad then \quad \boldsymbol{L[x] = \frac{1}{p^2}} \qquad ; \quad \boldsymbol{p > 0}$

Proof: *from the definition (6.1), we get:*

$$L[x] = \int_0^\infty xe^{-px}\,dx \qquad u = x \to du = dx$$

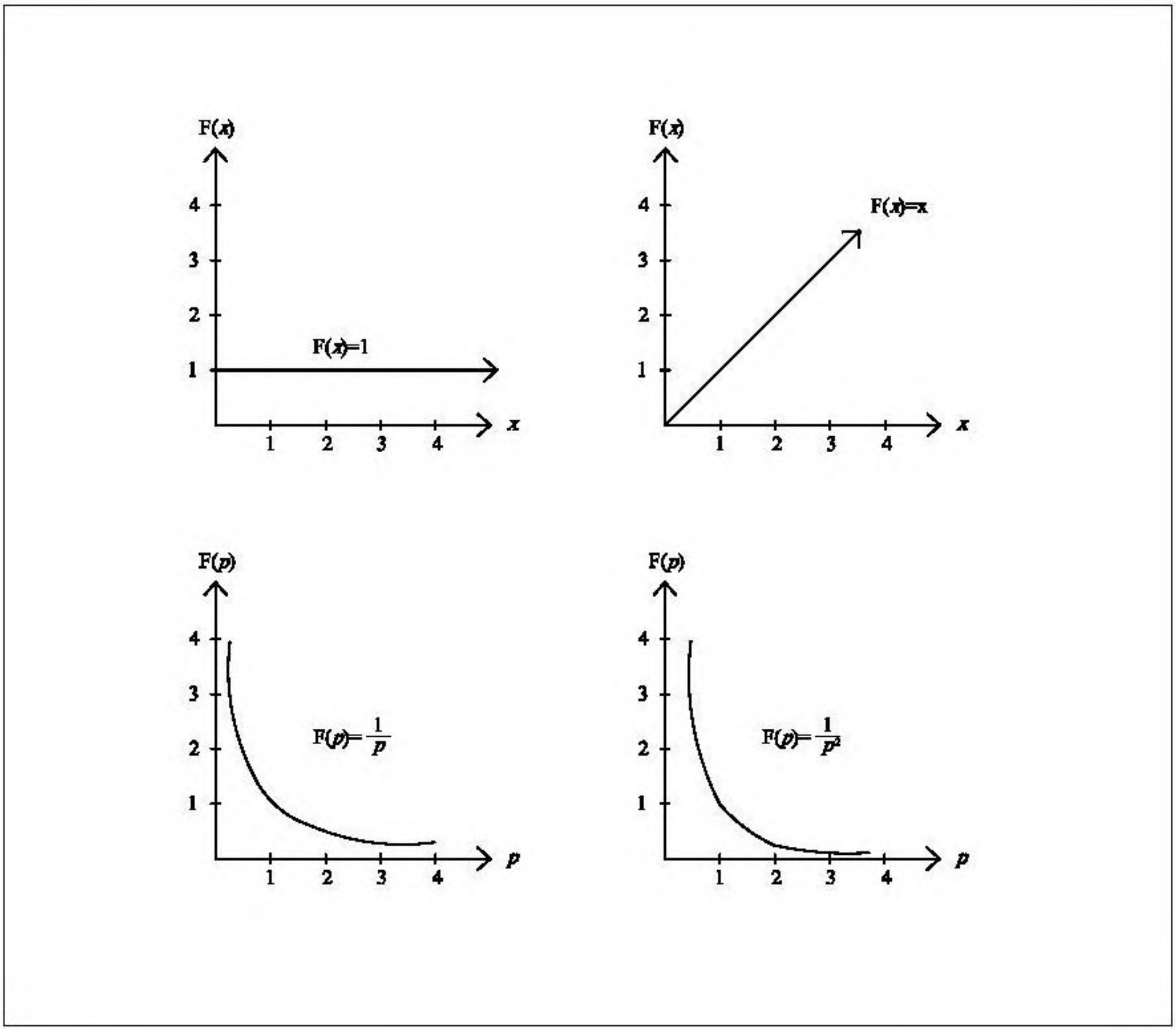
$$dv = e^{-px}dx \to v = \frac{-1}{p}e^{-px}$$

