- Truncation Error

Truncation error is defined on the error Caused by truncating a mathematical procedure.

For example, The Maclaurin series for is given as:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The series has infinite number of terms but when using this series to Calculate e^x, only a finite number of terms Can be used.

For example if one uses Three terms to Calculate e^x, then

$$e^x \approx 1 + x + \frac{x^2}{2!}$$

The Truncation error such an approximation is

$$e^x - \left[1 + x + \frac{x^2}{2!}\right] , \quad = \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

ثاني حاسبات
التحليل العددي
٢٠١٦-٢٠١٧

Chapter -1-

سهاد أحمد

مكتبة عمار
- هدايا - استنساخ
طباعة بحوث

Error Analysis :

- Absolute Error

if x' is approximation to x, The difference between the True Value and The approximate Value is called absolute error (E_x)

$$E_x = x - x'$$

- Relative Error

A number x' is considered to be an approximation to the True value x to d significant digits if d is the Largest positive integer for which

$$E_{rel} = \left| \frac{x - x'}{x} \right|$$

Ex. The number 3.1415927 is approximated as 3.1416 Find
- Absolute error
- Relative error

$$E_x = x - x' = 3.1415927 - 3.1416 = -0.000073$$

$$E_{rel} = \left| \frac{x - x'}{x} \right| = \left| \frac{-0.00073}{3.1415927} \right| = 0.0000023237$$

- Round off error

A Computer Can only represent a number approximately. For example a number Like $1/3$ may be represented as 0.333333 on a PC. Then The Round off error in this Case is

$$\frac{1}{3} - 0.333333 = 0.00000033$$

also Like π, $\sqrt{2}$, ----

- * Each calculated x is known as an instantaneous root : x_1, x_2, x_3, —. (in general x_i) { i may take 0-value }.
- * The more close x_i to the real root, the more accurate the solution is.
- * We may use the notations: x_i, x_{i+1}, x_{i-1}.
for example, if $i=3$:
$x_i = x_3$, $x_{i+1} = x_4$, $x_{i-1} = x_2$.
- * when the instantaneous root (x_i) gets closer to the real root. The function $f(x_i)$ gets closer to zero.
or, when $x_i \longrightarrow \bar{x}$, then, $f(x_i) \longrightarrow 0$.
- * In general, full accuracy is not obtained in numerical methods and we may consider the root as that value of (x_i) which makes $|f(x_i)| \leqslant \epsilon$. where ($\epsilon$) is a small quantity
- * For smaller (ϵ) we get higher accuracy.
- * Accuracy may be represented by different stopping conditions :-

- Absolute Error : Eabs : $|x_{i+1} - x_i| \leqslant \epsilon$
- Relative Error : Erel : $\left| \dfrac{x_{i+1} - x_i}{x_{i+1}} \right| \leqslant \epsilon$
- percentage Error : e% : $\left| \dfrac{x_{i+1} - x_i}{x_{i+1}} \right| * 100 \leqslant \epsilon$

- if $|f(x_i)| \leqslant \epsilon$
OR $|f(x_{i+1})| \leqslant \epsilon$

- Coincidence in Decimal digits for x_i-values (or x_{i+1}-values)

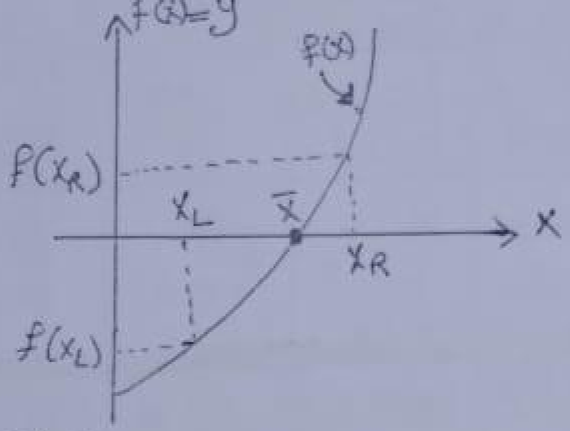
Correct to 2D
Correct to 3D

1. The Bisection Method [Half-Interval search].

* Let $f(x)$ be a continous function of (x).
* The equation $f(x) = 0$ has a root in the interval $[X_L, X_R]$ if this relation holds :

$$f(X_L) * f(X_R) < 0$$

$\{$ i.e $f(x_L)$ has a different sign of $f(x_R)\}$.



* Algorithm steps

1. choose an interval $[X_L, X_R]$ such that
$$f(X_L) * f(X_R) < 0$$
2. Find the Value of (X_i) by halfing the distance between X_L & X_R :
$$X_i = \frac{X_L + X_R}{2}$$
3. Calculate $f(X_i)$ $\{$using X_i- Value$\}$.
4. if $f(X_L) * f(X_i) < 0 \Rightarrow X_R = X_i$

   If $f(X_L) * f(X_i) > 0 \Rightarrow X_L = X_i$
5. Repeat the above procedure starting from step (2) to Calculate a new (x_i) .... and so on.
6. Terminate The calculations when The given accuracy condition is Satisfied.

Chapter-2-

1

ة بغداد / كلية ابن الهيثم
قسم علوم الحاسبات
المرحلة الثانية / التحليل العددي
د- سهاد

Numerical Analysis
Solving Non-Linear Equation with one Variable
{Root Finding}

General Notes

* our equations must be written in the form:

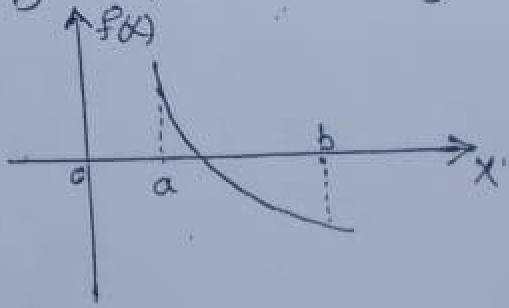
$$f(x) = 0$$

for example: $x^2 - 3x + 2 = 0$

$$e^{x^2} - 3\cos x = 0$$

* $f(x)$ is any function of the variable (x).
* the root of the equation $f(x)=0$ is the value of (x) which satisfies the equation., or, the root is the value of (x) which makes $f(x)$ equal to zero.
* we shall denote the root as $(\bar{x})$.
* The equation $f(x)=0$ may have more than one root.
* we shall take some iterative numerical methods for root finding.
* In many of these methods, we need to find the interval which contains the root { the root may be positive or negati
* For a positive root we make the table:

x	0	0.5	1	1.5	2
f(x)	+	+	+	-	-
			↑ a	↑ b	



There is a root in the interval $[a, b]$

* For a negative root, we take (-ve) values for (x) {start from zero if possible}.
* In iterative methods, we get closer and closer to the real root by calculating many values of (x).

Ex. 2 : Find the root of the equ. : $3x^2 = x + 5.25$ in the interval $[0.5, 3]$. Correct to $|f(x)| < 0.4$

Sol. :

$x_L = 0.5 \; ; \; x_R = 3$

$f(x) = 3x^2 - x - 5.25$

$x_i = \frac{x_L + x_R}{2}$, $f(x_L) * f(x_i) < 0 \Rightarrow x_R = x_i$

$> 0 \Rightarrow x_L = x_i$

x_L	x_R	x_i	$f(x_L)$	$f(x_i)$	$f(x_L)*f(x_i)$
0.5	3	1.75	-5	2.1875	−
0.5	1.75	1.125	-5	-2.5781	+
1.125	1.75	1.4375	-2.5781	-0.4882	+
1.4375	1.75	1.5937	-0.4882	0.7759	−
1.4375	1.5937	1.5156	-0.4882	0.1255	

$\therefore \bar{x} \simeq 1.5156$

* **Note :** Exact root is 1.5

Ex1 : Use bisection method to determine the (+ve) root of the equation :

$$e^{-x} = x \quad , \quad \text{Correct to } 6\%$$

Sol :

$$f(x) = e^{-x} - x$$

To Find the interval for (x) :

x	0	1
$f(x)$	1	-0.63

$$\Rightarrow \left\{ \begin{array}{ll} x_L = 0 & f(x_L) = 1 \\ x_R = 1 & f(x_R) = -0.63 \end{array} \right\}$$

$\therefore x_L = 0 \;\&\; x_R = 1$

we have $f(x_L) * f(x_R) < 0 \Rightarrow$ There is a root in the interval $[0, 1]$.

i	x_L	x_R	x_i	$e\%$	$f(x_L)$	$f(x_i)$	$f(x_L) * f(x_i)$
1	0	1	0.5	---	+	+	+
2	0.5	1	0.75	$\simeq 33\%$	+	−	−
3	0.5	0.75	0.625	$\simeq 20\%$	+	−	−
4	0.5	0.625	0.5625	$\simeq 11\%$	+	+	+
5	0.5625	0.625	0.5937	$\simeq 5.2\%$	+	−	−

$\therefore$ The root is : $\bar{x} \simeq 0.5937$

* **Note :**

We may check for the calculated root (0.5937) by substituting it in $f(x)$ and see whether $f(\bar{x})$ is a small quantity or not :

$f(\bar{x}) = f(0.5937) \Rightarrow e^{-0.5937} - 0.5937 = -0.0414$

$|f(\bar{x})| \simeq 0.0414$, which is small enough.

we have : $x_i = \dfrac{x_L f(x_R) - x_R f(x_L)}{f(x_R) - f(x_L)}$

i	x_L	x_R	$f(x_L)$	$f(x_R)$	x_i	$f(x_i)$	$f(x_L)\cdot f(x_i)$
1	1	2	-0.281	1.389	1.169	-0.288	+
2	1.169	2	-0.288	1.389	1.311	-0.223	+
3	1.311	2	-0.223	1.389	1.406	-0.138	+
4	1.406	2	-0.138	1.389	1.459	-0.075	+
5	1.459	2	-0.075	1.389	1.486	-0.038	+
6	1.486	2	-0.038	1.389	1.499	-0.019	+
7	1.499	2	-0.019	1.389	1.505	-0.008	+
8	1.505	2	-0.008	1.389	1.502	-0.004	+ ← <0.0

$\Rightarrow$ root $\bar{x} \simeq 1.509$

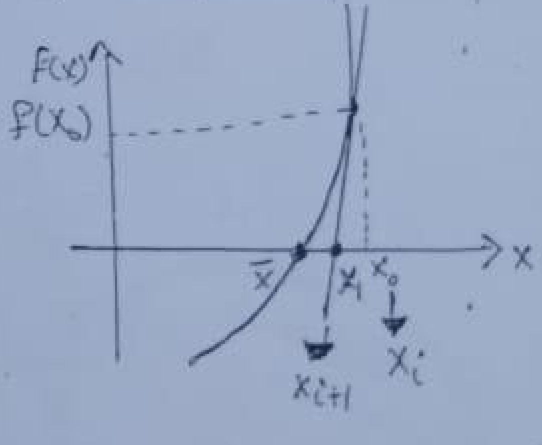
---

3. Newton - Raphson Method

* $f(x)$ is a function , and $\bar{x}$ is a root of this function
$\{ f(\bar{x}) = 0 \}$

* (x_0) is an initial guess.

* The algorithm of this method :



1. Take an initial guess (x_0)
2. Calculate $f(x_0)$ and $f'(x_0)$
3. Calculate the intersection (x_1)
4. Put $x_0 = x_1$ and calculate the new intersection x_2 by The same procedure.
5. Repeat The process to get x_3 , x_4, ---- untill reaching The required accuracy.

* In general $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Ex.1 : Find the root of the function $f(x)=e^{x}-3x$, in the interval $[0,1]$. Correct to $E_{abs}<0.005$ use Newton - Raphson method with $x_0=0$

Sol. :

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

$f(x_i)=e^{x_i}-3x_i$, $f'(x)=e^{x}-3$

$$x_{i+1}=x_i-\frac{e^{x_i}-3x_i}{e^{x_i}-3}$$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	E_{abs}
0	0	1	-2	0.5	—
1	0.5	0.1487	-1.3512	0.61	0.11
2	0.61	0.0164	-1.1595	0.6156	0.0056
3	0.6156	3.97×10^{-3}	-1.1492	0.619	0.0034

∴ The root $\bar{x}\simeq 0.619$

Ex.2 : Find the root of the eq. : $(x-2)^2=x+54$ by N-R method correct to 2D . Use $x_0=8$

Sol. :

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

$f(x_i)=x_i^2-5x_i-50$

$f'(x_i)=2x_i-5$

x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
8	-26	11	10.3636
10.3636	5.5862	15.7272	10.0084
10.0084	0.1260	15.0168	10.0000

∴ $\bar{x}=10.0000$

2. The False Position Method [Chard Method]

* This method is similar to the bisection method. It requires two initial guesses x_L & x_R.

* Algorithm steps :

1. choose an interval $[x_L, x_R]$ such that $f(x_L) * f(x_R) < 0$.
2. Find (x_i) {instantaneous root} :

$$x_i = \frac{x_L f(x_R) - x_R f(x_L)}{f(x_R) - f(x_L)}$$

3. Calculate $f(x_i)$ {using x_i - value}.
4. If $f(x_L) * f(x_i) < 0 \Rightarrow x_R = x_i$ & $f(x_R) = f(x_i)$

   $> 0 \Rightarrow x_L = x_i$ & $f(x_L) = f(x_i)$
5. Repeat the above procedure starting from step (2) to calculate a new (x_i) --- and so on.
6. Terminate The calculations when the given accuracy condition is satisfied.

Ex. :

Solve using Chard Method for :

$f(x) = e^x - 3x$ in the interval $[1, 2]$. Correct to $|f(x)| \leq 0.01$.

Sol :

let $x_L = 1$ & $x_R = 2$

- $f(x_L) = e^1 - 3(1) = 2.71 - 3 = -0.28 < 0$
- $f(x_R) = e^2 - 3(2) = 1.389 > 0$

$\therefore f(x_L) * f(x_R) < 0 \Rightarrow$ there is a root in the interval $[1, 2]$.

3. The Reciprocal of Any Number :-

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i}\right)^{-n}}{\left(-\frac{1}{x_i^2}\right)} \Rightarrow x_{i+1} = x_i\,(2 - n\,x_i)$$

for $i = 0, 1, 2, \ldots$

Ex.:- Find the reciprocal of 2, using N-R method, starting with ($x_1 = 0.1$) work to (4D)?

SoL:-

i	x_i	x_{i+1}
1	0.1000	0.1800
2	0.1800	0.2952
3	0.2952	0.4161
4	0.4161	0.4852
5	0.4852	0.4995
6	0.4995	0.4999
7	0.4999	0.4999

$\therefore \bar{x} = 0.4999$

Ex.:-

Find the reciprocal of 4, using The N-R method starting with ($x_0 = 0.2$) work to 3D.

SoL.:- $x_{i+1} = x_i\,(2 - n\,x_i)$

$\because n = 4 \Rightarrow x_{i+1} = x_i\,(2 - 4x_i)$

i	x_i	x_{i+1}
0	0.2	0.24
1	0.24	0.25
2	0.25	0.25

$\therefore \bar{x} = 0.25$

Special Cases For Newton-Raphson Method :-

1. square roots :-

$$X_{i+1} = \frac{1}{2}\left(X_i + \frac{n}{X_i}\right)$$

6

EX :- Find The square root of (10), using Newton-Raph method. starting with (3) as an initial Value.

i	X_i	X_{i+1}
1	3.0000	3.1667
2	3.1667	3.1623
3	3.1623	3.1623

$\therefore \bar{X} = 3.1623$

---

2. Roots of An arbitrary order :-

$$X_{i+1} = \left(1 - \frac{1}{K}\right) X_i + \frac{n}{K} X_i^{(1-K)} \quad , \text{ for } K = 2, 3, \ldots \quad i = 0, 1, 2, \ldots$$

EX :- Compute $\sqrt[3]{7}$, using N-R method, starting from $(X_0 = 1.5)$ take an accuracy 5D places.

i	X_i	X_{i+1}
1	1.50000	2.03704
2	2.03704	1.92034
3	1.92034	1.91296
4	1.91296	1.91293
5	1.91293	1.91293