$$= \frac{-x}{p}e^{-px}\Big]_0^\infty + \frac{1}{p}\int_0^\infty e^{-px}\,dx$$

من الخاصية الاولى

$$= \lim_{x\to\infty}\left(\frac{-x}{p}e^{-px}\right) - 0 + \frac{1}{p}\cdot\frac{1}{p}$$

$$= \frac{1}{p^2} \qquad ; \quad p > 0$$



3- If $\boldsymbol{f(x)=e^{ax}}$ where a is a constant then

$$\mathbf{L}[\boldsymbol{e^{ax}}]=\frac{\mathbf{1}}{\boldsymbol{p-a}} \qquad ; \ \boldsymbol{p>a}$$

Proof: *from equation (6.1)*

$$L[e^{ax}]=\int_0^{\infty} e^{ax}.e^{-px}dx$$

$$=\int_0^{\infty} e^{(a-p)x}\,dx$$

$$=\frac{1}{a-p}e^{(a-p)x}\Big]_0^{\infty}$$

$= \frac{1}{a-p}(\lim_{x\to\infty} e^{(a-p)x} - e^{0})$ Since $p > a$ then $a - p < 0$

$= \frac{1}{a-p}(-1)$ $\Rightarrow e^{-\infty} = 0$

$= \frac{1}{p-a}$

By the same way we can find the Laplace transform for another function as $cosax$, $sinax$, x^n , ...

Ex. 1: Let $\boldsymbol{f(x) = coshax , find\ L[f(x)]}$

Sol: $L[coshax] = L\left[\frac{1}{2}(e^{ax} + e^{-ax})\right]$

$$= \frac{1}{2}L[e^{ax}] + \frac{1}{2}L[e^{-ax}]$$

$$= \frac{1}{2}\left(\frac{1}{p-a} + \frac{1}{p+a}\right) = \frac{p}{p^2-a^2}$$

Ex. 2: Find $\boldsymbol{L[f(x)] when f(x)} = \begin{cases} \boldsymbol{x} & \boldsymbol{0 < x < 4} \\ \boldsymbol{5} & \boldsymbol{x > 4} \end{cases}$

Sol: $L[f(x)] = \int_0^\infty f(x)\, e^{-px} dx$

$$= \int_0^4 xe^{-px} dx + \int_4^\infty 5. e^{-px} dx$$

Using integration by parts, we get

$$L[f(x)] = \left[\frac{-x}{p}e^{-px} - \frac{1}{p^2}e^{-px}\right]_0^4 + \left[\frac{-5}{p}e^{-px}\right]_4^\infty$$

$$= -\frac{4e^{-4p}}{p} - \frac{e^{-4p}}{p^2} + 0 + \frac{1}{p^2} + \frac{5}{p}e^{-4p}$$

$$= \frac{1}{p^2} + \frac{e^{-4p}}{p} - \frac{e^{-4p}}{p^2}$$

Exercises 6.1

Find the Laplace transforms of the following:

1- $f(x) = 2x + 6$

2- $f(x) = \sin \pi x$

3- $f(x) = e^{a-bx}$

4- $f(x) = \sin(3x + 5)$

5- $f(x) = x^3 - x^2 + 4x$

6- $3e^{4x} - e^{-2x}$

7- $\cos^2 8x$

8- $f(x) = \begin{cases} 4, & 0 < x < 1 \\ 3, & x > 1 \end{cases}$

9- $x^2 e^{-3x}$

10- $5e^{2x} \sinh 2x$

Table of Laplace transform

	$f(x) = L^{-1}[F(p)]$	$F(P) = L[f(x)]$
1	1	$\frac{1}{p}$
2	e^{ax}	$\frac{1}{p-a}$
3	$x^n \qquad n = 1,2,3,\ldots$	$\frac{n!}{p^{n+1}}$
4	$\sin(ax)$	$\frac{a}{p^2+a^2}$
5	$\cos(ax)$	$\frac{p}{p^2+a^2}$
6	$x\, sin(ax)$	$\frac{2ap}{(p^2+a^2)^2}$
7	$x\, cos(ax)$	$\frac{p^2-a^2}{(p^2+a^2)^2}$
8	$sinh(ax)$	$\frac{a}{p^2-a^2}$
9	$cosh(ax)$	$\frac{p}{p^2-a^2}$
10	$e^{ax}\sin(bx)$	$\frac{b}{(p-a)^2+b^2}$
11	$e^{ax}\cos(bx)$	$\frac{p-a}{(p-a)^2+b^2}$
12	$e^{ax}\sinh(bx)$	$\frac{b}{(p-a)^2-b^2}$
13	$e^{ax}\cosh(bx)$	$\frac{p-a}{(p-a)^2-b^2}$
14	$x^n\, e^{ax} \qquad n = 1,2,3,\ldots$	$\frac{n!}{(p-a)^{n+1}}$