$\therefore \bar{X} \simeq 1.91293$

Finding Roots with Fixed point Iteration ④ ⑥

- Given $f(x)=0$ write x in terms of $x = ----$
- label left side as x_{n+1} and right side with x_n
- pick x_1 and plug into equation
- Repeat until Converges

Ex: Find where $x^2 - x - 1 = 0$, $x_0 = 1$

$$x^2 - x - 1 = 0$$

①

$x^2 = x + 1$

$x = 1 + \frac{1}{x}$

$x_{n+1} = 1 + \frac{1}{x_n}$

②

$x^2 - x = 1$

$x(x-1) = 1$

$x = \frac{1}{x-1}$

$x_{n+1} = \frac{1}{x_n - 1}$

③

$x^2 = x + 1$

$x = \mp\sqrt{x+1}$

Two an~~[illegible]~~ converge $\times \leftarrow$ $x_{n+1} = \mp\sqrt{x_n + 1}$

① $x_{n+1} = 1 + \frac{1}{x_n}$ $n = 0, 1, 2, ---$

Pick $x_1 = 2$

$x_2 = 1 + \frac{1}{2} = 1.5$

$x_3 = 1 + \frac{1}{1.5} = 1.6666$

$x_4 = 1 + \frac{1}{1.1666} = 1.6$

$x_5 = 1 + \frac{1}{1.6} = 1.625$

$x_6 = 1 + \frac{1}{1.625} = 1.612533462$

Converging to 1.613---

② $x_{n+1} = \frac{1}{x_n - 1}$

Pick $x_1 = 1.6$

$x_2 = \frac{1}{1.6-1} = 1.6666$

$x_3 = \frac{1}{1.6666-1} = 1.5$

$x_4 = \frac{1}{1.5-1} = 2$

$x_5 = \frac{1}{2-1} = 1$

not Converging

* when does it Converge?

$x_{n+1} = g(x_n)$

if $|g'(\text{root})| < 1$ $\longrightarrow$ Converges

secant method : ⑤

Ex// $x^3 = 20$; $x_0 = 4$, $x_1 = 5.5$ Find the estimate after 2 iteration.

Sol// $x^3 = 20$

$f(x) = x^3 - 20 = 0$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

i=1

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 5.5 - \frac{[(5.5)^3 - 20][5.5 - 4]}{[(5.5)^3 - 20] - [4^3 - 20]} = 3.353$$

$$|\epsilon_a| = \left|\frac{x_2 - x_1}{x_2}\right| * 100$$

$$= \left|\frac{3.353 - 5.5}{3.353}\right| * 100 = 63.92 !$$

i=2

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$x_1 = 5.5$, $x_2 = 3.353$

$$x_3 = 3.353 - \frac{[(3.353)^3 - 20][3.353 - 5.5]}{[(3.353)^3 - 20] - [(5.5)^3 - 20]} = 3.059$$

$$|\epsilon_a| = \left|\frac{x_3 - x_2}{x_3}\right| * 100$$

$$= \left|\frac{3.059 - 3.353}{3.059}\right| * 100$$

$= 9.691\%$

Convergence of

① $x_{n+1} = 1 + \frac{1}{x_n}$

$g(x) = 1 + \frac{1}{x}$

$g'(x) = -\frac{1}{x^2}$

$g'\left(\underbrace{\frac{1+\sqrt{5}}{2}}_{1.618\ldots}\right) = -\frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)^2}$

$= -0.3819660112501$

$|-0.3819660112501| < 1$

Converges

Convergence of

② $x_{n+1} = \frac{1}{x_n - 1}$

$g(x) = \frac{1}{x-1}$

$g'(x) = -\frac{1}{(x-1)^2}$

$g'\left(\frac{1+\sqrt{5}}{2}\right) = -\frac{1}{\left(\left(\frac{1+\sqrt{5}}{2}\right)-1\right)^2}$

$= -2.6180339887499$

$|-2.6180339887499| \geq 1$

Doesnt Converge

* about the order

- The order of fixed point iteration depends on $f(x)$
- Remember that $x_{n+1} = g(x_n)$
- if $|g'(r)| < 1 \rightarrow$ Converges ~~quadraticly (at least)~~
- if $g'(r) = 0 \rightarrow$ Converges quadraticly (at least)
- And if $g''(r) = 0 \rightarrow$ Converges order 3 (at least)
- And So on

Hw: using the point iteration metted Find the root of $x^2 - 3x + 1 = 0$, $x_0 = 1$

* order of secant method

$$\alpha = \frac{\ln\left(\frac{e_{n+1}}{e_n}\right)}{\ln\left(\frac{e_n}{e_{n-1}}\right)}$$

مهم جداً

for example; Find order

$e_1 = 2.09 \times 10^{-4}$

$e_2 = 2.146 \times 10^{-6}$

$e_3 = 1.35 \times 10^{-9}$

$$\alpha = \frac{\ln\left(\frac{e_3}{e_2}\right)}{\ln\left(\frac{e_2}{e_1}\right)} \simeq \underline{\underline{1.61}}$$

---

⑥ Aiken method :

تركز

* The procedure:

1. Give inital values for X_1, X_2, and X_3
2. Substitute in eqs. (1'), (2'), and (3') using old and new values (if possible) for the unknowns.
3. Repeat The procedure to get values of X_1, X_2, and X_3, untill some conditions are reached.

* Convergence Condition

The solution is Convergent when:

$$\boxed{|a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{N} |a_{kj}| \quad , \quad k = 1, 2, \ldots N}$$

* In iteration (i+1), The unknown $X_k^{(i+1)}$ is Calculated by:

$$X_k^{(i+1)} = \frac{C_k}{a_{kk}} - \sum_{j=1}^{k-1} \frac{a_{kj}}{a_{kk}} X_j^{(i+1)} - \sum_{j=k+1}^{N} \frac{a_{kj}}{a_{kk}} X_j^{(i)}$$

$k = 1, 2, \ldots N$

Ex.: Solve The following system of Linear eqs. Using Gauss-Seidel method, Correct to absolute error of 0.1.

$$8X_1 + X_2 - X_3 = 8$$
$$2X_1 + X_2 + 9X_3 = 12$$
$$X_1 - 7X_2 + 2X_3 = -4$$

Sol.:

Rearranging The eqs. for Convergence:

$$8X_1 + X_2 - X_3 = 8 \quad \text{---} ①$$
$$X_1 - 7X_2 + 2X_3 = -4 \quad \text{---} ②$$
$$2X_1 + X_2 + 9X_3 = 12 \quad \text{---} ③$$

٥١/
الخزيمي

CHapter -3- 1

ل العددي
ي حاسبات
د. سهاد فهد
٢٠١٧ - ٢٠١٦

Solving Systems of Linear Equations

* In a system of Linear eqs. we have :

No. of eqs. = No. of unknowns.

* For example, a system of three Linear eqs. :

$$6x + 4y - 2z = 20$$
$$x - 10y - 7z = 15$$
$$-x + 30y + z = -1$$

* In general, we may write :

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = C_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = C_2$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = C_3$$

A- Iterative Methode الطرق التكرارية

① The Gauss - Seidel Method

* If we have three Linear eqs. (N=3) for example :

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = C_1 \quad --- ①$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = C_2 \quad --- ②$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = C_3 \quad --- ③$$

* we solve each eqn. for one of the variables :

from ① : $x_1^{(i+1)} = \frac{C_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(i)} - \frac{a_{13}}{a_{11}} x_3^{(i)} \quad --- (1')$

from ② : $x_2^{(i+1)} = \frac{C_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(i+1)} - \frac{a_{23}}{a_{22}} x_3^{(i)} \quad --- (2')$

from ③ : $x_3^{(i+1)} = \frac{C_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(i+1)} - \frac{a_{32}}{a_{33}} x_2^{(i+1)} \quad --- (3')$

Definitions:-

* principle diagonal elements of a square matrix [A] of order (n) are:

$a_{11}, a_{22}, a_{33}, ---, a_{nn}$

* Diagonal Matrix: [D]

$$[D] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Leftarrow \{n=3\}$$

* Unit matrix: $[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
* if [B] = [A], Then: $b_{ij} = a_{ij}$
* Augmented Matrix {n=3, for example}

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & C_1 \\ a_{21} & a_{22} & a_{23} & C_2 \\ a_{31} & a_{32} & a_{33} & C_3 \end{array}\right] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13}, & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$$

↑
(3x4) matrix
nx(n+1) matrix.