The inverse transform (L^{-1}):

We said that before

$$F(p) = L[f(x)] \ then \quad L^{-1}[F(p)] = f(x)$$

So we can find $L^{-1}$for some function as follows:

1- $L[1] = \frac{1}{p} \to \ L^{-1}\left[\frac{1}{p}\right] = 1$

2- $L[x] = \frac{1}{p^2} \to \ L^{-1}\left[\frac{1}{p^2}\right] = x$

3- $L[e^{ax}] = \frac{1}{p-a} \to \ L^{-1}\left[\frac{1}{p-a}\right] = e^{ax}$

4- $L[\sin ax] = \frac{a}{p^2-a^2} \quad \to \ L^{-1}\left[\frac{a}{p^2-a^2}\right] = \sin ax$

.
.
.

And so on

وهكذا يمكن ايجاد تحويل لابلاس ومعكوسه لكثير من الدوال من خلال الجدول السابق وسنبين هنا ببعض الامثلة كيفية ايجاد معكوس تحويل لابلاس باستخدام بعض العمليات البسيطة بالاضافة الى الجدول.

Ex. 3: Evaluate (a) $L^{-1}\left[\frac{1}{p^5}\right]$ (b) $L^{-1}\left[\frac{1}{p^2+7}\right]$

Sol:

(a) $L^{-1}\left[\frac{1}{p^5}\right] = \frac{1}{4!}L^{-1}\left[\frac{4!}{p^5}\right] = \frac{1}{4!}x^4$

(b) $L^{-1}\left[\frac{1}{p^2+7}\right] = \frac{1}{\sqrt{7}}L^{-1}\left[\frac{\sqrt{7}}{p^2+7}\right] = \frac{1}{\sqrt{7}}\sin\sqrt{7}\,x$

Ex. 4: Find $L^{-1}\left[\frac{3p+2}{p^2+4}\right]$

Sol:

$$L^{-1}\left[\frac{3p+2}{p^2+4}\right] = L^{-1}\left[\frac{3p}{p^2+4} + \frac{2}{p^2+4}\right]$$

$$= 3L^{-1}\left[\frac{p}{p^2+4}\right] + L^{-1}\left[\frac{2}{p^2+4}\right]$$

$$= 3\cos 2x + \sin 2x$$

Ex. 5: Find $L^{-1}\left[\frac{p+5}{(p+2)^2+9}\right]$

Sol:

$$L^{-1}\left[\frac{p+5}{(p+2)^2+9}\right] = L^{-1}\left[\frac{p+2+3}{(p+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{p+2}{(p+2)^2+9}\right] + L^{-1}\left[\frac{3}{(p+2)^2+9}\right]$$

$$= e^{-2x}\cos 3x + e^{-2x}\sin 3x$$

Ex. 6: Evaluate $L^{-1}\left[\frac{3}{p(p+3)}\right]$

Sol:

$$L^{-1}\left[\frac{3}{p(p+3)}\right]$$

$$= L^{-1}\left[\frac{1}{p} - \frac{1}{p+3}\right]$$

$$= L^{-1}\left[\frac{1}{p}\right] - L^{-1}\left[\frac{1}{p+3}\right]$$

$$= 1 - e^{-3x}$$

توضيح: تم استخدام تجزئة الكسور

$$\frac{3}{p(p+3)} = \frac{A}{p} + \frac{B}{p+3}$$

$$= \frac{Ap+3A+Bp}{p(p+3)}$$

$$\therefore A + B = 0$$

$$3A = 3 \Rightarrow A = 1$$

$$\therefore B = -1$$

Exercises 6.2

Find $L^{-1}[F(p)]$for the following:

1. $L^{-1}\left[\frac{1}{p^2+2p+10}\right]$
2. $L^{-1}\left[\frac{3p}{p^2+4p+13}\right]$
3. $L^{-1}\left[\frac{1}{p^2+4p+4}\right]$
4. $L^{-1}\left[\frac{p^2-2p+3}{(p-1)^3}\right]$

Using Laplace Transform to Solve the (LODEs) With Constant Coefficients

One of the main applications of the Laplace transform (LT) is the solution of linear differential equations with constant coefficients in the existence of boundary and initial conditions, where the ordinary linear differential equation (OLDE) is transformed into an algebraic equation using the following theorem.

ان احد اهم تطبيقات تحويل لابلاس الرئيسية هو حل المعادلات التفاضلية الخطية ذات المعاملات الثابتة بوجود شروط حدودية وابتدائية حيث يتم تحويل المعادلة التفاضلية الخطية الاعتيادية الى معادلة جبرية باستخدام المبرهنة الاتية.

Theorem: if $f(x)$and its derivatives$f'(x),\ f^{(2)}(x), f^{(3)}(x), \dots, f^{(n)}(x),$ exist then:

$$L\left[f^{(n)}(x)\right] = p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - p f^{(n-2)}(0) - f^{(n-1)}(0) \qquad \dots\dots\ (6.3)$$

من هذا يتوضح ان

$$L[f'(x)] = pF(p) - f(0)$$

and

$$L[f''(x)] = p^2 F(p) - pf(0) - f'(0)$$

.
.
.

And so on

Now to solve the following diff. eq. by LT

$$\boxed{\begin{array}{l} a_0 y^n + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = g(x); \\ y(0) = \propto_0 \quad , \quad y'(0) = \propto_1 \ , \dots , \ y^{(n-1)}(0) = \propto_{n-1} \end{array}} \qquad \dots\dots\dots\dots\ (6.4)$$

Step 1:Taking LT for both sides and using eq. (6.2), we get:

$$a_0 L\left[y^{(n)}\right] + a_1 L\left[y^{(n-1)}\right] + \cdots + a_n L[y] = L[g(x)] \qquad \dots\dots\dots\dots\dots\dots\ (6.5)$$

Step 2: From eq. (6.3) we can find $L\left[y^{(n)}\right]$,

$$L\left[y^{(n-1)}\right], \dots, L[y''] \;, L[y']$$

For example: $L[y'] = pY(p) - y(0)$

$$L[y''] = p^2Y(p) - py(0) - y'(0)$$

And from the table of LT we can find $L[g(x)]$

Step 3: substituting the initial conditions $y(0) = \propto_{0,}$

$y'(0) = \propto_1 \;, \dots, y^{(n-1)}(0) = \propto_{n-1}$ in (6.5), we'll get an algebraic for $Y(p)$, after solving it we take the $L^{-1}$for both sides

Step 4: from the table we return all the functions to the original variables.

EX.7: **solve** $\boldsymbol{y' - y = 1 \quad , y(0) = 0 \; by \; LT}$

Sol. :

Taking LT for both sides

$$L[y'] - L[y] = L[1]$$

$$pY(p) - y(0) - Y(p) = \frac{1}{p}$$

$$(p-1)Y(p) - 0 = \frac{1}{p}$$

$$Y(p) = \frac{1}{p(p-1)}$$ بتجزئة الكسور

$$= \frac{-1}{p} + \frac{1}{p-1}$$

Taking $L^{-1}$for both sides we get:

$$L^{-1}[Y(p)] = -L^{-1}\left[\frac{1}{p}\right] + L^{-1}\left[\frac{1}{p-1}\right]$$

$\rightarrow \; y(x) = -1 + e^x$ from (table of LT)

EX.8: **solve $y'' + 2y' + 5y = 0$, $y(0) = 1$, $y'(0) = 5$**

Sol.: taking LT for both sides

$$L[y''] + 2L[y'] + 5L[y] = 0$$

$$[p^2Y(p) - py(0) - y'(0)] + 2(pY(p) - y(0)) + 5Y(p) = 0$$

$$p^2Y(p) - p - 5 + 2pY(p) - 2 + 5Y(p) = 0$$

$$(p^2 + 2p + 5)Y(p) = p + 7$$

$$Y(p) = \frac{p+7}{p^2 + 2p + 5}$$

$$= \frac{p + 1 + 6}{p^2 + 2p + 1 + 4}$$

$$= \frac{p + 1 + 6}{(p+1)^2 + 4} = \frac{p+1}{(p+1)^2 + 4} + \frac{6}{(p+1)^2 + 4}$$