B. Direct Methods

1. Gaussian Elimination:

a. Forward substitution:

$$\left.\begin{aligned} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 &= C_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 &= C_2 \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 &= C_3 \end{aligned}\right\} \Rightarrow \begin{aligned} &a_{11}X_1 = C_1 \text{ ---} ① \\ &a_{21}X_1 + a_{22}X_2 = C_2 \text{ ---} ② \\ &a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = C_3 \text{ ---} \end{aligned}$$

Forward substitution:-

From ① ⇒ Find (X_1)

From ② ⇒ Find (X_2) {using (X_1)}

From ③ ⇒ Find (X_3) {using (X_1) & (X_2)}

from ① : $X_1 = 1 - 0.125\,X_2 + 0.125\,X_3$

from ② : $X_2 = 0.571 + 0.143\,X_1 + 0.286\,X_3$

from ③ : $X_3 = 1.333 - 0.222\,X_1 - 0.111\,X_2$

Initial values : $X_1 = X_2 = X_3 = 0$

i	X_1	X_2	X_3	E_1	E_2	E_3
0	0	0	0	—	—	—
1	1	0.714	1.032	1	0.714	1.032
2	1.04	1.015	0.990	0.04	0.3	0.042
3	0.997	0.997	1.001	0.043	0.18	0.11
4	1.001	1.000	1	0.004	0.003	0.001

∴ The solutions are $X_1 = 1.001$, $X_2 = 1.000$, $X_3 = 1.000$. { Note : True solutions : $X_1 = X_2 = X_3 =$

Some Matrix Notations :

* if we have the following system of linear eqs.

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = C_1$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = C_2$$
$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = C_3$$

* The above system may be written in a matrix form :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

or,

$$\underset{\substack{\nearrow \\ \text{coefficient} \\ \text{matrix}}}{[A]}\;\underset{\substack{\uparrow \\ \text{unknowns} \\ \text{vector}}}{[x]} = [c]$$

* a_{ij} : Element of coefficient matrix.

$$\begin{array}{c}\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\end{array}\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

* Now The set of eqs. is :

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

or,

$$a_{11} X_1 = C_1$$
$$a_{21} X_1 + a_{22} X_2 = C_2$$
$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = C_3$$

* We Can Solve for X_1, X_2, and X_3 by forward substitution.

B. Backward Substitution :

$$\left.\begin{array}{l} a_{11} X_1 + a_{12} X_2 + a_{13} X_3 = C_1 \\ a_{21} X_1 + a_{22} X_2 + a_{23} X_3 = C_2 \\ a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = C_3 \end{array}\right\} \Rightarrow \begin{array}{r} a_{11} X_1 + a_{12} X_2 + a_{13} X_3 = C_1 \text{ --- ①} \\ a_{22} X_2 + a_{23} X_3 = C_2 \text{ -- ②} \\ a_{33} X_3 = C_3 \text{ --- ③} \end{array}$$

From ③ ⇒ Find (X_3)
From ② ⇒ Find (X_2) $\{$using $(X_3)\}$
From ① ⇒ Find (X_1) $\{$using (X_3) & $(X_2)\}$

$$* \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

* Elimination Procedure

* Augmented matrix :

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & C_1 \\ a_{21} & a_{22} & a_{23} & C_2 \\ a_{31} & a_{32} & a_{33} & C_3 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

* To eliminate a_{21} and a_{31}: pivot row is (R_1) and pivot element is (a_{11}):

* a_{21} elimination:
New $R_2 = R_2 - R_1\left(\frac{a_{21}}{a_{11}}\right)$

* a_{31} elimination:
New $R_3 = R_3 - R_1\left(\frac{a_{31}}{a_{11}}\right)$

* we get:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ 0 & a_{22} & a_{23} & c_2 \\ 0 & a_{32} & a_{33} & c_3 \end{array}\right]$$

* To eliminate a_{32}: pivot row is R_2, and pivot element is a_{22}:

New $R_3 = R_3 - R_2\left(\frac{a_{32}}{a_{22}}\right)$, we get:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ 0 & a_{22} & a_{23} & c_2 \\ 0 & 0 & a_{33} & c_3 \end{array}\right]$$

* Now, The eqs. system is:

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 &= C_1 \\ a_{22}X_2 + a_{23}X_3 &= C_2 \\ a_{33}X_3 &= C_3 \end{aligned}$$

* we can solve for X_3, X_2, and X_1 by backward substitution.

Ex.: Use backward Gaussian elimination to solve the following system of Linear equations.

$$\begin{aligned} 100X_1 + 80X_2 - 40X_3 &= 8 \\ 200X_1 - 40X_2 + 20X_3 &= 6 \\ 300X_1 + 340X_2 - 100X_3 &= -6 \end{aligned}$$

or, we convert the Coefficient matrix into a triangular form (leaving The lower elements):

$$\overset{U\text{-elements}}{\underset{L\text{-elements}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}} \Rightarrow \underset{\text{Triangular Matrix}}{\begin{bmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{bmatrix}}$$

* Elimination Procedure:

* $a_{11} X_1 + a_{12} X_2 + a_{13} X_3 = C_1$

$a_{21} X_1 + a_{22} X_2 + a_{23} X_3 = C_2$

$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 = C_3$

* we write The augmented Matrix:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & C_1 \\ a_{21} & a_{22} & a_{23} & C_2 \\ a_{31} & a_{32} & a_{33} & C_3 \end{array}\right] \begin{array}{ll} \text{Row 1} & (R_1) \\ \text{Row 2} & (R_2) \\ \text{Row 3} & (R_3) \end{array}$$

* To eliminate a_{13}: pivot row is (R_3), and pivot element is a_{33},

New $R_1 = R_1 - R_3 \left(\frac{a_{13}}{a_{33}}\right)$, we get:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & 0 & C_1 \\ a_{21} & a_{22} & a_{23} & C_2 \\ a_{31} & a_{32} & a_{33} & C_3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

* To eliminate a_{23}: pivot row is R_3, and pivot element is a_{33},

New $R_2 = R_2 - R_3 \left(\frac{a_{23}}{a_{33}}\right)$, we get:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & 0 & C_1 \\ a_{21} & a_{22} & 0 & C_2 \\ a_{31} & a_{32} & a_{33} & C_3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

* To eliminate a_{12}: Pivot Row is R_2, and Pivot element is a_{22}

New $R_1 = R_1 - R_2 \left(\frac{a_{12}}{a_{22}}\right)$, we get:

2. Gauss - Jordan Elimination

* In This method, The augmented matrix is Converted as shown:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1' \\ 0 & 1 & 0 & c_2' \\ 0 & 0 & 1 & c_3' \end{array}\right]$$

In This Case, we have:

$1 * x_1 = c_1' \Rightarrow x_1 = c_1'$

$1 * x_2 = c_2' \Rightarrow x_2 = c_2'$

$1 * x_3 = c_3' \Rightarrow x_3 = c_3'$

* Elimination procedure:

* The augmented matrix:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

* To eliminate a_{21} and a_{31}:
First divide R_1 by a_{11}: New $R_1 = R_1 / a_{11}$
we get:

$$\left[\begin{array}{ccc|c} 1 & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

* New $R_2 = R_2 - R_1 * a_{21}$
New $R_3 = R_3 - R_1 * a_{31}$
we get:

$$\left[\begin{array}{ccc|c} 1 & a_{12} & a_{13} & c_1 \\ 0 & a_{22} & a_{23} & c_2 \\ 0 & a_{32} & a_{33} & c_3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

* To eliminate a_{12} & a_{32}

First divide R_2 by a_{22}: New $R_2 = R_2 / a_{22}$
we get:

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & a_{12} & a_{13} & C_1 \\ 0 & 1 & a_{23} & C_2 \\ 0 & a_{32} & a_{33} & C_3 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

* New $R_1 = R_1 - R_2 * a_{12}$
New $R_3 = R_3 - R_2 * a_{32}$
we get:

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 0 & a_{13} & C_1 \\ 0 & 1 & a_{23} & C_2 \\ 0 & 0 & a_{33} & C_3 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