Taking $L^{-1}$for both sides we get:

$$L^{-1}[Y(p)] = L^{-1}\left[\frac{p+1}{(p+1)^2 + 4}\right] + 3L^{-1}\left[\frac{2}{(p+1)^2 + 4}\right]$$

From the table of LT (n.10) , (n.11), we get

$$y(x) = e^{-x}\cos 2x + 3e^{-x}\sin 2x$$

EX.9: Solve $y'' - 3y' + 2y = 1 - 4x + 2x^2$, y(0)=4 , y'(0)=5

Sol.: taking LT for both sides

$$L[y''] - 3L[y'] + 2L[y] = L[1] - 4L[x] + 2L[x^2]$$

$$p^2Y(p) - py(0) - y'(0) - 3\big(pY(p) - y(0)\big) + 2Y(p) = \frac{1}{p} - \frac{4}{p^2} + 2.\frac{2!}{p^3}$$

$$p^2Y(p) - 4p - 5 - 3pY(p) + 12 + 2Y(p) = \frac{1}{p} - \frac{4}{p^2} + \frac{4}{p^3}$$

$$(p^2 - 3p + 2)Y(p) = 4p - 7 + \frac{p^2 - 4p + 4}{p^3}$$

$$\Rightarrow Y(p) = \frac{4p-7}{p^2 - 3p + 2} + \frac{p^2 - 4p + 4}{p^3(p^2 - 3p + 2)}$$

$$= \frac{4p-7}{(p-2)(p-1)} + \frac{(p-2)^2}{p^3(p-1)(p-2)}$$

$Y(p) = \frac{4p-7}{(p-2)(p-1)} + \frac{p-2}{p^3(p-1)}$ تجزئة كسور

$= \frac{1}{p} + \frac{1}{p^2} + \frac{2}{p^3} + \frac{1}{p-2} + \frac{2}{p-1}$

Taking $L^{-1}$for both sides

$$L^{-1}[Y(p)] = L^{-1}\left[\frac{1}{p}\right] + L^{-1}\left[\frac{1}{p^2}\right] + L^{-1}\left[\frac{2}{p^3}\right] + L^{-1}\left[\frac{1}{p-2}\right] + 2L^{-1}[\frac{1}{p-1}]$$

$$\Longrightarrow y(x) = 1 + x + x^2 + e^{2x} + 2e^x$$

EX.10: solve $y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$

Sol.: taking LT for both sides

$$L[y''] - 3L[y'] + 2L[y] = L[e^{-4t}]$$

$$[p^2Y(p) - py(0) - y'(0)] - 3[pY(p) - y(0)] + 2Y(p) = \frac{1}{p+4}$$

$$(p^2 - 3p + 2)Y(p) - p - 5 + 3 = \frac{1}{p+4}$$

$$(p^2 - 3p + 2)Y(p) - p - 2 = \frac{1}{p+4}$$

$$(p^2 - 3p + 2)Y(p) = p + 2 + \frac{1}{p+4}$$

$$Y(p) = \frac{p+2}{p^2-3p+2} + \frac{1}{(p+4)(p^2-3p+2)}$$

$$= \frac{p^2+6p+9}{(p+4)(p^2-3p+2)}$$

$$= \frac{p^2+6p+9}{(p+4)(p-2)(p-1)}$$

$$= \frac{A}{p-1} + \frac{B}{p-2} + \frac{C}{p+4}$$

$A = \frac{-16}{5}$, $B = \frac{25}{6}$, $C = \frac{1}{30}$

Then

$$Y(p) = \frac{-\frac{16}{5}}{p-1} + \frac{25/6}{p-2} + \frac{1/30}{p+4}$$

Taking L^{-1} for both sides, we get:

$$L^{-1}[Y(p)] = -\frac{16}{5}L^{-1}\left[\frac{1}{p-1}\right] + \frac{25}{6}L^{-1}\left[\frac{1}{p-2}\right] + \frac{1}{30}L^{-1}\left[\frac{1}{p+4}\right]$$

$$y(t) = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Exercises 6.3

Solve the following initial-value problems by the Laplace transform method

1- $y'' - 9y = 6\cos 3x \quad , \quad y(0) = 0 \quad , \quad y'(0) = 3$

2- $\frac{dy}{dx} + 2y = \cos x \quad , \quad y(0) = 1$

3- $y'' + 4y = 10\sin 3x - 5\cos 3x \quad , \quad y(0) = 2 \quad , \quad y'(0) = -4$

4- $y'' - 6y' + 9y = 6x^2e^x \quad , \quad y(0) = y'(0) = 0$

5- $y'' + 9y = 40e^x \quad , \quad y(0) = 5 \quad , \quad y'(0) = -2$

6- $y'' - 2y' = -4 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 4$

7- $x''(t) - 2x'(t) = 6 - 4t \quad , \quad x(0) = 2 \quad , \quad x'(0) = 0$

8- $y' = 2e^t \quad , \quad y(0) = -1$

9- $y^{(3)} - y' = \sinh 2x \quad , \quad y(0) = y'(0) = y''(0) = 0$

10- $y''(t) - 6y'(t) + 9y(t) = 6t^2e^t \quad , y(0) = y'(0) = 0$

CHAPTER SEVEN

POWER SERIES SOLUTION OF THE LINEAR DIFFERENTIAL EQUATION

Introduction:

The power series method is a standard basic method for solving linear differential equation with variable coefficients. It gives solutions in the form of power series , this explains the name.

1.Basic definitions:

1. **Power series:** An infinite series of the form

$$\sum_{n=0}^{\infty} \boldsymbol{a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots} \qquad (\mathbf{1})$$

is called a **power series in (x-x_0)**.
where a_0,a_1,a_2,… are constants, called the coefficients of the series.
x_0 is called the center of the series.

for example: the power series
$\sum_{n=0}^{\infty}(x+1)^n$ is centered at $x_0 = -1$

2. **A power series in x:** It is an infinite series (1) when $x_0 = 0$

$$\sum_{n=0}^{\infty} \boldsymbol{a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots} \qquad (\mathbf{2})$$

3. **Ordinary point:** A point $x = x_0$ is called an ordinary point of the equation:

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

If both the functions P(x) and Q(x) are analytic at x=x_0.

4. **Singular point:** If the point x=x_0 is not an ordinary point of the diff. eq.(3) , then it is called a *Singular point* of eq.(3).
There are two types of singular points:
(i) *regular Singular point*
(**ii**)*irregular Singular point*
A singular point x=x_0 of the diff. eq. (3) is called **regular singular point** of the Diff. eq. (3) if both $(\boldsymbol{x-x_0})\boldsymbol{P(x)}$ and $(\boldsymbol{x-x_0})^2\boldsymbol{Q(x)}$ are analytic at x=x_0.

A singular point, which is not regular is called an **irregular singular point.**

5. Standard equation: equation (3) that in the form

$y'' + P(x)y' + Q(x)y = 0$

Is called a standard equation.

Example1.1: Determine whether x=0 is an ordinary point or a regular singular point of the differential equation $2x^2y'' + 7x(x+1)y' - 3y = 0$.

Sol. Dividing by $2x^2$, the given equation becomes,

$$y'' + \frac{7(x+1)}{2x}y' - \frac{3}{2x^2}y = 0 \qquad (4)$$

Comparing (4) with (3) we have

$P(x) = \frac{7(x+1)}{2x}$ and $Q(x) = -\frac{3}{2x^2}$

Since both $P(x)$ and $Q(x)$ are undefined at x=0, so both $P(x)$ and $Q(x)$ are not analytic at x= 0. Thus x=0 is not ordinary point so x=0 is a singular point

Also, $(x-0)P(x) = \frac{7(x+1)}{2}$ and $(x-0)^2Q(x) = -\frac{3}{2}$

Are analytic at x=0. Then x=0 is a regular singular point.

EXERCISES:

1.Show that x=0 is an ordinary point of $y'' - xy' + 2y = 0$.

2.Show that x=0 is an ordinary point of $(x^2-1)y'' + xy' - y = 0$, but x=1 is a regular singular point.