* To eliminate a_{13} & a_{23}:
New $R_3 = R_3 / a_{33}$, we get:

$$\begin{array}{cc} \left[\begin{array}{ccc|c} 1 & 0 & a_{13} & C_1 \\ 0 & 1 & a_{23} & C_2 \\ 0 & 0 & 1 & C_3 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

* New $R_1 = R_1 - R_3 * a_{13}$
New $R_2 = R_2 - R_3 * a_{23}$

* We get:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & C_1 \\ 0 & 1 & 0 & C_2 \\ 0 & 0 & 1 & C_3 \end{array}\right]$$

* The solution are: $X_1 = C_1$, $X_2 = C_2$, $X_3 = C_3$

Sol.:

Augmented matrix is :

$$\left[\begin{array}{ccc|c} 100 & 80 & -40 & 8 \\ 200 & -40 & 20 & 6 \\ 300 & 340 & -100 & -6 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_2 - R_1\left(\frac{200}{100}\right) \\ R_3 = R_3 - R_1\left(\frac{300}{100}\right) \end{array}$$

$$\left[\begin{array}{ccc|c} 100 & 80 & -40 & 8 \\ 0 & -200 & 100 & -10 \\ 0 & 100 & 20 & -30 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 = R_3 - R_2\left(\frac{100}{-200}\right) \end{array}$$

$$\left[\begin{array}{ccc|c} 100 & 80 & -40 & 8 \\ 0 & -200 & 100 & -10 \\ 0 & 0 & 70 & -35 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

The eqs. system is:

$$100 X_1 + 80 X_2 - 40 X_3 = 8 \quad \text{---} ①$$

$$-200 X_2 + 100 X_3 = -10 \quad \text{---} ②$$

$$70 X_3 = -35 \quad \text{---} ③$$

* Using backward substitution:

From ③: $X_3 = -35/70 \Rightarrow \boxed{X_3 = -0.5}$

From ②: $-200 X_2 = -10 - 100 X_3$

$$X_2 = \frac{-10 - 100 X_3}{-200} \Rightarrow X_2 = \frac{10 + 100(-0.5)}{200}$$

$$\boxed{X_2 = -0.2}$$

From ①: $100 X_1 = 8 - 80 X_2 + 40 X_3$

$$X_1 = \frac{8 - 80(-0.2) + 40(-0.5)}{100} \Rightarrow \boxed{X_1 = 0.04}$$

* The solution of the eqs. system are :

$X_1 = 0.04$, $X_2 = -0.2$, $X_3 = -0.5$

Ex.:= Find Eigen Value and Eigen Vector for the matrix [A]

$$A=\underset{\text{regular matrix}}{\begin{bmatrix}13&5\\2&4\end{bmatrix}}, \quad I=\underset{\text{identity matrix}}{\begin{bmatrix}1&0\\0&1\end{bmatrix}}, \quad \lambda=\underset{\text{scalar matrix}}{\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}}$$

$$(A-\lambda I)=\begin{bmatrix}13&5\\2&4\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}$$

$$=\underset{\text{singular matrix}}{\begin{bmatrix}(13-\lambda)&5\\2&4-\lambda\end{bmatrix}} \Rightarrow |A-\lambda I|=(13-\lambda)(4-\lambda)-10=0$$

$$=52-13\lambda-4\lambda+\lambda^2-10=0$$

$$=42-17\lambda+\lambda^2=0$$

Finally, solving the quadratic yields the eigen values

$\lambda_1=3, \quad \lambda_2=14$

<u>for $\lambda_1=3$</u>

$$(A-\lambda_1 I)=\begin{bmatrix}(13-3)&5\\2&(4-3)\end{bmatrix}=\begin{bmatrix}10&5\\2&1\end{bmatrix}$$

$$(A-\lambda_1 I)X_1=0 \quad \begin{bmatrix}10&5\\2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{matrix}10x_1+5x_2=0\\2x_1+x_2=0\end{matrix}$$

x_1	1	2	3	---
x_2	-2	-4	-6	---

$\Rightarrow X_1=\begin{bmatrix}2\\-4\end{bmatrix}$ This is an eigen vector for $\lambda_1=3$

<u>for $\lambda_2=14$</u>

$$(A-\lambda_2 I)=\begin{bmatrix}(13-14)&5\\2&(4-14)\end{bmatrix}=\begin{bmatrix}-1&5\\2&-10\end{bmatrix}$$

$$(A-\lambda_2 I)X_2=0 \quad \begin{bmatrix}-1&5\\2&-10\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{matrix}-x_1+5x_2=0\\2x_1-10x_2=0\end{matrix}$$

x_1	5	10	15	---
x_2	1	2	3	---

$\Rightarrow X_2=\begin{bmatrix}10\\2\end{bmatrix}$ This is an eigen vector for $\lambda_2=14$

Eigen Values and Eigen Vectors

- A variety of practical problems having to do with mechanical vibration lead to linear algebric system of the type.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \lambda x_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \lambda x_n$$

- In matrix form $[A][X] = \lambda [X]$
- By transfering the terms on the right side we obtain.

$$(a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n = 0$$

- In matrix form

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

- The value of $\underline{\underline{\lambda}}$ is called an eigen value or characteristi value of the matrix $[A]$ for $[X] \neq 0$
- The corresponding solution $[x] \neq 0$ are eigen vectors.

Ex.: Solve the following eqs. using Gauss-Jordan elimination method.

$$3x_1 - 6x_2 + 7x_3 = 3$$
$$9x_1 \qquad - 5x_3 = 3$$
$$5x_1 - 8x_2 + 6x_3 = -4$$

Sol.: The augmented matrix is:

$$\left[\begin{array}{ccc|c} 3 & -6 & 7 & 3\\ 9 & 0 & -5 & 3\\ 5 & -8 & 6 & -4 \end{array}\right]\begin{array}{l} R_1=R_1/3\\ R_2\\ R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2.333 & 1\\ 9 & 0 & -5 & 3\\ 5 & -8 & 6 & -4 \end{array}\right]\begin{array}{l} R_1\\ R_2=R_2-9R\\ R_3=R_3-51 \end{array}$$

$$\left[\begin{array}{ccc} 1 & -2 & 2.333\\ 0 & 18 & -25.997\\ 0 & 2 & -5.665 \end{array}\right]\begin{array}{l} R_1\\ R_2=R_2/18\\ R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2.333 & 1\\ 0 & 1 & -1.444 & -0.333\\ 0 & 2 & -5.665 & -9 \end{array}\right] R_2$$

$$R_1 = R_1 - (-2)R_2$$
$$R_3 = R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.555 & 0.334\\ 0 & 1 & -1.444 & -0.333\\ 0 & 0 & -2.777 & -8.334 \end{array}\right]\begin{array}{l} R_1\\ R_2\\ R_3=R_3/-2.777 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.555 & 0.334\\ 0 & 1 & -1.444 & -0.333\\ 0 & 0 & 1 & 3.001 \end{array}\right]\begin{array}{l} R_1=R_1-(-0.555)R_3\\ R_2=R_2-(-1.444)R_3\\ R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.999\\ 0 & 1 & 0 & 4\\ 0 & 0 & 1 & 3.001 \end{array}\right]$$

∴ The solution are:

$x_1 = 1.999$, $x_2 = 4$, $x_3 = 3.001$

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1

Interpolation (الاستكمال)

* تستخدم هذه الطريقة مع البيانات المجدولة، على سبيل المثال :

x	1	2	3	4	----
y	0.3	0.5	1	3	----

حيث لا تتوفر صيغة رياضية للدالة $\{y = f(x)\}$

* اذا ما اردنا استخراج قيمة y التي تقابل $x = 2.5$ مثلاً، فاننا نلجأ الى طريقة الاستكمال (interpolation)

* y : المتغير المعتمد dependent variable
x : المتغير المستقل Independent variable

* في الاستكمال يتم استخدام البيانات المتوفرة لصياغة متعددة حدود (Polynomial) $[P_n(x)]$ والتي تمثل الدالة $y = f(x)$ بصيغة متعددة حدود.