3. Determine the nature of the point x=0 for the equation $xy'' + y\sin x = 0$

2. Maclaurin Series:

Some functions can be expressed by the power series and are called Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (5)$$

Where $|x| < 1$,its called the geometric series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (6)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (7)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (8)$$

3.The power series method (the power series solution about x=0):

Suppose the second order linear diff. eq. in the standard form is:

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

and x=0 is an ordinary point. Therefore, to solve the above equation we take the following power series,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (9)$$

Differentiating (9) twice w.r.t. (x), we get

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \qquad (10)$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \qquad (11)$$

Substituting equations. (9), (10) and (11) in (3) and collecting the like terms into x (which have the same powers), equating to zero the coefficients of the smallest power of x starting with the constant terms, the terms containing x, the terms containing x^2, etc.

This gives us a relation between the coefficients which helps in determining the nature of the solution.

Remark: A solution of the form $y = \sum_{n=0}^{\infty}(x - x_0)^n$ is said to be a solution about the ordinary point x_0

Example 3.1: Solve $y' - y = 0 \quad about\ x = 0$ (12)

Sol: starting with (9) and (10)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{9}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \tag{10}$$

Substituting in (12), we get

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots \quad) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad) = 0 \tag{13}$$

Now we collect the terms of similar power to x:

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0 \tag{14}$$

Then

$$(a_1 - a_0) = 0 \ \rightarrow a_1 = a_0$$

$$(2a_2 - a_1) = 0 \rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2!}$$

$$(3a_3 - a_2) = 0 \rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3(2)} = \frac{a_0}{3!}$$

…

Substituting in (9), we get:

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots \tag{15}$$

Or

$$y = a_0\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) = a_0 e^x$$

Example3.2: Solve $y' = 2xy \quad about\ x = 0$ (16)

(use the power series method)

Sol: we arrange the equation as follows

$$y' - 2xy = 0 \tag{17}$$

From (9) and (10)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Substituting in (17) we get

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - 2x(\, a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = 0 \tag{18}$$

$$(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots) - (2\, a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + 2a_4 x^5 + 2a_5 x^6 + \cdots) = 0$$

Then

$a_1 = 0$ ⋇

$2a_2 = 2a_0 \ \rightarrow a_2 = a_0$

$3a_3 = 2a_1 \rightarrow a_3 = 0$ ⋇

$4a_4 = 2a_2 \rightarrow a_4 = \frac{a_2}{2} \rightarrow a_4 = \frac{a_0}{2!}$

$5a_5 = 2a_3 \rightarrow a_5 = 0$ ⋇

$6a_6 = 2a_4 \rightarrow a_6 = \frac{a_4}{3} \rightarrow a_6 = \frac{a_0}{3!}$

…

Note that the odd coefficients are equal to zero.

Substituting in (18) we get:

$$y = a_0 \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) \tag{19}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = a_0 e^{x^2} \qquad (20)$$

Example 3.3: Solve $y'' + y = 0$ about x=0 (21)

(use the power series method)

Sol: From (9) and (11)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (9)$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \qquad (11)$$

Substituting in (21), we get

$$(2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots) + (a_0 + a_1 x + a_2 x^2 + \cdots) = 0 \qquad (22)$$

Collecting like powers of x, we find

$$(2a_2 + a_0) + (3 \cdot 2a_3 + a_1)x + (4 \cdot 3a_4 + a_2)x^2 + \cdots = 0 \qquad (23)$$

Then

$$2a_2 + a_0 = 0 \quad \rightarrow \quad a_2 = -\frac{a_0}{2} = -\frac{a_0}{2!}$$

$$3 \cdot 2a_3 + a_1 = 0 \quad \rightarrow \quad a_3 = -\frac{a_1}{3 \cdot 2} \quad \rightarrow a_3 = -\frac{a_1}{3!}$$

$$4 \cdot 3a_4 + a_2 = 0 \quad \rightarrow a_4 = -\frac{a_2}{4 \cdot 3} \quad \rightarrow a_4 = \frac{a_0}{4!}$$

…

Substituting in (9) we get

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \qquad (24)$$

$$= a_0 \cos x + a_1 \sin x$$

4.Some properties of power series:

I. $\sum_{n=p}^{k} F(n) = F(p) + F(p+1) + \cdots + F(k) \quad ,k > p$ (25)

Where k and p are integer numbers.