* عدد الحدود في المتسلسلة يعتمد على عدد نقاط البيانات (x, y) المستخدمة

* في حالة استخدام $(n+1)$ من النقاط البيانية، فاننا نحصل على متسلسلة من الدرجة (n) $[P_n(x)]$ والتي تمر بجميع النقاط التي تكونها (نقاط الجدول) (x, y).

Lagrange Interpolation

* صيغة المتسلسلة $P_n(x)$:

* The form of the polynomial $P_n(x)$:

$$P_n(x) = \sum_{i=0}^{n} L_i(x)\, y_i \quad \text{---} ①$$

* على سبيل المثال: بالنسبة للجدول اعلاه، لدينا اربع نقاط [اي اربعة ازواج من (x, y)]، فنحصل على :

$$P_3(x) = L_0 y_0 + L_1 y_1 + L_2 y_2 + L_3 y_3$$

* لاستخدام هذه المتسلسلة [eq.①] يجب استخراج المعاملات $L_i(x)$.

$L_i(x)$: Lagrange polynomials

Eq.① : Lagrange interpolation formula

* $L_i(x)$ is given as:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \quad \text{----} ②$$

$$P_5(8) = 7 + (1.6)(4) - \frac{1}{2}(1.6)(1.6-1) + \frac{2}{6}(1.6)(0.6)(-0.4)$$
$$- \frac{1}{24}(1.6)(0.6)(-0.4)(-1.4) + \frac{0}{120}(1.6)(0.6)(-0.4) \times$$
$$(-1.4)(-2.4)$$

$$P_5(8) = 12.769$$

∴ The interpolated point (x, y) is :

$x = 8$ ، $y = 12.769$

---

2. Newton Backword Difference:

---

* نستخدم هذه الطريقة عندما تقع قيمة X المطلوب استكمالها في النصف الثاني من جدول البيانات .

* نكتب متعددة الحدود كما يلي :

$$P_n(x) = y_n + \frac{\nabla y_n}{1!} k + \frac{\nabla y_n^2}{2!} K(K+1) + \frac{\nabla y_n^3}{3!} K(K+1)(K+2) + \cdots$$

where $k = \frac{x - x_n}{h}$, ∇ : Backward Difference operator

* Difference Table :

If we have (5) points for example $(n = 4)$

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$
x_0	y_0				
x_1	y_1	∇y_1 $y_1 - y_0$			
x_2	y_2	∇y_2 $y_2 - y_1$	$\nabla^2 y_2$ $\nabla y_2 - \nabla y_1$		
x_3	y_3	∇y_3 $y_3 - y_2$	$\nabla^2 y_3$ $\nabla y_3 - \nabla y_2$	$\nabla^3 y_3$ $\nabla^2 y_3 - \nabla^2 y_2$	
x_4	$\boxed{y_4}$	∇y_4 $\boxed{y_4 - y_3}$	$\nabla^2 y_4$ $\boxed{\nabla y_4 - \nabla y_3}$	$\nabla^3 y_4$ $\boxed{\nabla^2 y_4 - \nabla^2 y_3}$	$\nabla^4 y_4$ $\boxed{\nabla^3 y_4 - \nabla^3 y_3}$

* Difference Table:

If we have (5) points for example ($n=4$):

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
x_0	$\boxed{y_0}$				
		Δy_0 $\boxed{y_1-y_0}$			
x_1	y_1		$\Delta^2 y_0$ $\boxed{\Delta y_1-\Delta y_0}$		
		Δy_1 y_2-y_1		$\Delta^3 y_0$ $\boxed{\Delta y^2_1-\Delta y^2_0}$	
x_2	y_2		$\Delta^2 y_1$ $\Delta y_2-\Delta y_1$		$\Delta^4 y_0$ $\boxed{\Delta^3 y_1-\Delta^3 y_0}$
		Δy_2 y_3-y_2		$\Delta^3 y_1$ $\Delta y^2_2-\Delta y^2_1$	
x_3	y_3		$\Delta^2 y_2$ $\Delta y_3-\Delta y_2$		
		Δy_3 y_4-y_3			
x_4	y_4				

Ex.: Use NDDIP to find (y) at $x=8$ from the following data:

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Sol.: $x=8$ is in the 1st halfe, Therefore we use The forward difference:

$$P_5(x) = y_0 + k\Delta y_0 + k(k-1)\frac{\Delta^2 y_0}{2!} + k(k-1)(k-2)\frac{\Delta^3 y_0}{3!} +$$

$$k(k-1)(k-2)(k-3)\frac{\Delta^4 y_0}{4!} + k(k-1)(k-2)(k-3)(k-4)\frac{\Delta^5 y_0}{5!}$$

$h = x_{i+1} - x_i = 5, \quad k = \frac{x - x_0}{h} \Rightarrow k = \frac{x}{5} \Rightarrow k = \frac{8}{5} = 1.6$

Difference Table:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
0	$\boxed{7}$					
		Δy_0 $\boxed{4}$				
5	11		$\Delta^2 y_0$ $\boxed{-1}$			
		3		$\Delta^3 y_0$ $\boxed{2}$		
10	14		1		$\Delta^4 y_0$ $\boxed{-1}$	
		4		1		$\Delta^5 y_0$ 0
15	18		2		-1	
		6		0		
20	24		2			
		8				
25	32					

Ex.: Find the value of y at x = 19 for the data given in the previous example:

x_i	0	5	10	15	20	25
y_i	7	11	14	18	24	32

Sol.: $n = 5$, $h = 5$, $k = \frac{x - 25}{5}$

for $x = 19 \Rightarrow k = \frac{19 - 25}{5} = -1.2$

$$P_5(x) = y_5 + \frac{\nabla y_5}{1!} k + \frac{\nabla y_5^2}{2!} k(k+1) + \frac{\nabla y_5^3}{3!} k(k+1)(k+2) + \frac{\nabla y_5^4}{!} k(k+1)(k+2)(k+3) + \frac{\nabla y_5^5}{5!} k(k+1)(k+2)(k+3)(k+4)$$

Difference Table:

x_i	y_i	∇y_i	∇y_i^2	∇y_i^3	∇y_i^4	∇y_i^5
0	7					
5	11	4				
10	14	3	-1			
15	18	4	1	2		
20	24	6	2	1	-1	
25	32	8	2	0	-1	0
	y_5	∇y_5	∇y_5^2	∇y_5^3	∇y_5^4	∇y_5^5

$$\therefore P_5(19) = 32 + 8(-1.2) + (-1.2)(-0.2) + \frac{(-1)}{24}(-1.2)(-0.2)(0.8)(1.8)$$

$$P_5(19) = 22.625$$

$\therefore$ The interpolated point (x,y) is:

x = 19 , y = 22.625

$$L_1 = \frac{x - x_0}{x_1 - x_0}$$

$$L_1 = \frac{x - 1.2}{1.3 - 1.2} = \frac{x - 1.2}{0.1} \Rightarrow L_1 = 10x - 12$$

$$\therefore P_1(x) = (-10x + 13) * (0.3849) + (10x - 12) * (0.4032)$$

$$P_1(x) = -3.849x + 5.0037 + 4.032x - 4.8384$$

$$P_1(x) = 0.183x + 0.1653$$

* This equation may be used to find the value of y at x = 1.22 :

$$y(1.22) = P_1(1.22)$$

$$= 0.183 * (1.22) + 0.1653$$

$$y(1.22) = 0.38856$$

$\therefore$ The interpolated point (x,y) is :

x = 1.22 , y = 0.38856

Ex.2 Find y(3) :

i	x	y
0	2	0.5
1	2.5	0.4
2	4	0.25

$\boxed{y(3)} \leftarrow$

$$P_n(x) = L_0 y_0 + L_1 y_1 + L_2 y_2$$

$$L_0(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) * \left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$= \left(\frac{x - 2.5}{2 - 2.5}\right) * \left(\frac{x - 4}{2 - 4}\right) = \frac{x - 2.5}{(-0.5)} * \frac{x - 4}{(-2)}$$

$$L_0(x) = \left(\frac{3 - 2.5}{-0.5}\right) * \left(\frac{3 - 4}{-2}\right) \Rightarrow \boxed{L_0(x) = -0.5}$$

$$L_1(x) = \left(\frac{x - x_0}{x_1 - x_0}\right) * \left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$= \left(\frac{x - 2}{2.5 - 2}\right) * \left(\frac{x - 4}{2.5 - 4}\right) = \left(\frac{3 - 2}{0.5}\right) * \left(\frac{3 - 4}{-1.5}\right)$$

$$\Rightarrow \boxed{L_1(x) = 1.3332}$$

* For example, if $n=2$ (Three points):

* using eq. ①

i	x	y
0	x_0	y_0
1	x_1	y_1
2	x_2	y_2

$$P_n(x)=\sum_{i=0}^{2} L_i(x)\, y_i$$

$$P_2(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2$$

* To Find $L_i(x)$, using eq. ②

① For $i=0$

$$L_i(x)=\prod_{\substack{j=0 \\ j\neq i}}^{2} \frac{x-x_j}{x_i-x_j}$$

$$L_0(x) = \left(\frac{x-x_1}{x_0-x_1}\right)*\left(\frac{x-x_2}{x_0-x_2}\right)$$

② For $i=1$

$$L_1(x) = \left(\frac{x-x_0}{x_1-x_0}\right)*\left(\frac{x-x_2}{x_1-x_2}\right)$$

③ For $i=2$

$$L_2(x) = \left(\frac{x-x_0}{x_2-x_0}\right)*\left(\frac{x-x_1}{x_2-x_1}\right)$$

Ex.1 Use Lagrange interpolation to estimate $y(1.22)$, if:
$y(1.2)=0.3849$, $y(1.3)=0.4032$

i	x	y
0	1.2	0.3849
$\boxed{1.22} \rightarrow$		
1	1.3	0.4032

SoL.: * $n=1$

$$P_1(x) = L_0(x)y_0 + L_1(x)y_1$$

$$L_0 = \frac{x-x_1}{x_0-x_1} = \frac{x-1.3}{1.2-1.3}$$

$$L_0 = \frac{x-1.3}{(-0.1)} = \frac{x}{-0.1} + \frac{1.3}{0.1}$$

$$L_0 = -10x+13$$

$$L_2(x) = \left(\frac{x - x_0}{x_2 - x_0}\right) * \left(\frac{x - x_1}{x_2 - x_1}\right)$$

$$= \left(\frac{x - 2}{4 - 2}\right) * \left(\frac{x - 2.5}{4 - 2.5}\right) \Rightarrow \left(\frac{3 - 2}{4 - 2}\right) * \left(\frac{3 - 2.5}{4 - 2.5}\right)$$

$$\boxed{L_2(x) = 0.1666}$$

$$y(3) = P_2(3) = L_0 y_0 + L_1 y_1 + L_2 y_2$$

$$= -0.5(0.5) + 1.3332(0.4) + 0.1666(0.25)$$

$$y(3) = 0.3248$$

∴ The interpolated point is :

$x = 3$, $y = 0.3248$

Newton Divided Difference Interpolation polynomial NDDIP for Equal spacing,

* نستخدم هذه الطريقه عندما تكون الفرق في القيم المتتاليه لـ x مقدار ثابت (h)

$x_{i+1} - x_i = h = \text{Constant}$

* تسمى هذه الطريقه ايضا طريقة Newton-Gregory

* هناك اسلوبان لهذه الطريقه استنادا على موقع (X) المطلوب اجراء الاستكمال عندها :

1. Newton forward difference :

* عندما تقع قيمة (X) المطلوبه في النصف الاول من جدول البيانات

* تكون متعدده الحدود $[P_n(x)]$ بالصيغه التاليه :

$$P_n(x) = y_0 + k\frac{\Delta y_0}{1!} + k(k-1)\frac{\Delta^2 y_0}{2!} + k(k-1)(k-2)\frac{\Delta^3 y_0}{3!} + \text{----}$$

where : $k = \frac{x - x_0}{h}$, Δ : Forward operator (عامل الفروقات الامامية)

* نعتبر ان لدينا $(n+1)$ نقطه

تحليل عددي
ثاني حاسبات
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الفصل الثاني
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Chapter 5-

سهاد شرف

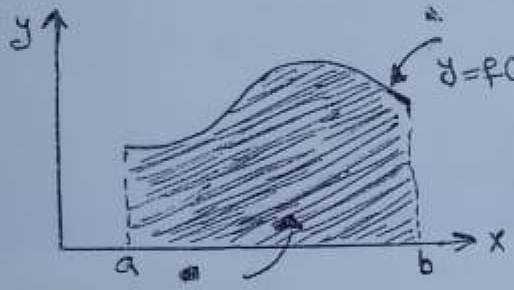
Numerical Integration

* Numerical integration is used to determine difinite integrals that can not be solved by analytical methods.

* يستخدم التكامل العددي لحساب قيم التكاملات المحدده (difinite integrals) والتي لا يمكن (او يصعب) حلها باستخدام الطرق التحليليه .

* التكامل بشكل عام يعطي المساحه تحت المنحني للداله ... اما التكامل العددي فيعطي المساحه التقريبيه تحت منحني الداله :

فاذا كانت (y) هي داله لـ (x)



المنحني : $y=f(x)$

$\int_a^b f(x)\,dx$: المساحه تحت المنحني

b,a : حدود قيم x (حدود التكامل)

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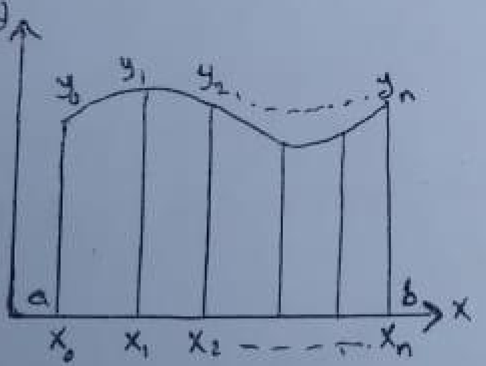
① Trapezoidal Rule (قاعده شبه المنحرف)



* المنحني يمثل الداله (y) وهي داله للمتغير (x) ، اي ان :

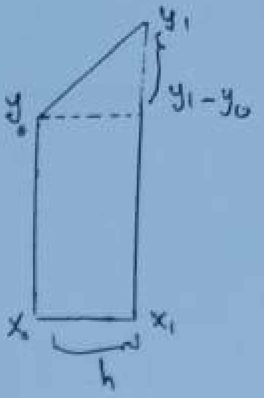
$$y \equiv y(x) \equiv f(x)$$

* اذا اردنا ان نكامل هذه الداله خلال الفتره من a الى b ، : $I=\int_a^b f(x)\,dx$

فاننا نقسم فتره التكامل {من a الى b} الى اجزاء متساويه عددها (n) وطول كل جزء (h) حيث : $h=\frac{b-a}{n}$

* في الشكل اعلاه ، تكون المساحه تحت المنحني (التكامل) عباره عن مجموعه من مساحات اشباه المنحرف ، وهذا يعطي قيمه تقريبيه للتكامل.

* كلما كان عدد اشباه المنحرف اكبر (n اكبر) كلما كان الدقه في قيمه التكامل اكبر



* قيمة التكامل للدالة هي مجموع مساحات اشباه المنحرف تحت منحني الدالة :

$$I=\frac{h}{2}\left[y_0+2(y_1+y_2+y_3+\cdots y_{n-1})+y_n\right] \quad ①$$

* قاعدة عامة : لاستخراج قيمة التكامل لدالة معينة [f(x)] أي {y(x)} فاننا نقوم بالخطوات التالية :

① تقسيم فترة التكامل الى (n) من الاجزاء واستخراج طول كل جزء h، حيث : $h=\frac{b-a}{n}$

② عمل جدول بالشكل : حيث تزيد كل قيمة لـ x عن القيمة التي تسبقها بالمقدار h، اي $x_{i+1}=x_i+h$

x_i	$y_i=f(x_i)$
x_1	y_1
x_2	y_2
x_3	y_3
⋮	⋮

③ استخدام قيم (x_i)، (y_i) التي حصلنا عليها من الجدول في المعادلة ① لاستخراج قيم التكامل

④ قيم (x_i) في الجدول تبدأ من القيمة ($x=a$) وتنتهي بالقيمة ($x=b$)، حيث (a)، (b) تمثلان حدود التكامل :

$$\int_a^b f(x)\,dx \qquad \{x_1=a \ \& \ x_n=b\}$$

Ex.1: Determine the Value of $\int_1^3 (2x^2-x+1)\,dx$, with $h=0.5$ Using The Trapezoidal Rule.