II. $\sum_{n=p}^{\infty} a_n F(n) x^{n+p} = \sum_{n=0}^{\infty} a_{n+p} F(n+p) x^{n+2p}$ (26)

III. $\sum_{n=k}^{\infty} a_n x^{n-k} + \sum_{n=m}^{\infty} a_n x^{n-m} = \sum_{n=0}^{\infty} a_{n+k} x^n + \sum_{n=0}^{\infty} a_{n+m} x^n$

$= \sum_{n=0}^{\infty} (a_{n+k} + a_{n+m}) x^n$ (27)

IV. If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \rightarrow a_n = b_n, \quad \forall n \geq 0$ (28)

For example: $if \quad \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n$ then

$$n a_n = a_{n-1} \rightarrow a_n = \frac{a_{n-1}}{n}$$

So , if n=1→$a_1 = a_0$

if n=2→$a_2 = \frac{a_1}{2} \rightarrow a_2 = \frac{a_0}{2}$

if n=3→$a_3 = \frac{a_2}{3} \rightarrow a_3 = \frac{a_0}{3\cdot 2} = \frac{a_0}{3!}$

V.(Identity property of power series):

If $\sum_{n=0}^{\infty} a_n x^n = 0$, for every x number in the interval of convergence (i.e. in a neighbourhood of 0), then $a_n = 0$, for all n.

Example 4.1: Solve $y'' + xy = 0$ (29)

Sol: Let $y = \sum_{n=0}^{\infty} a_n x^n$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting in (29) ,we get:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0 \qquad (30)$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \qquad (31)$$

$$2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \qquad (32)$$

From property II, we get,

$$2a_2 + \sum_{k=1}^{\infty} (k+1)(k+2)a_{k+2}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k = 0 \qquad (33)$$

Where for the second term we take k=n-2→ n=k+2, so if n=3 then k=1

And for the third term we take k=n+1→ n=k-1, so if n=0 then k=1

From property III, eq. (33) will be:

$$2a_2 + \sum_{k=1}^{\infty} [(k+1)(k+2)a_{k+2} + a_{k-1}]\, x^k = 0 \qquad (34)$$

From property V, we get:

$$2a_2 = 0 \to a_2 = 0 \qquad (35)$$

$$(k+1)(k+2)a_{k+2} + a_{k-1} = 0$$

Then

$$a_{k+2} = -\frac{a_{k-1}}{(k+1)(k+2)} \qquad (36)$$

So, if

K=1→$a_3 = -\frac{a_0}{(2)(3)}$

K=2→$a_4 = -\frac{a_1}{(3)(4)}$

K=3→$a_5 = -\frac{a_2}{(4)(5)} = 0$

K=4→$a_6 = -\frac{a_3}{(5)(6)} = \frac{a_0}{2{\cdot}3{\cdot}5{\cdot}6}$

K=5→$a_7 = -\frac{a_4}{(6)(7)} = \frac{a_1}{3{\cdot}4{\cdot}6{\cdot}7}$

K=6→$a_8 = -\frac{a_5}{(7)(8)} = 0$

K=7→$a_9 = -\frac{a_6}{(8)(9)} = \frac{a_0}{2\cdot3\cdot5\cdot6\cdot8\cdot9}$

…

Substituting in

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \quad (37)$$

$$y = a_0 + a_1x - \frac{a_0}{2\cdot3}x^3 - \frac{a_1}{3\cdot4}x^4 + \frac{a_0}{2\cdot3\cdot5\cdot6}x^6 + \frac{a_1}{3\cdot4\cdot6\cdot7}\ x^7 + \frac{a_0}{2\cdot3\cdot5\cdot6\cdot8\cdot9}x^9 + \cdots \quad (38)$$

EXERCISES:

Solve the following equations using the power series method:

1.$\frac{dy}{dx} = x^2 - y$

2. $(1 - x^2)y'' - 2xy' + 6y = 0$

3. $y'' + y' - xy = 0$

4.$y'' - y = 0$

5. $y' = x + y$

6. $(x^2 + 1)y'' + xy' - y = 0$

7. $y'' + x^2y' + xy = 0$

Chapter 7 : Power series solution of the linear differential equations