Sol.: $n=\frac{b-a}{h}=\frac{3-1}{0.5}=4$

x_i	$f(x_i)=2x_i^2-x_i+1$
1	2
1.5	4
2	7
2.5	11
3	16

$$I=\frac{h}{2}\left[y_0+2(y_1+y_2+y_3)+y_4\right]$$

$$=\frac{0.5}{2}\left[2+2(4+7+11)+16\right]$$

$$I=15.5$$

$$\therefore \int_1^3 (2x^2-x+1)\,dx=15.5$$

Ex.1: Use Euler method to approximate $y(x)$ for the differential equation: $dy/dx = -2x^3 + 12x^2 - 20x + 8.5$ from $x=0$ to $x=1.5$, with $h=0.25$, Given that $y(0)=1$

Sol.:

* القيم الابتدائية ل (x,y) هنا هي : $x_0=0$, $y_0=1$

* x_n هنا تساوي (1.5)

$x_0=0$, $y_0=1$, $h=0.25$

* باستخدام القانون :

$$y_{i+1} = y_i + h\,F(x_i, y_i)$$

i=0:

$$y_1 = y_0 + h[-2x_0^3 + 12x_0^2 - 20x_0 + 8.5]$$

$$y_1 = 1 + 0.25[8.5] \Rightarrow y_1 = 3.125$$

$$x_1 = x_0 + h \Rightarrow x_1 = 0.25$$

i=1:

$$y_2 = y_1 + h\,F(x_1, y_1)$$

$$y_2 = 3.125 + 0.25[-2(0.25)^3 + 12(0.25)^2 - 20(0.25) + 8.5]$$

$$y_2 = 4.1797, \quad x_2 = x_1 + h \Rightarrow x_2 = 0.5$$

i=2:

$$y_3 = 4.4922, \quad x_3 = x_2 + h \rightarrow x_3 = 0.75$$

i=3:

$$y_4 = 4.4922 + 0.25[-2(0.75)^3 + 12(0.75)^2 - 20(0.75) + 8.5]$$

$$y_4 = 4.3438, \quad x_4 = x_3 + h$$

$$x_4 = 1$$

i=4:

$$y_5 = 4.3438 + 0.25[-2(1)^3 + 12(1)^2 - 20(1) + 8.5]$$

$$y_5 = 4.2188, \quad x_5 = x_4 + h \Rightarrow x_5 = 1.25$$

$\underline{i=5}:$

$y_6 = 4.2188 + 0.25[-2(1.25)^3 + 12(1.25)^2 - 20(1.25) + 8.5]$

$y_6 = 3.8047$ ، $x_6 = x_5 + h \Rightarrow x_6 = 1.5$

* ملاحظة: نعرف عدد خطوات الحل مسبقاً نستفاد من القيمة الاولية والنهائية (X_0 ، X_n) للمتغير x حيث

$n = (X_n - X_0)/h \Rightarrow n = \frac{1.5 - 0}{0.25} = 6$

* كلما كانت (h) اصغر كلما كان الحل ادق.

* بالنسبة للمثال السابق، يمكن ترتيب الاجابة بصيغة جدول، وكما يلي:

i	x_i	y_i
0	0	1
1	0.25	3.125
2	0.5	4.1797
3	0.75	4.4922
4	1	4.3438
5	1.25	4.2188
6	1.5	3.8047

Ex.2 Use Euler method to approximate y(2) from the differential eq.: $dy/dx = x - y^2$, given that $y(1) = 0.75$ $h = 0.25$.

Sol.: $y_{i+1} = y_i + hF(x_i, y_i)$

$F(x_i, y_i) = x_i - y_i^2$

$\therefore y_{i+1} = y_i + h(x_i - y_i)^2$, $x_{i+1} = x_i + h$

$x_0 = 1$, $y_0 = 0.75$, $x_n = 2$

$\underline{i=0}:$ $y_1 = y_0 + h(x_0 - y_0^2)$

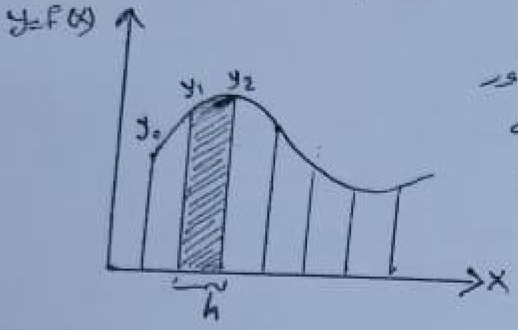
$y_1 = 0.75 + 0.25(1 - 0.75^2)$

$y_1 = 0.8593$, $x_1 = x_0 + h$

$x_1 = 1 + 0.25, \Rightarrow x_1 = 1.25$

② simpson's Rule 1/3

* تعتبر هذه الطريقة ادق من الطريقة السابقة حيث يتم هنا اعتماد عدد من المنحنيات من الدرجة الثانية بدلاً من الخط المستقيم في طريقة شبه المنحرف:



* لو اخذنا اي شريحة من الشكل المجاور فانه يمكن ملاحظة الفرق بين تقريب شبه المنحرف واسلوب المنحني الذي هو اقرب الى تمثيل الدالة:

* بصورة عامة، يعطى التكامل على عدد (n) بما يلي:

$$I = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + \cdots + \cdots 4y_{n-1} + y_n\right]$$

* في هذه الطريقة في التكامل يجب ان تكون قيمة (n) **زوجية**

Ex.2: Use Simpson's Rule to evaluate $\int_0^2 x^2 e^{-x^2} dx$, with $n=8$

sol.: $h = \frac{b-a}{n} = \frac{2-0}{8} = \frac{1}{4} = 0.25$

x_i	$f(x_i) = x_i^2 e^{-x_i^2}$	
0	0	y_0
0.25	0.0587	y_1
0.5	0.1947	y_2
0.75	0.3205	y_3
1	0.3679	y_4
1.25	0.3275	y_5
1.5	0.2371	y_6
1.75	0.1432	y_7
2	0.0733	y_8

$$\therefore I = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + y_8\right]$$

$$= \frac{0.25}{3}\left[0 + 4(0.0587) + 2(0.1974) + 4(0.3205) + 2(0.3679) \cdot 4(0.3275) + 2(0.2371) + 4(0.1432) + 0.6733\right]$$

$$I = 0.4227$$

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المحاضرة السابعة

Chapter 6

1

Numerical Solution of Ordinary Differential Equations [ODE]

* سيتم تناول المعادلات التفاضلية الاعتيادية {تحتوي على متغير مستقل واحد (X)} ومن المرتبة الاولى {اي تحتوي على المشتقة الاولى فقط}

* الصيغة العامة لهذه المعادلات: $dy/dx = F(x,y)$

* في هذه المسائل يتم اعطاء قيم اولية للمتغيرات (X), (Y) اي (X_0), (Y_0) والمطلوب هو استخراج القيم اللاحقة للدالة y عند قيم لاحقة للمتغير x اي:

x	y
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3

قيم اولية تعطى في السؤال

قيم لاحقة يتم حسابها بالاعتماد على الطريقة المتبعة.

* هنالك عدة طرق لحل هذه المعادلات التفاضلية:

① Euler Method:

$$\boxed{\begin{aligned} y_{i+1} &= y_i + hF(x_i, y_i) \\ x_{i+1} &= x_i + h \end{aligned}}$$

Euler general formula

* ملاحظات:
* لدينا دائماً نقطة اولية (نقطة بداية) $[x_0, y_0]$ تعطى في السؤال.
* السؤال يطلب استخراج قيمة y (الدالة) انطلاقاً من القيمة الاولية (y_0) عند $x = x_0$، وصولاً الى قيمة نهائية (y_n) عند $x = x_n$ [قيمة x_n تعطى في السؤال]
* التحرك من $x = x_0$ الى $x = x_n$ يتم من خلال خطوات Δx، حيث $\Delta x = h$
* في هذه الطريقة يتم حساب كل قيمة لـ y وذلك بالاستفادة من قيمة y السابقة و x السابقة.